FINITE FIELD MODELS OF ROTH'S THEOREM IN ONE AND TWO DIMENSIONS

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Derrick N. Hart

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Approved by:

Professor Michael Lacey, Committee Chair
School of Mathematics
Georgia Institute of Technology

Professor Prasad Tetali
School of Mathematics
Georgia Institute of Technology

Professor William Green
School of Mathematics
Georgia Institute of Technology

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SUMMARY

Recent work on many problems in additive combinatorics, such as Roth's Theorem, has shown the usefullness of first studying the problem in a finite field enviroment. Then using the techniques of Bourgain [2] to give a result in other settings such as general abelian groups. The author gives a walk through, including proof, of Roth's theorem in both the one dimensional and two dimensional cases (it would be more accurate to refer to the two dimensional case as Shkredov's Theorem). In the one dimensional case the argument is at its base Meshulam's [8] but the structure will be essentially Green's [5]. Let $\mathbb{F}_p^n, p \neq 2$ be the finite field of cardinality $N = p^n$. For large N, any subset $A \subset \mathbb{F}_p^n$ of cardinality

$$|A| \gtrsim \frac{N}{log\ N}$$

must contain a triple of the form $\{x, x+d, x+2d\}$ for $x, d \in \mathbb{F}_p^n, d \neq 0$. In the two dimensional case the argument is Lacey and McClain [7] who made considerable refinements to this argument of Green [5] who was bringing the argument to the finite field case from a paper of Shkredov [10]. Let $\mathbb{F}_2^n$ be the finite field of cardinality $N = 2^n$. For all large N, any subset $A \subset \mathbb{F}_2^n \times \mathbb{F}_2^n$ of cardinality

$$|A| \gtrsim N^2 (\log n)^{-\epsilon}\,, \qquad \epsilon <, 1\,,$$

must contain a corner $\{(x,y)\,, (x+d,y)\,, (x,y+d)\}$ for $x, y, d \in \mathbb{F}_2^n$ and $d \neq 0$.

CHAPTER I

INTRODUCTION

1.1 Statement of Purpose

Here we seek to give proofs of finite field models of Roth's theorem in both one and two dimensions. The exact statements and a full background will be given in the next section. In the one dimensional case the argument given will be at its base Meshelaum's [8] but the structure will be essentially Green's [5]. In the two-dimensional case the argument given will be that of Lacey and McClain [7] which is an expansion and improvement on Green's[5] exposition on the finite field model of an argument of Shkredov[10].

1.2 Background

In 1936, Erdős and Turán [3], asked the question – How large can a subset of $1, ..., N$ be and not contain a three term arithmetic progression? This was finally answered by Roth [9] in 1946. More precisely, Roth stated that letting $r_3(N)$ be the cardinality of the above implied set then $r_3(N) \ll \frac{N}{\text{loglog}N}$. This bound essentially survived for 40 years until Heath-Brown [6] and Szemeredi [12] working independently improved this to $r_3(N) \ll \frac{N}{(\text{log}N)^c}$. In 1999 Bourgain [2] brought us to what is currently the best known bound of $r_3(N) \ll N\left(\frac{\text{loglog}N}{\text{log}N}\right)^{1/2}$.

Bourgain's approach involved converting the argument from finite field models(in which the subject is somewhat cleaner) to the group $\mathbb{Z}_N$. Finite fields have a subspace structure and many other features that $\mathbb{Z}_N$ and abelian groups in general are lacking. Bourgain ingeniously managed to use an approximate of this structure to move to $\mathbb{Z}_N$. We will not be concerned with this method here and will stay squarely in the finite field world.

It is quite natural to ask as Gowers does in [4] whether a multidimensional version yields sensible bounds. For three dimensions and beyond the question is still open. In the case of two dimensions one is concerned with corners, i.e. triples of the form $(x, y), (x + d, y), (x, y + d)$. We may define the corresponding quantity for the size of the threshold set of $\mathbb{Z}_N \times \mathbb{Z}_N$ to be $r_\angle(\mathbb{Z}_N)$. In 1974 Ajtai and Szemerédi [1] gave $r_\angle(\mathbb{Z}_N) \ll N^2$. Vu in 2002 [13] and Solymosi [11] in 2003 gave bounds of the form $\frac{N}{(\log_* N)^c}$. Here $\log_* N$ is defined as the largest integer k such that $\log_{[k]} N \leq 2$ where $\log_{[l]} N = \log(\log_{[l-1]} N)$, i.e. inverse tower type. Shkredov [10] gave $r_\angle(\mathbb{Z}_N) \ll \frac{N}{(\text{logloglog} N)^c}$ in 2004 and refined his argument to $r_\angle(\mathbb{Z}_N) \ll \frac{N}{(\text{loglog} N)^c}$ in 2005[10], where c may be taken to be 1/73.

Green in [5] gave Shkredov's argument through the lense of the field model $\mathbb{F}_2^n$ and provided the result $r_\angle(\mathbb{F}_2^n) \ll \frac{N^2}{(\text{loglog} N)^c}$, where c may be taken to be 1/25. Very recently Lacey and McClain [7] have made considerable refinements to this argument giving a very nice bound of $r_\angle(\mathbb{F}_2^n) \ll \frac{N^2}{(\log N)^c}$ for $c < 1$. It is this argument that will be the principle subject of this document.

1.3 Preliminaries and Definitions

Let $\mathbb{F}_p^n$ be the finite field of cardinality p^n and let $H \subset \mathbb{F}_p^n$ denote a subspace. For a set A we define the characteristic or indicator function to be $A(x)$. We adopt the notations of probability with respect to the Haar measure on H. That is for a real valued function f we write

$$\mathbb{E}_{x\in H} f(x) := |H|^{-1} \sum_{x\in H} f(x), \tag{1.3.1}$$

$$\mathbb{V}_{x\in H} f(x) := \mathbb{E}\left[(X - \mathbb{E}[X])^2\right], \tag{1.3.2}$$

for the expectation and variance respectively; and

$$\delta_A(H) := \mathbb{E}_{x\in H} (A \cap H)(x) = \frac{|A \cap H|}{|H|}, \tag{1.3.3}$$

for the conditional density. If H is clear from context we will abuse notation and write δ_A for the sake of brevity.

The Fourier transform of a function $f : \mathbb{F}_n^p \to \mathbb{C}$ defined

$$\widehat{f}(\xi) := \mathbb{E}_{x\in H} g(x)\omega^{x\cdot\xi}, \tag{1.3.4}$$

where ω is a pth root of unity, will be central to our discussion. For two functions $f : \mathbb{F}_n^p \to \mathbb{C}$ and $g : \mathbb{F}_n^p \to \mathbb{C}$ we define convolution to be

$$(f * g)(d) := \mathbb{E}_{x\in H} f(x)g(d - x). \tag{1.3.5}$$

We will also refer to the L_p norms of a function and here we will mean

$$\|f\|_p = (\mathbb{E}_{x\in H}|f(x)|^p)^{1/p} \tag{1.3.6}$$

with the L_∞ norm being the implied sup norm. The orthogonality property of the Fourier Transform, i.e. that $\mathbb{E}_{x\in H}\omega^{x\cdot\xi}$ is equal to 1 for $\xi, x = 0$ and 0 otherwise yields many standard properties of the Fourier Transform. We summarize some of the properties of the Fourier Transform in the following note.

Lemma 1.3.7 (The Fourier Transform)**.** *Let* $f, g : \mathbb{F}_p^n \to \mathbb{C}$.

$$\hat{f}(0) = \mathbb{E}_{x\in H} f(x), \tag{1.3.8}$$

$$(\textit{Plancherel})\ \mathbb{E}_{x\in H} f(x)\overline{g(x)} = \sum_{\xi\in H} \hat{f}(\xi)\overline{\hat{g}(\xi)}, \tag{1.3.9}$$

$$(\textit{Inversion}) \quad f(x) = \sum_{\xi\in H} \hat{f}(\xi)\omega^{-x\cdot\xi}, \tag{1.3.10}$$

$$(\textit{Convolution})\ \widehat{(f * g)}(x) = \hat{f}(\xi)\hat{g}(\xi). \tag{1.3.11}$$

An important concept of this paper will be that of the "uniformity" of a set with respect to a subspace. The particular representation of this property that will concern us will be denoted

$$\|A\|_{\mathrm{Uni}} := \sup_{\xi\neq 0}|\widehat{A}(\xi)|\,. \tag{1.3.12}$$

If $\|A\|_{\mathrm{Uni}} \leq \eta$ then we say that A is *η–uniform.*

CHAPTER II

LEMMATA

2.1 *Deviation over a Partition*

A set A being highly uniform says very little about the density of A in H. However, it does tell us about the density of A in the members of a partition of H into affine subspaces. More specifically, it says a set being large in uniform norm implies that the density of A in the cosets of a hyperplane have a high average deviation in both the L_1 and L_2 sense.

Lemma 2.1.1 (Moments)**.** *Let H be a subspace of $\mathbb{F}_p^n$. Suppose that $A \subseteq H$. Consider $H' = \langle \xi \rangle^{\perp}$ the hyperplane taken with respect to H. Also enumerate the cosets H'_i with $i = 0, 1, ..., p-1$. We have*

$$\mathbb{E}_{i \in \mathbb{Z}_p} |\delta_A(H'_i) - \delta_A| \geq \|A\|_{\text{Uni}}, \tag{2.1.2}$$

$$\mathbb{V}_{i \in \mathbb{Z}_p} \delta_A(H'_i) \geq \|A\|^2_{\text{Uni}} . \tag{2.1.3}$$

Proof. (2.1.2) Let

$$\Delta_i = \delta_A(H'_i) - \delta_A.$$

Then partitioning the Fourier Tranform,

$$\widehat{A}(\xi) = \mathbb{E}_{x \in H} A(x) \omega^{x \cdot \xi} = |H|^{-1} \sum_i \sum_{x \in H'_i} A(x) w^{x \cdot \xi} = \widehat{\Delta}_i.$$

Giving us

$$\mathbb{E}_i |\Delta_i| \geq |\widehat{\Delta}_i| = \|A\|_{\text{Uni}}.$$

□

Proof. (2.1.3) First

$$\mathbb{E}_{i \in \mathbb{Z}_p} [\delta_A(H'_i)^2] = \tfrac{1}{p} \sum_{i \in Z_p} \tfrac{|A \cap H'_i|^2}{|H'_i|^2} = p^2 \mathbb{E}_{x \in H} (A * H')^2(x),$$

then from Plancherel,

$$= p^2 \sum_{\alpha} |\widehat{A}(\alpha)|^2 |\widehat{H'}(\alpha)|^2 \geq p^2(|\widehat{A}(0)|^2|\widehat{H'}(0)|^2 + |\widehat{A}(\xi)|^2|\widehat{H'}(\xi)|^2) =$$

$$\delta_A^2 + \|A\|_{\text{Uni}}^2.$$

Noting that $\mathbb{E}_{i\in\mathbb{Z}_p}\delta_A(H_i') = \delta_A$ we have that,

$$\mathbb{V}_{i\in\mathbb{Z}_p}\delta_A(H_i') = \mathbb{E}_{i\in\mathbb{Z}_p}\delta_A(H_i')^2 - \delta_A^2 \geq \|A\|_{\text{Uni}}^2.$$

□

2.2 *Uniform Convolution*

When counting arithmetic progression we are looking at when an element a a translate are in a set. Leading us quite naturally to convolutions. We seek some information about the control that uniformity of a set places on the the deviation of a convolution. We would certainly like to do this for the mean deviation. However, a direct result is not available and so we will first find bounds on the standard deviation.

The first result which may be found in [5] is:

Proposition 2.2.1. *Let $A \subset H$ and f be a function on H. Then*

$$\mathbb{V}_{d\in H}(f * A)(d) \leq \|A\|_{\text{Uni}}^2 \left[\mathbb{E}_x f(x)^2\right] \tag{2.2.2}$$

Proof. Using Plancherel,

$$\mathbb{E}_{d\in H}|(f*A)(d) - \delta_A \mathbb{E}_x f(x)|^2 = |H|^{-1}\mathbb{E}_\alpha|\widehat{A}(\alpha)\widehat{f}(\alpha) - \delta_A\mathbb{E}_x f(x)|^2 =$$

and then uniformity,

$$\sum_{\alpha\neq 0} |\widehat{A}(\alpha)|^2|\widehat{f}(\alpha)|^2 \leq \|A\|_{\text{Uni}}^2 \mathbb{E}_x f(x)^2.$$

□

A more complicated variant will allow us to deal with more entangled sums. Here we concern ourselves not with measuring the deviation from the mean but the deviation from the reconstructed average along the diagonal in the frequency domain. While this may not be immediately apparent it becomes natural after considering what type of uniformity is involved in two dimensional convolutions of the following type.

Proposition 2.2.3. *Let A be uniform and let f and g be functions on $\mathbb{F}_p^n$. Define*

$$\mathrm{L}(x) = \delta_A \sum_{\alpha \in H} \hat{f}(\alpha)\hat{g}(\alpha)\omega^{\alpha \cdot x}$$

and

$$\mathrm{B}(x, y) = \mathbb{E}_{s \in H} f(s + x)g(y - s)A(s)$$

then we have that,

$$\|\mathrm{B}(x, y) - \mathrm{L}(x + y)\|_2 \leq \|A\|_{\mathrm{Uni}}\|f\|_2\|g\|_2, \tag{2.2.4}$$

$$|\mathbb{E}_{x \in H} L(x)A(x) - \delta_A^2 \mathbb{E}_{x,y \in H} f(x)g(y)| \leq \delta_A\|A\|_{\mathrm{Uni}}\|f\|_2\|g\|_2. \tag{2.2.5}$$

Proof. (2.2.4) Expanding

$$\mathrm{B}(x, y) = \sum_{\alpha,\beta} \hat{f}(\alpha)\hat{g}(\beta)\omega^{\alpha \cdot x + \beta \cdot y} \mathbb{E}_s A(s)\omega^{(\alpha - \beta) \cdot s} =$$

$$\sum_{\alpha,\beta} \hat{f}(\alpha)\hat{g}(\beta)\omega^{\alpha \cdot x + \beta \cdot y} \widehat{A}(\alpha - \beta)\,.$$

Then we have $\widehat{\mathrm{B}}(\alpha, \beta) = \hat{f}(-\alpha)\hat{g}(-\beta)\widehat{A}(\alpha - \beta)$. Therefore from Plancherel,

$$\mathbb{E}_{x,y \in H}\left|\mathrm{B}(x, y) - \mathrm{L}(x + y)\right|^2 = \sum_{\alpha,\beta \in H} \left|\widehat{\mathrm{B}}(\alpha, \beta) - \delta_A \hat{f}(-\alpha)\hat{g}(-\alpha)\right|^2 =$$

$$\sum_{\alpha \neq \beta} \left|\hat{f}(-\alpha)\hat{g}(-\beta)\widehat{A}(\alpha - \beta)\right|^2 \leq \|A\|_{\mathrm{Uni}}^2 \left[\mathbb{E}_{x,y} f(x)^2 g(y)^2\right].$$

□

Proof. (2.2.5)

$$|\mathbb{E}_{x \in H} L(x)A(x) - \delta_A^2 \mathbb{E}_{x,y \in H} f(x)g(y)| = \sum_{\alpha \neq 0} |\hat{L}(\alpha)\hat{A}(\alpha)| \leq$$

$$\delta_A\|A\|_{\mathrm{Uni}}\mathbb{E}_\alpha|\hat{f}(\alpha)\hat{g}(\alpha)| \leq \delta_A\|A\|_{\mathrm{Uni}}\|f\|_2\|g\|_2$$

□

CHAPTER III

ROTH'S THEOREM IN FINITE FIELDS

3.1 Introduction

Roth's Theorem in Finite Fields states the following:

Theorem 3.1.1 (Roth's Theorem)**.** *We have* $r_3(\mathbb{F}_p^n) \ll \frac{N}{log\ N}$.

Our method of proof is as follows. We find a "Generalized Von Neumann"[1] lemma that states if our set A is sufficiently uniform it contains a nontrivial 3-term arithmetic progression assuming the pertinent subspace was initially large enough. If A is not originally uniform then Lemma 2.1.2 states that there exists a subspace on which A has increased density. Taking our subspace of increased density we apply this procedure iteratively. The density increase will be great enough that eventually the loop must terminate, else the density would exceed 1. At this point our subspace of A is uniform and contains a three term arithmetic progression. Counting the number of iterations necessary and imposing the original size condition yields Roth's Theorem.

3.2 Guaranteeing a 3-term AP

For a function $f, g, h : \mathbb{F}_p^n \to \mathbb{R}$ define a trilinear form for counting triples to be

$$\mathrm{T}_1(f, g, h) \stackrel{\text{def}}{=} \mathbb{E}_{x,s\in H} f(x)g(x+s)h(x+2s). \qquad (3.2.1)$$

The expected density of 3-term arithmetic progressions in a subspace H is given by $T(A, A, A)$. In the following lemma we seek to place a control on the density of triples within the context of uniformity.

Lemma 3.2.2 (Counting Lemma)**.** *Let* H *be a subspace of* $\mathbb{F}_p^n, p > 2$ *and* $A \subseteq H$. *Then the density of corners in* A *has the bound*

$$|\mathrm{T}_1(A, A, A) - \delta_A^3| \leq \|A\|_{\mathrm{Uni}}\delta_A. \qquad (3.2.3)$$

[1]This term, coined by Tao, is used to show the connections of theorems of this ilk to Ergodic theory.

Proof. We use the orthogonality of characters to move to the frequency domain,

$$|\mathrm{T}_1(A,A,A)-\delta_A^3| \leq |\mathbb{E}_{x,s\in H}A(x)A(x+s)A(x+2s)-\delta_A^3| = \tag{3.2.4}$$

$$\sum_{\alpha,\beta,\gamma\neq 0}|\hat{A}(\alpha)\hat{A}(\beta)\hat{A}(\gamma)|\mathbb{E}_{x,s\in H}\omega^{x\cdot(\alpha+\beta+\gamma)+s\cdot(\beta+2\gamma)} = \tag{3.2.5}$$

$$\sum_{\alpha\neq 0}|\hat{A}(\alpha)|^2|\hat{A}(2\alpha)| \leq \|A\|_{\mathrm{Uni}}\delta_A. \tag{3.2.6}$$

□

Now how large must H be to guarantee A has a 3-term arithmentic progression? The answer to this question is a simple corollary to (3.2.2).

Lemma 3.2.7 (Generalized Von Neumann)**.** *If in addition to the assumptions of (3.2.2) we have that A is η-uniform and $|H| > (\delta_A^2-\eta)^{-1}$ then A contains a* 3*-term arithmetic progression.*

Proof. The expected number of 3-term arithmetic progressions from (3.2.2) is $\geq \delta_A(\delta_A^2 - \eta)|H|^2$. This counts trivial 3-term progressions(points) as well of which there are $\delta_A|H|$. Hence we need $\delta_A|H| \leq \delta_A(\delta_A^2-\eta)|H|^2$. □

3.3 Roth's Theorem

Proof of 3.1.1. Let $A \subseteq \mathbb{F}_p^n$ with density $\delta_0 = \delta_A(\mathbb{F}_p^n)$ and $\kappa \in (0,1)$.

Initialize $A' \leftarrow A, H \leftarrow \mathbb{F}_p^n, \delta \leftarrow \delta_0$.

1. If A' is $(\kappa\delta^2)$-uniform then `STOP`.
2. If A' is not $(\kappa\delta^2)$-uniform then apply Lemma 2.1.2 to find a hyperplane $H' \leqslant H$ and $x \in H$ such that $|A' \cap (H'+x)|/|H'| > \delta + \kappa\delta^2$.
3. Update variables: $A' \leftarrow (A'-x)\cap H', \quad H \leftarrow H', \quad \delta \leftarrow \delta + \kappa\delta^2$.

This loop must be finite. Observe that after $\kappa^{-1}\delta_0^{-1}$ iterations the density doubles and after $\kappa^{-1}\delta_0^{-1}/2$ it quadruples, i.e. the density grows dyadically as the number of iterations grows geometrically. Proceeding this way δ must exceed one in $\lesssim \delta_0^{-1}$ steps. If N was large enough to guarantee that $|H| > (1-\kappa)^{-1}\delta^{-2}$ then we may apply the Generalized Von

Neumann (3.2.7) to show that A' contains a 3-term progression and hence A does. Since $|H|$ is now greater than $p^{-\delta_0^{-1}}N$ at this point we must have the condition $\delta_0 \gtrsim (logN)^{-1}$. $\square$

CHAPTER IV

TWO DIMENSIONAL ROTH'S THEOREM IN FINITE FIELDS

4.1 Introduction

One is tempted to think that two dimensional Roth may be solved in some sense by generalizing the one dimensional argument. However, there is no way to say that uniformity along one dimensional fibers causes a product set to be uniform in a way to guarantee a corner. With even less hope, in the case of non-uniform set, that we may pass to a sublattice that either has increased density and that acts nicely enough to yield a iteration argument. There simply is not enough control on products of independent sets to extract information about a two dimensional structure. Hence we will need to define a more apt version of two dimensional uniformity and use this to develop a Generalized Von Neumann which is the subject of the following section.

4.2 2D Generalized Von Neumann

We let $H \subset \mathbb{F}_2^n$ denote a subspace and $X, Y, D \subset H$. Define

$$S \stackrel{\text{def}}{=} X \times Y \cap X \stackrel{\text{diag}}{\times} D. \tag{4.2.1}$$

The first cartesian product is taken with respect to the to basis $(\mathrm{e}_1, \mathrm{e}_2)$and the second with respect to $(\mathrm{e}_1, \mathrm{e}_1 + \mathrm{e}_2)$. Hence the inspiration for the letter D for the 'diagonal' coordinate.

Define as in the one dimensional case an appropriate trilinear form to be

$$\mathrm{T}_2(f, g, h) \stackrel{\text{def}}{=} \mathbb{E}_{x,y,s \in H} f(x, y)g(x, y + s)h(x + s, y) = \mathbb{E}_{x,y,s \in H} f(x, y)g(x, x + s)h(y + s, y).$$

Uniformity will not provide enough control for us in this case. So we define some new norms. For a function $f : S \to \mathbb{R}$, define the following norm

$$\|f\|_{\square}^4 \stackrel{\text{def}}{=} \delta_D^{-4} \mathbb{E}_{\substack{x,x' \in X \\ y,y' \in Y}} f(x, y)f(x', y)f(x, y')f(x', y') \tag{4.2.2}$$

where we use the standard basis $(\mathrm{e}_1, \mathrm{e}_2)$. However, as we are looking at corners there are two more natural basis perspectives to consider. In the $(\mathrm{e}_1, \mathrm{e}_1 + \mathrm{e}_2)$ coordinate system we define the norm,

$$\|f\|_{\square,X}^4 := \delta_Y^{-4}\mathbb{E}_{\substack{x,x'\in X\\ d,d'\in D}} f(x,d)f(x',d)f(x,d')f(x',d')\,. \tag{4.2.3}$$

In the $(\mathrm{e}_2, \mathrm{e}_1+\mathrm{e}_2)$ coordinate system we define the norm,

$$\|f\|_{\square,Y}^4 := \delta_X^{-4}\mathbb{E}_{\substack{y,y'\in Y\\ d,d'\in D}} f(y,d)f(y',d)f(y,d')f(y',d')\,. \tag{4.2.4}$$

These norms will help us devise a two dimensional counting lemma.

Lemma 4.2.5 (Counting Lemma)**.** *Suppose that $A \subset S$. Let X, Y, D be η-uniform, f is the balanced function of A, and* $\max\{\|f\|_\square\,, \|f\|_{\square,X}\,, \|f\|_{\square,Y}\} \le \kappa\delta_A^{5/4}$. *Then*

$$\mathrm{T}_2(A,A,A) = O(\eta') + \kappa'\delta_A^3(\delta_X\delta_Y\delta_D)^2, \tag{4.2.6}$$

where η' is a function of η that will be unimportant.

Proof. First lets use a balanced function expansion of $f = A+\delta S$ to expand the appropriate quantity into manable parts,

$$\begin{aligned}
\mathrm{T}_2(A,A,A) = {} & \delta_A^3\,\mathrm{T}_2(S,S,S) && (4.2.7)\\
& + \delta_A^2\,\mathrm{T}_2(f,S,S) + \delta_A^2\,\mathrm{T}_2(S,f,S) + \delta_A^2\,\mathrm{T}_2(S,S,f) && (4.2.8)\\
& + \delta_A\,\mathrm{T}_2(f,f,S) + \delta_A\,\mathrm{T}_2(S,f,f) + \delta_A\,\mathrm{T}_2(f,S,f) && (4.2.9)\\
& + \mathrm{T}_2(f,f,f)\,. && (4.2.10)
\end{aligned}$$

For (4.2.7) we use Proposition 2.2.3,

$$\begin{aligned}
\mathrm{T}_2(S,S,S) = {} & \mathbb{E}_{x,s,y\in H} S(x,y)S(x+s,y)S(x,y+s) = && (4.2.11)\\
& \mathbb{E}_{x,s,y\in H} X(x)Y(y)X(x+s)Y(y+s)D(x+y)D(s) = && (4.2.12)\\
& \mathbb{E}_{x,y\in H} X(x)Y(y)D(x+y)\,\mathrm{L}(x+y) + \mathrm{O}(\eta') && (4.2.13)\\
& \delta_X\delta_Y\mathbb{E}_{x\in H} D(x)\,\mathrm{L}(x) + \mathrm{O}(\eta') && (4.2.14)\\
& \delta_D^2\delta_X^2\delta_Y^2 + \mathrm{O}(\eta')\,. && (4.2.15)
\end{aligned}$$

Now for the first term in (4.2.8)

$$\begin{aligned}
\mathrm{T}_2(S,S,f) =&\mathbb{E}_{x,y,s\in H} X(x)Y(x+s)D(x+y)f(y,y+s) = && (4.2.16)\\
&\delta_X \mathbb{E}_{x,y,s\in H} Y(x+s)D(x+y)f(y,y+s) + \mathrm{O}(\eta') = && (4.2.17)\\
&\delta_X\delta_Y\delta_D \mathbb{E}_{y,s\in H} f(y,s) + \mathrm{O}(\eta') = && (4.2.18)\\
&\mathrm{O}(\eta'), && (4.2.19)
\end{aligned}$$

and the other two follow similarly.

For (4.2.9) and (4.2.10) we will need to employ some control using the box norms. We will do this using the following lemma which completes the proof. □

Lemma 4.2.20. *For f, g, h supported on S with max$\{\|f\|_2, \|g\|_2, \|h\|_2\} \leq \delta_A(\delta_X\delta_Y\delta_D)^{1/2}$; we have the estimate*

$$|\mathrm{T}_2(f,g,h)| = O(\eta') + \delta_A(\delta_X\delta_Y\delta_D)^2 \min\{\|g\|_{\Box,X}\|h\|_{\Box,Y}, \|f\|_{\Box}\|g\|_{\Box,X}, \|f\|_{\Box}\|h\|_{\Box,Y}\}.$$

Proof. We prove

$$|\mathrm{T}_2(f,g,h)| = O(\eta') + \delta_A(\delta_X\delta_Y\delta_D)^2 \|g\|_{\Box,X} \cdot \|h\|_{\Box,Y}\,, \qquad (4.2.21)$$

and since the other two proofs are the same after a change of basis, this will prove the inequality.

Apply Cauchy Schwartz on (x,y), to get

$$|\mathrm{T}_2(f,g,h)| \leq \delta_A(\delta_X\delta_Y\delta_D)^{1/2}\ \left(\mathbb{E}_{x,y} D(x+y)\left|\mathbb{E}_s g(x,x+s)h(y+s,y)\right|^2\right)^{1/2}$$

Twinning the variables in the second term and applying Proposition 2.2.1,

$$\begin{aligned}
&\mathop{\mathbb{E}_{x,y}}_{s,s'} D(x+y)g(x,x+s)g(x,x+s')h(y+s,y)h(y+s',y) =\\
&\delta_D \mathbb{E}_{s,s'}\mathbb{E}_x g(x,x+s)g(x,x+s')\mathbb{E}_y h(y+s,y)h(y+s',y) + O(\eta').
\end{aligned}$$

Where we kept a factor of $D(x+y)$ to yield the correct power of δ_D in the end. Now applying Cauchy Schwartz again in the variables (s,s') and twinning x and y we have

$$(\mathbb{E}_{s,s'}\mathbb{E}_x g(x,x+s)g(x,x+s')\mathbb{E}_y h(y+s,y)h(y+s',y))^2 \leq U \cdot V$$

So changing variables we have,

$$U = \mathbb{E}_{\substack{x,x' \\ s,s'}} g(x, x+s)g(x', x'+s)g(x, x+s')g(x', x'+s) = \delta_X^2\delta_Y^2\delta_D^2\|g\|_{\Box,X}$$

$$V = \mathbb{E}_{\substack{y,y' \\ s,s'}} h(y+s, y)h(y'+s, y')h(y+s', y)h(y'+s', y') = \delta_X^2\delta_Y^2\delta_D^2\|h\|_{\Box,Y}$$

and we have proved the lemma. □

Lemma 4.2.22 (Generalized von Neumann)**.** *If in addition to the assumptions in (4.2.5) we have that* $\eta \stackrel{\text{def}}{=} (\delta\delta_X\delta_Y\delta_D)^c$*; and also*

$$N > C(\delta_A^2\delta_X\delta_Y\delta_D)^{-1}, \tag{4.2.23}$$

then A *has a corner.*

Proof. We wish to again show that A, given certain conditions hold, contains a nontrivial corner. Clearly, $\mathrm{T}_2(A, A, A)$ is the expected density of corners in A. Since the number of trivial corners in A is $\delta\delta_X\delta_Y\delta_D N^2$ and we have from the Counting Lemma 4.2.5 that $T_2(A, A, A) \gtrsim \delta\delta_X\delta_Y\delta_D N^2$(where the implied constant may depend upon X, Y, D, A, N) then A must have a corner for the stated restriction on N. □

4.3 Density Increment

We now have a reasonable condition for guaranteeing a corner if A is already uniform in the sense of the box norm. Next we show that if A has a large box norm then there exists a sublattice on which A has increased density. However, we must be careful that the resulting sublattice stays large enough to apply the Generalized Von Neumann (4.2.22). [1] We will the necessary conditions out of Lemma (4.3.3). We use the following Paley Zygmund type inequality. It states that a zero mean random variable, with L^∞ norm less than 1, must have the probability of the tail beyond a constant multiple of the standard deviation be at least as large as a constant multiple of the pth absolute moment.

[1]Note we have the additional worry that the resultant lattice may not be uniform which is also a necessity in applying the Generalized Von Neumann. We will deal with this in the next section.

Proposition 4.3.1 (Paley-Zygmund)**.** *Let $1 < p < \infty$. Let f be a balance function. Then,*

$$\mathbb{P}\left(\mathbb{E}_y f(x,y) > C_p\sqrt{\mathbb{V}_x|\mathbb{E}_y f(x,y)|}\right) \geq C_p \mathbb{E}_x|\mathbb{E}_y f(x,y)|^p. \tag{4.3.2}$$

In fact our subsequent key lemma may be viewed as an extension of this fact in our problem enviroment.

Lemma 4.3.3 (Density Increment)**.** *Let $\kappa, \epsilon \in (0,1)$ and $\delta = \delta_A$. If*

$$\max\{\|f\|_\square, \|f\|_{\square,X}, \|f\|_{\square,Y}\} > \kappa\delta^{5/4}\,, \tag{4.3.4}$$

then there exists a $\phi > 1$ and subsets $X' \subset X$, $Y' \subset Y$, $D' \subset D$ with corresponding $S' = X' \times Y' \cap X' \overset{\text{diag}}{\times} D'$ such that:

$$\text{One of the sets remains unrefined, e.g. } D = D'; \tag{4.3.5}$$

$$\delta_A(S') - \delta_A \gtrsim \delta^{2+\epsilon}\,; \tag{4.3.6}$$

$$\delta_{X'}(X)\,,\ \delta_{Y'}(Y)\,,\ \delta_{D'}(D) \gtrsim \delta^{\phi}; \tag{4.3.7}$$

where the implied constants are in $(0,1)$.

Proof of Lemma 4.3.3. We will prove the lemma in the case where D is not refined. The argument will be equivalent up to a change of coordinates thus proving the lemma as stated.

We begin by eliminating some of the ”extreme” cases.

Case 4.3.8 (High Variance)**.** Suppose that we have either of the following:

$$\mathbb{V}_{x\in X}\mathbb{E}_{y\in Y} f(x,y) \geq \kappa^2\delta^4\delta_D^2, \tag{4.3.9}$$

$$\mathbb{V}_{y\in Y}\mathbb{E}_{x\in X} f(x,y) \geq \kappa^2\delta^4\delta_D^2\,. \tag{4.3.10}$$

Proof. □

For a point $(x,y) \in A$, consider $N_x \overset{\text{def}}{=} \{y' \mid (x,y') \in A\}$ and $N_y \overset{\text{def}}{=} \{x' \mid (x',y) \in A\}$. Define

$$X'' \overset{\text{def}}{=} \{x \in X \mid \delta_{N_x}(Y) \geq \kappa\delta^5\delta_D\}, \tag{4.3.11}$$

$$Y'' \overset{\text{def}}{=} \{y \in Y \mid \delta_{N_y}(X) \geq \kappa\delta^5\delta_D\}, \tag{4.3.12}$$

$$S'' \overset{\text{def}}{=} X'' \times Y'' \cap X'' \overset{\text{diag}}{\times} D. \tag{4.3.13}$$

At this point since we have already covered case (4.3.8) we have $\delta_{X-X''}(X)\,,\, \delta_{Y-Y''}(Y) \leq \delta^2\,.$

We also have directly that the case when the density is to high If he wave that

$$\mathbb{E}_{\substack{x\in X''\\ y\in Y''}} f(x,y) \geq \kappa\delta^2\delta_D\,. \tag{4.3.14}$$

then the result is already proven.

Case 4.3.15 (Low Density)**.** Suppose that

$$\mathbb{E}_{\substack{x\in X''\\ y\in Y''}} f(x,y) \leq -\kappa\delta^4\delta_D^2\,. \tag{4.3.16}$$

Proof. We find a sublattice with increased density by looking at the complement,

$$\begin{aligned}\delta_A(S-S'')-\delta =& \frac{\delta-\delta_{S''}\delta_A(S'')}{\delta_{S-S''}}-\delta \geq \frac{\delta+\delta_{S''}(\kappa\delta^4\delta_D^2-\delta)}{\delta_{S-S''}}-\delta = \\ & \frac{\delta(1-\delta_{S''})+\kappa\delta^4\delta_D^2\delta_{S''}}{\delta_{S-S''}}-\delta = \kappa\delta^4\delta_D^2\frac{\delta_{S''}}{\delta_{S-S''}} \gtrsim \delta^2\,.\end{aligned}$$

Now $S-S''$ is a union of three sublattices and so one of these will A has the above increased density on S. Also we have that must have this sublattice to be at least $\frac{\delta^2}{2}$ coordinate-wise. ☐

We need to be able to control the lattice structure and so we define

$$\mathrm{Q}(f_0,f_1,f_2,f_3) \overset{\text{def}}{=} \mathbb{E}_{\substack{x,x'\in X''\\ y,y''\in Y''}} f_0(x,y)f_1(x',y)f_2(x,y')f_3(x',y')\,. \tag{4.3.17}$$

Lemma 4.3.18. *For the balance function $f = A+\delta S$ we have*

$$\mathrm{Q}(f,f,f,f) \geq c'\delta_D^4\delta^5\,. \tag{4.3.19}$$

Proof. This comes from the fact that the portion of the lattice off of each set X, Y, X'', Y' is small, e.g.

$$\begin{aligned}&\left|\mathbb{E}_{\substack{x\in X-X'',x'\in X\\ y,y'\in Y''}} f(x,y)f(x',y)f(x,y')f(x',y')\right| \\ &\qquad\leq \mathbb{E}_{\substack{x\in X-X'',x'\in X\\ y,y'\in Y''}} N_x(y)D(x+y')D(x'+y)D(x'+y') \\ &\qquad\leq \kappa\delta^5\delta_D^4 + \eta'.\end{aligned}$$

Then the four suggested inequalities along with the assumption that $\|f\|_\square^4 \geq c^4\delta^5$ give us the result with $0<\kappa<\frac{c^4}{8}$. ☐

To find the subset on which A has increased density,

$$\sup_{\substack{x\in X'',y\in Y''\\(x,y)\in A}} \delta_A(N_y \times N_x \cap S'') \geq \frac{\mathrm{Q}(A,A,A,A)}{\mathrm{Q}(A,A,A,S)} = \delta + \frac{\mathrm{Q}(A,A,A,f)}{\mathrm{Q}(A,A,A,S)}. \tag{4.3.20}$$

We begin with the numerator.

Lemma 4.3.21. *We have*

$$\mathrm{Q}(A,A,A,f) \geq c'\delta^5\delta_D^4. \tag{4.3.22}$$

Proof.

$$\mathrm{Q}(A,A,A,f) = \delta^3\mathbb{E}_{x,x'\in X'',y,y'\in Y''}D(x+y)D(x+y')D(x'+y)f(x',y') \tag{4.3.23}$$

$$+ \delta^2\mathbb{E}_{x,x'\in X'',y,y'\in Y''}f(x,y)D(x+y')D(x'+y)f(x',y') \tag{4.3.24}$$

$$+ \delta^2\mathbb{E}_{x,x'\in X'',y,y'\in Y''}D(x+y)f(x,y')D(x'+y)f(x',y') \tag{4.3.25}$$

$$+ \delta^2\mathbb{E}_{x,x'\in X'',y,y'\in Y''}D(x+y)D(x+y')f(x',y)f(x',y') \tag{4.3.26}$$

$$+ \delta\mathbb{E}_{x,x'\in X'',y,y'\in Y''}D(x+y)f(x,y')f(x',y)f(x',y') \tag{4.3.27}$$

$$+ \delta\mathbb{E}_{x,x'\in X'',y,y'\in Y''}f(x,y)D(x+y')f(x',y)f(x',y') \tag{4.3.28}$$

$$+ \delta\mathbb{E}_{x,x'\in X'',y,y'\in Y''}D(x+y)f(x,y')f(x',y)f(x',y') \tag{4.3.29}$$

$$+ \mathbb{E}_{x,x'\in X'',y,y'\in Y''}f(x,y)f(x,y')f(x',y)f(x',y') \tag{4.3.30}$$

(4.3.23) We have from the Low Density (4.3.15) and High Density (4.3.14) cases that the absolute value of this quanity may be taken to be $\leq \kappa\delta^5\delta_D^4 + \eta'$.

(4.3.24),(4.3.25),(4.3.26) These terms all contain two f's, and after using uniformity all the terms are positive and hence are negligible.

(4.3.30) This term we have already controlled by (4.3.19) which gives us a bound of $c'\delta^5\delta_D^4$.

(4.3.27),(4.3.28),(4.3.29) Here all the terms contribute a term

$$\eta' + \delta\delta_D\mathbb{E}_{x,x'\in X'',y,y'\in Y''}f(x,y)f(x,y')f(x',y).$$

We handle this with the following Lemma 4.3.31.

Lemma 4.3.31. *Fix $0 < c, \epsilon < 1$. If it is the case that*

$$\left|\mathbb{E}_{x,y}f(x,y)\mathbb{E}_{x'}f(x',y)\,\mathbb{E}_{y'}f(x,y')\right| \geq c\delta^5\delta_D^3 \tag{4.3.32}$$

Then, there is a constant $\phi \simeq \epsilon^{-1}$, $X' \subset X$, $Y' \subset Y$ with

$$\delta_{X'}(X)\,,\,\delta_{Y'}(Y) \geq \delta^{\phi}\,, \tag{4.3.33}$$

$$\delta_A(S') - \delta \geq c'\delta^{2+\epsilon}\,. \tag{4.3.34}$$

Proof. Let $p = 1+\epsilon$, and $1/2q = 1 - 1/p$. Then Holder's inequality yields

$$c\delta^5\delta_D^3 \leq \big|\mathbb{E}_{x,y} f(x,y) \mathbb{E}_{x'} f(x',y)\, \mathbb{E}_{y'} f(x,y')\big| \tag{4.3.35}$$

$$\leq \big[\mathbb{E}_{x,y} f(x,y)^p\big]^{1/p} \big[\mathbb{E}_{x,y}|\mathbb{E}_{x'} f(x',y)\, \mathbb{E}_{y'} f(x,y')|^{2q}\big]^{1/2q} \tag{4.3.36}$$

$$= \big[\mathbb{E}_{x,y} f(x,y)^p\big]^{1/p} [\mathbb{E}_{x,y}|\mathbb{E}_{x_1,x_2} f(x_1,y) f(x_2,y)\, \mathbb{E}_{y_1,y_2} f(x,y_1) f(x,y_2)|^q]^{1/2q} \tag{4.3.37}$$

Now from uniformity in D, we have

$$\mathbb{P}\big(\big|\mathbb{E}_y f(x,y)\big| > c\delta_D\big) \leq \eta'\,.$$

So

$$(4.3.37) \leq c\delta^{1-\epsilon}\delta_D^{1-\epsilon}\big[\mathbb{E}_x\big|\mathbb{E}_y f(x,y)\big|^q \mathbb{E}_y\big|\mathbb{E}_x f(x,y)\big|^q\big]^{1/q}. \tag{4.3.38}$$

Thus, we must have either

$$\big[\mathbb{E}_y\big|\mathbb{E}_x f(x,y)\big|^q\big]^{1/q} \geq \delta^{2+\epsilon/2}\delta_D \quad \text{or} \quad \big[\mathbb{E}_x\big|\mathbb{E}_y f(x,y)\big|^q\big]^{1/q} \geq \delta^{2+\epsilon/2}\delta_D.$$

In either case the conclusion follows from the Paley Zygmund inequality as in (4.3.8). □

□

Lemma 4.3.39. *We have*

$$0 < \mathrm{Q}(A,A,A,S) < 2\delta^3. \tag{4.3.40}$$

Proof. We have

$$\mathrm{Q}(A,A,A,S) = \delta^3 \mathbb{E}_{x,x'\in X'',y,y'\in Y''} D(x+y)D(x+y')D(x'+y)D(x'+y') \tag{4.3.41}$$

$$+\, 3\delta^2 \mathbb{E}_{x,x'\in X'',y,y'\in Y''} f(x,y)D(x+y')D(x'+y)D(x'+y') \tag{4.3.42}$$

$$+\, \delta \mathbb{E}_{x,x'\in X'',y,y'\in Y''} f(x,y)f(x,y')D(x'+y)D(x'+y') \tag{4.3.43}$$

$$+\, \delta \mathbb{E}_{x,x'\in X'',y,y'\in Y''} D(x+y)f(x,y')f(x',y)D(x'+y') \tag{4.3.44}$$

$$+\, \delta \mathbb{E}_{x,x'\in X'',y,y'\in Y''} f(x,y)D(x+y')f(x',y)D(x'+y') \tag{4.3.45}$$

$$+\, \mathbb{E}_{x,x'\in X'',y,y'\in Y''} f(x,y)f(x,y')f(x',y)D(x'+y') \tag{4.3.46}$$

(4.3.41) Uniformity gives us $\delta^3\delta_D^4 + \eta'$.

(4.3.42) Here from the low density case (4.3.15), at least $-3\kappa\delta^6\delta_D + \eta'$.

(4.3.43) This is approximately $\delta_D^2\delta\mathbb{E}_{x\in X''}\left|\mathbb{E}_{y\in Y''}f(x,y)\right|^2 + \eta'$ which is covered by the high variance result.

(4.3.44), Here we get $\delta_D^2\delta|\mathbb{E}_{x\in X,y\in Y}f(x,y)| + \eta' < \kappa^2\delta^5\delta_D^4 + \eta'$.

(4.3.45)This is approximately $\delta_D^2\delta\mathbb{E}_{y\in X''}\left|\mathbb{E}_{x\in Y''}f(x,y)\right|^2 + \eta'$ which is covered by the high variance result.

(4.3.45)The last term we have a term of $\eta' + \delta\delta_D\mathbb{E}_{x,x'\in X'',y,y'\in Y''}f(x,y)f(x,y')f(x',y)$ which is covered from (4.3.31). □

We now have the appropriate density increment. □

4.4 Uniformization

Now we fill in the last step of our iterative process.

Lemma 4.4.1. *Let* $0 < \eta, \tau < 1$. *Suppose that* $\dim(H) \geq C(\tau\eta^2)^{-1}$. *For* $X', Y', D \subset H$. *Then we have* $\mathcal{P} = \mathcal{U} \cup \mathcal{N}_1 \cup N_2 \cup \mathcal{N}_3$, *a collection of sets of* $H \times H$ *where* $H \times H = \cup_{\mathcal{P}}(G' \times G'')$ *and we have the following:*

1. *Every* $G' \times G'' \in \mathcal{P}$ *has the property that* $\dim(G') = \dim(G'') \geq \dim(H) - C'(\tau\eta^2)^{-1}$;
2. G', G'' *are translates of each other;*
3. *Every* $G' \times G'' \in \mathcal{U}$, *has the property that* $X' \cap G'$, $Y' \cap G''$, *and* $D \cap (G' + G'')$ *are* η*-uniform.*
4. *Let* $\mathcal{C}_{\mathcal{N}_1} = \bigcup_{G'\times G''\in\mathcal{N}_1} G'$, $\mathcal{C}_{\mathcal{N}_2} = \bigcup_{G'\times G''\in\mathcal{N}_2} G''$, *and* $C_{\mathcal{N}_3} = \bigcup_{G'\times G''\in\mathcal{N}_3}(G' + G'')$*;*
   Then $\delta_{\mathcal{C}_{\mathcal{N}_1}} \leq \tau\delta_{X'}$, $\delta_{\mathcal{C}_{\mathcal{N}_2}} \leq \tau\delta_{Y'}$, *and* $\delta_{\mathcal{C}_{\mathcal{N}_3}} \leq \tau\delta_D$.

Proof. Initialize $\mathcal{Q} \leftarrow \{H \times H\}$ and three counters $n_j \leftarrow 0$. Also,

$$\mathcal{N}_{n_1} \leftarrow \{G' \times G'' \in \mathcal{Q} \mid \|X' \cap G'\|_{\text{Uni}} \geq \eta\}, \tag{4.4.2}$$

$$\mathcal{N}_{n_2} \leftarrow \{G' \times G'' \in \mathcal{Q} \mid \|Y' \cap G''\|_{\text{Uni}} \geq \eta\}, \tag{4.4.3}$$

$$\mathcal{N}_{n_3} \leftarrow \{G' \times G'' \in \mathcal{Q} \mid \|D \cap (G' + G'')\|_{\text{Uni}} \geq \eta\}\,. \tag{4.4.4}$$

and

$$\mathcal{C}_{\mathcal{N}}(n_j) \leftarrow \bigcup_{G_1 \times G_2 \in \mathcal{N}_{n_j}} G_j.$$

`IF` $\left[\delta_{\mathcal{C}_{\mathcal{N}}(n_1)} \geq \tau\delta_{X'}\right]$ `THEN` update $n_1 \leftarrow n_1 + 1$.

`WHILE` $\mathcal{N}_{n_1} \neq \emptyset$

For each $G' \times G'' \in \mathcal{N}_{n_1}$

1. Apply Lemma 2.1.3 to have $G' = \cup G'_i$ with $\mathbb{V}_i \delta_{X'}(G'_i) \geq \|X' \cap G'\|^2_{\text{Uni}}$.
2. We have that G'' is a translate of G' Then we may find translates G''_i of G'_i so that $G'' = \cup G''_i$.
3. Update $\mathcal{Q} \leftarrow (\mathcal{Q} - G' \times G'') \cup \{G'_i \times G''_l\}_{i,l \in \{0,\ldots,p-1\}}$.
4. Update $\mathcal{N}_{n_1} \leftarrow \{G' \times G'' \in \mathcal{Q} \mid \|X' \cap G'\|_{\text{Uni}} \geq \eta\}$ and $\mathcal{C}_{\mathcal{N}}(n_1) \leftarrow \bigcup_{G' \times G'' \in \mathcal{N}_{n_1}} G'$.
5. Set $\upsilon_1(n_1) = n_1 + n_2 + n_3$ and $\mathcal{P}_{\upsilon_1(n_1)} = \mathcal{N}_{m_1} \cup (\mathcal{Q} - \mathcal{N}_{n_1})$.

`IF` $\left[\delta_{\mathcal{C}_{\mathcal{N}}(n_2)} \geq \tau\delta_{Y'}\right]$ `THEN` update $n_2 \leftarrow n_2 + 1$. `WHILE` $\mathcal{N}_{n_2}$ Apply the same process for each G'' with $G' \times G'' \in \mathcal{N}_{n_2}$. Then we have the values $\upsilon_2(n_2) = n_1 + n_2 + n_3$ and partitions $\mathcal{P}_{\upsilon_2(n_2)} = \mathcal{N}_{n_2} \cup (\mathcal{Q} - \mathcal{N}_{n_2})$.

`IF` $\left[\delta_{\mathcal{C}_{\mathcal{N}}(n_3)} \geq \tau\delta_{D}\right]$ `THEN` update $n_3 \leftarrow n_3 + 1$.

`WHILE` $\mathcal{N}_{n_3} \neq \emptyset$

For each $G' \times G'' \in \mathcal{N}_{n_3}$

1. Apply Lemma 2.1.3 to $F = G' + G''$ so that $F = \cup F_i$ with $\mathbb{V}_i \delta_D(F_i) \geq \|D \cap F\|^2_{\text{Uni}}$.
2. We have that F is a translate of both G' and G''. Then we may find translates G'_i and G''_i of F_i so that $G' = \cup G'_i$ and $G' = \cup G''_i$.
3. Update $\mathcal{Q} \leftarrow (\mathcal{Q} - G' \times G'') \cup \{G'_i \times G''_l\}_{i,l \in \{0,\ldots,p-1\}}$.
4. Update $\mathcal{N}_{n_3} \leftarrow \{G' \times G'' \in \mathcal{Q} \mid \|D \cap F\|_{\text{Uni}} \geq \eta\}$ and $\mathcal{C}_{\mathcal{N}}(n_3) \leftarrow \bigcup_{G' \times G'' \in \mathcal{N}_{n_3}} G'$.
5. Set $\upsilon_3(n_3) = n_1 + n_2 + n_3$ and $\mathcal{P}_{\upsilon_3(n_3)} = \mathcal{N}_{n_3} \cup (\mathcal{Q} - \mathcal{N}_{n_3})$.

When this loop terminates define $\mathcal{U} \stackrel{\text{def}}{=} \mathcal{Q} - \mathcal{N}_{n_1+n_2+n_3}$ and $\mathcal{C}_{\mathcal{N}} \stackrel{\text{def}}{=} \mathcal{C}_{\mathcal{N}}(m_1 + m_2 + m_3)$. Now each element of the partition has dimension $\geq \dim(H) - n_1 - n_2 - n_3$ and the other concllusions of the lemma hold by the construction. Therefore if we can estimate the run time we will be finished. Each time one of the three pieces of the iterate runs we get a mean square increase. We would like to use the cummulative effect of these mean square increases to give us a upper bound on the counters. However, one must be immediately wary that the "cutting" in a subsequent iterative pieces does not decrease the mean square density in a previous one. Here we appeal to the fact that a refinement of a partition of a probability space may only increase the mean square density. Now we may state that,

$$\sigma_{1,j} \stackrel{\text{def}}{=} \mathbb{E}\,\mathbb{E}(X' \mid \mathcal{P}_j)^2 = \sum_{H' \in \mathcal{P}_j} \delta_A(H')^2 \delta_{H'}(H) =$$

$$\sigma_{1,j-1} + \sum_{H' \in N_{n_1}} \delta_{H'}(H) \mathbb{V}_i \delta_{X'}(H'_i) \geq \sigma_{1,j-1} + \delta_{X'} \tau \eta^2.$$

Obviously the $\sigma_{1,j}$ are less than $\delta_{X'}$. Consequently, $n_1 \tau \eta^2 \delta_{X'} \leq \delta_{X'}$. Apply this to the other two counters and the proof is complete.

□

Lemma 4.4.5 (Uniformizing a Sublattice). *Suppose that* $0 < \epsilon < 1$, $\phi > 1$ *and that* X, Y, D *are* $(\delta_X \delta_Y \delta_D \delta)^C$*-uniform. If there are subsets* $X' \subset X, Y' \subset, D' \subset D$ *so that*

1. $\delta_{X'}(X) \geq c\delta^{\phi}$, $\delta_{Y'}(Y) \geq c\delta^{\phi}$, *and* $\delta_{D'}(D) \geq c\delta^{\phi}$;

2. *One of the subsets stay unrefined, e.g.* $D' = D$;

3. $\delta_A(S') = \delta + c\delta^{2+\epsilon}$;

4. $\dim(H) > C\delta^{-4}\eta^{-2}$, *where* η *is a fixed constant in* $(0, 1)$.

Then there exists $X'' \subset X'$, $Y'' \subset Y$, $D'' \subset D'$ *and* H', H'', *translates of a subspace* $H_t \leq H$, *so that*

1. $\|X''\|_{\text{Uni}}, \|Y''\|_{\text{Uni}} \|D''\|_{\text{Uni}} \leq \eta$,

2. $\delta_A(S'') - \delta \geq \frac{c}{2}\delta^{2+\epsilon}$,

3. $\dim(H_t) \geq \dim(H) - C\delta^{-4}\eta^{-2}$,

4. $\delta_{X''}(H') \geq \kappa\delta^2\delta_{X'}(H), \delta_{Y''}(H'') \geq \kappa\delta^2\delta_{Y'}(H), \delta_{D''}(H'+H'') \geq \kappa\delta^2\delta_{D'}(H)$.

Proof of Lemma 4.4.5. As before lets work in the case D is not refined, i.e $D' = D$. Apply the Lemma (4.4.1) with $\tau = \frac{c}{16}\delta^{2+\epsilon}$. First let us define the so called "empty" sets,

$$\mathcal{E}_1 \stackrel{\text{def}}{=} \{G' \times G'' \in \mathcal{P} \mid \delta_{X'}(G') \leq \tfrac{c}{16}\delta^{2+\epsilon}\},$$
$$\mathcal{E}_2 \stackrel{\text{def}}{=} \{G' \times G'' \in \mathcal{P} \mid \delta_{Y'}(G'') \leq \tfrac{c}{16}\delta^{2+\epsilon}\},$$
$$\mathcal{E}_3 \stackrel{\text{def}}{=} \{G' \times G'' \in \mathcal{P} \mid \delta_{D}(G'+G'') \leq \tfrac{c}{16}\delta^{2+\epsilon}\}.$$

Also

$$X_b \stackrel{\text{def}}{=} \{X' \cap G' \mid G' \times G'' \in E_1 \cup \mathcal{N}_1\},$$
$$Y_b \stackrel{\text{def}}{=} \{Y' \cap G'' \mid G' \times G'' \in E_2 \cup \mathcal{N}_2\},$$
$$D_b \stackrel{\text{def}}{=} \{D \cap (G'+G'') \mid G' \times G'' \in E_3 \cup \mathcal{N}_3\}.$$

Clearly, we have that $\delta_{X_b \cap X'} \leq c\frac{\delta^2}{8}$. Then from uniformity we have that

$$\mathbb{E}_{x,y\in H} X_b(x)D(x+y)Y'(y) = \eta' + \delta_{X_b}\delta_D\delta_{Y'}.$$

And so for $S_{X_b} \stackrel{\text{def}}{=} X_b \times Y' \cap X_b \stackrel{\text{diag}}{\times} D$ we may assume that

$$\delta_{S_{X_b}}(S') \leq c\tfrac{\delta^2}{4}.$$

Simililarly, we have that

$$\delta_{S_{Y_b}}(S') \leq c\tfrac{\delta^2}{4}.$$

Letting $X_c = X - X_b$, $Y_c = Y - Y_b$, $D_c = Y - D_b$ and $S_c \stackrel{\text{def}}{=} X_c \times Y_c \cap X_c \stackrel{\text{diag}}{\times} D$ we have

$$\delta_A(S_c) - \delta \geq c\tfrac{\delta^2}{2} .$$

Let $G' \times G'' \in \mathcal{P} - \mathcal{N}_2 - \mathcal{N}_2 - \mathcal{E}_1 - \mathcal{E}_2$. Then

$$\mathbb{E}_{x\in G', y\in G''} X(x)G'(x)Y(y)G''(y)D(x+y) = \eta' + \delta_X(G')\delta_Y(G'')\delta_D(G'+G'') .$$

Then we have for $S_c^b \stackrel{\text{def}}{=} X_c \times Y_c \cap X_c \stackrel{\text{diag}}{\times} D_b$ that,

$$\delta_{S_c^b}(S_c) \leq c\frac{\delta^2}{4}.$$

And so for $S_c^c \stackrel{\text{def}}{=} X_c \times Y_c \cap X_c \overset{\text{diag}}{\times} D_c$,

$$\delta_A(S_c^c) - \delta \geq c\frac{\delta^2}{4}.$$

From this we have that there exists a $H' \times H'' \in \mathcal{U}$ so that,

$$\delta_A(H' \times H'' \cap S_c^c) - \delta \geq c\frac{\delta^2}{4}\,.$$

Taking $X'' = X_c \cap G'$, $Y'' = Y_c \cap G''$, $D'' = D_c \cap H' + H''$ and we have proved the lemma. $\square$

4.5 Roth's Theorem in Two Dimensions

We may now apply the Density Increment (4.3.3) and Uniformity 4.4.1 Lemmas in succession to form an appropriate set to apply the Generalized Von Neumann (4.2.22). Hence we have an iterative proof of two dimensional Roth just as in the case of one dimensional Roth.

Theorem 4.5.1 (2D Roth)**.** *For all* $0 < \epsilon < 1$*, we have* $r_\angle(\mathbb{F}_2^n) \ll N^2(\log\log N)^{-1+\epsilon}$.

Proof. Initialize $X, Y, D, H \leftarrow \mathbb{F}_2^n$ and $S \leftarrow \mathbb{F}_2^n \times \leftarrow \mathbb{F}_2^n$. Also $\delta_X, \delta_Y, \delta_D \leftarrow 1$. Fix a set A with density δ_0 in $\mathbb{F}_2^n \times \mathbb{F}_2^n$. Initialize $A' \leftarrow A$ and $\delta \leftarrow \delta_{A'}$.

1. If $\max\{\|f\|_\Box, \|f\|_{\Box,X}, \|f\|_{\Box,Y}\} < \kappa\delta^{5/4}$ then `STOP`.

2. Otherwise, apply the Density Increment Lemma 4.3.3 so that we have $X' \subset X$, $Y' \subset Y$, $D' \subset D$ with corresponding S' such that $\delta_{X'}(X)\,, \delta_{Y'}(Y)\,, \delta_{D'}(D) \gtrsim \delta^\phi$ and $\delta_{A'}(S') - \delta \gtrsim \delta^{2+\epsilon}$.

3. If one of the sets X', Y' or D' fails to be $(\delta\delta_{X'}\delta_{Y'}\delta_D)^C$ - uniform, apply the Uniformization Lemma. 4.4.5. Then have the sets and subpaces from the lemma, $X'' \subset X'$, $Y'' \subset Y'$, $D'' \subset D'$ and $H', H'' \subset H$ containing X'', Y'', D''.

4. Update variables:

$$X \leftarrow X'',\ Y \leftarrow Y'',\ D \leftarrow D'',\ H \leftarrow H'\,, \tag{4.5.2}$$

$$\delta_X \leftarrow \delta_{X''}(H'),\ \delta_Y \leftarrow \delta_{Y''}(H')\,, \delta_D \leftarrow \delta_{D''}(H')\,, \tag{4.5.3}$$

$$S \leftarrow X \times Y \cap X \overset{\text{diag}}{\times} D\,, \quad \delta \leftarrow \delta_{A'}(S). \tag{4.5.4}$$

Things work here similarly to the one dimensional case. Observe that the density of the incremented A' on the set S has increased to by at least $\kappa\delta_0^{2+\epsilon}$ while δ_X, δ_Y, and δ_D have have decrease no more than $(\kappa\delta_0)^C$. The loop must terminate in less than $\lesssim \delta_0^{-1-\epsilon}$ or the density will exceed one. Once this loop stops, if the initial sets were large enough we may apply the 3.2.7 and conclude that A has a corner. How large must X and Hbe so that we have sufficient size upon termination of the loop? Well the loss of dimension from the Uniformity Lemma (4.4.1) gives us that $\delta_X \geq (\kappa\delta_0)^{(\kappa\delta_0)^{-1-\epsilon}}$. Then to apply the Von Neumann (3.2.7) we need the stated bound. □

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