A Mechanics Framework for Modeling Fiber Deformation on Draw Rollers and Freespans

A Thesis
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

School of Polymer, Textile and Fiber Engineering
Georgia Institute of Technology
August 2006
A Mechanics Framework for Modeling Fiber Deformation on Draw Rollers and Freespans

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Date Approved April 27, 2006
ACKNOWLEDGEMENTS

I would like to thank Prof. Karl I. Jacob, Prof. Stephen E. Bechtel and Center for Advanced Engineering Fibers and Films (CAEFF) at Clemson for providing partial funding for the project.
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Case 1 of fiber behavior on the draw pin, in which the fiber attaches to the pin with speed \( v_5 \) less than the preceding second roller speed \( r \omega_2 \). Subcase 1a: The fiber does not fully reload on the pin \( (v_5 < r \omega_2) \), where \( v_5 \) denotes the departure speed from the pin.

Subcase 1b: The fiber fully reloads on the pin but does not draw into its final stiff state \( (r \omega_2 < v_5 < v_b) \).

Subcase 1c: The fiber draws into its final stiff state on the pin \( (v_b > v_5) \).

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Subcase 2b: The fiber draws to its final stiff state on the pin \( (v_5 > v_b) \).

Isothermal draw process without draw pins: On the left speed \( (v) \) vs. arclength \( (s) \) and on the right tension \( (T) \) vs. arclength. Locations denoted by \( (\times) \) are where the fibers either attach to or depart from a roller, \( (\circ) \) are where initiation or cessation of slip on the rollers takes place, \( (\triangle) \) indicated locations where the transitions between stiff and soft fiber behavior happen. In the velocity and tension profiles, \( (- - -) \) are zones of no slip, and \( (. . . .) \) are free spans.

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Isothermal draw process with a draw pin: tension vs. speed, 1: \( v = r \omega_1 \) in \( x_0 \leq s \leq y_1 \), 2: \( v = v_a \) at \( s = y_a \), 3: \( v = r \omega_2 \) in \( x_1 \leq s \leq y_3 \), 6: \( v = v_3 = 550 \) cm s\(^{-1} \) in \( x_3 \leq s \leq x_4 \), 7: \( v = v_b \) at \( s = y_b \), 8: \( v = r \omega_2 \) for \( s \geq x_5 \). Note that the fiber now draws partly on the first roller (from 1 to 3), partly on the second roller (from 3 to 6), and partly on the draw pin (from 6 to 8).
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 Isothermal draw process with a draw pin: In the plot of tension vs. speed 1: \( v = r_1 \) in \( x_0 \leq x \leq y_1 \), 2: \( v = v_2 \) at \( s = y_a \), 3: \( v = r_2 \) in \( x_1 \leq s \leq x_4 \), 7: \( v = v_5 \) at \( s = y_b \), 8: \( v = r_3 \) for \( s \geq x_5 \). The fiber now draws only on the first roller (from 1 to 3), and on the pin (from 3 to 8). 

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Frictional force per unit length $f(s)$ from the pulleys on the belt for stiff belt ($k = 25\text{kN}$, top) and a compliant belt ($k = 0.2\text{kN}$, bottom): Full solution (-----), engineering solution (----), alternate solution (· · · ·), and capstan solution (---). The full, engineering, and alternate solutions are indistinguishable for the stiff belt case.
SUMMARY

Polymeric fibers are generally produced by a process called melt spinning. The resulting fiber, called as spun fiber, is relatively weak, showing a large central plastic zone or natural draw, in its stress-strain response. In order to improve its mechanical behavior the as-spun fiber is drawn beyond the plastic region in various drawing zones during the fiber drawing process. Fiber drawing is usually done by a process called multistage draw which consists of wrapping the fiber over a series of rollers having incrementally higher rotational speeds. The speed difference between successive rollers causes the fiber to draw on each roller due to the resulting friction between the fiber and roller surfaces. The draw produces greater orientation of the polymer chains in the axial direction which enhances mechanical characteristics of the fiber.

The objective of this work is to develop a new model for studying a two stage fiber draw process. Morphological aspects related to draw are not addressed in detail, this approach is mainly directed to understanding the constitutive behavior in a two stage draw process. This task is accomplished in several stages. The set of conservation equations for the fiber on a roller and free span are developed first. These equations are obtained by using the classical control volume approach of continuum mechanics. By augmenting the conservation equations with a constitutive model of the fiber and solving the resulting system of equations, the velocity and tension of the fiber can be predicted.

The equations of motion are first applied to fiber draw on a roller under isothermal conditions using a simple linearly elastic fiber material model. The predicted response of the fiber is developed as a function of three non-dimensional parameters reflecting fiber stiffness, fiber tension, and the length of the deformation zone. The new model takes into account both the centrifugal acceleration and stretching acceleration. For a certain range of the three non-dimensional parameters, the full solution can be approximated either by
the quasi-static solution by neglecting both acceleration terms in the momentum equation, commonly referred to as the capstan solution, or by the engineering solution which retains only the centrifugal acceleration.

A multistage draw consists of several rollers which draw the fiber sequentially. A two stage draw process is a particular case of the multistage draw and consists of three draw rollers. By using a constitutive characterization of the fibers to a piecewise linearly elastic-plastic to capture the softening behavior of as-spun fibers and by incorporating temperature dependence, the fiber behavior in the two stage draw is found by assembling the governing equations for fibers on rollers and in free spans, together with matching boundary conditions, to produce a comprehensive model for a two stage, non-isothermal industrial draw process. The model is then employed to simulate representative draw lines having isothermal conditions, heated free span, and heated rollers.

In a commercially relevant fiber drawing process, draw pins are used for localizing and enhancing the fiber draw. Inclusion of a draw pin introduces the possibility of mechanical unloading in the fiber, resulting in additional complexity for the elastic-plastic fiber characteristics in the analysis. In the absence of unloading, the stress in the fiber monotonically increases, so that only the elastic response of the fiber enters into the analysis. In analyzing an isothermal two stage draw process with a draw pin in the second free span, it is observed that the total draw is generally distributed between the first roller, the second roller, and the draw pin. The offset location of the pin is shown to have a significant effect on the distribution of the draw as well as the tension profile. As this offset is increased, the nature of the draw, stress, or tension on the second roller changes from loading, to neither loading nor unloading at a critical offset, to unloading. Under isothermal conditions the unloading of the fiber was entirely mechanical in nature. For non-isothermal conditions however the unloading is supplemented by thermal unloading due to the softening of the fiber as the temperature is increased. By adjusting the draw pin offset and the heating in the second free span it is possible to achieve an optimum setting in which there is no unloading on the second roller and all the draw takes place on the draw pin. This of course has economic benefits for the process as the damage on the second roller due to friction is mitigated and
shifted to the draw pin which is made of a cheaper and replaceable material. An additional factor which permits the draw to be transferred to the draw pin is the final take up speed. With the amount of loading or unloading on the second roller established by a given thermal profile and draw pin offset, it is possible to change the draw profile by altering the final take up speed. As the final roller speed is increased the amount of unloading on the second roller decreases. By fine tuning the final roller speed the unloading on the second roller can be decreased to zero and all the loading or draw, is transferred to the draw pin. The effectiveness of the draw pin depends on the compliance of the fiber. For a given increment in tension the amount of draw induced is greater for soft fibers than for stiff fibers. So it is more effective to introduce draw pins at locations where the fiber velocity in the fiber tow corresponds to the tension values lying in the soft plateau.

The model depends on the friction profile. With the creep rate dependent friction model the magnitude of the friction force is directly proportional to the relative velocity between the roller and the fiber in contrast to the Coulomb friction model where the friction force is dependent on the the sign of the relative velocity. Whereas the Coulomb friction model graphically resembles a jump function, the creep rate dependent friction regularizes the step function to a linear relation. The fiber velocity profiles show that at sufficiently large values of the creep rate coefficient the solution profiles approach that obtained by using Coulomb friction.

The significant new contribution of this work to existing literature in contrast to all existing solutions, is the inclusion of inertia and stretching effects in the governing equations. It is observed that the effectiveness of draw is dependent on the set of parameters mentioned earlier. With this normalization it is possible to ascertain what process conditions result in optimal draw.
CHAPTER I

INTRODUCTION

1.1 Process Outline

The manufacturing process for melt spun polymer fibers consists of first extruding molten polymer through a spinneret and then cooling it into filaments as it undergoes extensional flow before being wound on wind-up rollers. Melt spinning is followed by drawing which involves uniaxial extension of the solidified filaments above the glass transition temperature, see Figure 1. Drawing of spun fibers is designed to produce filaments with enhanced mechanical properties, which is accomplished by inducing sufficient orientation of the polymer molecules along the axial direction of the filament. During the spinning process polymer exits the capillary of the spinneret usually with a die swell, which removes most of the molecular orientation in the polymer when it passes through the capillary of the spinneret. The mostly isotropic polymeric melt is then stretched in the spinline downstream of the die swell while it cools, inducing some molecular orientation prior to solidification. The relaxation time of the polymer in the melt is sometimes comparable to the time available for the fiber before it solidifies, however, thereby undoing much of this orientation, and hence the amount of orientation that can be induced in the fiber during spinning is insufficient to produce good mechanical properties.

The orientation of polymer molecules can be increased after the spinning process by a subsequent drawing process. During the drawing process the solidified, as-spun fiber is heated above the glass transition temperature and drawn over a series of rollers. The purpose of the draw process is to convert relatively weak as-spun fibers to fibers with greater molecular orientation and crystallinity resulting in better mechanical properties. Draw enhanced morphology and micro-structure is responsible for improved properties in fibers and films [13, 14, 18, 19]. Using Nuclear Magnetic Resonance (NMR), Botev et al. [13] showed that for Polyamide-6 fibers, drawing temperature plays a key role in determining
Figure 1: Schematic representation of the manufacturing process of melt spun polymeric fiber.

whether the draw is effective. Cansfield et al. [14] studied the effect of windup speeds and
draw ratios on the mechanical properties, molecular orientation, and shrinkage behavior of
PET fibers. They report that the effect of increasing the windup speed of the spun fiber
is similar to an increase in molecular weight and hence the viscosity of the polymer. The
effect of drawing produces the same network structure for the different molecular weight
fibers however the amount of amorphous material is less in high molecular weight polymer,
due to more but smaller crystallites, than low weight polymer. Postema et al. [29] found
that homogeneous drawing, achieved by drawing at low deformation rate, is most efficient
in improving strength for Poly (L-Lactide) fibers. Inhomogeneous drawing under high
deformation rates with temperature gradients resulted in reduced fiber properties. However,
in a manufacturing process it is important to use a draw process with high deformation rate
in order to improve productivity, but the temperature strain rate combinations that optimize
fiber properties need to be identified.

Figure 2: A two-stage draw process. The rollers run at angular speeds $\omega_1 < \omega_2 < \omega_3$ and the draw pin is stationary.

During draw the degree of crystallinity increases due to the formation of new crystalline regions as well as the redistribution of existing crystalline regions. Salem [32] investigated the influence of uniaxial draw strain on crystallization in PET films, under both constant extension rate and constant strain rate conditions. It was found that at high strain rates the crystallization rate increases, while the pattern gradually reverses as strain rate is reduced. Hermanutz et al. [19] studied the effect of draw, and the resulting micro-structure, on surface properties. They found that the wettability of an unrestrained PET film decreases with increase in draw ratio. This is attributed to two factors; an increase in amorphous orientation and crystallization. Amorphous orientation of polymer chains along the draw axis produces greater molecular interaction between the chains which reduces the availability of sites capable of interacting with a liquid. Likewise crystallization also results in decrease of polymer-liquid interactions due to alignment and close packing arrangement of crystalline regions. Hence a decrease in polymer-liquid interaction on the surface indicates an increased order in underlying the polymer structure. Gohil and Salem [18] studied the effect of bi-axial drawing of PET film on amorphous orientation. They investigated the effect of simultaneous drawing on producing a “balanced” bi-axial orientation to enhance strength in both directions which they found to be entirely dependent on a balanced distribution of amorphous orientation - with the orientation of the crystallites playing no detectable role.

A wide variety of drawing processes are used in industry. The main reason for differences
in drawing process is based on whether the product is a continuous filament yarn or a staple tow (where filaments will be cut to short fiber segments at a later stage). In drawing, filament yarn consists of fewer filaments than a staple yarn, which contains thousands of filaments in the tow. As a consequence, PET filament yarn is generally drawn with radiant heating in the free span between the rollers. The rollers are not heated, and the draw occurs in the freespan. Staple PET tow is sometimes heated using a hot water bath during drawing. As the rollers are also enclosed in the chamber, they are also heated. Due to the large number of filaments present in a staple tow, a set of feed rollers and a set of take-up rollers are used in the drawing process. Each set of feed rollers are moving with the same angular velocity, and draw is achieved by rotating each set of rollers faster than previous set.

In some processes all the draw is introduced in one step, in a single stage of feed and the take-up rollers. This single-stage draw can result in fiber breakage; hence, a two-stage draw is mostly used in the manufacturing process, (Figure 2). In the two-stage process most of the draw (generally between 2.2 to 2.7 draw ratio) is induced in the first stage and a relatively smaller draw (1.1 to 1.2) is applied in the second stage to improve filament strength and modulus. One could use multiple draw stages beyond the two to introduce additional draw. The optimal number of stages in the draw is generally determined from extensive experimental effort. As the number of stages is increased, it is possible to keep each free span and roller at a different temperature in order to induce the maximum possible draw in each stage in order to obtain the maximum molecular orientation in the fiber. This is the idea behind the new Incremental Draw Process (IDP) [36], in which draw is induced progressively in the fiber using a large number of incremental stages.

A comprehensive treatment of modeling the thermomechanical response of fibers under different combinations of thermal histories and windup speeds in a drawing process is lacking in the literature. In this work a model to quantify the drawing process will be developed, with the development of a fundamental theoretical basis to analyze draw on rollers and in the free spans between rollers. The restriction to draw in two stages is for specificity only; models for drawlines with more than two stages can straightforwardly be constructed
following the procedure that will be demonstrated for two stages in this work.

1.2 Melt Spinning

Melt spinning is the most common and versatile process for producing synthetic fibers for different applications, [34]. Properties of spun fibers depend on the conditions during the spinning process and on the material properties of the polymer which depend on the chemical composition, rheological properties, and crystallization kinetics of the polymer. Over the last four decades predictive models have been developed that provide a fairly reasonable understanding of the influence of process conditions and material properties on the structure and properties of the fiber.

The modeling of the melt spinning process is fairly complicated since the process variables, such as - extrusion temperature, mass flow rate through the spinneret, take-up velocity, spinline cooling, length of spin line, shape and spacing of orifice are coupled. For instance the length of the spinline is dependent on the rate of cooling in the spinline, so at high cooling rates shorter spinlines are developed and vice versa. Furthermore the rate of spinline cooling in turn is dependent on the the conditions of the cooling air. The most important variable is the take-up velocity which has a significant effect on the structure and properties of the spun fiber and therefore the fiber behavior during subsequent drawing and texturing.

The process variables discussed above basically enforce the boundary conditions for the spinning process and the material variables define the constitutive behavior. Material variables can be split into two categories, one effects the rheological properties and the other the solidification behavior. The variables effecting rheology (i.e. the viscoelastic behavior) of the melt are: polymer molecular weight and its distribution, chain stiffness and branching, presence of additives and fillers. Melt spinning involves the solidification of the polymer as it is quenched by the cooling air. In semicrystalline polymers solidification and crystallization depends on; temperature, crystallinity, crystallization kinetics, composition, stereo regularity, molecular weight, influence of additives such as nucleating agents, antioxidants, pigments etc. In amorphous polymers solidification or vitrification is strongly dependent
on the glass transition temperature.

The analysis of the spinning process using mass, momentum, and energy conservation equations coupled to an appropriate constitutive equation and evolution equations for orientation and crystallinity has been under steady development since the 1960s.

Kase [23] proposed a process model which captures the essential features of a typical melt spinning process. To keep the model reasonably simple it was assumed that the process is steady, fiber section is circular, polymer density is constant, flow is purely extensional with uniform velocity across the fiber section, specific heat of the polymer is constant, temperature across fiber section is uniform, and axial heat conduction is neglected. Despite these assumptions reasonable predictions can be made about the effect of different process variables and their interaction.

Next, the spin line model is briefly outlined starting with the conservation equations in one dimension. The mass flow rate $G$ from the conservation of mass requires

$$G = \rho Av,$$

where $\rho$ is the filament density, $A$ is the cross sectional area, and $v$ is the fiber speed.

The conservation of momentum for a one dimensional flow gives, see Ziabicki [37],

$$G \frac{dv}{dx} = \frac{dT}{dx} - \frac{1}{2} \rho_a v^2 C_d \sqrt{4\pi A} + \rho Ag,$$

where $T = \sigma A$ is the fiber tension obtained by assuming uniform stress $\sigma$ across the fiber section. The drag force on the fiber is determined from the density $\rho_a$ of the cooling air, the relative velocity of the fiber with respect to the air is $v_r$ which here is taken to equal $v$ by assuming that it has no component parallel to the fiber speed. The drag coefficient $C_d$ is related to the Reynolds number ($Re = \frac{2\rho_a v^2}{\mu_a}$, where $\nu_a = \frac{\mu_a}{\rho_a}$ is the kinematic viscosity of air, $r$ is the fiber radius, and $\mu_a$ is the dynamic viscosity) by $C_d = \frac{K}{(Re)^n}$. Shimizu et al. [33] have taken $K = 0.37$ and $n = 0.61$. The contribution from surface tension $\frac{d}{dx}(\gamma \sqrt{\pi A})$ is usually neglected except for spinning of low molecular weight polymers. Gravitational force is small compared to the other forces except when spinning thick fibers. So for high molecular weight polymers under usual spinning conditions the drag and inertial forces dominate.
The conservation of energy that accounts for the heat transfer to the fiber and heat generation due to crystallization is given by, see [33, 24],

\[ Gc \frac{d\theta}{dx} = \sqrt{4\pi Ah(\theta - \theta_a)} + G\Delta H \frac{d\chi}{dx}, \]

due to crystallization is given by, see [33, 24],

\[ Gc \frac{d\theta}{dx} = \sqrt{4\pi Ah(\theta - \theta_a)} + G\Delta H \frac{d\chi}{dx}, \]

here \( c \) is the heat capacity of the polymer, \( \theta \) is the fiber temperature, and \( \theta_a \) is the ambient air temperature. The first term gives the amount of heat transfer from the fiber to the cooling air. The expression for convective heat transfer coefficient \( h \) was obtained experimentally by Kase [23] by relating the Nusselt number \( (Nu = \frac{2h}{ka}) \) to the Reynolds number \( (Re = \frac{2\nu}{\nu_a}) \) empirically by \( Nu = 0.42(Re)^{\frac{1}{2}}(1 + (\frac{8\nu_a}{\nu})^2)^{\frac{1}{2}}, \) where \( \nu_a \) is the cross flow velocity of the cooling air, \( ka \) and \( \nu_a \) is the thermal conductivity and kinematic viscosity of air. Since the polymer is crystallizing, the heat released is captured by the second term with heat of fusion is given by \( \Delta H \) and the degree of crystallinity by \( \chi \).

In general viscoelastic constitutive equations are used to model molten polymers. However most of the spinning processes use modest molecular weight polymers which can be modeled by a purely viscous constitutive equation as opposed to a viscoelastic equation used to model high molecular weight polymers. In addition, at high rates of elongation observed in the spinning process a viscoelastic material will show elastic behavior, however the decrease of temperature along the spin line increases viscosity which becomes more dominant than elastic effects thus justifying the use of a purely viscous constitutive equation. For such a purely viscous model of the melt the viscosity is taken to be a function of temperature \( \theta \), the strain rate \( \dot{\varepsilon} \), and degree of crystallinity \( \chi \). The general form of the constitutive equation is

\[ \sigma = \eta(\theta, \dot{\varepsilon}, \chi) \frac{dv}{dx}, \]

where \( \sigma \) is the stress along the axial direction of the fiber, \( \eta(\theta, \dot{\varepsilon}, \chi) \) is the viscosity, and \( \frac{dv}{dx} \) is the velocity gradient in the axial direction of the fiber.

The viscosity of a Newtonian fluid is independent of the strain rate i.e. \( \eta = \eta_0 \), where \( \eta_0 \) is a constant. For a Non-Newtonian fluid the viscosity is rate dependent. One frequently used relation is a power law which gives \( \eta = \frac{\eta_0}{1 + a(\dot{\varepsilon})^b} \). Where \( \eta_0 \) is the zero strain viscosity and \( a \) and \( b \) are empirical constants.
The temperature dependence of viscosity on temperatures much greater than the melting point is captured by an Arrhenius type relation i.e. \( \eta = \eta_0 e^{\frac{B}{T}} \), where \( \eta_0 \) and \( B \) are empirical constants. For ranges of temperature close to the glass transition a WLF type relation given by \( \eta = \eta_0 e^{\frac{C_1[\theta - \theta_g]}{C_2[\theta - \theta_g]}} \) is used, where \( \eta_0, C_1, C_2 \) are empirical constants and \( \theta_g \) is the glass transition temperature.

When crystallization sets in the viscosity undergoes a rapid increase, Shimizu et al. [33] have used the following relation

\[
\eta(\theta, \dot{\varepsilon}, \chi) = \eta(\theta, \dot{\varepsilon})e^{c\chi^4}.
\]

(5)

to incorporate the effect of crystallinity on viscosity. Ziabicki in [40] has proposed a model of crystallization based on the idea that as the polymer crystallizes the polymer chains get interconnected through crystallites i.e. as physical crosslinks. At a certain critical value of the number of these physical crosslinks, the polymer loses its fluidity and with further increase in crystallinity it becomes more rigid. The proposed viscosity relation is given by,

\[
\eta(\theta, \dot{\varepsilon}, \chi) = \eta(\theta)(1 - \frac{\chi}{\chi_{cr}})^{-\alpha}.
\]

(6)

where \( \eta(\theta) \) is a function of temperature only and \( \chi_{cr} \) is the critical crystallinity at which complete solidification is reached, i.e. viscosity approaches infinity.

Polymers have low nucleation rates close to their melting temperature and a low growth rate close to their glass transition temperature due lack of mobility. So the maximum crystallization rate happens at some intermediate temperature. Also irrespective of the temperature, there is always a finite amount of time needed to crystallize the material. Hence in a spinline the amount of crystallization will depend on crystallization kinetics and the cooling rate. In addition, the presence of molecular orientation also increases rate of crystallization of a polymer. The effect of cooling rate and crystallization kinetics is shown in Figure 3. The 'c-curve' gets shifted to the left either with increased stress or faster crystallization kinetics resulting in faster crystallization compared to the stress free case.

Equations for crystallization rates for melt spinning are difficult to develop due to the non-isothermal conditions and development of molecular orientation due to stress. The
Figure 3: The influence of cooling rate and stress on crystallization behavior, (figure from [34]).

theory for crystallization even in the absence of molecular orientation is not well established and even less is known in the presence of molecular orientation. Taking the Avrami equation, which is used for isothermal crystallization with, \( k = K(\theta)^n \) the crystallization rate constant and \( n \) is the Avrami index,

\[
\chi = 1 - e^{-(K(\theta)t)^n},
\]

as the basis; a model developed by Nakamura et al. [27, 26] is found to be the best for non-isothermal conditions, the model is given by

\[
\chi = 1 - e^{-\int_0^t K(\theta)dt'^n},
\]

where \( \chi \) is the degree of crystallinity. The effect of molecular orientation is included by extending the crystallization factor to include molecular orientation i.e. \( K = K(\theta, f) \), where \( f \) is the Hermann’s orientation function. A form of \( K \) proposed by Ziabicki [37] is

\[
K(\theta, f) = K_{\text{max}}e^{-\frac{\theta-\theta_{\text{max}}}{D^2}}e^{A(\theta)f^2}
\]

where \( A(\theta) \) is an empirical parameter to be determined, \( K_{\text{max}} \) is the rate constant at \( \theta_{\text{max}} \) the temperature at which the crystallization rate is the maximum, and \( D \) is the half width of \( K \) vs \( \theta \) plot. Another relation in use was proposed by Katayama et al. [24] has the
following form

\[ K(\theta, f) = K_0 e^{\left( -\frac{U^*}{R(\theta - \theta_{inf})} \right)} e^{\left( -\frac{C_3}{\theta \Delta \theta + C_2 \theta^2} \right)}, \]

where \( U^* \) is the activation energy, \( R \) is the gas constant, \( \theta_{inf} = \theta_g - 30K \), where \( \theta_g \) is the glass transition temperature, \( \Delta \theta = \theta^0_m - \theta \), here \( \theta^0_m \) is the melting temperature under quiescent conditions. The values of \( K_0 \) and \( C_3 \) are obtained from quiescent crystallization kinetics. The value of \( C \) is adjusted to make the simulation agree with the experimental results.

The above discussion pertains to a single filament. The actual industrial process however is a **multi-filament** set-up i.e. a bundle of fibers is spun. Hence both the velocity and temperature of the cross flow air changes within the fiber bundle which imposes different boundary conditions on the individual fibers. Typically the cross flow air velocity will decrease and temperature increase over the width of the bundle while both these effects decrease away from the spinneret. As a result different amount of heat is transferred from the fibers in the bundle resulting in different structure and properties.

Another simplification adopted was the assumption of uniform radial distribution of temperature and velocity across the filament section. The assumption of pure extensional flow resulting in uniform velocity across the filament section is valid under most melt spinning conditions. The temperature and crystallinity distribution however have been found by Katayama et al. [24] to show a radial variation.

Numerical simulation to solve the set of mass, momentum, energy equations coupled to the constitutive equation along with crystallization has been done over the years to predict evolution of several process variables that permits the prediction of fiber properties to be made. Spruell [34] presents simulation results for representative polymer with physical properties like polyamide 6 with average molecular weight of 25000, and mass flow rate of 2.5g/min from a capillary. Figure 4 shows the results that claim to be consistent with the experimental observations, for instance crystallization takes place only at take-up speeds in excess of 5000m/min. The output shown in Figure 4 is obtained by using the mass, momentum, and energy equations listed earlier. For the Constitutive behavior the following
Figure 4: Simulation results for velocity, temperature, crystallinity, and birefringence for a representative polymer with properties like polyamide-6 (figure from [34]).

The relation is used

\[
\sigma = \begin{cases} 
A(M_w)^{3.55} \frac{H}{\eta T^{2/3} \pi} \frac{\partial u}{\partial x} & \text{if } \theta > \theta_m \\
A(M_w)^{3.55} \frac{H}{\eta T^{2/3} \pi} \left( a \left( \frac{x}{\lambda_{int}} \right)^b \right) \frac{\partial u}{\partial x} & \text{if } \theta < \theta_m
\end{cases}
\]

1.3 Microstructure

In the previous section the melt spinning process was described. During melt spinning the polymer microstructure evolves under the influence of process parameters which effect the fiber orientation, temperature, and the amount of process time.  

Flow: Primary factor that determines fiber morphology of is the nature of flow. A quiescent isotropic melt results in a folded chain (spherulitic) type of crystalline structure whereas under dynamic conditions, with either a high mass flow rate or high strain gradients,
an oriented melt will result which will crystallize into an extended chain (fibrillar) crystalline structure. Hence the degree of crystalline and amorphous orientation and therefore the type of crystal structure is directly dependent on the kind of flow the melt is subjected to.

The take-up velocity determines the strain gradient in the melt and thus controls the flow or deformation of the spinline. Crystallization induced by changes in take-up velocity are shown in Figure 5. A steep increase in birefringence and crystallinity starts at 4000m/min

![Birefringence, Density, Tenacity vs. Take-up Velocity](image)

**Figure 5:** Properties of PET yarn as function of take-up velocity (figure from [34] pp. 52).

then stabilizes before decreasing beyond 7000m/min. In Figure 6 the effect of take-up speed on % shrinkage (from boiling in water) shows an initial increase at low speeds due to formation of shrinkable oriented amorphous phase. At higher take-up speeds due to the formation of crystallites the orientation is solidified and shrinkage is restricted. The X-ray patterns for PET spun at different speeds were interpreted by Shimizu et al. [33] to have microfibrillar structure. The microfibrils are parallel to the spinline and are made of alternating crystalline blocks separated by amorphous regions as shown in Figure 7. At higher speeds the size of the crystalline blocks increases and the stagger angle between two adjacent crystalline blocks on neighboring fibrils decreases. The amorphous separation between two crystalline blocks in the same fibril decreases with increasing take-up speeds. Necking in the PET spinline is observed above 4500m/min and becoming more pronounced
as the speed is increased. The observed necking is like one seen during tensile testing though the physics is thought to be different, however no definite explanation has been found. Hypothesis advanced by Ziabicki [39] offers the best explanation, he suggests that the rapid rise of viscosity due to stress induced crystallization is responsible for the neck like flow. The morphological model suggested by Shmizu et al. [33] corresponds very well with the above hypothesis. The schematic model of the neck in Figure 8 shows the transition from the un-oriented melt to oriented mesophase and the subsequent development of a neck followed by crystallization into a microfibrillar structure.

In addition changes in mass flow rate also effect flow induced crystallization, though higher take-up speeds would be needed to counteract the effect increased mass flow rate. Also a high molecular weight polymer has a higher viscosity which results in increased stress resulting in higher orientation.

**Temperature:** Crystalization takes place as the polymer melt cools in the spinline. The temperature range in which this occurs lies between the glass transition temperature \( \theta_g \) and the melting point \( \theta_m \). The rate of crystallization reaches a maximum somewhere in this
\[ \tan \theta = 1.4 \]
\[ \tan \phi = 0.5 \]
\[ 9,000 \text{ m/min} \]
\[ 5,000 \text{ m/min} \]

**Figure 7:** Microfibrillar models for PET spun at 5000 and 9000 m/min (figure from [34] pp. 56).

As the take-up speed is increased the time available for the melt to crystallize decreases however at higher take-up speeds the crystallization rate increases which overwhelms the effect of reduced available time. The result is increased crystallinity at higher take-up speeds.

The combined effect of deformation (or stress), temperature, and time (rate of cooling) can be seen in the CCT (Continuous Cooling Transformation) curve like in Figure 3. Under quiescent conditions a longer time is needed for crystallization to take place at a given temperature. However under stress, the crystallization curve is shifted so that at the same temperature now the crystallization takes less time. From another point of view at higher stress levels higher cooling rates are possible but at low stress levels however the cooling curve misses the crystallization curve thus resulting in a amorphous structure.

Although a certain degree of orientation is achieved by increasing the take-up velocity, as
will be explained later it is still not sufficient for several applications. Hence the conventional process of melt spinning at low speeds followed by drawing at little over the glass transition temperature to produce increased molecular orientation, that will be discussed in the next section, is widely prevalent.

1.4 Draw of spun fibers

The objective of a fiber processing operation is to produce fibers with specific properties in the material for specific end uses by controlling the microstructure of the polymer. One of the most important properties is the mechanical strength, determined for instance by the ultimate strength of the fiber. The factors that contribute favorably to the increase in strength are stretch, alignment, and orientation of the polymer molecules. The theoretical value of axial modulus of $100 - 350$ GPa for commonly used fiber forming polymers can be achieved (hypothetically) by fully disentangling and aligning all the polymer molecules. Molecular orientation is only developed to a certain degree during melt spinning and to a greater degree during a sequential draw process. During spinning if the time available to the polymer molecules is less than the time it takes for them to relax then molecular
orientation will take place. Thus by altering the time available to the molecules to relax, done by changing the process conditions such as the take-up speed, the degree of molecular orientation can be controlled. Melt spun fibers produced at low take up speeds (less than 3000 m/min) have little orientation - so an additional sequential draw operation is necessary to produce required molecular orientation. Fibers produced at high speeds (above 3000 m/min) show significant molecular alignment and formation of oriented crystalline regions. However orientation of amorphous regions can be enhanced by an additional draw process and significant the chain folding seen in crystalline regions can also be mitigated. Hence a drawing process on the spun fibers becomes necessary. Apart from better strength achieved by drawing the as spun fiber, other properties that are intimately tied to the microstructure can be suitably enhanced by drawing. For instance a disordered microstructure in the amorphous component provides chain mobility above the glass transition temperature which is necessary for dye diffusion, the same disorder in the amorphous region facilitates recovery from large strains. By controlling the crystalline structure the degree stress relaxation in the amorphous regions can be controlled during heat setting. Process conditions such as strain rate, temperature, and initial microstructure determine the development and final state of the microstructure. Hence an optimal control of process variables allows the final polymer properties to be determined.

**Constant Extension Rate** (CER) experiment for fiber deformation is done in laboratories on a tensile test machine. One end of the sample is fixed and the other is made to move at a constant speed $v$. The true strain rate of the sample is then given by

$$\dot{\varepsilon} = \frac{1}{t + \frac{L_0}{v}},$$

where $\dot{\varepsilon}$ is the strain rate, $t$ is the time, $L_0$ is the initial length of the sample. The strain rates attained during a CER experiment are simpler (compared to constant force deformation) and orders of magnitude lower than those attained in a commercial drawing process. The experimental results nevertheless could shed light on the factors influencing the formation of microstructure under stress.
Crystallizable amorphous polymers - Crystallizable amorphous polymer like PET crystallizes as it is drawn. The effect of changing the strain rate on the stress versus draw ratio curve is shown in Figure 9 for a sample of amorphous PET film drawn at 90 C° [15]. As the

\[ \text{TRUE STRESS (MPa)} \]

\[ \lambda \]

Figure 9: True stress versus draw ratio of PET film at two strain rates and a draw temperature of 90 C°. The onset of crystallization E1 and the onset of regime 2 crystallization E2 are indicated, (figure from [32]).

film is film is extened the stress increases and an inflection in the curve is observed at E1 this is the point where the material starts crystallizing. With further draw crystallinity and crystalline orientation see a rapid increase between E1 and E2. At E2 the rate of growth of crystallinity and amorphous orientation decreases and the amount of crystallinity reaches a characteristic level for that temperature. The reason being that by the time E2 is reached a crystallite network has formed which provides rigid crosslinks, unlike entanglements which allow slippage. The above mentioned decrease in the orientation rate is due to the formation of tout intercrystalline chains which prevent the uncoiling and hence orientation of the neighboring tie chains. Furthermore, the increase in stress beyond E2 is mainly due to increase in viscosity resulting from the interconnections between crystallites. Deformation now proceeds by the translational slippage between fibrils formed by crystallites held.
together by extended tie chains.

At higher strain rates there is less time available for molecular chain relaxation resulting in increased stress and molecular orientation. The fast development of orientation results in an earlier onset of crystallization $E_1$ and shift of the final stiff zone to lower draw ratios. At lower strain rates the behavior is reversed. At strain rates lower than a critical value for a given temperature the time available for relaxation is large enough so that little or no orientation occurs, hence no orientation induced crystallization and no increased stiffness is seen. This kind of deformation behavior is termed as the so called flow drawing.

Increasing the draw temperature has a similar effect as decreasing the strain rate. As temperature is decreased the initial stiff part becomes stiffer and as the stress values increase the final stiff region gets shifted to the left. At draw temperatures lower than $\theta_g$ the initial stiff region sees a softening after reaching a maxima, this becomes more pronounced as the draw temperature is lowered further as shown in Figure 10. At temperatures much lower than $\theta_g$ (around 60 ° lower than $\theta_g$) the polymer behaves like a brittle solid. At higher temperatures the chains are more mobile resulting in lower molecular orientation with flow. Also at higher temperatures, due to higher mobility, the rate of crystallization is higher since the interaction between mobility and rate of crystallization plays an important role.

The effect increasing the molecular weight is similar to decreasing the temperature or increasing the strain rate. With increased molecular weight the number of entanglements increase which slows down the rate of molecular relaxation. Hence the molecules show more rapid orientation, so that the start of orientation $E_1$ and the final stiff zone are shifted to the left.

Crystalline polymers - The stress-strain response to temperature and strain rate of crystalline polymers is similar to amorphous polymers though the causes differ. In the early stages of draw the spherulites become elongated and at the yield point the chains tilt and slip within the chain folded crystals. If the draw temperature is high enough the chains partially unfold and lamellae break into crystallites connected by tie molecules, a microfibrillar structure. The material deforms with the microfibrils sliding past each other.

Increase in temperature results in a less stiff initial zone and a lower yield stress as
Figure 10: Typical stress strain curve for an unoriented polymer, showing the influence of temperature and strain rate, (figure from [32]).

the thermal energy added to the material will permit the breakup and reorientation of the lamellae with less mechanical energy. Chain slippage increases the extent of the plastic plateau before the final stiff zone.

Increasing strain rates produces higher initial stiffness, higher yield, and a shorter plastic flow zone. This strain hardening effect is due to the large resistance provided by the fibrillar structure to sliding motion.

Noncrystallizable amorphous polymers - like PMMA show a a low stiffness final section of the stress strain curve due to the absence of strain-induced crystallization. The rate of strain stiffening is also decreases drastically with increasing temperature above \( \theta_g \) due to increasing entanglement slippage. This is in contrast to crystallizable polymers which form crystallites that act as junctions for the polymer network thus increasing the viscosity associated with entanglement slippage.

*Constant Force:* In Constant Force (CF) deformation the force applied on the fiber is constant and the stress distribution within the fiber is similar to the stress strain curve
observed in CER. Difference between CER and CF is that in CF the microstructural evolution is reflected by the strain rate, for instance high viscosity polymers are difficult to deform, hence the strain rate for a given force will be low and as the viscosity changes with deformation so will the strain rate.

Since the viscosity changes with the microstructural parameters such as entanglement slippage, molecular orientation, and crystallization, the strain rate will change accordingly. The stress within the fiber develops according to both the strain and strain rate profiles in the free span. With a non uniform strain rate profile a neck will be formed with the stress value that will minimize the area above and below the stress line and the nominal stress-strain curve.

The deformation kinetics for a amorphous, unoriented PET film follows the following evolution with time; At a given load initially resistance of a coherent entanglement network results in low deformation rate. With further deformation entanglement slippage results in softening and increase in strain rate. At some point orientation induced crystallization leads to strain hardening and reduction in strain rate which finally reaches a limiting draw ratio where the resistance of the polymer to deformation is equal to the load. The effect of increasing the temperature is to accelerate the kinetics of deformation.

Monomers are strongly bonded along the chain direction but show weak interaction between chains. This opens up the possibility stretching out all the chains along the same direction. The purpose of drawing is to achieve this state.

1.5 Objectives, Assumptions and Limitations

The main objective of the thesis is to develop a mechanics based framework for modeling the draw of polymer fibers. A model for a two stage draw process is developed. This model does not fully incorporate and predict all the possible features observed in an actual industrial draw process but provides a basic methodology that applies to a simplified model for a typical polymeric material. The solutions obtained are a first approximation to the drawline behavior which can be further refined by using a better material model.
There are several simplifying assumptions that will be made while developing the equations of motion for the fiber and material properties of the fiber. The fiber is assumed to be a one dimensional line segment. Hence the fiber velocity, temperature, and tension are vary only along the axis of the fiber. Clearly due to the one dimensional assumption quantities such as heat that flow into the fiber from the surface are treated mathematically as "source" terms with a linear density variation along the length of the fiber. This assumption simplifies the calculations required to obtain the solution. The limitation being that the radial variations are neglected which according to [33] do exist.

It is also assumed that the draw process is taking place under steady conditions. This assumption seems acceptable since in an industrial setting the draw process would have been going for long enough for the transients to have died out. The limitation of this assumption is that any time dependent perturbations along the draw line cannot be predicted. Again most the the perturbations are observed during the spinning stage in which the polymer solidifies under a set of complex interactions between the temperature, crystallization, and fiber stress. The draw stage is less complicated since the fiber has already solidified.

Regarding the behavior of the polymer material several simplifying assumptions have also been made. The crystallinity induced for the amount of draw to be considered does not produce a substantial change in volumetric density. Hence the volumetric density of the polymer material is assumed to remain constant during the draw process. The fiber is assumed to have a piecewise linear relation between tension and strain (or velocity). This simplifying assumption mimics the behavior of uniaxial deformation of a typical polymer like PET with its initial stiff region, a soft plastic plateau, and a final stiff region. The assumption of piecewise linearity able to account for the prominent trends in the fiber deformation and also makes the solution process easier.
CHAPTER II

FORMULATION OF THE PROBLEM

A theoretical framework to predict the thermo-mechanical behavior of undrawn fibers during extensional strains is developed by satisfying the necessary conditions of continuum mechanics (i.e. conservation of mass, momentum, and energy). The formulation is fairly flexible, capable of accommodating any constitutive relationship. The necessary equations are developed in the following sections.

Consider an extensible fiber on a roller of radius $r$ moving at an angular speed $\omega$. The direction of increasing arclength $s$, roller angular speed $\omega$, and positive fiber speed $v$ are taken to be in the same direction. An Eulerian formulation is adopted with the point $s$ fixed in space. The development that follows is flexible that it can be used for analyzing torque transmission in conveyor belts, see Bechtel et al. [11], in addition to studying fiber drawing. The set of time dependent one dimensional conservation equations is developed based on the outline presented by Hodge [20].

2.1 Conservation of Mass

Figure 11 shows a free-body diagram of a space-fixed portion of the fiber of length $ds$ at location $s$, subtending an angle $d\theta$. The fiber in general is extensible so that its speed entering the control volume at $s$ is $v$ and the speed exiting it at $s + ds$ is $v + dv$. The mass per unit volume and area of cross section of the fiber are $\rho$ and $A$ respectively. The velocity, cross section area, and density are in general functions of both the independent variables arclength $s$ and time $t$, this dependence will not be explicitly indicated to simplify the presentation.

For conservation of mass, the control volume requires that

$$\text{(rate of accumulation of mass)} = \text{(net rate of mass influx)}.$$

At a given instant of time the total mass in the control volume is $\rho A ds$. The rate at which
mass accumulates in the control volume is then given by

$$\frac{\partial (\rho A)}{\partial t} ds.$$ 

The rate of influx of mass at $s$, where the fiber enters the control volume, is

$$(7) \quad G = \rho A v.$$ 

At $s + ds$ where the fiber exits the control volume the rate of mass efflux is given by

$$G + \frac{\partial G}{\partial s} ds.$$ 

Here $\frac{\partial G}{\partial s}$ accounts for the spatial change of the mass flux over the control volume. Applying the principle of conservation of mass to the arbitrarily small control volume of length $ds$ results in the following equation for the conservation of mass:

$$(8) \quad \frac{\partial (\rho A)}{\partial t} + \frac{\partial G}{\partial s} = 0.$$ 

For a steady process the time dependent term drops out and on integration Equation (8) gives

$$(9) \quad G = \rho_0 A_0 v_0 = \text{constant},$$
where $\rho_0$, $A_0$, and $v_0$ are values of density, section area, and fiber velocity at a reference state.

The conservation of mass for an element in the free-span, shown in Figure 12, also generates Equations (8) and (9) for unsteady and steady conditions respectively.

![Figure 12: A free body diagram of a section of fiber $ds$ in the free-span.](image)

### 2.2 Conservation of Momentum

Figure 11 shows the free-body diagram of a section fiber under tension $T$ at $s$ and $T + dT$ at $s + ds$. The force per unit length projected by the roller on the fiber in the normal and tangential directions is $n$ and $f$ respectively. The tangential frictional force $f$ is positive in the direction of decreasing $s$ and the normal force $n$ is positive radially outward (compressive). Where there is no slip the sign and magnitude of friction depends on the demands of the momentum equation. Where there is slip it is assumed that the magnitude of $f$ is given by

\begin{equation}
|f| = \mu(\theta)n,
\end{equation}

where, for specificity, the coefficient of friction is a function of temperature $\theta$ (as in Amijima [2], $\mu$ could also be considered as a function of filament speed $v$, resulting in only minor changes to the following analysis). According to the sign convention on friction, $f$ is positive if the fibers are moving faster than the roller surface (as in draw on a feed roller) hence Equation (10) becomes

\begin{equation}
f = \mu n,
\end{equation}
and \( f \) negative if the fibers are moving slower (as in draw on a take-up roller) hence Equation (10) becomes

\[
(12) \quad f = -\mu n.
\]

Aerodynamic forces and viscous heating are ignored at this stage of development.

For the fixed control volume of length \( ds \) the conservation of momentum states:

\[
(\text{rate of accumulation of momentum}) = (\text{net rate of momentum influx}) + (\text{sum of body and contact forces}).
\]

Since momentum is a vector quantity, its conservation must be considered in both the tangential and radial directions.

First consider the tangential direction, at a given instant the mass in the control volume of length \( ds \) is \( \rho Ads \). Using the entry velocity \( v \cos(\frac{d\theta}{2}) \) as the approximation to the fiber velocity in the control volume, the rate of change of momentum in the control volume calculated to be

\[
\frac{\partial(\rho Av)}{\partial t} \cos(\frac{d\theta}{2}) ds.
\]

The rate of influx of momentum in the tangential direction at \( s \) due to the motion of the fiber into the control volume is

\[
Gv \cos(\frac{d\theta}{2}).
\]

The fiber leaves the control volume at \( s + ds \) with a rate of momentum efflux of

\[
(Gv + \frac{\partial(Gv)}{\partial s} ds) \cos(\frac{d\theta}{2}).
\]

Here \( \frac{\partial(Gv)}{\partial s} \) accounts for the change in momentum flux over the length of the control volume.

The sources of momentum are from: the tension acting on the fiber element from the part of the fiber external to the control volume and friction force from the roller surface acting tangentially on the fiber element. The fiber tension in the tangential direction is \( -T \cos(\frac{d\theta}{2}) \) at \( s \) and \( (T + \frac{dT}{ds} ds) \cos(\frac{d\theta}{2}) \) at \( s + ds \). The frictional force is modeled as a body force with
a linear density of $f$ which gives a tangential force of $fds$ acting on the fiber element.

Applying the momentum conservation principle to the control volume gives

$$\frac{\partial (\rho Av)}{\partial t} \cos\left(\frac{\theta}{2}\right) ds = Gv \cos\left(\frac{\theta}{2}\right) - (Gv + \frac{\partial (Gv)}{\partial s} ds) \cos\left(\frac{\theta}{2}\right)$$

$$- T \cos\left(\frac{\theta}{2}\right) + (T + \frac{\partial T}{\partial s} ds) \cos\left(\frac{\theta}{2}\right) - fds$$

Using $|d\theta| \ll 1$ so that $\cos\left(\frac{\theta}{2}\right) \approx 1$ and the definition of $G$ from Equation (7) the above equation simplifies for an arbitrarily small control volume of length $ds$ to

(13) $$\frac{\partial G}{\partial t} + \frac{\partial (Gv)}{\partial s} = -f + \frac{\partial T}{\partial s}.$$ 

For a steady process $G$ is constant, so Equation (13) reduces to

(14) $$G \frac{dv}{ds} = \frac{dT}{ds} - f.$$ 

The rate at which momentum accumulates in the radial direction is given by

$$\frac{\partial (\rho Av)}{\partial t} \sin\left(\frac{\theta}{2}\right) ds.$$ 

The rate of momentum influx at $s$ is $Gv \sin\left(\frac{\theta}{2}\right)$ and the at $s + ds$ due to the fiber leaving the control volume is $-(Gv + \frac{\partial (Gv)}{\partial s} ds) \sin\left(\frac{\theta}{2}\right)$. The sign convention for the radial direction is positive away from the roller center and negative toward the roller center. Hence there is a negative sign because the direction momentum flux at $s + ds$ is toward the roller center.

The external forces which act as sources of momentum in the radial direction are; radial component of tension force acting on the fiber element and normal force exerted on the fiber element by the roller surface. The force acting on the fiber element is $-T \sin\left(\frac{\theta}{2}\right)$ at $s$ and $-(T + \frac{\partial T}{\partial s} ds) \sin\left(\frac{\theta}{2}\right)$ at $s + ds$. With density of the normal force per length equal to $n$, the total normal force in the radial direction acting on the control volume is $nds$. Application of the momentum conservation principle to the control volume gives

$$\frac{\partial (\rho Av)}{\partial t} \sin\left(\frac{\theta}{2}\right) ds = Gv \sin\left(\frac{\theta}{2}\right) + (Gv + \frac{\partial (Gv)}{\partial s} ds) \sin\left(\frac{\theta}{2}\right)$$

$$- T \sin\left(\frac{\theta}{2}\right) - (T + \frac{\partial T}{\partial s} ds) \sin\left(\frac{\theta}{2}\right) + nds.$$
Using $|d\theta| \ll 1$ so that $\sin(\frac{d\theta}{2}) \approx \frac{d\theta}{2}$ and neglecting terms with products of $ds$ (note $r d\theta = ds$), the above expression for an arbitrarily small control volume length $ds$ may be written as

\begin{equation}
(15) \quad n = \frac{T - Gv}{r}.
\end{equation}

With adhesive forces neglected, the normal force per unit length $n$ from the roller on the fibers must be compressive or zero (non-negative according to the adopted sign convention), which in combination with Equation (15) demands

\begin{equation}
(16) \quad T - Gv \geq 0; \nonumber
\end{equation}

as insufficiently-tensioned fibers will fly off the roller.

The control volume of a section of the free span of length $ds$ is shown in Figure 12. The rate of accumulation of momentum for the fiber element, which for the free span is only in the axial direction, is given by

\[ \frac{\partial(pAv)}{\partial t} ds. \]

The rate of momentum flux is $Gv$ at $s$ and $(Gv + \frac{\partial Gv}{\partial s} ds)$ at $s + ds$ where the fiber leaves the control volume. With air drag neglected, tension is the only force acting on the fiber taking a value $T$ at $s$ and $(T + \frac{\partial T}{\partial s} ds)$ at $s + ds$. Applying the momentum conservation principle to the control volume gives

\[ \frac{\partial G}{\partial t} ds = Gv - (Gv + \frac{\partial(Gv)}{\partial s} ds) - T + (T + \frac{\partial T}{\partial s} ds), \]

which for an arbitrarily small control volume of length $ds$ becomes

\begin{equation}
(17) \quad \frac{\partial G}{\partial t} + \frac{\partial(Gv)}{\partial s} = \frac{\partial T}{\partial s}. \nonumber
\end{equation}

For a steady process $G$ is constant, so Equation (17) reduces to

\begin{equation}
(18) \quad G \frac{dv}{ds} = \frac{dT}{ds}. \nonumber
\end{equation}
### 2.3 Conservation of Energy

The total energy $E$ in the control volume is assumed to be the sum of kinetic energy $K$ and internal $U$ energy. The gravitational potential energy $P$ is neglected. The conservation of energy for the control volume requires:

\[
\text{(rate of accumulation of energy)} = (\text{net rate of energy influx}) + (\text{sum of working and heating terms}),
\]

where working is the rate at which mechanical work is done on the fiber and heating is the rate at which heat is added.

Let $u$ be the internal energy per mass, then the rate of accumulation of total energy per mass $e$ in the control volume is given by

\[
\frac{\partial}{\partial t}(\rho Au + \frac{1}{2}\rho Av^2) ds.
\]

The rate of influx of energy at $s$ is

\[
Gu + \frac{1}{2}Gv^2,
\]

and the rate of efflux of energy at $s + ds$ where the fiber exits the control volume is

\[
Gu + \frac{1}{2}Gv^2 + \frac{\partial}{\partial s}(Gu + \frac{1}{2}Gv^2) ds.
\]

The rate at which energy is added to the fiber element from external mechanical work done on the fiber element by tension is $-Tv$ at $s$ and $(T + \frac{\partial T}{\partial s} ds)(v + \frac{\partial v}{\partial s} ds)$ at $s + ds$. There is a negative sign is the first term because tension and fiber velocity are vectors acting in opposite directions, hence the rate of mechanical work defined as their vector dot product acquires a negative sign. For friction the rate of work done is $-fv ds$. Normal force does no work since there is no corresponding radial velocity component of the fiber. Assuming constant heat conductivity $k$ of the fiber, the rate at which heat diffuses into the fiber along the axial direction at $s$ is $-kA\frac{\partial T}{\partial s}$ and the rate at which it is diffusing out at $s + ds$ is $-kA\frac{\partial T}{\partial s} - \frac{\partial}{\partial s}(kA\frac{\partial T}{\partial s}) ds$. The heat sources per length are lumped into a single term $q$ whose constituents will be discussed later.
Using the conservation of energy principle

\[ \frac{\partial}{\partial t}(\rho Au + \frac{1}{2} \rho Av^2)ds = \frac{1}{2} Gv^2 + Gu - (Gu + \frac{1}{2} Gv^2 + \frac{\partial}{\partial s}(Gu + \frac{1}{2} Gv^2)ds) \]

\[ - T v + (T + \frac{\partial T}{\partial s} ds)(v + \frac{\partial v}{\partial s} ds) - f v ds \]

\[ - k A \frac{\partial \theta}{\partial s} - (-k A \frac{\partial \theta}{\partial s} - \frac{\partial}{\partial s}(k A \frac{\partial \theta}{\partial s})ds) + q ds \]

for an arbitrary small control volume of length \( ds \) and using Equations (8) and (13) the expression above simplifies to

\begin{equation}
(19) \quad \rho A \frac{\partial u}{\partial t} + G \frac{\partial u}{\partial s} = T \frac{\partial v}{\partial s} + k \frac{\partial}{\partial s}(A \frac{\partial \theta}{\partial s}) + q.
\end{equation}

For the steady case the time dependent term drops out and equation (19) reduces to

\begin{equation}
(20) \quad G \frac{\partial u}{\partial s} = T \frac{\partial v}{\partial s} + k \frac{\partial}{\partial s}(A \frac{\partial \theta}{\partial s}) + q.
\end{equation}

For the free span shown in Figure 12 the same equations as above are produced. Since there is no friction in the freespan there will not be any frictional heating but radiant heat from an infra-red heating source may be included if present.

The heat source term \( q \) consists of heat absorbed by the fiber through the skin and heat generated throughout the volume of the fiber. There are several modes through which heat enters the fiber element through the fiber skin; by conduction from the heated draw roller with which the fiber is in contact with, by convection from heated ambient fluid which could be hot air, steam, or hot water, and by radiant heat from an infra-red heating source, for instance.

### 2.4 Constitutive Behavior

A constitutive form which idealizes the typical behavior of an undrawn PET fiber, rather than for a particular fiber, will be used. The fiber tow is characterized as thermo-elastic-plastic, i.e., the increment \( dT \) of tensile force in a fiber at a point depends on the tow’s axial strain \( \varepsilon \), strain increment \( d\varepsilon \), and temperature \( \theta \) at that point. Viscous effects are not included in this idealized constitutive model for fiber response during drawing. Viscous effects are highly dependent on the processing temperature, and the draw temperatures that
will be simulated are below the range where viscous effects are important. Moreover, the
draw processes to be simulated are relatively fast (greater than 450 cm/s in drawing spans
of about 450 cm), so that the residence times in the draw processes are relatively short
compared to the relaxation time of the polymer. Hence during these processes viscous
effects are likely to be insignificant (although there may be significant relaxation after the
draw).

Let the length of an infinitesimal element in the undeformed state be $dl_{ref}$ which deforms
to a length of $dl(s)$ in the deformed configuration. The axial strain $\varepsilon$ at a point $s$ of the
tow is defined to be the change in the length of the fiber from the undeformed to deformed
state divided by the original length in the undeformed state. Hence the strain $\varepsilon$ is given by

$$
\varepsilon(s) = \frac{dl(s) - dl_{ref}}{dl_{ref}} = \frac{dl(s)}{dl_{ref}} - 1.
$$

The state of the fiber tow as it attaches to the first roller is selected as the reference state,
i.e. the state to which zero strain is assigned. The tensile force in the tow as it attaches
to the first roller is labeled $T_0$. In the draw process $T_0$ is controlled. From constraint (16),
this upline tension $T_0$ must be greater than $Gr\omega_1$ as an insufficiently tensioned tow will fly
off the roller.

A thermoelastic-plastic constitutive equation for the fiber tow which models the behavior
of a soft draw plateau between stiffer regions is typical of as-spun, undrawn polymeric fibers.
For small strains the tow is stiff and elastic, with large modulus $K_1$. When the loading and
strain are monotonically increasing throughout the draw line, at a threshold value $\varepsilon_a$ of
strain the modulus abruptly softens to a value $K_2$ less than $K_1$, and beyond a second
transition strain $\varepsilon_b$ the modulus again stiffens to a value $K_3$ greater than the plateau $K_2$.
At strain $\varepsilon_c$ the filament breaks. The moduli $K_1, K_2,$ and $K_3$ and the strains $\varepsilon_a, \varepsilon_b,$ and $\varepsilon_c$
are specified functions of temperature $\theta$.

The three moduli $K_1, K_2, K_3$ and three transition strains $\varepsilon_a, \varepsilon_b, \varepsilon_c$ are specified functions
of temperature $\theta$. The resulting thermoelastic-plastic constitutive relation for positive strain
Table 1: Values of material constants used in Equation (22) to characterize the fiber.

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</tr>
<tr>
<td>(K_2)</td>
<td>0.002945E+07</td>
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<td>(\varepsilon_a)</td>
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</tr>
<tr>
<td>(\varepsilon_b)</td>
<td>0.6990</td>
<td></td>
</tr>
<tr>
<td>(\varepsilon_c)</td>
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</tr>
</tbody>
</table>

Increments \((d\varepsilon > 0)\) at maximum current strain \((\varepsilon = \varepsilon_{\text{max}})\) is

\[
dT = \begin{cases} 
K_1 d\varepsilon + \varepsilon \frac{dK_1}{d\theta} d\theta & \text{if } 0 \leq \varepsilon < \varepsilon_a, \\
K_2 d\varepsilon + \left( \varepsilon_a \left( \frac{dK_1}{d\theta} - \frac{dK_2}{d\theta} \right) + (K_1 - K_2) \frac{d\varepsilon_a}{d\theta} + \varepsilon \frac{dK_3}{d\theta} \right) d\theta & \text{if } \varepsilon_a \leq \varepsilon < \varepsilon_b, \\
K_3 d\varepsilon + \left( \varepsilon_a \left( \frac{dK_1}{d\theta} - \frac{dK_2}{d\theta} \right) + (K_1 - K_2) \frac{d\varepsilon_a}{d\theta} + \varepsilon_b \left( \frac{dK_2}{d\theta} - \frac{dK_3}{d\theta} \right) + (K_2 - K_3) \frac{d\varepsilon_b}{d\theta} + \varepsilon \frac{dK_3}{d\theta} \right) d\theta & \text{if } \varepsilon_b \leq \varepsilon \leq \varepsilon_c.
\end{cases}
\]

When the strain increment is negative \((d\varepsilon < 0)\) or the value of current strain is less than the maximum strain \((\varepsilon < \varepsilon_{\text{max}})\) then

\[
dT = K_1 d\varepsilon + \varepsilon \frac{dK_1}{d\theta} d\theta;
\]

The fiber is loading when the increment in tension is positive i.e., \(dT > 0\). The fiber unloads with the initial modulus \(K_1\) when \(dT < 0\). If there is no change in temperature for the section of the tow under consideration then for unloading it is sufficient for \(d\varepsilon < 0\) to be satisfied. If however there is some temperature change then \(K_1 d\varepsilon + \left[ \varepsilon \frac{dK_1}{d\theta} \right] d\theta < 0\) must be satisfied. Since \(K_1 > 0\), \(\varepsilon > 0\), and \(\frac{dK_1}{d\theta} < 0\) (instance due to the choice of parameters so that the fiber becomes slack with rise of temperature) to ensure that there is unloading in the fiber the increment of temperature \(d\theta\) must be such that \(dT\) is always negative.

In the simulations to follow, a linear dependence of these functions on temperature is
Table 2: Values of material constants of Equation (24) and (25) used to characterize the fiber tow.

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<td>K$^{-1}$</td>
</tr>
</tbody>
</table>

adopted:

\[
K_1(\theta) = k_{11} + k_{12}(\theta - \theta_0),
\]

(24)

\[
K_2(\theta) = k_{21} + k_{22}(\theta - \theta_0),
\]

(25)

\[
K_3(\theta) = k_{31} + k_{32}(\theta - \theta_0),
\]

(unless the value of temperature is such that the function evaluates to a negative value, in which case the modulus is zero), and

\[
\varepsilon_a(\theta) = \varepsilon_{11} + \varepsilon_{12}(\theta - \theta_0),
\]

\[
\varepsilon_b(\theta) = \varepsilon_{21} + \varepsilon_{22}(\theta - \theta_0),
\]

\[
\varepsilon_c(\theta) = \varepsilon_{31} + \varepsilon_{32}(\theta - \theta_0).
\]

In Equations (24) and (25), $\theta_0$ is the temperature of the fibers as they attach to the first roller.

If there is no unloading in the fiber, the increments $dT$ and $d\varepsilon$ of fiber tension and strain are positive everywhere, i.e. $T$ and $\varepsilon$ monotonically increase down the drawline. For such processes, the incremental thermoelastic-plastic constitutive model (22) is integrated to produce a stress-strain relation for the thermoelastic-plastic fiber, giving tension as a
function of strain and temperature,

\[ T(\varepsilon, \Theta) = \begin{cases} 
T_0 + K_1\varepsilon & \text{if } 0 \leq \varepsilon \leq \varepsilon_a \\
T_0 + K_1\varepsilon_a + K_2(\varepsilon - \varepsilon_a) & \text{if } \varepsilon_a \leq \varepsilon \leq \varepsilon_b \\
T_0 + K_1\varepsilon_a + K_2(\varepsilon_b - \varepsilon_a) + K_3(\varepsilon - \varepsilon_b) & \text{if } \varepsilon_b \leq \varepsilon \leq \varepsilon_c 
\end{cases} \]  

(26)

This is a special case of the result in plasticity that in uniaxial tension with no unloading (or, most generally, 3-D loading in which the stress components monotonically increase during the experiment in proportion to one another, called ‘simple,’ ‘radial,’ or ‘proportional’ loading) the incremental theory relating increment of strain, strain, and increment of stress integrates to function relating stress to strain, i.e. the equations behave as equations of nonlinear elasticity, even though there is permanent deformation [22, 25].

For a moving fiber the speed of the fiber in the undeformed state is \( v_{\text{ref}} \). In the deformed state at point \( s \) the fiber speed is \( v(s) \). During a small interval of time \( dt \) the particle in the undeformed state would have moved a distance of \( dl_{\text{ref}} = v_{\text{ref}} dt \) while in the deformed state at point \( s \) it would have moved by an amount \( dl = v(s) dt \). Recalling the definition of strain \( \varepsilon(s) \) at any point \( s \) from Equation (21) the following is obtained

\[ \varepsilon(s) = \frac{dl(s)}{dl_{\text{ref}}} - 1 = \frac{v(s)}{v_{\text{ref}}} - 1. \]  

(27)

If the reference state of the fiber is taken to be the state at which it attaches to the roller, then \( v_{\text{ref}} \) is set equal to the surface speed \( r\omega_1 \) of the roller. Hence, the incremental constitutive Equation (22) can be recast as an incremental relation between tension and fiber speed. The resulting thermoelastic-plastic constitutive relation for positive speed
increments \((dv > 0)\) at \(v = v_{\text{max}}\) is

\[
dT = \begin{cases} 
\frac{K_1}{r \omega_1} dv + k_{12} \left( \frac{v}{r \omega_1} - 1 \right) d\Theta & \text{if } v < v_a, \\
\frac{K_2}{r \omega_1} dv + \left( k_{12} \left( \frac{v_a}{r \omega_1} - 1 \right) + (K_1 - K_2)\epsilon_{12} + k_{22} \left( \frac{v_b}{r \omega_1} - \frac{v_a}{r \omega_1} \right) \right) d\Theta & \text{if } v_a \leq v < v_b, \\
\frac{K_3}{r \omega_1} dv + \left( k_{12} \left( \frac{v_a}{r \omega_1} - 1 \right) + (K_1 - K_2)\epsilon_{12} + k_{22} \left( \frac{v_b}{r \omega_1} - \frac{v_a}{r \omega_1} \right) + (K_2 - K_3)\epsilon_{22} + k_{33} \left( \frac{v}{r \omega_1} - \frac{v_b}{r \omega_1} \right) \right) d\Theta & \text{if } v_b \leq v \leq v_c.
\end{cases}
\]

When the speed increment is negative \((dv < 0)\) or \(v < v_{\text{max}}\) then the following is used

\[
dT = \frac{K_1}{r \omega_1} dv + k_{12} \left( \frac{v}{r \omega_1} - 1 \right) d\Theta.
\]

The values of \(k_{11}, k_{12}, k_{21}, k_{22}, k_{31}, k_{32}, \epsilon_{11}, \epsilon_{12}, \epsilon_{21}, \epsilon_{22}, \epsilon_{31},\) and \(\epsilon_{32}\) used to characterize the fiber are given in below in table 2.

The relation (27) for a moving fiber relates the strain \(\varepsilon(s)\) at any point \(s\) is to the fiber speed \(v(s)\) at that point and the speed \(r \omega_1\) of the fiber in the reference state where it attaches to the first roller. Using this relation, the constitutive Equation (26) can be recast as tension as a function of fiber speed and temperature,

\[
\tilde{T}(v, \Theta) = \begin{cases} 
T_0 + K_1 \left( \frac{v}{r \omega_1} - 1 \right) & \text{if } r \omega_1 \leq v \leq v_a, \\
T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v}{r \omega_1} - \frac{v_a}{r \omega_1} \right) & \text{if } v_a \leq v \leq v_b, \\
T_0 + K_1 \left( \frac{v_b}{r \omega_1} - 1 \right) + K_2 \left( \frac{v_a}{r \omega_1} - \frac{v_b}{r \omega_1} \right) + K_3 \left( \frac{v}{r \omega_1} - \frac{v_b}{r \omega_1} \right) & \text{if } v_b \leq v \leq v_c,
\end{cases}
\]

where \(v_a = r \omega_1 (1 + \varepsilon_a)\), \(v_b = r \omega_1 (1 + \varepsilon_b)\), and \(v_c = r \omega_1 (1 + \varepsilon_c)\) are specified functions deduced from Equations (25). From Equations 24 and (30) an increment of force \(dT\) is related to increments \(dv\) and \(d\Theta\) of speed and temperature, respectively, through

\[
dT = \frac{\partial \tilde{T}}{\partial v} dv + \frac{\partial \tilde{T}}{\partial \Theta} d\Theta,
\]
where

\[
\left\{ \begin{array}{lcl}
\frac{1}{r^2} \left( k_{11} + k_{12}(\Theta - \Theta_0) \right) & \text{if } v \leq v_a \\
\frac{1}{r^2} \left( k_{21} + k_{22}(\Theta - \Theta_0) \right) & \text{if } v_a \leq v \leq v_b \\
\frac{1}{r^2} \left( k_{31} + k_{32}(\Theta - \Theta_0) \right) & \text{if } v_b \leq v \leq v_c,
\end{array} \right.
\]

(32)

and

\[
\left\{ \begin{array}{lcl}
k_{12} \left( \frac{v}{r^2} - 1 \right) & \text{if } v \leq v_a \\
k_{12} \left( \frac{v_a}{r^2} - 1 \right) + k_{22} \left( \frac{v}{r^2} - \frac{v_a}{r^2} \right) & \text{if } v_a \leq v \leq v_b \\
k_{12} \left( \frac{v_b}{r^2} - 1 \right) + k_{22} \left( \frac{v}{r^2} - \frac{v_b}{r^2} \right) + k_{32} \left( \frac{v}{r^2} - \frac{v_b}{r^2} \right) & \text{if } v_b \leq v \leq v_c.
\end{array} \right.
\]

(33)

The stiffness $K_1$ of the initial elastic zone decreases, and the stiffnesses $K_2$ and $K_3$ of the subsequent zones and width of the compliant intermediate draw plateau increase with increasing temperature. Typically, the initial elastic stiffness $K_1$ of an undrawn fiber decreases with increasing temperature, Ziaicki [37]. Beyond the elastic range the stiffness of the fibers is strongly influenced by the amorphous orientation and the degree of crystallinity induced by the draw, and the relative proportion of extended chain crystals vs. folded chain crystals present in the fiber, which in turn are influenced by temperature. Hence, morphological changes in the fiber could explain the experimental observations that a higher stiffness is observed for the third stage (i.e., the stiff stage after the draw plateau) when the uniaxial tension test is carried out at higher temperatures. The choices in Table 1 qualitatively account for the temperature dependence of these morphological factors affecting fiber stiffness.

The values of material constants used in the simulations to follow are given in Table 1. Figure 13 displays the relation (30) between fiber tension and speed with the values of Table 2 at several temperatures.

2.5 Solution Technique

In a fully coupled thermomechanical problem, the mass, momentum, and constitutive equations of the this section are combined with the energy equation, and these coupled equations
Figure 13: Constitutive relation between axial force and speed employed in the simulations as temperature increases from 293 K to 323 K (top to bottom) in increments of 20 K.

are solved simultaneously for the fiber speed \( v(s) \), tension \( T(s) \), and temperature \( \theta(s) \). An alternative to solving for temperature from the coupled problem is to measure it on-line; with \( \theta(s) \) known, the mechanical equations decouple. This latter approach will be adopted here initially: \( \theta(s) \) is specified, and the mass, momentum, and constitutive equations are solved for \( v(s) \) and \( T(s) \).

In the draw process the fibers are either in a freespan or on a roller; on a roller the possible conditions are no-slip, draw on a feed roller (i.e. slip with the fibers moving faster than the roller surface), and draw on a take-up roller (i.e. slip with the fibers moving slower than the roller surface). The momentum, friction, and constitutive equations in each of these cases are used to obtain increments in fiber speed and tension.

When draw occurs on the feed roller, in that slip zone the fiber is moving faster than the roller surface. Friction is kinetic due to the slip, and positive according to the adopted sign convention. The relevant equations for draw on a feed roller are thus Equations (9), (14), (28) or (29), and (11); combining them produces the increment in \( dv \). Inserting the velocity increment back into the constitutive Equation (28) gives the increment in fiber tension \( dT(s) \). For draw on the take-up roller, the only difference is that the fiber is moving
slower than the roller surface, so for friction equation (12) is used instead. In the no-slip zone the fiber speed matches that of the underlying roller surface, and hence is constant. The complete solution is produced by assembling the solutions to these cases all of which or a combination may occur on a roller. In the free span there is no friction so equations (9), (18) and (28) or (29) are combined to obtain the increment in velocity. Inserting this result into the constitutive equation gives the increment in tension.
CHAPTER III

STRETCHING AND SLIPPING OF FIBERS IN
ISOTHERMAL DRAW PROCESSES

In this chapter the drawing of fibers on a roller is modeled using equations of motion in which centrifugal acceleration and acceleration due to stretching are retained in the momentum equations. An important distinction is that the this solution fundamentally couples fiber tension to fiber stretch, in contrast to other solutions, in which either stretching acceleration or both centrifugal acceleration and stretching acceleration are neglected. The tension and draw ratios between the point of exit to the point of attachment of fibers with the roller depend on three non-dimensional parameters; \( \frac{G_m}{\gamma_0} \), \( \frac{G_m}{k} \), and \( \mu \beta \). For certain values of these parameters, the exact solution can be approximated either by the quasistatic solution (which neglects the effect of both acceleration terms in the momentum equation), or by the engineering solution (which includes only the effect of centrifugal acceleration).

3.1 Introduction

Recall from Chapter 1 that in a typical fiber manufacturing process, polymeric fibers are drawn through a series of feed and take-up rollers to improve their mechanical properties. When the viscous heating is negligible, in a cold draw process (i.e., where no external heat is supplied) the draw occurs on the rollers, accompanied by stretching and sliding of fibers. For the torque transmission problem as shown in Appendix A the existing models for belts on pulleys either altogether neglect both centrifugal and stretching acceleration or just stretching acceleration, since belts do not stretch much. In fiber drawing processes, however, usually the speeds are high and the amount of stretching large; hence the effects of centrifugal acceleration and acceleration due to stretching likely have large, if not dominant, roles in momentum considerations.

The equation of motion for a moving, stretching fiber on a roller were developed in
Chapter 2. These equations included the same centrifugal acceleration term in the normal projection of the momentum equation present in existing studies of high-speed belts, but also a stretching term in the tangential projection not seen elsewhere. In Appendix A the full momentum equations are applied to model power transmission by a belt between two pulleys, the investigation includes the stretching acceleration term. The effect of this stretching term can be important even with belts, becoming more pronounced with decreasing belt stiffness and increasing belt speeds. It is essential to use the full equations when modeling the drawing of polymer fibers and films, in which the stiffnesses are much smaller and the speeds typically much greater than in belt applications.

In this chapter the equations of motion are used to solve the boundary value problem of draw on a feed roller. To highlight the importance of accelerations in the draw process, this full solution which includes both centrifugal acceleration and stretching acceleration is contrasted with the approximate solution neglecting centrifugal acceleration and stretching acceleration, and the approximate solution including centrifugal acceleration but neglecting stretching acceleration. The full solution for draw on a take-up roller is also be presented. In a later section it will also be demonstrated that a linearly elastic tow cannot undergo steady, isothermal draw in a free span.

3.2 Equations of motion for a fiber on a roller and free span

In this section the mass and momentum equations developed in Chapter 2 are recalled and compared to equations used in literature. For an extensible fiber with mass per volume \( \rho \) and cross-sectional area of \( A \) under steady motion (independent of time) with speed \( v \) on a roller. The conservation of mass given by Equation (9) requires that the mass flow rate \( G \) be constant.

With fiber tension \( T \), normal force per length \( n \), and tangential force per length \( f \) given (ignoring aerodynamic forces). The conservation of momentum in the tangential and normal directions given by Equations (14) and (15) can be rewritten as;

\[
(34a) \quad dT - fds = Gdv,
\]
\[ n = \frac{T - Gv}{r}. \]

As is noted in Amijima [1], when adhesive forces are neglected the normal force per unit length \( n \) from the roller on the fiber must be compressive (nonnegative according to the adopted sign convention), which in combination with (15) demands the inequality (16) be satisfied. Hence, at any point on the fiber there must be a tension greater than or equal to the mass flow rate of the fiber times its speed. If not, the roller would have to pull on the fiber to keep it in contact and accelerating in a circular path. This the roller cannot do, and the insufficiently-tensioned fiber will fly off the roller.

In free spans, shown in Figure 1 for a typical draw process, the conservation of mass and momentum are given by Equations (9) and (18) respectively.

### 3.2.1 Comparison with fiber equations in the literature

The mass and momentum equations for a deformable fiber on a roller are Equations (9), (14), and (15). A review of the literature revealed that all studies employ the same mass Equation (9), but all use momentum equations which are simplified forms of the momentum Equations (14) and (15). For instance, Amijima [1, 2], Rothbart [31], and Fazekas [16] employ

\[ dT - f ds = 0, \]

\[ n = \frac{T - Gv}{r}. \]

Comparing Equations (35) with Equations (34), it is observed that the centrifugal term \( Gv \) is included in the normal projection (35b), but the inertia term \( Gdv \) is absent from the tangential projection (35a). Equations (35) are an exact special case of Equations (34) if and only if the stretch in the fiber is uniform \((dv = 0)\). Johnson [21], Firbank [17], and all statics textbooks employ

\[ dT - f ds = 0, \]
(36b) \n = \frac{T}{r},

where the inertia terms are absent from both projections. Equations (36) are an exact reduction of Equations (34) if and only if the fiber is motionless \((v = 0, \ dv = 0)\).

3.3 The Constitutive Model: A Linearly Elastic Belt

Unlike the momentum formulations (35) and (36), the formulation (34) couples the evolution \(dT\) of fiber tension explicitly to the evolution \(dv\) of fiber stretch. Hence fiber tension along the roller surface cannot be computed from momentum considerations alone; one must complement Equations (34) with a constitutive model relating the fiber stretch to fiber tension. Many such models are possible, depending on the application. In this chapter the fiber is taken to be linearly elastic.

The axial strain \(\varepsilon\) at a point \(s\) of the fiber is given by Equation (21). Consider a fiber for which the tension at a point on the fiber depends only on the fiber's axial strain \(\varepsilon\) at that point (i.e. the fiber is elastic), and further that this dependence is linear. Hence

\[
(37) \quad T(s) = T_{ref} + k\varepsilon(s),
\]

where \(k\) is the elastic modulus (units of force) and \(T_{ref}\) is the tension in the reference state. (The linearly elastic assumption is for the sake of definiteness; alternatively introduce a nonlinear elastic characterization, as in Amijima [2, 16] could be considered.)

At any point \(s\) along the fiber, the strain \(\varepsilon(s)\) is related to the fiber speed \(v(s)\) at that point and the speed \(v_{ref}\) of the fiber in the reference state by Equation (27). The relation between strain and speed in the fiber permits the tension to be expressed as a function of fiber speed, so that the constitutive assumption Equation (39) becomes

\[
(38) \quad T(s) = T_{ref} + k \left( \frac{v(s)}{v_{ref}} - 1 \right).
\]

3.4 Draw on the Feed Roller

In this section the equations of motion for a linearly elastic belt are applied to the boundary value problem of draw on a feed roller with slip zone of angle \(\beta\). (Section 3.5 addresses draw
Figure 14: The boundary value problem for draw on a feed roller.

on a take-up roller.) The location of the onset of draw is labeled as $s = 0$ and the point of
departure as $s = r \beta$. See Figure 14.

The tension $T_0$ and speed $v_0 = r \omega$ at the onset of draw $s = 0$ are specified; these are
chosen to be $T_{ref}$ and $v_{ref}$, respectively. With this choice, the linear constitutive Equation
(39) used in $A$ becomes

$$T(s) = T_0 + k \left( \frac{v(s)}{v_0} - 1 \right).$$

The value of the elastic modulus $k$ is assumed known, thereby completing the characteriza-
tion of the fibers. The slip angle $\beta$ is taken as specified. The solution of the boundary value
problem is the fiber tension $T(s)$ and speed $v(s)$ as functions of $s$, from $s = 0$ to $s = r \beta$. For
the purpose of graphical display of the solutions, these functions are evaluated at the depart-
ure location, thereby focusing on the predictions of the departure tension $T_1 = T(s = r \beta)$
and departure speed $v_1 = v(s = r \beta)$, relative to the prescribed values $T_0$ and $v_0$ at $s = 0.$
(In this formulation, the slip angle $\beta$ is specified and departure speed $v_1$ is deduced as part
of the solution; referring ahead to the parenthetical note after Equation (40) for a discussion
of the alternative formulation in which $v_1$ is specified and $\beta$ is deduced.)
In draw on a feed roller the tow is moving faster than the underlying roller surface. Then, according to the sign convention and assuming Coulomb friction, \( f \) is given by Equation (11). As a slight digression, Equation (11) models each the following physical situations:

**a.** The fiber tow is going faster than a moving roller. In this case \( \mu = \mu_k, \omega \neq 0, v > r \omega \), and \( dv \) can be zero or nonzero.

**b.** The fibers are slipping on a static roller. In this case \( \mu = \mu_k, \omega = 0, v > 0 \), and \( dv \) can be zero or nonzero.

**c.** Clockwise motion (slip) is impending on a static roller; \( \mu = \mu_s, \omega = 0, v = 0 \), and \( dv = 0 \).

**d.** Slip in the direction of motion is impending; \( \mu = \mu_s, \omega > 0, v = r \omega \), and \( dv = 0 \).

With no approximations, the possible boundary value problems are, from Equations (9), (34) (i.e. Equations (14) and (15)) and (11):

**Problem 1:** Static inextensible fibers: if \( v = 0 \) and \( dv = 0 \),

\[
dT - f ds = 0, \quad n = \frac{T}{r}, \quad f = \mu n, \quad T(0) = T_0.
\]

**Problem 2:** Moving inextensible fibers: if \( v \neq 0 \) and \( dv = 0 \),

\[
dT - f ds = 0, \quad n = \frac{T-Gv}{r}, \quad f = \mu n, \quad T(0) = T_0, \quad v(0) = v_0 \geq r \omega.
\]

**Problem 3:** Moving stretching fibers: if \( v \neq 0 \) and \( dv \neq 0 \),

\[
dT - f ds = G dv, \quad n = \frac{T-Gv}{r}, \quad f = \mu n, \quad T(0) = T_0, \quad v(0) = v_0 \geq r \omega.
\]

For cases a and b note that \( v \neq 0 \) and either \( dv \neq 0 \) or \( dv = 0 \), depending on if the fibers is stretching or not; hence in these cases either Problem 2 or Problem 3 is solved. For case c \( v = 0 \) and \( dv = 0 \) must be true, so Problem 1 is the problem to be solved. For case d, \( v \neq 0 \) and \( dv = 0 \), so Problem 2 needs to be solved. For draw on a feed roller, the relevant case is a only, and the relevant problem is Problem 3.
In section 3.4.1 this full boundary value problem is solved for fiber tension and speed, in which both fiber accelerations (centrifugal and stretching) are present in the momentum equations. Two approximate problems are also solved for the purposes of connecting with the belt literature and isolating the effects of centrifugal and stretching accelerations in the full solution: In section 3.4.2 the standard capstan problem in which both fiber inertia terms are neglected from the momentum equations is solved, and in section the intermediate approximate problem in which centrifugal acceleration is retained but stretching acceleration in neglected is solved.

3.4.1 Full solution for moving, stretching fibers

The full coupled problem for fiber tension and speed consists of equations (34) (i.e. Equations (14) and (15)), (39), and (11), subject to the two boundary conditions \( T(0) = T_0 \) and \( v(0) = v_0 = r\omega \). The solution \( T(s) \) and \( v(s) \) depends on the three nonnegative dimensionless parameters \( \mu \beta, \frac{G_m}{T_0}, \) and \( \frac{G_m}{k} \). Except for degenerate cases that will be discussed below, this solution evaluated at \( s = r\beta \) is

\[
\frac{T_1}{T_0} = 1 + \left( 1 - \frac{G_m}{T_0} \right) \left( \frac{1 - \mu \beta}{1 - \frac{G_m}{k}} \right) (\mu \beta - 1),
\]

\[
\frac{v_1}{v_0} = 1 + \left( \frac{T_0}{\frac{G_m}{k}} - 1 \right) \left( \frac{1 - \mu \beta}{1 - \frac{G_m}{k}} \right) (\mu \beta - 1).
\]

(Recall that \( T_0, v_0, \) and \( \beta \) have been identified as known and solved for are the departure tension \( T_1 \) and departure speed \( v_1 \). One can also invert relationship (40b) and deduce, say, departure tension \( T_1 \) and slip angle \( \beta \) from \( T_0, v_0, \) and take-up speed \( v_1. \))

The parameter \( \frac{G_m}{T_0} = \frac{\rho A_0 r^2}{T_0} \) is increased by either increasing the fibers mass per length \( \rho A_0 \) or speed \( v_0 \), or decreasing the initial tensioning \( T_0 \). The non-negative parameter \( \frac{G_m}{T_0} \) cannot be increased indefinitely; the constraint \( T - Gv \geq 0 \) restricts it to be less than or equal to one. At the limiting value \( \frac{G_m}{T_0} = 1 \), Equations (40) demand \( T_1 = T_0 \) and \( v_1 = v_0 \). In this limit the normal force per length \( n \) on the fiber tow from the roller is zero, so that no friction is developed and the tension in the fibers is constant. If \( \frac{G_m}{T_0} \) is greater than 1 the fibers will fly off the roller. The dependence of the magnification \( \frac{T_1}{T_0} \) of tension in the
Figure 15: Draw on a feed roller: Tension ratio $T_1/T_0$ and draw ratio $\nu_1/\nu_0$ vs. the inertia to tension ratio $Gv_0/T_0$, for several inertia to stiffness ratios, at $\mu\beta = 0.5\pi$.

draw zone and the amount of draw $\nu_1/\nu_0$ on parameter $Gv_0/T_0$ for fixed $\mu\beta$ and $Gv_0/k$ is shown in Figure 15. This figure confirms that both $T_1/T_0$ and $\nu_1/\nu_0$ are 1 when $Gv_0/T_0 = 1$. The tension ratio $T_1/T_0$ increases linearly as $Gv_0/T_0$ decreases, reaching a finite maximum when $Gv_0/T_0 = 0$. The draw ratio $\nu_1/\nu_0$ increases nonlinearly as $Gv_0/T_0$ decreases, becoming unbounded for $Gv_0/T_0 = 0$. Note, however, that this blowing up of the draw ratio as $Gv_0/T_0 \to 0$ with $Gv_0/k$ nonzero and finite corresponds to the physically awkward limits of either $T_0$ finite, $Gv_0 \to 0$, and $k \to 0$, or $k$ and $Gv_0$ finite and $T_0 \to \infty$. The solution for the physically realizable limit of $k$ and $T_0$ finite and $Gv_0 = 0$ (implying $Gv_0/T_0 = Gv_0/k = 0$) is the static capstan relation, given in section
3.2. Figure 15 shows the linear dependence of the tension ratio \( \frac{T_1}{T_0} \) on \( \frac{G_{m0}}{k} \). A three orders of magnitude change in the draw ratio \( \frac{u_1}{v_0} \) from a tight or slow fibers (\( \frac{G_{m0}}{k} = 0.1 \)) to a slack or fast fibers (\( \frac{G_{m0}}{k} = 0.9 \)) is also observed.

The parameter \( \frac{G_{m0}}{k} = \frac{\rho A_0 \sigma_0^2}{k} \) can be increased either by increasing the fibers mass per length \( \rho A_0 \) or speed \( v_0 \), or decreasing the tow stiffness \( k \). Figure 16 shows the dependence of \( \frac{T_1}{T_0} \) and \( \frac{u_1}{v_0} \) on \( \frac{G_{m0}}{k} \) for fixed \( \mu \beta = 0.5 \pi \) and \( \frac{G_{m0}}{k} \). As can be seen from Figure 16, as \( \frac{G_{m0}}{k} \) increases from zero, \( \frac{T_1}{T_0} \) increases nonlinearly from a finite value greater than one, and \( \frac{u_1}{v_0} \) increases nonlinearly from one. Both quantities become large as \( \frac{G_{m0}}{k} \) approaches one. For

\[ \frac{T_1}{T_0} \]

\[ \frac{u_1}{v_0} \]
\[ G_{m} / k = 1, \] a solution to the coupled problem only exists if also \( G_{m} / T_{0} = 1. \) Above this resonant condition (i.e. for \( G_{m} / k > 1 \)), the solution predicts \( T_{1} / T_{0} \) and \( u_{1} / u_{0} \) are less than one.

Figures 17-20 show predictions of fiber behavior in the physically meaningful processing range of both \( G_{m} / T_{0} \) and \( G_{m} / k \) between 0 and 1. The tension ratio \( T_{1} / T_{0} \) and draw ratio \( u_{1} / u_{0} \) depend exponentially on the product of friction coefficient \( \mu \) and slip angle \( \beta \). Figures 17 and 18 show this exponential dependence of \( T_{1} / T_{0} \) and \( u_{1} / u_{0} \) on \( \mu \beta \) for a stiff or slow fibers \( (G_{m} / k = 0.1) \) and a compliant or fast fibers \( (G_{m} / k = 0.9) \), respectively, at several values of \( G_{m} / T_{0} \). In going from \( G_{m} / k = 0.1 \) to 0.9 (either by making the fibers less stiff or increasing the speed), the tension ratio \( T_{1} / T_{0} \) increases in general by \textit{one} order of magnitude but the draw ratio \( u_{1} / u_{0} \) increases by \textit{two} orders of magnitude. This is because the multiplying factor \( \frac{1 - G_{m} / k}{1 - G_{m} / T_{0}} \) in \( T_{1} / T_{0} \) increases by a factor of 9, and the multiplying factor \( \frac{G_{m} / T_{0} - 1}{G_{m} / k - 1} \) in \( u_{1} / u_{0} \) increases by a factor 81.

Figures 19 and 20 show the exponential dependence of \( T_{1} / T_{0} \) and \( u_{1} / u_{0} \) on \( \mu \beta \) for a tight or slow fibers \( (G_{m} / T_{0} = 0.1) \) and a slack or fast fibers \( (G_{m} / T_{0} = 0.9) \), respectively, at several values of \( G_{m} / k \). In going from \( G_{m} / T_{0} = 0.1 \) to 0.9 (by increasing the speed or and/or decreasing the initial tension), the tension ratio \( T_{1} / T_{0} \) in general decreases by \textit{one} order of magnitude but the draw ratio \( u_{1} / u_{0} \) decreases by \textit{three} orders of magnitude.

To summarize the behavior (40) of draw on a feed roller: The relative increase \( T_{1} / T_{0} \) of tension in the draw and the draw \( u_{1} / u_{0} \) itself depend on three dimensionless groups, namely the product \( \mu \beta \) of the friction coefficient and slip angle, the inertia to initial tension ratio \( G_{m} / T_{0} \), and the inertia to stiffness ratio \( G_{m} / k \). Tension magnification \( T_{1} / T_{0} \) increases as either \( \mu \beta \) increases or \( G_{m} / T_{0} \) decreases, with an exponential dependence on \( \mu \beta \) and a linear dependence on \( G_{m} / T_{0} \), and increases with increasing \( G_{m} / k \), proportional to \( (1 - G_{m} / k)^{-1} \). The draw ratio \( u_{1} / u_{0} \) also increases as \( \mu \beta \) increases, \( G_{m} / k \) increases, or \( G_{m} / T_{0} \) decreases; the changes brought by \( G_{m} / T_{0} \) and \( G_{m} / k \) are in general orders of magnitude greater in \( u_{1} / u_{0} \) than \( T_{1} / T_{0} \).

To quantify separately the contributions of centrifugal and stretching acceleration in the draw process, the next two subsections compare the full solution (40) to solutions that neglect one or both of these contributions.

In fiber drawing operations there is considerable variation of the values of fiber densities, process speeds, and fiber tensions. Data obtained for a typical drawing process, [30], show
Figure 17: Draw on a feed roller: Tension ratio $T_1/T_0$ and draw ratio $V/V_0$ vs. the product $\mu \beta$ of friction coefficient and slip angle, for several inertia to tension ratios, at inertia to stiffness ratio $Gv_0/\kappa = 0.1$. 
Figure 18: Draw on a feed roller: $\frac{T_1}{T_0}$ and $\frac{V_1}{V_0}$ vs. $\mu \beta$, for several values of $\frac{Gv_0}{T_0}$, at $\frac{Gv_0}{k} = 0.9$. 
Figure 19: Draw on a feed roller: $T_1/T_0$ and $V_1/V_0$ vs. $\mu \beta$, for several values of $\frac{Gv}{k}$, at $\frac{Gm}{T_0} = 0.1$. 
Figure 20: Draw on a feed roller: $\frac{T_1}{T_0}$ and $\frac{v_1}{v_0}$ vs. $\mu \beta$, for several values of $\frac{Gv_0}{k}$, at $\frac{Gm}{T_0} = 0.9$. 
the following range of values for a fiber bundle; linear densities vary between 200 – 5000 deniers, velocities 200 – 5000 m/min, tension of 1000 gf, and stiffness depending on the fiber material. Using these values the nondimensional parameter \( \frac{Gv}{k} \) takes on values between \( 0.25 \times 10^{-6} \) to 0.39 and nondimensional parameter \( \frac{Gd}{k} \) will in general have values depending on the stiffness of the fiber.

3.4.2 Approximate solution in which all acceleration terms are neglected from the momentum equations

First the standard approximation in the modeling of low-speed belts on pulleys is followed, and all inertia terms in the momentum equations are ignored. Neglecting \( Gv \) from Equation (34a) and \( Gdv \) from Equation (34b) results in the following decoupled boundary value problem for fiber tension, as employed by Johnson [21] and Firbank [17]:

\[
(41) \quad dT - fd\beta = 0, \quad n = \frac{T}{r}, \quad f = \mu n, \quad T(0) = T_0.
\]

Note that the quasistatic formulation (41) of draw on a feed roller differs from the static Problem 1 earlier in this section in that for the drawing process being modeled here \( v \) and \( dv \) are not zero, but \( Gv \) and \( Gdv \) are nonetheless neglected from the momentum equation; problem 1 is the formulation modeling static inextensible fibers, for which \( v = 0 \) and \( dv = 0 \) identically.

The constitutive Equation (39) does not appear in this approximate problem for fiber tension, and only the boundary condition on tension is demanded by the governing equations. The decoupling of fiber tension from fiber speed is due to the removal of the \( Gdv \) and \( Gv \) terms from Equations (34). The solution of problem (41) evaluated at the point of detachment \( s = r\beta \) is the capstan relation for static belts found in statics textbooks:

\[
(42) \quad \frac{T_1}{T_0} = e^{\mu \beta}.
\]

In this approximate, quasistatic solution the tension ratio \( \frac{T_1}{T_0} \) depends only on the single dimensionless parameter \( \mu \beta \).

Although exactly valid only for the case with \( v(s) = 0 \), solution (42) has been applied in the belt literature to situations in which the belt is moving and stretching [21]. In this
approximate formulation, speed \( v(s) \) can be determined by inserting the solution \( T(s) \) of problem (41) into the constitutive Equation (39) and solving for \( v(s) \). Evaluating at \( s = r\beta \):

\[
\frac{v_1}{v_0} = \frac{T_0}{k} e^{\mu \beta} + 1 - \frac{T_0}{k} = 1 + \frac{Gv_0}{k} \left( e^{\mu \beta} - 1 \right).
\]

Note that solution (42) is the special case of the full solution (40a) for fiber tension with \( \frac{Gv_0}{T_0} = \frac{Gv_0}{k} = 0 \), but solution (43) is not a special case of the full solution (40b) for fiber speed; i.e. the quasistatic solution for tension is a valid reduction of the full solution, but the quasistatic solution for speed is not.

### 3.4.3 Approximate solution in which stretching acceleration is neglected from the momentum equations

The standard approximation in the high-speed belt literature is to retain the effect of the centrifugal acceleration of the belt but neglect that of stretching acceleration, i.e. to retain \( Gv \) in Equation (34b) but delete \( Gdv \) from Equation (34a). Since stretching is ignored, the \( v \) in Equation (34b) is set equal to the roller speed \( v_0 \). The resulting problem for fiber tension is (Amijima [1, 2], Rothbart [31], and Fazekas [16])

\[
dT - f ds = 0, \quad n = \frac{T - Gv_0}{r}, \quad f = \mu n, \quad T(0) = T_0.
\]

Again, it needs to be emphasized that the formulation (44) is an approximate formulation for the draw process, in which the stretching term \( Gdv \) is neglected from the momentum equation (although \( dv \) is not zero), whereas problem 2 earlier in this section is the exact formulation for moving, inextensible fibers (\( v \neq 0 \) and \( dv = 0 \)).

The problem for fiber tension has been decoupled from the problem for fiber speed because the \( Gdv \) term has been removed from Equation (34a) and \( v \) has been replaced with the constant \( v_0 \) in Equation (34b). Its solution is

\[
\frac{T_1}{T_0} = 1 + \left( 1 - \frac{Gv_0}{T_0} \right) (e^{\mu \beta} - 1).
\]

Although problem (44) is exactly valid only for the inextensible case \( v(s) = v_0 \), a varying speed can be backed out by inserting the solution \( T(s) \) of problem (44) into the constitutive Equation (39) and solving for \( v(s) \). Evaluating at \( s = r\beta \):

\[
\frac{v_1}{v_0} = 1 + \frac{Gv_0}{k} \left( \frac{T_0}{Gv_0} - 1 \right) (e^{\mu \beta} - 1).
\]
Note that solution (45) is the special case of the full solution (40a) for fiber tension when \( \frac{G_{\text{mu}}}{k} = 0 \), but solution (46) is not a special case of the full solution (40b) for fiber speed.

### 3.4.4 Comparison

The two approximate solutions are compared to the full solution in Figures 21 and 22, where the dependence (or lack of dependence) of the three predictions of increase of tension \( \frac{T_1}{T_0} \) and amount of draw \( \frac{v_1}{v_0} \) on the inertia to stiffness ratio \( \frac{G_{\text{mu}}}{k} \), and inertia to tension ratio \( \frac{G_{\text{mu}}}{T_0} \), respectively are displayed. The tension increase \( \frac{T_1}{T_0} \) has no dependence on \( \frac{G_{\text{mu}}}{k} \) in both approximate solutions, missing the nonlinear dependence of the full solution. There is linear dependence of draw \( \frac{v_1}{v_0} \) on \( \frac{G_{\text{mu}}}{T_0} \) in both approximate solutions, compared to nonlinear dependence in the full solution. In the quasistatic solution of 3.4.2 neglecting both stretching and centrifugal accelerations, there is no dependence of \( \frac{T_1}{T_0} \) on \( \frac{G_{\text{mu}}}{T_0} \); this dependence is linear in both the full solution and the approximate solution given in 3.4.3 neglecting stretching acceleration but including centrifugal acceleration. All three solutions have nonlinear dependence of \( \frac{v_1}{v_0} \) on \( \frac{G_{\text{mu}}}{T_0} \). Seen from Figures 21 and 22 is that the approximate solution neglecting stretching acceleration but including centrifugal acceleration always underpredicts the tension increase \( \frac{T_1}{T_0} \) and amount of draw \( \frac{v_1}{v_0} \). For \( \frac{G_{\text{mu}}}{k} \to 0 \) and \( \frac{G_{\text{mu}}}{T_0} \to 1 \) this approximation approaches the full solution. The quasistatic approximation in 3.4.2 underpredicts the tension ratio \( \frac{T_1}{T_0} \) and draw ratio \( \frac{v_1}{v_0} \) if \( k < T_0 \), and overpredicts them if \( k > T_0 \).

### 3.5 Draw on the take-up roller

When fibers are drawing on the take-up roller the tow is moving faster than the underlying roller surface, and according to the sign convention the frictional force per length \( f \) is negative i.e. Equation (11) is used. The full and approximate boundary value problems for draw on the take-up roller are therefore the same as in the previous section except with \( f = -\mu n \) instead of \( f = \mu n \). Corresponding to equations (40), the full solution for draw on the take-up roller is

\[
\frac{T_1}{T_0} = 1 + \left( \frac{1 - \frac{G_{\text{mu}}}{T_0}}{1 - \frac{G_{\text{mu}}}{k}} \right) \left( e^{-\mu \beta} - 1 \right),
\]

(47a)
Figure 21: Draw on a feed roller: Comparison of the full solution in Equation (40), Capstan solution in Equations (42) and (43) neglecting both centrifugal and stretching accelerations, and Engineering solution in Equations (45) and (46) neglecting stretching acceleration but including centrifugal acceleration. Tension ratio $\frac{T_1}{T_0}$ and draw ratio $\frac{V_1}{V_0}$ vs. inertia to stiffness ratio $\frac{Gv_0}{k}$, for inertia to tension ratio $\frac{Gv_0}{T_0} = 0.9$ and product of friction coefficient and slip angle $\mu \beta = 0.5\pi$. 
Figure 22: Draw on a feed roller: Comparison of the full solution in Equation (40), Capstan solution in Equations (42) and (43) neglecting both centrifugal and stretching accelerations, and approximate solution in Equations (45) and (46) neglecting stretching acceleration but including centrifugal acceleration. Tension ratio $\frac{T_1}{T_0}$ and draw ratio $\frac{v_1}{v_0}$ vs. inertia to tension ratio $\frac{G_{\text{in}}}{T_0}$, for $\frac{G_{\text{in}}}{k} = 0.5$ and $\mu \beta = 0.5\pi$. 
\[ \frac{v_1}{v_0} = 1 + \left( \frac{\frac{J_0}{\Omega_0^2}}{k} - 1 \right) (e^{-\mu \beta} - 1). \]

Since on the take-up roller the fiber accelerates to the roller speed \( r \omega \) rather than from it as on the feed roller, \( v_1 \), not \( v_0 \), equals \( r \omega \).

### 3.6 Draw in free spans

Integration of Equation (18), the momentum equation in the free span, gives

\[ T = Gv + c, \]

where \( c \) is a constant of integration. For a linear elastic fiber the constitutive Equation (39) may be rewritten as

\[ T = \left( \frac{k}{v_{ref}} \right) v + T_{ref} - k. \]

Comparing the last two equations one notes that the mass flow rate \( G \) (which is dependent on the process) will not in general equal the elastic modulus divided by the reference velocity. Hence the only relevant solution of Equation (18) in the freespan for isothermal motion is

\[ T = \text{constant}, \quad v = \text{constant}, \]

This implies that for an isothermal draw process no draw can occur in the free span. Thus, draw occurs on the feed or the take-up rollers. The draw on the rollers satisfy the equations derived in the previous two sections.

### 3.7 Conclusion

In this chapter the equations for draw on rollers which include all inertia terms, due to both stretching and centrifugal accelerations are solved. The resulting solution for tension and velocity magnification fundamentally couples the momentum and fiber constitutive equations. Fiber behavior in draw depends on the three dimensionless numbers \( \mu \) (coefficient of friction), \( \frac{\Omega_0}{T_0} \) (inertia to initial tension ratio), and \( \frac{G \Omega_0}{k} \) (inertia to stiffness ratio).

To understand the effects of centrifugal and stretching accelerations during draw on rollers, the full solution is compared to two approximate solutions, one in which stretching
acceleration is neglected and the other in which both stretching and centrifugal accelerations are neglected. It is found that neglecting stretching acceleration but retaining centrifugal acceleration results in the underprediction of tension magnification and draw, for all process conditions. It can be observed that the quasistatic solution neglecting both centrifugal and stretching accelerations underpredicts both tension magnification and draw if tow stiffness $k$ is less than the initial tension $T_0$, is the same if $k = T_0$, and overpredicts if $k > T_0$. 
CHAPTER IV

TWO-STAGE DRAW

In this chapter a comprehensive model for a two stage, non-isothermal industrial draw process is produced by assembling of governing equations for fibers on rollers and in free spans, together with matching conditions. The fibers are characterized as piecewise linear elastic-plastic material to capture the softening behavior of as-spun fibers. The model is then employed to simulate three representative draw lines i.e. isothermal draw, draw with a heated freespan, and draw with a heated roller.

4.1 Introduction

During a two-stage draw process the fiber tow passes over three rollers. The two-stage draw process is modeled with no guides; see Figure 23. For simplicity all three rollers are assumed to have the same radius \( r \), although this restriction is unnecessary. These rollers rotate at specified constant angular velocities \( \omega_1, \omega_2, \) and \( \omega_3 \), respectively, each faster than the one before \( (\omega_1 < \omega_2 < \omega_3) \). Roller 1 functions as the feed roller of the first draw stage; roller 2 functions as both the take-up roller of the first draw stage and the feed roller of the second draw stage; roller 3 functions as the take-up roller of the second draw stage.

The path of the fibers is described with space-fixed (Eulerian) arclength \( s \). In the notation that will be used, \( x \) indicates a specified location, and \( y \) and \( v \) indicate an unknown location and speed, respectively. As shown in Figure 23, the fibers attach to the first roller at specified location \( s = x_0 \) with speed \( r \omega_1 \) matching the surface speed of the roller. The fibers detach from the first roller at specified location \( s = x_1 \) with a speed \( v_1 \) that will be deduced by the model; \( v_1 \) is in general greater than or equal to the roller surface speed \( r \omega_1 \), indicating the possibility of a draw (i.e. slip) zone on the first roller, in which the fibers are moving faster than the underlying roller surface. The model will also predict the location on the first roller where this draw begins, denoted by \( y_1 \). Note the inequality \( y_1 \leq x_1 \) must
Figure 23: The possible draw zones for the two-stage draw process: 1) on the feed roller of the first stage (between \( y_1 \) and \( x_1 \)), 2) in the first free span (between \( x_1 \) and \( x_2 \)), 3) on the take-up roller of the first stage (between \( x_2 \) and \( y_2 \)), 4) on the feed roller of the second stage (between \( y_3 \) and \( x_3 \)), 5) in the second free span (between \( x_3 \) and \( x_4 \)), and 6) on the take-up roller of the second stage (between \( x_4 \) and \( y_4 \)).

be satisfied: If the model predicts solutions \( y_1 < x_1 \) and \( v_1 > r \omega_1 \) then there is draw on the first roller; the region \( x_0 \leq s \leq y_1 \) is the no-slip zone, and \( y_1 < s \leq x_1 \) is the draw zone. If the model predicts \( y_1 = x_1 \) and \( v_1 = r \omega_1 \) there is no draw on roller 1, i.e. there is no draw on the feed roller of the first stage.

The first freespans extends \( s = x_1 \) to \( s = x_2 \), where \( s = x_2 \) is the specified point of attachment to the second roller. The speed \( v_2 \) at which the fibers attach to the second roller, to be deduced by the model, is in general greater than or equal to the speed \( v_1 \) with which they depart the first roller, so that there is possibly draw in the first free span. If the model predicts \( v_2 > v_1 \), then there is draw in the free span; if \( v_2 = v_1 \), there is no draw.

Further, the speed \( v_2 \) of attachment to the second roller deduced by the model is in general less than or equal to the roller speed \( r \omega_2 \), so that there is also the possibility of a draw zone on the second roller, where the fiber is moving slower than the underlying roller.

The draw on the roller terminates at a location \( s = y_2 \) to be deduced by the model. In this draw zone \( x_2 \leq s < y_2 \) the second roller is functioning as the take-up roller of the first stage. If the model predicts \( y_2 = x_2 \) and \( v_2 = r \omega_2 \) then there is no take-up draw on roller 2.

Moving down the draw line, the fibers detach from the second roller at specified location.
\(s = x_3\) with a speed \(v_3\) deduced by the model that is in general greater than or equal to the roller speed \(r\omega_2\). Hence the fibers move without slip at roller speed \(r\omega_2\) from \(s = y_2\) (where the first stage draw ceases) to some point \(s = y_3\) where a possible second draw zone on the middle roller begins. In this second draw zone \(y_3 < s \leq x_3\) the second roller is functioning as the feed roller of the second stage, and the fibers are moving faster than the underlying roller. Again, a solution \(y_3 = x_3\) and \(v_3 = r\omega_2\) would indicate that there is no feed draw on roller 2.

The second free span extends from the location \(s = x_3\) of departure from second roller to the location \(s = x_4\) of attachment to the third and final roller. The fibers attach to the final roller with a speed \(v_4\) deduced by the model which in general is greater than or equal to the speed \(v_3\) with which they depart the second roller, so that draw is possible in the second free span.

The speed \(v_4\) of attachment to the third roller deduced by the model is also in general less than or equal to its surface speed \(r\omega_3\), creating the possibility of a take-up draw zone in which the speed of the fibers is less than the underlying roller speed. The slip terminates at \(s = y_4\), at which point the fibers have reached the speed \(r\omega_3\) of the roller. The fibers maintain this speed until they exit the final roller at \(s = x_5\).

Summarizing, in the two-stage draw process without guides draw is possible in six zones, depending on process conditions and the response of the as-spun fiber (see Figure 23):

1. on the feed roller of the first stage, if \(v_1 > r\omega_1\) and \(y_1 < x_1\),
2. in the free span of the first stage, if \(v_2 > v_1\),
3. on the take-up roller of the first stage, if \(v_2 < r\omega_2\) and \(y_2 > x_2\),
4. on the feed roller of the second stage, if \(v_3 > r\omega_2\) and \(y_3 < x_3\); (note that roller 2 is both the take-up roller for the first draw stage and the feed roller of the second stage),
5. in the free span of the second stage, if \(v_4 > v_3\), and
6. on the take-up roller of the second stage, if \(v_4 < r\omega_3\) and \(y_4 > x_4\).
For each of these six possible places, draw does not occur if the corresponding inequality above is instead an equality, e.g., there is no draw on the take-up roller of the second stage if \( v_4 = r \omega_3 \) and \( y_4 = x_4 \). In the modeling of the two-stage draw, \( x_1, x_2, x_3, x_4, r \omega_1, r \omega_2, \) and \( r \omega_3 \) are specified, and \( y_1, y_2, y_3, y_4, v_1, v_2, v_3, \) and \( v_4 \) are deduced.

### 4.2 Equations of Motion

The conservation equations under steady conditions presented in Chapter 3 are retained. The conservation of mass is given by Equation (9). The conservation of momentum on the roller is given by Equations (14) and (15), and by Equation (18) in the free span. To obtain the equations of motion the conservation of mass and momentum are combined with Equation (31).

In the draw process the fibers are either in a freespans or on a roller; on a roller the possible conditions are no-slip, draw on a feed roller (i.e. slip with the fibers moving faster than the roller surface), and draw on a take-up roller (i.e. slip with the fibers moving slower than the roller surface). In the freespans the relevant equations are Equations (9), (18), and (31). Combining them results in

\[
\frac{dv}{ds} = \frac{\frac{\partial T}{\partial \phi}}{G - \frac{\partial T}{\partial \phi}},
\]

where \( \frac{\partial T}{\partial \phi} \) and \( \frac{\partial T}{\partial \phi} \) are given explicitly in Equations (32) and (33). When draw occurs on the feed roller, in that slip zone the fiber is moving faster than the roller surface. Friction is kinetic due to the slip, and positive according to the adopted sign convention so Equation (11) is used. The relevant equations for draw on a feed roller are thus Equations (9), (14), (15), (31), and (11); combining them produces

\[
\frac{dv}{ds} = \frac{\frac{\partial T}{\partial \phi}}{G - \frac{\partial T}{\partial \phi}} \frac{\partial \phi}{\partial s} - \frac{\mu}{r} (T - Gv)
\]

When draw occurs on the take-up roller, in that slip zone the fiber is moving slower than the roller surface. Friction is kinetic due to the slip, and negative according to the adopted sign convention. Hence Equation (10) becomes (12). The relevant equations for draw on a
take-up roller are Equations (9), (14), (15), (31), and (12); combining them produces

\[ \frac{dv}{ds} = \frac{\partial T}{\partial \Theta} \frac{d\Theta}{ds} + \frac{\mu (T - Gv)}{G - \frac{\partial T}{\partial \Theta}}. \]

This equation is valid in the take-up roller draw zones \( x_2 \leq s < y_2 \) and \( x_4 \leq s < y_4 \). In the no-slip zones \( x_0 \leq s \leq y_1, y_2 \leq s \leq y_3, \) and \( y_4 \leq s \leq x_5 \), the fiber speed matches that of the underlying roller surface, and hence is constant. Therefore in these ranges of \( s \),

\[ \frac{dv}{ds} = 0. \]

### 4.3 Solution

In this section the thermo-mechanical behavior of the fibers during drawing is predicted. The conservation of mass and momentum must be satisfied and the response of the fiber under uniaxial tension for various temperatures must be input. A fiber model of a form which idealizes the typical behavior of undrawn PET fiber as outlined in Chapter 2 will be used.

In a fully coupled thermomechanical problem, the mass, momentum, and constitutive equations of the this section are combined with the energy equation, and these coupled equations are solved simultaneously for the fiber speed \( v(s) \), tension \( T(s) \), and temperature \( \Theta(s) \). An alternative to solving for temperature from the coupled problem is to measure it on-line; with \( \Theta(s) \) known, the mechanical equations decouple. This latter approach is followed here with \( \Theta(s) \) specified, and mass, momentum, and constitutive equations are solved for \( v(s) \) and \( T(s) \). Thus the equations of motion in each of the cases i.e. slip on feed roller, slip on take-up roller, freespan, and no slip on roller are integrated to obtain closed form solutions for fiber speed and tension.

Referring to Figure 23, in the two-stage process Equation (49) is valid in the free spans \( x_1 < s < x_2 \) and \( x_3 < s < x_4 \). Integrating Equation (49) from the beginning to end of each freespan gives:

\[ v_2 = v_1 + \int_{x_1}^{x_2} \frac{\partial T}{\partial \Theta} \frac{d\Theta}{ds} ds. \]

\[ v_4 = v_3 + \int_{x_3}^{x_4} \frac{\partial T}{\partial \Theta} \frac{d\Theta}{ds} ds. \]
Equations (53) involve the four unknowns \( v_1, v_2, v_3, \) and \( v_4 \); once they are known, the speed as a function of arclength within each freespans is given by

\[
v(s) = \begin{cases} 
    v_1 + \int_{x_1}^{s} \frac{\phi_1}{G - \frac{\partial T}{\partial v}} \, ds & \text{if } x_1 < s < x_2, \\
    v_3 + \int_{x_3}^{s} \frac{\phi_1}{G - \frac{\partial T}{\partial v}} \, ds & \text{if } x_3 < s < x_4.
\end{cases}
\]

Inserting this function into the constitutive Equation (30) produces fiber tension \( T(s) \).

Referring to Figure 23, in the two-stage process Equation (50) is valid in the feed roller draw zones \( y_1 < s \leq x_1 \) and \( y_3 < s \leq x_3 \). Equation (50) is integrated from point of slip initiation to point departure from the roller for each feed roller draw zone to obtain

\[
(55a) \quad x_1 = y_1 + \int_{r\omega_1}^{v_1} \frac{G - \frac{\partial T}{\partial v}}{\frac{\partial v}{\partial T} \frac{\partial T}{\partial v} - \frac{\partial v}{\partial T} \frac{\partial T}{\partial v}} \, dv,
\]

\[
(55b) \quad x_3 = y_3 + \int_{r\omega_2}^{v_3} \frac{G - \frac{\partial T}{\partial v}}{\frac{\partial v}{\partial T} \frac{\partial T}{\partial v} - \frac{\partial v}{\partial T} \frac{\partial T}{\partial v}} \, dv;
\]

the departure speeds \( v_1 \) and \( v_3 \) deduced by the model must be in the ranges \( r\omega_1 \leq v_1 \leq r\omega_2 \) and \( r\omega_2 \leq v_3 \leq r\omega_3 \). Solutions \( y_1 \) and \( y_3 \) of Equation (55) must satisfy the conditions

\[
(56a) \quad y_1 \leq x_1,
\]

\[
(56b) \quad y_3 \leq x_3,
\]

i.e. the draw zone must be on the roller. If for a particular constitutive relation and temperature profile condition equation (56a) is not satisfied by the solution of Equation (55a), then it is replaced by \( y_1 = x_1 \); i.e. there is no draw on the feed roller of the first stage. If the solution of Equation (55b) does not satisfy condition (56b) then it is replaced by \( y_3 = x_3 \).

Equations (55) involve the four unknowns \( y_1, y_3, v_1, \) and \( v_3 \); once they are known, the fiber speed as a function of \( s \) within the draw zones is

\[
v(s) = \begin{cases} 
    v_1 + \int_{x_1}^{s} \frac{\phi_1}{G - \frac{\partial T}{\partial v}} \, ds & \text{if } y_1 < s \leq x_1, \\
    v_3 + \int_{x_3}^{s} \frac{\phi_1}{G - \frac{\partial T}{\partial v}} \, ds & \text{if } y_3 < s \leq x_3.
\end{cases}
\]

Again, inserting this function into the constitutive Equation (30) produces fiber tension \( T(s) \).
Equation (51) for a take up roller is integrated from the point of roller attachment to the point of slip termination,

\begin{equation}
    y_2 = x_2 + \int_{v_2}^{r} \left( \frac{G - \frac{\partial f}{\partial v}}{\frac{\partial f}{\partial \theta} \frac{d\theta}{ds} + \frac{\mu(T - Gv)}{v}} \right) dv,
\end{equation}

\begin{equation}
    y_4 = x_4 + \int_{v_4}^{r} \left( \frac{G - \frac{\partial f}{\partial v}}{\frac{\partial f}{\partial \theta} \frac{d\theta}{ds} + \frac{\mu(T - Gv)}{v}} \right) dv;
\end{equation}

the attachment speeds \( v_2 \) and \( v_4 \) deduced by the model must be in the ranges \( r\omega_1 \leq v_2 \leq r\omega_2 \) and \( r\omega_2 \leq v_4 \leq r\omega_3 \). Solution of \( y_2 \) and \( y_4 \) of Equations (58) must be consistent with the conditions

\begin{equation}
    y_2 \geq x_2,
\end{equation}

\begin{equation}
    y_4 \geq x_4;
\end{equation}

otherwise they are replaced by \( y_2 = x_2 \) and \( y_4 = x_4 \), respectively. Equations (58) involve the four unknowns \( y_2, y_4, v_2 \) and \( v_4 \); once they are known, the fiber speed as a function of \( s \) within the draw zones is

\begin{equation}
    v(s) = \begin{cases} 
        v_2 + \int_{x_2}^{s} \left( \frac{\partial T}{\partial \theta} \frac{d\theta}{ds} + \frac{\mu(T - Gv)}{v} \right) ds & \text{if } x_2 \leq s < y_2, \\
        v_4 + \int_{x_4}^{s} \left( \frac{\partial T}{\partial \theta} \frac{d\theta}{ds} + \frac{\mu(T - Gv)}{v} \right) ds & \text{if } x_4 \leq s < y_4.
    \end{cases}
\end{equation}

Using Equation (52) for no slip on a roller, the momentum projection (14) reduces to an equation for the friction force,

\begin{equation}
    f = -\frac{\partial T}{\partial \theta} \frac{d\theta}{ds}.
\end{equation}

### 4.4 The Two-Stage Draw Process

The basic solutions of the previous section are now assembled to construct a model of the entire two-stage draw process. To simulate a particular process, the tow must be characterized with an explicit constitutive equation \( T = \tilde{T}(v, \Theta) \), and a temperature profile \( \Theta = \Theta(s) \) must be provided.

In the solution procedure the four free boundaries \( y_1, y_2, y_3, y_4 \) between slip and no-slip and four transition speeds \( v_1, v_2, v_3, v_4 \), labeled in Figure 23 are first solved for.
Equations (53), (55), (58) are six equations for the eight unknowns $y_1, y_2, y_3, y_4, v_1, v_2, v_3$, and $v_4$, and hence the problem is under-determined. The two free parameters in the problem exist because the steady formulation is does not distinguish stable steady states from unstable steady states. To select which of the two-parameter family of steady solutions is the solution that will be physically sustained, either the stability of the steady solutions in the context of the time dependent equations must be examined, or energy considerations must be appealed to. The latter approach is followed here.

The velocities $v_2$ and $v_4$ are identified as the free parameters, noting that they must satisfy $r_1 \leq v_2 \leq r_2$ and $r_2 \leq v_4 \leq r_3$, and solve Equations (53), (55), (58) for $y_1, y_2, y_3, y_4, v_1$, and $v_3$ in terms of $v_2$ and $v_4$.

With $y_1, y_2, y_3, y_4, v_1$, and $v_3$ all known in terms of the free parameters $v_2$ and $v_4$, Equations (54), (57), and (60) are solved individually for $v(s)$ within each slip zone and free span. The assembled solution along the draw line is labeled by

$$v = \bar{v}(s; v_2, v_4),$$

and the dependence of $v$ on parameters $v_2$ and $v_4$ is denoted. With $v(s)$ computed, the tensile force,

$$T = \bar{T}(s; v_2, v_4),$$

is obtained as a function of position everywhere in the draw line by inserting $v(s; v_2, v_4)$, together with the posited temperature profile $\Theta(s)$, into the constitutive Equation (30). The normal and frictional components $\bar{n}(s; v_2, v_4)$ and $\bar{f}(s; v_2, v_4)$ are also obtained from the momentum equations of the previous section once $\bar{v}(s; v_2, v_4)$ has been computed. A different set of solutions $v$, $T$, $f$, and $n$ are produced by each pair of free parameters $v_2$ and $v_4$. The stable set is selected as the one corresponding to minimum energy.

An energy function which includes kinetic, thermal, and material contributions is proposed. The kinetic contribution to this function is $\frac{1}{2}Gv^2$, and the thermal contribution is $Gc\Theta$, where $c$ is the specific heat of the fiber material. The material contribution is the recoverable power, shown schematically as the shaded region in Figure 24. Recall it is assumed that the fiber unloads along a line parallel to the first segment of the constitutive
relation (30), which has a slope of \( \frac{K_1}{r\omega_1} \). From Figure 24 it is seen that

\[
(64) \quad \frac{K_1}{r\omega_1} = \frac{T}{v - v_r}, \quad \text{or} \quad v_r = v - \frac{T r \omega_1}{K_1}.
\]

where \( v_r \) is the speed of permanently stretched unloaded fiber. Hence the recoverable power is \( \frac{1}{2} T (v - v_r) = \frac{1}{2} \frac{T^2 r \omega_1}{K_1} \). Thus the energy function is constructed from an expression \( p \) for power,

\[
(65) \quad p = \frac{1}{2} G v^2 + G c \Theta + \frac{1}{2} \frac{T^2 r \omega_1}{K_1}.
\]

**Figure 24:** The shaded area indicates the power that can be recovered from the fiber at speed \( v \) and corresponding tension \( T \); \( v_r \) is the speed of the permanently stretched, unloaded fiber.

The energy of the draw line is obtained by integrating the power over the residence time of a material particle in the draw line,

\[
(66) \quad e = e(v_2, v_3) = \int_{t_0}^{t_f} p dt = \int_{x_0}^{x_f} \frac{p}{v} ds = \int_{x_0}^{x_f} \left( \frac{1}{2} G v + \frac{G c \Theta}{v} + \frac{1}{2} \frac{T^2 r \omega_1}{K_1 v} \right) ds.
\]
The correct values of the free parameters \(v_2\) and \(v_4\) are those which minimize the energy (66).

### 4.5 Simulations

To demonstrate the model, three different two-stage draw processes are simulated. The specification of fiber behavior used in the three simulations, in the context of the constitutive function (30) is given in Table 2.

The first process that will be simulated is isothermal, the second has a heater in the second free span, and the third has a heated third roller. These three processes are distinguished by the temperature profile that is input. In Simulation 1 the temperature is a constant 298K throughout the process (see Figure 25); note that in this simulation passive heating due to internal dissipation from friction between the fibers and rollers and stretching is neglected. In Simulation 2 the tow temperature is specified to be the same constant 298K until the second free span, over which it ramps to 323K (due to a free span heater), then ramps back down to 298K over the first 5 cm on the unheated roller 3 (see Figure 26). In Simulation 3 the temperature is a constant 298K until the fiber contacts the heated third roller, then over the first 5 cm of roller 3 the temperature ramps up to 513K, thereafter remaining at the roller temperature 513K (see Figure 27). The remaining process conditions for the three simulations are displayed in Table 3. Note that, aside the temperature profiles, the processes of Simulations 1 and 2 are identical.

In addition to the temperature profile, Simulation 3 differs from the other two in that there is a lower coefficient of friction between the fiber and the rollers (\(\mu = 0.05\) rather than 0.2), and there are two and a quarter wraps on the first roller rather than one and a quarter, necessitated by the low friction coefficient. (If there were only one and a quarter wrap in this simulation, the solution of Equation (55) would yield \(y_1 = -33\) cm; the negative value signifies that the necessary draw surface is greater than what is available with just one and a quarter wrap, so the fibers would slip over the entire contact with the roller).

In each simulation the fiber speed \(v(s)\) and tension \(T(s)\) throughout the draw line are computed as described in section 4.4, and these functions are displayed in the figure.
Table 3: Input for Simulation 1 (isothermal draw), Simulation 2 (draw with a heated second free span), and Simulation 3 (draw with a heated third roller). The posited temperature profiles are displayed in Figures 25, 26, and 27, respectively.

<table>
<thead>
<tr>
<th>input</th>
<th>sim.1</th>
<th>sim.2</th>
<th>sim.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>reference temperature $\Theta_0$ (K)</td>
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<td>298</td>
<td>298</td>
</tr>
<tr>
<td>coefficient of friction $\mu$</td>
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<td>0.2</td>
<td>0.05</td>
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<tr>
<td>linear density of fiber $\rho$ (denier)</td>
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<td>450</td>
<td>450</td>
</tr>
<tr>
<td>length of free span 1 (cm)</td>
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<td>50.0</td>
<td>50.0</td>
</tr>
<tr>
<td>length of free span 2 (cm)</td>
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<td>50.0</td>
<td>50.0</td>
</tr>
<tr>
<td>roller radius $r$ (cm)</td>
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<td>12.7</td>
<td>12.7</td>
</tr>
<tr>
<td>number of wraps on roller 1</td>
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<td>1.25</td>
<td>2.25</td>
</tr>
<tr>
<td>number of wraps on roller 2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>number of wraps on roller 3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>roller 1 surface speed $r\omega_1$ (cm s$^{-1}$)</td>
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<td>396</td>
<td>396</td>
</tr>
<tr>
<td>roller 2 surface speed $r\omega_2$ (cm s$^{-1}$)</td>
<td>476</td>
<td>476</td>
<td>476</td>
</tr>
<tr>
<td>roller 3 surface speed $r\omega_3$ (cm s$^{-1}$)</td>
<td>740</td>
<td>740</td>
<td>740</td>
</tr>
</tbody>
</table>

above the specified temperature profile. With the computed functions $T(s)$ and $v(s)$, Equations (14) and (15) are employed to deduce the frictional and normal forces per length $f$ and $n$ from the rollers on the fiber as functions of position; these are also displayed in the figures. In these plots, the superimposed symbols “×” label the locations where the fibers attach to and depart from the rollers (these locations are input to the model), and the symbols “○” label where slip starts in feed draw and ceases in take-up draw (these locations are deduced by the model). The abrupt bends in the speed plot not at an × or ○ indicate the transitions into and out of the draw plateau of the fiber.

Referring to Figures 25, 26, and 27, the top left plot displays the computed fiber speed $v$ as a function of arclength $s$, and the top right plot displays the computed fiber tension $T(s)$. Below this pair are the computed normal force per length $n(s)$ from the roller on fiber, computed frictional force per length $f(s)$ from the roller on the fiber, and specified temperature profile $\Theta(s)$, respectively. Proceeding from left to right in each plot, the first × (and beginning of the plot) is the point of attachment $x_0 = 0$ cm of the fiber to roller 1, the second × is the point of departure $x_1$ from roller 1, the third × is the point of attachment $x_2$ to roller 2, the fourth × is the point of departure $x_3$ from roller 2, the fifth × is the point of attachment to the third roller, and the sixth × (and termination of plot) is the point of
detachment from the roller 3. The first \( \circ \) is the point of initiation \( y_1 \) of feed slip on roller 1, the second \( \circ \) is the point of cessation \( y_2 \) of take-up slip on roller 2, the third \( \circ \) is the point of initiation \( y_3 \) of feed slip on roller 2, and the fourth \( \circ \) is the point of cessation \( y_4 \) of take-up slip on roller 3.

In the isothermal simulation (Figure 25) it is found that conditions (59) are violated by the solutions of Equations (58) for all allowable speeds \( v_2 \) and \( v_4 \) of attachment with the take-up rollers, \( r\omega_1 \leq v_2 < r\omega_2 \) and \( r\omega_2 \leq v_4 < r\omega_3 \). Hence for the given fiber response function and process conditions draw does not occur on either take-up roller, i.e., the model demands \( y_2 = x_2 \), \( y_4 = x_4 \). Also, since the fiber is isothermal in both free spans \( (\frac{d\theta}{ds} = 0) \) and there are no intervening guides, Equation (49) dictates that speed is constant in both free spans. Therefore in the isothermal process of Simulation 1, of the six possible places of draw listed at the end of section 4.1 and in Figure 23, draw occurs only in the first and fourth, i.e., on the two feed rollers. This is due to specifying of both process conditions and filament constitutive response.

Referring to Figure 25, the fiber attaches to roller 1 at the first \( \times (s = 0 \text{ cm}) \) with tension \( T_0 = 4.00 \times 10^5 \text{ dyn} \) and at the surface speed \( r\omega_1 = 396 \text{ cm s}^{-1} \). The fibers proceed without slip on roller 1 until the first \( \circ \) at \( s = 67 \text{ cm} \); in this 67 cm of no slip the speed, tension, and normal force per length are all constant, and the friction force per length \( f \) is zero \( (\frac{d\theta}{ds} = 0 \text{ implies } f = 0 \text{ in Equation (61)}) \). The draw zone on roller 1 is the 33 cm from the first \( \circ (y_1 = 67 \text{ cm}) \) to the second \( \times (x_1 = 100 \text{ cm}) \), where the fibers depart the roller. In this draw zone the fiber speed increases from the surface speed 396 cm s\(^{-1}\) of roller 1 to the surface speed 476 cm s\(^{-1}\) of roller 2, and the tension increases from 4.00 \( \times 10^5 \text{ dyn} \) to 6.76 \( \times 10^5 \text{ dyn} \). Note from the presence of the kink at \( s = 99 \text{ cm} \) in the \( v(s) \) plot that the fiber is drawn on the first feed roller from its initial stiff response into its soft plateau.

For 67 cm < \( s < 99 \text{ cm} \) the draw is in the stiff portion of the filament response, where a large increase in tension accompanies a small increase in strain and hence speed. For 99 cm < \( s \leq 100 \text{ cm} \) the draw is in the soft portion of the filament response, where a small increase in tension accompanies a large increase in strain and speed. In the draw zone the frictional force on the fiber is increasing and in the direction opposite to fiber motion.
Figure 25: Speed ($v$), tension ($T$), normal force per length ($n$), frictional force per length ($f$), and temperature profiles ($\Theta$) for an isothermal draw process. Crosses ($\times$) denote locations where the fibers either attach to or depart from a roller, and circles ($\circ$) denote locations of the initiation or cessation of slip. Solid lines (——) indicate zones of draw on rollers, dashed-dotted lines (-----) indicate zones of no slip, and dotted lines (·····) indicate freespans.
The 50 cm between the second \( x_1 = 100 \) cm and third \( x_2 = 150 \) cm is the first free span. In this isothermal free span without guides, speed and tension are constant.

Note that the second \( \circ \) coincides with the third \( \times \), indicating the aforementioned result that there is no take-up draw on roller 2: the fiber attaches to roller 2 at \( x_2 = y_2 = 150 \) cm already at its surface speed \( r\omega_2 = 476 \) cm s\(^{-1}\). The fiber proceeds on roller 2 without slip until the third \( \circ \), where it begins to slip; this draw zone on roller 2 is the 14 cm from the third \( \circ \) at \( y_3 = 216 \) cm to the fourth \( \times \) at \( x_3 = 230 \) cm, where the fiber departs from roller 2 already at the surface speed of roller 3. Note from the kink in \( v(s) \) that the fiber has been drawn out of its soft plateau on the second feed roller.

Between the fourth \( \times \) and fifth \( \times \) (at \( x_4 = 280 \) cm) is the second free span, again isothermal with no guides and hence without draw. The fourth \( \circ \) coincides with the fifth \( \times \), again indicating that there is no take-up draw in this isothermal process. The fiber proceeds without slip on its entire path on roller 3.

Figure 25 shows that at the location \( y_1 = 67 \) cm where slip begins, there is a rapid buildup of normal and frictional forces. As the tow stretches the normal and the frictional forces increase, until they vanish when the tow leaves the roller. The tension in the tow increases uniformly down the draw line. The fiber velocity increases slightly when the extension of the fiber is within the initial stiffer region of the constitutive relation. At \( s = 99 \) cm, the fiber behavior moves from the first stiff region to the compliant region, and correspondingly the speed of the tow increases rapidly up to the surface speed of roller 2. Between 100 cm and 216 cm the tow moves through the free span and the no-slip zone on roller 2 with no changes in the tension and velocity. At \( y_3 = 216 \) cm, the tow starts slipping by moving faster than the roller surface, the normal and the frictional forces build up, and the tension increases monotonically. The velocity increases rapidly when the fibers are still in the compliant plateau, but when they enter the second stiffer region the increase in velocity slows down. There are no changes in the tension or velocity in the second free span, as the process is set up.

In Figure 26 depicting Simulation 2, again note the upward ramp in the temperature profile \( \Theta(s) \) between the fourth \( \times \) and the fifth \( \times \), modeling the heated second free span,
and the downward ramp in \( \Theta(s) \) for the first 5 cm after the fifth \( x \), modeling the cooling of the fibers upon contact with the unheated roller 3. In this process there is draw in *three* of the six possible zones, namely draw on the two feed rollers (as in Simulation 1) and also draw in the second free span due to the presence of the heater. There is no draw in the other three possible zones: The model indicates there is no draw on the two take-up rollers in the same manner as in Simulation 1; we find that conditions (59) are violated by solutions of Equations (58) for all allowable speeds \( v_2 \) and \( v_4 \). There is no draw in first free span since it is isothermal and without guides.

In Figure 26 the behavior of the tow is identical to that of the isothermal process of Simulation 1 until the heated second free span. When the fibers experience higher temperature, they behave in accordance with the corresponding constitutive equations for that temperature, resulting in reduced tension in the tow as the fibers become softer. Since the fiber can stretch with lower tension in this region, the speed of the fibers increases in this zone in order to match the the surface speed of the third roller. The tow enters the third roller without slip, but the decreasing temperature of the fiber induces an increased tension, accompanied by frictional and normal forces immediately after the tow attaches to the roller.

In the third simulation (Figure 27), the differences are a lower friction coefficient, two and a quarter wraps on roller 1 (note that the distance between the first \( x \) and the second \( x \) is now 180 cm rather than 100 cm), no free span heaters, and a heated roller 3. For this process there is draw on both feed rollers and no draw on first (unheated) take-up roller and unheated free spans. Due to the presence of the heated roller 3, however, there are for the first time in these simulations steady solutions with take-up draw, in which the tow attaches to roller 3 with speed less than the surface speed of roller 3.

When Equation (58b) is solved, the solutions of \( y_1 \) satisfying \( y_1 \geq x_4 \) are found for all values of \( v_4 \) between 687.5 and 740 cm s\(^{-1}\). Each of these values produces a steady solution \( v(s), T(s), f(s), \) and \( n(s) \). Those with \( 687.5 \leq v_4 < 740 \) cm s\(^{-1}\) exhibit take-up on the heated roller; that with \( v_4 = 740 \) cm s\(^{-1}\) has no take-up draw. As discussed before, only the steady solution with the lowest energy will be stable and hence persist in the real world. It
Figure 26: Speed, tension, normal force per length, frictional force per length, and temperature profiles for a draw line with heated second freespan.
Figure 27: Speed, tension, normal force per length, frictional force per length, and temperature profiles for a draw line with heated final roller and with $\mu = 0.05$. The draw line length is greater than the previous simulations due to an added wrap on the first roller.
is found that the solution with no take-up draw has the lowest value of the energy function $e$ defined in Equation (66), and this is the solution shown in Figure 27.

The reason for the absence of take-up draw on the heated roller 3 is fundamentally different than the reason for no take-up draw on roller 2: On the unheated roller 2 take-up draw is not possible, since Equation (58a) has no solution satisfying $y_2 > x_2$. On the heated roller 3, take-up draw is possible but unstable.

The effect of the heated roller as shown in Figure 27 is a large decrease in fiber tension and a large frictional force from the roller when the fiber temperature is increasing. Also note that the lower friction coefficient has produced a much larger draw zone on the first feed roller, and reduced the magnitude of friction. The difference in behavior shown between Figure 26 and Figure 27 is indicative of the differences in response of filament yarn and staple tow draw processes.

### 4.6 Conclusion

A general method of characterizing fiber behavior in a multi-stage draw process is described in this chapter. The efficacy of the model is demonstrated for three different two-stage draw process, with different drawing conditions. The model is flexible to address any fibrous materials, as long as its piecewise linear elastic-plastic behavior is known for a range of temperature conditions. The analysis presented allows one to compare the effectiveness of different drawing conditions, enabling fiber manufacturers to identify optimum processing conditions. The outcome of this analysis is the velocity and tension at any point along the length of the yarn, and the forces from the rollers on the yarn, for specific drawing conditions. These predictions can in turn serve as the input data for investigating micro-structure development in the fiber during drawing. As mentioned earlier, the micro-structure developed in the fiber is responsible for the fiber properties. As the velocity and tension profile is altered, vastly different micro-structure in the fiber could result.
CHAPTER V

FIBER DRAW USING DRAW PINS UNDER ISOTHERMAL CONDITIONS

A commercially relevant fiber drawing process with draw pins which are used for localizing and enhancing the fiber draw is analyzed in this chapter. Inclusion of a draw pin introduces the possibility of unloading in the fiber, resulting in the additional complexity of plastic fiber characteristics in the analysis. (In the absence of unloading, the stress in an elastic-plastic fiber monotonically increases, so that only the elastic response of the fiber enters into the analysis.) In this chapter, an isothermal two stage drawing process with a draw pin in the second draw zone is analyzed. Various possibilities of draw are analyzed which show that the total draw is in general distributed between the first and the second roller as well as on the pin. The offset location of the pin is shown to have a significant effect on the distribution of the draw. As this offset is increased, the nature of the draw on the second roller changes from loading, to neither loading nor unloading at a critical offset, to unloading.

5.1 Introduction

The insertion of a stationary draw pin between rollers, however, introduces the possibility of unloading of the filament during the draw process, i.e. decreasing tension along the draw line. This unloading adds the complication of inelastic behavior of spun fiber into the analysis of the draw process. In this chapter, the cause of unloading is restricted to draw pins, rather than radiant heaters or heated rollers. By restricting to isothermal conditions the effect of unloading is restricted to that due to draw pins. The generic draw process with a draw pin that will be analyzed is shown in Figure 2.
5.2 Equations of Motion

The conservation equation of mass is given by equation (9) and momentum by equations (14) and (15) on the roller and (18) in the free span, respectively. For a steady draw process these equations are then combined with the Coulomb friction model of either equation (11) or (12). To calculate the solutions these equations are combined with the constitutive equation of the fiber which accounts for fiber unloading.

In the free span the relevant equations are (9), (18), and (71). The only way in which both equations (18) and (71) are jointly satisfied at all times for an isothermal process is when

\[ dv = 0. \]

Therefore the speed \( v \), and hence from equation (28) the tension \( T \), are constant in the free spans. This result is a consequence of the assumptions of elastic-plastic fiber characterization and isothermal, steady conditions. Draw in the free span is predicted by other models if the conditions are not isothermal (see [8] for an example), or if effects other than elasticity and plasticity are incorporated in the model. On the feed roller the fiber is moving faster than the roller surface, friction \( f \) acting on the fiber is kinetic due to the slip, and positive according to the adopted sign convention. Hence equation (11) is combined with momentum equations to give

\[ (67) \quad dT - \frac{\mu}{r}(T - Gv)ds = Gdv, \]

where \( T \) and \( v \) are the values of tension and speed, respectively, at the upstream boundary of the increment \( ds \) in arclength, and the particular \( dT \) depends on \( v \) and the sign of \( dv \) as described in equation (71). On the take up roller the fiber is moving slower than the roller surface, friction is kinetic, and negative according to the sign convention. Hence equation (12) is combined with equations of mass and momentum to give produces

\[ (68) \quad dT + \frac{\mu}{r}(T - Gv)ds = Gdv, \]

Over a no-slip zone on a roller, the fiber speed matches that of the underlying roller surface,
and hence is constant. Therefore in these ranges of $s$,

\[(69) \quad dv = 0.\]

When passing over a **draw pin**, the fiber is always moving faster than than the stationary draw pin surface, and hence the governing equation for the fiber on a draw pin is equation (67).

### 5.3 Constitutive Model: Linearly Elastic-Plastic

In the modeling of this draw process an elastic-plastic constitutive equation for fiber behavior which captures the behavior of a soft draw plateau between stiffer regions encountered in as-spun, undrawn polymeric fibers is adopted. Further it is assumed that the processing is done at high speeds, allowing little time for viscoelastic effects, so that viscoelasticity can be reasonably neglected without resulting is significant errors.

The incremental elastic-plastic constitutive model, equation (22), for the isothermal case with increment of temperature $d\theta = 0$ can be expressed as

\[(70) \quad dT = \begin{cases} 
K_1 d\varepsilon & \text{if } \varepsilon < \varepsilon_a \text{ or } d\varepsilon \leq 0 \text{ or } \varepsilon < \varepsilon_{\text{max}}, \\
K_2 d\varepsilon & \text{if } \varepsilon_a \leq \varepsilon < \varepsilon_b \text{ and } d\varepsilon > 0 \text{ and } \varepsilon = \varepsilon_{\text{max}}, \\
K_3 d\varepsilon & \text{if } \varepsilon_b \leq \varepsilon \leq \varepsilon_c \text{ and } d\varepsilon > 0 \text{ and } \varepsilon = \varepsilon_{\text{max}},
\end{cases}\]

where $K_1$, $K_2$, and $K_3$ are in general specified functions of strain $\varepsilon$, the transition strains $\varepsilon_a$, $\varepsilon_b$, and $\varepsilon_c$ are specified constants, and $\varepsilon_{\text{max}}$ is the maximum value of strain attained in history of deformation since the state at $s = 0$. This chapter restricts the development to the isothermal case. A schematic representation of the constitutive behavior exhibiting three possible paths in load/strain space for the filament is given in Figure 28. In each path depicted on this figure the filament stretches monotonically to some strain $\varepsilon_{\text{max}}$ and then unloads. If the maximum stretch of the filament is less than the elastic limit $\varepsilon_a$ (path $\text{a}$) then the loading and unloading are along the same line. If the maximum stretch exceeds the elastic limit, either into the soft plateau (path $\text{b}$) or through the plateau into the second stiff region (path $\text{c}$), the filament unloads with the initial elastic stiffness $K_1$, revealing the permanent deformation. For the moving fiber the strain $\varepsilon(s)$ at any point $s$ is related
Figure 28: Possible paths in filament load/strain space in which the filament stretches monotonically to some maximum strain $\epsilon_{\text{max}}$ and then unloads.  

- a $\epsilon_{\text{max}} < \epsilon_a$: the unloading is elastic, along the same line as the loading.
- b $\epsilon_{\text{max}} < \epsilon_a < \epsilon_b$ and
- c $\epsilon_b < \epsilon_{\text{max}} < \epsilon_c$: filament unloads with slope $K_1$.  


to the fiber speed \( v(s) \) at that point through equation (27). The incremental constitutive equation between fiber tension and fiber strain is now rewritten as an incremental relation between tension and fiber speed,

\[
dT = \begin{cases} 
\frac{K_1}{r_1}dv & \text{if } v < v_a \text{ or } dv \leq 0 \text{ or } v < v_{\text{max}}, \\
\frac{K_2}{r_2}dv & \text{if } v_a \leq v < v_b \text{ and } dv > 0 \text{ and } v = v_{\text{max}}, \\
\frac{K_3}{r_3}dv & \text{if } v_b \leq v \leq v_c \text{ and } dv > 0 \text{ and } v = v_{\text{max}},
\end{cases}
\]

(71)

where \( K_i \) \( (i = 1, 2, 3) \) are now functions of \( v \). The transition speeds \( v_a, v_b, \) and \( v_c \) follow from the transition strains \( \varepsilon_a, \varepsilon_b, \) and \( \varepsilon_c, \) i.e. \( v_a = r\omega_1(1 + \varepsilon_a), v_b = r\omega_1(1 + \varepsilon_b), \) and \( v_c = r\omega_1(1 + \varepsilon_c); v_{\text{max}} \) is the maximum speed achieved by the fiber since it attached to the first roller at \( s = 0. \)

5.4 The Two Stage Draw Process with a Draw Pin

The isothermal two-stage process with an intermediate draw pin in the second free span is modelled. It is assumed all three rollers have the same radius, labeled \( r \). The angular velocities of the first, second, and third rollers are specified as \( \omega_1, \omega_2, \) and \( \omega_3, \) respectively, with \( \omega_1 < \omega_2 < \omega_3. \)

In Figure 29 the conceivable slip and no-slip zones on the rollers and pin for this process are indicated, and labels for important locations along the draw line are introduced. The terminology from the last chapter carried over. No change in the tow line solution from the point of fiber attachment to the first roller at location \( x_0 \) to the as-yet-undetermined location \( y_3, \) where the fiber speed first departs from the roller speed, to \( x_3, \) where it leaves the roller. In this zone due to the the presence of the draw pin, both \( v_3 < r\omega_2 \) and \( v_3 > r\omega_2 \) are possible on the second roller.

The second free span is divided into two sections by the draw pin. In the first section the fiber has the speed \( v_3. \) Since the pin is fixed, there is necessarily slip between fiber and pin throughout the contact region, and, as shown in the next section, slip is always accompanied by draw. Hence there is draw over the entire contact with the pin, from \( x_4 \) to \( x_5; \) this draw zone is labeled \( \Delta \) in Figure 29. The fiber therefore departs the pin with a
Figure 29: Conceivable draw zones for the two-stage draw process with a draw pin. 1: on the feed roller of the first stage (between \(y_1\) and \(x_1\)), 2: on the take-up roller of the first stage (between \(x_2\) and \(y_2\)), 3: on the feed roller of the second stage (between \(y_3\) and \(x_3\)), 4: on the draw pin (between \(x_4\) and \(x_5\)), and 5: on the take-up roller of the second stage (between \(x_6\) and \(y_6\)).
speed $v_5$ different than the speed $v_4 = v_3$ with which it attaches to the pin. The fiber now enters the second section of the second free span; in this section the fiber stays at a speed $v_5$.

Draw takes place on the third roller marked as zone 5 extending from $x_6$ to $y_4$ if the speed of the fiber $v_6 = v_5$ at entry to the roller is less than the speed $r\omega_3$ of the roller. The fiber reaches the speed $r\omega_3$ at $y_4$, and stays at that speed until it exits the final roller at $x_7$.

5.5 Solution

In this section the basic equations of the second section are integrated to obtain relations between the process conditions and unknown slip boundaries $y_1, y_2, y_3, y_4$, unknown fiber departure speeds $v_1, v_3, v_5$, and unknown attachment speeds $v_2, v_4, v_6$, and to obtain relations that determine the speed and tension profiles $v(s)$ and $T(s)$ everywhere along the draw line in terms of these ten quantities.

Introducing a draw pin creates many new conceivable paths of fiber behavior, and hence this section necessarily contains a large number of equations and conditional statements. However in the next section it will be seen that the solution of these equations and conditions is straightforward and easy to understand.

To reduce the number of possibilities and equations to consider, a case where the roller speeds and fiber properties are such that in the no-slip zone on the second roller the fiber has been drawn into its soft plateau $(r\omega_1 < v_a < r\omega_2 < v_b)$, and in the no-slip zone on the third roller the fiber has been drawn into its second stiff state but below its breaking strain $(v_b < r\omega_3 < v_c)$ is considered. Although this is invariably the situation in industrial practice, note that the solution in this case can be readily extended to other cases.

Since there is no draw pin in the first free span, the departure speed $v_1$ of the fiber from the first roller must be in the range between the surface speeds of the first and second rollers, $r\omega_1 \leq v_1 \leq r\omega_2$. If there is no draw on the first roller then $v_1 = r\omega_1$, and if there is a draw zone on the first roller then $v_1 > r\omega_1$; part of the solution procedure is to identify which of these two situations occurs for specific draw line conditions.

If there is a draw zone on the first roller then equation (67) applies, and in this draw
zone the fiber speed is monotonically increasing, so that \( dv > 0 \) and \( v = v_{\text{max}} \) everywhere in the zone. For such monotonically increasing loading, the incremental constitutive equation (28) integrates to

\[
T(v) = \begin{cases} 
T_0 + K_1 \left( \frac{v}{r \omega_1} - 1 \right) & \text{for } v \leq v_a, \\
T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v}{r \omega_1} - \frac{v_a}{r \omega_1} \right) & \text{for } v_a < v \leq v_b, \\
T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r \omega_1} - \frac{v_a}{r \omega_1} \right) + K_3 \left( \frac{v}{r \omega_1} - \frac{v_b}{r \omega_1} \right) & \text{for } v_b < v \leq v_c.
\end{cases}
\]

Equation (72) combined with equation (67) can be integrated from the beginning \( (y_1) \) to the end \( (x_1) \) of the draw zone, yielding a relation between the unknowns \( v_1 \) and \( y_1 \):

\[
y_1 = x_1 - \begin{cases} 
\frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_1}{r \omega_1} - 1 \right) - G v_1}{T_0 - G r \omega_1} & \text{if } r \omega_1 \leq v_1 \leq v_a, \\
\frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v_1}{r \omega_1} - \frac{v_a}{r \omega_1} \right) - G v_1}{T_0 - G r \omega_1} & \text{if } v_a < v_1 \leq r \omega_2.
\end{cases}
\]

The solution to equation (73) obviously must satisfy the condition \( y_1 \leq x_1 \), since the slip zone \( y_1 \leq s \leq x_1 \) must be on the pulley (see Figure 29). If in a specific process condition \( y_1 \leq x_1 \) is satisfied by the solution of equation (73) and if the fiber departs the first roller still in the initial stiff state (i.e., \( v_1 \leq v_a \)), then integrating equations (67) and (72) from \( y_1 \) to arbitrary \( s \leq x_1 \) gives the speed profile

\[
v(s) = \frac{(T_0 - G r \omega_1)}{K_1} \frac{\mu (s - y_1)}{r \omega_1} - \frac{(T_0 - K_1)}{K_1} - G \]

for \( y_1 \leq s \leq x_1 \).

If the condition \( y_1 \leq x_1 \) is satisfied and the fiber departs the first roller in the soft plateau (i.e., \( v_1 \geq v_a \)), then
\[ v(s) = \begin{cases} 
\frac{\mu (s - y_1)}{r} - \frac{T_0 - K_1}{r_\omega 1 - G} & \text{for } y_1 \leq s \leq y_a, \\
\frac{T_0 + K_1 \left( \frac{v_a}{r_\omega 1} - 1 \right) - Gv_a}{T_0 + K_1 \left( \frac{v_a}{r_\omega 1} - 1 \right) - Gv_a} & \text{for } y_a < s \leq x_1,
\end{cases} \]

where \( y_a \) is the location with the draw zone where the fiber draws into the soft plateau,

\[ y_a = x_1 - \frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r_\omega 1} - 1 \right) + K_2 \left( \frac{v_1}{r_\omega 1} - \frac{v_a}{r_\omega 1} \right) - Gv_1}{T_0 + K_1 \left( \frac{v_a}{r_\omega 1} - 1 \right) - Gv_a}. \]

The tension profile \( T(s) \) is computed by inserting the pertinent equations, equations (74) or (75) into equation (72).

If the solution to equation (73) predicts the inadmissible result \( y_1 > x_1 \) for all possible \( v_1 \), then there is no draw zone on the first roller, and \( y_1 = x_1 \) and \( v_1 = r_\omega 1 \).

The fiber leaves the first roller at \( x_1 \) with a speed \( v_1 \). After crossing the free span the fiber attaches itself to the second roller at \( x_2 \) with a speed of \( v_2 \); for the isothermal free span there is no draw in the free span and \( v_2 = v_1 \).

Since there is no pin in the first free span, the attachment speed \( v_2 \) of the fiber to the second roller must be in the range \( r_\omega 2 \leq v_2 \leq r_\omega 2 \). If \( v_2 < r_\omega 2 \) then there is a take-up draw zone on the second roller with the fiber moving slower than the roller surface, so that equation (68) applies. In this draw zone, the fiber speed is monotonically increasing, with \( dv > 0 \) and \( v = v_{\text{max}} \) everywhere, so that the incremental constitutive equation (28) integrates to equation (72). Combining equations (72) and (68), and integrating from \( x_2 \) to
the point \( y_2 \) of slip termination where \( v \) first reaches \( r\omega_2 \) produces

\[
y_2 = x_2 - \begin{cases} 
\frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Gr\omega_2}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - Gv_2} & \text{if } v_2 \leq v_a, \\
\frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Gr\omega_2}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Gv_2} & \text{if } v_a < v_2 \leq r\omega_2.
\end{cases}
\]

The solution of equation (77) must satisfy the condition \( y_2 \geq x_2 \). If this condition is satisfied and the fiber attaches to the second roller still in its initial elastic state \( (v_2 < v_a) \), then the speed profile is given by

\[
v(s) = \begin{cases} 
\left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - Grv_2}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - Gv_a} \right) e^{-\frac{\mu}{r} (s - x_2)} - (T_0 - K_1) & \text{for } x_2 \leq s \leq y_a, \\
\frac{K_1}{r\omega_1} - G & \text{for } y_a < s < y_2, \\
\left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - Grv_2}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - Gv_2} \right) e^{-\frac{\mu}{r} (s - y_a)} - \left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - K_2 \frac{v_a}{r\omega_1}}{r\omega_1} \right) & \text{for } y_a < s < y_2.
\end{cases}
\]

where the transition location \( y_a \) into the soft plateau of fiber behavior is now given by

\[
y_a = x_2 - \frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - Gv_a}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - Gv_2}.
\]

If the condition \( y_2 \geq x_2 \) is satisfied and the fiber attaches to the second roller already drawn into its soft plateau \( (v_2 \geq v_a) \), then the speed profile is given by

\[
v(s) = \frac{\left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - Grv_a}{K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - Gv_2} \right) e^{-\frac{\mu}{r} (s - x_2)} - \left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - K_2 \frac{v_a}{r\omega_1}}{r\omega_1} \right)}{K_2 \frac{v_a}{r\omega_1} - G}.
\]

If equation (77) nonsensically predicts \( y_2 < x_2 \) (i.e., the slip zone on the roller is not on the roller) for all possible \( v_2 \), then there is no take-up draw on the first roller, and \( x_2 = y_2 \) and \( v_2 = r\omega_2 \).
In the no-slip zone on the second roller fiber speed is given by \( v(s) = r\omega_2 \). It is conceivable that at some location before departure from the second roller the fiber will begin to slip and hence the departure speed \( v_3 \) will be different from the roller speed \( r\omega_2 \). If there is no draw pin present in the second free span then the departure speed must be somewhere between the surface speeds of the second and third rollers, \( r\omega_2 \leq v_3 \leq r\omega_3 \) (for a model of two-stage draw without draw pins, see [8]). If there is a pin, however, then \( v_3 \) can be either greater than or equal to the second roller speed \( r\omega_2 \) (qualitatively the same as if there were no pin) or less than \( r\omega_2 \) (possible because of the pin). Each of the three cases \( v_3 = r\omega_2, v_3 > r\omega_3, \) and \( v_3 < r\omega_2 \) is handled separately; which of these cases happens depends on the draw process conditions and is part of the solution to the model equations.

If \( v_3 > r\omega_2 \) then there is a feed draw zone before departure from the second roller in which the fiber is moving faster than the roller surface, for which equation (67) applies.

The fiber speed in this draw zone would be monotonically increasing \( (dv > 0 \) and \( v = v_{\text{max}} \) everywhere), so that equation (72) is valid (i.e. the incremental constitutive equation (28) integrates to \( T \) as a function of \( v \)). Combining equations (67) and (72) and integrate from the beginning \( (y_3) \) to the end \( (x_3) \) of the draw zone one obtains

\[
y_3 = x_3 - \left\{ \begin{array}{ll} \frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_3}{r\omega_1} - \frac{v_a}{r\omega_1} \right)}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r\omega_1} - \frac{v_a}{r\omega_1} \right)} - Gv_3 & \text{if } r\omega_2 < v_3 \leq v_b, \\
\frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_3 \left( \frac{v_3}{r\omega_1} - \frac{v_b}{r\omega_1} \right)}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_3}{r\omega_1} - \frac{v_a}{r\omega_1} \right)} - Gv_3 & \text{if } v_b < v_3 \leq r\omega_3; \\
\end{array} \right. \]

it is useful to recall that draw processes with the transition speed \( v_b \) of the fiber from soft plateau to stiff behavior is between the second and third roller speeds, \( r\omega_2 < v_b < r\omega_3 \) are being modeled. The solution of equation (81) obviously must satisfy \( y_3 \leq x_3 \). If this condition is satisfied and the fiber departs the second roller still in its soft plateau \( (v_3 < v_b) \),
then the speed profile is given by

\begin{equation}
(82) \quad v(s) = \frac{\mu}{K_1} \frac{(T_0 - G\omega_2)}{r\omega_1} (s - y_1) - \frac{(T_0 - K_1)}{K_1} - G \quad \text{for } y_3 \leq s \leq x_3.
\end{equation}

If the condition \( y_3 \leq x_3 \) is satisfied and the fiber departs the first roller already drawn into its second stiff phase \( (v_3 \geq v_b) \) then

\begin{equation}
(83) \quad v(s) = \begin{cases} 
\frac{1}{K_2} \left[ \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - G\omega_2 \right) \frac{\mu}{r} (s - y_3) 
\right. \\
\left. - \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - K_2 \frac{v_a}{r\omega_1} \right) \right] 
\end{cases}
\end{equation}

\begin{equation}
\text{for } y_3 < s \leq y_b,
\end{equation}

\begin{equation}
\frac{1}{K_3} \left[ \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - G\omega_2 \right) \frac{\mu}{r} (x_3 - y_3) 
\right. \\
\left. - \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - K_3 \frac{v_b}{r\omega_1} \right) \right]
\end{equation}

\begin{equation}
\text{for } y_b < s \leq x_3,
\end{equation}

where the location \( y_b \) of transition into the final stiff region of the fiber behavior is given by

\begin{equation}
(84) \quad y_b = x_3 - \frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Gv_3}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - G\omega_2}.
\end{equation}

If the solution to equation (81) predicts the nonsensical result \( y_3 > x_3 \) for all possible \( v_3 \), then there is no feed draw zone on the second roller and \( y_3 = x_3 \) and \( v_3 = r\omega_2 \).

If the fiber departs the second roller with a speed slower than the underlying roller surface \( (v_3 < r\omega_2) \), a situation possible if and only if there is a draw pin present between the second and third rollers) then \( d\nu < 0 \) and \( u < v_{max} = r\omega_2 \) for some length of fiber on the second roller. In this case the incremental constitutive equation (28) is integrated to obtain

\begin{equation}
(85) \quad T(v) = T_0 + K_1 \left( \frac{v_a}{r\omega_1} - \frac{v_{max}}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_{max}}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_1 \left( \frac{v}{r\omega_1} \right).
\end{equation}
In the departing slip zone of the second roller the fibers speed and tension decrease, according to equation (85), from their maximum values,

\[
(86) \quad v_{\text{max}} = r\omega_2, \quad T_{\text{max}} = T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right),
\]

(attained in the no-slip zone on the second roller) to \( v = v_3 < r\omega_2 \) and \( T = T(v_3) = T_0 + K_1 \left( \frac{v_a}{r\omega_1} - \frac{r\omega_2}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_1 \left( \frac{v_3}{r\omega_1} \right) \). Integrating the coupled equations 85 and (68) from beginning \( (y_3) \) to end \( (x_3) \) of this unloading zone produces

\[
(87) \quad y_3 = x_3 + \frac{r}{\mu} \ln \left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - \frac{r\omega_2}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_1 \left( \frac{v_3}{r\omega_1} \right) - Gv_3}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Gr\omega_2} \right).
\]

The corresponding speed profile is given by

\[
(88) \quad v(s) = \frac{1}{K_1 \frac{1}{r\omega_1} - G} \left[ \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Gr\omega_2 \right) e^{-\frac{\mu}{r} (s - y_3)} \right. \\
\left. - \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - \frac{r\omega_2}{r\omega_1} - 1 \right) + K_2 \left( \frac{r\omega_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) \right) \right],
\]

which decreases from \( r\omega_2 \) at \( y_3 \) to the departure speed \( v_3 \) from the second roller at \( x_3 \).

The fiber then passes through the first section of the second isothermal free span and attaches to the draw pin at \( x_4 \) with speed \( v_4 = v_3 \).

On the draw pin the speed of the fiber is always greater than the speed of the stationary pin surface, so that momentum equation (67) applies. Throughout contact with the pin the fiber speed is monotonically increasing \( (dv > 0) \), and hence the fiber exits the pin with a speed greater than that with which it entered. Two different cases and (and five) of fiber behavior on the draw pin are possible, depending on whether or not the fiber has unloaded on the immediately preceding second roller (see Figure 30 to 32 and 33 to 34).

The more interesting Case 1, sketched in Figures 30 to 32, is investigated first where the fiber has unloaded on the second roller (indicated by the condition \( v_3 = v_4 < r\omega_2 \)). In this case there is always at least a portion of the contact interval with the pin where the fiber’s speed and tension are increasing, but are not yet up to the maximum values \( v_{\text{max}} \) and \( T_{\text{max}} \) that they had on the second roller before the unloading, given by equation (86).

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In this portion, the incremental constitutive equation (28) integrates to equation (85); until it returns to its previous maximum, the fiber loads along the path it just unloaded. Three subcases are possible, depending on if the fiber leaves the pin still below its previous maximum tension (Subcase 1a), the fiber fully reloads on the pin and leaves with a speed greater than \( rv_2 \) but less than the the transition speed \( v_b \) (Subcase 1b), or it fully reloads and addition draws into the final stiff region (Subcase 1c). For each subcase there are different equations for fiber speed and tension, and different relations between locations of attachment and departure:

**Subcase 1a:** The entry speed \( v_4 \) to the pin and exit speed \( v_5 \) are related through

\[
x_5 = x_4 + \frac{r}{\mu} \ln \left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - \frac{rv_2}{r\omega_1} - 1 \right) + K_2 \left( \frac{rv_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_1 \left( \frac{v_5}{r\omega_1} \right) - Gv_5}{T_0 + K_1 \left( \frac{v_a}{rv_1} - \frac{rv_2}{r\omega_1} - 1 \right) + K_2 \left( \frac{rv_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_1 \left( \frac{v_4}{r\omega_1} \right) - Gv_4} \right),
\]

with speed profile on the pin given by

\[
v(s) = \frac{1}{K_1 \frac{r}{r\omega_1} - G} \left[ \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - \frac{rv_2}{r\omega_1} - 1 \right) + K_2 \left( \frac{rv_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_1 \left( \frac{v_4}{r\omega_1} \right) - Gv_4 \right)^{\mu} \left( s - x_4 \right)
- \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - \frac{rv_2}{r\omega_1} - 1 \right) + K_2 \left( \frac{rv_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) \right) \right] \text{ for } x_4 < s \leq x_5.
\]

**Subcase 1b:**

\[
x_5 = x_4 + \frac{r}{\mu} \ln \left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{rv_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Gv_5}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - \frac{rv_2}{r\omega_1} - 1 \right) + K_2 \left( \frac{rv_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_1 \left( \frac{v_4}{r\omega_1} \right) - Gv_4} \right),
\]

and where \( y_p \) is the intermediate location on the pin where drawing in the soft plateau resumes,

\[
y_p = x_4 + \frac{r}{\mu} \ln \left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{rv_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Grv_2}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - \frac{rv_2}{r\omega_1} - 1 \right) + K_2 \left( \frac{rv_2}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_1 \left( \frac{v_4}{r\omega_1} \right) + Gv_4} \right).
\]
Figure 30: Case 1 of fiber behavior on the draw pin, in which the fiber attaches to the pin with speed $v_4$ less than the preceding second roller speed $r\omega_2$. Subcase 1a: The fiber does not fully reload on the pin ($v_5 < r\omega_2$, where $v_5$ denotes the departure speed from the pin).

Figure 31: Subcase 1b: The fiber fully reloads on the pin but does not draw into its final stiff state ($r\omega_2 < v_5 < v_b$).
Subcase 1c:

\[ x_5 = x_4 + \frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r \omega_1} - \frac{v_a}{r \omega_1} \right) + K_3 \left( \frac{v_5}{r \omega_1} - \frac{v_b}{r \omega_1} \right) - Gv_5}{T_0 + K_1 \left( \frac{v_a}{r \omega_1} - \frac{r \omega_2}{r \omega_1} - 1 \right) + K_2 \left( \frac{r \omega_2}{r \omega_1} - \frac{v_a}{r \omega_1} \right) + K_1 \left( \frac{v_4}{r \omega_1} \right) - Gv_4}, \]

and

\[
v(s) = \begin{cases} 
\frac{1}{K_1} - G \left[ \left( T_0 + K_1 \left( \frac{v_a}{r \omega_1} - \frac{r \omega_2}{r \omega_1} - 1 \right) + K_2 \left( \frac{r \omega_2}{r \omega_1} - \frac{v_a}{r \omega_1} \right) + K_1 \left( \frac{v_4}{r \omega_1} \right) - Gv_4 \right) \frac{\mu}{r} (s - x_4) 
\right] 
\end{cases}
\]

\[
- \left( T_0 + K_1 \left( \frac{v_a}{r \omega_1} - \frac{r \omega_2}{r \omega_1} - 1 \right) + K_2 \left( \frac{r \omega_2}{r \omega_1} - \frac{v_a}{r \omega_1} \right) \right) \quad \text{for } x_4 < s \leq y_p,
\]

\[
\frac{1}{K_2} - G \left[ \left( T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{r \omega_2}{r \omega_1} - \frac{v_a}{r \omega_1} \right) - Gr \omega_2 \right) \frac{\mu}{r} (s - y_p) \right] 
\]

\[
- \left( T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{r \omega_2}{r \omega_1} - \frac{v_a}{r \omega_1} \right) \right) \quad \text{for } y_p < s \leq y_b,
\]

\[
\frac{1}{K_3} - G \left[ \left( T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{r \omega_2}{r \omega_1} - \frac{v_a}{r \omega_1} \right) - Gv_b \right) \frac{\mu}{r} (s - y_b) \right] 
\]

\[
- \left( T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{r \omega_2}{r \omega_1} - \frac{v_a}{r \omega_1} \right) - K_3 \left( \frac{v_b}{r \omega_1} \right) \right) \quad \text{for } y_p < s \leq x_5,
\]

where the intermediate location \( y_p \) on the pin where drawing in the soft plateau resumes is again given by equation (92), and \( y_b \) is the intermediate location on the pin where fiber draws out of its soft plateau and into its final stiff state,

\[ y_b = x_5 - \frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r \omega_1} - \frac{v_a}{r \omega_1} \right) + K_3 \left( \frac{v_5}{r \omega_1} - \frac{v_b}{r \omega_1} \right) - Gv_5}{T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r \omega_1} - \frac{v_a}{r \omega_1} \right) - Gv_b}. \]

Next Case 2, sketched in Figures 33 and 34 is investigated. For this case the fiber has not unloaded on the second roller (indicated by the condition \( v_3 = v_4 \geq r \omega_2 \)). Two subcases are possible, depending on whether the fiber leaves the pin at a speed less than the transition speed \( v_b \) (Subcase 2a), or greater than \( v_b \) so that it draws on the pin into the
**Figure 32:** Subcase 1c: The fiber draws into its final stiff state on the pin \((v_6 > v_5)\).

final stiff region (Subcase 2b). As with the previous case different equations for fiber speed
and tension are obtained:

**Subcase 2a:** The entry and exit speeds to the pin, \(v_4\) and \(v_5\) respectively, are related
through

\[
x_5 = x_4 + \frac{r}{\mu} \ln \left[ \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_5}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Gv_5}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_4}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Gv_4} \right],
\]

and the speed profile on the pin is given by

\[
v(s) = \frac{1}{\frac{K_2}{r\omega_1} - G} \left[ \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_4}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - Gv_4 \right) e^{\frac{\mu}{r}(s - x_4)} - \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - K_2 \left( \frac{v_a}{r\omega_1} \right) \right) \right]
\]

for \(x_4 < s \leq x_5\).
Subcase 2b:

\[
(98) \quad x_5 = x_4 + \frac{r}{\mu} \ln \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_3 \left( \frac{v_5}{r\omega_1} - \frac{v_b}{r\omega_1} \right) - G v_5}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_4}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - G v_4},
\]

with

\[
(99) \quad v(s) = \begin{cases} 
\frac{1}{K_2} \left[ \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_4}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - G v_4 \right) e^{\frac{s-x_4}{r}} 
- \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) - K_2 \left( \frac{v_a}{r\omega_1} \right) \right) \right] & \text{for } x_4 < s \leq y_b, \\
\frac{1}{K_3} \left[ \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - G v_b \right) e^{\frac{s-y_b}{r}} 
- \left( T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - K_3 \left( \frac{v_b}{r\omega_1} \right) \right) \right] & \text{for } y_b < s \leq x_5.
\end{cases}
\]

with the location \( y_b \) on the pin where the fiber draws out of its soft plateau is given by equation (95).
Figure 34: Subcase 2b: The fiber draws to its final stiff state on the pin \((v_5 > v_b)\).

After exiting the draw pin the fiber enters the second section of the second free span. The fiber does not undergo a change in speed in this isothermal free span, and hence attaches to the third roller at \(x_6\) with a speed \(v_6 = v_5\).

If the attachment speed to the third roller is less than the roller speed \((v_6 < r\omega_3)\) then draw takes place on the roller, with the draw zone extending from \(x_6\) to some \(y_4\), where the fiber speed reaches \(r\omega_3\). Integrating equation (68) from \(x_6\) to \(y_4\) produces

\[
\frac{r}{\mu} \ln \left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_3 \left( \frac{r\omega_3}{r\omega_1} - \frac{v_b}{r\omega_1} \right) - G r \omega_3}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_6}{r\omega_1} - \frac{v_a}{r\omega_1} \right) - G v_6} \right) 
\]

if \(v_b \leq v_6\),

\[
\frac{r}{\mu} \ln \left( \frac{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_3 \left( \frac{r\omega_3}{r\omega_1} - \frac{v_b}{r\omega_1} \right) - G r \omega_3}{T_0 + K_1 \left( \frac{v_a}{r\omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r\omega_1} - \frac{v_a}{r\omega_1} \right) + K_3 \left( \frac{v_6}{r\omega_1} - \frac{v_b}{r\omega_1} \right) - G v_6} \right) 
\]

if \(v_b < v_6 \leq r\omega_3\),
and integrating to some intermediate location $s$ yields the speed profile

$$v(s) = \begin{cases} 
\frac{1}{K_3} \left[ \left( T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v_6}{r \omega_1} - \frac{v_a}{r \omega_1} \right) - G v_6 \right) e^{-\frac{\mu}{r} (s - x_6)} \right. \\
\left. - \left( T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r \omega_1} - \frac{v_a}{r \omega_1} \right) - K_3 \left( \frac{v_b}{r \omega_1} \right) \right) \right] 
\end{cases}$$

for $s \leq y_b$, \[ \frac{1}{K_3} \left[ \left( T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v_6}{r \omega_1} - \frac{v_a}{r \omega_1} \right) + K_3 \left( \frac{r \omega_3}{r \omega_1} - \frac{v_b}{r \omega_1} \right) - G v_6 \right) e^{-\frac{\mu}{r} (s - x_6)} \right. \\
\left. - \left( T_0 + K_1 \left( \frac{v_a}{r \omega_1} - 1 \right) + K_2 \left( \frac{v_b}{r \omega_1} - \frac{v_a}{r \omega_1} \right) - K_3 \left( \frac{v_b}{r \omega_1} \right) \right) \right] 
\]

If equation (100) nonsensically predicts $y_1 < x_6$ for all possible $v_6$, then there is no draw on the final roller, i.e. $v_6 = r \omega_3$ and $x_6 = y_4$.

Beyond the point of termination of draw at $y_4$ the fiber stays at the surface speed of the roller of $r \omega_3$, marking the end of the two stage draw.

Solution procedure

Recall that the slip boundaries $y_1$, $y_2$, $y_3$, $y_4$ and the fiber departure and attachment speeds $v_1$, $v_2$, $v_3$, $v_4$, $v_5$, $v_6$ are unknown. These ten unknowns are related by the five equations (73), (77), (81), (100), and one of equations (89), (91), (93), (96), and (98), and the three constraints $v_1 = v_2$, $v_3 = v_4$, $v_5 = v_6$ due to the free spans being isothermal. Hence there are eight equations for ten unknowns, and the problem is under-determined. $v_2$ and $v_6$ are identified as the free parameters, noting that they must satisfy $r \omega_1 \leq v_2 \leq r \omega_2$ and $v_6 \leq r \omega_3$, and solve the set of equations mentioned earlier (i.e. (73), (77), (81), (100), and one of equations (89), (91), (93), (96), and (98), and the three constraints $v_1 = v_2$, $v_3 = v_4$, and $v_5 = v_6$) for $y_1$, $y_2$, $y_3$, $y_4$, $v_1$, $v_3$, $v_4$, and $v_5$ in terms of $v_2$ and $v_6$. In the isothermal simulations of this chapter it is found that the free parameters $v_2$ and $v_6$ are uniquely determined by the conditions that the solutions of equations (77) and (100) must satisfy $y_2 \geq x_2$ and $y_4 \geq x_6$, respectively. In simulations of non-isothermal draw line
this is not the case, and the values of $v_2$ and $v_6$ are selected so as to minimize energy, as demonstrated by [8].

Once the transition locations $y_1, y_2, y_3, y_4$ and fiber departure and attachment speeds $v_1, v_2, v_3, v_4, v_5, v_6$ are determined, the speed profile $v(s)$ along the entire draw line from equations (74), (75), (78), (80), (82), (83), (88), (90), (94), (97), (99), and (101) is constructed and the tension evolution $T(s)$ by inserting this speed profile into equations (72) and (85).

5.6 Simulations

In this section four draw processes are simulated. To highlight the effect of the addition of a draw pin, first a draw process without a draw pin (Simulation 1) is simulated. Three processes with a draw pin in the second free span are simulated. These processes demonstrate the three qualitatively different draw line behaviors that are possible with a pin (Simulations 2, 3, and 4). To facilitate study of the effect of the pin, the same elastic moduli $K_1, K_2, K_3$, transition strains $\varepsilon_a, \varepsilon_b$, and breaking strain $\varepsilon_c$ are employed in the four simulations (listed in Table 1), as well as the same process conditions of upstream fiber tension $T_0$, friction coefficient $\mu$, fiber linear density $\rho$, roller radius $r$, and roller speeds $r\omega_1, r\omega_2, r\omega_3$ (listed in Table 4). Simulations 2, 3, and 4 differ only in the offset $h$ of the draw pin, labeled in Figure 2. Note in Table 4 that due to geometrical constraints the length of contact of the fiber with roller 2, roller 3, and the pin vary with pin offset $h$.

Simulation 1: No draw pin

In [8] the solution of Simulation 1 is obtained using the isothermal special case of the model for two-stage draw without draw pin. This solution is shown in Figures 35 and 36. In all plots crosses ($\times$) have been used to denote points where the fiber either enters or departs from a roller, circles (o) to indicate the locations of initiation or cessation of slip, and triangles ($\triangle$) to indicate the transitions between the initial stiff zone and soft plateau of the fiber, and the soft plateau and final stiff zone. On first two plots (fiber speed vs.
Table 4: Process conditions for the four simulations.

<table>
<thead>
<tr>
<th>process condition</th>
<th>sim.1</th>
<th>sim.2</th>
<th>sim.3</th>
<th>sim.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>upstream tension $T_0$ (dyn)</td>
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<td>40000.0</td>
<td>40000.0</td>
<td>40000.0</td>
</tr>
<tr>
<td>coefficient of friction $\mu$</td>
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<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>linear density of fiber $\rho$ (denier)</td>
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<td>450</td>
<td>450</td>
<td>450</td>
</tr>
<tr>
<td>roller radius $r$ (cm)</td>
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<td>12.7</td>
<td>12.7</td>
<td>12.7</td>
</tr>
<tr>
<td>draw pin radius $r_p$ (cm)</td>
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<td>3.18</td>
<td>3.18</td>
<td>3.18</td>
</tr>
<tr>
<td>wrap on roller 1 (cm)</td>
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<td>99.75</td>
<td>99.75</td>
<td>99.75</td>
</tr>
<tr>
<td>wrap on roller 2 (cm)</td>
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<td>86.39</td>
<td>86.65</td>
<td>91.96</td>
</tr>
<tr>
<td>wrap on draw pin (cm)</td>
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<td>3.3</td>
<td>3.43</td>
<td>6.08</td>
</tr>
<tr>
<td>wrap on roller 3 (cm)</td>
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<td>106.34</td>
<td>106.6</td>
<td>111.91</td>
</tr>
<tr>
<td>free span, roller 1 to roller 2 (cm)</td>
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<td>50.0</td>
<td>50.0</td>
<td>50.0</td>
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<tr>
<td>free span, roller 2 to draw pin (cm)</td>
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<td>20.0</td>
<td>20.7</td>
</tr>
<tr>
<td>free span, draw pin to roller 3 (cm)</td>
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<td>20.0</td>
<td>20.0</td>
<td>20.7</td>
</tr>
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<td>roller 1 surface speed $r\omega_1$ (cm s$^{-1}$)</td>
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<td>396</td>
<td>396</td>
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<tr>
<td>roller 2 surface speed $r\omega_2$ (cm s$^{-1}$)</td>
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<td>476</td>
<td>476</td>
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<td>roller 3 surface speed $r\omega_3$ (cm s$^{-1}$)</td>
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<td>draw pin offset $h$ (cm)</td>
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<td>8.71</td>
<td>9.17</td>
<td>20.17</td>
</tr>
</tbody>
</table>

areal length and fiber tension vs. areal length) these points are connected with either solid lines (---) that indicate zones of draw on rollers, dashed-dotted lines (-----) that indicate zones of no slip, or dotted lines (· · · ·) that indicate free spans. In Figure 36 (tension vs. speed), the arrows show the loading path of the fiber, and the numbers indicate important points on this loading path.

As represented by the solution depicted in Figure 35, in an isothermal two-stage draw without pins, draw occurs only as feed draw on the first roller and feed draw on the second roller (zones 1 and 2 in Figure 29). For the process in Simulation 1, the fiber attaches to the first roller at the first $x \times (s = 0 \text{ cm})$ with speed $r\omega_1 = 396 \text{ cm s}^{-1}$. The fiber then proceeds without slip on the first roller until the first $\circ$ at $s = 67 \text{ cm}$; in this $67 \text{ cm}$ of no slip the speed and tension are constant. This no-draw portion of the process occurs at point 1 in the speed vs. tension plot in Figure 36.

The draw zone on the first roller is the $33 \text{ cm}$ from the first $\circ$ ($y_1 = 67 \text{ cm}$) to the second $\times (x_1 = 100 \text{ cm})$, where the fiber departs the roller. In this draw zone, lying between points 1 and 3 of the speed vs. tension plot, the fiber speed increases from the surface speed $396 \text{ cm s}^{-1}$ of the first roller (at point 1 on the speed vs. tension plot) to the surface
Figure 35: Isothermal draw process without draw pins: On the left speed ($v$) vs. arclength ($s$) and on the right tension ($T$) vs. arclength. Locations denoted by ($\times$) are where the fibers either attach to or depart from a roller, ($\circ$) are where initiation or cessation of slip on the rollers takes place, ($\triangle$) indicated locations where the transitions between stiff and soft fiber behavior happen. In the velocity and tension profiles, (——) are zones of draw on rollers, (-----) are zones of no slip, and (······) are free spans.
speed 476 cm $s^{-1}$ of the second roller (at point 3). The corresponding tension increases from $4.00 \times 10^5$ dyn to $6.76 \times 10^5$ dyn. Note by kink at $s = 99$ cm (marked by a $\Delta$ and numerical 2) that the fiber is drawn on the first feed roller from its initial stiff response into its soft plateau: For $67 \text{ cm} < s < 99 \text{ cm}$ the draw is in the initial stiff portion of the filament response, where a large increase in tension accompanies a small increase in strain and hence speed. For $99 \text{ cm} < s \leq 100 \text{ cm}$ the draw is in the soft portion of the filament response, where a small increase in tension accompanies a large increase in strain and speed.

The fiber departs the first roller already at the surface speed of the second roller for two reasons: The free span between the first and second rollers is an isothermal free span without draw pins, and hence speed and tension are constant; the 50 cm in the top two plots between the second $\times (x_1 = 100 \text{ cm})$ and third $\times (x_2 = 150 \text{ cm})$ is the free span. Then, for all possible attachment speeds $v_2$ to the second roller (i.e. all $r\omega_1 \leq v_2 < r\omega_2$), equation (77) predicts $y_2 < x_2$; hence, as discussed after equation (80), for this process there is no takeup draw on the second roller (the second $\circ$ and third $\times$ on the plots coincide).

The fiber attaches to the second roller at $x_2 = y_2 = 150 \text{ cm}$ already at its surface speed $r\omega_2 = 476 \text{ cm} s^{-1}$ and proceeds without slip until the third $\circ$ at $y_3 = 216 \text{ cm}$; from its departure from the first roller until this location the fiber is at point 3 on the tension vs. speed diagram. Beyond $y_3 = 216 \text{ cm}$ the fiber begins to slip; this draw zone on the second roller is the 14 cm from the third $\circ$ to the fourth $\times$ at $x_3 = 230 \text{ cm}$, where the fiber departs from the second roller already at the surface speed of the third roller. This draw on the second roller extends from point 3 to 8 of the speed vs. tension plot. Note from the kink marked by the second triangle at 217 cm (numeral 7) that the fiber is drawn out of its soft plateau during this feed draw.

Between the fourth $\times$ and fifth $\times$ (at $x_4 = 280 \text{ cm}$) is the second free span, again isothermal with no draw pins and hence without draw. In this process equation (100) spuriously predicts $y_4 < x_6$ for all possible $v_6$, and hence there is no takeup draw at the end of the second draw stage, indicated 35 and 36 by the fourth $\circ$ coinciding with the fifth $\times$. The fiber proceeds without slip on the third roller.
Figure 36: Isothermal draw process without draw pins: tension vs. speed, 1: \( v = r \omega_1 \) in \( x_0 \leq s \leq y_1 \), 2: \( v = v_a \) at \( s = y_a \), 3: \( v = r \omega_2 \) in \( x_1 \leq s \leq y_3 \), 7: \( v = v_b \) at \( s = y_b \), 8: \( v = r \omega_2 \) in \( s \geq x_6 (= y_4) \). Note that 1 through 3 happen in the feed draw on the first roller, and 3 through 8 happen in to feed draw on the second roller.
**Simulation 2**: Pin adjusted so that there is draw on both rollers and the pin

When a draw pin is present in the second free span, three different types of behavior are possible in the draw line, depending on the pin offset \( h \) (all other process conditions being held fixed): There can be (i) draw on the both the first and second rollers and the pin, (ii) draw on the first roller and pin, and not on the second roller, with no unloading of the fiber on the second roller, and (iii) draw on the first roller and pin with unloading on the second roller. Note that case (ii) is the boundary between cases (i) and (iii).

![Graph: Isothermal draw process](image)

**Figure 37**: Isothermal draw process with a draw pin: the draw pin offset \( h = 8.71 \) cm is such that the exit speed \( v_3 \) from the second roller is greater than the roller surface speed \( r \omega_2 \), so that there is takeup draw on that roller. The left plots speed vs. arclength, and the right tension vs. arclength.

In Simulation 2 \( h \) is selected so that behavior (i) is obtained (specifically \( h = 8.71 \) cm). In this simulation, shown in Figures 37 and 38, again it is found that for all \( v_2 \) and \( v_6 \) in the ranges \( r \omega_1 < v_2 \leq r \omega_2 \) and \( v_6 \leq r \omega_3 \) other than \( v_2 = r \omega_2 \) and \( v_6 = r \omega_3 \), equations (77)
and (100) produce the spurious results \( y_2 < x_2 \) and \( y_4 < x_6 \), respectively. Hence, as in Simulation 1 without a draw pin, there is no takeup draw on either the second or third roller.

The solution in Simulation 2 is identical to that of Simulation 1 for the first stage of the draw process, since the process conditions are identical and there is no draw pin in that stage. The first notice of the presence in Simulation 2 of the second-stage draw pin is the migration of the onset of slip on the second roller (the third o) downstream in the draw process, closer to the fiber exit \( x_3 \) from the second roller (the fourth \( \times \)). The effect of the pin is to transfer draw from the second roller to the pin. Referring to the speed vs. tension plots for the two simulations, the draw on the second roller between points 3 and 8 in Figure 36 has been split in Figure 38 into 3 to 6 on the second roller and 6 to 8 on the pin. In this simulation the transition from the soft plateau to final stiff zone (point 7) occurs on the pin (Subcase 2b shown in Figure 34).

Quantitatively, on the feed zone of the second roller, starting at the third o, the fiber has been drawn to a speed \( v_3 = 550 \text{ cm s}^{-1} \) when it exits the roller at \( x_3 = 236 \text{ cm} \), marked by the fifth \( \times \). The fiber stays at \( 550 \text{ cm s}^{-1} \) in the first section of the free span before attaching to the draw pin at \( x_4 = 256 \text{ cm} \). The fiber is at the surface speed of the third roller when it leaves the pin \( (v_5 = r_\omega_3 = 740 \text{ cm s}^{-1} \text{ at } x_5 = 259 \text{ cm}) \). There is no additional draw in the second part of the second free span and on the third roller.

**Simulation 3:** Pin adjusted so that there is neither draw nor unloading on the second roller

All other process conditions held constant, as the pin offset \( h \) increases the location of the onset of slip on the second roller migrates downstream toward the exit from the roller into the free span, until at a critical value \( h = h_c \) the two points coincide. In the family of simulations, \( h_c = 9.17 \text{ cm} \). For \( h < h_c \) (such as \( h = 8.71 \text{ cm} \) in the Simulation 2 above) there is a feed draw zone on the second roller immediately upstream of the pin. For \( h > h_c \) (e.g. Simulation 4 to follow, with \( h = 20.17 \text{ cm} \)) the fiber unloads on the second roller, exiting the second roller with a speed less than the surface speed of the roller.
Figure 38: Isothermal draw process with a draw pin: tension vs. speed. 1: $v = r\omega_1$ in $x_0 \leq s \leq y_1$, 2: $v = v_a$ at $s = y_a$, 3: $v = r\omega_2$ in $x_1 \leq s \leq y_3$, 6: $v = v_3 = 550 \text{cm s}^{-1}$ in $x_3 \leq s \leq x_4$, 7: $v = v_b$ at $s = y_b$, 8: $v = r\omega_3$ for $s \geq x_5$. Note that the fiber now draws partly on the first roller (from 1 to 3), partly on the second roller (from 3 to 6), and partly on the draw pin (from 6 to 8).
Figure 39: Isothermal draw process with a draw pin: the draw pin offset is adjusted to the critical value $h_c = 9.17$ cm so that the speed $v_3$ of the fiber exiting the second roller is exactly $r \omega_2$, and hence there is no draw or unloading on the second roller. On the left is the plot of speed vs. arclength and on the right tension vs. arclength.
Figure 40: Isothermal draw process with a draw pin: In the plot of tension vs. speed 1: 
v = r\omega_1 in x_0 \leq s \leq y_1, 2: v = v_a at s = y_a, 3: v = r\omega_2 in x_1 \leq s \leq x_4, 7: v = v_b at s = y_b, 8: v = r\omega_3 for s \geq x_5. The fiber now draws only on the first roller (from 1 to 3), and on the pin (from 3 to 8).
In Simulation 3, shown in Figures 39 and 40, \( h = h_c = 9.17 \text{ cm} \). At this critical value there is no draw on the second roller (it all having been transferred to the pin), but also no unloading. The point \( y_3 \) of initiation of draw on the second roller (the third \( o \)) coincides with the departure location \( x_3 \) (the fourth \( x \)). At this point the fiber enters the second free span with a speed of \( 476 \text{ cm s}^{-1} \). On the draw pin the fiber is drawn from \( 476 \text{ cm s}^{-1} \) to \( 740 \text{ cm s}^{-1} \) (the final roller speed) over a length extending from \( x_4 = 256 \text{ cm} \) to \( x_5 = 259 \text{ cm} \). Referring to Figure 40, the fiber draws from 1 to 3 on the first roller, and 3 to 8 on the pin.

**Simulation 4:** Pin adjusted so that there is unloading on the second roller

In Simulation 4, depicted in Figure 41 and 42, the draw pin is lowered past the critical location to \( h = 20.17 \text{ cm} \). In this supercritical case \( y_3 = 231 \) (the third \( o \)) marks the start of the unloading zone on the second roller. The fiber unloads to an exit speed \( 472 \text{ cm s}^{-1} \) less than the surface speed \( r\omega_2 = 476 \text{ cm s}^{-1} \). The fiber maintains this exit speed up to the point of attachment \( x_4 = 263 \text{ cm} \) to the draw pin. On the pin the fiber first stretches back to the second roller speed and then further draws to the final roller speed \( 740 \text{ cm s}^{-1} \) when it departs the pin at \( x_5 = 272 \text{ cm} \).

In Figure 42, the fiber loads on the first roller from 1 to 3. From 3 to 4 the fiber unloads on the second roller and then from 4 to 5 reloads on the draw pin. The direction of the arrows indicates the unloading and reloading. After returning on the draw pin to the second roller speed the fiber undergoes new draw on the pin to the third roller speed. This new draw, from 5 to 8 on the speed vs. tension plot, passes out of the soft plateau and into the final stiff zone at point 7.

### 5.7 Conclusion

Using the governing equations developed earlier, incorporating full radial and tangential inertial effects, the analysis of draw for a commercially relevant draw process where a draw pin is included for localizing and enhancing fiber draw is described. In the absence of the draw pin, axial stress in the fiber increases monotonically. However, the presence of
Figure 41: Isothermal draw process with a draw pin height \( h = 20.17 \text{ cm} \), which results in an exit speed \( v_3 \) from the second roller less than \( r\omega_2 \), and hence unloading of the fiber on the second roller. On the left is the plot of speed vs. arclength, and on the right is tension vs. arclength.
Figure 42: Isothermal draw process with a draw pin height $h = 20.17$ cm, in the tension vs. speed plot 1: $v = r\omega_1$ for $x_0 \leq x \leq y$, 2: $v = v_a$ at $s = y_a$, 3: $v = r\omega_2$ for $x_1 \leq s \leq y_3$, 4: $v = v_3 < r\omega_2$ for $x_3 \leq s \leq x_4$, 5: $v = r\omega_2$ at $s = y_p$, 7: $v = v_b$ at $s = y_b$, 8: $v = r\omega_3$ for $s \geq x_5$. The fiber tension decreases on the second roller from 3 to 4, and then increases on the pin from 4 to 8 (recovering from 4 to 5, and then completing the draw from 5 to 8).
a non-rotating draw pin introduces the possibility of unloading in the fiber and additional complexity in the analysis, since the modulus in the unloading region is not the same as the modulus realized in continuous loading for the soft plateau and the second stiff region.

The methodology presented is very general, and can be applied to any isothermal drawing process with one or more draw pin. The results of simulations of two-stage isothermal draw processes are presented, here the fiber is at the soft plateau when moving over the draw pin, thereby limiting the number of possible combinations in the analysis. Within this class of processes that are shown the position of the draw pin (i.e., the offset from the feed/take-up roller) has a significant effect on the distribution of draw. For small offsets of the draw pin, no unloading occurs before the draw pin. Hence, the fiber stress monotonically increases, with draw present on the first two rollers as in a two stage draw process without the presence of a draw pin; a portion of the draw is transferred from the second roller to the pin, however. At a critical offset of the draw pin, neither draw nor unloading occurs on the second roller. Further increase in offset results in unloading in the fiber on the second roller followed by a significant draw on the pin.
CHAPTER VI

MODELING OF FIBER DRAW PROCESSES WITH MECHANICAL AND THERMAL UNLOADING

The governing equations for fiber draw developed in chapter 4 are used in this chapter for a two stage draw process augmented by a draw pin for drawing an elastic-plastic fiber with a temperature dependent response. The effect of varying draw conditions such as temperature profile, draw pin height, and final takeup speeds are explored. Consider the generic draw process shown schematically in Figure, 23 which shows a two stage draw process with a possibly heated freespans, draw pin, and rollers. For altering and/or entirely removing the amount of unloading or loading of the fiber on the second roller, the following actions can be taken: heating the drawpin, altering the offset of the draw pin, or changing the speed of the final roller.

6.1 Constitutive Equation

The constitutive equation is characterized as thermally responsive elastic-plastic, from the development in 2 the increment of force \( dT \) is related to increments \( dv \) and \( d\Theta \) of speed and temperature, respectively, through

\[
dT = \bar{B} dv + \bar{C} d\Theta,
\]

where

\[
(102) \quad \bar{B} = \begin{cases} 
\frac{K_1}{m} & \text{if } v < v_a \text{ or } dv \leq 0 \text{ or } v \leq v_{\text{max}}, \\
\frac{K_2}{m} & \text{if } v_a \leq v < v_b \text{ and } dv > 0 \text{ and } v = v_{\text{max}}, \\
\frac{K_3}{m} & \text{if } v_b \leq v < v_c \text{ and } dv > 0 \text{ and } v = v_{\text{max}}, 
\end{cases}
\]

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and

\[
\hat{C} = \begin{cases} 
  k_{12} \left( \frac{v}{v_0} - 1 \right) & \text{if } v \leq v_a, \\
  k_{12} \left( \frac{v_a}{v_0} - 1 \right) + (K_1 - K_2) \varepsilon_{12} + k_{22} \left( \frac{v}{v_0} - \frac{v_a}{v_0} \right) & \text{if } v_a \leq v \leq v_b, \\
  k_{12} \left( \frac{v_a}{v_0} - 1 \right) + (K_1 - K_2) \varepsilon_{12} + k_{22} \left( \frac{v}{v_0} - \frac{v_a}{v_0} \right) + (K_2 - K_3) \varepsilon_{22} + k_{32} \left( \frac{v}{v_0} - \frac{v_a}{v_0} \right) & \text{if } v_b \leq v \leq v_c.
\end{cases}
\]

The solution for fiber speed and tension are obtained by integrating the combined momentum, friction, and constitutive equations in each of the above cases.

### 6.2 Equations of Motion

In the draw process the fibers are either in a free span or on a roller (or pin); on a roller the possible conditions are no-slip, slip with the fibers moving faster than the roller surface (this is also valid for a draw pin, the draw pin is stationary hence fiber speed is always greater than the draw pin surface speed), and slip with the fibers moving slower than the roller surface. By combining the conservation of mass (9), conservation of momentum (14) and (15) on rollers and (18) in free spans with the friction relations (11) or (12) and the constitutive equation (102) for each type of draw zone the equations of motion are obtained.

For the general case of fiber not slipping on a roller the equation of motion is given by (52). Hence the velocity \( v(s) \) in a no slip region remains constant. The friction force in this zone is determined by combining Equation (52), the constitutive Equation (102), and the momentum projection from Equation (34a) to give

\[
f = \hat{C} \frac{d\Theta}{ds}.
\]

The normal force in all cases with fiber on roller is given by Equation (34b). Referring to
Figure 29, the no slip zones are \( x_0 \leq s \leq y_1, y_2 \leq s \leq y_3, \) and \( y_4 \leq s \leq x_7. \)

When the fiber is moving faster than the roller, friction is kinetic due to slip and positive according to the adopted sign convention. Hence Equation (11) is used. The equations governing draw on a roller where the fiber is moving faster than the roller surface are thus Equations (9), (34), (102), and (11); combining them produces

\[
\frac{dv}{ds} = \frac{\dot{C} \frac{d\theta}{ds} - \mu(T - Gv)}{G - B}.
\]

The zones with slip where the fiber is moving faster than the roller are (see Figure, 29) \( y_1 < s \leq x_1, y_3 < s \leq x_3, \) and \( x_4 \leq s \leq x_5. \)

The fixed draw pin is a special case of a roller with zero surface speed. On the pin there is always slip, since the fiber is moving faster than the pin surface.

When the fiber is moving slower than the roller surface, friction is kinetic due to slip and according to the adopted sign convention friction is negative so Equation (12) is used. Combining the relevant equations for draw, i.e. (9), (34), (102), and (12) produces

\[
\frac{dv}{ds} = \frac{\dot{C} \frac{d\theta}{ds} + \mu(T - Gv)}{G - B}.
\]

Equation (107) is valid for \( x_2 \leq s < y_2 \) and \( y_3 < s \leq x_3 \) if the pin is placed such that the fiber unloads on the third roller.

In the freespan the relevant equations are (9), (18), and (102). Combining them gives

\[
\frac{dv}{ds} = \frac{\dot{C} \frac{d\theta}{ds}}{G - B}.
\]

Referring to Figure, 29, Equation (108) is valid in for \( x_1 < s < x_2, \) \( x_3 < s < x_4, \) and \( x_5 < s < x_6. \) In the freespan both the friction and normal forces are zero.

### 6.3 Solution

To find the velocity profile \( v(s) \) the boundary value problems by the equations of motion are integrated over their operational range. Two possibilities may arise: In the first alternative
the velocity at one end of the boundary and the extent of the zone of integration \( s \) is known. The unknown velocity at the other boundary is found by integrating \( \frac{dv}{ds} \) i.e.

\[
(109) \quad v_f = v_i + \int_{s_i}^{s_f} \frac{dv}{ds} ds,
\]

here \( s_i \) and \( s_f \) are the start and end of the integration zone respectively. The velocity \( v_f \) is the speed of the fiber at the end of the slip zone after having started with a speed of \( v_i \).

In the other alternative the velocities \( v_i \) and \( v_f \) at both ends of the zone respectively are known. The initial location \( s_i \) of the integration zone is known and the other end of the zone is determined to be,

\[
(110) \quad s_f = s_i + \int_{v_i}^{v_f} \frac{ds}{dv} dv.
\]

With fiber speed \( v(s) \) obtained and temperature profile \( \theta(s) \) known, the increment in drawline tension can be obtained from the incremental thermal elastic-plastic constitutive model (28) and (29) by using the increments in \( v \) and \( \theta \).

The normal \( n \) and friction \( f \) force per length are then obtained by substituting the values of tension \( T(s) \) and fiber speed \( v(s) \) in Equations (34b) and (11) or (12).

### 6.4 Simulations and Discussion

In this section the effects of temperature, drawpin height \( h \), and final roller speed \( \omega_3 \) on selected drawline variables from velocity, tension, and friction are studied.

#### 6.4.1 Effect of Heating

The following temperature profiles are used for the simulations: isothermal, heated freespans, heated third roller, heated freespans and third roller, and heated drawpin shown in Figures 47 to 62.

The results for the isothermal case with drawline temperature maintained at 293 K are presented in Figures 43-46. This discussion will concern itself with the region between the second and third roller since the differences in the velocity and tension profiles produced by the different heating conditions become apparent in this region. On the second roller beyond the no slip zone the fiber unloads and the velocity drops from \( v = 476 \) cm s\(^{-1}\) at \( s = 232 \) cm
to \( v = 473 \text{ cm s}^{-1} \) at of 239 cm. The corresponding tension in the fiber drops from \( T = 675564 \text{ dyn} \) to \( T = 598690 \text{ dyn} \). Note the unloading is purely a mechanical effect since the temperature is maintained at 293 K. As far as friction is concerned its value is negative on the second roller due to unloading. Friction increases from \( f = -10637 \text{ dyn cm}^{-1} \) to \( f = -9427 \text{ dyn cm}^{-1} \). The unloading is shown also in the tension \( T(s) \) vs. velocity \( v(s) \) curve. In the freespan following the second roller due to isothermal conditions the fiber does not undergo any changes. On the drawpin to which the fiber attaches to next there is a substantial amount of draw. The velocity increases from \( v = 473 \text{ cm s}^{-1} \) to \( v = 740 \text{ cm s}^{-1} \) with a corresponding increase in tension from \( T = 598690 \text{ dyn} \) to \( T = 838350 \text{ dyn} \). The value of the frictional force acting on the fiber also increases from \( f = 37710 \text{ dyn cm}^{-1} \) to \( f = 52800 \text{ dyn cm}^{-1} \). On exit from the drawpin the same velocity and tension are maintained to the point of draw termination. Note that on the \( T(s) \) vs. \( v(s) \) curve the last point indicates the point of exit from the final roller and does not indicate fiber breakage.

Shown in Figure 47 is the temperature profile for a heated freespan. Heating is done prior to the drawpin where the temperature ramps up linearly from 293 K to 323 K and then cools to 293 K.

The fiber draw with a heated freespan is shown in Figures 48-51. As before the fiber unloads on the second roller with the fiber tension dropping from \( T = 675560 \text{ dyn} \) to \( T = 597880 \text{ dyn} \) and the corresponding speed drops from \( v = 476 \text{ cm s}^{-1} \) to \( v = 473 \text{ cm s}^{-1} \). The value of friction changes from \( f = -10637 \text{ dyn cm}^{-1} \) to \( f = -9414 \text{ dyn cm}^{-1} \). Due to the temperature gradient in the freespan the fiber further unloads to \( T = 405130 \text{ dyn} \) before loading up again to \( T = 598690 \text{ dyn} \) this is because the temperature gradient changes signs. On the drawpin the fiber tension increases \( T = 838350 \text{ dyn} \). In the remaining freespan following the drawpin the fiber unloads again, this time to a value of \( T = 710751 \text{ dyn} \), the frictional force responsible for this shows an increase from \( f = 37707 \text{ dyn cm}^{-1} \) to \( f = 52800 \text{ dyn cm}^{-1} \). The fiber leaves the drawpin and maintains the same speed and tension for the rest of the tow until the point of exit from the third roller. Note that the unloading as shown in the \( T(s) \) vs. \( v(s) \) in 51 indicates a straightening of the unloading curve, this is because the plot shows the sideview of the progress of the drawpoint on the
Figure 43: Fiber velocity $v(s)$ versus arclength $s$ for isothermal draw. The $\times$ indicate points of attachment and detachment from the rollers and $\Delta$ indicates start of slip.
Figure 44: Tension $T(s)$ versus arc length $s$ for isothermal draw.
Figure 45: Friction $f(s)$ versus arclength $s$ for isothermal draw.
Figure 46: Tension $T(s)$ vs velocity $v(s)$ for isothermal draw.
Figure 47: The fiber temperature in the tow ($\Theta(s)$) vs arclength ($s$). The points of entry and exit to rollers are labeled with $\times$. A heated freespan with a triangular temperature gradient.
tension $T(s)$, velocity $v(s)$ and temperature $\theta(s)$ surface.

The temperature profile for a heated third roller is shown in Figure 52, the roller temperature is maintained at 513 K while rest of the drawline is kept at 293 K. With a heated third roller the tension profile is identical to the isothermal case right up to the point of attachment to the third roller. On the third roller, due to heating, the fiber tension drops from $T = 838352$ dyn to $T = 586189$ dyn. Since the fiber is heats up over small region downstream from the point of attachment to the roller a thermal gradient exits, The frictional force thus generated to satisfy equilibrium increases from $f = -12138$ dyn cm$^{-1}$ at the point of attachment to the third roller to zero at the end of the cooldown zone.

The combination of the last two temperature profiles gives a heated freespan and heated third roller, shown in Figure 57. With a heated freespan and heated roller the response of the fiber is shown in Figures 58 to 61 which is a linear combination of the previous two profiles.

In Figure 62 is the temperature profile for a heated drawpin at 513 K the fiber heats up linearly from 293 K to 513 K as it slides on the drawpin and then after exiting the drawpin it cools down to 293 K in the freespan. When a heated drawpin is used, fiber responds as shown in Figures 63 to 66, the fiber tension drops marginally from $T = 675564$ dyn to $T = 665667$ dyn on the second roller. However on attaching to the draw pin the fiber unloads from $T = 665667$ dyn to $T = 475010$ dyn and then loads back up to $T = 568840$ dyn and then exits the drawpin, note the that the fiber loads up on the drawpin can be because the stiffness transitions into the final stiff part in the constitutive equation. As the fiber cools in the remaining part of the freespan, due to the temperature gradient, it loads back up to a final value of $T = 838350$ dyn before attaching to the final roller. The friction force on the fiber when its on the drawpin first decreases from $f = 209500$ dyn cm$^{-1}$ to $149566$ dyn cm$^{-1}$ due to the temperature gradient and then increases to $179116$ dyn cm$^{-1}$ to maintain equilibrium with the increased fiber stiffness as the fiber transitions into the final stiff portion of the constitutive curve. The tension drops as the velocity increases, this is due to the increasing temperature along the drawline. The sudden dip seen in tension as the velocity increases is due to the change of stiffness of the fiber. The tension then
Figure 48: Fiber velocity $v(s)$ versus arclength $s$ due to a heated freespan between second roller and the drawpin.
Figure 49: Fiber tension $T(s)$ versus arclength $s$ due to a heated freespan between second roller and drawpin.
Figure 50: Friction force $f(s)$ versus arclength $s$ due to a heated freespan between second roller and drawpin.
Figure 51: Fiber tension $T(s)$ versus velocity $v(s)$ due to a heated freespans between second roller and drawpin.
Figure 52: The temperature profile of the draw line with a heated final takeup roller.
Figure 53: Fiber velocity $v(s)$ versus arclength $s$ due to the heated third roller.
Figure 54: Fiber tension $T(s)$ versus arclength $s$ due to the heated third roller.
Figure 55: Friction force $f(s)$ versus arclength $s$ due to the heated third roller.
Figure 56: Fiber tension $T(s)$ versus velocity $v(s)$ due to the heated third roller.
Figure 57: The temperature profile of the draw line with a heated freespan and heated final takeup roller.
Figure 58: Fiber velocity $v(s)$ versus arclength $s$ for a heated freespans and heated third roller.
Figure 59: Fiber tension $T(s)$ versus arc length $s$ for a heated free span and heated third roller.
Figure 60: Friction force $f(s)$ versus arclength $s$ for a heated freespan and heated third roller.
Figure 61: Fiber tension $T(s)$ versus velocity $v(s)$ for a heated freespan and heated third roller.
Figure 62: The temperature profile of the drawline with a heated drawpin at a temperature of 513 K.
increases as the fiber cools in the freespans.

Shown together in Figure 67 are all the tension profiles obtained earlier. There are remarkable differences in the $T(s)$ vs. $v(s)$ plots due to different temperature profiles used. The response for heated freespans, heated third roller, and heated drawpin are plotted. The tension plot for the isothermal case has not been plotted as it is very much like the one for the heated third roller, (- - - ) in Figure 67, except that the tension maintains its freespans value on the third roller. Also the plot for and heated freespans & third roller has not been shown since it is a linear combination of the plot for a heated freespans and heated third roller. All three responses plotted are identical up to the (un)loading zone of the second roller where the least amount of unloading is shown by the drawline with a heated drawpin. The other two profiles i.e. heated freespans (-----) and heated third roller (- - - ) show identical amounts of unloading. In the freespans the profile using a heated freespans shows the most unloading and resembles the shape of the temperature profile itself. On the drawpin the profiles for the heated freespans and heated third roller are identical whereas the profile using a heated drawpin shows unloading over an initial part of the drawpin and loading on the rest of the drawpin, this is because of the stiffer part of the constitutive relation has been reached. On the freespans beyond the drawpin the fiber loads up and all three profiles reach the same tension. Once on the third roller, the profile for the heated third roller unloads to a lower tension level due to heating whereas the other two profiles maintain the same tension levels.

In Figure 68 the plot of friction force profiles obtained using the the three temperature profiles as was done in the last paragraph. Differences are seen on the second roller where the extent of the unloading zone is the smallest for the profile using the heated drawpin. Also on the drawpin the initial value for frictional force for the heated drawpin is greater than for the heated freespans and heated third roller profiles but it shows a drop and then an increase before dropping back to zero at the end of the drawzone on the drawpin. The heated third roller a negative spike in friction is seen due to a temperature gradient resulting from the steep increase of temperature.
Figure 63: Fiber velocity $v(s)$ versus arclength $s$ obtained using a heated drawpin in the draw line.
Figure 64: Fiber tension $T(s)$ versus arclength $s$ obtained by using a heated drawpin in the draw line.
Figure 65: Friction $f(s)$ versus arclength $s$ obtained by using a heated drawpin in the draw line.
Figure 66: Fiber tension $T(s)$ versus velocity $v(s)$ obtained by using a heated drawpin in the draw line.
Figure 67: Comparison of drawline tension obtained using the following temperature profiles: heated freespan (—), heated third roller ( - - ), and heated drawpin ( - - - - ).
Figure 68: Comparison of drawline friction forces obtained using the following temperature profiles: heated freespan (----), heated third roller (- - -), and heated drawpin (--- - -).
6.4.2 Effect of drawpin height \((h)\)

In the previous section the profiles for the case that produced fiber unloading on the second roller for isothermal conditions was presented. The drawpin height was set at \(h = 30\) cm. In this section the drawpin is raised to \(h = 10\) cm and a comparison between the two tension profiles is made.

For the *isothermal* response shown in Figure 69 the drawpin is at \(h = 10\) cm and the fiber loads up on the second roller from \(T = 675560\) dyn to \(T = 799320\) dyn compared to the case where the drawpin is at \(h = 30\) cm in which case the fiber unloads. So by lowering the drawpin unloading can be induced on the second roller and by raising it the amount unloading reduces and reverses to loading beyond a critical height of the drawpin. Remember that on the drawpin itself the fiber always loads up.

The response from using a *heated second free span* is shown in Figure 70, when the drawpin is at \(h = 10\) cm, the fiber loads up on the second roller from \(T = 675560\) dyn to \(T = 802740\) dyn. Since the freespan is heated then as expected the fiber unloads, it does so from \(T = 802740\) dyn to \(T = 534150\) dyn. and then loads up again as it cools down to \(T = 799290\) dyn. Next the fiber attaches to the drawpin and loads up to \(T = 838350\) dyn which the fiber retains as its final tension value. Compared to the case where the drawpin height is \(h = 30\) cm the fiber undergoes a lesser tension variation.

With a *heated third roller*, the response is shown in Figure 71, the tension profile matches the isothermal profile to the point of attachment to the third roller, at which point the fiber is subjected to a thermal gradient due to a short heat up zone (329 cm to 332 cm) which takes the fiber temperature to 513 K. Due to the high temperature gradient the fiber unloads to a tension of \(T = 586190\) dyn and stays at this valued until it exits from the third roller. Compared to the case of \(h = 30\) cm the tension first drops on the second roller and then continuously increases to the final value on the drawpin before dropping down on the third roller.

The fourth case, shown in Figure 72, is with a *heated freespan and heated third roller*. The tension response is a combination of responses due to a heated freespan only and due to heated third roller only.
Figure 69: The fiber tension $T(s)$ vs arclength $s$ under isothermal conditions for (---) $h = 10$ cm and (---) $h = 30$ cm.
Figure 70: The fiber tension $T(s)$ vs arc length $s$ with a heated second free span for (——) $h = 10$ cm and (---) $h = 30$ cm.
Figure 71: The fiber tension $T(s)$ vs arclength $s$ with a heated third roller for (——) $h = 10 \text{ cm}$ and (---) $h = 30 \text{ cm}$. 
Figure 72: The fiber tension $T(s)$ vs arclength $s$ with a heated second free span and heated third roller for (——) $h = 10$ cm and (· · ·) $h = 30$ cm.
With the heated draw pin, the response is shown in Figure 73, the fiber unloads on the drawpin and then loads up again in the partly on the pin and the latter half of the freespan, as it cools. For the drawpin at \( h = 10 \) cm the fiber loads up on the second roller from \( T = 675560 \) dyn to \( T = 809450 \) dyn which is maintained to the point of attachment to the drawpin. On the drawpin the fiber unloads to 559440 dyn and then loads up due to transition in stiffness of the constitutive relation before exiting the drawpin with a tension of \( T = 568840 \) dyn. The fiber tension increases as it cools down, reaching a value of 838350 dyn which is maintained for the rest of the journey. Compared to the case with \( h = 30 \) cm there is no loading on the second roller, the fiber unloads on the second roller and the drawpin before loading up again as it cools in the freespan.

The response of the tow due to drawpin height was just presented. It is observed that loading and unloading on the second roller can be entirely eliminated by adjusting the drawpin height to a critical value which allows all loading to happen on the drawpin. This is a useful behavior allowing the process to be controlled to minimize unloading on the second roller.

### 6.4.3 Effect of final takeup roller speed (\( \omega_3 \))

In this section the effect of increasing final roller speed (\( \omega_3 \)) on the tension profile is presented. Figures 74 to 78 show the tension profiles for a towline with the drawpin at a height of \( h = 30 \) cm subjected to the five temperature profiles discussed earlier. The tension profiles are produced with the third roller surface speed (\( r\omega_3 \)) is maintained at 770 cm s\(^{-1}\), 740 cm s\(^{-1}\), and 710 cm s\(^{-1}\) with curves produced being indicated by (——), (---), and (- - - - -) respectively. In all the cases above it is noted that increasing the final roller speed reduces the amount of unloading on the second roller, a further increase actually leads to a reversal from unloading to loading. A critical value of \( \omega_3 \) allows all loading to take place on the drawpin and none on the second roller. From a design consideration by eliminating draw on the roller and moving it to a drawpin has the benefit of cost saving in terms of frequent replacement of rollers as drawpins are cheaper.
Figure 73: The fiber tension $T(s)$ vs arclength $s$ with a heated draw pin for (——) $h = 10$ cm and (---) $h = 30$ cm.
Figure 74: The fiber tension in the tow $T(s)$ vs arclength $s$ under isothermal conditions for $h = 30$ cm with final roller surface speeds (—) 770, (---) 740, and (---•---) 710 cm s$^{-1}$. 
Figure 75: The fiber tension in the tow $T(s)$ vs arclength $s$ with a heated second free span for $h = 30$ cm with final roller surface speeds (——) 770, (---) 740, and (----) 710 cm $s^{-1}$. 
Figure 76: The fiber tension in the tow $T(s)$ vs arclength $s$ with a heated third roller for $h = 30$ cm with final roller surface speeds (---) 770, ( - - ) 740, and ( - - - - - ) 710 cm $s^{-1}$.
Figure 77: The fiber tension in the tow $T(s)$ vs arclength $s$ with a heated freespan and heated third roller for $h = 30$ cm with final roller surface speeds (——) 770, (---) 740, and (- - - -) 710 cm s$^{-1}$. 

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Figure 78: The fiber tension in the tow $T(s)$ vs arclength $s$ with a heated draw pin for $h = 30$ cm with final roller surface speeds (——) 770, (- - -) 740, and (· · · · ·) 710 cm $s^{-1}$.
CHAPTER VII

THE EFFECT OF TWO DRAWPINS IN A TWO STAGE DRAW PROCESS.

The two stage fiber draw problem for a piecewise elastic fiber was introduced in Chapter 4. Recall that in an industrial fiber draw process draw rollers experience degradation due to friction between the roller surface and the drawn fiber. So to mitigate this problem a non rotating draw pin made of a relatively inexpensive material is introduced in the free span. It was shown in Chapter 5 that by adjusting the height of the draw pin inserted in the second free span of a two stage draw process the draw zone on the preceding roller was shifted to the draw pin. In this article the effect of an additional draw pin positioned in the first free span will be explored.

7.1 Equations of Motion

The development proceeds exactly as in Chapter 5 with equations for mass, momentum on rollers, momentum in free spans, and friction given by (9), (34), (18), and (116) respectively. The conservation equations and friction relation are combined, depending on the relative speed of the fiber, to give one of the following equations of motion: if the fiber is moving faster than the roller surface then the resulting equation is (67), if the fiber is slower than the equation is (68). If there is no slip then (69) is used. The equations of motion are coupled to the piece-wise elastic-plastic constitutive equation in (71) which relates the tension and velocity increments for loading or unloading, see Figure 28.

Now that the equations of motion have been established an isothermal two-stage process with intermediate draw pins in the first and second free span will be modeled. The addition of a drawpin to the first free span opens the possibility of loading and unloading on the first roller. The basic two-stage draw process discussed in chapter 4 does not change.
7.2 Solution

In a typical two stage draw process the introduction draw pins with adjustable offsets creates many new conceivable paths of fiber behavior. The speed and tension profiles $v(s)$ and $T(s)$ obtained as solutions by integrating the equations of the previous section everywhere along the drawline contain a large number of equations and conditional statements. To make this analysis easy to understand the draw profiles that will be considered are only the ones that are feasible for a typical two stage draw process. The improbable case of the fiber attaching a roller at less than the roller speed will be ignored based on results obtained in Chapter 5. Secondly to calculate the fiber velocity profiles on rollers (either fiber loading and unloading) and freespans the solution to the equations of motion for each piecewise section of the constitutive equation is used to generate a general form of the solution for each of the several draw zones that are then strung together to obtain the solution profiles over the entire draw line.

Consider a draw zone on either a roller or drawpin. The fiber speed is monotonically increasing so that $dv > 0$ and $v = v_{max}$ everywhere in the zone, where $v_{max}$ is the maximum speed attained by the fiber. For monotonically increasing speed the incremental constitutive equation (71) integrates to

$$T(v) = \begin{cases} 
K_1 \left( \frac{v}{v_0} - 1 \right) & \text{for } v \leq v_a, \\
K_1 \left( \frac{v_a}{v_0} - 1 \right) + K_2 \left( \frac{v}{v_0} - \frac{v_a}{v_0} \right) & \text{for } v_a < v \leq v_b, \\
K_1 \left( \frac{v_a}{v_0} - 1 \right) + K_2 \left( \frac{v_b}{v_0} - \frac{v_a}{v_0} \right) + K_3 \left( \frac{v}{r_0} - \frac{v_b}{r_0} \right) & \text{for } v_b < v \leq v_c.
\end{cases}$$

Integrating the combination of (111) and (67) gives the solution $v(s)$ on the roller or drawpin. The solution starts at some location $s_i$ with initial speed $v_i$, where $v_i$ lies in either of the intervals delineated by the transition speeds i.e. $v_0 \leq v_i < v_a$, or $v_a \leq v_i < v_b$, or
\( v_b \leq v_i < v_c \). The possible solutions then are,

\[
(112) \quad v(s) = \begin{cases} 
\frac{v_i + v_0}{1 - \frac{Gv_0}{K_1}} \left( \frac{v_i}{v_0} - 1 \right) - \frac{Gv_i}{K_1} \left( e^{s-s_1} - 1 \right) \quad \text{for} \quad v_0 \leq v_i \leq v_a , \\
\frac{v_i + v_0}{1 - \frac{Gv_0}{K_1}} \left[ \frac{K_1}{K_2} \left( \frac{v_a}{v_0} - 1 \right) + \left( \frac{v_i}{v_0} - \frac{v_a}{v_0} \right) - \frac{Gv_i}{K_2} \right] \left( e^{s-s_1} - 1 \right) \quad \text{for} \quad v_a \leq v_i \leq v_b , \\
\frac{v_i + v_0}{1 - \frac{Gv_0}{K_1}} \left[ \frac{K_1}{K_3} \left( \frac{v_a}{v_0} - 1 \right) + \frac{K_2}{K_3} \left( \frac{v_b}{v_0} - \frac{v_a}{v_0} \right) + \left( \frac{v_i}{v_0} - \frac{v_b}{v_0} \right) - \frac{Gv_i}{K_3} \right] \left( e^{s-s_1} - 1 \right) \quad \text{for} \quad v_b \leq v_i \leq v_c ,
\end{cases}
\]

The values of \( v_i \) and \( s_i \) may or may not be known in advance, in which case they will be the quantities that need to be solved for the entire process.

If the fiber speed on a roller is monotonically decreasing so that \( dv < 0 \) and \( v < v_{\text{max}} \), the incremental constitutive equation (71) integrates to

\[
(113) \quad T(v) = \begin{cases} 
K_1 \left( \frac{v}{v_0} - 1 \right) \quad \text{for} \quad v < v_{\text{max}} , \\
K_1 \left( \frac{v_a}{v_0} - 1 \right) + K_2 \left( \frac{v_{\text{max}}}{v_0} - \frac{v_a}{v_0} \right) - K_1 \left( \frac{v_{\text{max}}}{v_0} - \frac{v}{v_0} \right) \quad \text{for} \quad v < v_{\text{max}} , \\
K_1 \left( \frac{v_a}{v_0} - 1 \right) + K_2 \left( \frac{v_b}{v_0} - \frac{v_a}{v_0} \right) + K_3 \left( \frac{v_{\text{max}}}{v_0} - \frac{v_b}{v_0} \right) - K_1 \left( \frac{v_{\text{max}}}{v_0} - \frac{v}{v_0} \right) \quad \text{for} \quad v < v_{\text{max}} .
\end{cases}
\]

Combining equation (68) with (113) the velocity profiles for an unloading fiber are obtained
and are given by

\[
\begin{align*}
\theta(s) &= \begin{cases}
\frac{v_i - v_0}{v_0 - v_0} & \text{for } v_0 \leq v_{\max} \leq v_a \text{ and } v_i \geq v_0,
\frac{v_a - v_0 - (v_a - v_0) - (v_{\max} - v_0) - (v_{\max} - v_0)}{v_0 - v_0} & \text{for } v_a \leq v_{\max} \leq v_b \text{ and } v_i \geq v_{\max} - (v_a - v_0) - (v_{\max} - v_a),
\frac{v_b - v_0 - (v_b - v_0) - (v_{\max} - v_0) - (v_{\max} - v_b)}{v_0 - v_0} & \text{for } v_b \leq v_{\max} \leq v_c \text{ and } v_i \geq v_{\max} - (v_b - v_0) - (v_{\max} - v_b).
\end{cases}
\end{align*}
\]

7.3 Simulations

The general solutions obtained in the previous section are used to generate the drawline profiles for all classes of possible isothermal drawlines. Both drawpins can be adjusted so that fiber loads on both the first and second rollers. It is also possible to cause the fiber to unload on the first roller and load on the second roller and vice versa. The fiber can be made to unload on both the first and second rollers. The special case of the fiber not drawing on both rollers and the draw taking place entirely on the draw pins is vital to the economics of the process due to the savings obtained from reduced wear on draw rollers.

To reduce the number of possibilities and equations to consider, it is assumed that the roller speeds and fiber properties are such that in the no-slip zone on the second roller the fiber has been drawn into its soft plateau \((r\omega_1 < v_a < r\omega_2 < v_b)\), and in the no-slip zone on the third roller the fiber has been drawn into its second stiff state but below its breaking strain \((v_b < r\omega_3 < v_c)\). Although this is invariably the situation in industrial practice, the solution in this case can be readily extended to the other cases.
Figure 79: Plot of $v$ versus $s$ for loading on both rollers, drawpin offsets are (10cm,10cm).

Figure 79 shows the velocity profile obtained by adjusting both drawpins so that the fiber speed increases on both the first and second rollers. The draw pin offsets for both first and second draw pins are 10cm. Figure 80 shows the velocity profile with the drawpins adjusted so that the fiber speed increases on the first roller and the fiber does not draw on the second roller. The drawpin offsets are 10cm for the first and 1.68cm for the second. Figure 81 shows the velocity profile obtained with drawpins adjusted to obtain no draw the first roller and loading on the second roller. The drawpin offsets are -38.31cm on the first and 10cm on the second. The negative sign on the first drawpin offset indicates that the drawpin has been lowered below the centerline connecting the draw rollers. The final plot in Figure 82 shows the velocity profile obtained with drawpins adjusted so as to obtain no draw on both rollers. The draw pin offsets are -38.31cm on the first and 1.68cm on the second.
Figure 80: Plot of $v$ versus $s$ for loading first roller and no draw on second, drawpin offsets are (10cm,1.68cm).
Figure 81: Plot of $v$ versus $s$ for no draw on first roller and loading on second, drawpin offsets are (-38.31cm,10cm).
Figure 82: Plot of v versus s for no draw on both rollers, drawpin offsets are (-38.31cm, 1.68cm).
7.4 Conclusion

From the plots obtained it can be concluded that the effectiveness of the draw pin depends where the draw zone produced by the drawpin lies on the constitutive curve. This implies that the amount of draw for the same increment in tension is less for the initial stiff zone compared to intermediate soft plateau. Hence to maximize draw the drawpins should be introduced at locations where the fiber tension is in the soft plateau.
CHAPTER VIII

CONTRASTING THE PREDICTIONS FOR COULOMB AND CREEP-RATE DEPENDENT FRICTION MODELS.

The theoretical predictions of the behavior of the fibers in an industrial fiber drawing process depend on the particular friction model employed for quantifying friction forces between filament tow and draw rollers. In this chapter the the usual Coulomb model for friction is replaced with a creep-rate dependent friction model. The Coulomb friction model, as was shown in earlier chapters, predicts adhesion zones on the rollers, however no adhesion zones are developed when the creep-rate dependent friction model is used. For creep-rate dependent friction the fiber velocity at the point of attachment to the draw roller must be greater than the roller surface speed for the equations of momentum to be satisfied. This nature of the velocity profile can be interpreted as the formation of a neck in the fiber just upstream of the point of attachment to the roller.

8.1 Introduction

The draw process in general involves sets of feed rollers and take-up rollers; draw is achieved by rotating the take-up rollers faster than the feed rollers. This creates frictional forces between the fiber and the roller surface. Friction forces are responsible for producing the stress field within the fiber which causes its draw and resulting molecular orientation.

Studies on fiber draw using a Coulomb model for the friction between the fiber and the rollers was done in the previous chapters. In this chapter the behavior predicted by two different friction models i.e. Coulomb friction and creep-rate dependent friction [28]-[12] is compared. To focus on the effect of the choice of friction model the special case of isothermal draw processes without draw pins depicted in Figure 23 is simulated.
8.2 Friction Models

Two different friction models: the Coulomb friction model, and the creep-rate dependent friction model are compared and contrasted. In the Coulomb friction model the frictional force per unit length \( f \) acting on the fibers has magnitude \( \mu n \), where \( \mu \) is the static or kinetic coefficient of friction depending whether there is no relative motion between fibers and the roller and when there is relative motion, respectively, and \( n \) is the normal force per unit length. The relative motion between the fiber and a roller is given by the relative velocity \( v_{rel} \), i.e. the difference between the speed \( v \) of the fiber and the surface speed \( r\omega \) of the roller,

\[
v_{rel} = v - r\omega.\tag{115}
\]

The coefficient of friction \( \mu \) is in general is considered to be a function of temperature. The friction is in the direction that opposes the relative motion: given the adopted sign conventions on \( f \), \( v \), and \( \omega \) in Figure 11, \( f = \mu n \) if \( v_{rel} > 0 \) and \( f = -\mu n \) if \( v_{rel} < 0 \).

If there is no relative motion (\( v_{rel} = 0 \)) then \( f = \frac{dT}{ds} - G\frac{dv}{ds} \) is calculated from the tangential projection (34a) of the momentum equation. Subject to the condition that the magnitude of \( f \) does not exceed \( \mu n = \frac{\mu}{r}(T - Gv) \), where the normal projection (34b) of the momentum equation is used. Here \( \mu \) is the static coefficient of friction.

In a zone of no slip (i.e. \( v_{rel} = 0 \)), fiber speed \( v = r\omega = \) constant, so that \( dv = 0 \). For an isothermal process, tension \( T \) is a function only of \( v \), and hence \( dv = 0 \) implies \( dT = 0 \). Inserting \( dv = 0 \) and \( dT = 0 \) in eq. (34a) demands \( f = 0 \), and so for an isothermal process friction is zero in zones of no slip, and the Coulomb friction model reduces to

\[
f = \begin{cases} 
+\mu n & \text{if } v_{rel} > 0, \\
0 & \text{if } v_{rel} = 0, \\
-\mu n & \text{if } v_{rel} < 0.
\end{cases} \tag{116}
\]

The above relationship of friction on relative velocity is shown schematically in Figure 83.

In creep-rate dependent friction, the friction force per unit length \( f \) for fibers moving faster than the roller surface (i.e. \( v_{rel} > 0 \)) is related linearly to the relative velocity \( v_{rel} \).
Figure 83: Coulomb friction model; frictional force per length vs. the relative speed between the fiber and roller surface.

through a constant $\nu$, i.e. $f = \nu v_{rel}$ to an upper limit where $v_{rel} = \frac{\mu n}{\nu}$. Beyond $v_{rel} = \frac{\mu n}{\nu}$ the magnitude of $f$ reverts to its Coulomb friction model value of $+\mu n$. For fibers moving slower than the roller, i.e. $v_{rel} < 0$, the value of $f$ is the same, i.e. $\nu v_{rel}$, in the range $v_{rel} = 0$ to $v_{rel} = -\frac{\mu n}{\nu}$, where it reverts to the Coulomb friction model value of $-\mu n$. The creep-rate model is thus summarized by

$$f = \begin{cases} +\mu n & \text{if } v_{rel} > \frac{\mu n}{\nu}, \\ v_{rel} & \text{if } -\frac{\mu n}{\nu} \leq v_{rel} \leq \frac{\mu n}{\nu}, \\ -\mu n & \text{if } v_{rel} < -\frac{\mu n}{\nu}, \end{cases}$$

(117)

shown schematically in Figure 84.

8.3 Equations of Motion

The conservation equations developed earlier are reused again. The conservation of mass is given by Equation (9), momentum equations in the tangential and radial direction are given by (34a) and (34b), and the momentum equation for the freespan is (18).
Figure 84: Creep-rate dependent friction model.

For the processes modelled in this chapter, there are no draw-pins, and consequently there is no unloading since the process is also isothermal. Hence for such processes, the incremental thermoelastic-plastic constitutive model (70) when combined with (27) can be integrated to produce the tension-velocity relation given by Equation (72) for the fiber.

The combined momentum, friction, and constitutive equations in the following cases: freespan, no-slip on a roller, draw on a feed roller (i.e. slip with the fibers moving faster than the roller surface), and draw on a take-up roller (i.e. slip with the fibers moving slower than the roller surface), are integrated to obtain the closed form solutions for fiber speed and tension along the drawline, as well as the frictional and normal forces in the case of the fiber a the roller.

In the free span the Equations (9), (18), and (72) combine with the isothermal condition \( \frac{d\Theta}{ds} = 0 \) to give (49) from which follows

\[ v(s) = \text{constant.} \] 

(118)

The relevant equations for draw on a roller are Equations (9), (34a), (72), and either
(116) or (117), which combine to produce

\[
\frac{dv}{ds} = \frac{f}{dT} - \frac{G}{dv}
\]  

(119)

where \( \frac{dT}{dv} \) is dictated by (32), and \( f \) is given by (116) or (117), depending on which friction model is employed, Coulomb friction or creep-rate dependent friction.

8.4 Solution

The solution to the governing equations using the Coulomb model and the Creep rate dependent friction model will be discussed in this section.

8.4.1 Coulomb friction

By solving the governing equations using the Coulomb friction model, for which the frictional force per unit length is given by equations (116), and inserting (15) and (116) into (119) (for a generic roller with surface speed \( r\omega \)), the following relationship is obtained

\[
\frac{dv}{ds} = \begin{cases} 
\frac{\mu T - Gv}{r \frac{dT}{dv} - G} & \text{if } v > r\omega, \\
0 & \text{if } v = r\omega, \\
-\frac{\mu T - Gv}{r \frac{dT}{dv} - G} & \text{if } v < r\omega.
\end{cases}
\]  

(120)

Referring back to Figure 2, it is assumed that the fiber attaches to the first roller at location \( s_0 \) in its reference state with a speed \( v_0 = r\omega_1 \), i.e. without slip. Likewise it departs the third roller at \( s_3 \) without slip with a speed \( r\omega_3 \). The fiber detaches from the first roller at specified location \( s_1 \) with a speed \( v_1 \) which must be determined. The exit speed \( v_1 \) is in general greater than or equal to the roller speed \( r\omega_1 \), indicating the possibility of a draw zone on the first roller. The point where draw begins is denoted as \( y_1 \) and is yet to be determined. Hence the region \( s_0 \leq s \leq y_1 \) is the no-draw zone on the first roller, and \( y_1 < s \leq s_1 \) is the feed-draw zone on the first roller.

The first free-span extends from \( s_1 \) to \( s_2 \), the point of attachment to the second roller. On account of (118) there is no draw in the free-span, and \( v_1 = v_2 \), where \( v_2 \) is the attachment
speed to second roller. For this model, i.e. an isothermal, steady process for an elastic-plastic fiber using the Coulomb friction model, in chapter 4 it was shown that the velocity profile with \( v_2 = r\omega_2 \) and \( y_2 = s_2 \) (i.e. no take-up draw on the second roller) minimizes the energy of the fiber and hence is the solution.

The speed \( v_3 \) of the fiber as it departs the second roller at \( s_3 \) is in general greater than the roller speed \( r\omega_2 \), so that there exists a feed draw zone from a point \( y_3 \) (to be determined) to \( s_3 \). Due to (118) the departure speed \( v_3 \) is equal to the attachment speed \( v_4 \) to the third roller. Finally, the velocity profile with \( v_4 = r\omega_3 \) and \( y_4 = s_4 \) (i.e. no take-up draw on the third roller) minimizes the energy of the fiber, and hence is the solution.

To summarize, with Coulomb friction the draw in the isothermal fiber occurs on the first and second rollers, from from \( y_1 \) to \( s_1 \) (feed draw on the first roller) and from \( y_3 \) to \( s_3 \) (feed draw on the second roller).

The boundaries \( y_1 \) and \( y_3 \) of the draw zones, and the velocity profiles within the draw zones, are found by integrating the \( v > r\omega \) branch of equation (120) on the first and second rollers backwards from the location of departure, where the fiber speed is known \( (v(s_1) = r\omega_2 \) on the first roller and \( v(s_3) = r\omega_3 \) on the second roller), to determine the point \( (y_1 \) on the first roller and point \( y_3 \) on the second roller) where the fiber velocity becomes equal to the surface speed of the first and second roller respectively.

The velocity profile on the draw line are grouped as follows: From \( s_0 \) to \( y_1 \), in the no-draw zone, the fiber velocity is equal to the roller surface speed, \( v(s) = r\omega_1 \). In the draw zone from \( y_1 \) to \( s_1 \) the velocity profile is the solution of the \( v > r\omega \) branch of (120), increasing from \( v = r\omega_1 \) at \( y_1 \) to \( v = r\omega_2 \) at \( s_1 \). In the freespans extending from \( s_1 \) to \( s_2 \) and on to the draw-zone boundary \( y_3 \), the fiber speed is a constant \( v(s) = r\omega_2 \). Draw occurs on the second roller from \( y_3 \) to \( s_3 \), with the velocity determined by integrating the \( v > r\omega \) branch of (120), increasing from \( v = r\omega_2 \) at \( y_3 \) to \( v = r\omega_3 \) at \( s_3 \). In the freespans extending from \( s_3 \) to \( s_4 \) and on the entire third roller, the fiber speed is the constant \( v(s) = r\omega_3 \).

Tension profile \( T(s) \) is computed from the velocity profile \( v(s) \) using constitutive equation (72). The normal force per length \( n(s) \) and frictional force per length \( f(s) \) on the fiber from the roller are computed from \( v(s) \) and \( T(s) \) using momentum equations (34).
8.4.2 Creep-rate dependent friction

The solution procedure for the governing equations incorporating the creep-rate dependent friction model is as follows. Combining the creep-rate dependent friction model (117) with the momentum equations (34) produces the following equations for fiber behavior on a roller with surface speed $r\omega$:

\[
\frac{dv}{ds} = \begin{cases} 
\frac{\mu}{r} \frac{T - Gv}{dT - G} & \text{if } v_R < v , \\
\nu(v - r\omega) & \text{if } v_L \leq v \leq v_R , \\
\frac{-\mu}{r} \frac{T - Gv}{dT - G} & \text{if } v < v_L ,
\end{cases}
\]

(121)

where the transition speeds in the creep rate dependent friction model are

(122a) \[ v_L = r\omega - \frac{\mu}{r\nu}(T(v_L) - Gv_L), \]

(122b) \[ v_R = r\omega + \frac{\mu}{r\nu}(T(v_R) - Gv_R). \]

The following observations are made:

1. From the middle branch of (121) it is observed that if $v = r\omega$ at one point then $\frac{dv}{ds} = 0$ at that point, and therefore $v = r\omega$ everywhere if continuity is demanded. That is, if there is one point of no slip between the roller and fiber, then there is no slip anywhere on the roller, according to the creep-rate dependent friction model.

2. If the fiber speed is greater than the roller speed $r\omega$ but less than the transition speed $v_R$ then middle branch (121) is the governing equation, and if the fiber speed exceeds $v_R$ then the top branch of equation (121) is the governing equation. For both cases the gradient $\frac{dv}{ds}$ is positive, since for the draw process $T - Gv \geq 0$ (by inequality (16)) and $\frac{dT}{dv} > G$, so that the fiber speed will increase monotonically.

3. If the fiber speed is less than the roller surface speed but greater than the transition speed $v_L$, the governing equation is the middle branch of (121), and if the fiber speed
is less than \( v_L \) then bottom branch of equation (121) is used. In both, the gradient \( \frac{dv}{ds} \) is negative, so that the fiber speed will decrease monotonically.

As when using the Coulomb friction model, it is assumed that the fiber speed at the point of detachment from the third (final roller) matches the roller surface speed \( r\omega_3 \). From observation 1 it can be concluded that there is no draw anywhere on the third roller (as was with the case of the Coulomb friction model). On account of equation (118) no draw occurs in the freespans as well. Thus the fiber draws either on the first roller, or the second roller, or on both first and second rollers.

If all draw is to take place on the first roller, the fiber must exit the first roller at \( s_1 \) with \( v_1 = r\omega_3 \), the final fiber speed. This would entail that the fiber attaches to the second roller with a speed greater than the roller surface speed \( r\omega_2 \), since \( r\omega_3 > r\omega_2 \). By observation 2 this demands that \( \frac{dv}{ds} > 0 \), which contradicts the assumption that there is no draw on the second roller.

If all draw is to take place on the second roller the fiber must attach to the second roller with speed \( r\omega_1 < r\omega_2 \) and detach with speed \( r\omega_3 > r\omega_2 \). From observation 3, the first condition demands \( \frac{dv}{ds} < 0 \) everywhere on the second roller, which predicts an exit speed less than \( r\omega_2 \) a contradiction with the second condition. Therefore the assumption that all draw takes place entirely on the second roller is incorrect.

Hence the third case, in which (as with Coulomb friction) draw takes place on both the first and second rollers is the only option. To obtain the velocity profile according to the creep-rate friction model (121) is integrated backwards from the known exit speed \( r\omega_3 \) at location \( s_3 \), the point of departure from the second roller. Since the fiber detaches the second roller with a speed \( r\omega_3 \) greater than the roller surface speed \( r\omega_2 \), observation 2 demands \( \frac{dv}{ds} > 0 \) everywhere on the second roller. Therefore integrating the top and middle branch of (121) the fiber speed of attachment \( v_2 \) to the second roller at location \( s_2 \) is deduced to be greater than the roller surface speed, i.e. \( v(s_2) = r\omega_2 + \delta_2 \) with \( \delta_2 > 0 \). Similarly a fiber speed of attachment \( v_0 \) to the first roller at location \( s_0 \) is computed and found to be greater than the roller surface speed, i.e. \( v(s_0) = r\omega_1 + \delta_1 \) with \( \delta_1 > 0 \).

Since the fiber speed never equals the roller surface speed anywhere on the first and
second rollers, it is to be noted that the creep-rate dependent friction model does not predict a no-slip zone on either of the rollers. The fiber draws along its entire contact with the rollers. There are however two distinct draw zones on each roller, the first due to the linear part of the friction vs. relative velocity curve and the second due to the constant part of the friction vs. relative velocity curve (see Figure 84).

The speeds $v_{R1}$ on the first roller and $v_{R2}$ on the second roller mark the transitions from the linear to constant part of the friction curve. The transition speeds $v_{R1}$ and $v_{R2}$ are determined as the solutions to (122b) with $r\omega = r\omega_1$ and $r\omega = r\omega_2$, respectively, and are dependent on the magnitude of $\nu$. If the magnitude of $\nu$ is less than a certain critical value $\nu_c$ then the transition from linear to constant friction is never achieved hence the draw zone is entirely due to the linear part of the friction curve. The critical $\nu_c$ is calculated by setting the transition speed equal to the speed of detachment from the roller in the expression (122b) and solving for $\nu$.

### 8.5 Simulation

The effect of the different friction models is contrasted by comparing the velocity, tension, and friction profiles along the tow line predicted by the Coulomb and creep-rate dependent friction models for a specific isothermal draw process. In particular the two-stage draw process defined in Tables 2 and simulation 1 in 3 is simulated.

#### 8.5.1 Coulomb friction

In the integration of the Coulomb friction (116) backwards from the point of departure $s_1 = 99.7$ cm of the fiber from the first roller to the slip boundary $y_1$, it is to be noted that the fiber at $s_1$ has speed $476.0$ cm $s^{-1}$, which is greater than both the transition speed $v_a = 406.4$ cm $s^{-1}$ between the initial stiff regime and the soft plateau, and the roller surface speed $r\omega_1 = 396.0$ cm $s^{-1}$. Therefore the relevant equation is the $v > r\omega$ branch of (120), with constitutive equation $\dot{T}(v)$ given by the $v_a < v < v_b$ segment of (72). Integration of
this equation gives

\[
v(s) = r \omega_2 + \left[ \frac{T_0 + K_1 r(a - r \omega_1) + K_1 r(\omega_2 - v_a) - Gr \omega_2}{K_2} \right] \left( \exp \left( \frac{\mu}{r} (s - s_1) \right) - 1 \right).
\]

This equation is valid for all \( s < s_1 \) until the attachment location \( s_0 \) or the location \( s_a \) where the transition speed \( v_a \) is achieved, whichever is larger. The transition location \( s_a \) is obtained by setting \( v(s) = v_a = 406.4 \) cm \( s^{-1} \) in (123), and solving for \( s_a \):

\[
s_a = s_1 - \frac{r}{\mu} \ln \left| \frac{T_0 + K_1 r(a - r \omega_1) + K_1 r(\omega_2 - v_a) - Gr \omega_2}{T_0 + K_1 r(a - r \omega_1) - Gv_a} \right| = 99.2 \text{ cm}.
\]

Note that for this process \( s_0 = 0.0 < s_a = 99.2 < s_1 = 99.7 \) cm. To obtain the remainder of the velocity profile \( v(s) \) for \( s_0 < s < s_a \) the \( v > r \omega \) branch of (120) is integrated using the \( v < v_a \) segment of the constitutive equation (72) producing

\[
v(s) = v_a + \left[ \frac{T_0 + K_1 r(a - r \omega_1) - Gv_a}{K_2} \right] \left( \exp \left( \frac{\mu}{r} (s - s_1) \right) - 1 \right).
\]

The slip boundary \( y_1 \) is obtained by setting \( v(s) = r \omega_1 = 396.0 \) cm \( s^{-1} \) in (125), giving

\[
y_1 = s_a - \frac{r}{\mu} \ln \left| \frac{T_0 + K_1 r(a - r \omega_1) - Gv_a}{T_0 - Gr \omega_1} \right| = 68.9 \text{ cm}.
\]

From \( y_1 \) back to the point of attachment \( s_0 \) is the no-slip zone on the first roller. The velocity in this zone is \( v(s) = r \omega_1 = 396.0 \) cm \( s^{-1} \), the surface speed of the roller.

In the freenspan extending from \( s_1 = 99.7 \) cm to \( s_2 = 149.8 \) cm, and in the no drawn zone on the second roller from \( s_2 \) to the slip boundary \( y_3 \), the fiber speed is the constant \( v(s) = r \omega_2 = 476.0 \) cm \( s^{-1} \). On the second roller draw takes place between \( y_3 \) and \( s_3 \). The fiber velocity in the draw zone increases monotonically from \( r \omega_2 = 476.0 \) cm \( s^{-1} \) to \( r \omega_3 = 740.0 \) cm \( s^{-1} \) while transitioning through \( v_b = 672.8 \). By integrating the \( v > r \omega \)
branch of (120) the velocity profile between \( y_3 \) and \( s_3 \) is obtained,

\[
(127) \quad v(s) = \begin{cases} 
    r_\omega_3 + \left[ T_0 + \frac{K_1}{r_\omega_1} (v_a - r_\omega_1) + \frac{K_2}{r_\omega_1} (v_b - v_a) + \frac{K_3}{r_\omega_1} (r_\omega_3 - v_b) - Gr_\omega_3 \right] \times \\
    \exp\left(\frac{\mu}{r} (s - s_3)\right) - 1 & \text{for } v_b < v \leq r_\omega_3, \\
    v_b + \left[ T_0 + \frac{K_1}{r_\omega_1} (v_a - r_\omega_1) + \frac{K_2}{r_\omega_1} (v_b - v_a) - G v_b \right] \times \\
    \exp\left(\frac{\mu}{r} (s - s_b)\right) - 1 & \text{for } r_\omega_2 < v \leq v_b.
\end{cases}
\]

The transition location \( s_b \) in the fiber between the soft plateau and the second stiff segment is found by setting \( v(s) = v_b \) in the first branch of (127), and solving for \( s \):

\[
(128) \quad s_b = s_3 - \frac{r}{\mu} \ln \left| \frac{T_0 + \frac{K_1}{r_\omega_1} (v_a - r_\omega_1) + \frac{K_2}{r_\omega_1} (v_b - v_a) + \frac{K_3}{r_\omega_1} (r_\omega_3 - v_b) - Gr_\omega_3}{T_0 + \frac{K_1}{r_\omega_1} (v_a - r_\omega_1) + \frac{K_2}{r_\omega_1} (v_b - v_a) - G v_b} \right| = 216.8 \text{cm.}
\]

The slip boundary \( y_3 \) is obtained setting fiber speed \( v(s) = r_\omega_2 \) in the second branch of (127),

\[
(129) \quad y_3 = s_b - \frac{r}{\mu} \ln \left| \frac{T_0 + \frac{K_1}{r_\omega_1} (v_a - r_\omega_1) + \frac{K_2}{r_\omega_1} (v_b - v_a) - G v_b}{T_0 + \frac{K_1}{r_\omega_1} (v_a - r_\omega_1) + \frac{K_2}{r_\omega_1} (r_\omega_2 - v_a) - Gr_\omega_2} \right| = 215.3 \text{cm.}
\]

In the freespan extending from \( s_3 = 229.5 \text{ cm} \) to \( s_4 = 279.5 \text{ cm} \) and over the entire third roller before exiting the final roller at \( s_5 = 379.3 \text{ cm} \) the fiber maintains constant speed \( v(s) = r_\omega_3 = 740 \text{ cm s}^{-1} \).

The velocity profile for the entire draw line is plotted in Figure 85 with magnified pictures of critical regions in figures 86-87. The tension profile computed from this velocity and equation (72) is shown in Figures 90-92. The normal force per length and the frictional force per length on the fiber obtained from equations (34) are shown in Figures 93-95 and 96-98, respectively.

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8.5.2 Creep-rate dependent friction

As with Coulomb friction, there is no draw with creep-rate dependent friction from the beginning \( s_3 = 229.5 \) cm of the second free span through the departure location \( s_5 = 379.3 \) cm from the third roller. Hence the fiber speed is the constant \( v(s) = r\omega_3 = 740.0 \text{ cm s}^{-1} \) for \( s_3 = 229.5 < s < s_5 = 379.3 \) cm. To obtain the remainder of the velocity profile predicted from the creep-rate dependent friction model, the governing equation (121) is integrated backwards in \( s \) from \( s_3 = 229.5 \) cm.

The transition speed \( v_{R2} \) in the creep-rate dependent friction model on the second roller is calculated from (122b) with \( r\omega \) set to second roller surface speed \( r\omega_2 \),

\[
v_{R2} = \frac{\mu}{r} \left( \frac{T_0}{r\omega_2} - \frac{K_1}{r\omega_2} \left( v_a - \frac{v_a}{v_0} - 1 \right) - \frac{K_2}{r\omega_2} \left( \frac{v_3}{v_0} - 1 \right) \right) + \nu \left[ \frac{4.1 + \nu}{-1.2 + \nu} \right] \text{ cm s}^{-1}.
\]

Similarly \( v_{R1} \), the transition speed on the first roller, is calculated from (122b) with \( r\omega \) set to \( r\omega_1 \),

\[
v_{R1} = \frac{\mu}{r} \left( \frac{T_0}{r\omega_1} - \frac{K_1}{r\omega_1} \right) + \nu \left[ \frac{\nu - 354.7}{\nu - 370.6} \right] \text{ cm s}^{-1}.
\]

On the second roller integration of equation (121) backwards from \( s_3 = 229.5 \) cm where \( v(s_3) = 740.0 \text{ cm s}^{-1} \) to \( s_2 = 149.8 \) cm yields three zones with the following velocity profiles:

\[
v(s) = \begin{cases} 
  r\omega_3 + \frac{T_0 + \frac{K_1}{r\omega_1} (v_a - r\omega_1) + \frac{K_2}{r\omega_1} (v_b - v_a) + \frac{K_3}{r\omega_1} (r\omega_3 - v_b) - G r\omega_3}{r\omega_1 G} \times 
  & \left( \exp \left( \frac{\mu}{r} (s - s_3) \right) - 1 \right) \text{ for } s_b < s \leq s_3, \\
  v_b + \frac{T_0 + \frac{K_1}{r\omega_1} (v_a - r\omega_1) + \frac{K_2}{r\omega_1} (v_b - v_a) - G v_b}{r\omega_1 G} \times 
  & \left( \exp \left( \frac{\mu}{r} (s - s_b) \right) - 1 \right) \text{ for } s_{R2} < s \leq s_b, \\
  r\omega_2 + [v_{R2} - r\omega_2] \exp \left( \frac{\nu}{r\omega_1 G} (s - s_{R2}) \right) \text{ for } s_2 < s \leq s_{R2}. 
\end{cases}
\]
The location \( s_b \) on the second roller is found by setting the speed \( v(s) = v_b \) in the first branch of (132) and solving for \( s \), yielding \( s_b = 216.8 \) cm. The location \( s_{R2} \) on the second roller where friction changes from linear to constant is found by setting speed \( v(s) = v_{R2} \) in the first branch of (132) and solving for \( s \), yielding

\[
(133) \quad s_{R2} = \left( 216.8 - 63.5 \ln \left| \frac{0.8 \nu + 6.4}{\nu + 4.1} \right| \right) \text{ cm}.
\]

In the freespan extending from \( s_1 = 99.7 \) cm to \( s_2 = 149.8 \) cm the fiber speed is unchanged.

Evaluating (121)3 at the point of attachment \( s_2 \) to the second roller yields \( v_2 = v(s_2) = r \omega_2 + \delta_2 \), where \( r \omega_2 = 740.0 \) cm s\(^{-1} \) and

\[
(134) \quad \frac{\delta_2}{r \omega_2} = \frac{5.3}{\nu - 1.2} \exp \left( -0.1 \nu \left( 9.1 - 8.7 \ln \left| \frac{0.8 \nu + 6.4}{\nu + 4.1} \right| \right) \right).
\]

On the first roller integrating (121) backwards from \( s_1 = 99.7 \) cm to the point of attachment \( s_0 = 0.0 \) cm yields

\[
(135) \quad v(s) = \begin{cases} 
 v_1 + \left[ \frac{T_0 + K_1 (v_a - r \omega_1) + K_2 (v_1 - v_a) - G v_1}{r \omega_1} \right] \times \\
 \left( \exp \left( \frac{\mu}{r} (s - s_1) \right) - 1 \right) & \text{for } s_a < s \leq s_1, \\
 v_a + \left[ \frac{T_0 + K_1 (v_a - r \omega_1) - G v_a}{r \omega_1} \right] \times \\
 \left( \exp \left( \frac{\mu}{r} (s - s_a) \right) - 1 \right) & \text{for } s_{R1} < s \leq s_a, \\
 r \omega_1 + [v_{R1} - r \omega_1] \exp \left( \frac{\nu}{r} (s - s_{R1}) \right) & \text{for } s_0 < s \leq s_{R1},
\end{cases}
\]

The location \( s_a \) is obtained by setting the speed \( v(s_a) = v_a \) in the first branch of (135) and solving for \( s \), yielding \( s_a = 216.6 \) cm. The location \( s_{R1} \) on the first roller where friction changes from linear to constant is found by setting \( v(s_{R1}) = v_{R1} \) in the second branch of (135) and solving for \( s \), yielding

\[
(136) \quad s_{R1} = \left( 99.2 - 63.5 \ln \left| \frac{1.6 \nu - 954.9}{\nu - 354.7} \right| \right) \text{ cm}.
\]

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Evaluating the bottom branch of (121) at the point of attachment $s_i$ to the first roller yields
\[ v_0 = v(s_0) = r \omega_2 + \delta_1, \] where $r \omega_1 = 396.0 \text{ cm s}^{-1}$ and
\[
\frac{\delta_1}{r \omega_1} = \frac{15.9}{\nu - 370.6} \exp \left( -0.001\nu \left( \frac{4.2 - 2.7 \ln \left| \frac{1.6\nu - 954.9}{\nu - 354.7} \right| \right) \right). 
\]

In the simulations that follow the fiber velocity is assumed to be continuous everywhere except there is necessarily a discontinuity at $s_0$ the point of attachment to the first roller. Alternatively a discontinuity in $\delta_2$ could be allowed however for this $\nu$ the value of $\delta_1$ is not small and this discontinuity becomes large as $\nu$ becomes small.

### 8.6 Discussion

In the solution procedure first the velocity profile $v(s)$ is calculated from which the fiber tension $T(s)$, friction force per length $f(s)$, and normal force per length $n(s)$ are calculated. A comparison is made between the tow profiles obtained using coulomb friction and creep-rate dependent friction generated with different values of creep-rate coefficient $\nu$. The operating parameters used on this work are presented in table 3.

Figure 85 shows velocity $v$ with the position on the tow $s$ using Coulomb friction with $\mu = 0.2$ and creep-rate dependent friction for several values of creep-rate coefficient $\nu$ (1.0, 50.0, 100.0, 200.0, 1000.0 and 100000.0). The velocity profile on the first roller is shown in Figure 86. The general trend observed is the decrease in $\delta$ with increasing $\nu$. Thus, as $\nu$ is increased the creep rate dependent behavior approaches that of the Coulomb behavior. At $\nu = 1$ very little draw is observed, where $\delta_1$ is very high with $v_1 = 715 \text{ cm s}^{-1}$. For values of $\nu$ from 10 to 10000 dyn cm$^{-1}$ cm s$^{-1}$ the velocity profile steadily approaches the profile obtained using Coulomb friction. Figure 87 shows that the values of $\delta_2$ are less for the same value of $\nu$ compared to what is observed on the first roller. Also of interest is the near superimposition to the Coulomb curve seen at $\nu = 1.0E5$. The plot of $\delta_2 = v_3 - r \omega_2$ vs. $\nu$ with all other process variables fixed is given in Figure 88. At small values of $\nu$ the amount of fiber draw for a given length of roller contact is small, hence we would expect a large value of $\delta_2$, as shown in the figure. At higher $\nu$, more draw takes place and hence the value of $\delta_2$ also decreases. At $\nu = 8.0$ dyn cm$^{-1}$ cm s$^{-1}$ a sudden drop is observed, this is attributed to shift from the final stiff region to the central soft plateau in the constitutive
equation. In the middle branch of (117) the denominator term \( \frac{dT}{dv} - G \) decreases nearly thirty times (see Table 2) so the corresponding increment of speed for a given increment of wrap will be large, hence the speed drops more rapidly as we integrate resulting in a nearly negligible \( \delta_2 \). A similar behavior is observed in Figure 89 where \( \nu \) is plotted against \( \delta_1 = \nu_1 - \nu_0 \). In this case the dramatic change which we attribute to the same reason happens at \( \nu = 3 \text{ dyn cm}^{-1} \text{ cm s}^{-1} \).

The tension profile is shown in Figure 90. As with velocity profiles of the previous paragraph the behavior of tension profiles obtained using creep-rate dependent friction show a drift toward Coulomb friction profile with increasing values \( \nu \). Figures 91 and 92 show a close up of the tension on the first and second roller. On the first roller a considerable difference in the fiber tension at the point of attachment to the roller compared to the tension obtained using Coulomb friction is observed. On the second roller shown in Figure 92 the tension drops off quite fast for higher values of \( \nu \).

The profiles for normal force per length and friction force per length are shown in Figures 93-98. The normal force profiles show nothing remarkable and follow the general trend observed for the velocity and tension profiles.

The friction profiles on both the first and second rollers in Figures 97-98 show a magical trend where the curve generated by coulomb friction is a limiting case of the creep-rate dependent friction curves.
Figure 85: Velocity profile $v(s)$ for the entire drawline using Coulomb friction with $\mu = 0.2$ and several values of the creep-rate dependent friction coefficient $\nu$. 
Figure 86: The velocity profile $v(s)$ on the first roller.
Figure 87: The velocity profile $v(s)$ on the second roller.
Figure 88: Creep-rate dependent friction coefficient $\nu$ versus $\delta_2$ on the second roller.
Figure 89: Creep-rate dependent friction coefficient $\nu$ versus $\delta_1$ on the first roller.
**Figure 90:** The fiber tension $T(s)$ plotted against arclength $s$ along the tow.
Figure 91: Fiber tension $T(s)$ on the first roller.
Figure 92: Fiber tension $T(s)$ on the second roller.
Figure 93: Normal force per length $n(s)$ acting on the fiber for the entire drawline.
Figure 94: Normal force \( n(s) \) on the first roller.
Figure 95: Normal force \( n(s) \) on the second roller.
Figure 96: Friction force per length $f(s)$ versus arc length $s$ along the tow.
Figure 97: Friction force $f(s)$ on the first roller.
Figure 98: Friction force $f(s)$ on the second roller.
CHAPTER IX

PREDICTION OF FIBER TEMPERATURE PROFILES
FOR A TWO-STAGE DRAW PROCESS

In this chapter the coupled energy and momentum equations are solved for both the temperature and velocity profiles. By assuming a thermo-elastic constitutive relation for the fiber the solution for fiber velocity and temperature along the tow for different heating conditions is obtained.

9.1 Governing Equations

In chapter 4 the model for the two stage fiber draw process is obtained by combining the equations of conservation i.e. mass (9), momentum on the roller (34), and momentum in the freespan (18) with the friction relations (116) or (117). With the temperature profiles in the drawline known, the equations of motion augmented with the constitutive equation (30) are then directly integrated to generate the velocity profile along the drawline.

In this chapter however the fiber temperature is an unknown. The conservation of energy in Equation (20) provides the additional relation between the fiber temperature and other draw variables that is required to complete the set of equations necessary to solve the system.

Some simplifying assumptions are made in the energy equation. The heat conduction along the fiber axis is negligible compared to other terms in the equation. The internal energy \( u \) of the fiber per mass is assumed to be related to the fiber temperature \( \theta(s) \) through,

\[
(138) \quad u = c\theta(s),
\]

where \( c \) is the specific heat of the fiber which is taken to be constant. All heat entering the fiber per unit length is lumped into a single term \( q \), the energy equation under steady
conditions (20) reduces to

\[ Gc \frac{d\theta}{ds} = T \frac{dv}{ds} + q. \]

Here \( T \) and \( v \) are the tension and velocity of the fiber at location \( s \).

The heat source term, \( q \) is due to any one or a combination of conduction, convection, radiation, and friction heating, these will be discussed in the paragraphs that follow. First consider the heat conducted into the fiber due to contact with the roller. The amount of heat conducted depends on the thermal conductivity of the fiber \( K \), the temperature gradient between the fiber and roller, and the area of contact between the fiber and the roller. Furthermore the temperature at the center of the fiber is assumed to be \( \theta \). It is also assumed, because the roller is a much larger heat sink than a fiber, that at the contact boundary between the roller and the fiber the fiber temperature is at the roller temperature \( \theta_r \). The temperature gradient is assumed to be linear and is obtained by taking the quotient of the temperature difference between the roller and fiber and the radius of the fiber. For a semi-contact width \( a \), see [21], the amount of heat conducted per unit fiber length into the fiber is then given by

\[ q_{\text{cond}} = 2aK \frac{\theta_r - \theta}{\sqrt{r^2 - a^2}}. \]

Since the fiber is also in contact with the ambient fluid medium which is usually air that adds heat to the fiber by convection. The entire surface of the fiber except the part in contact with the roller is exposed to air at constant temperature \( \theta_a \). The amount of heat absorbed by the fiber per length is given by

\[ q_{\text{conv}} = 2(\pi - \arcsin \left( \frac{a}{r} \right))rh(\theta_a - \theta). \]

\(^1\)The polymeric fiber is assumed to be a circular in cross section with radius \( r = \frac{G}{\pi \rho} \). The contact between the fiber and roller produces, see Johnson [21], a contact patch of semi-contact width \( a \) given by

\[ a = \sqrt{\frac{4.0nR^2}{\pi E^*}}, \]

where \( \frac{1}{R} = \frac{1}{r} + \frac{1}{R} \) and \( \frac{1}{E^*} = \frac{1 - \nu^2}{E} + \frac{1 - \nu_R^2}{E_R} \). The poisson ratio for fiber and roller are \( \nu \) and \( \nu_R \) and stiffness moduli is \( E \) and \( E_R \) respectively. The modulus of the fiber is \( E = \frac{1}{A} \frac{dT}{d\varepsilon} = \frac{v_0}{A} \frac{d\varepsilon}{dv} \), where the values used are given in table 5.
Where \( h \) is the convective heat transfer coefficient obtained from the empirical relation developed by Kase et al. [38] given by,

\[
h = 0.42 \frac{K_f}{2.0} \left( \frac{2.0 \rho v^2 \pi \eta_f}{G} \right)^{3/4}.
\]

Here \( K_f \) and \( \eta_f \) are the conductivity and viscosity of the fluid (air). The above relation was originally developed for convective cooling of melt during fiber spinning. So the expression is only an approximation of convective heating a fiber wrapped around a roller.

Radiant heating governed by the Stefan-Boltzmann law is an additional source of heat. The entire surface of the fiber except the region of contact with the roller is assumed to be exposed to radiant heat where present. The radiant heat absorbed per length by the fiber is given by,

\[
q_{rad} = \epsilon \sigma 2(\pi - \arcsin \left( \frac{a}{r} \right) )r(\theta_h^4 - \theta^4).
\]

Here \( \epsilon \), the emissivity of the fiber, accounts for the departure from black body radiation. The temperature of the radiant source is \( \theta_h \) and \( \sigma \) is the Stefan-Boltzmann constant.

Sliding of the fiber on the roller also generates heat, it is due to friction. The heat diffusivity of the fiber and the roller material are not identically hence the frictional heat gets partitioned. A fraction going into the fiber and the rest into the roller. From Suh [35] for low Peclet numbers \( (Pe = \frac{va}{\alpha} \approx 0.5 \), valid for the simulations to follow), the partition function given by

\[
\phi = \frac{K_r}{K_r + K}.
\]

Here \( \phi \) gives the fraction of the heat flowing into the roller and \( K_r \) and \( K \) are the thermal conductivities for the roller and fiber respectively. For a fiber sliding at a speed \( v \) with respect the roller turning at \( r\omega \) the amount of heat diffusing into the fiber is given by

\[
q_{fric} = (1 - \phi) \mu f(v - r\omega).
\]
9.2 Simulations for a Thermo-elastic fiber

The momentum and energy equations for the fiber on a roller form a system of two coupled ordinary differential equations,

\begin{align}
G \frac{dv}{ds} &= \left( \frac{\partial T}{\partial v} \right) \frac{dv}{ds} + \left( \frac{\partial T}{\partial \theta} \right) \frac{d\theta}{ds} - f \\
Gc \frac{d\theta}{ds} &= T \frac{dv}{ds} + q.
\end{align}

The system of equations is solved for velocity \( v \) and temperature \( \theta \) using rate-dependent friction (117) and the constitutive relation 30. The value of the fiber tension \( T \) is obtained by substituting the velocity and temperature profiles obtained into 30, the normal force per length \( n \) is obtained from (34b), and friction force per length \( f \) from (117). In the freespan the system of equations is like (144) but with the \( f \) term dropping out.

The two stage draw is simulated with fiber and roller properties given in Table 5 and the geometric layout used is in simulation 1 of Table 3.

To produce isothermal conditions in the drawline the temperature gradient \( \frac{d\theta}{ds} \) is set to zero. The momentum equation (144a) is used to solve for the velocity profile along the drawline. By substituting the velocity gradient into the the energy equation (144b) the amount of heating (or cooling) \( q \) required to generate isothermal conditions along the drawline can be determined. The result of isothermal simulation is given in Figures 99 to 102. The main reason for this isothermal simulation is to confirm that the twostage draw simulations like the one in chapter 4 can be reproduced.

In the next simulation shown in Figures 103 to 106 the rollers and air are kept at an ambient temperature of 298 K. There are no perceptible changes in velocity observed in Figure 103. In 104 the temperature change along the draw line is plotted. Several interesting features can be observed. The fiber temperature rises by about 10 K in the draw zone on the first roller due frictional heating. In the first freespan the fiber cools down and continues to cool down in the no slip zone of the second roller since an additional amount of heat is extracted from the fiber because it is in contact with the roller that acts as a heat sink. Due to frictional heating in the draw zone on the second roller there is a rise around 36 K which
is broken into two segments, the first almost vertical and the second with a lesser gradient. The reason for the transition is due to the transition from the soft plateau to the final steep section of the constitutive curve. After detaching from the second roller and entering the second free span the fiber undergoes convective cooling. Additional cooling due to heat being conducted away from the fiber takes place on the third and final roller. The fiber tension curve in Figure 103 shows the marked effect of temperature on fiber tension. In the first and second draw zones temperature increases due to heat generation which results in softening of the fiber. The friction relation shown in Figure 106 shows a in the first draw zone a steep increase in friction followed by lower gradient. On the second roller the draw zone shows a steep increase then a drop and increase in friction. The drop and increase in friction is because the fiber tension shows a similar drop and rise in value due to frictional heating which raises the temperature softening the fiber and subsequent cooling in the free span which increases the tension again.

In the next simulation whose result is shown in Figures 107 to 110, the amount of heat exchanged $q$ is set to zero. The velocity profile in Figure 107 is quite the same as the previous ones. The temperature profile in Figure 108 shows steep increases in the temperature in both the draw zones. Since there is no heat exchange to the surroundings the values of temperature don’t show any change. The tension profile in 109 shows a rather unusual behavior it is still not clear why the fiber tension shows a drop in value after each draw zone and a subsequent increase in tension only beyond the second draw zone. The friction force in Figure 110 mirrors the drop and rise in the tension value.

In the next simulation the second free span is is heated to 323 K. In Figure 112 is the temperature profile, which resembles the temperature profile obtained from the process with ambient temperature at 298 K of Figure 104. The only difference is that because of heating in the second free span the fiber temperature does not decrease at high rate seen in Figure 104 which has a marked effect on the tension profile. In Figure 113 the tension profile shows a rise in tension at a much slower rate compared to what was seen in Figure 113 which allows stresses to relax.
Table 5: Material constants for roller and fiber materials.

<table>
<thead>
<tr>
<th>material properties</th>
<th>fiber (PET)</th>
<th>roller (steel)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson ratio ($\nu$)</td>
<td>0.34</td>
<td>0.29</td>
</tr>
<tr>
<td>Modulus ($E$) dyne cm$^{-2}$</td>
<td>2.7E9 (nominal)</td>
<td>2000.0E9</td>
</tr>
<tr>
<td>Thermal conductivity ($K$) erg cm$^{-1}$ K$^{-1}$</td>
<td>0.02E6</td>
<td>5.19E6</td>
</tr>
<tr>
<td>Coefficient of friction ($\mu$)</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>Creep rate coefficient ($\nu$)</td>
<td>10.0E3</td>
<td></td>
</tr>
<tr>
<td>Specific heat ($c$) erg g$^{-1}$ K$^{-1}$</td>
<td>20.0E6</td>
<td>4.81E6</td>
</tr>
<tr>
<td>Density ($\rho$) g cm$^{-3}$</td>
<td>1.33</td>
<td>7.872</td>
</tr>
<tr>
<td>Thermal diffusivity ($\alpha = \frac{K}{\rho c}$)</td>
<td>0.75188E-3</td>
<td>137.1E-3</td>
</tr>
</tbody>
</table>
Figure 99: Fiber velocity $v(s)$ versus arclength $s$ for an isothermal process.
Figure 100: Fiber temperature $\theta(s)$ versus arclength $s$ for an isothermal process.
Figure 101: Fiber tension $T(s)$ versus arclength $s$ for an isothermal process.
Figure 102: Friction force $f(s)$ versus arclength $s$ for an isothermal process.
Figure 103: Fiber velocity $v(s)$ versus arc length $s$ for a draw process carried out at ambient temperature 298 K.
Figure 104: Fiber temperature $\theta(s)$ versus arclength $s$ for a draw process carried out at ambient temperature 298 K.
Figure 105: Fiber tension $T(s)$ versus arclength $s$ for a draw process carried out at ambient temperature 298 K.
Figure 106: Friction force $f(s)$ versus arclength $s$ for a draw process carried out at ambient temperature 298 K.
Figure 107: Fiber velocity $v(s)$ versus arc length $s$ for a draw process carried out under adiabatic conditions.
Figure 108: Fiber temperature $\theta(s)$ versus arclength $s$ for a draw process carried out under adiabatic conditions.
Figure 109: Fiber tension $T(s)$ versus arclength $s$ for a draw process carried out under adiabatic conditions.
Figure 110: Friction force $f(s)$ versus arclength $s$ for a draw process carried out under adiabatic conditions.
Figure 111: Fiber velocity $v(s)$ versus arclength $s$ for a draw process with a heated second free span at 323 K.
Figure 112: Fiber temperature $\theta(s)$ versus arclength $s$ for a draw process with a heated second free span at 323 K.
Figure 113: Fiber tension $T(s)$ versus arc length $s$ for a draw process with a heated second free span at 323 K.
Figure 114: Friction force $f(s)$ versus arc length $s$ for a draw process with a heated second freespan at 323 K.
CHAPTER X

CONCLUDING REMARKS

A general framework of characterizing fiber behavior in a multi-stage draw process is presented in this thesis. The model can be used to simulate a two-stage draw process with different drawing conditions and is flexible enough to address any fibrous materials, as long as its constitutive behavior, either piecewise elastic or viscoelastic, is known for a range of temperature conditions. The analysis allows one to compare the effectiveness of different drawing conditions, enabling fiber manufacturers to identify optimum processing conditions. The outcome of such an analysis are the velocity and tension at any point along the length of the fiber, and the forces from the rollers on the fiber, for specific drawing conditions.

10.1 Results and Discussion

The equations of motion for an extensible belt on a pulley in which all effects of inertia, including acceleration due to stretching, are retained in the momentum balance are derived in [11] (see Appendix A). These equations are also valid for fibers and films on rollers undergoing cold draw. The solution to the problem of torque transmission by a linearly elastic belt between two pulleys is compared to solutions in which centrifugal acceleration is included but stretching acceleration is neglected (the common engineering practice), and the solution in which both centrifugal and stretching accelerations are neglected. It is found that ignoring both centrifugal and stretching accelerations results in an overprediction of the maximum moment that can be transmitted, and, for a given transmitted moment, underprediction of the slip angles on the driving and driven pulleys and overprediction of belt strain rates and normal and frictional forces from the pulley on the belt in the slip zones. The common engineering practice of including the effects of centrifugal acceleration but neglecting stretching acceleration also results in errors, for example underpredicting
the maximum moment that can be transmitted, overpredicting the slip angles, and under-
predicting belt strain rates and normal and frictional forces on the driving pulley. The
percentage error increases as the ratios of belt stiffness to centrifugal acceleration or initial
belt tension decrease. The conservation equations developed in [11] lay the foundation on
which modeling of multi-stage fiber draw will be based on.

The proposed model for draw on rollers retains both centrifugal acceleration and acceler-
ation due to stretching in the equations of motion. The solution fundamentally couples fiber
tension to fiber stretch, in contrast to all existing solutions, which either neglect stretching
acceleration or neglect both centrifugal acceleration and stretching acceleration. Hence it is
found that the tension and draw ratios between the point of exit to the point of attachment
of fibers with the roller depend on three non-dimensional parameters: \( \frac{G_m}{T_0} \), \( \frac{G_w}{k} \), and \( \mu \beta \).
For certain values of these parameters, the exact solution can be approximated either by
the quasistatic solution neglecting both acceleration terms in the momentum equation, or
by the solution including only the centrifugal acceleration. The quasistatic approximation
underpredicts the tension magnification and draw ratio when the stiffness \( k \) of the fibers is
less than the initial tension \( T_0 \), and overpredicts when \( k \) is greater than \( T_0 \). The solution
including only the centrifugal acceleration always underpredicts the tension magnification
and the amount of draw. The response of the fiber draw on a roller obtained by varying
fiber parameters such as initial fiber tension, fiber stiffness and wrap angle (angle for which
the fiber stretches) are presented in [9] (see Chapter 3).

Up until now a model for slip of linearly elastic belts on pulleys [11] was created, and then
adapted to isothermal draw of fibers on rollers [9]. The model for two stage draw (special
case of multistage draw) is developed in [8] (see Chapter 4) by refining and augmenting the
previous two developments by:

- incorporating temperature dependence,
- extending the constitutive characterization of the fibers to piecewise linearly elastic-
  plastic to capture the softening behavior of as-spun fibers,
- assembling the governing equations for fibers on rollers and in free spans, together
with matching boundary conditions, to produce a comprehensive model for a two stage, nonisothermal industrial draw process.

The model is employed to simulate three representative drawlines i.e. isothermal, heated freespan, and heated roller, with speed profiles of the fiber in the tow being obtained for several draw geometries. Note that although the fiber was characterized as elastic-plastic, due to the draw conditions chosen only monotonic loading of the fiber was encountered.

Using the governing equations developed in [9] and [8], a commercially relevant fiber drawing process where draw pins are used for localizing and enhancing the fiber draw is analyzed. Inclusion of a draw pin introduces the possibility of unloading in the fiber, resulting in the additional complexity of elastic-plastic fiber characteristics in the analysis. In the absence of unloading, the stress in the fiber monotonically increases, so that only the elastic response of the fiber enters into the analysis. In analyzing an isothermal two stage draw process with a draw pin in the second free span, it was observed that the total draw is in general distributed between the first roller, the second roller, and the draw pin. The offset location of the pin is shown to have a significant effect on the distribution of the draw. As this offset is increased, the nature of the draw on second roller changes from loading, to neither loading nor unloading at a critical offset, to unloading. The effect of a drawpin under isothermal conditions is discussed in [10] (see Chapter 5).

Effect of the position of draw pins on the draw is subject of further investigation. For anisothermal two stage draw process with one draw pin the development is carried out in [5] (see Chapter 6). The response of two draw pins, one in each freespan of the two-stage draw process is discussed in [6] (see Chapter 7). It is observed that the effectiveness using the draw pin depends where the draw zone developed by the draw pins is located on the constitutive curve. Since the amount of draw for the same increment in tension is less for the initial stiff zone compared to intermediate soft plateau. So to maximize the amount of draw the draw pin should ideally be introduced at locations where the fiber tension is in the soft plateau.

Theoretical predictions of the behavior of the fibers in an industrial fiber drawing process depend on the particular friction model employed for quantifying the friction forces between
filament tow and draw rollers. When the usual Coulomb model for friction is replaced with a creep-rate dependent friction model as in [4] (see Chapter 8) it is observed that whereas the Coulomb friction model predicts adhesion zones on the rollers, it is found that no adhesion zones are present with the creep-rate dependent friction model. For creep-rate dependent friction the fiber velocity at the point of attachment to the draw roller must be greater than the roller surface speed for the equations of momentum to be satisfied. This nature of the velocity profile can be interpreted as the formation of a neck in the fiber just upstream of the point of attachment to the roller.

The general solution procedure adopted so far has been to assume a temperature distribution thus uncoupling the momentum and energy equations. The momentum equation is then solved for the velocity profile and the fiber tension is obtained by substituting the now known velocity and temperature distributions into the constitutive equation for tension. In [7] (see Chapter 9) the full set of steady conservation equations is solved to obtain both the velocity and temperature profiles. Using this approach multistage draw is modeled.

10.2 Conclusions

There are several advantages of the work presented in this thesis. A mechanics based framework is introduced to the analysis of the fiber draw process. Thus adding the predictive power of modeling and simulation to a largely empirical field of study.

The equations of motion developed here account for both the centrifugal acceleration and acceleration due to the stretching of the fiber. The inclusion of these acceleration terms results in solution profiles which show additional features not seen in solutions obtained by earlier models i.e. capstan model that disregarded both stretching and centrifugal or the engineering model that disregarded stretching but accounted for centrifugal effects.

The non dimensional parameters identified and used to carry out parametric studies provide an excellent tool for investigating actual drawing processes.

The placement of a draw pin in the freespan of the draw line allows the fiber to unload on the draw roller upstream from the draw pin. At an ideal height the draw pin forces all the fiber draw to take place on the draw pin instead of the draw roller upstream from the
draw pin. This saves the draw roller from abrasion. The location of the draw pin along the
draw line is decided by the fiber modulus at the location. More draw can be developed if the
draw pin is placed where the fiber speed corresponds to the soft region on the constitutive
curve. Thus placing the draw pin in a free span where fiber has been previously deformed
beyond the initial stiff zone is preferable.

The micro-structure developed in the fiber is responsible for the fiber properties. Vastly
different micro-structure in the fiber result as the velocity and tension profiles are altered.
The results obtained for a representative run using a given constitutive equation for a draw
process can in turn serve as the input data for investigating micro-structure development
in the fiber during drawing.

10.3 Future Work

The work presented in this thesis establishes a method to model the draw of polymer
fibers. The conservation equations were developed assuming 1-D fiber geometry. This was
applied to a simplified piecewise linear elastic constitutive equation for the fiber. Due to its
simplicity the accuracy of the predictions made by this model are somewhat limited.

To refine the draw model a 2-D axisymmetric or a full 3-D fiber geometry can be
considered in the future. These refinements will be able to account for the radial variations
of draw line variables.

The constitutive equation needs to be more realistic. A step in this direction is to use an
equation with a more accurate loading-unloading profile and strain rate dependence giving
a suitable viscoplastic phenomenological equation. The constitutive equation can be further
refined by using the microstructure and its evolution as the basis of the model. Realistic
predictions can be anticipated from simulations obtained from microstructure based models.
These draw line predictions need to be verified experimentally.

To understand and optimize fiber behavior during the draw process, exact stress-strain
curves need to be used. By transforming the results obtained from a real fiber spinning
process into the non-dimensional parameters developed earlier a parametric study can be
carried out to establish optimal draw conditions.
APPENDIX A

THE STRETCHING AND SLIPPING OF BELTS AND FIBERS ON PULLEYS

In the equations of motion for an extensible fiber on a roller or belt on a pulley derived Chapter 2 all effects of inertia, including acceleration due to stretching, are retained in the momentum balance. These equations are valid for fibers and films on rollers undergoing cold draw and are also applicable to the problem of torque transmission by a belt between two pulleys. The resulting solution is compared to solutions in which centrifugal acceleration is included but stretching acceleration is neglected (the common engineering practice), and the solution in which both centrifugal and stretching accelerations are neglected. It is discovered that ignoring both centrifugal and stretching accelerations results in an over prediction of the maximum moment that can be transmitted, and, for a given transmitted moment, over prediction of the slip angles on the driving and driven pulleys and over prediction of belt strain rates and normal and frictional forces from the pulley on the belt in the slip zones. The common engineering practice of including the effects of centrifugal acceleration but neglecting stretching acceleration also results in errors, for example under prediction the maximum moment that can be transmitted, over prediction the slip angles, and under predicting belt strain rates and normal and frictional forces on the driving pulley. The percentage error increases as the ratios of belt stiffness to centrifugal acceleration or initial belt tension decrease.

A.1 Introduction

The stretching and sliding of belts on pulleys or fibers and films on rollers have significant industrial implications. Torque transmission between pulleys is affected by the stretching and slipping of the belt. In a fiber manufacturing process, polymeric fibers are drawn between feed and take-up rollers in order to improve their mechanical properties. In a
cold draw process (i.e., where no external heat is supplied) the draw occurs on the rollers, accompanied by stretching and sliding of fibers. Similar behavior is also true for films. In this chapter the problem of belts on pulleys is addressed and this nomenclature is adopted, the formulation is equally applicable to the drawing of fibers and films.

Any examination of a belt-pulley system which takes into account the compliance or elasticity of the belt, for prediction of the slip angles on the pulleys, analysis of creep, etc., must, to be consistent, include the effects of changing belt stretch in the momentum equations. These analyses recognize that the belt tension is not uniform; there is a tight side and a slack side. The elastic characterization of the belt indicates that this change of tension is accompanied by a change of strain, so there must be a changing stretch in the belt. This rate of change of stretch is an acceleration that results in a change of momentum.

From the momentum equations for a moving, stretching belt on a pulley or roller derived in 2. it is found that in the normal projection of the momentum equation there is a centrifugal acceleration term \((Gv\) in (34b)), and in the tangential projection there is a change of stretch term \((Gdv\) in equation (34a)) A review of the literature has revealed that some studies of stretching belts neglect both of these inertial terms in the momentum equations, and others incorporate only the centrifugal acceleration term in the normal projection. This work includes the stretching acceleration in the tangential direction.

Without stretching acceleration, the two momentum projections reduce to a single differential equation for the evolution of belt tension with arclength, decoupled from belt stretch. When the stretching acceleration term is included, the two momentum projections become coupled differential equations for the evolution of both belt tension and belt stretch, and to close the mathematical problem statement a constitutive equation relating belt tension to belt stretch must be adjoined. Since belts are relatively stiff and elastic it is characterized as linearly elastic here.

In this section four solutions for the problem of torque transmission of a belt between two pulleys are obtained and compared. The pulley radii, transmitted moment, angular velocity of the driving pulley, initial tension in the belt, stiffness of the belt, and coefficient
of friction between the belt and the pulleys are considered to be specified, and the problem is solved for the angular velocity of the driven pulley, the angles over which the belt is slipping on the driving and driven pulleys, the belt tension and speed at all locations along the belt, and the normal and frictional forces per length from the pulleys on the belt at all locations of contact, as well as the maximum moment that can be transmitted by the belt-pulley system. The first solution, the full solution, which accounts for both stretching acceleration and centrifugal acceleration in the momentum equations. The next two solutions include centrifugal acceleration but neglect stretching acceleration (the common engineering practice). The fourth solution, recalled from the literature (Johnson [21]), neglects both centrifugal acceleration and stretching acceleration.

Beyond having a consistent mathematical formulation, there is a quantitative advantage to the full solution. It is found that the solution in the literature which neglects both centrifugal and stretching accelerations under predicts the slip angles on the driving and driven pulleys, over predicts the normal and frictional forces on the belt from the pulleys, over predicts the strain rate of the belt in the slip zones, and over predicts the maximum moment that can be transmitted. The common engineering simplification of including centrifugal acceleration but neglecting stretching acceleration decreases the magnitude of the errors, and reverses the direction of some. The quantitative differences between the full solution and the common engineering practice is slight for most power transmission applications, but these differences increase as the stiffness of the belt decreases and the speeds increase. It is essential to use the new equations in applications such as the drawing process in the manufacturing of polymer fibers and films, in which the stiffnesses are small and the speeds are great.

A.2 The Torque Transmission Problem

Next the steady torque transmission problem of figure 115, with an extensible, linearly elastic belt, and pulleys mounted at fixed center distance is solved. The problem is solved in three ways: (i) using the full momentum equations (34) for the belt on the pulley, which account for both centrifugal acceleration and the acceleration due to stretching, (ii) using
Figure 115: Schematic diagram of a belt transmitting a torque $M$ between two pulleys.

momentum equations in the form of equation (35), where centrifugal acceleration is included but stretching acceleration is neglected, and (iii) using equations (36) on the pulley surface, which neglect both accelerations. The quasi-static solution (iii) appears in the literature in Johnson [21]. It has long been held that the effect of centrifugal acceleration is significant, so that in belt applications solution (ii) is used, never (iii). It is also widely conjectured that the effect of the stretching inertia term is insignificant, although it has never before been computed. The full solution (i) for the calculates this effect. It is found that

- computationally it is less problematic to retain both acceleration terms than to include one and neglect the other; equations (35) and the elastic constitutive equation are inconsistent (the momentum equations (35) predict that the tension is changing in the belt, and hence the elastic constitutive equation demands that the stretch of the belt is changing, but this changing stretch is absent from equations (35)), resulting in an artificially decoupled and, as will be seen later, ambiguous problem statement,

- for relatively stiff belts (in a sense that will be made explicit later) solutions (i) and (ii) coincide, with solution (iii) noticeably different, supporting in such applications the common engineering wisdom that that the effect of centrifugal acceleration is significant and the effect stretching acceleration is insignificant, but
• for compliant belts, and fibers undergoing draw, solution (i) is significantly different from solution (ii), indicating that the effect of stretching in these applications is important.

The pulley radii \( r \), transmitted moment \( M \), driving pulley angular velocity \( \omega_1 \), initial belt tension \( T_{\text{init}} \), belt stiffness \( k \), and coefficient of friction \( \mu \) between the belt and the pulleys are assumed to be specified. The unknowns that constitute the solution are the angular velocity \( \omega_2 \) of the driven pulley, the subtended angles \( \beta_1 \) and \( \beta_2 \) over which the belt is slipping on the driving and driven pulleys, respectively (see figure 115), the belt tension \( T(s) \) and speed \( v(s) \) at all locations \( s \) along the belt, and the normal and frictional forces per length \( n(s) \) and \( f(s) \) from the pulleys on the belt at all locations of contact. It will be seen that solutions (i) and (ii) depend on the specified quantities through the four dimensionless combinations

\[
A = \frac{M}{2r k}, \quad B = \frac{T_{\text{init}}}{k}, \quad C = \frac{G \omega_1 r}{k}, \quad \mu.
\]

Solution (iii) depends only on \( \frac{M}{2r}, \frac{T_{\text{init}}}{k} \) and \( \mu \). (The combinations (145) rather than, say, \( \frac{M}{2}T_{\text{init}}, \frac{k}{T_{\text{init}}}, \frac{G \omega_1 r}{k} \) are selected, since it is commonly argued that the fractional change due to including the stretching acceleration is proportional to \( \frac{G \omega_1 r}{k} \), or, stated differently, solution (ii) likely can be obtained from solution (i) by letting \( G \omega_1 r \ll k \). Combinations (145) permits the investigation of this conjecture by considering the limit of small \( C = \frac{G \omega_1 r}{k} \).

In the free spans the tensions and velocities are constant (see equation (48)). When a constant torque \( M \) is transmitted between two pulleys of the same radius \( r \) as shown in figure 115, the tensions \( T_i \) in the tight free span and \( T_s \) in the slack free span are (see Amijima [3], Johnson [21])

\[
T_i = T_{\text{init}} + \frac{M}{2r} = T_{\text{init}}(1 + \frac{A}{B}), \quad T_s = T_{\text{init}} - \frac{M}{2r} = T_{\text{init}}(1 - \frac{A}{B}),
\]

obtained by assuming the bearings are frictionless and summing the moments on either pulley to zero; \( T_{\text{init}} \) is the tensile force in the belt when the moment \( M \) is zero.

Assuming, as do Amijima [3], Firbank [17], Johnson [21], that the belt is not slipping on either the driving or driven pulleys where it first attaches, so that the speeds \( v_i \) and \( v_s \),

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of the tight and slack freespans are given by

\begin{equation}
(147) \quad v_t = r\omega_1, \quad v_s = r\omega_2.
\end{equation}

The reference state is chosen as the one in which the tension \( T_{\text{ref}} \) is zero, so that the linearly elastic constitutive equation (39) reduces to

\begin{equation}
(148) \quad T(s) = k \left( \frac{v(s)}{v_{r\text{ef}}} - 1 \right),
\end{equation}

the same constitutive equation employed by Johnson [21]. Evaluating this constitutive equation on the tight and slack freespan speeds and recalling eqs. (146) gives

\begin{equation}
(149) k \left( \frac{v_t}{v_{r\text{ef}}} - 1 \right) = T_t = T_{\text{init}}(1 + \frac{A}{B}), \quad k \left( \frac{v_s}{v_{r\text{ef}}} - 1 \right) = T_s = T_{\text{init}}(1 - \frac{A}{B}).
\end{equation}

These two equations can be inverted to give the angular velocity \( \omega_2 \) of the driven pulley and the reference speed \( v_{r\text{ef}} \) in terms of specified quantities,

\begin{equation}
(150) \quad \omega_2 = \left( \frac{B - A + 1}{B + A + 1} \right) \omega_1, \quad v_{r\text{ef}} = \frac{r\omega_1}{A + B + 1}.
\end{equation}

Note that \( \omega_2 < \omega_1 \). Then the constitutive equation (148) can be recast as

\begin{equation}
(151) \quad T(s) = \frac{T_{\text{init}}}{B} \left( (1 + A + B) \frac{v(s)}{r\omega_1} - 1 \right).
\end{equation}

In the torque transmission problem the belt passes through six zones. In the direction of increasing arclength \( s \) (clockwise), referring to figure 115, they are the tight freespan, no-slip zone on the driving pulley, the slip zone on the driving pulley, the slack freespan, the no-slip zone on the driven pulley and the slip zone on the driven pulley. The location \( s = 0 \) is selected as the start of the tight freespan. In the following subsections, for each of the three formulations, the belt tension, speed, and friction and normal force per length through these zones as a function of arclength \( s \), as well as the angular extents \( \beta_1 \) and \( \beta_2 \) of the slip zones are obtained.

A.2.1 Full solution retaining all inertia terms

First the problem using equations (34) when the belt is on the pulley is solved.
The tight freespan: Belt behavior is governed by equations (48), so that

$$T(s) = T_i = T_{init} \left( 1 + \frac{A}{B} \right), \quad v = v_i = r\omega_1.$$  

No-slip zone on the driving pulley: The belt attaches without slip to the driving pulley with the tight-span speed $v_t = r\omega_1$, and maintains this constant speed through a no-slip zone of yet-to-be-determined length. Since speed is constant so is strain, and hence in the elastic belt so is the tension, $T = T_i$. With constant speed ($dv = 0$) and constant tension ($dT = 0$), equation (34a) demands $f = 0$: there is no friction between belt and pulley in the no-slip zone. The normal force per length $n$ is computed using equation (34b).

Summarizing,

$$T(s) = T_{init} \left( 1 + \frac{A}{B} \right), \quad v(s) = r\omega_1, \quad f(s) = 0, \quad n(s) = \frac{T_{init}}{r} \left( 1 + \frac{A}{B} - \frac{C}{B} \right).$$

Slip zone on the driving pulley: Since the belt leaves the driving pulley with the slack-span speed $v_s = r\omega_2$ less than the tight-span speed $v_t = r\omega_1$ that it attaches (recall from eq. (150) that $\omega_1 > \omega_2$), there must be a slip zone on the driving pulley. In this zone the friction is kinetic, and, since the belt is moving slower than the pulley surface, the direction of this friction is in the direction of motion. Therefore, according to the chosen sign convention, $f = -\mu n$. Equations (9), (34), and this friction relation reduce to

$$\frac{dT - Gv}{T - Gv} = -\mu d\theta.$$  

Integrating eq. (154) over the entire slip zone yields a relation for the angle of slip $\beta_1$ on the driving pulley,

$$\beta_1 = \frac{1}{\mu} \ln \left( \frac{T_i - Gr\omega_1}{T_s - Gr\omega_2} \right) = \frac{1}{\mu} \ln \left( \frac{B + A - C}{B - A - \frac{B}{B-A+1} C} \right).$$

The boundary between the no-slip and slip zones on the driving pulley is therefore at $s = L + r(\pi - \beta_1)$, where $L$ is the length of the freespan. Integrating eq. (154) from this boundary to an arbitrary $s$ within the slip zone gives:

$$T(s) - Gv(s) = (T_i - Gv_i)e^{-\mu s} = \frac{T_{init}}{r} \left( 1 + \frac{A}{B} - \frac{C}{B} \right) e^{-\mu s},$$

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where \( \bar{s} = s - l - r(\pi - \beta_1) \) is the arclength from the start of the slip zone. To obtain the belt tension \( T(s) \) and speed \( v(s) \) in the slip zone separately as functions of arclength, eq. (156) is combined with the elastic constitutive equation (151) to obtain
\[
T(s) = T_{init} \left[ \left(1 + \frac{4}{B - \frac{C}{B + A + 1}}\right) e^{-\mu \bar{s}} + \frac{C}{B + A + 1} \right] ,
\]
\[
v(s) = r\omega_1 \left[ \left(1 - \frac{1}{B + A + 1}\right) e^{-\mu \bar{s}} + \frac{1}{B + A + 1} \right] .
\]
The normal and frictional forces per length \( n(s) \) and \( f(s) \) are then obtained algebraically from \( T(s) \) and \( v(s) \) via equation (34b) and the friction relation:
\[
n(s) = \frac{T(s) - Gv(s)}{r} = \frac{T_{init}}{r} \left(1 + \frac{4}{B - \frac{C}{B + A + 1}}\right) e^{-\mu \bar{s}} ,
\]
\[
f(s) = -\mu n(s) = -\mu \frac{T_{init}}{r} \left(1 + \frac{4}{B - \frac{C}{B + A + 1}}\right) e^{-\mu \bar{s}} .
\]

**Slack freespan:** In this zone, as in the tight freespan, the speed and the tension remain constant,
\[
T(s) = T_s = T_{init} \left(1 - \frac{A}{B}\right) , \quad v(s) = v_s = r\omega_2 = \left(\frac{B - A + 1}{B + A + 1}\right) r\omega_1 .
\]

**No-slip zone on the driven pulley:** The belt attaches to the driven pulley at the slack-span speed \( v_s = r\omega_2 \) of the pulley surface. The belt continues with this speed for a yet-to-be-determined distance on the pulley. As in the no-slip zone on the driving pulley, there is no friction between belt and pulley. Summarizing,
\[
T(s) = T_{init} \left(1 - \frac{A}{B}\right) , \quad v(s) = \left(\frac{B - A + 1}{B + A + 1}\right) r\omega_1 , \quad f(s) = 0 , \quad n(s) = \frac{T_{init}}{r} \left[1 - \frac{A}{B} - \left(\frac{B - A + 1}{B + A + 1}\right) \frac{C}{B}\right] .
\]

**Slip zone on the driven pulley:** In this zone the belt speed increases from \( r\omega_2 \) to \( r\omega_1 \). The belt is moving faster than the pulley surface speed \( r\omega_2 \), so that according the sign convention \( f = \mu n \). Combining this friction relation with equations (9) and (34) gives
\[
\frac{d(T - Gv)}{T - Gv} = \mu d\theta .
\]
Integrating over the slip zone yields (since the friction coefficients on both pulleys are the same) the same expression for the angle of slip \( \beta_2 \) on the driven pulley as was obtained in
eq. (155) for the angle $\beta_1$ on the driving pulley,

$$\beta_2 = \beta_1 = \frac{1}{\mu} \ln \left[ \frac{B + A - C}{B - A - \left( \frac{B - A + 1}{B + A + 1} \right) C} \right].$$

Within the slip zone of the driven pulley,

$$T(s) = T_{\text{init}} \left[ \left( 1 - \frac{A}{B} - \frac{\frac{C}{B}}{B + A + 1} \right) e^{\mu^{*} s} + \frac{\frac{C}{B}}{B + A + 1} \right],$$

$$v(s) = r \omega_1 \left[ \left( \frac{B - A + 1}{B + A + 1} - \frac{1}{B + A + 1} \right) e^{\mu^{*} s} + \frac{1}{B + A + 1} \right],$$

$$n(s) = \frac{T_{\text{init}}}{r} \left( 1 - \frac{A}{B} - \frac{B - A + 1}{B + A + 1} \frac{C}{B} \right) e^{\psi s^{*}},$$

$$f(s) = \mu \frac{T_{\text{init}}}{r} \left( 1 - \frac{A}{B} - \frac{B - A + 1}{B + A + 1} \frac{C}{B} \right) e^{\psi s^{*}},$$

where $s^{*} = s - 2l - \pi r - r(\pi - \beta_2)$ is the arclength measured from the start of the slip zone.

The maximum value $M_{\text{max}}$ of torque that can be transmitted by the extensible belt is found by setting the angles $\beta_1 = \beta_2$ of the slip zones equal to the maximum allowable value $\pi$ in eq. (155) and solving for $M_{\text{max}}$ which gives;

$$M_{\text{max}} = \frac{r T_{\text{init}}}{(e^{\mu^{*} + 1})} \left\{ (e^{\mu^{*}} + 1) \left( \frac{C}{B} \right) - 2 + \left\{ (1 - \frac{C}{B})^2 + 4 \left( 1 + \frac{1}{B} \right) \left( 1 - \frac{C}{B} \right) \right\} e^{\mu^{*} 2} \right. $$

$$\left. + \left\{ 2 \left( \frac{1 - C}{B} \right) \left( 2 + \frac{1}{B} - \frac{C}{B} \right) \right\} e^{\mu^{*}} + \left( \frac{1 + C}{B} \right)^2 \right\}^{1/2},$$

(A.2.2) Formulations neglecting stretching acceleration in the momentum equations

If the effect of centrifugal acceleration is included in the momentum equation, but not the tangential acceleration, i.e. if equations (35) are employed instead of equation 34, the mass conservation, momentum, and friction equations in the slip zones on the driving and driven pulleys reduce to

$$\frac{dT}{T - Gv} = -\mu d\theta , \quad \frac{dT}{T - Gv} = \mu d\theta ,$$

respectively, rather than equations (154) and (159). The modeling inconsistency of this approximation (assuming $dv = 0$ in the momentum equations but coupling change of speed with change of tension in the elastic constitutive equation) allows for two possible ways to interpret equations (162), and hence two different solutions.
A.2.2.1 The engineering solution

Consistent with the assumption in the momentum equation that change of speed $dv$ is negligible, the common engineering practice is to consider $v$ in eqs. (162) to be the surface speed $r\omega_1$ of the driving pulley. Integrating over the slip zone of the driving pulley the angle of slip is obtained,

\begin{equation}
\beta_1 = \frac{1}{\mu} \ln \left( \frac{T_1 - Gr\omega_1}{T_0 - Gr\omega_1} \right) = \frac{1}{\mu} \ln \left( \frac{B + A - C}{B - A - C} \right).
\end{equation}

Then, eq. (162) is a decoupled differential equation for the evolution of belt tension in the slip zone of the driving pulley, which integrates to give

\begin{equation}
T(s) = (T_i - Gr\omega_1)e^{-\mu \beta_1} + Gr\omega_1 = T_{init} \left[ (1 + \frac{A}{B} - \frac{C}{B}) e^{-\mu \beta_1} + \frac{C}{B} \right],
\end{equation}

where $s = s - l - r(\pi - \beta_1)$ is the arclength from the start of the slip zone. The belt normal force per length $n(s)$ and frictional force per length $f(s)$ are obtained algebraically from $T(s)$,

\begin{align*}
n(s) &= \frac{1}{r} (T(s) - Gr\omega_1) = \frac{T_{init}}{r} \left( 1 + \frac{A}{B} - \frac{C}{B} \right) e^{\mu \beta_1}, \\
f(s) &= -\mu n(s) = -\mu \frac{T_{init}}{r} \left( 1 + \frac{A}{B} - \frac{C}{B} \right) e^{\mu \beta_1}.
\end{align*}

If $v$ is also set to $v_i = r\omega_1$ in the slip zone of the driven pulley, one obtains

\begin{equation}
\beta_2 = \beta_1 = \frac{1}{\mu} \ln \left( \frac{B + A - C}{B - A - C} \right), \quad T(s) = T_{init} \left[ (1 - \frac{A}{B} - \frac{C}{B}) e^{\mu \beta_1} + \frac{C}{B} \right],
\end{equation}

\begin{align*}
n(s) &= \frac{T_{init}}{r} \left( 1 - \frac{A}{B} - \frac{C}{B} \right) e^{\mu \beta_1}, \\
f(s) &= \mu \frac{T_{init}}{r} \left( 1 - \frac{A}{B} - \frac{C}{B} \right) e^{\mu \beta_1},
\end{align*}

where $s^* = s - 2l - \pi r - r(\pi - \beta_2)$ is the arclength from the start of the slip zone. Note that belt elasticity $k$ does not appear in this solution unless one chooses to back out belt speed from the constitutive equation. (The problem for belt tension, normal force, and friction decouples from the constitutive equation; each of the non-dimensional parameters $A = \frac{M}{2\pi k}$, $B = \frac{T_{init}}{k}, C = \frac{Gr\omega_1}{k}$ have $k$ in their denominators, but solutions (164) and (A.2.2.1) depend only on the ratios $\frac{A}{B} = \frac{M}{2\pi T_{init}}$ and $\frac{C}{B} = \frac{Gr\omega_1}{T_{init}}$, so that there is no $k$ dependence.)

The maximum torque that can be transmitted according to this solution is

\begin{equation}
M_{max} = 2r T_{init} \left( 1 - \frac{C}{B} \right) \left( \frac{e^{\mu \pi} - 1}{e^{\mu \pi} + 1} \right) = 2r \left( T_{init} - Gr\omega_1 \right) \left( \frac{e^{\mu \pi} - 1}{e^{\mu \pi} + 1} \right).
\end{equation}
A.2.2.2 Alternate solution

Although eqs. (162) follow from setting \( \mathrm{d}v = 0 \) in the momentum equations, it is arguable that \( v \) in eqs. (162) can be considered as a dependent variable, related to tension \( T \) through the constitutive equation (151). With this viewpoint, eq. (162) becomes

\[
\frac{dT}{\left(1 - \frac{C}{B+A+1} \right)T - \frac{Gr\omega_1}{B+A+1}} = -\mu d\theta.
\]

Integrating this cumbersome expression over the slip zone produces the angle of slip on the driving pulley,

\[
\beta_1 = \frac{1}{\mu \left(1 - \frac{C}{B+A+1} \right)} \ln \left( \frac{B + A - \frac{C}{B+A+1}}{B - A - \frac{C}{B+A+1}} \right),
\]

and integrating to arbitrary location with the slip zone produces

\[
T(s) = T_{\text{init}} \left[ \left(1 + \frac{A}{B} - \frac{C}{B} \right) e^{-\mu (1-\frac{C}{B+A+1}) s} + \frac{C}{B+A+1} \right],
\]

where \( s = s - l - r(\pi - \beta_1) \) is the arc length from the start of the slip zone. Belt speed \( v(s) \), normal force per length \( n(s) \), and frictional force per length \( f(s) \) in the slip zone are then obtained algebraically from \( T(s) \) using the elastic constitutive equation, the normal projection of momentum, and the friction relation,

\[
v(s) = r\omega_1 \left[ \left(1 - \frac{1}{B+A+1} \right) e^{-\mu (1-\frac{C}{B+A+1}) s} + \frac{1}{B+A+1} \right];
\]

\[
n(s) = \frac{T_{\text{init}}}{r} \left(1 + \frac{A}{B} - \frac{C}{B} \right) e^{-\mu (1-\frac{C}{B+A+1}) s};
\]

\[
f(s) = -\mu \frac{T_{\text{init}}}{r} \left(1 + \frac{A}{B} - \frac{C}{B} \right) e^{-\mu (1-\frac{C}{B+A+1}) s}.
\]

The solutions for slip angle on the driven pulley, and belt tension, belt speed, and normal and frictional forces in the slip zone of the driven pulley are obtained in a similar fashion by integrating eq. (162)2,

\[
\beta_2 = \beta_1 = \frac{1}{\mu (1-\frac{C}{B+A+1})} \ln \left[ \frac{B+A - \frac{B+A+1-C}{C}}{B-A - \frac{B+A+1-C}{C}} \right];
\]

\[
T(s) = T_{\text{init}} \left[ \left(1 - \frac{A}{B} - \frac{C}{B} \right) e^{\mu (1-\frac{C}{B+A+1}) s} + \frac{C}{B+A+1} \right];
\]

\[
v(s) = r\omega_1 \left[ \left(1 - \frac{B-A+1}{B+A+1} \right) e^{\mu (1-\frac{C}{B+A+1}) s} + \frac{1}{B+A+1} \right];
\]

\[
n(s) = \frac{T_{\text{init}}}{r} \left[1 - \left( \frac{B-A+1}{B+A+1} \right) \frac{C}{B} \right] e^{\mu (1-\frac{C}{B+A+1}) s};
\]

\[
f(s) = -\mu \frac{T_{\text{init}}}{r} \left[1 - \left( \frac{B-A+1}{B+A+1} \right) \frac{C}{B} \right] e^{\mu (1-\frac{C}{B+A+1}) s}.
\]

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where $s^* = s - 2l - \pi r - r(\pi - \beta_2)$ is the arclength from the start of the slip zone.

The prediction of the maximum possible torque that can be transmitted is found by solving for $M_{\text{max}}$ after setting $\beta_1 = \beta_2 = \pi$ in (168);

\begin{equation}
M_{\text{max}} = 2r T_{\text{init}} \left[ \frac{\gamma(B+1) - \mu \pi (B+1-C)}{B(\mu \pi - \gamma)} \right],
\end{equation}

where $\gamma$ is the root of

\begin{equation}
2(B + 1) e^\gamma \gamma^2 + \mu \pi [(C - 3 - 2B) e^\gamma + 1 + C] \gamma + \mu^2 \pi^2 (e^\gamma - 1) = 0,
\end{equation}

that produces the least value of $M_{\text{max}}$.

### A.2.3 Capstan solution neglecting inertia in the momentum equations

The solution of Johnson [21], in which inertia is neglected in the momentum equations (i.e. equations (36) and (151) are solved instead of equations (34) and (151)), is recovered by setting $C = \frac{G r \omega_i}{k} = 0$ with $A = \frac{M}{2 \pi k}$ and $B = \frac{T_{\text{init}}}{k}$ finite in either the full solution of section A.2.1, the engineering solution of section A.2.2.1, or the alternate solution of section A.2.2.2; all three collapse to the same solution. (Alternatively, the solution can be obtained by setting $\frac{C}{B} = \frac{G r \omega_i}{k T_{\text{init}}} = 0$, with $A = \frac{M}{2 \pi T_{\text{init}}}$ and $B^{-1} = \frac{k}{T_{\text{init}}}$ finite.) In this formulation the predictions of behavior in the two free spans are the same as those in the previous solutions, but the prediction in all zones on the pulley surfaces are altered. For instance, in the absence of inertia the momentum and friction relations in the driving pulley slip zone reduce to the capstan equation,

\begin{equation}
\frac{dT}{T} = -\mu d\theta,
\end{equation}

the slip angles are given by

\begin{equation}
\beta_1 = \beta_2 = \frac{1}{\mu} \ln \left( \frac{B + A}{B - A} \right),
\end{equation}

and the maximum moment $M_{\text{max}}$ that can be transmitted is

\begin{equation}
M_{\text{max}} = 2r T_{\text{init}} \left( \frac{e^{\mu \pi} - 1}{e^{\mu \pi} + 1} \right).
\end{equation}
A.2.4 Comparison

Four solutions to the torque transmission problem have been presented:

- the full solution (developed in section A.2.1), which for the first time includes the
effect of stretching acceleration in the momentum equations;

- two solutions which neglect stretching acceleration in the momentum equations (con-
tained in sections A.2.2.1 and A.2.2.2, and referred to as the engineering solution and
alternate solution, respectively); and

- the solution that neglects all inertia contributions in the momentum equations (con-
tained in section A.2.3 and referred to as the capstan solution).

First, note that neither of the two solutions which include centrifugal acceleration and
neglect stretching acceleration correspond to a specialization of the full solution to small
$C = \frac{Gr\omega_1}{k}$. These solutions follow from keeping the $C$ terms in some of the governing
equations (specifically the normal projection of momentum and, in the alternate solution, the
constitutive equation) but dropping the $C$ terms from others (e.g., the tangential projection
of momentum) before solving them, and hence do not represent some limit of the full
solution. As noted in section A.2.3, setting $C = \frac{Gr\omega_1}{k} = 0$ with $A = \frac{M}{2\pi k}$ and $B = \frac{T_{init}}{k}$
finite (neglecting inertia with respect to elastic stiffness), or setting $\frac{C}{B} = \frac{Gr\omega_1}{T_{init}} = 0$ with
$B^{-1} = \frac{k}{T_{init}}$ and $A = \frac{M}{2\pi T_{init}}$ finite (neglecting inertia with respect to initial belt tension) in
the full, engineering, and alternate solutions reduces all three to the capstan solution.

To examine the differences between the predictions of the four formulations for nonzero
values of $C = \frac{Gr\omega_1}{k}$ and $\frac{C}{B} = \frac{Gr\omega_1}{T_{init}}$, two cases with the same initial tension $T_{init} = 50$ N,
pulley radius $r = 0.05$ m, driving pulley angular velocity $\omega_1 = 500$ rad/s, belt mass flow
rate $G = 0.5$ kg/s, and coefficient of friction $\mu = 0.6$, but with differing elastic moduli $k$ are
considered. The modulus $k = 25$ kN in the first case corresponds to a stiff belt; the value
$k = 0.2$ kN of the second case is much smaller, approaching that of a textile tow in a drawing
process. The dimensionless combinations are $B = \frac{T_{init}}{k} = 2 \times 10^{-3}$, $C = \frac{Gr\omega_1}{k} = 5 \times 10^{-4}$
for the stiff belt case ($k = 25$ kN) and $B = \frac{T_{init}}{k} = 2.5 \times 10^{-1}$, $C = \frac{Gr\omega_1}{k} = 6.25 \times 10^{-2}$
for the compliant belt case \((k = 0.2 \text{ kN})\). In both cases the length \(L\) of each free span is taken to be \(\pi r\), so that the length of the total belt circuit is \(2\pi r + 2L = 4\pi r = 0.2\pi\) m (see figure 115).

The maximum moments \(M_{\max}\) that can be transmitted in either case and the corresponding dimensionless critical value \(A_{\max} = \frac{M_{\max}}{2\pi k}\), as predicted by the four solutions, are given in Table 6. The capstan solution (170) gives the same value \(M_{\max} = 3.682\) Nm for both \(k = 25\) kN and \(k = 0.2\) kN since extensibility effects are decoupled from the momentum equations in this approximation; in both cases the capstan solution severely over predicts the maximum possible moment, by 33\% and 23\%, respectively. When centrifugal acceleration is included in the formulation, but not stretching acceleration (the engineering and alternate solutions) the maximum moment is under predicted. For stiff belts relative to inertia or initial tension (small values of \(C = \frac{G r w}{k}\) or \(C = \frac{G r w}{T_{\text{init}}}\)) the error is slight, in agreement with the common prejudice in engineering practice, but as the belt becomes relatively more compliant the error due to neglecting stretching acceleration increases: The errors are 0.11\% and 0.14\% for \(k = 25\) kN, and 7.8\% and 2.8\% for \(k = 0.2\) kN in the engineering and alternate solutions, respectively. Note that the alternate solution is worse than the engineering solution in the stiff belt case, but better than the engineering solution for the compliant belt.

Table 6 also displays the slip angles \(\beta_1 = \beta_2\) on the driving and driven pulleys for \(M = 2.0\) Nm \((A = \frac{M}{2\pi k} = 8 \times 10^{-4}\) for the stiff belt case and \(A = \frac{M}{2\pi k} = 1 \times 10^{-1}\) for the compliant belt case), a value selected so as to be less than the maximum moment \(M_{\max}\) that can be transmitted for either value of \(k\), as predicted by all four solutions. Figures 116, 117, 118, and 119 display the four predictions of belt tension, belt speed, normal force per length, and frictional force per length as functions of arclength \(s\) for the complete circuit of the belt, for this specified subcritical moment. For the stiff belt the predictions of the engineering and alternate solutions are graphically indistinguishable from those of the full solution, whereas the capstan solution neglecting all effects of inertia significantly under predicted the lengths of the slip zones and significantly over predict the strain rate of the belt and the normal and frictional forces on the belt in the slip zones. In the compliant belt
Table 6: Maximum moment $M_{max}$ that can be transmitted by the stiff ($k = 25$ N) and compliant ($k = 0.2$ N) belts and corresponding critical values of the dimensionless parameter $A = \frac{M}{2r k}$, and the slip angles $\beta_1 = \beta_2$ for the subcritical moment $M = 2.0$ Nm, as predicted by the four solutions.

<table>
<thead>
<tr>
<th>Solution</th>
<th>$k = 25$ kN</th>
<th>$k = 0.2$ kN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_{max}$ (Nm)</td>
<td>$A_{max} = \frac{M_{max}}{2r k}$</td>
</tr>
<tr>
<td>Full (sec. A.2.1)</td>
<td>2.764</td>
<td>1.106 x 10^{-3}</td>
</tr>
<tr>
<td>Engineering (sec. A.2.2.1)</td>
<td>2.761</td>
<td>1.104 x 10^{-3}</td>
</tr>
<tr>
<td>Alternate (sec. A.2.2.2)</td>
<td>2.760</td>
<td>1.104 x 10^{-3}</td>
</tr>
<tr>
<td>Capstan (sec. A.2.3)</td>
<td>3.682</td>
<td>1.473 x 10^{-3}</td>
</tr>
</tbody>
</table>
case, the engineering and alternate solutions depart noticeably from the full solution as well. On the driving pulley the errors are in the opposite directions of the inertia-less capstan solution, over predicting the slip zones and under predicting strain rates and frictional and normal forces. On the driven pulley the errors are more complicated to describe: Both the engineering and alternate solutions over predict the extent of the slip zone, and in this zone the alternate solution over predicts the normal and frictional forces; the engineering solution predictions for normal and frictional forces coincide with the full solution where the zones coincide. Since the engineering solution incorrectly sets the belt speed in the no-slip zone on the driven pulley to be the surface speed of the driving pulley, it over predicts the normal force there.

A.3 Discussion

The quantitative differences just observed between the solution including stretching acceleration in the momentum equations and the solutions neglecting stretching acceleration increase as the stiffness of the belt decreases and the speeds increase. The equations that were derived in this chapter are also applicable to the drawing process in the manufacturing of polymer fibers and films. In these processes the fiber or film is routed through a series of rollers, each with faster surface speeds than the one before. Much if not most of the stretching (i.e. draw) can occur on the roller surfaces. In applications of the equations of this chapter to fiber and film drawing, the pulleys become the rollers and the belt becomes the fiber or film. Stiffnesses are much less and speeds usually much greater than in the torque transmission problem considered here, and it will be essential to use equations (34) rather than equations (35) or (36).
Figure 116: Belt tension $T(s)$ as a function of arc length $s$ for a stiff belt ($k = 25\text{kN}$, top) and a compliant belt ($k = 0.2\text{kN}$, bottom): Circles (○) indicate locations of attachment to the pulleys, and boxes (□) indicate locations of departure. Full solution (——), engineering solution (−−−−), alternate solution (· · · · ·), and capstan solution (−−−−). The full, engineering, and alternate solutions are indistinguishable for the stiff belt case.
Figure 117: Belt speed \( v(s) \) as a function of arc length \( s \) a stiff belt \((k = 25\, \text{kN}, \text{top})\) and a compliant belt \((k = 0.2\, \text{kN}, \text{bottom})\): Circles (○) indicate locations of attachment to the pulleys, and boxes (□) indicate locations of departure. Full solution (——), alternate solution (····), and capstan solution (−−−−). The full and alternate solutions are indistinguishable for the stiff belt case.
Figure 118: Normal force per unit length \( n(s) \) from the pulleys on the belt for a stiff belt \( (k = 25\text{kN}, \text{top}) \) and a compliant belt \( (k = 0.2\text{kN}, \text{bottom}) \): Full solution (-----), engineering solution (-- --), alternate solution (⋅⋅⋅⋅⋅), and capstan solution (-----). The full, engineering, and alternate solutions are indistinguishable for the stiff belt case.
Figure 119: Frictional force per unit length $f(s)$ from the pulleys on the belt for stiff belt ($k = 25$kN, top) and a compliant belt ($k = 0.2$kN, bottom): Full solution (---), engineering solution (-- --), alternate solution (-----), and capstan solution (----). The full, engineering, and alternate solutions are indistinguishable for the stiff belt case.
REFERENCES


