

THE EFFECT OF ACCELERATION AND DECELERATION ON THE
DYNAMIC STABILITY OF A MISSILE

A THESIS

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the Faculty of the Graduate Division

by
William J. Steinmetz

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Master of Science in Aeronautical Engineering

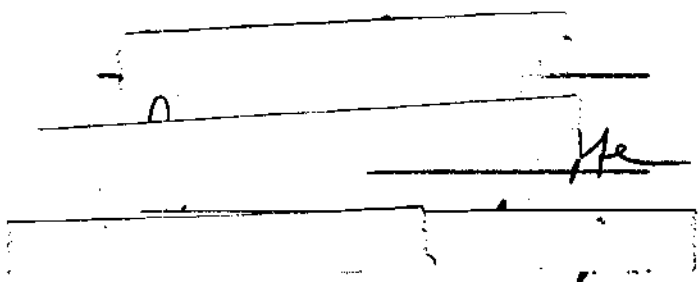
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DYNAMIC STABILITY OF A MISSILE

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LIST OF SYMBOLS

English

a	constant coefficient in exponential variation of f
b	constant coefficient in exponential variation of dimensionless parameter, M
C_D	aerodynamic drag coefficient based on missile frontal area
C_L	aerodynamic lift coefficient based on missile frontal area
C_m	aerodynamic pitching moment coefficient, about the center of gravity, based on missile frontal area and characteristic length, L
$C_{L\alpha}$	rate of change of lift coefficient with angle of attack, $\frac{\partial C_L}{\partial \alpha}$
$C_{m\alpha}$	rate of change of pitching moment coefficient with angle of attack, $\frac{\partial C_m}{\partial \alpha}$
$C_{m\dot{\alpha}}$	rate of change of pitching moment coefficient with time rate of change of angle of attack parameter $\frac{\dot{\alpha}L}{2V}$, $\frac{\partial C_m}{\partial (\dot{\alpha}L/2V)}$
$C_{m\dot{\theta}}$	rate of change of pitching moment coefficient with pitching velocity parameter $\frac{\dot{\theta}L}{2V}$, $\frac{\partial C_m}{\partial (\dot{\theta}L/2V)}$
$C_{m\delta}$	rate of change of pitching moment coefficient with control deflection, $\frac{\partial C_m}{\partial \delta}$
D	aerodynamic drag, pounds
f	dimensionless dynamic stability function
F	function of dimensionless parameter, M
g	gravitational acceleration, 32.17 feet per second per second
h	dimensionless static stability function
I	moment of inertia of missile, slug-ft. ²

L	aerodynamic lift, pounds, or, characteristic length, feet
m	mass of missile, slugs
M	aerodynamic pitching moment, about the center of gravity, foot-pounds, or, dimensionless function of f and h
q	local dynamic pressure, pounds per ft. ²
r	distance from center of earth to missile, feet
S	frontal area of missile, ft. ²
t	time, seconds
T	thrust, pounds
V	velocity of missile, feet per second
V_0	velocity of missile at $t = 0$, feet per second
W	weight of missile, pounds
y	altitude or vertical co-ordinate of missile location, feet

Greek

α	angle of attack, radians
δ	flight-path angle, radians
δ	control deflection angle, radians
θ	angle of pitch measured from axis fixed in space, radians
θ	angle of pitch measured from local horizontal, radians
ρ	local atmospheric density, slugs per cubic foot
τ	non-dimensional time, $\frac{tV_0}{L}$
ϕ	angular displacement of missile from fixed space axis, $\theta - \theta_0$, radians
χ	function of dimensionless parameter, M

Subscripts

o	condition at $t = 0$ or oscillatory variable
s	non-oscillatory variable

Superscripts

- . first derivative with respect to time, t
- .. second derivative with respect to time, t

SUMMARY

Although much work has been done concerning the dynamic stability of both aircraft and missiles, until recently few analyses have considered variable aerodynamic coefficients. The purpose of this research was to establish an approximate criterion for dynamic stability of a vehicle traversing an arbitrary path within the atmosphere. The aerodynamic coefficients, atmospheric density, vehicle velocity, moment of inertia, and mass were time variable.

Tobak and Allen (Reference 9) derived the equations of motion and separated them into static and oscillatory equations. It was desired to solve the oscillatory equations of motion and establish a criterion by which the dynamic stability of a missile could be predicted at any instant of time.

Wintner (Reference 8) found an asymptotic (time $\rightarrow \infty$) solution to this equation. The mode of oscillation was found to be trigonometric. To establish a stability criterion, the envelope function was differentiated and set less than zero. If this condition is satisfied, the oscillations will not diverge.

The approximate stability criterion was applied to a special case in which the coefficients of the oscillatory equation of motion vary exponentially with time. It was found that when these functions were increasing functions of time, the Wintner solution became more accurate with increasing time, whereas, when these functions were decreasing functions of time, the Wintner solution became less accurate with time. However,

for the two cases studied here, the error was not large.

A missile, during first and second stage burnout, was studied and an exact solution of the oscillatory equation of motion was obtained. The motion was approximated as vertical and the translational equations of motion were uncoupled from the oscillatory equations. They were solved numerically using a digital computer.

The Wintner solution was compared with the exact solution, for these two cases, over several cycles and shown to be very accurate. The envelopes of the Wintner, exponential and exact solutions were compared and the accuracy of the Wintner and exponential solutions decreased with time as was predicted from the theory. The Wintner and exponential solutions were conservative.

CHAPTER I

INTRODUCTION

The dynamic stability of aircraft and missiles has been studied extensively. Until recently, however, little work has been done that does not utilize one or more of the restrictive assumptions of constant aerodynamic coefficients, constant velocity, or high supersonic speeds. The stability criteria previously established have been based on an oscillatory equation of motion with constant coefficients.

A. R. Collar (Reference 1) showed that while a given differential equation with constant coefficients might be stable, instability may result, even for positive damping, if the coefficients are decreasing functions of the independent variable.

In 1957 Allen and Eggers (Reference 2) made a simplified analysis of the velocity and deceleration history of missiles entering the earth's atmosphere at high supersonic speeds. Allen (Reference 3) gave an analysis of the oscillating motion of such a missile, angularly misaligned upon entering the atmosphere, using the same assumptions of constant drag and constant gravitational acceleration, with the additional assumption that the rates of change of the aerodynamic coefficients with respect to angle of attack, time derivative of angle of attack, and time derivative of angular displacement, are constant.

Tobak and Allen (Reference 4) extended the previous analyses to predict the dynamic stability of vehicles traversing ascending or descending paths through the atmosphere, still utilizing the assumptions of

constant drag coefficient and aerodynamic coefficients independent of Mach number. A distinguishing feature of their solution is the appearance of the Bessel function rather than the trigonometric function as the characteristic mode of oscillation.

Several other investigators, including Oswald (Reference 5), Friedrich and Dore (Reference 6), and Laitone (Reference 7) have made studies of the dynamic motion of missiles. They made assumptions, however, such as constant speed, constant altitude, and constant aerodynamic coefficients, which are not applicable to the present investigation of an accelerated missile with variable aerodynamic coefficients.

Wintner (Reference 8), in 1958, developed a theory which gave the general solution to the second order differential equation with coefficients which are functions of the independent variable. Sommer and Tobak (Reference 9) applied this solution to the oscillatory motion of vehicles which traverse arbitrarily prescribed trajectories through the atmosphere to obtain an approximate convergence criterion.

Force (Reference 10) discussed a deceleration instability experienced by the Deacon-Arrow and Viper-Arrow sounding rockets of the Sandia Corporation. The vehicles, described as slightly unstable at second-stage burnout, appeared stable up to the time optical coverage was lost at second-stage burnout. The motion then diverged, according to a theory developed at the time, but due to the decreasing Mach number and rate of change of dynamic pressure, the motion again began to converge. The vehicles attained only about half the predicted altitudes.

The purpose of this research is to establish an approximate criterion for dynamic stability of a vehicle traversing an arbitrary path

within the atmosphere. The aerodynamic coefficients, atmospheric density, vehicle velocity, moment of inertia, and mass will be considered time variable. The criterion obtained will be limited by the assumption of positive static stability and the assumed small disturbance variations of the lift and moment coefficients.

CHAPTER II

EQUATIONS OF MOTION

The equations of motion for a vehicle traversing an arbitrary path, within the measurable atmosphere ($\rho > 0$), were shown by Sommers and Tobak (Reference 9) to be as follows:

$$m \frac{dV}{dt} = -D + W \sin \gamma + T \quad (1)$$

$$mV\dot{\gamma} = -L + W \cos \gamma - m \frac{V^2}{r} \cos \gamma \quad (2)$$

$$I\ddot{\theta} = M \quad (3)$$

where

I is the moment of inertia

m is the mass

V is the velocity

W is the weight of the vehicle

D is the drag force

L is the lift force

M is the pitching moment about the center of gravity of the vehicle

T is the thrust

r is the distance from the center of the earth to the vehicle

t is time

γ is the flight-path angle and

θ is the angle of pitch measured from the axis fixed in space
(Figure 1).

Assuming small disturbances, i.e., small deviations from the equilibrium, lift and pitching moment coefficients may be defined as follows:

$$C_L = \frac{L}{qS} = \frac{\partial C_L}{\partial \alpha} \alpha + \frac{\partial C_L}{\partial \dot{\alpha}} \dot{\alpha} + \dots + \frac{\partial C_L}{\partial \theta} \theta + \frac{C_L}{\theta} \dot{\theta} + \dots \quad (4)$$

$$C_m = \frac{M}{qSL} = \frac{\partial C_m}{\partial \alpha} \alpha + \frac{\partial C_m}{\partial \dot{\alpha}} \dot{\alpha} + \dots + \frac{\partial C_m}{\partial \theta} \theta + \frac{\partial C_m}{\partial \dot{\theta}} \dot{\theta} + \dots + \frac{\partial C_m}{\partial \delta} \delta \quad (5)$$

where C_L , C_m , α , $\dot{\alpha}$, θ , $\dot{\theta}$, δ , etc. are measured from the equilibrium values and δ is the control deflection angle. These equations may be simplified considerably by noting pertinent physical characteristics. For instance, the lift coefficient may be considered to be independent of the pitching angle, θ , and its derivatives. Also, derivatives of the lift coefficient with respect to time derivatives of the angle of attack, α , may be neglected. Finally, the moment coefficient may be considered to be independent of θ and time derivatives of θ and α of higher order than the first. Hence,

$$C_L = \frac{\partial C_L}{\partial \alpha} \alpha \quad (6)$$

$$C_m = \frac{\partial C_m}{\partial \alpha} \alpha + \frac{\partial C_m}{\partial \dot{\alpha}} \dot{\alpha} + \frac{C_m}{\delta} \dot{\delta} \quad (7)$$

Tobak and Allen (Reference 4) argued that the three equations of motion above may be separated into two sets of equations, one set (γ_s ,

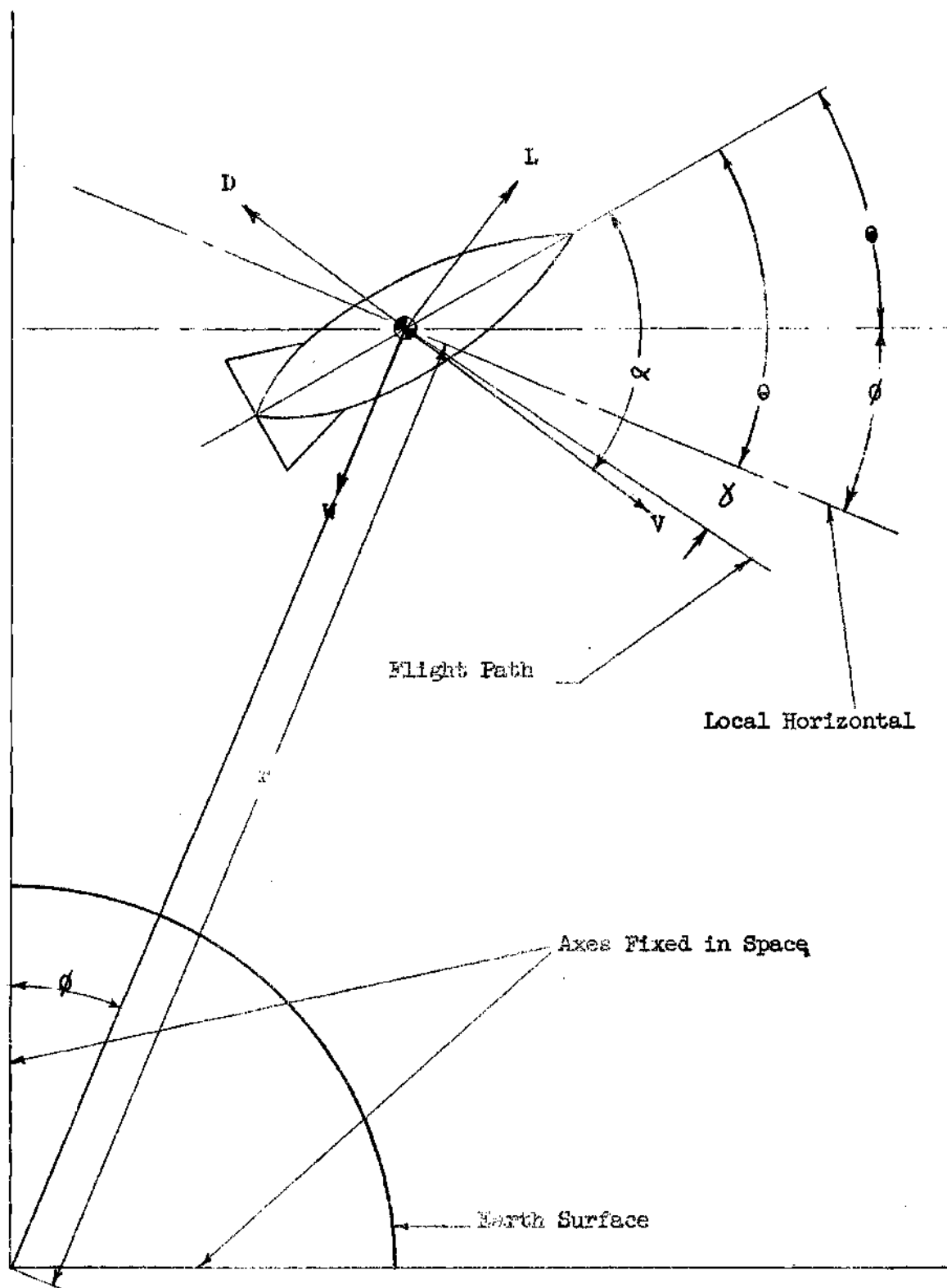


Fig. 1 Sketch of Dynamics of a Missile in Flight

α_s, θ_s) defining the static trajectory of the center of gravity of the vehicle, the other ($\delta_o, \alpha_o, \theta_o$) defining the oscillatory motions of the vehicle about the static trajectory. Hence, with the restriction that $|\delta_o/\delta_s| \ll 1$, the static trajectory equations are:

$$m \frac{dV}{dt} + qSC_D - W \sin \delta_s - T = 0 \quad (8)$$

$$mV \dot{\delta}_s + qSC_{L\alpha} \alpha_s + m \left(\frac{V^2}{r} - g \right) \cos \delta_s = 0 \quad (9)$$

$$I \ddot{\theta}_s - qSL \left(C_{m\alpha} \alpha_s + C_{m\dot{\theta}} \frac{\dot{\theta}_s L}{2V} + C_{m\alpha} \frac{\alpha_s L}{2V} + C_{m\delta} \delta \right) = 0 \quad (10)$$

and for the oscillatory motion,

$$mV \dot{\delta}_o + qSC_{L\alpha} \alpha_o = 0 \quad (11)$$

$$I \ddot{\theta}_o - qSL \left(C_{m\alpha} \alpha_o + C_{m\dot{\theta}} \frac{\dot{\theta}_o L}{2V} + C_{m\alpha} \frac{\alpha_o L}{2V} \right) = 0 \quad (12)$$

where,

$$C_D = \frac{D}{qS} = \text{aerodynamic drag coefficient}$$

$$C_{L\alpha} = \frac{\partial C_L}{\partial \alpha}$$

$$C_{m\alpha} = \frac{\partial C_m}{\partial \alpha}$$

$$C_{m\delta} = \frac{\partial C_m}{\partial \delta}$$

$$C_{m\dot{\theta}} = \frac{\partial C_m}{\partial \left(\frac{\dot{\theta} L}{2V} \right)}$$

$$C_{m\dot{\alpha}} = \frac{\partial C_m}{\partial \left(\frac{\dot{\alpha}L}{2V}\right)}$$

Note that ϕ , as defined in Figure 1, is a non-oscillatory quantity. Hence,

$$\theta_s = \theta_s - \phi$$

$$\theta_o = \theta_o$$

By use of the relation $\gamma_o = \alpha_o - \theta_o$, Equations (11) and (12) may be combined to give a single equation for the oscillatory angle of attack:

$$\begin{aligned} \ddot{\alpha}_o(t) + \left[\frac{qS}{mV} C_{L\alpha} - (C_{m\dot{\theta}} + C_{m\dot{\alpha}}) \frac{qSL^2}{2IV} \right] \dot{\alpha}_o(t) \\ + \left[-\frac{qSL}{I} C_{m\alpha} + \frac{d}{dt} \left(\frac{qS}{mV} C_{L\alpha} \right) - \left(\frac{qSL}{V} \right)^2 \frac{1}{2Im} C_{L\alpha} C_{m\dot{\theta}} \right] \alpha_o(t) = 0 \end{aligned} \quad (13)$$

The subscript o will now be dropped, it being understood that in referring to α , the subscript o is implied.

Define a non-dimensional time variable, $\tau = \frac{tV}{L}$. Substituting this new variable into Equation (13), the following non-dimensionalized angular equation of motion results:

$$\frac{d^2 \alpha}{d\tau^2} + f(\tau) \frac{d\alpha}{d\tau} + h(\tau) \alpha = 0 \quad (14)$$

where,

$$f(\tau) = \frac{qSL}{mV_0} C_{L\alpha} - \frac{qSL^3}{2IV_0} (C_{m\alpha} + C_{m\alpha}) \quad (15)$$

$$h(\tau) = -\frac{qSL^3}{IV_0^2} C_{m\alpha} + \frac{d}{d\tau} \left(\frac{qSL}{mV_0} C_{L\alpha} \right) \quad (16)$$

$$- \left(\frac{qSL^2}{VV_0} \right)^2 \frac{1}{2Im} C_{L\alpha} C_{m\alpha}$$

CHAPTER III

ANALYSIS

Equation (14) may be put into normal form by defining the transformation:

$$\alpha(\tau) = \bar{\alpha}(\tau) e^{-1/2 \int f(\tau) d\tau} \quad (17)$$

Substituting this into the angular equation of motion, the normalized form is obtained:

$$\ddot{\bar{\alpha}} + M(\tau) \bar{\alpha} = 0 \quad (18)$$

where,

$$M(\tau) = h - \frac{1}{2} \frac{df}{d\tau} - \frac{1}{4} f^2 \quad (19)$$

Wintner (Reference 8) showed that any differential equation of the form Equation (18), whose coefficient $M(\tau)$ satisfies the conditions:

$$M(\tau) > 0 \quad \text{for all } \tau \quad (20)$$

$$\int_{-\infty}^{\infty} \sqrt{M(\tau)} |F(\tau)| d\tau < \infty \quad (21)$$

where,

$$F(\tau) = \frac{1}{4} \left[\frac{5}{4} \frac{1}{M^3} \left(\frac{dM}{d\tau} \right)^2 - \frac{1}{M^2} \frac{d^2 M}{d\tau^2} \right] \quad (22)$$

will have a general solution which approaches asymptotically ($\tau \rightarrow \infty$) the form:

$$\bar{\alpha}(\tau) = \frac{1}{M^{1/4}(\tau)} \left[c_1 \cos \psi(\tau) + c_2 \sin \psi(\tau) \right] \quad (23)$$

where,

$$\psi(\tau) = \int \sqrt{M(\tau)} \, d\tau \quad (24)$$

Equation (23) is the exact solution to the equation:

$$\ddot{\bar{\alpha}} + M(1 - F) \bar{\alpha} = 0 \quad (25)$$

When F is much smaller than unity, Equation (23) should provide a good approximate solution to Equation (18). Equation (21) guarantees "sufficient smallness" of F in the limit, as $t \rightarrow \infty$; then, Equation (23) is an asymptotic solution to Equation (18).

For a flight terminating in a finite time, the condition

$$F(\tau) \ll 1 \quad (26)$$

is more restrictive than Equation (21). Equation (21) may be satisfied while Equation (26) is not satisfied and hence, the Wintner solution, Equation (23), would be invalid for finite time.

Combining Equations (23) and (17), an approximate solution to Equation (14) becomes:

$$\alpha(\tau) = \frac{c_1}{M^{1/4}(\tau)} e^{-1/2 \int f(\tau) \, d\tau} \cos \left[\psi(\tau) - \psi_1 \right] \quad (27)$$

where ψ_1 is a phase angle.

As Sommer and Tobak (Reference 9) have pointed out, the solution obtained by Tobak and Allen in Reference 4, involving zero-order Bessel functions of the first and second kind, approaches its asymptotic form very rapidly. This suggests that the asymptotic solution should be accurate for most practical cases, the advantage being that the assumptions of constant flight-path angle, γ , and constant $C_{m\infty}$ need not be employed.

A stability criterion which requires that the envelope of Equation (27) shall always decrease may be written:

$$\frac{d\alpha_{\max}}{dT} < 0 \quad (28)$$

or,

$$\frac{d}{dT} \left[\frac{e^{-1/2 \int f(T) dT}}{M^{1/4}(T)} \right] < 0 \quad (29)$$

Differentiating this expression gives the more explicit stability criterion:

$$f(T) > - \frac{1}{2M(T)} \frac{dM}{dT} \quad (30)$$

This approximate criterion is valid for any f and M , with M satisfying Equation (20) and F satisfying Equation (26). M is essentially a measure of static stability and the dynamic stability is indicated by f . Since M must be positive for this solution to be valid, if $\frac{dM}{dT} > 0$, f may be negative and still satisfy the criterion. That is,

if M increases with time, the vehicle may be dynamically stable even with negative damping ($C_{m_{\dot{\theta}}} + C_{m_{\dot{\alpha}}} > 0$). However, if M decreases with time, the vehicle may possess positive damping ($C_{m_{\dot{\theta}}} + C_{m_{\dot{\alpha}}} < 0$) and yet be dynamically unstable.

It is of interest to examine the stability criterion resulting from a prescribed variation of f and M . Assume the following exponential variations:

$$f = f_0 e^{-a\tau} \quad M = M_0 e^{-b\tau} \quad (31)$$

where the subscript 0 indicates the value of the function at $\tau = 0$ and constants a and b may be either positive or negative.

For this case, Equations (20) and (26) become:

$$M_0 > 0 \quad (32)$$

$$F(\tau) = \frac{b^2}{16 M_0} e^{b\tau} \ll 1 \quad (33)$$

When b is negative, F decreases with time and, hence, the accuracy of the exponential solution,

$$\alpha(\tau) = C_0 \exp \left[\frac{f_0}{2a} e^{-a\tau} + \frac{b}{4}\tau \right] \cos \left[\frac{2}{b} \sqrt{M_0} e^{-\frac{b}{2}\tau} + \psi_0 \right] \quad (34)$$

increases with time. However, when b is positive, F increases with time and the accuracy of the exponential solution decreases with time. At very large τ , $F(\tau)$ becomes large and the asymptotic solution is invalid.

The stability criterion, for these exponential variations of f and M , takes the form:

$$f_0 e^{-a\tau} > \frac{b}{2} \quad (35)$$

or,

$$a\tau < \ln \frac{2f_0}{b} \quad (36)$$

Figure 2 illustrates the relative influence of $\frac{f_c}{2a}$ and $\frac{b}{a}$ on the dynamic stability of a missile. The envelope of Equation (34) is plotted versus a for $f_0/2a = .1, 1, 3, 5,$ and 10 , and $b/a = 1, 2,$ and 3 . Note that for values of $f_0/2a$ of interest, the influence of b/a on stability is less than that of $f_0/2a$.

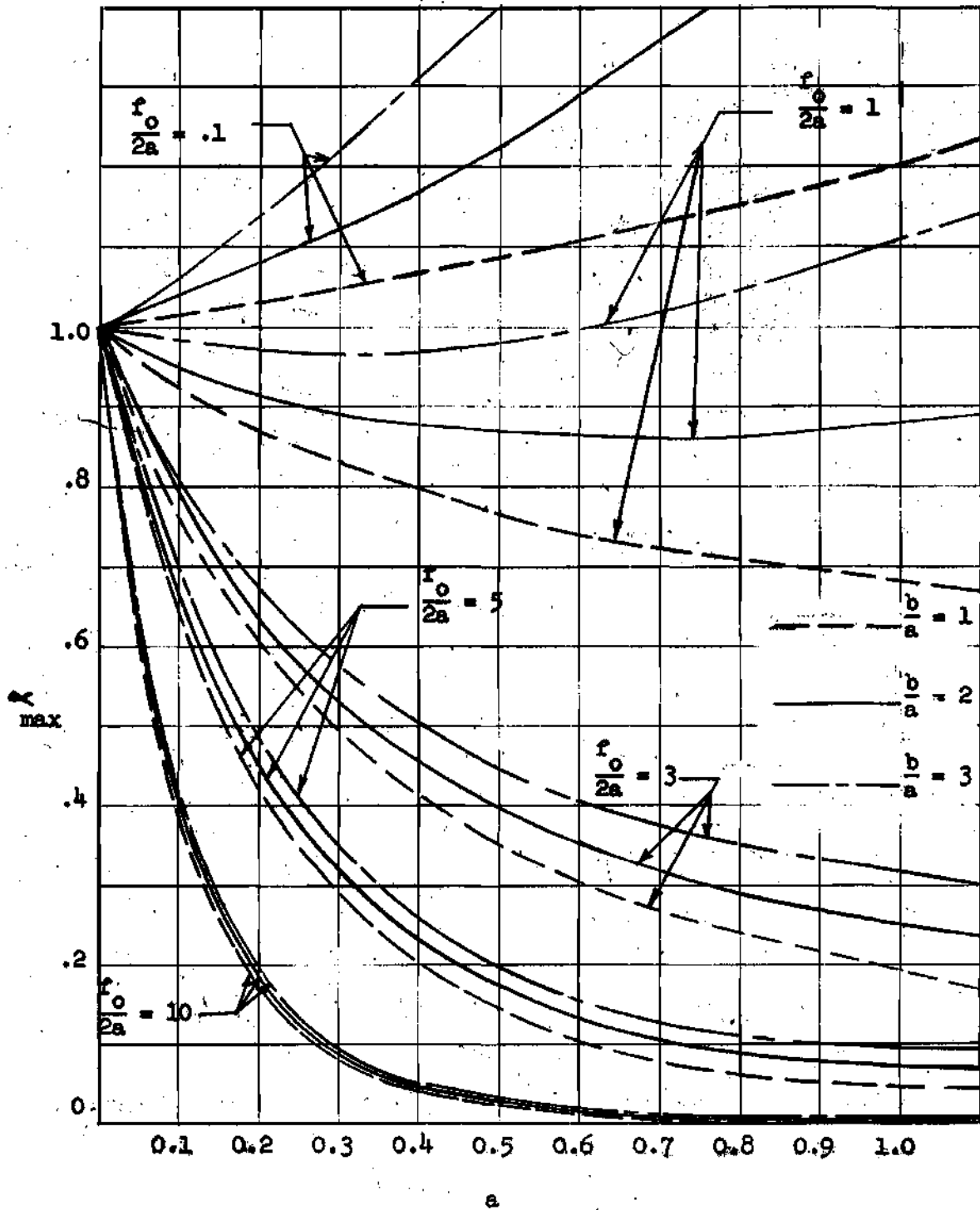


Fig. 2 Influence of $\frac{f_0}{2a}$ and $\frac{b}{a}$ on the Envelope of the Exponential Solution

CHAPTER IV

APPLICATION OF THEORY

The theory developed in Chapter III will now be applied to a specific missile, under two operating conditions. These conditions are first and second stage burnout. The properties of this missile, for these two cases, are listed in Table 1. The variations of the aerodynamic coefficients with Mach number are shown in Figures 3, 4, 5, and 6. These curves have been approximated, over the range of Mach numbers attained in the two cases considered here, by polynomials (Table 2).

The exact solution of the oscillatory motion was obtained by means of a digital computer. The translational equations of motion, (8), (9), and (10), can be uncoupled from the oscillatory equations, (11) and (12), by assuming that the motion is essentially vertical. That is,

$$\gamma = -\frac{\pi}{2} \quad (37)$$

$$\dot{y} = V \quad (38)$$

Since the burnout phase of flight will be investigated, the thrust term in Equation (8) is zero and the moment of inertia and mass are constant. For the missile considered here, lack of data requires the additional assumption that $C_{m\dot{\alpha}} = C_{m\alpha}$. Hence, the equations of motion may be written:

$$\frac{d^2 y}{d\tau^2} + \frac{\rho S}{2m} C_D \left(\frac{dy}{d\tau}\right)^2 + \frac{gI^2}{V_0^2} = 0 \quad (39)$$

Table 1

Properties and Initial Conditions for Cases 1 and 2

	<u>Case 1</u>	<u>Case 2</u>
Characteristic Length (feet)	11.5	11.5
Weight (pounds)	1140	645
Frontal Area (ft. ²)	1.37	1.37
Moment of Inertia (slug-ft. ²)	278	240
Initial Altitude (feet)	2300	27000
Initial Velocity (feet per second)	1550	3960
Initial Mach Number	1.40	3.93

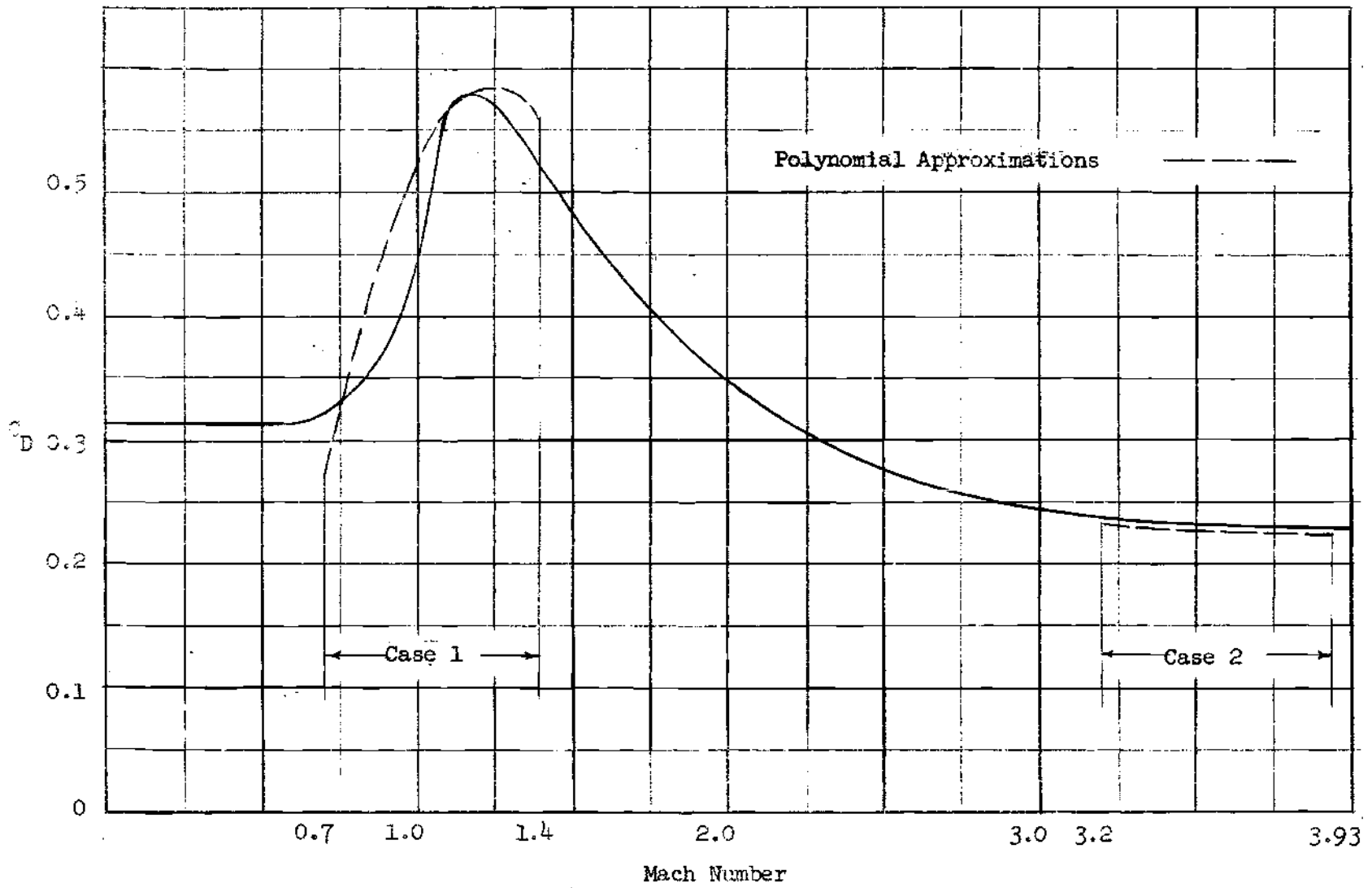


Fig. 3 Drag Coefficient for Cases 1 and 2

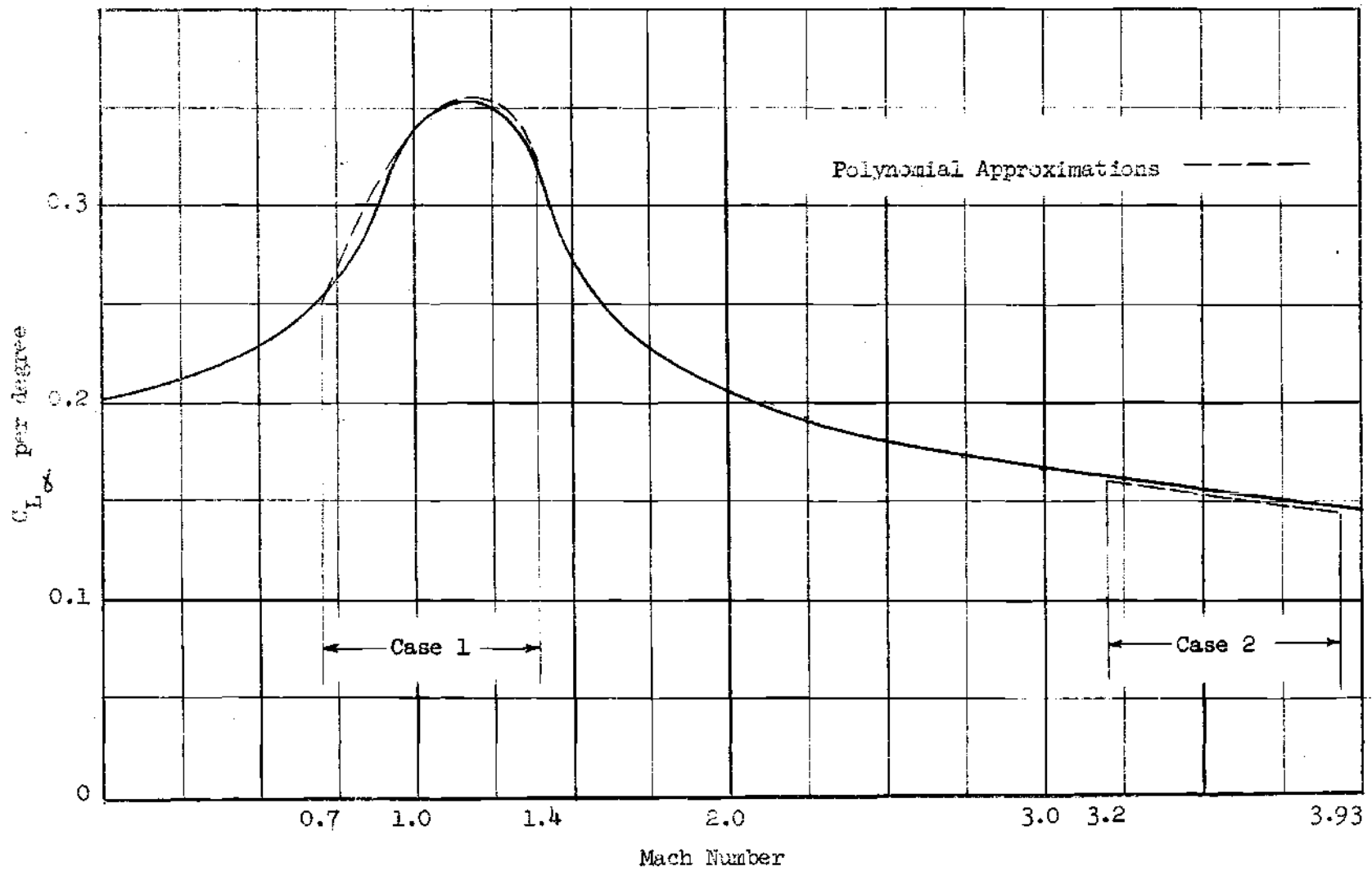


Fig. 4 Lift Curve Slope for Cases 1 and 2

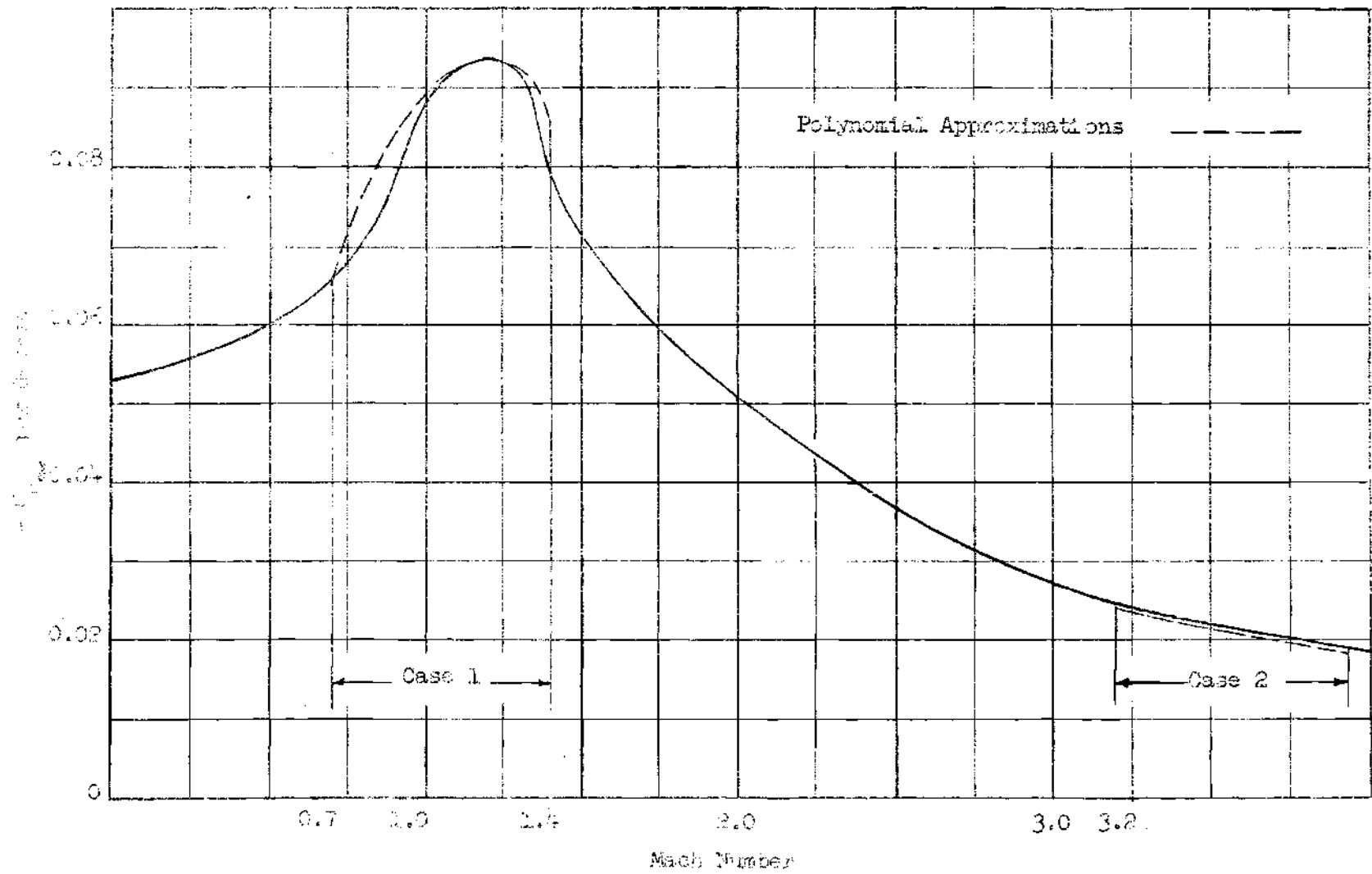


Fig. 5 Static Stability Coefficient for Cases 1 and 2

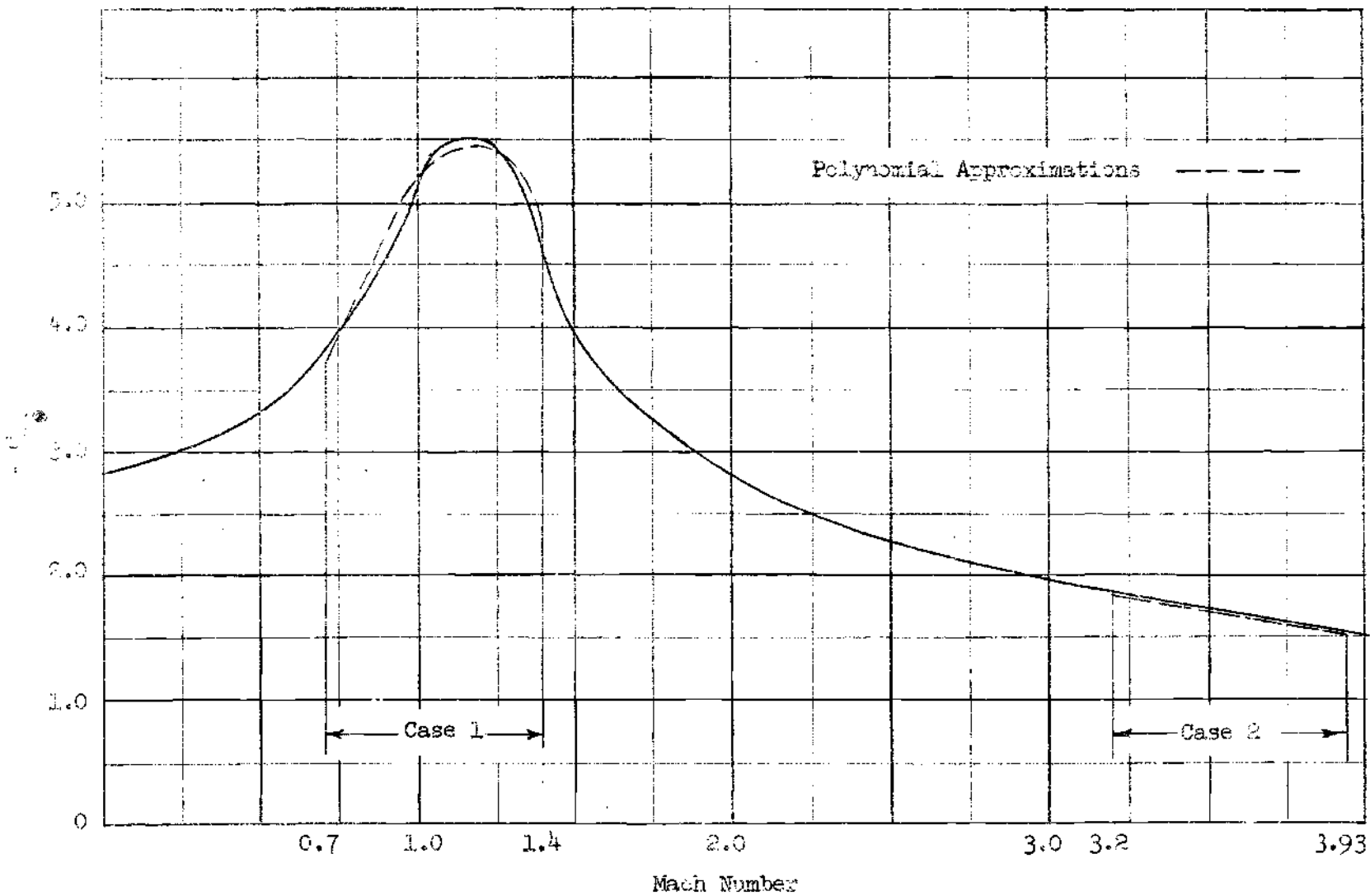


Fig. 6 Dynamic Stability Coefficient for Cases 1 and 2

Table 2

Polynomial Approximations to Mach Number Variations of
Aerodynamic Coefficients for Cases 1 and 2

$$\frac{C_D}{C_{D_0}} = A_1 M^2 + A_2 M + A_3$$

$$\frac{C_{m\alpha}}{C_{m\alpha_0}} = B_1 M^2 + B_2 M + B_3$$

$$\frac{C_{L\alpha}}{C_{L\alpha_0}} = C_1 M^2 + C_2 M + C_3$$

$$\frac{C_{m\dot{\alpha}}}{C_{m\dot{\alpha}_0}} = D_1 M^2 + D_2 M + D_3$$

	<u>Case 1</u>	<u>Case 2</u>
A_1	- 1.838794	0.05715617
A_2	4.587093	- 0.5069814
A_3	- 1.817893	2.1096658
B_1	- 1.684511	0.11740257
B_2	3.874374	- 1.2019164
B_3	- 1.1224822	3.9102604
C_1	- 1.6073441	0.022603568
C_2	3.696891	- 0.30053425
C_3	- 1.0252534	1.8323823
D_1	- 2.0009154	0.16037681
D_2	4.602105	- 1.4065991
D_3	- 1.5211538	4.0509307

$$\frac{d^2 \alpha}{d\tau^2} + f(\tau) \frac{d\alpha}{d\tau} + h(\tau) \alpha = 0 \quad (40)$$

The time variations of altitude, $y(\tau)$, and velocity, $\dot{y}(\tau)$, may now be calculated from the translational equation (39). In this manner, the variation of Mach number with time and, indirectly, the time variations of the aerodynamic coefficients may be calculated. Substitution of the calculated time variations of the aerodynamic coefficients in Equation (40) gives the solution, $\alpha(\tau)$.

The predictor-corrector method was used to solve the translational equation of motion. The oscillatory equation was solved utilizing the method of Runge and Kutta.

The applicability of exponential variations of $f(\tau)$ and $M(\tau)$, for these two cases, is illustrated in Figures 7 and 8. The assumption of exponential variations is better for Case 2 than for Case 1 because the curvatures of the plots of aerodynamic coefficients versus Mach number for Case 1 change sign.

A comparison of the envelope curves obtained from the Wintner solution, Equation (27), the exponential solution, Equation (34), and the exact solution of Equations (39) and (40) is made in Figures 9 and 10 for the two cases studied here. It is interesting to note that the Wintner and exponential solutions become less accurate with increasing time. Since, for this study, the constant b is negative for both cases, this occurrence could have been predicted from the discussion in Chapter III.

The first several oscillations of $\alpha(\tau)$ obtained from the exponential solution are compared with the numerical solution in Figures 11.

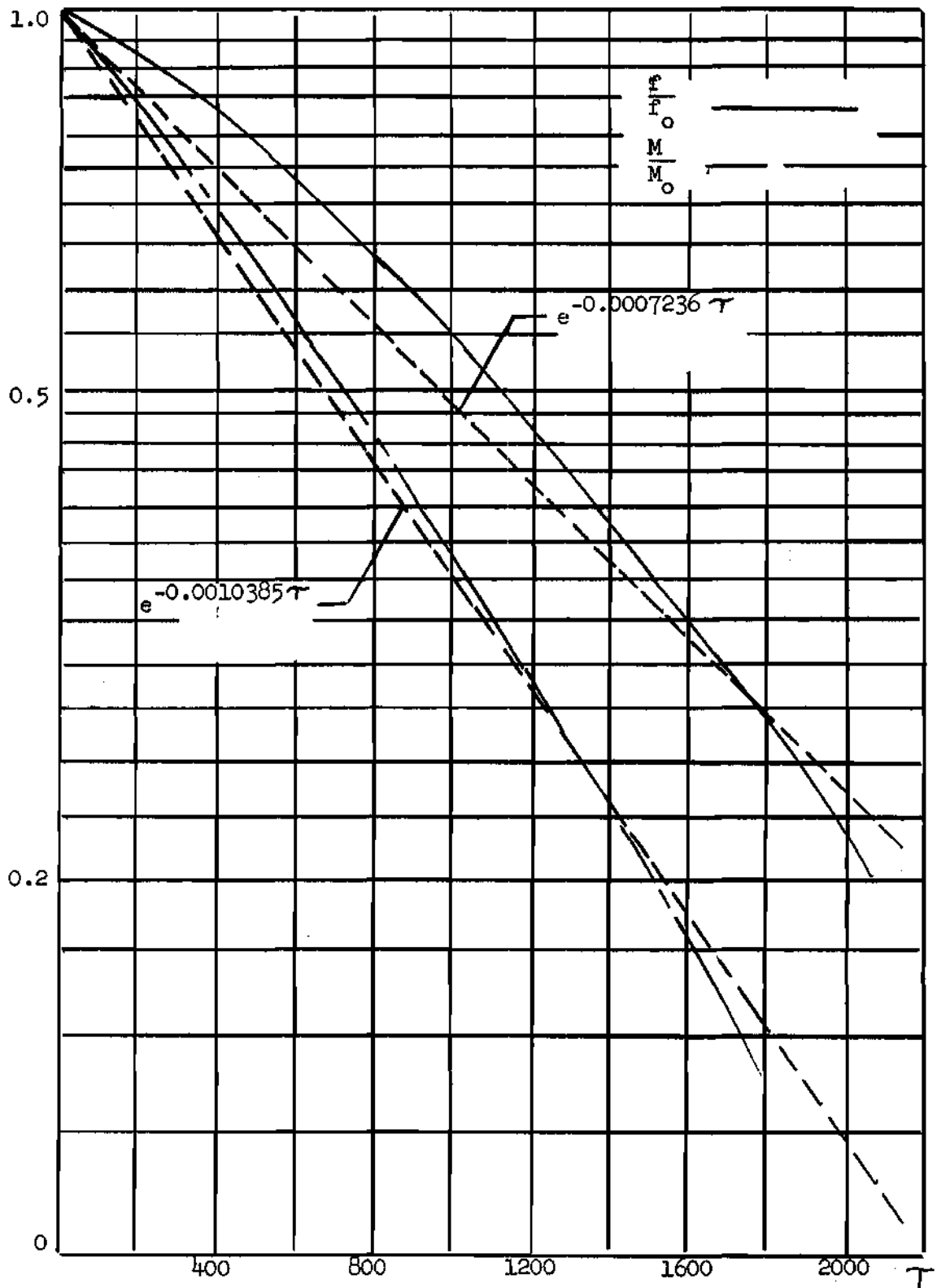


Fig. 7 Variations of f and M for Case 1

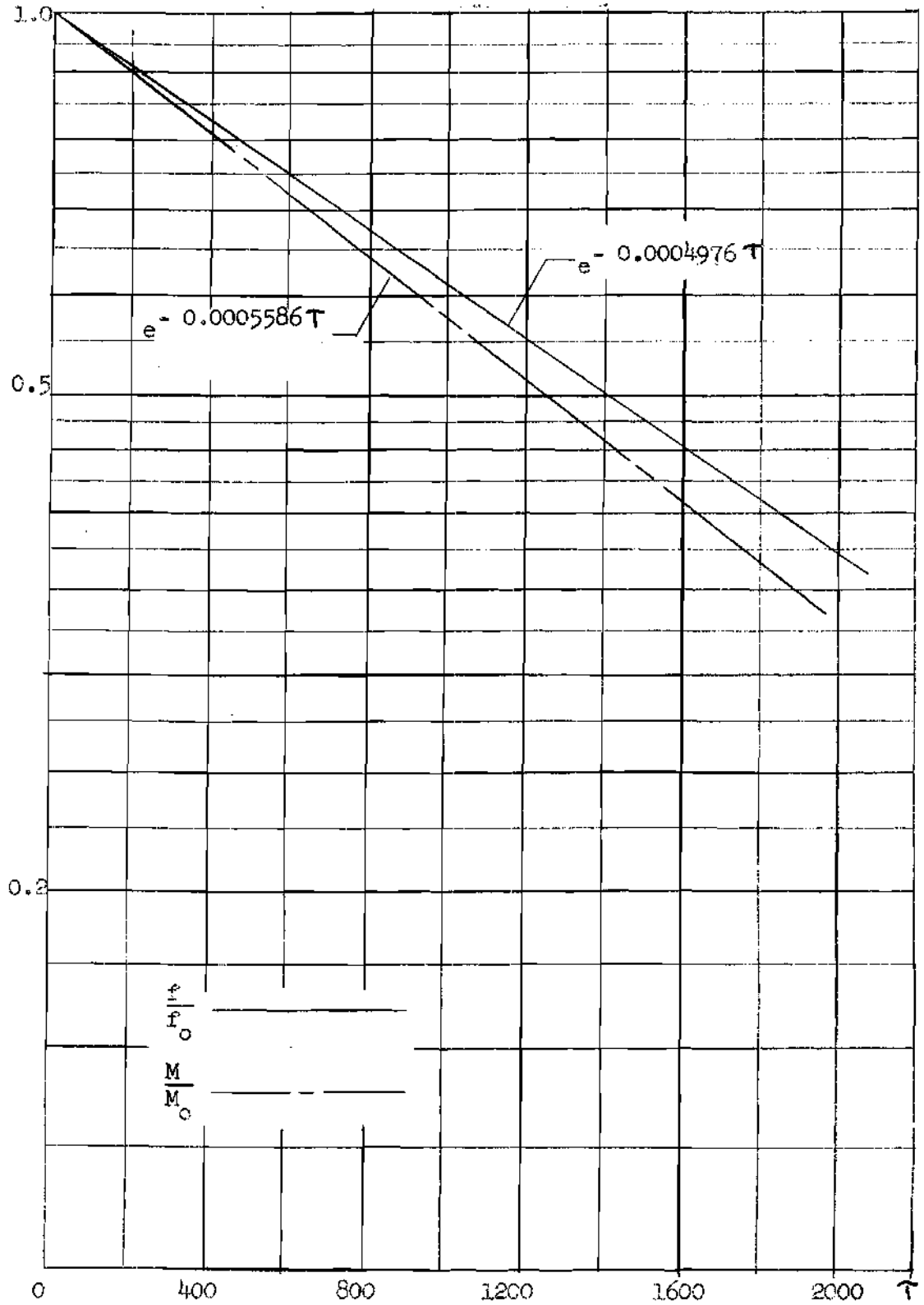


Fig. 8 Variations of f and M for Case 2

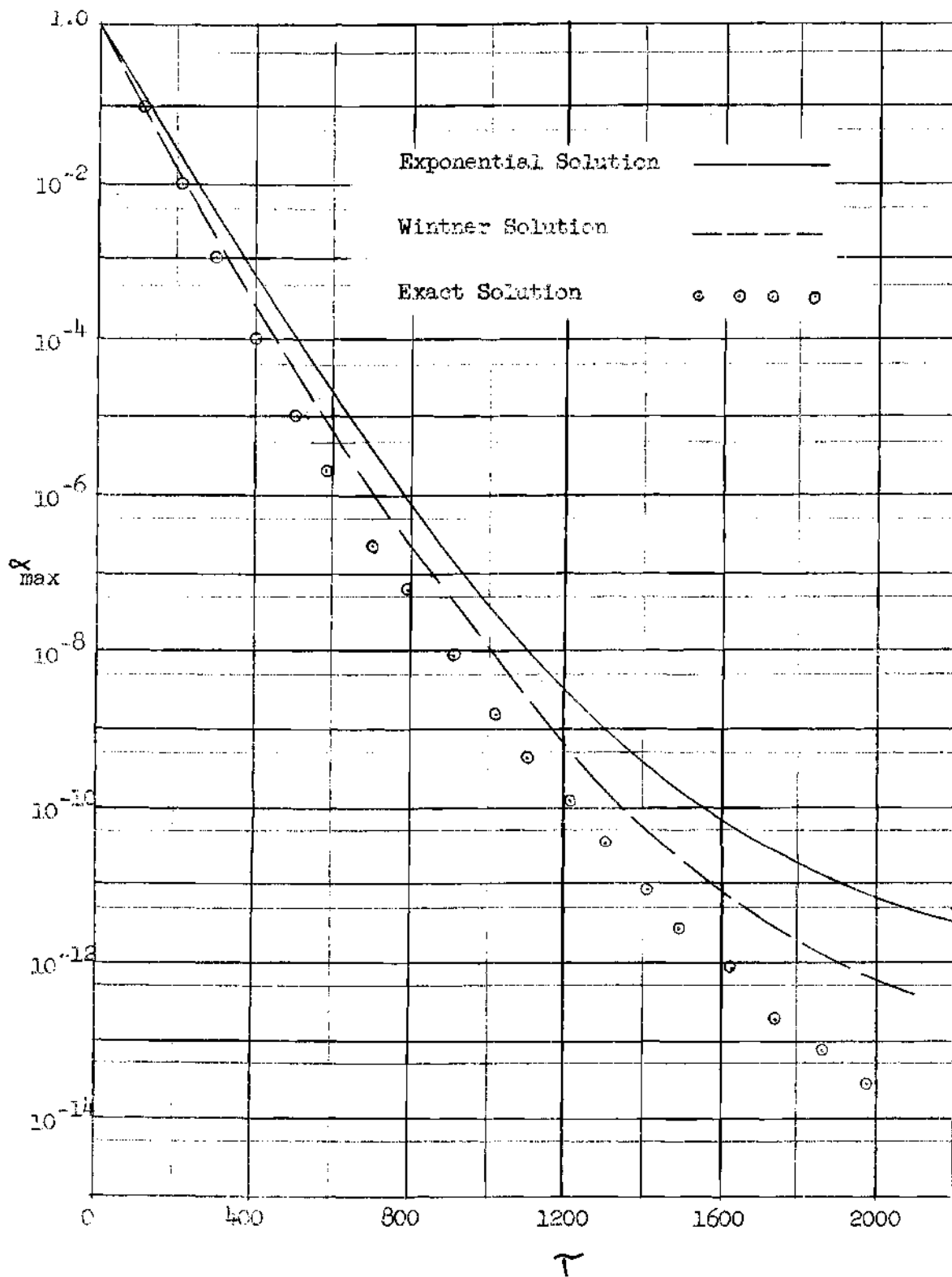


Fig. 9 Envelopes of Exponential Solution, Wintner Solution, and Exact Solution for Case 1.

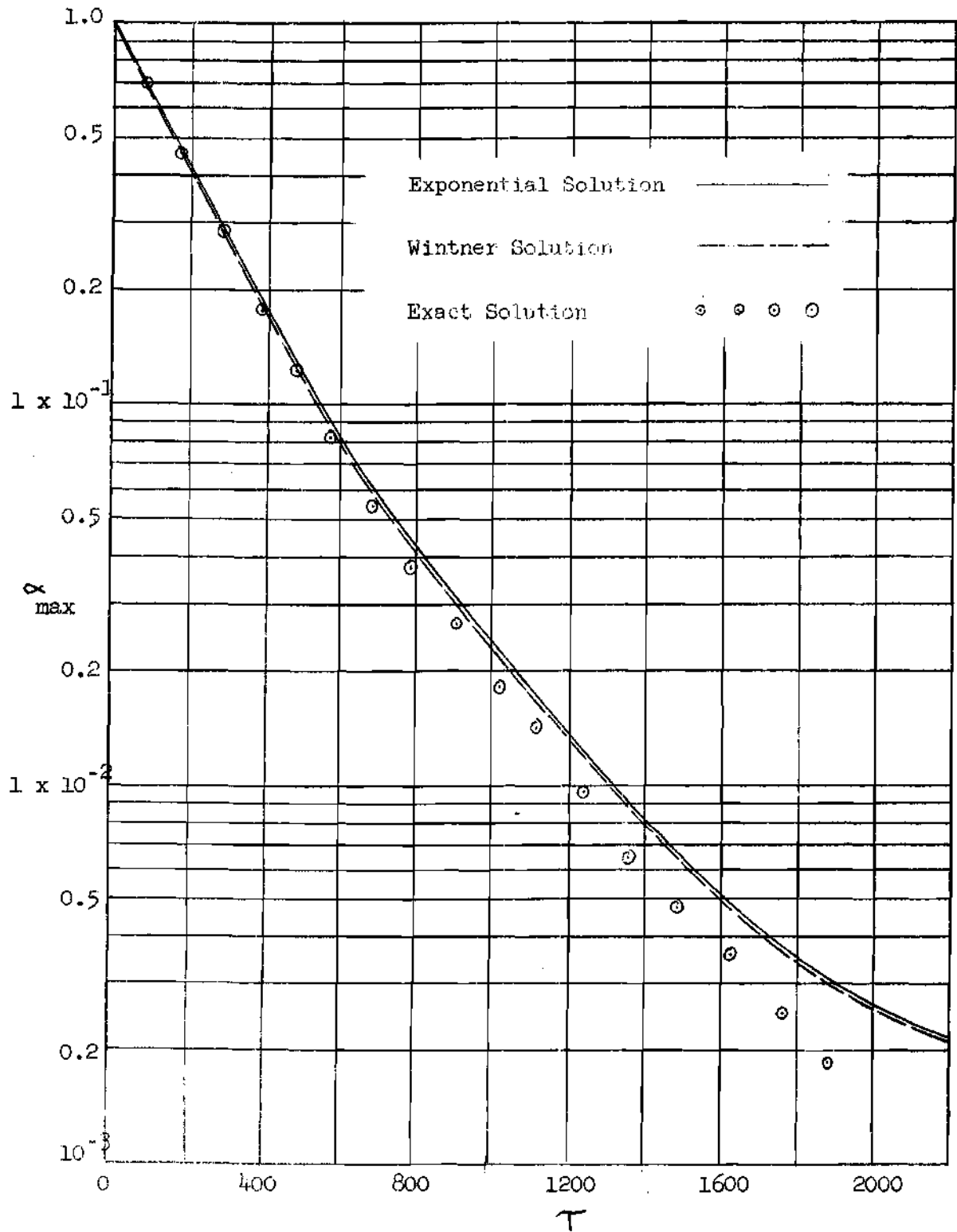


Fig. 10 Envelopes of Exponential Solution, Wintner Solution, and Exact Solution for Case 2

and 12. The exponential solution is seen to be quite accurate over this range of time.

The time, τ_{div} , at which divergence begins is predicted by the exponential solution as follows:

$$\tau_{div} = \frac{1}{a} \ln \frac{2f_0}{b} \quad (41)$$

For Case 1, $\tau_{div} = 6303$ and for Case 2, $\tau_{div} = 7154$. Figures 9 and 10 indicate that these estimates are conservative.

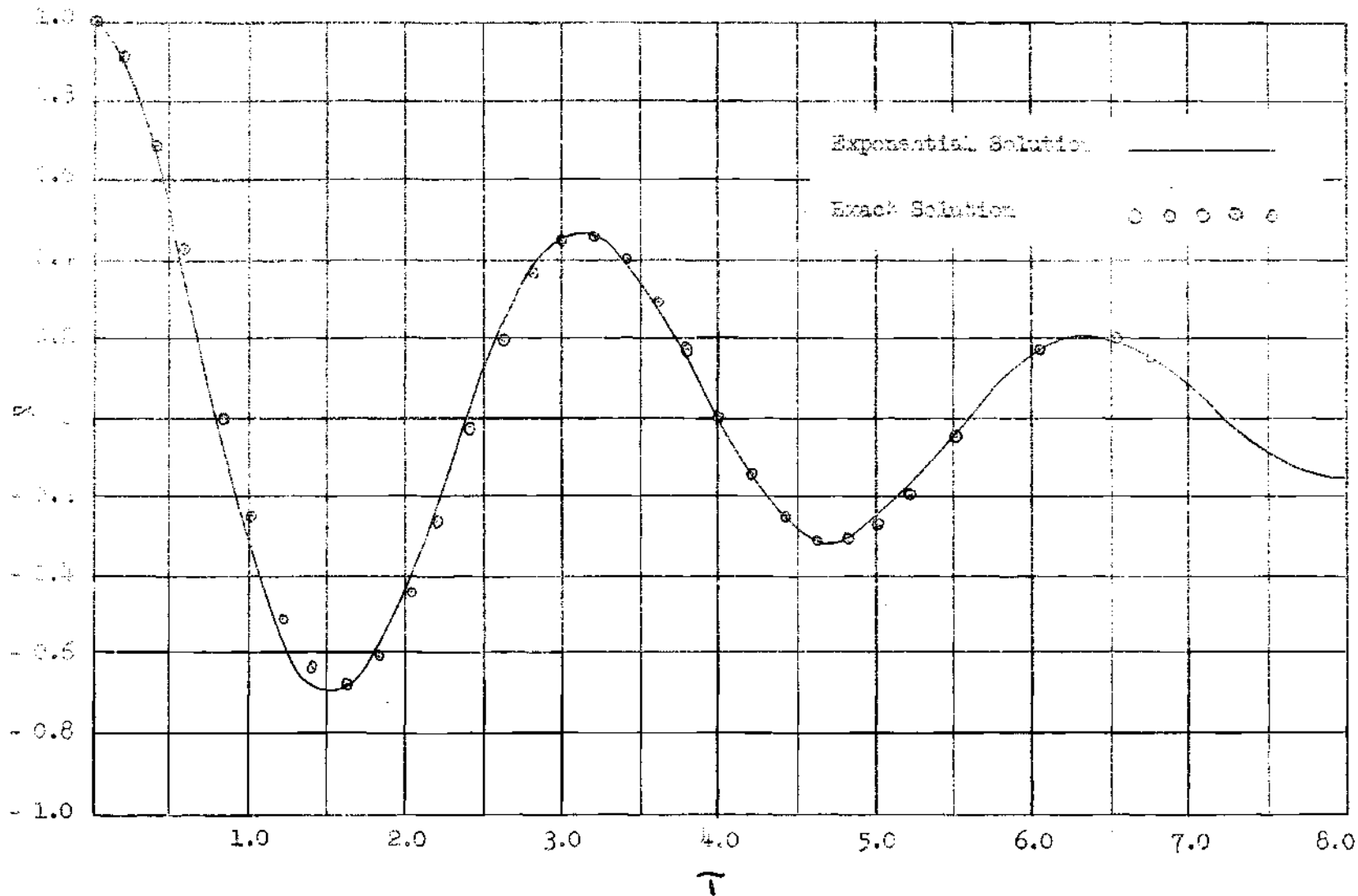


Fig. 11 Exponential and Exact Solutions of Oscillatory Motion for Case 1

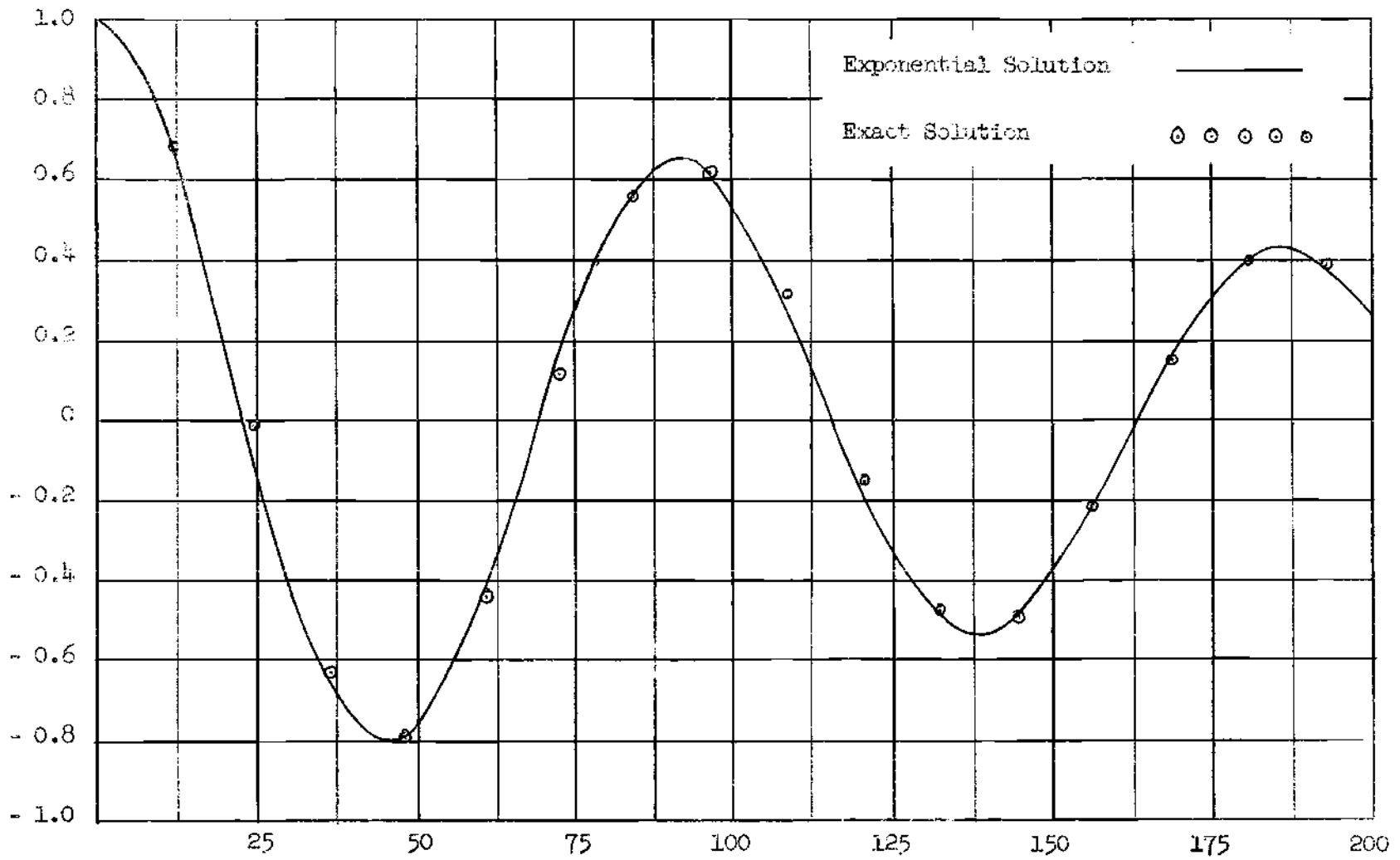


Fig. 12 Exponential and Exact Solutions of Oscillatory Motion for Case 2

CHAPTER V

RESULTS

Under the assumptions of a vehicle traversing an arbitrary path within the measurable atmosphere ($\rho > 0$), small deviations from equilibrium, and that the oscillatory flight path angle is much smaller than the static flight path angle, the equations of motion were established. The oscillatory equations of motion were combined into a single equation of the form:

$$\frac{d^2\alpha}{d\tau^2} + f(\tau) \frac{d\alpha}{d\tau} + h(\tau)\alpha = 0 \quad (14)$$

The asymptotic ($\tau \rightarrow \infty$) solution of this equation was found utilizing a theory developed by Wintner (Reference 8). A criterion for dynamic stability ($\frac{d\alpha_{\max}}{d\tau} < 0$) was established for general $f(\tau)$ and $M(\tau)$. It was stated as follows:

$$f(\tau) > - \frac{1}{2M(\tau)} \frac{dM}{d\tau} \quad (30)$$

where,

$$M(\tau) = h - \frac{1}{2} \frac{df}{d\tau} - \frac{1}{4} f^2 \quad (19)$$

Equation (30) is considered the most important result of this research because of the mildness of the conditions affecting its generality.

It was of interest to prescribe specific variations for $f(\tau)$ and $M(\tau)$ as an example. It was assumed that:

$$f = f_0 e^{-a\tau} \qquad M = M_0 e^{-b\tau} \qquad (31)$$

Because of the conditions on Wintner's solution, it was shown that the sign of the constant b determined the accuracy of the solution over a wide range of time. That is, if $b < 0$, the exponential solution becomes more accurate with increasing time, whereas, if $b > 0$, the opposite is true. This reasoning was verified in Figures 9 and 10 for the two cases considered in Chapter IV.

The criterion for dynamic stability for these exponential variations of $f(\tau)$ and $M(\tau)$ takes the simple form:

$$a\tau < \ln \frac{2f_0}{b} \qquad (36)$$

For the two specific cases studied here, Equation (36) predicted conservatively.

CHAPTER VI

CONCLUSIONS

Within the limitations of the assumptions of a vehicle traversing an arbitrary path within the measurable atmosphere ($\rho > 0$), small deviations from equilibrium, $M(\tau) > 0$ for all τ , $\int_0^{\infty} \sqrt{M(\tau)} |F(\tau)| d\tau < \infty$, and $F(\tau) \ll 1$, it is concluded that:

1. An approximate criterion for dynamic stability of a missile is:

$$f(\tau) > - \frac{1}{2M(\tau)} \frac{dM}{d\tau}$$

For the special case of $f = f_0 e^{-a\tau}$ and $M = M_0 e^{-b\tau}$, it is concluded that:

2. An approximate criterion for dynamic stability of a missile is:

$$a\tau < \ln \frac{2f_0}{b}$$

3. For large τ , the Wintner solution to the angular equation of motion:

$$\alpha(\tau) = \frac{C_1}{M^{1/4}(\tau)} e^{-1/2 \int f(\tau) d\tau} \cos \left[\int \sqrt{M(\tau)} d\tau - \psi_1 \right]$$

is more accurate for $b < 0$ than for $b > 0$.

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Note: "NACA" refers to National Advisory Committee for Aeronautics;
 "NASA" refers to National Aeronautics and Space Administration.