Measurement of Applied Stresses Using the Polarization of Rayleigh Surface Waves

A Thesis
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Master of Science in
Engineering Science and Mechanics

School of Civil and Environmental Engineering
Georgia Institute of Technology
August 2003

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Acknowledgements

My special gratitude goes to Prof. Laurence Jacobs for being an excellent advisor and friend. I am very grateful for his great support and advise not only regarding my thesis but also during my whole stay at Georgia Tech. He contributed in many ways to make my year abroad a wonderful experience.

I would like to extend my gratitude to Prof. Jianmin Qu of the School of Mechanical Engineering for his valuable advise and the fruitful discussions.

My special thanks goes to Prof. Jacek Jarzynski for his great practical support and advise. It was a great experience to work together with him in the lab.

I would like to express my thanks to Kritsakorn Luangvilai for his helpful suggestions and for letting me share so much of his valuable knowledge. Furthermore, I would like to thank my colleagues in the lab, Frank Blum, Monica Hochstein and Prof. Valeria LaSaponara, for their support and for the good time we shared together.

I also would like to express my gratitude to Prof. Lothar Gaul from the University of Stuttgart for giving me the opportunity to study at Georgia Tech. I would also like to thank DAAD (German Academic Exchange Service) for providing generous financial support for my studies in the U.S.

Last but not least, I would like to deeply thank my family and Ute for their support and encouragement during this great year.

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List of Variables

α	Stroh's formalism, coefficient for linear combination
δ_{ij}	Kronecker delta
ε^i	initial strain for small deformation
ρ	mass density
θ_w	critical angle for the generation of a Rayleigh wave with a wedge
λ, μ	Lamé constants
$\lambda_l, \lambda_r, \lambda_s$	longitudinal, Rayleigh and shear wavelength
ν	Poisson ratio
ν_1, ν_2, ν_3	third-order elastic constants
Π	polarization of Rayleigh wave
ϕ	scalar potential of Helmholtz decomposition
ϕ	strain energy function
ϕ^*	potential function in Stroh's formalism
ψ	vectorial potential for Helmholtz decomposition
$\mathbf{a}$	displacement vector of a Rayleigh wave
$\mathbf{a}, \mathbf{b}$	eigenvectors in Stroh's formalism
$\mathbf{A}$	matrix containing the displacement vectors $\mathbf{a}$
$\mathbf{B}$	matrix notation of boundary conditions
c_l	longitudinal wave speed
c_s	shear wave speed
c_r	Rayleigh wave speed
C_{mnpq}	second order stiffness tensor

$C_{\alpha\beta\delta\gamma\epsilon\eta}$	third order stiffness tensor
$\hat{C}_{IJKL}$	adapted stress tensor of a prestressed body
d	unit displacement vector for plane wave
D	matrix used in Stroh's formalism solving Rayleigh wave equations
E^i, E^f, E	Lagrangian strain tensors
f	linear combination vector in Stroh's formalism
f_m	internal body forces
$G(X_2)$	matrix containing decaying function of Rayleigh wave
i	complex number
I_1, I_2, I_3	strain invariants of 1st, 2nd and 3rd order
k	wave number
k^c	proportionality factor between applied stress and rel. speed
k^p	proportionality factor between applied stress and rel. polarization
N	outer normal vector on free surface of half-space
p	unit propagation vector for plane wave
p_i	decaying parameter for plane Rayleigh wave
P	matrix containing decaying parameters p_i
q, s	decaying parameters for unstressed Rayleigh wave
Q, R, S	matrices used in Stroh's formalism
$\hat{Q}, \hat{R}, \hat{S}$	matrices used in Stroh's formalism solving Rayleigh wave equations
t	time variable
t_{JK}^i, t_{JK}^f	initial and final Cauchy stress tensor
T_{mn}	stress-tensor used in introduction
$T_{\alpha\beta}^i, T_{\alpha\beta}^f$	initial and final second Piola Kirchhoff stress tensor
$T_{\alpha\beta}^i$	initial second Piola Kirchhoff stress tensor
u	incremental displacement vector

u^i	initial displacement vector
u^f	final displacement vector
U	matrix used in singular value decomposition
v	velocity vector of a particle
V	matrix used in singular value decomposition
w_i	variable used in singular value decomposition
x, X, η	coordinate vector of a particle
z	linear combination factor of x_1 and x_2

Summary

This research develops a procedure to use the polarization of Rayleigh surface waves to measure applied stresses. An analytical model (that uses Stroh's formalism) is developed to predict the effect of applied stresses on the wave speed and on the polarization of Rayleigh surface waves. This model is then numerically simulated, and the influence of uncertainties in the third-order elastic constants on the wave speed and the polarization of a Rayleigh wave is investigated. The experimental procedure uses a laser Doppler vibrometer (LDV) to (nearly) simultaneously measure both the in-plane and out-of-plane displacement components of a Rayleigh wave, as a function of applied stress. Note that the proposed technique is a pointwise and reference free measurement technique. Further advantages of this procedure, when compared to standard time-of-flight measurements, are that the change in polarization is an order of magnitude greater than the change in wave speed, and that the proposed measurement is more robust in regard to uncertainties in the third-order elastic constants.

CHAPTER 1

Introduction

The objective of this research is to determine if it is possible to use the polarization of ultrasonic Rayleigh waves as a practical measurement technique to determine the residual or applied stresses in an engineering component. The polarization of a Rayleigh wave is defined as the ratio between the maximum in-plane, and the maximum out-of-plane displacement components.

The characterization of material properties by various types of ultrasonic waves – such as the determination of material constants, flaw detection in specimens, or the determination of applied stresses – has been extensively studied. Crecraft showed that *acoustoelasticity* can be used as a practical method with reasonable accuracy [5, 6], measuring stress-induced velocity changes of both longitudinal and shear waves. Using the analogy of the stress dependence of optical wave velocities in solids (known as the photo-elastic effect), the *acoustoelastic effect* is the dependency of the ultrasonic wave speed on stress.

The acoustoelastic effect could not be explained with standard linear elasticity theory. With the introduction of higher order terms, up to triple strain products in the definition of the strain energy function, and with the definition of the *third-order elastic constants (TOE-constants)*, the acoustoelastic effect could then be explained with a theoretical model. Murnaghan developed a nonlinear elastic theory for isotropic materials [28]. Toupin and Bernstein applied the nonlinear elastic theory to the propagation behavior of acoustic waves in a deformed, perfectly elastic

material [39].

Various other authors also measured the acoustoelastic effect for several materials, and determined the TOE-constants of the materials. Eagle and Bray [11] investigated Rail Steel, Fukuoaka and Toda [12] tested aluminum, copper and steel specimens, and Smith [35] performed measurements on various aluminum alloys, as well as molybdenum and tungsten. Hughes and Kelly [17] investigated the non-metals, polystyrene and pyrex glass.

The acoustoelastic effect on *Rayleigh waves* in a homogeneous material was investigated by Hirao et. al. [16] — they present the frequency dispersion of a Rayleigh wave for a simple bending beam. Duquennoy et. al. also used Rayleigh waves to investigate residual stresses [9, 10].

In many of the earlier work, other influences on the wave speed than applied stress are neglected. However, other factors such as temperature, micro-inhomogeneity, texture and weak anisotropy have a similarly large effect on wave speed, and should be included in a complete analysis. In recent years, the use of Rayleigh surfaces waves for the characterization of surface roughness and for the effect of shot-peening has been studied [22, 23].

The acoustoelastic effect is very small, and thus special techniques for the precise measurement of the wave speed are needed. The two main techniques are the *sing-around technique* and the *pulse-echo technique*. In the sing-around technique, one or two transducers (depending on the setup) are coupled to the specimen. The first received echo at the receiving transducer is used to re-trigger the sending transducer. The observed frequency of the re-triggering is directly related to the travel-time of the ultrasonic wave through the specimen [30]. A sensitivity of 5×10^{-5} is published for this technique. In the pulse-echo technique, the time delay is measured between any two echoes arising from one transmitted pulse. Then the sweep of the digitizing

oscilloscope is triggered at a frequency which is an integral multiple of the time-of-flight of the pulse through the specimen. If the repetition rate is exactly an integral multiple of the time-of-flight, the echo appears to be stationary on the screen. Even very small changes will then be observed by a shift of the overlapping peaks of the echoes of the pulse.

Both techniques are integral measurements, which means the influence of material properties is accumulated along the propagation path of the wave. This also implies that the properties determined by an ultrasonic wave speed measurement reflect only an average of the properties along the propagation path. Another property of both the sing-around technique and the pulse-echo technique is that they are not reference free measurements, very precise knowledge of the propagation distance is needed for both techniques.

This thesis investigates if another measurement technique can be established, one that does not have the disadvantages of being integral, and non-reference free. The proposed measurement technique measures the state of stress in a specimen using the polarization of a Rayleigh surface wave in a pointwise, reference free fashion, using a non-contact measurement. An analytical model is develop to predict the change in the polarization of a Rayleigh wave as a function of applied stresses — this model is developed using Stroh's formalism. This model is then numerically simulated and analyzed. The changes in wave speed and polarization is calculated for various materials. Additionally, the influence of uncertainties in the third-order elastic constants on the wave speed and the polarization of a Rayleigh wave is investigated.

An experimental setup is developed to measure the polarization of the Rayleigh waves by a laser Doppler Vibrometer (LDV) in order to validate the analytical model.

CHAPTER 2

Theoretical Background

This chapter provides a brief description of the theoretical background needed to understand the analytical model developed in this thesis. The mathematical formulation for wave propagation in solids (with a special emphasis on the Rayleigh surface wave) is introduced first. Then, a short review of the Stroh's formalism is given, including the method to represent the solution of two-dimensional elasticity problems in a matrix notation.

2.1 Wave-propagation in solids

2.1.1 Equations of motion

The Navier governing equations for an elastic solid are given by

$$\frac{\partial T_{mn}}{\partial x_n} + \rho f_m = \rho \frac{\partial^2 u_m}{\partial t^2} \quad (2.1)$$

where u_m denotes the displacement in the x_m direction of a cartesian coordinate system, ρ represents the material mass density and f_m are the internal body forces. The summation convention is used in the Eq. above, and is also applied throughout the rest of this thesis. In the summation notation, a sum is applied over a repeated index, e.g. $T_{ii} = \sum_{i=0}^3 T_{ii}$. Eq. (2.1) can be derived from the equations for the conservation of energy, mass and momentum in a body. Using Hooke's law for a

linear elastic material

$$T_{mn} = C_{mnpq} \frac{\partial u_p}{\partial x_q} \quad (2.2)$$

and neglecting internal body forces, the equations of motion can be written as

$$C_{mnpq} \frac{\partial^2 u_p}{\partial x_n \partial x_q} = \rho \frac{\partial^2 u_m}{\partial t^2}. \quad (2.3)$$

In an isotropic material, C_{mnpq} is

$$C_{mnpq} = \lambda \delta_{mn} \delta_{pq} + \mu (\delta_{mp} \delta_{nq} + \delta_{mq} \delta_{np}) \quad (2.4)$$

and (2.3) can be further simplified to

$$\mu \nabla^2 u_m + (\lambda + \mu) \frac{\partial}{\partial x_m} \left(\frac{\partial u_n}{\partial x_n} \right) = \rho \frac{\partial^2 u_m}{\partial t^2} \quad (2.5)$$

In the Eq. above, ∇ denotes the nabla-operator $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

2.1.2 Plane Waves in isotropic material

Propagating plane waves in an unbounded solid are a physical phenomenon that can be described mathematically. Waves of arbitrary shape $f(\cdot)$ that can be expressed in the form

$$\mathbf{u} = f(\mathbf{x} \cdot \mathbf{p} - ct) \mathbf{d} \quad (2.6)$$

satisfy the equations of motion (2.3), where $\mathbf{u}$ denotes the displacement vector at the location $\mathbf{x}$. $\mathbf{p}$ and $\mathbf{d}$ represent the unit propagation, and unit displacement vectors, respectively. Waves in which the polarization vector is parallel to the displacement vector ($p_i = d_i$) are called longitudinal waves. By plugging (2.6) into the equations of motion (2.5), the wave speed of the longitudinal wave can be determined as

$$c_l = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad (2.7)$$

For shear waves, where the polarization vector is perpendicular to the displacement vector ($p_i d_i = 0$), the propagation speed is

$$c_s = \sqrt{\frac{\mu}{\rho}} \quad (2.8)$$

Longitudinal and shear waves are the two basic waves that can propagate in an unbounded solid, but there are various other waves possible when boundaries are introduced. One of these waves is the Rayleigh surface wave that is discussed in the next section (2.2). Readers interested in more details in wave propagation in solids are referred to either [32] or [1].

2.1.3 Helmholtz decomposition

The basic idea of the Helmholtz decomposition is to separate the coupled terms of longitudinal and shear waves. For this purpose, $\mathbf{u}$ is expressed in terms of the gradient of a scalar function, ϕ , and the curl of a vector field, $\boldsymbol{\psi}$, or

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\psi} \quad (2.9)$$

Substitution of (2.9) into (2.5) leads to the decoupled Eq.

$$\nabla^2 \phi = \frac{1}{c_l^2} \frac{\partial^2 \phi}{\partial t^2} \quad \nabla^2 \boldsymbol{\psi} = \frac{1}{c_t^2} \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2}$$

2.2 Rayleigh waves

The existence of Rayleigh surface waves (also called surface waves) was first theoretically demonstrated by the English scientist, Lord Rayleigh, in 1885. Rayleigh showed the existence of a wave that can propagate along the stress free boundary between an elastic half-space and a vacuum. This wave has an amplitude that is rapidly decaying with depth. Experimental evidence was first obtained in studies on earthquakes, when a wave was observed propagating along the earth's surface. An animation of a propagating Rayleigh wave can be found in [33].

Figure 2.1: Coordinate system. The origin is on the free surface of the half-space

2.2.1 Mathematical description

The most common approach to formulate the mathematical solution of a Rayleigh wave is using the Helmholtz decomposition [42] (see section 2.1.3). It is assumed that the wave is propagating in a half-space in the positive x_1 -direction and decaying in the x_2 -direction as shown in Fig. 2.1. For the plane-strain-case, the displacement in the x_3 -direction is zero, which implies that $\psi_1 = \psi_2 = 0$, and leads to

$$\begin{aligned}\phi &= A_1 e^{-kq x_2} e^{ik(x_1 - c_r t)} \\ \psi_3 &= B_1 e^{-ks x_2} e^{ik(x_1 - c_r t)},\end{aligned}\tag{2.10}$$

where the variables are defined as

- k : the wavenumber
- c_r : phase velocity of the surface wave
- q, s : amplitude decaying parameters

In order to satisfy the equations of motion (2.5), the amplitude decaying parameters are given by

$$q = \sqrt{1 - \left(\frac{c_r}{c_t}\right)^2} \quad s = \sqrt{1 - \left(\frac{c_r}{c_s}\right)^2}$$

The surface at $x_2 = 0$ has to be stress free, which leads to the boundary condition

$$T_{12}|_{x_2=0} = T_{22}|_{x_2=0} = 0 \quad (2.11)$$

Eqs. (2.10) and (2.11) yield the characteristic Rayleigh Eq., whose roots are equal to the the wave speed (phase velocity) of the Rayleigh wave, c_r

$$\left(2 - \frac{c_r^2}{c_s^2}\right)^2 - 4\sqrt{\left(1 - \frac{c_r^2}{c_t^2}\right)}\sqrt{\left(1 - \frac{c_r^2}{c_s^2}\right)} = 0 \quad (2.12)$$

The Rayleigh wave speed is always smaller than the shear wave speed. This can be seen in Fig. 2.2, where the ratio of Rayleigh and shear wave speed is plotted, as a function of the Poisson's ratio.

2.2.2 Polarization

Following [32], the displacement of the Rayleigh waves that satisfy the boundary conditions, (2.11), are given by

$$\begin{aligned} u_1 &= A(re^{-qx_2} - 2sqe^{-sx_2})e^{ik(x_1 - c_rt)} \\ u_2 &= iAq(-re^{-qx_2} + 2sqe^{-sx_2})e^{ik(x_1 - c_rt)}, \end{aligned} \quad (2.13)$$

where $r = 2 - (c_r/c_s)^2$ and $A = kB_1/2q$. By considering the movement of a particle at one fixed location, it is obvious from (2.13) that the particle will move on an ellipse as can be seen in Fig. 2.3. (The actual displacement for a given time and a given location is obtained by taking only the real part of the results of Eq. (2.13).) The

Figure 2.2: Ratio of the Rayleigh wave speed to the shear wave speed as a function of the Poisson-ratio

axes of this ellipse indicate the polarization direction of the Rayleigh Wave. The ratio of the maximum displacements,

$$\Pi = \frac{\bar{u}_1}{\bar{u}_2}, \quad (2.14)$$

along the ellipse's axes is a measure for the polarization, Π , of the Rayleigh wave. Note that the rotation direction of a particle changes as a function of depth.

2.3 Stroh's formalism

Stroh's formalism is a mathematically elegant and technical powerful method. With this formalism, it is possible to solve two-dimensional, anisotropic material problems. Initially, Stroh's formalism was applied to solve anisotropic elasticity problems, but it was extended in [37] to steady state motion problems, which will be used later in chapter 3. This section presents a brief description of Stroh's formalism. A

Figure 2.3: Trajectory plot of a particle at a fixed location for various depths

Figure 2.4: Typical displacement values of a Rayleigh wave in the x_1 - and x_2 - direction as a function of the depth x_2 . Values are normalized with respect to u_2 at $x_1 = 0$

more detailed introduction Stroh's formalism can be found in [38]. The basic idea of Stroh's formalism is to represent the solution of a problem in a matrix-notation, starting with an initial assumed solution of the displacement and then deriving the stresses out of this assumed solution.

Given a two-dimensional problem that is represented in a fixed cartesian coordinate system (as in Fig. 2.1), the stress-displacement relation is given by (2.2) and the equations of equilibrium for a static problem is the special case of (2.3), by setting the right-hand side to zero

$$C_{mnpq} \frac{\partial^2 u_p}{\partial x_n \partial x_q} = 0 \quad (2.15)$$

For the two-dimensional problem, the displacements depend on x_1 and x_2 only. The solution of second order differential Eq. (2.15) can be represented by a composite variable z , that is a linear combination of x_1 and x_2 and is assumed to be

$$u_i = a_i f(z), \text{ with } z = x_1 + \alpha x_2, \quad (2.16)$$

where a_i and α are unknown constants to be determined. By this, the derivative of u can be written as

$$\frac{\partial u_i}{\partial x_q} = (\delta_{q1} + \alpha \delta_{q2}) a_i f'(z) \quad (2.17)$$

f' denotes the differentiation of f with respect to its argument (in this case z). Applying the differentiation twice (2.15) can be written as

$$\{C_{m1p1} + \alpha(C_{m1p2} + C_{m2p1}) + \alpha^2 C_{m2p2}\} a_p = 0 \quad (2.18)$$

or with appropriately chosen matrices

$$\underbrace{\{Q + \alpha(R + R^T) + \alpha^2 S\}}_M a = 0 \quad (2.19)$$

Eq. (2.19) has only a solution (and thus a solution for (2.15)) if the matrix M has a rank deficiency. Three pairs of conjugate-complex solutions $(\alpha_i, \alpha_{i+3}), i = 1, 2, 3$ with

$\alpha_{i+3} = \bar{\alpha}_i$ can be obtained, with the associated eigenvectors $\mathbf{a}_i$. The overall solution for the displacement can be written as a sum

$$\mathbf{u} = \sum_{i=0}^3 \{ \mathbf{a}_i f_i(z_i) + \bar{\mathbf{a}}_i f_{i+3}(\bar{z}_i) \} \quad (2.20)$$

with $z_i = x_1 + \alpha_i x_2$. So far, a solution for the displacement has been formulated. To proceed, the state of stress is related to the displacement by (2.2), which yields

$$T_{m1} = (Q_{mp} + \alpha R_{mp}) a_k f'(z) = -\alpha b_m f'(z) \quad (2.21)$$

$$T_{m2} = (R_{mp} + \alpha S_{mp}) a_k f'(z) = b_m f'(z) \quad (2.22)$$

with

$$\mathbf{b} = (\mathbf{R}^T + \alpha \mathbf{S}) \mathbf{a} = -\frac{1}{\alpha} (\mathbf{Q} + \alpha \mathbf{R}) \mathbf{a}. \quad (2.23)$$

Eq. (2.23) is satisfied because of Eq. (2.19) and Eq. (2.22). It can be shown, that there exists a potential stress function, ϕ^* , that links T_{m1} and T_{m2} . This stress function is assumed to be given by

$$\phi^* = \mathbf{b} f(z) \quad (2.24)$$

The stresses are represented in terms of this stress function

$$T_{m1} = -\frac{\partial \phi_m^*}{\partial x_2} \quad T_{m2} = \frac{\partial \phi_m^*}{\partial x_1} \quad (2.25)$$

Analogous to the sum of the overall solution of the displacement in (2.20), the overall solution for a stress function can be formulated. A even more compact notation can be derived by bundling the obtained vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ in matrices

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3] \quad (2.26)$$

By this, the overall solution can be formulated by

$$\begin{bmatrix} \mathbf{u} \\ \phi^* \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} \begin{bmatrix} f_1(z) & f_2(z) & f_3(z) & f_4(\bar{z}) & f_5(\bar{z}) & f_6(\bar{z}) \end{bmatrix}^T \quad (2.27)$$

Thus, both the solution for the displacement, as well as for the stress function are described by one large matrix and represent an efficient, compact solution for handling this problem — this is especially true if the matrix is solved using a matrix oriented computer language like Matlab.

2.4 Natural, initial and final state of a body — notation convention

There are three different states of a body that are considered in this work, and each of these states must be carefully distinguished. A body that is in a totally stress- and strain-free state is said to be in the *natural state*. No forces are applied, and no residual stresses exist within the body. In reality, this state almost never exists in a real solid material. The state of stress or strain of a material is changed due to the fabrication processes (like heating), or due to an external loading. This state is called the *initial state*. An ultrasonic wave propagating in a solid superimposes a further displacement, and causes another change of stress and strain. The resulting state is called the *final state*. In Fig. 2.5, the positions of a single particle is shown for the different states. The position of the particle in the natural, initial and final states are referred to by the vectors ξ , X and x , respectively. All vectors are used in the same Cartesian coordinate system. Following the convention proposed in [30], quantities that refer to the natural state are denoted by a Greek subscript, quantities that refer to the initial state are written with capital Roman letters as a subscript, and quantities of the final state are denoted with lower case Roman letters. The displacements that results from the change of state are called $u^i(\xi)$, $u^f(\xi)$ and $u(\xi)$.

Figure 2.5: Coordinates for a material point at the natural (ξ), initial (X) and final (x) state, following [30]

The following equations hold

$$u^i(\xi) = X - \xi \quad (2.28)$$

$$u^f(\xi) = x - \xi \quad (2.29)$$

$$u(\xi) = x - X = u^f - u^i \quad (2.30)$$

The partial derivatives of the position vectors of one state with respect to another state, are given by the following equations

$$\frac{\partial X_K}{\partial \xi_\beta} = \delta_{K\gamma} \left[\delta_{\gamma\beta} + \frac{\partial u_\gamma^i}{\partial \xi_\beta} \right] \quad (2.31)$$

$$\frac{\partial x_i}{\partial X_K} = \delta_{iM} \left[\delta_{MK} + \frac{\partial u_M^f}{\partial X_K} \right] \quad (2.32)$$

$$\frac{\partial x_i}{\partial \xi_\alpha} = \delta_{i\gamma} \left[\delta_{\alpha\gamma} + \frac{\partial u_\gamma^f}{\partial \xi_\alpha} \right] \quad (2.33)$$

The deformation around a particle can be specified by the Langrangian strain tensor. The strain in the initial and final state is [13]

$$E_{\alpha\beta}^i = \frac{1}{2} \left(\frac{\partial u_\alpha^i}{\partial \xi_\beta} + \frac{\partial u_\beta^i}{\partial \xi_\alpha} + \frac{\partial u_\lambda^i}{\partial \xi_\alpha} \frac{\partial u_\lambda^i}{\partial \xi_\beta} \right) \quad (2.34)$$

$$E_{\alpha\beta}^f = \frac{1}{2} \left(\frac{\partial u_\alpha^f}{\partial \xi_\beta} + \frac{\partial u_\beta^f}{\partial \xi_\alpha} + \frac{\partial u_\lambda^f}{\partial \xi_\alpha} \frac{\partial u_\lambda^f}{\partial \xi_\beta} \right) \quad (2.35)$$

Assuming that the displacement from the inital to the final state is very small compared to the the displacement from the natural to the initial state ($\|u_\alpha\| \ll \|u_\alpha^i\|$, and $\|E_{\alpha\beta}^f - E_{\alpha\beta}^i\| \ll \|E_{\alpha\beta}^i\|$), the strain tensor for the strain from the initial to the final state is given by

$$E_{\alpha\beta} = \frac{1}{2} \left(\frac{\partial u_\alpha}{\partial \xi_\beta} + \frac{\partial u_\beta}{\partial \xi_\alpha} + \frac{\partial u_\lambda^i}{\partial \xi_\alpha} \frac{\partial u_\lambda}{\partial \xi_\beta} + \frac{\partial u_\lambda^i}{\partial \xi_\beta} \frac{\partial u_\lambda}{\partial \xi_\alpha} \right) \quad (2.36)$$

Eq. (2.36) is obtained by rewriting the final displacement $u_\lambda^f = u_\lambda^i + u_\lambda$ and discarding all terms that are quadratic small (e.g. u^2). The mass density changes due to the deformation. By conservation of mass, it follows that

$$\rho^0 = \rho^i \left| \frac{\partial \mathbf{X}}{\partial \boldsymbol{\xi}} \right| = \rho^f \left| \frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} \right| \quad (2.37)$$

$$\rho^i = \rho^f \left| \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right| \quad (2.38)$$

2.5 Third-order elastic (TOE) constants for isotropic materials

Many problems in mechanics are solved adequately by assuming the well known Hooke's Law, which proposes a linear stress-strain relationship. However, some phenomena in wave propagation can not be adequately explained with this linear stress-strain model. As a result, a theory for higher order nonlinear effects was developed. In addition to the Lamé constants, third-order elastic constants (TOE-constants) are

introduced. Unfortunately, the nomenclature for these TOE-constants is not unique, and several different definitions exist. Murnaghan was one of the first to define the TOE-constants for an isotropic medium in [28]. Toupin and Bernstein [39] also develop TOE-constants, and their notation will be adopted in this thesis. Both definitions are similar, and there is a simple transformation from one to the other. A conversion table is given in [34] and further conversions can be found in [14]. The now following definition of the TOE-constants for anisotropic materials is a combination of the derivations given in [40, 39, 14] and is later specialized for the isotropic case. As in the linear-elastic model, a strain-energy function is assumed, that can be described by a polynomial

$$\rho_0\phi = \phi_1 + \phi_2 + \phi_3 + \text{H.O.T} \quad (2.39)$$

This strain-energy function describes the stored energy, ϕ , per unit of the deformed mass as a function of the displacement gradient $\frac{\partial u_i}{\partial x_j}$. ϕ_1, ϕ_2, ϕ_3 , are homogeneous polynomials of the first, second and third degree in the displacement gradients $\frac{\partial u_i}{\partial x_j}$. It can be shown, that $\phi_1 = 0$. For small strains, the strain-energy function can also be expressed as a function of the Lagrangian strain tensor (2.34)

$$\rho_0\phi = \frac{1}{2!}C_{\alpha\beta\gamma\delta}E_{\alpha\beta}E_{\gamma\delta} + \frac{1}{3!}C_{\alpha\beta\gamma\delta\epsilon\eta}E_{\alpha\beta}E_{\gamma\delta}E_{\epsilon\eta} + \text{H.O.T.} \quad (2.40)$$

$C_{\alpha\beta\gamma\delta}$ and $C_{\alpha\beta\gamma\delta\epsilon\eta}$ are the second order and third order elastics constants. Obviously, the components of the strain-energy function ϕ_2 and ϕ_3 can now be expressed in terms of $C_{\alpha\beta\gamma\delta}$, $C_{\alpha\beta\gamma\delta\epsilon\eta}$ and the displacement gradients which links the physical definition of the strain-energy function (to be a function of the displacement gradients) and the definition of the second- and third-order elastic constants together. Eq. (2.40) is the general case for an anisotropic material. The second order, nonlinear constitutive stress strain relation for the initial and final case is then obtained by

$$T_{\alpha\beta}^i = C_{\alpha\beta\gamma\delta}E_{\gamma\delta}^i + \frac{1}{2}C_{\alpha\beta\gamma\delta\epsilon\eta}E_{\gamma\delta}^iE_{\epsilon\eta}^i \quad (2.41)$$

$$T_{\alpha\beta}^f = C_{\alpha\beta\gamma\delta}E_{\gamma\delta}^f + \frac{1}{2}C_{\alpha\beta\gamma\delta\epsilon\eta}E_{\gamma\delta}^fE_{\epsilon\eta}^f. \quad (2.42)$$

For the more specific case of an isotropic material, the strain-energy function needs to satisfy the following Eq. [28]

$$\phi(E) = \phi(R^T E R), \quad (2.43)$$

with R being a rotation matrix. In other words, this means that the strain-energy function is invariant as a function of rotation. In order to satisfy this invariance, ϕ has to depend only on the strain invariants [40]

$$I_1 = E_{\alpha\alpha} \quad (2.44)$$

$$I_2 = \frac{1}{2}[(E_{\alpha\alpha})^2 - E_{\alpha\beta}E_{\beta\alpha}] \quad (2.45)$$

$$I_3 = \frac{1}{6}[2E_{\alpha\gamma}E_{\gamma\beta}E_{\beta\alpha} - 3E_{\gamma\gamma}E_{\alpha\beta}E_{\beta\alpha} + (E_{\alpha\alpha})^3] \quad (2.46)$$

By this, the isotropic second- and the third- order elastic constants reduce to [39]

$$C_{\alpha\beta\gamma\delta} = \lambda\delta_{\alpha\beta}\delta_{\gamma\delta} + \mu(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \quad (2.47)$$

and

$$\begin{aligned} C_{\alpha\beta\gamma\delta\epsilon\eta} = & \nu_1\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\epsilon\eta} + \nu_2[\delta_{\alpha\beta}(\delta_{\gamma\epsilon}\delta_{\delta\eta} + \delta_{\gamma\eta}\delta_{\delta\epsilon}) + \delta_{\gamma\delta}(\delta_{\alpha\epsilon}\delta_{\beta\eta} + \\ & \delta_{\beta\epsilon}\delta_{\alpha\eta}) + \delta_{\epsilon\eta}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})] \\ & + \nu_3[\delta_{\alpha\gamma}(\delta_{\beta\epsilon}\delta_{\delta\eta} + \delta_{\beta\eta}\delta_{\delta\epsilon}) + \delta_{\beta\delta}(\delta_{\alpha\epsilon}\delta_{\gamma\eta} + \delta_{\alpha\eta}\delta_{\epsilon\gamma}) + \\ & \delta_{\alpha\delta}(\delta_{\beta\epsilon}\delta_{\gamma\eta} + \delta_{\beta\eta}\delta_{\epsilon\gamma}) + \delta_{\beta\gamma}(\delta_{\alpha\epsilon}\delta_{\delta\eta} + \delta_{\alpha\eta}\delta_{\delta\epsilon})], \end{aligned} \quad (2.48)$$

where μ and λ are the well known lamé constants and ν_i , $i = 1, 2, 3$ are the TOE-constants in the notation of Toupin and Bernstein. It is worth noting that an elastic material initially in any state of stress other than a simple hydrostatic pressure or tension cannot be elastically isotropic [28]. Thus, residual stresses cause a so called "stress-induced anisotropy."

In literature the TOE-constants for several material were determined. In the appendix B tables with the numbers of TOE-constants for various materials is given.

CHAPTER 3

Analytical Model

This chapter develops an analytical model that predicts the relationship between the polarization of a Rayleigh waves and the state of stress. This relationship will then be used to measure applied stress. The first section presents the equations of motion for a prestressed body, following Pao et al. [30]. Then the main model is introduced assuming a homogeneous stress.

The following rules will be applied in the analytical development to ensure consistency:

Rule 3.1

- *Small displacements or small displacement gradients of order higher than one are discarded. This is due to the assumption made in section 2.4 that the superimposed displacements from the initial to the final state are small when compared to the displacement from the natural to the initial state.*
- *Products of small stress tensors with small displacements or displacement gradients are discarded (but not products of small stress tensors with normal displacements, or vice versa)*

These rules are consistent with the derivation of the incremental Lagrangian strain tensor, $\mathbf{E}$, in section 2.4.

3.1 Equations of motion for a prestressed body

The Cauchy stress tensor, $t^i(\mathbf{X})$, represents the state of stress at a given material point as a function of $\mathbf{X}$. The Kirchhoff stress tensor (second Piola Kirchhoff stress tensor, also called the Lagrangian stress tensor), $T^i(\boldsymbol{\xi})$, is used when expressing the state of stress (at the same material point), but instead referring to the initial configuration. Both tensors are related by

$$t_{JK}^i = \left| \frac{\partial \mathbf{X}}{\partial \boldsymbol{\xi}} \right|^{-1} \frac{\partial X_K}{\partial \xi_\beta} \frac{\partial X_J}{\partial \xi_\alpha} T_{\alpha\beta}^i \quad (3.1)$$

Note, that the notation introduced in section 2.4 is applied here. The tensor on the left hand side refers to the initial configuration, while the tensor on the right hand side refers to the natural state. In an analogous fashion, the Cauchy stress tensor, t^f , for the final state, and the Kirchhoff tensor are related by

$$t_{ij}^f = \left| \frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}} \right|^{-1} \frac{\partial x_i}{\partial \xi_\alpha} \frac{\partial x_j}{\partial \xi_\beta} T_{\alpha\beta}^f = \left| \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right|^{-1} \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_L} T_{KL}^f \quad (3.2)$$

It will become important to express the incremental stress tensor that reflects the stresses caused by the superimposed ultrasonic wave when referring to the natural or initial configuration

$$T_{JK} = T_{JK}^f - t_{JK}^i \quad (3.3)$$

$$T_{\alpha\beta} = T_{\alpha\beta}^f - T_{\alpha\beta}^i \quad (3.4)$$

Both incremental Kirchhoff tensor are related by

$$T_{JK} = \left| \frac{\partial \mathbf{X}}{\partial \boldsymbol{\xi}} \right|^{-1} \frac{\partial X_J}{\partial \xi_\alpha} \frac{\partial X_K}{\partial \xi_\beta} T_{\alpha\beta} \quad (3.5)$$

With all these preliminary definitions, the equations of motion for the final state can now be formulated in different forms. Referring to the final configuration, the equations of motion for the final state is given by [1]

$$\frac{\partial t_{ij}^f}{\partial x_j} = \rho^f \left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} \right) \quad (3.6)$$

Applying Eqs. (3.2) and (2.32) to Eq. (3.6), the equations of motion can be expressed in reference to the initial configuration as

$$\frac{\partial}{\partial X_K} \left[T_{KJ}^f + T_{KL}^f \frac{\partial u_J}{\partial X_L} \right] = \rho^i \frac{\partial^2 u_J^f}{\partial t^2} \quad (3.7)$$

So far, information from the initial state has not been considered, and only knowledge of the final state has been applied. Since the (pre-) deformation associated with the initial state is static, the following equations of equilibrium need to be satisfied

$$\frac{\partial t_{JK}^i}{\partial X_K} = 0 \quad (3.8)$$

$$\frac{\partial}{\partial \xi_\beta} \left[T_{\beta\gamma}^i \left(\delta_{\alpha\gamma} + \frac{\partial u_\alpha^i}{\partial \xi_\gamma} \right) \right] = 0 \quad (3.9)$$

Eq. (3.9) is obtained in a similar as before, by converting the Cauchy stress tensor in Eq. (3.8) to the Kirchhoff stress tensor using Eq. (3.1). Now, the conditions for the initial and the final cases can be combined. Subtracting Eq. (3.8) from Eq. (3.7) yields

$$\frac{\partial}{\partial X_J} \left[T_{IJ} + t_{JK}^i \frac{\partial u_I}{\partial X_K} \right] = \rho^i \frac{\partial^2 u_I}{\partial t^2}, \quad (3.10)$$

where the rule 3.1 is applied, and the term $T_{KL} \frac{\partial u_L}{\partial X_L}$ is neglected. Note that the argument is no longer in terms of the final displacement, $\mathbf{u}^f$ (as in Eq. (3.7)), but is written instead in terms of the incremental displacement, $\mathbf{u}$. The incremental stress tensor, $\mathbf{T}_{IJ}$, in Eq. (3.10) needs to be replaced by a stress strain relationship in order to get the final equations of motion for a prestressed body. In the article of Pao et al. [30], the incremental stress tensor is obtained from the difference of the stress tensors for the initial (Eq. (2.41)) and final states (Eq. (2.42)), or

$$T_{\alpha\beta} = C_{\alpha\beta\gamma\delta} E_{\gamma\delta} + C_{\alpha\beta\gamma\delta\epsilon\eta} \epsilon_{\gamma\delta}^i \epsilon_{\epsilon\eta}, \quad (3.11)$$

where $\epsilon_{\alpha\beta} = \frac{1}{2} \left(\frac{\partial u_\alpha}{\partial \xi_\beta} + \frac{\partial u_\beta}{\partial \xi_\alpha} \right)$. Eq. (3.11) can be expressed just in terms of displacement gradients, again retaining only the terms that are either linear in the displacement

gradient of $\mathbf{u}$ or $\mathbf{u}^i$, or

$$T_{\alpha\beta} = C_{\alpha\beta\gamma\delta} \left(\delta_{\rho\gamma} + \frac{\partial u_p^i}{\partial \xi_\gamma} \right) \frac{\partial u_p}{\partial \xi_\delta} + C_{\alpha\beta\gamma\delta\epsilon\eta} \frac{\partial u_\gamma^i}{\partial \xi_\delta} \frac{\partial u_\epsilon}{\partial \xi_\eta} \quad (3.12)$$

By transforming this constitutive equation, Eq. (3.12), to the initial state with Eq. (3.5) and by then plugging the result into the previous version of the equations of motion, Eq. (3.10), the final form of the equations of motion is obtained as

$$\frac{\partial}{\partial X_J} \left[(\delta_{IK} t_{JL}^i + \hat{C}_{IJKL}) \frac{\partial u_K}{\partial X_L} \right] = \rho^0 (1 - \epsilon_{NN}^i) \frac{\partial^2 u_I}{\partial t^2}, \quad (3.13)$$

since for small deformations, Eq. (2.37) can be approximated by

$$\rho^i \approx \rho^0 (1 - \epsilon_{NN}^i) \quad (3.14)$$

For an anisotropic material, $\hat{C}$ is a long expression that is given in Eq. (27) of [30]. For an isotropic material, $\hat{C}$ can be expressed using the second and third order elastic constants as

$$\begin{aligned} \hat{C}_{IJKL} = & \lambda \delta_{IJ} \delta_{KL} + \mu (\delta_{IK} \delta_{JL} + \delta_{IL} \delta_{JK}) \\ & + [(-\lambda + \nu_1) \delta_{IJ} \delta_{KL} + (-\mu + \nu_2) (\delta_{IK} \delta_{JL} + \delta_{IL} \delta_{JK})] \epsilon_{MM}^i \\ & + 2(\lambda + \nu_2) (\epsilon_{IJ}^i \delta_{KL} + \epsilon_{KL}^i \delta_{IJ}) \\ & + 2(\mu + \nu_3) (\epsilon_{IK}^i \delta_{JL} + \epsilon_{IL}^i \delta_{JK} + \epsilon_{JK}^i \delta_{IL} + \epsilon_{JL}^i \delta_{IK}) \end{aligned} \quad (3.15)$$

It is important to note that even though this development of the equations of motion for a prestressed body uses the nonlinear definition of Hooke's law, it was assumed that the displacements are small, and thus only terms that are linear in either the gradient of $\mathbf{u}$ or $\mathbf{u}^i$ are retained — this can be seen in the definition of $\hat{C}$. This is why the initial strain, ϵ_{KL}^i , can be calculated by an inversion of the linear Hooke's law [32].

$$\epsilon_{KL}^i = \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \delta_{KL} t_{NN}^i + \frac{1}{2\mu} t_{KL}^i \quad (3.16)$$

3.2 Rayleigh wave speed and polarization — constant stress case

Section 2.2 presents the mathematical development for Rayleigh waves in an unstressed medium. This description will now be extended to the prestressed case. The solution is presented using Stroh's formalism that was introduced in section 2.3. Since a Rayleigh wave propagates on the surface and its motion decays exponentially with depth the displacement is selected to be of the following form

$$\mathbf{u} = \mathbf{a} e^{ik(X_1 + pX_2 - c_r t)}, \quad (3.17)$$

where $\mathbf{a}$ is the displacement vector, k the wavenumber, p the decay parameter and c_r the Rayleigh wave speed. The derivatives are given by

$$\frac{\partial u_K}{\partial X_L} = ik(\delta_{L1} + p\delta_{L2})u_K \quad (3.18)$$

$$\frac{\partial u_K}{\partial t} = -ikc_r u_K \quad (3.19)$$

3.2.1 Boundary conditions

The surface is stress free in both the initial and the final states. Thus, the boundary conditions can be expressed by (see picture 3.1)

$$N_J t_{IJ}^i = 0, \text{ at } X_2 = 0 \quad (3.20)$$

$$n_j t_{ij}^f = 0, \text{ on the wavy surface} \quad (3.21)$$

It is worth noting that the first boundary condition restricts the initial stresses as: $t_{21}^i = t_{12}^i = t_{22}^i = t_{23}^i = t_{32}^i = 0$. The two boundary conditions can be combined. First, the Cauchy stress tensor, $\mathbf{t}^f$, that refers to the final state can be transformed to a Kirchhoff stress tensor in the initial state by using Eq. (3.2). The normal unit vector

Figure 3.1: Boundary conditions for a Rayleigh wave in a predeformed half-space

that refers to the final coordinates, $\mathbf{n}$, also needs to be transformed to the normal unit vector, $\mathbf{n}$, that refers to initial coordinates. Both vectors are related by

$$\mathbf{n} = \frac{dA}{da} \left| \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right| \left(\left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right]^{-1} \right)^T \mathbf{N} \quad (3.22)$$

Plugging Eqs. (3.22) and (3.2) into Eq. (3.21) yields

$$\frac{\partial x_I}{\partial X_K} (T_{KJ}^f N_J) = 0 \quad (3.23)$$

It follows from Eq. (2.31) that

$$(T_{IJ}^f + T_{KJ}^f \frac{\partial u_I}{\partial X_K}) N_J = 0, \text{ at } X_2 = 0 \quad (3.24)$$

As mentioned before, the final Kirchhoff stress tensor can be written in terms of the initial Cauchy stress tensor and the incremental Kirchhoff stress tensor: $\mathbf{T}^f = \mathbf{t}^i + \mathbf{T}$, Eq. (3.3). This relationship can be applied to the last equation. Considering the result of Eq. (3.20), and neglecting $T_{KJ} \partial u_I / \partial X_K$ gives

$$N_J T_{IJ} = N_J \hat{C}_{IJKL} \frac{\partial u_K}{\partial X_L} = 0, \text{ at } X_2 = 0 \quad (3.25)$$

By applying $N_J = [0 \quad -1 \quad 0]$ Eq. (3.21) can finally expressed as

$$T_{I2} = \hat{C}_{I2KL} \frac{\partial u_K}{\partial X_L} = 0, \text{ at } X_2 = 0 \quad (3.26)$$

3.2.2 Conditions for the Existence of a Rayleigh Wave

If the initial stress is homogeneous, the equations of motion simplify to

$$(\delta_{IK}t_{JL}^i + \hat{C}_{IJKL})\frac{\partial u_K}{\partial X_J X_L} = \rho^0(1 - \varepsilon_{NN}^i)\frac{\partial^2 u_I}{\partial t^2} \quad (3.27)$$

In reality, the assumption of a constant initial stress might only be valid for a Rayleigh wave with a large frequency, where the stress does not change over the small penetration depth of the Rayleigh wave.

Applying the equations of motion: The surface wave needs to satisfy the equations of motion. By plugging Eq. (3.17) into Eq. (3.27), it follows that

$$\left\{ (\delta_{IK}t_{22}^i + \hat{C}_{i2k2})p^2 + (\delta_{IK}t_{12}^i + \hat{C}_{I1K2} + \delta_{IK}t_{21}^i + \hat{C}_{I2K1})p + (\delta_{IK}t_{11}^i + \hat{C}_{I1K1}) - \delta_{IK}\rho_0(1 - \varepsilon_{NN}^i)c_r^2 \right\} a_K = 0 \quad (3.28)$$

Since the stress tensor t^i is assumed to be constant and t_{22}^i and t_{12}^i has to be zero at the surface due to Eq. (3.20), these stress components are zero everywhere. Eq. (3.28) can then be written in a more compact form by defining three matrices

$$\hat{S}_{IK} = \hat{C}_{I2K2} \quad \hat{R}_{IK} = \hat{C}_{I1K2} \quad \hat{Q}_{IK} = \delta_{IK}t_{11}^i + \hat{C}_{I1K1}$$

It follows

$$\underbrace{\{ p^2 \hat{S} + p(\hat{R} + \hat{R}^T) + \hat{Q} - \rho_0(1 - \varepsilon_{NN}^i)c_r^2 \mathbf{I} \}}_{D(c_r, p)} \mathbf{a} = 0 \quad (3.29)$$

and $\mathbf{I}$ is the identity matrix. For a non-trivial solution of this homogeneous equation, the matrix D must have a rank deficiency, thus

$$\|D(c_r, p)\| = 0 \quad (3.30)$$

The solution of Eq. (3.30) yields three pairs of complex conjugate roots for p (for a given c_r). It is physically reasonable to select only those roots that have displacements

that decrease, for an increasing X_2 . Thus, the roots with a negative imaginary part are discarded. The roots with positive real parts are denoted with p_n , $n = 1, 2, 3$. It is worth noting that for this problem, all values of p_i are purely imaginary [37]. The displacement vector, $\mathbf{a}_i$, for each p_i can be determined by solving for the null-space of $\mathbf{D}$. This allows the displacement to be written as a linear combination of the single solutions using the matrix notation, or

$$\mathbf{u} = \mathbf{A}\mathbf{G}(X_2)\mathbf{f}e^{ik(X_1-ct)}, \quad (3.31)$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \quad (3.32)$$

$$\mathbf{G}(X_2) = \begin{bmatrix} e^{ikp_1X_2} & 0 & 0 \\ 0 & e^{ikp_2X_2} & 0 \\ 0 & 0 & e^{ikp_3X_2} \end{bmatrix} \quad (3.33)$$

and $\mathbf{f}$ is an vector whose elements are the factors for the linear combination.

Applying the boundary conditions: $\mathbf{f}$ will be determined by applying the boundary conditions. Plugging Eq. (3.31) into the boundary conditions, Eq. (3.26), gives

$$\underbrace{(\hat{\mathbf{R}}^T \mathbf{A} + \hat{\mathbf{S}} \mathbf{A} \mathbf{P})}_{\mathbf{B}(c_r, p)} \mathbf{f} = 0 \quad (3.34)$$

with

$$\mathbf{P} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \quad (3.35)$$

The condition of a nontrivial solution to Eq. (3.34) gives

$$\|\mathbf{B}(c_r, p)\| = 0 \quad (3.36)$$

The vector $\mathbf{f}$ is then obtained by solving for the null-space of $\mathbf{B}(c_r, p)$.

3.2.3 Conclusion

A solution that follows this development yields

$$\begin{aligned}\|D(c_r, p)\| &= 0 \\ \|B(c_r, p)\| &= 0\end{aligned}$$

The speed c_r that satisfies these two equations is the waves speed of the Rayleigh wave in a prestressed body. The displacements are then determined with Eq. (3.31), and the polarization vector is given by

$$\hat{u}(X_2) = AG(X_2)f \quad (3.37)$$

The polarization is calculated using the maximum displacements in the X_1 and X_2 directions. At the surface ($X_2 = 0$), the polarization, Π , is given by

$$\Pi = \frac{(Af)_1}{(Af)_2} \quad (3.38)$$

Unfortunately, there is not an easy analytical solution for this problem. A numerical procedure to solve this problem is presented in chapter 4.

The displacement of the Rayleigh wave, given by Eq. (3.31), can be related to the displacement that is presented in Eq. (2.13) in section 2.2. For the special case when the initial stresses are zero ($t_{IJ}^i = 0$), the material has ideally isotropic properties since the third TOE -constants disappear in Eq. (3.15). It is shown in [37] that for this case, all the roots, p_i , are not only purely imaginary, but additionally two roots are the same (assume $p_1 = p_2$). Thus, Eq. (3.31) can be written as a sum of two exponential -functions of the form $2e^{ikp_1X_2}e^{ik(X_1-c_r t)}$ and $e^{ikp_3X_2}e^{ik(X_1-c_r t)}$ multiplied by a displacement vector and a scalar factor. This is actually the same form that was obtained in Eq. (2.13).

CHAPTER 4

Numerical Simulation

The numerical solution of the problem of Rayleigh waves in a prestressed medium is presented in this chapter. Only the results for the constant stress case are shown. First, an iterative algorithm that solves the relevant equations is presented, followed by an analysis of the numerical problem. Then the simulation results are presented. The chapter concludes with results from the sensitivity analysis, that investigates the influence of uncertainties in the TOE-constants on the wave speed and the polarization, plus a finite-element method (FEM) simulation that justifies the uniaxial stress assumption.

4.1 Iterative algorithm for numerical solution

An iterative algorithm to solve for the unknown surface waves speed c_r and the polarization vector in an isotropic half space under the influence of a initial constant stress field can be formulated in the following way:

1. Specify an initial guess for the Rayleigh wave speed c_{init} . It is known from section 2.2.1 that $c_r \leq c_s$. An initial guess can be obtained from the approximation for the solution of the characteristic Rayleigh equation (for the unstressed specimen) which is given by [42]

$$c_{r0} = \frac{0.87 + 1.12\nu}{1 + \nu} c_{s0} \quad (4.1)$$

The subscript index 0 for the speeds represents the unstressed case. c_{s0} is given by Eq. (2.8). This initial guess has been shown to be close enough for the final solution, since the relative changes of the wave speed are in the range of less than 0.05% /MPa.

2. Solve for the p_i that makes the determinant of $\mathbf{D}$ in Eq. (3.30) equal to zero.
3. Construct the matrix $\mathbf{P}$ according to Eq. (3.35).
4. Solve for the null-space, $\mathbf{a}_i$, of the linear Eq. (3.29) for each p_i .
5. Construct the matrix $\mathbf{A}$ according to Eq. (3.32)
6. Construct the matrix $\mathbf{B}$ for the boundary conditions according to Eq. (3.34).
7. If the determinant of $\mathbf{B}$ is not equal to zero as stated in Eq. (3.36), then
 - (a) Specify another guess for c_{init} .
 - (b) Go back to step 2.
8. The value for which the boundary conditions are satisfied is now denoted as c_r . This c_r is the Rayleigh wave speed under the influence of the initial stress, t^i .
9. Use this c_r to solve for the vector $\mathbf{f}$ in Eq. (3.34).
10. Compute the polarization vector using Eq. (3.37).

This iterative algorithm can be easily implemented in any programming language that can deal with matrices. Matlab is selected in this research since Matlab is a matrix oriented programming language that has a symbolic toolbox — this symbolic toolbox makes it possible to calculate the solution for the decaying factors p_i symbolically.

4.2 Analysis of the numerical problem

The changes in wave speed and polarization due to initial or residual stresses are small. A numerical solution will never give an exact solution, since the results are limited by the numerical accuracy of the hardware and software. The speed of convergence to a specific solution will depend on the numerical conditioning (well-conditioned or ill-conditioned) of a particular problem. In this section the numerical problem for solving Rayleigh waves in a prestressed body is analyzed.

4.2.1 $\text{Det}(\mathbf{D})$: Big numbers, small solution

In step 2 of the iterative algorithm, proposed in section 4.1, the decaying factors, p_i , are calculated. From the unstressed case, it is known that a Rayleigh wave decays within less than two wavelengths, (Fig. 2.4). As a result of this decay, the values of the decaying parameters, p_i , are expected to be smaller than 1. The p_i s are calculated by setting the determinant of $\mathbf{D}$ to zero. A closer look at the matrix $\mathbf{D}$ reveals that the values of its elements are extremely big. As described in Eq. (3.29), the elements of $\mathbf{D}$ contain the second and third order elastic constants. The values of the second and third order elastic constants are typically in the giga-pascal range. Additionally, the diagonal elements of $\mathbf{D}$ contain the initial stresses. Normally, the initial stresses are in the mega-pascal or giga-pascal range. When calculating the determinant, the elements of $\mathbf{D}$ are again multiplied with each other which yields coefficient values higher than $[\text{GPa}]^3 = [10^{27}]$. Setting the determinant to zero in Eq. (3.30), results in a polynomial equation of the sixth order in the variable p . The solution to this polynomial equation is numerically tricky, since the equation contains very large numbers, while the p_i s (the solution variable) are expected to be less than unity.

4.2.2 $\text{Det}(\mathbf{B})$: Complex numbers, undefined sign

When implementing the iterative algorithm shown in section 4.1, the main goal is to change c_r until the determinant of $\mathbf{B}$ is zero — this is a typical numerical zero-search problem. The calculation of the determinant of $\mathbf{B}$ can be encapsulated into a “black box.” The solution algorithm then calls this “black-box”, with the Rayleigh wave speed as the input, and the value of the determinant of $\mathbf{B}$ as the output. This determinant can be a complex number. For most zero-searching algorithms, complex numbers have to be treated as two variables, which creates a more complex two-dimensional problem. Another issue that must be considered results from the definition of $\mathbf{B}$ — $\mathbf{B}$ is related to $\mathbf{A}$ by Eq. (3.34). The matrix $\mathbf{A}$ itself is assembled out of the null-spaces that are calculated for each of the decaying parameters, p_i . The null-space of a linear homogeneous equation is a matrix, or a vector that can be determined up to a scalar. By norming this vector, the solution can be further specified, and the null-space is now specified up to the sign. A property of determinants is that the sign of the value of the determinant changes when either two rows or columns are flipped, or one column or row is multiplied by -1 [3]. Since the sign of the null-space vectors in $\mathbf{A}$ are not specified, this leads to the problem that the sign of the determinant might change from one calculation to the next, although the determinant was never zero in between. Unfortunately, a search algorithm would search for a zero-crossing in between those two steps.

Two decaying parameters, p_i , are the same when the initial stresses are zero. For this case, the null-space is a 2-by-3 matrix and how to sort the columns is not defined, which again might cause randomly sign changes of the determinant of $\mathbf{B}$.

A typical plot of how the determinant might look is shown in Fig. 4.1. This Fig. clearly shows how the imaginary part apparently randomly changes sign. Note the very small interval of the horizontal wave speed axis, and the high values of the determinant (in this small wave speed interval), even though the determinant values

Figure 4.1: Values for the determinant of B for mild steel in a very small interval around the actual solution for the Rayleigh wave speed. The values are already normalized (see section 4.3.2). The value of $\det(B)$ is purely imaginary in this case.

have already been normalized (see section 4.3.2).

4.2.3 f : determination of null-space

Now that the Rayleigh wave velocity has been obtained, the next step in the iterative algorithm is to determine the vector f (in step 9) which is the null-space of B in Eq. (3.34). Since the Rayleigh wave speed, c_r , is numerically determined, the determinant of B will be close to, but not exactly, zero. A standard Gaussian

elimination algorithm fails in this case, since the matrix is ill-conditioned and a null-space does not exactly exist, since the determinant is not exactly zero. If a matrix is ill-conditioned, a singular value decomposition (SVD) of the matrix (the SVD will be introduced in section 4.3.4) can be used to determine its eigenvectors. A matrix is defined as being ill-conditioned if the ratio between its smallest and largest elements, $w_{\min}$ and $w_{\max}$, approaches the floating point precision of the computer [31]. This ratio $w_{\min}/w_{\max}$ is the reciprocal of the *condition number*. Consider some values for the matrix $\mathbf{B}$ — this example uses mild steel that has a uniaxial stress of $t_{11}^i = -50$ MPa.

$$\begin{bmatrix} +246.929726636661i & 0 & 141.229757646434i \\ -164.025075014806 & 0 & -93.813012745789 \\ 0 & 31.160812338551i & 0 \end{bmatrix} \quad (4.2)$$

These values are already the normed parameters (see section 4.3.2), and its determinant is -1.0935×10^{-10} . This matrix has a reciprocal condition number, $w_{\min}/w_{\max}$ of 4.039×10^{-17} , which indicates that a numerical problem might arise when using a standard Gaussian elimination algorithm.

4.3 Solution methods

This section considers solution techniques. The proposed solutions are in the same sequential order as the numerical problems are introduced in the previous section.

4.3.1 Reduce order of polynomial

Step 2 of the iterative algorithm sets the determinant of $\mathbf{D}$ equal to zero. A closer look for isotropic materials reveals that the resulting polynomial only has components of the order 6,4,2 and 0 in p [30]. By substituting $p^* = p^2$, this polynomial can be

reduced to the order of three. This not only increases the numerical accuracy, but also creates a polynomial where the roots can be solved by an explicit formula [3].

4.3.2 Normalization

Normalization is an effective method to deal with the large numbers of a matrix. The units that are used in Eq. (3.29) are

$$p^2[\text{GPa}] + p([\text{GPa}] + [\text{GPa}]) + [\text{GPa}] + [\text{kg/m}^3]([1])[\text{m/s}]^2 = 0 \quad (4.3)$$

For most solids, the mass density is higher than 1000 [kg/m³] and the wave speed is around 1000 [m/s]. This is why it makes sense to divide Eq. (3.29) by the factor of 10⁹ and then specify the parameters accordingly. This means that all Lamé- and TOE-constants are divided by 10⁹, and c_r and ρ are each divided by 10³. This normalization significantly reduces any numerical issues.

4.3.3 The use of the magnitude of $\det(\mathbf{B})$

$\mathbf{B}$ normally is a complex number and its sign is not clearly defined, since the null-space can only be determined up to a scalar. This research takes the following approach: instead of searching for the zero of a complex number with a randomly flipping sign, select the magnitude of the determinant of $\mathbf{B}$ as the output of the “black box”. Fig. 4.1 shows that the magnitude values of the determinant of $\mathbf{B}$ can be connected to 2 straight lines. These two lines meet at the only zero point. Numerically, it is much easier to search for the zero of this solid line. However, convergence problems were initially encountered when running the simulation. The standard search algorithm in Matlab 6.1 is the “Gauss-Newton Method.” Unfortunately, the Gauss-Newton algorithm did not ensure convergence of this numerical problem — convergence only occurred when the initial starting value was close enough to the

final solution. This research does not quantitatively examine the numerical conditioning of the matrices $\mathbf{D}$ and $\mathbf{B}$, but the discontinuous first derivative of $\text{abs}(\det(\mathbf{B}))$ might be one cause of these convergence problems. Fortunately, the latest release of Matlab, Matlab 6.5, contains a more robust algorithm — “Trust-Region Dogleg Method” [25]. This trust-region dogleg algorithm has greatly improved robustness when the starting point is further away from the final solution, and it shows much better convergence for the current problem.

4.3.4 Singular Value Decomposition (SVD)

Any $m \times n$ -matrix $\mathbf{B}$ with $m > n$ can be decomposed into three matrices of the following form [31]

$$\mathbf{B} = \mathbf{U} \cdot \begin{bmatrix} w_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & w_n \end{bmatrix} \cdot \mathbf{V}^T, \quad (4.4)$$

where $\mathbf{U}$ is a $m \times n$ column-orthonormal matrix, $\mathbf{W}$ is a $n \times n$ diagonal matrix, and $\mathbf{V}$ is a orthogonal $n \times n$ matrix. This decomposition can be made, even if the matrix $\mathbf{B}$ is singular. For the case that $n = m$, $\mathbf{B}$ is a square matrix and its inverse can be easily calculated from Eq. (4.4) as

$$\mathbf{B}^{-1} = \left[\mathbf{V}^T \right]^{-1} \cdot \begin{bmatrix} w_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & w_n \end{bmatrix}^{-1} \cdot \mathbf{U}^{-1} \quad (4.5)$$

Using the properties that $\mathbf{U}$ and $\mathbf{V}$ are orthogonal, and $\mathbf{W}$ is a diagonal matrix, Eq. (4.5) can be rewritten as

$$\mathbf{B}^{-1} = \mathbf{V} \cdot \begin{bmatrix} \frac{1}{w_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{w_n} \end{bmatrix} \cdot \mathbf{U}^T \quad (4.6)$$

The SVD is critical when solving equations of the form

$$\mathbf{B}\mathbf{f} = \mathbf{c} \quad (4.7)$$

In the iterative algorithm $\mathbf{c} = 0$ and thus

$$\mathbf{B}\mathbf{f} = 0 \quad (4.8)$$

with $\mathbf{f}$ being the null-space of $\mathbf{B}$. Ideally, $\det(\mathbf{B})$ is exactly zero, and thus $\mathbf{f}$ is the null-space of $\mathbf{B}$ with a dimension d , $0 \leq d \leq n$ and a range of $n - d$. The nice thing about SVD is that SVD explicitly constructs orthonormal bases for the null-space and the range of the matrix $\mathbf{B}$. The columns of $\mathbf{U}$ whose same-numbered w_i are nonzero are the orthonormal basis vectors that span the range of $\mathbf{B}$. The columns of $\mathbf{V}$ with the same-numbered w_i that are zero, are the orthonormal basis for the null-space. However, in the numerical solution $\det(\mathbf{B})$ is not totally zero, and the null-space of $\mathbf{B}$ does not exist — at least not analytically. This is the point where the SVD has its main advantage in comparison to other methods. The column of the $\mathbf{V}$ whose same-numbered w_i is very close to zero, gives a good numerical stable approximation for the null-space of a similar matrix that satisfies $\det(\mathbf{B}^*)=0$. Sometimes, an even better solution with respect to the norm of $\mathbf{B}\mathbf{f}$ might be obtained by setting the small $w_i = 0$ [31]. Setting $w_i = 0$ does not improve the solution significantly for the simulations in this research.

4.4 Simulation results

This research examines the following materials: mild steel, pure aluminum (99.5%), aluminum alloy B53S (2.8% Mg, 0.8% Mn, 0.2% Cr), aluminum alloy D54S (4.5% Mg, 0.8% Mn, 0.1% Cr), aluminum alloy JH77S (7% Zn, 2% Mg, 2% Cu), brass (70% Cu, 30% Zn), Pyrex glass and polystyrene. This wide range of materials with different

TOE-constants is selected in order to demonstrate how the TOE-constants and the applied stress influence the behavior of the wave speed and the polarization. For each material, the wave speed and the polarization vector is calculated for different states of stress. The main focus in these simulations is on the uniaxial stress case, $t_{11}^i \neq 0$, since this state of stress is the one that is examined in the experiments. The applied stress is varied in all metallic materials from a tension of $t_{11}^i = 50$ MPa, up to a compression of $t_{11}^i = -50$ MPa. These upper and lower limits are much lower than the respective yield strengths of each of the materials. The weakest metal, brass, has a yield strength in tension of 105 MPa [41]. For Pyrex glass and polystyrene the stress is varied between $t_{11}^i = -25$ MPa and $t_{11}^i = 25$ MPa, taking the yielding strengths of the materials into account.

The change of Rayleigh wave speed, and the change of the polarization vectors due to the initial stresses are very small. This is why in literature such as [16, 9], the *relative change of wave speed*, and not the absolute wave speed is plotted when visualizing the acoustoelastic effect. The relative change of Rayleigh wave speed is defined by

$$\Delta c_r = \frac{c_r - c_{r0}}{c_{r0}} \quad (4.9)$$

Analogous to the wave speed-plot, the *relative change of polarization*, and not the absolute value, is used to show the effect of initial or applied stress on polarization. The relative change of polarization is defined as

$$\Delta \Pi = \frac{\Pi - \Pi_0}{\Pi_0} \quad (4.10)$$

Π is the a measure for the polarization and was earlier introduced in Eq. (2.14).

Table 4.1: Mass density, Lamé constants and the TOE-constants of the simulated materials. References can be found in appendix B

material	ρ [kg/m ³]	λ [GPa]	μ [GPa]	ν_1 [GPa]	ν_2 [GPa]	ν_3 [GPa]
Mild Steel	7837.0	107.4	81.9	-13.0	-200.0	-200.0
Al (99.3%)	2710.0	50.9	26.1	-27.7	-91.7	-89.3
Al Alloy B53S	2677.0	58.0	26.0	-249.0	-99.0	-69.0
Al Alloy D54S	2719.0	49.1	26.0	-379.0	-198.0	-80.0
Al Alloy JH77S	2864.0	57.5	26.8	-320.0	-177.0	-109.0
Polystyrene	1040.0	2.8	1.3	-18.9	-13.3	-10.0
Pyrex Glass	2240.0	13.5	27.5	14.0	92.0	42.0
Brass	8640.0	85.3	33.5	-687.0	-298.0	-71.0

4.4.1 Mild steel

The simulation results of the wave speed for mild steel are shown in Fig. 4.2(a). The mass density, the Lamé constants, and the TOE-constants used in these simulations are given in table 4.1. The references for those numbers can be found in the Appendix B. As can be seen from Fig. 4.2(a), the Rayleigh wave speed increases during compression, and decreases as a function of increasing tension, but these changes are relatively small.

Figure 4.2(a) shows that the relation between Δc_r and applied uniaxial stress is approximately a linear relation

$$\Delta c_r = k_{\text{steel}}^c \cdot t_{11}^i \quad (4.11)$$

where k_{steel}^c is a proportionality factor. For mild steel, the simulation predicts

$$k_{\text{steel}}^c = -5.7 \times 10^{-7} / \text{MPa}$$

This proportionality factor k^c will be used to measure the sensitivity of the Rayleigh wave speed on the initial stress t_{11}^i . It represents the slope of the line in Fig. 4.2(a).

Fig. 4.2(b) presents the change in the relative polarization, as a function of uniaxial load, t_{11}^i . The changes in the polarization vector are also small. However, a

(a) Relative change of the wave speed

(b) Relative change of polarization

(c) Convergence indicator: absolute value of $\det(B)$

Figure 4.2: Relative change of wave speed and polarization for mild steel as a function of applied uniaxial stress along the propagation direction

comparison of the two upper plots in Fig. 4.2 shows that the change of the relative polarization is an order of magnitude higher than the change of the relative wave speed. The linear approximation for the relative polarization is given by

$$\Delta\Pi = k_{\text{steel}}^p \cdot t_{11}^i \quad (4.12)$$

with $k_{\text{steel}}^p = (9.81 \times 10^{-6}/\text{MPa})$. Like k^c , this proportionality factor k^p will be used to measure the sensitivity of the polarization of the Rayleigh wave on the initial stress t_{11}^i . k^p represents the slope of the line in 4.2(b).

Recall that the proposed numerical solution is computationally difficult, and that a means of verifying its convergence is required. Since in the real implementation, the first part is solved with the symbolic toolbox of Matlab, it is assured that the determinant of $\mathbf{A}$ is zero. The problem that is most numerically challenging is to set the determinant of $\mathbf{B}$ to zero. To check the convergence, the absolute value of the determinant of $\mathbf{B}$ are plotted in Fig. 4.2(c). The smaller the value of $\text{abs}(\det(\mathbf{B}))$, the more accurate is the resulting null-space $\mathbf{f}$, and thus the solution of wave speed and the polarization vector. For all states of stress, the values of $\text{abs}(\det(\mathbf{B}))$ are small enough to give an accurate solution. By the experience of earlier simulation it can be stated that the solutions of the wave speed and the polarization vector do not significantly change once the value of the determinant is lower than $\det(\mathbf{B}) < 0.5 - 0.01$. This value varies for different materials.

4.4.2 Aluminum

Aluminum is the second material for which the Rayleigh wave speed and the polarization are calculated as a function of uniaxial stress. Several different, independent measurements of the TOE-constants exist for aluminum and various aluminum alloys. Table 4.1 shows that the TOE-constants vary for different aluminum alloys and thus simulations have to be run for each alloy. It needs to be emphasized that

(a) Relative change of the wave speed

(b) Relative change of polarization

(c) Convergence indicator: absolute value of $\det(B)$

Figure 4.3: Change of wave speed and polarization for different aluminum alloys as a function of applied uniaxial stress along the propagation direction

both Fukuoka in [12] and Smith [35] point out that the TOE-constants (especially ν_1) have a high degree of scatter. As a result, this simulation uses the mean value of the TOE-constants and section 4.5 of this thesis will then study the sensitivity of polarization and wave speed to variations in TOE-constant values. Furthermore, it is important to note that the TOE-constants in aluminum are strongly dependent upon the rolling direction. In fact, aluminum is not an isotropic material, at least not in the TOE-constants. In order to be able to apply the isotropic model, the TOE-constants must be exchanged depending on whether the Rayleigh wave is propagating along the rolling direction or perpendicular to it. Figure 4.3 shows the simulation results for different aluminum alloys. The simulation results for pure aluminum are plotted with the solid line. The dashed line represents the simulation results for the alloy B53S. The dotted-dashed line illustrates the results for the alloy D54S, and the dotted line stands for the alloy JH77S. Figure 4.3 demonstrates that for aluminum, both the relative change of wave speed and the relative change of polarization is larger per applied stress, than for mild steel. Again, the change in relative polarization is more sensitive to t_{11}^i , than the change in relative wave speed. The influence of the supplementary materials to the aluminum alloy is quite large, and the proportionality factors k_{A1}^c and k_{A1}^p vary but remain on the same order of magnitude. The proportionality factors can be found in table 4.2. The smallest value of k_{A1}^c is for the alloy JH77S with $k_{JH77S}^c = -2.37 \times 10^{-5} / \text{MPa}$. The biggest value is obtained for B53S with $k_{B53S}^c = -2.88 \times 10^{-6} / \text{MPa}$.

In [9], Duquennoy et al. obtained the same proportionality factor k_{A1}^c in a set of experiments. He called this factor $A_{R12}^{(1)}$. For the aluminum alloy 2214-T6 this factor was measured $A_{R12}^{(1)} = -1.59 \times 10^{-5} / \text{MPa}$. Of course, it is not possible to compare this value to the constants that were calculated in the simulation since it is a different alloy. But at least it is shown by this comparison, that the constants are of the same order of magnitude.

Table 4.2: Comparison of the simulation results for the different materials. Values are changes per MPa applied stress

material	$k^c = \Delta c_r / c_{r0}$ [1/MPa]	Pol-change k^p [1/MPa]	Ratio k^p to k_{steel}^p	Ratio of k^p to k^c
x Mild Steel	-5.7217E-07	9.1822E-06	1.00	-16.05
Al (99.3%)	-1.0023E-05	3.5809E-05	3.90	-3.57
Al Alloy B53S	-2.8842E-06	2.7893E-05	3.04	-9.67
Al Alloy D54S	-2.0447E-05	6.6730E-05	7.27	-3.26
Al Alloy JH77S	-2.3684E-05	6.0955E-05	6.64	-2.57
Polystyrene	-6.4805E-05	7.9229E-04	86.29	-12.23
Pyrex Glass	5.5364E-05	-2.6748E-05	-2.91	-0.48
Brass	-6.4805E-06	3.8480E-05	4.19	-5.94

4.4.3 Polystyrene

Polystyrene is selected for simulation because it has been demonstrated previously that its longitudinal and shear wave is highly sensitive to applied stress, [17]. Since polystyrene is a relatively weak material, the stress t_{11}^i is only varied between ± 25 MPa . The simulation shows that the two proportionality factors k_{PS}^c and k_{PS}^p are quite large (in a fashion similar to longitudinal waves, [17]). The results are plotted in Fig. 4.4. Considering Fig. 4.4 and the fourth column of table 4.2, the factor of k_{PS}^p is eighty-six times larger than the k_{steel}^p . However, the linear relationship between the relative wave speed and the relative polarization to the stress t_{11}^i is no longer a good approximation. As can be seen in the Figs. 4.3(a) and 4.4(b), the curves are not straight lines. For polystyrene, the computed changes in the polarization of the Rayleigh wave are large enough that they can be easily visualized in a trajectory plot. Fig. 4.5 shows the changes of the “orbit” of a particular particle on the surface, when the stress changes from tension (25 MPa) to compression (-25 MPa).

(a) Relative change of the wave speed

(b) Relative change of polarization

(c) Convergence indicator: absolute value of $\det(B)$

Figure 4.4: Change of wave speed and polarization for Polystyrene as a function of applied uniaxial stress along the propagation direction

Figure 4.5: Trajectory plot: Motion of a particle on the surface in polystyrene. Solid line shows motion for uniaxial compression, dashed line represents uniaxial compression.

4.4.4 Conclusion

Table 4.2 summarizes the complete simulation results for all eight materials. Not all materials listed in this table are mentioned in this chapter. The simulation results for all material are shown in Appendix A, including the absolute calculated values for the wave speed and the polarization, as opposed to the relative values shown in the previous figures. In table 4.2, the proportionality factors k^c and k^p for the different materials are listed in the second and the third columns. The sensitivity

of the relative wave speed and the relative polarization of a Rayleigh wave can be easily compared with these two proportionality factors. The fourth column shows the ratio between the proportionality factor of the polarization of steel, k_{steel}^p , and the proportionality factors of the various other materials. These results lead to the following conclusion — the stronger the material, the smaller are the changes in both wave speed and polarization. The last column shows the ratio between the change in relative wave speed and the change in relative polarization. It is important to note that the proportionality factor of the relative polarization, k^p , is larger than the proportionality factor of the relative wave speed, k^c for all materials. For mild steel, Al-B53S and polystyrene, k^p is even an order of magnitude larger than k^c . The only exception is Pyrex glass. This is the only material where the proportionality factors are of opposite sign and k^c is larger than k^p . This means that the simulation predicts the following: The determination of t^i by the measurement of the polarization is for most materials much more sensitive than the determination of t^i by the measurement of the wave speed.

The values in table 4.2 are used to select aluminum and Polystyrene as the materials to be experimentally evaluated. Polystyrene is chosen because the predicted changes in polarization are the largest, while aluminum is a commonly available material which should be relatively sensitive to the applied stress.

4.5 Sensitivity analysis for TOE-constants

Prior to this section, the values of the TOE-constants were assumed to be known exactly, while in reality, the “actual” value of the TOE-constants will have a large degree of scatter [35, 12]. For example, [35] specifies the TOE-constants for Al-B53S

Table 4.3: Variation of the TOE-constants used in the sensitivity analysis

	ν_1	ν_2	ν_3
min	-298.80	-188.80	-82.80
mean	-249.00	-99.00	-69.00
max	-199.20	-118.80	-55.20

as

$$\nu_1 = -249 \pm 60 \text{ GPa}$$

$$\nu_2 = -99 \pm 10 \text{ GPa}$$

$$\nu_3 = -69 \pm 5 \text{ GPa},$$

while the scatter in the TOE-constants in other materials is on this order, or even higher (as measured for aluminum alloy in [12] and for pure aluminum in [35]). This high level of uncertainty in the actual value of the TOE-constants necessitates a study of the influence of the TOE-constants on the numerically predicted Rayleigh wave speed and polarization vector. A simulation is implemented to calculate this dependency, by examining the proportionality factors, k^c and k^p , as a function of the TOE-constants. Al-B53S is selected as the candidate material, and the TOE-constants are changed by $\pm 20\%$ of their respective mean values, which leads to the values in table 4.3. Note that the Lamé-constants of a material can be determined very precisely (with an error less than 1%), so their values are assumed to be constant throughout this simulation. Mathematically, the variation of the TOE-constants spans a 3-dimensional space, and has the shape of a cube. This cube must be discretized for the simulation. Since the problem is 3-dimensional, doubling the number of steps in each dimension will increase the total number of calculations by a factor of eight. Therefore, the discretization step size is critical for numerical efficiency — this simulation divided each dimension into nine equal parts. When searching for the maximum and minimum values of the proportionality factors k^c and k^p , it is of great

Table 4.4: Maximum, minimum and original value of the proportionality factors k_{B53S}^c and k_{B53S}^p for an uncertainty of 20% in the TOE-constants. Right columns shows associated TOE-constants.

	k_{B53S}^c %		ν_1	ν_2	ν_3
min	-9.9600E-06	-245.33%	-298.80	-118.80	-82.80
real	-2.8840E-06		-249.00	-99.00	-69.00
max	4.1920E-06	245.33%	-199.20	-79.20	-55.20
	ν_1	ν_2	ν_3	k_{B53S}^p	%
min	1.7238E-05	38.20%	-199.20	-79.20	-55.20
real	2.7893E-05		-249.00	-99.00	-69.00
max	3.8549E-05	-38.20%	-298.80	-118.80	-82.80

interest to determine exactly where the maximum and the minimum is located; is the minimal and maximal extremum on the border of the spanned space (which symbolically speaking means on the vertices of the cube) or within the spanned space? If it is analytically shown that there is a strictly monotonic relationship between each of the TOE-constants and the proportionality factors, k^c and k^p , then the maximum and the minimum value will be located on the border of the cube. [9] develops an analytical description for k^c for non-isotropic materials (Eq. (31) in [9]). However, this relation is rather complicated. Furthermore, an analytical description for the factor, k^p does not exist. This is why the behavior of the two proportionality factors k^c and k^p is simply determined numerically.

The simulation results show that there is a strictly monotonic relationship between each of the TOE-constants, and the proportionality factors k^c and k^p — Fig. 4.6 shows this relation for ν_3 . In both figures, eighty-one lines are drawn which are too close to each other to be resolved separately. Each of these eighty-one lines reflects the value of k^c and k^p dependant, on ν_3 , while ν_1 and ν_2 are fixed. Similar plots are obtained for ν_1 and ν_2 , but these figures are not presented. This strict monotony forces the maximum and the minimum of the proportionality factors to one of the vertices of the cube.

(a)

(b)

Figure 4.6: Proportionality factors k_{B53S}^c and k_{B53S}^p dependant on ν_3 . Each of the 81 plotted lines represent the behavior of k_{B53S}^c and k_{B53S}^p for constant ν_1 and ν_2 .

Table 4.5: Influence of each TOE-constant on k_{B53S}^c and k_{B53S}^p . By changing the ν_1, ν_2 or ν_3 by 1% the proportionality factors change by the percentages shown.

	$\Delta\nu_1 = 1\%$	$\Delta\nu_2 = 1\%$	$\Delta\nu_3 = 1\%$
Δk_{B53S}^c	0.11%	3.35%	8.80%
Δk_{B53S}^p	0.06%	0.87%	0.98%

Table 4.4 shows the minimum and the maximum values of k_{B53S}^c and k_{B53S}^p for the simulated material aluminum B53S. It is seen that the proportionality factors for the relative wave speed, k_{B53S}^c , varies by more than $\pm 240\%$, while the proportionality factor for the relative polarization, k_{B53S}^p , only varies by $\pm 38\%$. This means that the relative polarization is not only *more sensitive* to applied stress than the relative wave speed (since the proportionality factor, k^p , is larger than k^c), but that the relative polarization is also *more robust*, because it is less dependent on knowing the exact values of the TOE-constants. As a result, the polarization of Rayleigh waves should be a more accurate, and robust indicator of applied stress. Table 4.5 shows the influence of each TOE-constant on the proportionality factors — this table shows how the proportionality factors change (in %), when either ν_1, ν_2 or ν_3 varies by 1%. These results again indicates the stronger influence of changes in the TOE-constants to the Rayleigh wave speed. This table also shows the exact influence of each TOE-constant. Note that the influence of ν_1 is the weakest — this is a helpful result since ν_1 is usually known with the least amount of accuracy.

These results are summarized in Fig. 4.7. This figure shows the maximum and the minimum expected relative wave speed and relative polarization, and their dependence on t_{11}^i . Assuming that the TOE-constants will not be exactly known for a test specimen, the results of the experiments might vary within the grey-shaded area.

(a)

(b)

Figure 4.7: Effect of uncertainties in the TOE-constants on the relative wave speed and the relative polarization. If an uncertainty of 20% in the TOE-constants is assumed the relative wave speed and the relative polarization changes within the gray shaded area. The relative polarization is more robust against uncertainties in the TOE-constants.

4.6 FEM simulation of loaded specimen

The previous section 4.4 presents the influence of a homogeneous uniaxial stress on the wave speed and the polarization of a Rayleigh surface wave. The proposed experimental procedure necessitates determining whether such an uniaxial stress can be easily generated by simply applying an uniaxial force. A more detailed understanding of the actual state-of-stress in the finite-sized, experimental component will (hopefully) validate the uniaxial stress assumption. The commercial finite element (FEM) program, ANSYS, is used to develop an FEM-model to study the static state of stress.

4.6.1 The FEM-model

The specimen is modeled as a simple block that is divided into $21 \times 13 \times 9$ elements of the type Solid 45, the standard cubic element of ANSYS. The displacements of the block are limited by fixtures located on each node of the smaller lateral surface of the block as shown in Fig. 4.8. Note that the nodes of the right surface are fixed (no displacement) in all three coordinate directions, while the nodes of the left surface are restricted in the X_2 - and X_3 -directions. The force is applied as a homogeneous pressure on the left surface. This model closely approximates the real situation of the applied pressure by the vise.

4.6.2 The numerical solution

The FEM-simulation is run for both aluminum and polystyrene. Both materials have a similar Poisson's ratio. Thus, the ratio between the applied stress, t_{11}^i , and the induced stresses will be similar for both materials. When applying a uniaxial load, there is an effect, called the "barrelling out effect" — the compression of a material along the axis of the applied load causes the cross-section perpendicular to this loading

Figure 4.8: FEM-model for polystyrene: The nodes of the left surface can move in the X_1 -direction. On the left surface a homogeneous load of 25 MPa is applied.. The nodes of the right surface are fixed for all translation directions.

axis to increase. Figure 4.9 shows the displacement along the centerline of the top surface. The horizontal axis is the X_1 -axis. Recall an earlier assumption that the Rayleigh surface wave is propagating in the X_1 direction. The vertical axis represents the values of the displacement, u_2^i . Notice that this section uses the nomenclature of Fig. 2.5. The maximum displacement on the top surface due to deformation is 0.145 mm, which is 0.23% of the height of 64 mm. This number demonstrates that the “barrelling out effect” is not significant at this stress level.

The most important result of the FEM analysis is the state of stress on the top surface, and in the underlying 2-wavelength-thick layer. This state of stress is needed in order to justify the uniaxial stress assumption, and it is not crucial to know the absolute values of the stress tensor, $\mathbf{t}^i$, but rather the ratio of the applied stress, t_{11}^i , to the other induced stresses. Figures 4.10(a) and 4.10(b) show the ratio of the applied stress, t_{11}^i , to the stress components t_{22}^i and t_{33}^i induced by the deformation $\mathbf{u}^i$. It can be clearly seen in 4.10(a) that the stress component, t_{22}^i is zero almost everywhere, and satisfies the stress free surface and Eq. (3.20). Additionally, Fig. 4.10(b) shows

Figure 4.9: Displacement u_2^i due to the applied stress t_{11}^i . Plot shows the displacements on the top surface along the centerline parallel to the X_1 direction.

that around the region of $(X_1, X_3) = (65, 20)$, the initial stress is more than fifty times higher than t_{33}^i . This is the region where the Rayleigh waves will be detected in the experiment.

4.6.3 Conclusion

Figures 4.10(a) and 4.10(b) show, the applied stress is clearly uniaxial, but there are also other stresses present. The value of the other stresses (other than t_{11}^i) are large in the regions where the load is applied, but this FEM simulation shows that the St-Venant principle can be applied here — the Rayleigh surface wave measurements made in the middle of the specimen will propagate in a region of uniform, uniaxial stress.

4.7 Conclusion

This chapter investigates the influence of applied homogeneous uniaxial stresses on the wave speed and the polarization of a Rayleigh wave. The simulation results

Figure 4.10: Ratio of the induced stresses t_{22}^i and t_{33}^i to the applied stress of t_{11}^i at the surface of polystyrene. Values ≥ 0.3 are colored black.

predict that for all the materials tested (except Pyrex glass) the *polarization is much more sensitive to applied stresses than the wave speed*. Additionally, the sensitivity analysis predicts that the measurement of applied stress using the polarization of a Rayleigh wave is *more robust*, when considering the amount of uncertainty in the values of the TOE-constants. Finally, the FEM-model confirms that the experimental specimen will be in a state of uniaxial stress.

CHAPTER 5

Experimental Procedure

The experimental procedure is divided into three sections. The first part describes the theoretical and practical issues of how to generate Rayleigh waves in both aluminum and polystyrene. The second section discusses the experimental procedure to measure the Lamé-constants, while the final section describes the integrated laser detection, Rayleigh wave generation setup.

5.1 Generation of Rayleigh waves

Rayleigh waves can be generated through a variety of different techniques, and a number of them (both theoretical and experimental) are discussed in [42]. Given these results, only the two most suitable techniques are presented in this thesis — the generation of Rayleigh waves with either a wedge or with a comb.

5.1.1 The wedge technique

For the wedge technique, a longitudinal transducer is mounted on a wedge. The wedge has a the cross-section of a prism. The surface on which the transducer is mounted is rotated from the surface where the Rayleigh wave is going to be generated by an angle of θ_w , as can be seen in Fig. 5.1. The angle θ_w is chosen such that the

Figure 5.1: Wedge with attached longitudinal wave transducer for generation of Rayleigh wave.

following equation between the incident angle, θ_w , and refracted angle, $\theta_r = 90^\circ$, holds

$$\frac{\sin \theta_w}{\sin \theta_r} = \frac{c_l^{(1)}}{c_r^{(2)}} = \frac{\lambda_l^{(1)}}{\lambda_r^{(2)}} \quad (5.1)$$

This leads to the angle θ_w of

$$\sin \theta_w = \frac{c_l^{(1)}}{c_r^{(2)}} \quad (5.2)$$

Of course, this technique only works for a material combination where $c_r^{(2)} > c_l^{(1)}$. An important property of this technique is that the mode conversion from the longitudinal wave to Rayleigh wave is theoretically perfect; neither a longitudinal or a shear wave is transmitted. This is due to the property that the Rayleigh wave speed is always smaller than the shear waves speed. Thus, with the incident angle specified in Eq. (5.2), the angle of the transmitted longitudinal and shear waves would be

$$\sin \theta_l = \sin \theta_w \cdot \frac{c_l^{(2)}}{c_l^{(1)}} = \frac{c_l^{(2)}}{c_r^{(2)}} > 1 \quad (5.3)$$

$$\sin \theta_t = \sin \theta_w \cdot \frac{c_t^{(2)}}{c_l^{(1)}} = \frac{c_t^{(2)}}{c_r^{(2)}} > 1 \quad (5.4)$$

Both terms on the right hand side are larger than one, so a solution for these two equations does not exist. Another advantage of this wedge technique compared to most of the others discussed in [42] is that the generated Rayleigh wave only propagates

in one direction. Furthermore, this wedge technique is applicable for all frequencies without changing the angle of the setup. The last advantage concerns the input-output behavior of the wedge technique. The waveform of the longitudinal wave does not change when it is mode converted to the Rayleigh wave. A sinusoidal longitudinal wave will be mode converted into a sinusoidal Rayleigh wave.

Considering the material properties of aluminum and polystyrene, it becomes clear, that the wedge technique is only applicable for aluminum but not for polystyrene, since there is no commonly available material which can be used for the wedge whose longitudinal wave speed is as low as 1000 m/s. For aluminum, the typical Rayleigh wave speed is 2900 m/s which yields an angle of $\theta_w = 70.9^\circ$ for a wedge made out of plexiglass ($c_t = 2740$ m/s [32]).

Signal generation The transducer is driven by a sinusoidal signal, generated by a Wavetek 50 Mhz Pulse/Function generator model 81. The number of cycles is set between 2 and 20 cycles, and the exact number of cycles is indicated in the applicable plots. The frequency is varied in between 0.5 MHz and 3 MHz to determine the best measurement frequency of the overall setup. The signal is amplified by the an ENI Model 240 L RF amplifier.

Transducers/Receivers For the generation of the Rayleigh waves in the experiments, the commercially available wedge ABWSL-2056 by Panametrics with the attached transducer Panametrics A403-S is used. The transducer has a center frequency of 2.25 MHz. Small pin-transducers are used as a receiver. Theses pin transducers are self-fabricated and show a broad-band behavior.

Waveform acquisition The signals collected with the Digital Wave transducer are directly measured by the Tektronix TDS420 oscilloscope without any additional amplification. The signals are averaged either thirty or two-hundred times (the exact

Figure 5.2: Systematic design of a comb for the generation of Rayleigh waves. The width of each strip and the spacing in between is $\lambda_r/2$ each.

number is indicated in the relevant figures) to improve the SNR.

5.1.2 The comb technique

Experimental observation shows that Rayleigh waves can be generated by a combination of a longitudinal wave transducer and a number of small strips that form a comb. This comb is then placed in between a longitudinal transducer and the material in which the Rayleigh waves will be generated. Each strip of the comb must have a width equal to the half-wavelength of the Rayleigh wave to be generated [42]. The spacing in between each strip should again be equal to the half-wavelength, as shown in Fig. 5.2. If the frequency is chosen correctly, all the strips will act in constructive interference and create a Rayleigh surface wave. Measurement results for Rayleigh waves in aluminum using the comb technique can be found in [18].

The advantage of the comb technique compared to the wedge technique is its flexible applicability to nearly all materials, but there are several disadvantages. Each comb only works for one particular wavelength, which implies that the frequency cannot be chosen freely. In [42], the theoretical relationship between the variation of the frequency and the amplitude of the created Rayleigh wave is examined. According

to this theory, a variation of only 5% off of the ideal frequency leads from perfectly constructive interference to totally destructive interference — this means that the generation frequency and the comb spacing needs to be known very precisely. Accordingly, the Rayleigh wave speed also needs to be predicted very precisely, which can be done with the exact knowledge of the Lamé-constants.

As opposed to the wedge technique which has a linear input-output behavior, the shape of a Rayleigh wave generated with the comb technique does not exactly match the input signal. This is due to the fact that each strip of the comb is located at a different distance from the measurement point. When the transducer generates a sinusoidal longitudinal wave, the Rayleigh waves generated by each strip arrives at the measurement point at a different time. The wave from the closest strip arrives first. This is why the measured signal shows an increasing amplitude at the beginning of the signal, and a decaying amplitude at the end of the generated signal. This physical behavior was modeled in [42].

A limitation of the comb technique is that a number of different waves are generated (besides the Rayleigh wave). The comb generates not only Rayleigh waves. Chapter 6 of [27] presents a theoretical model that shows the contribution of the different waves due to an impulse signal on a line source. The wavefield generated includes longitudinal, head, shear and Rayleigh waves. Although the Rayleigh wave dominates the wavefield, the influence of the other waves cannot be neglected. Of particular importance are the reflected longitudinal waves, because they are in the time window of interest (the same time window as the direct Rayleigh wave).

Four combs are designed to generate Rayleigh waves. The main focus while designing these combs is to generate Rayleigh waves in polystyrene. Table 5.1 shows the main measures and theoretical frequency for the generation of Rayleigh waves. The comb calculations are performed using the results in section 6.2 and the Rayleigh wave speed is calculated using Eq. (2.12).

Table 5.1: Measures of constructed combs and theoretical frequency for the generation of Rayleigh waves in polystyrene

No.	$\lambda/2$ [in]	$\lambda/2$ [cm]	f [kHz]	material
1	0.030	0.076	540.571	aluminum
2	0.063	0.160	257.415	aluminum
3	0.135	0.343	120.127	aluminum
4	0.060	0.152	270.285	PMMA

Signal generation The Wavetek 50 Mhz Pulse/Function generator model 81 is again used to generate sinusoidal signals. The signal is amplified by the an ENI Model 240 L RF amplifier. The frequency for a particular comb is fixed, and the theoretically computed frequency shown in table 5.1 is first used. Then the frequency is slightly varied until a maximum amplitude is obtained. The number of cycles is chosen with respect to the previously described considerations. On the one hand, a large number of cycles is needed to ensure that each strip of the comb contributes to the generation of the Rayleigh wave. On the other hand, if the number of comb strips exceeds a certain value, the reflected longitudinal waves will interfere with the Rayleigh wave, and thus a polarization measurement is no longer possible.

Transducers/Receiver Several transducer are tested to generate the longitudinal waves. For the results that are shown in chapter 6, a longitudinal transducer of Aerotech is used. The Aerotech-transducer has a center frequency of 500 kHz, but also works well at lower frequencies, all the way down to 150 kHz. In order to generate Rayleigh waves in aluminum higher frequencies are needed. For frequencies > 1 MHz the transducer Panametrics A403-S is used.

A Digital Wave (DW) B-1025 transducer is used as the receiver. The frequency response of this transducer is presented in [2], and a broadband behavior was observed.

Waveform acquisition The signal collected with the Digital Wave transducer is directly acquired with the Tektronix TDS420 oscilloscope without additional amplification. Signal averaging is again used to increase SNR.

5.2 Determination of material constants

Since the material constants of the polystyrene are not exactly specified by the manufacturer, and the longitudinal wave showed a significantly different speed in a preliminary test than expected by standard values taken out of literature, the Lamé constants and the density are determined in an experiment. The precise knowledge of the Lamé constants is necessary for the generation of the Rayleigh waves, as shown in section 5.1. By measuring the longitudinal and shear wave speeds, μ and λ can be determined. From Eq. (2.7) and (2.8) follows

$$\lambda = c_l^2 \rho - 2\mu \quad (5.5)$$

$$\mu = c_s^2 \rho \quad (5.6)$$

The actual procedure measures the time-of-flight (t_l and t_s) and the propagation distance and not the actual wave speed. For measuring the time-of-flight, the arrival time of the impulse at the receiving end has to be determined very precisely. Physically, the arrival of the impulse is accompanied by a sudden increase in the acceleration of the particles, but this information is not available during the experiment. Thus, the arrival time is determined at the point in the displacement plot where the particle starts moving. Determination of this point can be difficult. That's why, to verify these results another additional method for measuring the time-of-flight is applied. The received signal of different propagation distances are plotted in one plot. Since the waveforms all look similar, the time-difference between the maximum peaks of the waveform of different propagation distances can be determined. This gives very

precise results, since the peak of a signal can be easily and precisely determined. This method can only be applied for an isotropic material, since the different distances are measured along different axes of the material.

Besides the Lamé constants, the third order elastic (TOE) constants need to be determined. Unfortunately, the experimental determination of these TOE-constants requires a precise (and expensive) experimental setup, so this research uses previously published values (see Tab. B.1 for the TOE-constant values). Note that the influence of these TOE-constants is examined in section 4.5.

Signal generation The source signal for this experiment is an impulse. The impulse is generated by a Panametrics 500PR impulse generator which generated pulses of 110 Volts. The pulse generator has a decay parameter, D , that can be adjusted to change the pulse shape. The higher the value of D , the shorter is the decay time and the sharper is the peak of the impulse in time domain. For these experiments, D is adjusted such that the received signal has both a high amplitude and a narrow maximum, which leads to a trade-off adjustment. Additionally, the pulse repetition frequency PRF can be adjusted between 500 and 5000Hz. The pulse generator is operated in through transmission mode. Various plots in both the time- and frequency-domains for different settings of the pulse generator are given in [2].

Transducer/Receiver The Aerotech transducers are used to generate and detect the longitudinal waves. The Aerotech transducer has a diameter of 1 inch, and a center frequency of 0.5 MHz.

The shear waves are generated and detected by Utran V153 shear wave transducers with a center frequency of 1 MHz.

Waveform acquisition The signal from the detection transducer is amplified by the pulse-generator's built-in amplifier by 40dB. The amplification gain is set to 3.5

Figure 5.3: Experimental Setup of the polarization measurement in an aluminum specimen.

in full mode. The amplified signal is then visualized on the Tektronix TDS420 oscilloscope, averaging over 20 samples.

5.3 The main experimental setup

Figure 5.3 shows the main experimental setup. In this particular drawing, the experiment is prepared to measure the polarization of Rayleigh waves in aluminum. The wedge transducer generates a Rayleigh wave which propagates along the surface of the aluminum sample. The velocity of the particle at a small spot is then measured by a laser Doppler vibrometer (LDV). The specimen is exposed to a uniaxial stress generated by a hydraulic vise. In the next sections, the components of this setup are described in detail.

5.3.1 The laser doppler vibrometer (LDV)

The interferometer that is used for these experiments is a heterodyne laser Doppler vibrometer (LDV). The basic principle of this system is to detect the frequency shift of laser light that is reflected from a vibrating surface. The velocity of the particles

on the surface cause a Doppler shift of the incident laser light. A very small spot is focused on the surface, and it is assumed that all particles within this spot have the same particle velocity. A more detailed description of the LDV-system can be found in [20]. A setup of the LDV system is shown in Fig. 5.4.

Laser bench configuration An Argon-Ion Laser with a wavelength of $\lambda = 514 \text{ nm}$ is used as the sources. This laser generates a linear polarized laser beam with a coherence length about 15 cm. Its power reaches up to 1 W, but is normally operated at a level of 100-200 mW. The laser beam passes through an acoustic-optic modulator (AOM), often also called Bragg-cell, and is split into two main frequency shifted beams – beam 1 and beam 2. The frequency shift is the same as the driving frequency of the Bragg-cell of 40 MHz. Beam 2 is reflected by mirror M3 and M4 before it passes through a glass plate which deflects 4% of the light generating beam 3. All three beams are now coupled into optical single-mode fibers of a diameter of $4 \mu\text{m}$. For the coupling, a special unit is used assembled out of microscope objective and a fiber coupler Newport F-916. In the best case, 25-30% of the light are coupled into the fibers. Fibers 1 and 2 end at the probe head. By using the fibers, the probe head is decoupled from the optical bench. The light of the fibers at the probe head is then focused to tiny spot on the testing surface by GRIN (graded-index cylindrical) lenses. The two GRIN lenses enclose an angle of $2\alpha=40^\circ$. The position of the GRIN lenses can be adjusted. The rough (compared to the wavelength) surface scatters the light. Part of the light is collected through a third GRIN lens mounted in between the two others and is coupled into a multi-mode fiber, fiber 4. A multi-mode fiber of diameter $62.5/125 \mu\text{m}$ is used, since it collects more of the scattered light compared to a single-mode fiber [7]. The light in the multi-mode fiber 4 is mixed with the reference signal in fiber 3 in a beam splitter. GRIN lenses are again used to focus the light into the beam splitter. After the beam splitter, the interfered light is then

Figure 5.4: Optical Setup of the LDV system. This Fig. is the modified version of Fig. 14 in [8]

focused again before it hits an avalanche photo diode. The photo diode generates a signal with a carrier frequency of 40 MHz (which is the operating frequency of the Bragg-cell) that is phase modulated by the velocity of the particles at the testing surface [7]. Since the coherence length of the laser is much less than the distance, the light travels in the whole system, the differences of pathlength of the different beams have to be very carefully considered. [24] shows how this pathlength problem is addressed for this LDV-system. The LDV system can be operated in two different modes — for either in-plane or out-of-plane detection.

In-plane detection For the in-plane detection, beam three is blocked. The detected phase change due to the in-plane measurement is totally decoupled from the out-of-plane behavior. The phase shift that is generated by the displacement velocity is given by

$$\Delta\Phi = \frac{4\pi}{\lambda} \sin(\alpha) u_1 \quad (5.7)$$

The scattered light causes a frequency modulated signal at the photodetector. The instantaneous frequency of this signal is related to the time derivative of phase shift caused by the in-plane displacement

$$f_i = \frac{1}{2\pi} \frac{d(\Delta\Phi)}{dt} \quad (5.8)$$

A FM-discriminator demodulates the carrier signal. The voltage of the output signal is then proportional to the surface velocity. For the in-plane setup, the voltage is given by

$$V_1 = C_{\text{FM}} \left[\frac{2}{\lambda} \sin(\alpha) \frac{du_1}{dt} \right], \quad (5.9)$$

where C_{FM} is a constant that is characteristic for the FM-discriminator. For this particular FM-discriminator, the constant is given by $C_{\text{FM}} = 260.46 \frac{\text{mV}}{\text{MHz}}$. Inverting the equation above then yields to the surface velocity for the in-plane motion

$$v_1 = \frac{\lambda}{2C_{\text{FM}} \sin(\alpha)} V_1 \quad (5.10)$$

Out-of-plane detection In the out-of-plane configuration, beam 3 is opened again, and beam 2 is blocked instead. In this setup, the incident beam of fiber 1 is not perpendicular to the testing surface which results in a collected light signal where the out-of-plane is *not* totally decoupled from the in-plane component. The phase change caused by the particle's velocity is given by the coupled term

$$\Delta\Phi = \frac{2\pi}{\lambda} [\sin(\alpha)u_1 + (1 + \cos(\alpha))u_2] \quad (5.11)$$

But since the in-plane component is known, the out-of-plane signal can be decomposed. Following the same steps as in the in-plane configuration leads to the out-of-plane particle velocity of

$$v_2 = \frac{\lambda}{C_{FM}(1 - \cos(\alpha))} (V_2 - 0.5V_1) \quad (5.12)$$

In the present setup, the noise generated during the various steps is significant. In order to get a better indication of the sensitivity of the system, the calculated signal-to-noise-ratio (SNR) is used to express a noise-equivalent surface displacement, that would generate the same voltage [21]. It is found that the sensitivity for the in-plane measurement is approximately 2.5 times greater than the out-of-plane component [36]. In [36], the minimum detectable displacements are specified by the values given in Tab. 5.2

Table 5.2: Minimal detectable displacements of the LDV system at 100 kHz [36].

$u_1^{\min}$ [nm]	$u_2^{\min}$ [nm]
0.32	0.13

Data acquisition Before the data is acquired with an oscilloscope, it is processed in several steps. Fig. 5.5 shows the consecutive processing steps. The signal that comes from the avalanche photo diode of the LDV-system is passed through the

FM-discriminator, which creates an output signal that is proportional to the surface velocity. This signal is then filtered by a low-pass filter with a pass-band up to 2.5 MHz. The filter is a Mini-circuits BLP 2.5. Then the signal is amplified by 20 dB with the amplifier, built into the FM-discriminator. After that, the signal is again filtered with a Mini-circuits BLP 1.9 (1.9 MHz low-pass frequency). Finally, it is acquired by the Tektronix TDS420 oscilloscope. Since the noise level is still very high, the signal averaged over 2000 different waveforms.

Figure 5.5: Signal processing steps of the measured signal

5.3.2 Pressure device

A hydraulic vise is designed to provide a static compressive pressure field (stress field) in the specimens. This vise is designed to load specimens up to a size of 10.16x7.62x22.23 cm (4x3x8.75 in). A hydraulic cylinder is used to generate forces up to 100 kN. This force is equally distributed by two rigid, steel blocks onto the specimen. The vise is designed such that, if the specimen is centered on the symmetry axes, no torsional moments are generated. The hydraulic cylinder is of the type Enerpac RSM-100. The maximum stroke is 1.12 cm (0.44 in). The vise itself can be further manually adjusted, and thus the size of the specimen can vary from 0 cm till the maximum length.

5.3.3 Samples

Two samples are prepared for the experiments. One sample is made out of aluminum, Alcoa Al-6061, of the size 16.03x8.71x1.27 cm (6.3125x3.43x0.5 in). Al-6061 is a commonly used aluminum alloy, which does have exactly the same supplementary materials as the aluminum alloys that are simulated in section 4.4.2. This limits the later experiment to a rather qualitative verification of the polarization changes. The thickness of the plate is 1.27 cm, which corresponds to 5.7 wavelength of the Rayleigh wave at 1.25 Mhz.

The other sample is a block of polystyrene, of the size 6.45 x 4.01 x 10.3 cm (2.54 x 1.57 x 4.06 in). For this work, measurements were only taken for the aluminum sample. Fig. 5.6 shows a picture of the aluminum sample fixed in the vise.

Figure 5.6: Experimental setup with aluminum sample

CHAPTER 6

Experimental Results

6.1 Rayleigh waves in an unstressed medium

6.1.1 Rayleigh waves in aluminum

Both techniques presented in the previous chapter – the wedge technique and the comb techniques – are used to generate Rayleigh waves in an aluminum specimen.

6.1.1.1 The wedge technique

Measurement description Several input frequencies are used for the wedge experiments. Experimental data is obtained for a 2-cycle, sinusoidal signal with the frequencies of 0.5, 0.75, 1.0, 1.5, 2, 2.25, 2.5 and 3.0 MHz. The signal amplitude of the function generator is set such that the amplified input signal at the transducer is 100 V, peak-to-peak. The signal repetition rate is set to 50 ms, and the received signal is averaged 30 times to increase SNR. The Rayleigh waves are launched into the aluminum specimen that is described in section 5.3.3.

Results and discussion Figure 6.1 shows the measurement results for 3 selected frequencies — 1.5, 2.25 and 2.5 MHz. The left-hand set of plots shows the results in the time-domain. The vertical line in each time-domain plot indicates the theoretical arrival time of the Rayleigh wave at the measured location — 0.1 m away from the wedge. There is good agreement between the theoretical and the measured arrival

times. As expected, the Rayleigh wave is the only wave received — neither longitudinal nor shear waves are generated in this technique. The detecting transducer is a pin-transducer, and these are not shielded. This might be one reason for the relatively high noise level in the signal at 50 kHz.

The right-hand set of plots in Fig. 6.1 shows the Fast Fourier Transform (FFT) of these measured signals. The cleanest frequency response is seen in the signal with a drive frequency of 2.25 MHz — the center frequency of the transducer attached to the wedge. All three frequency responses show a typical windowing effect: the input signal contains only two cycles of a sine-wave, which can be mathematically described as a sinusoidal signal multiplied by a rectangular window of the length of two cycle durations. As described in [29], this leads to leakage (like a sinc-function) in the frequency domain. A comparison of the three signals leads to the conclusion that this setup can be used for all of the frequencies shown. The results of these measurements with the lower and higher frequencies show that the signal quality decreases at frequencies below 0.75 MHz and above 2.5 MHz.

Additional experiments are performed to determine the best couplant type to be used between the wedge and the aluminum specimen. Both a vacuum grease couplant, and commercial shear couplant (honey) are tested in exactly the same experimental setup. No significant differences are observed between the different couplants, and note that these results are not included in this work.

6.1.1.2 The comb technique

Measurement description The input signal is again a sinusoidal signal of 20 cycles. The input signal is amplified to get the amplitude of the input signal equal to 120 V peak-to-peak. Using the Rayleigh wave speed for aluminum in [32] ($c_r = 2820$ m/s), the calculated frequency for comb No. 1 is $f = 1.855$ MHz. The signal is varied around this frequency until a “local maximum” is observed in the detected

Figure 6.1: Rayleigh waves in aluminum using a wedge. Time and frequency domain for different frequencies

time-domain signal. The detected signal is averaged over 200 measurements. The aluminum specimen that is used for this measurement is a block 11.9x9.06x5.08 cm (4.69x2.00x3.57 in). The Rayleigh wave is launched on the larger lateral surface.

Results and discussion Figure 6.2 shows the results of the comb tests. The upper three plots show the detected Rayleigh wave (time-domain signal) for different frequencies. The maximum amplitude is observed for the signal at a frequency of 1.84 MHz, which is plotted in the second plot of Fig. 6.2. This is a deviation of -8% of the calculated optimal frequency value. This result is in good agreement with the result in [42]. In [42], the maximum amplitude is measured approximately 4.5% lower than the theoretically computed optimal frequency. Figure 6.3 shows the Fast Fourier transform (FFT) of this signal, and the peak at 1.84 MHz dominates the rest of the frequency spectrum.

As predicted in [42], a change in the frequency results in a decrease in amplitude. The time-domain signals at a frequency of 1.8 MHz and 1.9 MHz show a lower amplitude than the optimal frequency. However, this decrease is not as significant as theoretically predicted in [42]. In all the upper three plots, the detected signal between 12 μs and 30 μs has an increasing and then a decreasing part, as theoretically discussed in section 5.1.2. After 30 μs , the longitudinal wave that is reflected at the bottom of the specimen arrives and the waveform changes significantly. The theoretically computed propagation time through the specimen, and back, is 28.9 μs . Taking into account that there is an additional propagation time through the comb, and that the received longitudinal wave has a slightly longer way through the specimen than twice the specimen's thickness, a very good match between the expected and the actual arrival of the longitudinal wave is observed. Note that the shortest path for the longitudinal wave to travel to the receiver is to propagate along the two equal length sides of an isosceles triangle. It is impossible to definitely conclude from

Figure 6.2: Rayleigh waves in aluminum using comb No. 1. Top three plots show different frequencies. Bottom plot shows the received signal when the comb is turned by 90°

Figure 6.3: FFT of the signal shown in the second plot of Fig. 6.2 (1.84 MHz)

these figures that the increase in amplitude is only due to the change in frequency of the input signal, and the thus a more effective generation of the Rayleigh wave with the comb. This is because the frequency response of both the generating transducer and the receiving transducer are not known and thus their frequency response cannot be de-convoluted from the measured signal.

To double-check the effectiveness of the comb, the comb is rotated by 90° , and the measurement is repeated. Now the strips of the comb no longer function as a line source. The received signal for this setup is shown in the bottom plot in Fig. 6.2. As can be clearly seen, the amplitude of the longitudinal wave at a $t > 30 \mu\text{s}$ stays almost the same, but the amplitude of the Rayleigh wave decreases by more than a factor of five. This result demonstrates the effectiveness of the comb setup to generate Rayleigh waves.

Note that these waveforms are similar to the results presented in [18], but the experiments in [18] are measured at much higher frequencies.

6.1.1.3 Conclusion

Each of the candidate techniques to generate Rayleigh waves in aluminum show good results — both signals have a clean frequency spectrum, and in both cases, the Rayleigh wave is clearly separated from other waves in the time-domain signal. However, a rather thick (9.06 cm) material must be used for the comb technique — this thick specimen will limit the stress that can be applied to the specimen. This restriction is not an issue in the wedge technique. As a result, the following Rayleigh wave polarization measurements in the aluminum specimen will use the wedge technique.

6.1.2 Rayleigh waves in polystyrene

Since the wedge technique cannot be applied to the polystyrene specimen, an effort is made to generate Rayleigh waves using the comb technique

6.1.2.1 The comb technique

Measurements in the polystyrene are performed with all four combs described in Tab. 5.1. The best results are obtained with comb No. 2 and No. 4, both made for frequencies around 260 kHz. The results for comb No. 1 and No. 3 are not presented in this work, because of excessive damping (comb No.1) and inadequate separation in the time-domain (comb No.3).

Measurement description The input signal for both measurements is a sinusoidal signal with ten cycles. In the measurement setup with comb No. 2 (aluminum base) the signal is amplified such that an amplitude of the input signal is 190 V peak-to-peak. In the measurement setup with comb No. 4 (PMMA base) the input signal is amplified up to a signal level of 182 V peak-to-peak. For both measurements, the

received signal is directly detected with the Digital Wave (DW) B-1025 transducer — no pad or comb is placed in between the surface and the receiver. The distance between the transmitting and receiving transducer is approximately 3.5 cm (measured center to center).

The Rayleigh waves are launched on the larger lateral surface of the polystyrene block which is described in section 5.3.3.

Results and discussion Figures 6.4 and 6.5 show the measurement results for Rayleigh waves generated with comb No. 2 (aluminum) and comb No. 4 (PMMA). These figures have the same structure as in Fig. 6.2 — the top 3 plots show the measured signals for different frequencies, the bottom plot shows the signal for the comb turned by 90°. For both combs, the typical frequency dependency can be observed. The maximum Rayleigh wave amplitudes for comb No. 2 is determined at 240 kHz. For comb No. 4, the maximum amplitude is located at 246 kHz. The variation of the frequency results in a strong decrease of amplitude. In a fashion similar to the aluminum specimen, the observed optimal frequencies reflect a deviation of -6% and -8% from the theoretically calculated value (shown in Tab. 5.1). These three measurements show the clear trend that the optimal frequency is slightly lower than the theoretically predicted frequency.

For the polystyrene specimen, the calculated propagation time for the longitudinal wave that propagates through the specimen, and back, is 61.2 μ s. In the top three plots of Figs. 6.4 and 6.5, the arrival of the reflected longitudinal wave is located close to this calculated time. The actual arrival time is slightly later, which again can be explained by the additional propagation time through the comb and by the slightly longer propagation distance of the reflected longitudinal wave.

Before the arrival of the reflected longitudinal wave, the received signal shows the expected shape. Driven with the optimal frequency (240 kHz for comb No. 2

Figure 6.4: Rayleigh waves in polystyrene using comb No. 2 (aluminum base). Top three plots show different frequencies. Bottom plot shows the received signal when the comb is turned by 90°

Figure 6.5: Rayleigh wave in polystyrene using comb No. 4 (PMMA base). Top three plots show different frequencies. Bottom plot shows the received signal when the comb is turned by 90°

and 246 kHz for comb No. 4), the amplitude increases first, and then stabilizes at a certain level. The signal decays after 10 cycles (which is the number of cycles of the input signal).

Comparing the detected signals generated by comb No. 2 to those generated by comb No. 4, there is a two and a half times greater amplitude of the Rayleigh wave generated with comb No. 4, though the input signal is varied by only 4 %. This fact might be explained by the better overall transmission coefficient of the transducer-PMMA base-comb-specimen setup. Additionally, the Rayleigh waves generated with comb No. 4 (PMMA) have a much cleaner shape.

The last plots in Figs. 6.4 and 6.5 again demonstrate the effectiveness of the comb. Though the input signal is set at the optimal frequencies, the generated signals show only a low amplitude at a time where the Rayleigh wave is supposed to arrive, while the reflected longitudinal wave is still present.

Additional measurements are performed to determine the optimal receiver configuration. In one experiment, an additional comb is placed in between the specimen and the receiving transducer, without changing the rest of the setup. In a second experiment, a small plastic pad is used in between the specimen and the receiving transducer. Both items are supposed to increase the sensitivity of receiving transducer. These measurements show that the best configuration is the direct coupling of the receiving transducer to the specimen. No plots of this experiments are shown in this work.

6.1.2.2 Conclusion

Due to the higher amplitudes and the cleaner waveforms of the waves received with comb No. 4 (PMMA base), the Rayleigh waves for the later polarization measurements will be generated with this comb No. 4.

6.2 Determination of material constants

Measurement description For the longitudinal wave measurements, the damping decay parameter, D , is set to 6, for the shear waves it is set to 2. During the entire experiment, the PRF is set to 7.5 (the scale on the impulse generator), ensuring that the distortion of one pulse is almost completely decayed before the next wave is generated. The polystyrene used for the determination of the Lamé constants is the same block as described earlier in section 5.3.3. The weight of this specimen is 0.283 kg. On each surface, three independent measurements are performed at different locations, and the time of flight for traversing the specimen is obtained for both the longitudinal and the shear wave. As a variation of this experiment, a sinusoidal signal is generated and excited with the transducers. Since the results are very similar, they are not presented in this thesis.

Results and discussion The measurement results are shown in Fig. 6.6 for the longitudinal waves, and in Fig. 6.7 for the shear waves. Both figures show the transmitted signals, and the received signals for different propagation distances. In all plots, the horizontal axis represents the time. In the subplots showing the received signals (Figs. 6.6(b) and 6.6(b)) the vertical axis represents the propagation distance. This type of plot enables a visualization of the wave speed. The slope of the straight line in the figure is the wave speed, which is obtained from the measurements (Tab. 6.2). For all propagation distances, the wave form of the signal stays almost the same. However, the amplitudes decrease with increasing distance due to attenuation.

In Tab. 6.2, the values determined and calculated from the data in both figures are listed. The error of the wave speed both for the longitudinal and the shear wave is very small. Comparing the results of the through-transmission, to the peak-to-peak measurements, reveals the difficulty in determining the arrival of the impulse for the through-transmission technique. The times determined (especially for the shear wave)

(a) Transmitted signal

(b) Received signal

Figure 6.6: Longitudinal wave in polystyrene: Input signal ($D=6$), Received signal for thickness $d_1 = 4.071$ cm, $d_2 = 6.465$ cm and $d_3 = 10.344$ cm The straight line shows the wave speed determined from the data in Tab. 6.2.

(a) Transmitted signal

(b) Received signal

Figure 6.7: Shear wave in polystyrene: Input signal ($D=6$), Received signal for thickness $d_1 = 4.071$ cm, $d_2 = 6.465$ cm and $d_3 = 10.344$ cm The straight line shows the wave speed determined from the data in Tab. 6.2

Table 6.1: Material properties for General purpose and High Impact Polystyrene provided by Boedeker Plastics

	General Purpose PS (GPPS)	High Impact PS (HIPS)
density [kg/m^3]	1050	1040
tensile modulus [GPa]	3.10	1.79

tend to be too large. Thus the wave-speeds determined by measuring the time for the through-transmission tend to be lower than those of the peak-to-peak measurement.

Out of these measurements, μ and λ are obtained as

$$\mu_{PS} = 0.795735 \text{ GPa}$$

$$\lambda_{PS} = 3.023210 \text{ GPa}$$

Using the conversion, $E = \mu \frac{3+2\mu/\lambda}{1+\mu/\lambda}$, the elastic-modulus is calculated as $E_{PS} = 2.22 \text{ GPa}$. This result is within the range that is provided by the manufacturer. Boedeker Plastics provides relatively imprecise values that are shown in Tab. 6.1. Unfortunately, the values given in [17] greatly differ from the values determined in this experiment. This implies that the simulation results will never exactly match the results of the experiment. In order to get a better match between simulation and experiments, the TOE-constants for the polystyrene in this measurement would need to be determined in a series of measurements as described in [17]. However, by measuring the polarization dependence on uniaxial stress, one should be able to observe a trend and obtain a qualitative verification of the model.

Table 6.2: Time-of-flight and wave speeds for longitudinal and shear waves in Polystyrene

	longitudinal waves		
d [m]	t [s]	v [m/s]	% off mean
transmit trough			
0.06465	3.0610E-05	2112.05	-0.242%
0.06465	3.0630E-05	2110.68	-0.176%
0.06465	3.0690E-05	2106.55	0.020%
0.04071	1.9280E-05	2111.51	-0.216%
0.04071	1.9340E-05	2104.96	0.095%
0.04071	1.9320E-05	2107.14	-0.009%
0.10344	4.9140E-05	2105.01	0.093%
0.10344	4.9200E-05	2102.44	0.215%
0.10344	4.9240E-05	2100.73	0.296%
peak to peak			
0.02394	1.1400E-05	2100.00	0.330%
0.06273	2.9750E-05	2108.57	-0.076%
0.03879	1.8350E-05	2113.90	-0.329%
Average		2106.96	
standard dev		4.54	
	shear waves		
d [m]	t [s]	v [m/s]	% off mean
transmit trough			
0.06465	7.3260E-05	882.47	-0.863%
0.06465	7.3300E-05	881.99	-0.808%
0.06465	7.3300E-05	881.99	-0.808%
0.04071	4.5760E-05	889.64	-1.682%
0.04071	4.5880E-05	887.31	-1.416%
0.04071	4.6040E-05	884.23	-1.064%
0.10344	1.1810E-04	875.87	-0.108%
0.10344	1.1800E-04	876.61	-0.193%
0.10344			
peak to peak			
0.02394	2.8000E-05	855.00	2.277%
0.06273	7.3400E-05	854.63	2.319%
0.03879	4.5400E-05	854.41	2.345%
Average		874.92	
standard dev		13.59	

6.3 Polarization measurements

These experiments combine the results of the previous experiments to measure the polarization of a Rayleigh surface wave.

Measurement description for aluminum specimen As described in the previous chapter, the Rayleigh wave is generated with the wedge technique. The input signal is a ten-cycle sinusoidal wave with a frequency of $f=1.25$ MHz. The signal amplitude is adjusted such that the amplified input signal has an amplitude of 200 V peak-to-peak. The signal repetition rate is set to 50 ms ensuring that the previously generated wave has completely disappeared before the next Rayleigh wave is generated. In order to get good scattering of the incident laser beam (from the LDV-system), a small aluminum pad is glued on the specimen. This pad is cut out of a standard kitchen aluminum foil, and the dull side faces out. The Rayleigh wave is detected 5 cm away from the generating wedge. These Rayleigh waves are detected with the LDV-system, and are averaged over 2000 measurements.

While preparing and adjusting the measurement setup, the strength of the laser signal is closely watched. For these measurements, the power of the laser is set to 100 mW. After the signal is split and frequency shifted its signal strength is measured again. After mirror M4 and M5, the signal strength is 27.2 mW and 43.4 mW respectively. The signal strength of the reflected, collected light is then again measured for the in-plane and the out-of-plane detection at the exit of fiber 4, after the beam splitter. For in-plane detection, the signal strength of the reflected, collected signal of both fiber 1 and fiber 2 is together $0.46 \mu\text{W}$. For the out-of-plane detection, the signal strength of the reflected, collected signal is $0.41 \mu\text{W}$, while the signal strength of the reference beam (fiber 3) is $1.4 \mu\text{W}$.

The polarization of the Rayleigh wave is first measured in the unstressed aluminum specimen. Then, in a second set of measurements, a force of 29 kN is applied to the

specimen, creating a the state of stress $t_{11}^i = 26.16$ MPa.

During the whole experiment, all other constants that are known to influence the propagation properties of a Rayleigh wave are kept constant. The temperature is kept at 20.1 °C. The aluminum sample remains unchanged and thus the influence of texture and surface roughness stay the same. Furthermore, the frequency of the Rayleigh wave is fixed at 1.25 MHz.

6.3.1 Results for the unstressed case

The raw measured data detected with the LDV-system is presented in Fig. 6.8. The signal in the upper plot of Fig. 6.8 is proportional to the in-plane particle velocity at the measured point. The signal in the lower plot of Fig. 6.8 is proportional to the out-of-plane particle velocity superimposed, by the in-plane velocity (see section 5.3.1).

At 33 μ s, the Rayleigh wave arrives, which is in good agreement with the theoretically calculated propagation time of 32.2 μ . As can be seen in Fig. 6.8, the signal to noise ratio (SNR) is very low. It is only 4.8 dB for the in-plane-component, and 7.8 dB for the out-of-plane component.

Figure 6.9 shows the magnitude of the Fourier transform of the measured signal. The highest peak is clearly located at 1.25 MHz — the input frequency of the signal. The height of this peak is almost two time higher for the out-of-plane signal. This also indicates the higher SNR of the out-of-plane signal. A second peak is observed close 2.25 MHz, which is the center frequency of the A403-S transducer.

This raw measurement data has to be post-processed. In order to decompose the in-plane component from the out-of-plane signal, Eq. (5.12) is applied to the raw, out-of-plane signal. Additionally, the raw in-plane signal is scaled by applying Eq. (5.10). Furthermore, a low-pass filter is designed to reduce the high frequency noise. In Fig. 6.10, the frequency response of this 100-th order least square low-pass

Figure 6.8: Raw measurement data of the LDV for Rayleigh wave in aluminum. Out-of-plane component contains still in-plane signal. Specimen is unstressed.

Figure 6.9: Magnitude of the FFT of the of raw measurement data shown in Fig. 6.8.

filter is shown. The passband frequency f_p is set to 1.25 MHz, and its magnitude drops below 20 dB around 1.7 MHz.

After decomposition, scaling, and filtering of the raw measured data, the post-processed signal is plotted in Fig. 6.11. Notice that the time axis is zoomed in to the relevant part of the signal.

By scaling and decomposing the signals, the ratio of the amplitudes change dramatically. The maximum in-plane *velocity* of the measured particles is two times higher than the maximum out-of-plane velocity. This ratio also applies for the absolute maximum *displacements* of the measured particle, since the integration of the plotted sinusoidal signals in Fig. 6.11 does not change the ratio between the then calculated maximum in- and out-of-plane displacements.

This ratio of $\frac{\bar{u}_1^{\text{measured}}}{\bar{u}_2^{\text{measured}}} = \frac{1}{1.99}$ does not fit the theoretically predicted results. The ratio that is obtained in the numerical simulation is $\frac{\bar{u}_1^{\text{sim}}}{\bar{u}_2^{\text{sim}}} = \frac{1}{0.6}$ (as shown in Tab. A.3 in Appendix A). In [4] a similar experiment is performed and the polarization of the Rayleigh wave is determined. A rather good agreement between theoretical computed

Figure 6.10: 100-th order least square low-pass filter with a pass-band up to 1.25 MHz

and the measured polarization is reported.

This large discrepancy between simulation and experimental results might be explained by the influence of the aluminum pad glued to the surface for better detection of the in-plane signal. Most likely, the influence of this pad cannot be neglected, as initially thought. Two factors might contribute to this discrepancy. First, the movement of the particle on the aluminum specimen is affected by the aluminum pad, and must now be modeled as a layered media. Second, it might be that the aluminum pad is not perfectly bonded onto the aluminum specimen. For bonding an epoxy glue was used, specially designed for bonding metals together. Though the displacement of the ultrasonic waves are small (in this measurement the maximum displacement is around 100 nm), the accelerations at a frequency of 1.25 MHz becomes extremely high (around $6 \times 10^6 \text{ m/s}^2$). Normally, glue transmits shear forces much better than it does normal forces. This property, combined with the extremely high accelerations might cause the pad to not move in a synchronization with the aluminum specimen, and thus the measured displacements reflect only partially the movement of the Rayleigh wave on the aluminum specimen.

Figure 6.11: In-plane and out-of-plane signal of Rayleigh wave in aluminum. Filtered and decomposed signal. Specimen is unstressed.

6.3.2 Results for stressed case

The measurement of polarization of the stressed aluminum specimen uses exactly the same measurement setup and post-processing. At a stress level of $t_{11}^i = 26.16$ GPa, the relative polarization is expected to be around 0.094 % (see Tab. A.3). The exact number cannot be predicted since the TOE-constants for the aluminum specimen (Alcoa Al-6061) are not known.

Figure 6.12 shows the post-processed results of the measurement with the stressed specimen. The signal looks very similar to the unstressed signal. The main goal here is to determine a difference of the polarization between the measurements of the unstressed and the stressed specimen, as it is defined in Eq. (4.10).

As mentioned earlier, the measured signals have a very low SNR. The numerical model demonstrates that the predicted changes in Rayleigh wave amplitude induced by an applied stress field are very small — these amplitude changes are especially small when compared to the experimental error of the system. This error, which is mainly due to the noise of the LDV-system, makes it clear that the changes in polarization due to changes in applied stress are at the limit of the measurable precision of this LDV-system. Figure 6.13 shows the difference in the raw measured data of the two measurements (the difference between the unstressed and stressed displacement components). The signals plotted in Fig. 6.13 indicate that both signals (unstressed and stressed) are very similar to each — the plots in Fig. 6.13 basically contain noise. The noise level is relatively high, so high that there is not a discernable change in amplitude during the critical time window between 35 μs and 43 μs .

So far, all the plots contained the measured in- and out-of-plane *velocities*. The absolute particle *displacement* can be obtained by integration. Figure 6.14 shows this integrated signal for the stressed specimen. Both plots show an increasing or decreasing tendency. This is a result of the measurement noise, which does not have a mean of zero. The maximum displacements measured are 125 nm for the in-plane,

Figure 6.12: In-plane and out-of-plane signal of Rayleigh wave in aluminum. Filtered and decomposed signal. Specimen is stressed.

Figure 6.13: Difference of raw measurement data of unstressed and stressed case.

and 63 nm for the out-of-plane signal.

The predicted relative polarization, which is around 0.094 % for the applied stress of $t_{11}^i = 26.16$ MPa, cannot be converted directly into an expected change in absolute displacements, since only the changes of the displacement ratios are predicted by the numerical simulation (and not the absolute change of each of the in-plane and out-of-plane displacement components). There are three combinations of how a change of 0.094% of the ratio $\Pi = \frac{\bar{u}_1}{\bar{u}_2}$ can occur. The first two possibilities are that just the u_1 displacement changes, and u_2 stays the same, or that u_2 changes, and u_1 stays the same. The third possibility is that both u_1 and u_2 change. The largest change in amplitude of either the maximum in-plane or the maximum out-of-plane displacements at a change of 0.094 % of the ratio $\Pi = \frac{\bar{u}_1}{\bar{u}_2}$ occurs if the larger amplitude is changed and the smaller amplitude remained unchanged. In the experimentally measured case, 0.094 % of the larger amplitude of 125 nm (which is the in-plane component) is 0.1 nm. This number (0.1 nm) is exactly at the edge of the minimum detectable displacement of the LDV-system as shown in Tab. 5.2. Though the numbers in Tab. 5.2 are experimentally determined for a frequency of 100 kHz (and can't be directly applied to the sensitivity of the system at 1.25 MHz), they still indicate that the changes in polarization might be measurable if the measurement system is further tuned, or the input signal (Rayleigh wave amplitude) is increased.

A helpful visualization of the particle motion can be made by plotting the filtered and decomposed, out-of-plane signal over the in-plane signal, which yields a trajectory plot. Since the ratio of the in-plane to the out-of-plane *displacement* is the same as the ratio of the in-plane to the out-of-plane *velocity*, the out-of-plane *velocity* is plotted over the in-plane velocity in Fig. 6.15. This avoids the problems that arise with a non-zero-mean noise-signal. As expected, both signals are phase-shifted by 90°, and thus form the shape of an ellipse whose axes are located almost parallel to the coordinate axes. The fact that the curves are located closer to each other at the top and the

Figure 6.14: Absolute in-plane and out-of-plane displacement of Rayleigh wave in aluminum. Specimen is stressed

Figure 6.15: Movement of a particle. Out-of-plane velocity plotted over in-plane velocity.

bottom portion of the ellipse (as compared to the left and the right part of the ellipse) again help visualize the higher SNR for the out-of-plane measurement.

6.3.3 Conclusion

This series of measurements demonstrate that the numerically predicted changes in polarization can not be experimentally validated with the measurements made in this research. There are three main reasons why these experimental measurements are not successful:

- The experimental SNR is too low, so that the expected changes in polarization are smaller than the noise level itself.
- The numerically predicted changes in polarization are extremely small, and are in the range of the minimum of the experimentally detectable displacements.
- The aluminum pad (used to get a better reflection and thus an increase the SNR of the in-plane-measurements) complicated the behavior of the surface waves, making the comparison invalid.

However, it still might be possible to experimentally measure these changes in polarization. By changing the setup, the experiment might be successful. The following changes are suggested for future work:

- Increase the SNR. This can be done by using a higher amplitude input signal, and by obtaining a higher carrier signal in the LDV-system. The higher carrier signal can be achieved by further tuning of the LDV-system. Previous experience shows that the carrier signal can still be significantly improved.
- Increase the applied load in the specimen. The force applied to the specimen is far lower than the yield strength of aluminum. The stress could be increased

by 10 times (up to more than 250 MPa), without either buckling or yielding occurring. These higher stresses would be useful to validate the numerically predicted behavior.

- Replace the aluminum pad. Instead of using the aluminum pad, the surface of the aluminum specimen can be treated to give the required scattering reflection. Preliminary experiments on the aluminum sample show that almost the same signal strength can be achieved by roughening up the specimen surface with fine sandpaper and steel wool.

By making these improvements, it should be possible to significantly increase the quality of the signal (increase SNR) and to measure the changes in the polarization of the Rayleigh wave.

CHAPTER 7

Conclusion

This research investigates the influence of applied stresses on the polarization of Rayleigh surface waves. An analytical model is developed which is used to predict the changes in both the wave speed and the polarization of a Rayleigh surface wave. Stroh's formalism is used to express the analytical model in a compact matrix notation. This formulation makes it possible to implement the numerical solution of the model in a straight forward way. Several solution methods are applied to solve this numerically ill conditioned problem. The simulation results are promising, and indicate that the expected changes in the polarization of a Rayleigh wave are larger than the analogous changes in wave speed. For some materials, the changes in polarization are an order of magnitude higher than the corresponding changes in wave speed. A sensitivity analysis investigates the influence of uncertainties in the values of the TOE-constants on the resulting changes in polarization and wave speed — small changes in the values of the TOE-constants have a greater effect on the wave speed than on the polarization. This gives the proposed polarization technique a natural advantage (in terms of robustness) over the wave speed technique, since it is difficult to independently measure accurate values of the TOE-constants. These simulation results also demonstrate that the polarization of a Rayleigh wave is a more sensitive and more robust indicator of the state of applied stress level than the wave speed.

The proposed polarization measurement technique shows several advantages when

compared to more commonly used wave speed techniques like the sing-around technique and the pulse-echo technique. The polarization measurements are truly point-wise, which enables (with a suitable inversion algorithm) the determination of the applied stress at a particular local point, instead of only obtaining an average applied stress over the propagation path of the Rayleigh wave (as is the case for wave speed measurements). The polarization measurement is independent of the wave source, and the exact location of the source does not necessarily need to be exactly known — the measurement of applied stresses with the polarization of Rayleigh waves is thus a reference-free measurement.

The practical use of the proposed polarization measurement has not been definitely demonstrated in this research — this research shows that the changes in polarization due to “realistic stress levels” are at the limit of (or possibly just below the limit of) the accuracy of the LDV used in this study. However, improvements to the existing experimental procedure (which are beyond the scope of this research) would enable definitive proof of the viability of the proposed technique.

Further work can be done to investigate the influence of non-homogeneous stresses on the polarization of a Rayleigh wave. Another area of interest is the characterization of the influence of other material properties on the polarization of a Rayleigh wave. Candidate material properties that could also effect Rayleigh wave polarization include texture, temperature and surface roughness. Combining the effect of these properties into one model would demonstrate which material factor (if any) dominates the polarization, and would give an additional indication of the robustness of the proposed technique.

APPENDIX A

Complete simulation results

The complete simulation results for various materials are listed in the following tables. Besides the *relative values* for the speed and polarization as used in section 4.4, the *absolute values* for the wave speed and the polarization vector are listed in the tables. In section 4.2 it was pointed out, that the polarization can be determined by Eq. (3.38) up to a scalar. In the tables below the polarization vector is normed to the magnitude of one.

Table A.1: Simulation results for mild steel

t_{11}^i	c_r	$\text{abs}(\det(\mathbf{B}))$	Polarization vector			Δc_r	$\Delta \Pi$
-50.0	2989.83	3.33E-011	0.553310	0.832975	0.000000	2.854E-05	-4.585E-04
-40.0	2989.82	1.18E-012	0.553346	0.832952	0.000000	2.284E-05	-3.669E-04
-30.0	2989.80	3.82E-011	0.553381	0.832928	0.000000	1.714E-05	-2.752E-04
-20.0	2989.78	1.42E-010	0.553416	0.832905	0.000000	1.143E-05	-1.835E-04
-10.0	2989.76	3.79E-009	0.553451	0.832882	0.000000	5.719E-06	-9.180E-05
0.0	2989.75	1.96E-010	0.553487	0.832858	0.000000	0.000E-00	0.000E-00
10.0	2989.73	1.05E-010	0.553522	0.832835	0.000000	-5.725E-06	9.185E-05
20.0	2989.71	1.17E-010	0.553557	0.832811	0.000000	-1.145E-05	1.837E-04
30.0	2989.70	6.91E-011	0.553592	0.832788	0.000000	-1.719E-05	2.757E-04
40.0	2989.68	3.49E-011	0.553628	0.832764	0.000000	-2.293E-05	3.677E-04
50.0	2989.66	1.05E-005	0.553663	0.832741	0.000000	-2.868E-05	4.598E-04
1/MPa	1.711E-03		3.525E-06	-2.343E-06	0.000E+00	-5.722E-07	9.182E-06

Table A.2: Simulation results for pure aluminum

t_{11}^i	c_r	$\text{abs}(\det(\mathbf{B}))$	Polarization vector			Δc_r	$\Delta \Pi$
-50.0	2894.11	4.90E-012	0.538602	0.842560	0.000000	4.997E-04	-1.780E-03
-40.0	2893.83	5.02E-012	0.538738	0.842473	0.000000	4.000E-04	-1.426E-03
-30.0	2893.54	4.98E-012	0.538874	0.842386	0.000000	3.002E-04	-1.071E-03
-20.0	2893.25	1.48E-012	0.539010	0.842299	0.000000	2.002E-04	-7.146E-04
-10.0	2892.96	6.45E-012	0.539147	0.842212	0.000000	1.002E-04	-3.577E-04
0.0	2892.67	7.11E-011	0.539284	0.842124	0.000000	0.000E-00	0.000E-00
10.0	2892.38	2.22E-012	0.539421	0.842036	0.000000	-1.003E-04	3.585E-04
20.0	2892.09	3.34E-013	0.539558	0.841948	0.000000	-2.007E-04	7.178E-04
30.0	2891.80	2.20E-012	0.539696	0.841860	0.000000	-3.012E-04	1.078E-03
40.0	2891.51	5.30E-012	0.539833	0.841772	0.000000	-4.019E-04	1.439E-03
50.0	2891.22	3.07E-014	0.539972	0.841683	0.000000	-5.026E-04	1.801E-03
1/MPa	2.899E-02		1.369E-05	-8.770E-06	0.000E+00	-1.002E-05	3.581E-05

Table A.3: Simulation results for Al B53S

t_{11}^i	c_r	$\text{abs}(\det(\mathbf{B}))$	Polarization vector			Δc_r	$\Delta \Pi$
-50.0	2912.16	4.09E-012	0.534054	0.845450	0.000000	1.435E-04	-1.388E-03
-40.0	2912.07	2.48E-013	0.534160	0.845383	0.000000	1.149E-04	-1.111E-03
-30.0	2911.99	3.79E-012	0.534266	0.845316	0.000000	8.626E-05	-8.344E-04
-20.0	2911.90	3.93E-012	0.534372	0.845249	0.000000	5.756E-05	-5.568E-04
-10.0	2911.82	4.10E-012	0.534478	0.845182	0.000000	2.881E-05	-2.787E-04
0.0	2911.74	6.26E-009	0.534585	0.845115	0.000000	0.000E-00	0.000E-00
10.0	2911.65	1.04E-012	0.534691	0.845048	0.000000	-2.887E-05	2.792E-04
20.0	2911.57	5.00E-012	0.534798	0.844980	0.000000	-5.780E-05	5.589E-04
30.0	2911.48	2.55E-012	0.534905	0.844912	0.000000	-8.679E-05	8.392E-04
40.0	2911.40	6.06E-012	0.535012	0.844844	0.000000	-1.158E-04	1.120E-03
50.0	2911.32	5.41E-007	0.535119	0.844777	0.000000	-1.450E-04	1.401E-03
1/MPa	8.398E-03		1.065E-05	-6.737E-06	0.000E+00	-2.884E-06	2.789E-05

Table A.4: Simulation results for polystyrene

t_{11}^i	c_r	$\text{abs}(\det(\mathbf{B}))$	Polarization vector			Δc_r	$\Delta \Pi$
-25.0	1045.51	1.06E-011	0.528659	0.848834	0.000000	1.491E-03	-1.855E-02
-20.0	1045.22	1.63E-011	0.530026	0.847981	0.000000	1.211E-03	-1.502E-02
-15.0	1044.92	1.65E-010	0.531423	0.847106	0.000000	9.229E-04	-1.140E-02
-10.0	1044.61	8.99E-012	0.532851	0.846209	0.000000	6.251E-04	-7.697E-03
-5.0	1044.29	4.10E-014	0.534310	0.845289	0.000000	3.176E-04	-3.897E-03
0.0	1043.96	3.52E-014	0.535801	0.844345	0.000000	0.000E-00	0.000E-00
5.0	1043.62	6.00E-014	0.537325	0.843375	0.000000	-3.279E-04	3.997E-03
10.0	1043.26	4.03E-016	0.538883	0.842380	0.000000	-6.664E-04	8.099E-03
15.0	1042.90	2.13E-016	0.540477	0.841359	0.000000	-1.016E-03	1.231E-02
20.0	1042.52	5.58E-016	0.542107	0.840309	0.000000	-1.377E-03	1.663E-02
25.0	1042.13	5.05E-017	0.543775	0.839231	0.000000	-1.750E-03	2.107E-02
1/MPa	6.765E-02		3.023E-04	-1.921E-04	0.000E+00	-6.480E-05	7.923E-04

Table A.5: Simulation results for pyrex glass

t_{11}^i	c_r	$\text{abs}(\det(\mathbf{B}))$	Polarization vector		Δc_r	$\Delta \Pi$
-25.0	3165.95	1.85E-012	0.585032	0.811011	-1.386E-03	6.700E-04
-20.0	3166.83	4.25E-012	0.584980	0.811048	-1.109E-03	5.358E-04
-15.0	3167.71	1.90E-012	0.584928	0.811085	-8.312E-04	4.017E-04
-10.0	3168.59	2.68E-012	0.584877	0.811122	-5.540E-04	2.677E-04
-5.0	3169.47	3.83E-012	0.584825	0.811159	-2.769E-04	1.338E-04
0.0	3170.35	1.24E-012	0.584774	0.811196	0.000E-00	0.000E-00
5.0	3171.23	1.40E-008	0.584722	0.811233	2.767E-04	-1.337E-04
10.0	3172.10	2.48E-013	0.584671	0.811271	5.533E-04	-2.673E-04
15.0	3172.98	1.42E-012	0.584620	0.811308	8.297E-04	-4.007E-04
20.0	3173.85	7.09E-013	0.584568	0.811345	1.106E-03	-5.341E-04
25.0	3174.73	3.18E-007	0.584517	0.811382	1.382E-03	-6.674E-04
1/MPa	-1.755E-01		-1.029E-05	7.420E-06	5.536E-05	-2.675E-05

Table A.6: Simulation results for brass

t_{11}^i	c_τ	$\text{abs}(\det(\mathbf{B}))$	Polarization vector		Δc_τ	$\Delta \Pi$
-50.0	1844.33	2.65E-012	0.529354	0.848401	3.227E-04	-1.912E-03
-40.0	1844.21	4.47E-012	0.529499	0.848310	2.583E-04	-1.532E-03
-30.0	1844.09	4.69E-012	0.529645	0.848219	1.939E-04	-1.150E-03
-20.0	1843.97	1.15E-011	0.529791	0.848128	1.294E-04	-7.677E-04
-10.0	1843.85	7.55E-012	0.529937	0.848037	6.476E-05	-3.843E-04
0.0	1843.73	5.19E-004	0.530084	0.847945	0.000E-00	0.000E-00
10.0	1843.61	6.95E-005	0.530230	0.847854	-6.485E-05	3.853E-04
20.0	1843.49	6.55E-012	0.530377	0.847762	-1.298E-04	7.715E-04
30.0	1843.37	1.25E-012	0.530525	0.847669	-1.949E-04	1.159E-03
40.0	1843.25	3.30E-012	0.530673	0.847577	-2.601E-04	1.547E-03
50.0	1843.13	7.35E-012	0.530821	0.847484	-3.254E-04	1.936E-03
1/MPa	1.195E-02		1.467E-05	-9.168E-06	-6.481E-06	3.848E-05

APPENDIX B

Third-order elastic constants (TOE) for selected materials

The Lamé-constants and TOE-constants for selected materials are listed in the following tables. Table B.1 lists isotropic TOE-constants, as opposed to anisotropic TOE-constants in table B.2.

Table B.1: Lamé-constants and TOE-constants, isotropic material

Material	λ	μ	ν_1	ν_2	ν_3	Reference
Cu-Zn	85.3	33.5	-687.0	-298.0	-710.0	[19]
Aluminum 2S	-57.0	27.6	-228.0	-197.0	-102.0	[35]
Aluminum D54S	-49.1	26.0	-379.0	-198.0	-80.0	[35]
Aluminum B53S	58.0	26.0	-249.0	-99.0	-69.0	[35]
Aluminum JH77S	57.5	26.8	-320.0	-177.0	-109.0	[35]
Molybdenum (sintered)	157.0	110.0	-51.0	-283.0	-193.0	[35]
Polystyrene	2.9	1.4	-21.2	-8.3	-2.5	[17]
Pyrex Glass	13.5	27.5	264.0	-118.0	105.0	[17]
Steel Hecla 17	110.5	82.0	-134.0	-261.0	-167.0	[35]
Steel Hecla 37	111.0	82.1	-358.0	-282.0	-177.0	[35]
Rail steel	115.8	79.9	36.0	-266.0	178.5	[11]
Tungsten (sintered)	75.0	73.0	-215.0	-143.0	-124.0	[35]

Table B.2: TOE-constants, anisotropic material

	C_{111}	C_{112}	C_{123}	C_{144}	C_{166}	C_{456}	Reference
Ag	-84.3	-52.9	18.9	5.6	-63.7	8.3	[15]
Au	-172.9	-92.2	-23.3	-1.3	-64.8	-1.2	[15]
Cu	-127.1	-81.4	-5.0	-0.3	-78.0	-9.5	[15]
Germanium	-71.0	-38.9	-1.8	-2.3	-29.2	-5.3	[26]
Silicon	-82.5	-45.1	-6.4	1.2	-31.0	-6.4	[26]

Bibliography

- [1] J.D. Achenbach. *Wave propagation in solids*. Elsevier Science Publishers B.V., 8th edition, 1999.
- [2] Jens Becker. Investigation of the microstructure of heterogeneous materials using ultrasonic waves. Master's thesis, School of Civil and Environmental Engineering, Georgia Institute of Technology, August 2002.
- [3] Ilja N. Bronstein, Konstantin A. Semendjajew, Gerhard Musiol, and Heiner M'uhlig. *Taschenbuch der Mathematik*. Verlag Harry Deutsch, 1997. in German.
- [4] A. Cand and J.-P. Monchalin. Detection of in-plane and out-of-plane ultrasonic displacements by a two-channel confocal fabry-perot interferometer. *Applied Physics Letter*, 64(4):414–416, 1994.
- [5] D.I. Crecraft. Ultrasonic wave velocities in stresses nickel steel. *Nature*, 195(4847):1193, 1962.
- [6] D.I. Crecraft. The measurement of applied and residual stresses in metals using ultrasonic waves. *Journal of Sound and Vibration*, 5(1):173–192, 1967.
- [7] L.E. Drain. *Laser Doppler Technique*. John Wiley & Sons, 1980.
- [8] Marc Dressler. Theoretical and experimental modeling modeling of ultrasonic waves in multi-layered media. Master's thesis, School of Civil and Environmental Engineering, Georgia Institute of Technology, August 2002.

- [9] M. Duquennoy, M. Ouaftough, and M. Ourak. Ultrasonic evaluation of stress in orthotropic materials using rayleigh waves. *NDT&E International*, 32(4):189–199, 1999.
- [10] M. Duquennoy, M. Ouaftough, M.L. Qian, and M. Ourak. Ultrasonic evaluation of stress in orthotropic materials using rayleigh waves. *NDT&E International*, 34(5):355–362, 2001.
- [11] D.M. Egle and E.D. Bray. Measurement of acoustoelastic and third-order elastic constants for rail steel. *Journal of the Acoustical Society of America*, 60:741–744, 1976.
- [12] H. Fukuoka and A. Toda. Preliminary experiment on acoustoelasticity for stress analysis. *Archives of Mechanics*, 29(5):673–686, 1977.
- [13] Phillip L. Gould. *Introduction to Linear Elasticity*. Springer Verlag, 1994.
- [14] Mark F. Hamilton and David T. Blackstock, editors. *Nonlinear Acoustics*. Academic Press, San Diego, CA, 1998.
- [15] Yosio Hiki and A.V. Granato. Anharmonicity in noble metals; higher elastic constants. *Physical Review*, 144(2):411–419, 1966.
- [16] M. Hirao, H. Fukuoka, and K. Hori. Acoustoelastic effect of rayleigh surface wave in isotropic material. *Journal of Applied Mechanics*, 48:119–124, 1981.
- [17] D.S. Hughes and J.L. Kelly. Second-order elastic deformation of solids. *Physical Review*, 92(5):1145–1149, December 1953.
- [18] D.C. Hurley. Non-linear propagation of narrow-band rayleigh waves excited by a comb transducer. *Journal of the Acoustical Society of America*, 106(4):1782–1788, 1999.

- [19] Shoji Kanemochi and Michijiro Akaboshi. The third-order elastic constants for cu-zn alloy from ultrasonic velocity measurements. *Japan. J. Appl. Phys.*, 17(7):1277–1278, 1978.
- [20] Hyun-Gwon Kil, Jacek Jarzynski, and Yves H. Berthelot. Wave decomposition of the vibrations of a cylindrical shell with an automated scanning laser vibrometer. *Journal of the Acoustical Society of America*, 104(6):3161–3168, 1998.
- [21] M.B Klein, G.D. Bacher, A. Grunnet-Jepsen, D. Wright, and W.E. Moerner. Homodyne detection of ultrasonic surface displacements using two-waves mixing in photorefractive polymers. *Optics communication*, 162:79, 1999.
- [22] V.V. Krylov and Z.A. Smirnova. Experimental study of the dispersion of a rayleigh wave on a rough surface. *Sov. Phys. Acoust.*, 36:583–585, 1990.
- [23] A.I. Lavrentyev and W.A. Veronesi. Ultrasonic measurement of residual stress in shot peened aluminum alloy. *Review of Progress in Quantitative Nondestructive Evaluation*, 20:1472–1479, 2001.
- [24] Q. Li. Measurement of acoustic properties of materials using torsional waves. Master's thesis, Georgia Institute of Technology, 1998.
- [25] Mathworks. Nonlinear systems of equations. Internet, 2003. <http://www.mathworks.com/access/helpdesk/help/toolbox/optim-tutor11b.shtml>.
- [26] H.J. McSkimin and P. Andreatch Jr. Measurements of third-order moduli of silicon and germanium. *Journal of Applied Physics*, 35(11):3312–3319, 1964.
- [27] Julius Miklowitz. *The theory of elastic waves and waveguides*. North-Holland Pub. Co, Amsterdam, New York, 1978.

- [28] Francis D. Murnaghan. *Finite deformation of an elastic solid*. Wiley, New York, 1951.
- [29] Alan V. Oppenheim and Ronald W. Schaffer. *Discrete-Time Signal Processing*. Prentice Hall, 1998.
- [30] Yih-Hsing Pao, Wolfgang Sachse, and Hidekazu Fukuoka. Acoustoelasticity and ultrasonic measurements of residual stresses. *Physical Acoustics*, 17:61–143, 1984.
- [31] William H Press, Brian P. Flannery, Saul A. Teukolsky, and William T. Vetterling. *Numerical recipes*. Cambridge University Press, 1986.
- [32] Joseph L. Rose. *Ultrasonic Waves in Solid Media*. Cambridge University Press, 1999.
- [33] Daniel A. Russel. Longitudinal and transverse wave motion, 1999. <http://www.gmi.edu/~drussell/Demos/waves/wavemotion.html>.
- [34] R.T. Smith. Stress-induced anisotropy in solids, the acousto-elastic effect. *Ultrasonics Journal*, 1:135–147, 1963.
- [35] R.T. Smith, R. Stern, and R.W.B Stephens. Third-order elastic moduli of polycrystalline metals from ultrasonic velocity measurements. *Journal of the Acoustical Society of America*, 40(5):1002–1008, 1966.
- [36] Jens Christian Stolzenburg. Non-contact method to measure the material properties of layered media. Master's thesis, Georgia Institute of Technology, 2002.
- [37] A.N. Stroh. Steady state problems in anisotropic elasticity. *Journal of Mathematics & Physics*, 41:77–103, 1962.
- [38] Thomas C.T. Ting. *Anisotropic Elasticity*. Oxford University Press, 1996.

- [39] R.A. Toupin and B. Bernstein. Sound waves in deformed perfectly elastic materials. acoustoelastic effect. *Journal of the Acoustical Society of America*, 33(2):216–225, 1961.
- [40] R.A. Toupin and R.A. Rivlin. Dimensional changes in crystals caused by dislocation. *Journal of Mathematical Physics*, 1(1):8–15, 1960.
- [41] Ansel C. Ugural and Saul K. Fenster. *Advanced Strength and Applied Elasticity*. Prentice Hall PTR, 1995.
- [42] I.A. Victorov. *Rayleigh and Lamb waves: Physical theory and applications, with a foreword by Warren P. Mason*. Plenum Press, New York, 1967.