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ADDITIONAL MOMENTS OF INERTIA OF
A FULL-SCALE AIRPLANE AND ITS EFFECTS
ON DYNAMIC LATERAL STABILITY

A THESIS
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the Faculty of the Graduate Division
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of the Requirements for the Degree
Master of Science in Aeronautical Engineering

By
Robert Earl Lucas
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ADDITIONAL MOMENTS OF INERTIA OF
A FULL-SCALE AIRPLANE AND ITS EFFECTS
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# LIST OF SYMBOLS

The following symbols are used in the calculations for the additional-mass corrections of the moments of inertia of airplanes.

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<th>Symbol</th>
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<tr>
<td>$b$</td>
<td>span of surface</td>
</tr>
<tr>
<td>$S$</td>
<td>area of surface (for wing and horizontal tail includes area of fuselage between surface panels)</td>
</tr>
<tr>
<td>$A$</td>
<td>aspect ratio of surface $(b^2/a)$</td>
</tr>
<tr>
<td>$c_r$</td>
<td>root chord of surface</td>
</tr>
<tr>
<td>$c_t$</td>
<td>tip chord of surface $\left( \frac{2S}{b} - c_r \right)$</td>
</tr>
<tr>
<td>$\bar{c}$</td>
<td>mean chord of surface $(b/s)$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>plan-form taper ratio of surface $\left( c_r/c_t \right)$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>dihedral of wing or horizontal tail in degrees</td>
</tr>
<tr>
<td>$L_f$</td>
<td>length of fuselage</td>
</tr>
<tr>
<td>$v$</td>
<td>geometric average width of fuselage</td>
</tr>
<tr>
<td>$d$</td>
<td>geometric average depth of fuselage</td>
</tr>
<tr>
<td>$l$</td>
<td>component in plane of surface of perpendicular distance between axis of rotation and centroid of area of surface</td>
</tr>
<tr>
<td>$l_{fx}$</td>
<td>distance from centroid of side area of fuselage to axis of rotation parallel to and in the plane of the $X$-axis (conveniently referred to herein as axis of $X$ swinging)</td>
</tr>
<tr>
<td>$l_{fy}$</td>
<td>component of distance in the $X$-$Y$ principal plane of fuselage of the perpendicular distance between the centroid of plan area of fuselage and the axis of rotation parallel to and in the plane of the $Y$-axis</td>
</tr>
<tr>
<td>$l_{fz}$</td>
<td>distance from centroid of side area of fuselage to axis of rotation parallel to and in the plane of the $Z$-axis</td>
</tr>
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</table>
\( l_{tX} \) Distance from centroid of vertical-tail area to axis of rotation parallel to and in the plane of the X-axis

\( l_{tY} \) component of distance in the X-Y plane of fuselage of the perpendicular distance between the centroid of horizontal-tail area and the axis of rotation parallel to and in the plane of the Y-axis

\( l_{tZ} \) distance from centroid of vertical-tail area to axis of rotation for Z swinging

\( l_{wY} \) component of distance in the X-Y plane from the centroid of area of wing to axis of rotation parallel to and in the plane of the Y-axis

\( k \) coefficient of additional mass of a flat rectangular plate

\( k' \) coefficient of additional moment of inertia of a flat rectangular plate

\( D_{\lambda} \) taper-ratio correction factor

\( D_{\phi} \) dihedral correction factor

\( k_{X}, k_{Y}, k_{Z} \) coefficients of additional mass of equivalent ellipsoids for motion along the X-, Y-, and Z-axes, respectively

\( k'_{X}, k'_{Y}, k'_{Z} \) coefficients of additional moments of inertia of equivalent ellipsoids about the X-, Y-, and Z-axes, respectively

\( m_{a} \) additional mass of a body

\( I \) true moment of inertia of a body

\( I_{a} \) additional moment of inertia of a body

\( I_{v} \) virtual moment of inertia of a body

\( I_{X}, I_{Y}, I_{Z} \) moments of inertia about X-, Y-, and Z-body axes, respectively

\( I_{xa}, I_{ya}, I_{za} \) total additional moments of inertia about X, Y, and Z swinging axes, respectively

\( I_{5} \) 5 per cent increase in true moment of inertia
$I_{25}$  
25 per cent increase in true moment of inertia

$\rho$  
density of air, slug per cubic foot

Subscripts:

$v$  
w  
wing

$fus$  
 fuselage

$ht$  
h  
horizontal tail

$vt$  
v  
vertical tail

The following symbols are used in the stability investigations for effects of additional moments of inertia.

$L$  
$C_L$  
$\mu$  
$a$  
plane relative density factor

$\kappa$  
$K$  
radius of gyration

$C_y\beta$  
$Cy\beta$  
side force derivative

$C_n\beta$  
$\Delta \beta$  
directional stability derivative

$C_{1\beta}$  
dihedral effect

$C_{1p}$  
damping derivative in roll

$C_{n\gamma}$  
damping derivative in yaw

$C_{1\gamma}$  
rolling moment due to the yawing velocity

$C_{np}$  
yawing moment due to the rolling velocity

$C_{n\delta r}$  
rudder power parameter

$C_{n\delta r}$  
yawing moment due to rudder angular velocity

$C_{\alpha}$  
hinge moment parameter

$C_{\delta}$  
rate of change of rudder hinge moment with rudder angle

$C_{\delta \psi}$  
change in rudder hinge moment due to yawing velocity
\( \text{Chd} \delta \) 
rudder damping derivative

\( \text{Cl} \delta a \) 
rolling moment due to flap deflection

\( \text{Chd} \delta a \) 
change in aileron hinge moment due to rate of flap deflection

\( \text{Cl}d \delta a \) 
change in rolling moment due to rate of flap deflection

\( \text{Chd} \phi \) 
change in aileron hinge moment due to rolling velocity

\( \lambda \) 
root of characteristic equation

\( \tau \) 
time parameter
In stability investigations involving free-flight tests of dynamically scaled models or full-scale airplanes, it is generally assumed that air reactions on a moving body depend on the instantaneous state of velocity only, not on the accelerations. It is obvious that this assumption can only be an approximation and that some influence of the accelerations must exist. Since a body moving in a fluid originally at rest behaves like a body of increased inertia, it is necessary to determine the degree of influence of the fluid accelerations in terms of increased inertia in order to verify the validity of the simplifying assumptions.

No theory or method exists for determining the effects of ambient air on an airplane in flight. However, a method does exist for calculating the moments of inertia of a full-scale airplane being swung as an integral part of a pendulum and likewise the additional moments of inertia. Therefore, the additional moments of inertia are calculated for a full-scale airplane and the results are applied to several cases of lateral dynamic stability in a parametric study.

The results of the study show that the nonconservative forces, such as lift, hinge moments, and dihedral effects, play a decisive part in problems of dynamic stability. Large increases in moments of inertia, up to twenty-five per cent, have little effect on damping and slight influence on the nature of the modes of lateral motions.

Thus it can be concluded that no sufficient basis exists for
taking into account the influence of accelerations on the forces exerted by the air on a moving airplane.
CHAPTER I

INTRODUCTION

According to the theory of irrotational flow of a perfect fluid, a body moving in a fluid originally at rest behaves like a body of increased inertia. This behavior has the effects of increased mass with attendant increases in moments and products of inertia depending on the density and shape of the body. Since the air flow about an airplane is a discontinuous irrotational motion, it is difficult to apply this theory. However, the influence of the increased inertia must have some effect on the stability of the airplane and it is the purpose of this thesis to determine to what degree.

In most stability investigations, simplifying assumptions are made neglecting the influence of acceleration terms on the forces exerted by the air on a moving airplane. When the moments of inertia of dynamically scaled models or full-scale airplanes are determined by swinging the model or the airplane as an integral part of a pendulum, these virtual moments of inertia are corrected to the true moments of inertia for purposes of stability investigations. This is done because there is no correlation between the ambient air effects on an airplane in flight and a model or airplane being swung as an integral part of a pendulum.

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Consideration of the effects of ambient air has shown that the virtual moment of inertia of the airplane about any given axis of oscillation must be regarded as made up of two parts: namely, the true moments of inertia and the additional moments of inertia. The true moment of inertia is the moment of inertia of the structure and the entrapped air. The additional moment of inertia is that due to the increased mass effects of the external air influenced by the airplane's motion.

Since no theory or method exists for determining the effects of ambient air on an airplane in flight, it is desired to determine the additional moments of inertia of a full-scale airplane being swung as an integral part of a pendulum and to apply the results to stability investigations. For the purpose of this paper, the effects of increased inertia on stability has been limited to three conditions of dynamic lateral stability. It is intended to verify whether or not the simplifying assumption that the influence of acceleration terms on the forces exerted by the air on a moving airplane can be neglected is valid.

The fundamental basis of the effect of the ambient air mass on bodies undergoing acceleration has been known since early in the nineteenth century. Experimental data on the effect of the ambient air mass on the moments of inertia of flat plates obtained by swinging the

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plates as an integral part of a pendulum was provided in the mid-1930's. These data were used to estimate moment-of-inertia corrections for a full-scale airplane by considering the projected areas of various components of the airplane in planes normal and parallel to the plane of symmetry. Further corrections to the moment of inertia of a full-scale airplane were determined for the fuselage theoretically from the motion of an ellipsoid in a three-dimensional flow. Formulas have been developed for the rapid evaluation of the additional moment-of-inertia corrections for airplanes that are swung as an integral part of a pendulum. The additional moments of inertia estimated by these formulas have been correlated with experimental data obtained by swinging dynamically scaled models in air and in a vacuum chamber. The results indicate that satisfactory estimations can be obtained by use of these formulas.


CHAPTER II

DETERMINATION OF ADDITIONAL MOMENTS OF INERTIA OF A
FULL-SCALE AIRPLANE

Procedure.--The formulas for additional-mass corrections to the moments of inertia are applied to the Grumman F6F-3 airplane. The formulas estimate the total additional moments of inertia about the X, Y, and Z swinging axes of an airplane swung as an integral part of a pendulum. The swinging axes refer to the axes of rotation of the pendulum and are parallel to the airplane X, Y, and Z axes. These formulas are:

\[ I_{X_a}' = \frac{k_D}{48} \left( k^r D a y S_b^2 \right)_v + \rho \left( k_{f_Y} L_p \omega d \, l_{f_Y}^2 \right)_{fus} \]

\[ I_{Y_a}' = \left[ \frac{\rho}{2} \, k \cdot r_Y \cdot L_p \cdot \omega d \left( \frac{L_p^2}{4} + \frac{3d^2}{2x^2} \right) \right]_{fus} \]

\[ + \rho \left( k_{f_Z} L_p \omega d \, l_{f_Z}^2 \right)_{fus} + \frac{\rho_0}{4} \left( k S_b^2 \, l_{f_Y} \right)_{ht} \]

\[ I_{Z_a}' = \left[ \frac{\rho}{2} \, k \cdot r_Z \cdot L_p \cdot \omega d \left( \frac{L_p^2}{4} + \frac{3d^2}{2x^2} \right) \right]_{fus} \]

\[ + \rho \left( k_{f_Y} L_p \omega d \, l_{f_Z}^2 \right)_{fus} + \frac{\rho_0}{4} \left( k S_b^2 \, l_{f_Z} \right)_{vt} \]

The formulas for additional-mass corrections to the moments of inertia entail the application of the moment-of-inertia equation to
various portions of the airplane, which can be considered either as flat plates or ellipsoidal bodies, and then transferring the values to the reference swinging axis by use of the moment-of-inertia transfer equation:

\[ I_{a'} = I_a + m_a l^2 \]

The methods of determining the various dimensions and areas are as indicated in Appendix I. The centroids of the various parts of the airplane were found by plotting the parts to scale and using a planimeter to measure the areas and the moment arms. The distances between the swinging axes and the corresponding body axes were selected as being typical lengths for swinging full-scale airplanes as an integral part of a pendulum. The coefficients and correction factors for taper ratio and dihedral have been determined empirically and substantiated experimentally.

The virtual moment of inertia, \( I_v \), is made up of the true moment of inertia, \( I \), plus the additional moment of inertia, \( I_a \). However, the following equations show that the virtual moment of inertia of a body about its body axis is made up of the true moment of inertia about the body axis plus the additional moment of inertia about the swinging axis. Using the transfer equation

\[ I_v = I_{v'} + m l^2 \]
Substituting \[ I_v' = I' + I_a' \]

\[ I_v = I' + I_a' + m l^2 \]

Rearranging the equation, it becomes

\[ I_v' = (I' + m l^2) + I_a' \]

Substituting \[ I = I' + m l^2 \]

\[ I_v = I + I_a' \]

Therefore the additional moment-of-inertia equations were derived to determine the additional moments of inertia about the swinging axes. It can be seen that the additional moment of inertia is a function of the pendulum length, or the distance from the swinging axis to the corresponding body axis.

Results—Using the aforementioned procedure, the additional moments of inertia about the swinging axes are calculated for the Grumman F6F-3 airplane. The F6F-3 has a gross weight of 11,423 pounds giving it a mass of 355 slugs. The true moments of inertia for this gross weight have been determined by Grumman Aircraft Engineering Corporation and are listed below.

Values for the formula

\[ I_v = I + I_a' \]
are listed for comparing the results of this portion of the thesis.

All values are in slug feet.

\[ I_x = 9,050 \]

\[ I_{x_a} = 487 \]

\[ I_{y} = 9,537 \]

\[ \frac{I_{x_a}}{I_x} = 0.0539 = 5.39 \text{ per cent increase for the additional moment of inertia about the X axis} \]

\[ I_y = 11,900 \]

\[ I_{y_a} = 314 \]

\[ I_{y_a} = 12,214 \]

\[ \frac{I_{y_a}}{I_y} = 0.0264 = 2.64 \text{ per cent increase for the additional moment of inertia about the Y axis} \]

\[ I_z = 20,550 \]

\[ I_{z_a} = 1,719 \]

\[ I_{z_a} = 22,269 \]

\[ \frac{I_{z_a}}{I_z} = 0.0835 = 8.35 \text{ per cent increase for the additional moment of inertia about the Z axis} \]
CHAPTER III

EFFECTS OF ADDITIONAL MOMENTS OF INERTIA ON AIRPLANE STABILITY

Procedure

In deriving the formulas for additional-mass corrections to the moments of inertia of airplanes, vacuum-chamber tests were made on forty different spin-tunnel models. It was found that the effect of ambient air on the apparent moments of inertia were usually small but may be as large as twenty-five per cent of the true moments of inertia for airplanes with low wing loading. Therefore, the effects of aircraft stability due to additional moments of inertia should be determined.

The effects of additional moments of inertia on airplane stability are determined parametrically by varying the terms affected by the moment of inertia in the airplane's characteristic equations of motion. Thus it can be seen how the additional moments of inertia affect the nature of the motions which characterize the response of the airplane to a disturbance from some equilibrium flight condition and the nature of the transient motions of the airplane in response to the movement of its controls.

Three cases of lateral dynamic stability are investigated. First, the characteristic modes of lateral motion for the airplane with

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6 Malvestuto and Gale, op.cit., p. 1
controls locked are studied; secondly, the dynamic lateral stability with the rudder free to float under the influence of rudder hinge moment parameters; lastly, the dynamic lateral stability with the ailerons free but the rudder locked. In each of the three cases the stability derivatives are held constant while the moments of inertia are varied for three different values; the true moment of inertia, a five per cent increase in the moments of inertia, and a twenty-five per cent increase in the moments of inertia.

Characteristic motions of the airplane with controls locked.—If the controls of the airplane are assumed locked, the equations of motion for level flight are: 7

\[
\begin{align*}
\left( c_{y\beta} - 2 \frac{d}{2} \right) \beta - 2 d \dot{\psi} + c_{L} \phi &= 0 \\
\mu c_{1\beta} \beta + \left( \frac{c_{1r}}{2} d \right) \dot{\psi} + \left( \frac{c_{1p}}{2} d - J_x d^2 \right) \phi &= 0 \\
\mu c_{n\beta} \beta + \left( \frac{c_{nr}}{2} - J_z d \right) d \dot{\psi} + \frac{c_{np}}{2} d \phi &= 0
\end{align*}
\]

where
\[
\begin{align*}
J_x &= 2 \left( \frac{k_x}{b} \right)^2 \\
J_z &= 2 \left( \frac{k_z}{b} \right)^2
\end{align*}
\]

and $k$ is the airplane's radius of gyration about each of the airplane axes and is determined for each of the airplane axes by:

$$K_X = \sqrt{\frac{I_X}{m}} \quad ; \quad K_Z = \sqrt{\frac{I_Z}{m}}$$

The solution to these equations of motion are obtained by assuming the solution in the form:

$$\beta = \beta_i e^{\lambda t/\kappa} \quad ; \quad \psi = \psi_i e^{\lambda t/\kappa} \quad ; \quad \text{etc.}$$

Substituting the assumed solutions into these equations of motion and dividing out the term $e^{\lambda t/\kappa}$, the equations reduce to three simultaneous equations in $\lambda$. The value of $\lambda$ is found by equating the determinant of the coefficients to zero and expanding the determinant to obtain a quartic equation in $\lambda$:

$$A \lambda^4 + B \lambda^3 + C \lambda^2 + D \lambda + E = 0$$

where

$$A = 1$$

$$B = -\frac{1}{2} \left( C_{y/\beta} + \frac{C_{n/\gamma}}{I_X} + \frac{C_{p/\gamma}}{I_Z} \right)$$
\[
c = \frac{1}{4} \frac{1}{J_x J_Z} \left( c_{lp} c_{n_r} - c_{l_r} c_{n_p} \right) + \frac{c_y \beta}{4} \left( \frac{c_{n_r}}{J_Z} + \frac{c_{l_p}}{J_x} \right) + \frac{\mu c_{n \beta}}{J_Z}
\]

\[
D = -\frac{\mu}{2 J_x J_Z} \left( c_{n \beta} c_{l_p} - c_{l \beta} c_{n_p} \right) - \frac{\mu}{2 J_x} c_L c_{l \beta} - \frac{c_y \beta}{8 J_x J_Z} \left( c_{l_p} c_{n_r} - c_{l_r} c_{n_p} \right)
\]

\[
E = \frac{\mu}{4 J_x J_Z} c_L \left( c_{l \beta} c_{n_r} - c_{n \beta} c_{l_r} \right)
\]

In the above equations for the coefficients for the quartic, the only terms involving the moments-of-inertia are \( J_x \) and \( J_Z \). Since the "J" values are a function of the radius of gyration and the wing span, and the radius of gyration is a function of moment-of-inertia and mass, the following relationship exists:

\[
J = 2 \left( \frac{k}{b} \right)^2
\]

and

\[
k = \sqrt{\frac{I}{m}}
\]

Substituting, we get the relation:

\[
J = 2 \frac{I}{m b^2}
\]
Inasmuch as the additional mass was used to determine the corrections to the moments of inertia, or additional moments of inertia, the true mass of the airplane configuration is used in the parametric substitutions for the coefficients and is held constant. Therefore, the values of $J$ vary in direct proportion to the values of $I$, since the mass and wing span are constant.

Since the stability derivative information for the Grumman F6F-3 is not at hand, an airplane similar to the F6F-3 with typical stability derivatives will be used to demonstrate the effects of increased moments of inertia on the aforementioned lateral modes of motion. Substituting the derivatives with the assumed flight $C_L$ and density factor, $\mu$, in the equations for the coefficients, values for the coefficients in the quartic equation are found. The real roots of the quartic are extracted by Horner's method, in this case two real roots, and the remaining roots are found by using the quadratic equation. These roots can then be examined for the characteristics of the state of motion.

In examining the roots of the quartic equation for the state of motion, the negative real roots indicate a stable motion, or convergence, while positive roots indicate an unstable motion, or divergence. The degree of convergence or divergence is in direct relation to the value of the root. In the case of complex roots the motion will be stable if the real parts of the roots have the negative sign and unstable if the real parts of the roots have a positive sign.

The period and damping of the oscillatory modes of motion are determined from the complex roots. If the values of $\lambda$ for the complex roots take the form:
\[ \lambda = \varepsilon + i n \]

the period and damping are determined by the formulas:

\[ \text{Period} = \frac{2 \pi}{n} \text{ seconds} \]

\[ \text{Time to damp to 1/2 amplitude} = \frac{\ln 2}{\varepsilon} \text{ seconds} \]

Dynamic lateral stability, rudder free (ailerons locked).—To continue the study of the effects of increased inertia on the airplane's lateral dynamics, it is necessary to investigate the new modes of motion introduced through freeing the rudder. For this condition, only the "snaking" mode is of design importance to the aerodynamicist. To simplify the mathematics, the unimportant modes which slightly affect the "snaking" mode are eliminated. The simplified equations of motion are:

\[ (-\mu c_n \beta + \frac{c_{nr}}{2} d - J_z \ddot{d}) \dot{\psi} \]

\[ + \mu (c_n \delta_r + c_{nd} \delta_r d) \delta_r = 0 \]

\[ (-c_n \beta + c_{nd} \psi \dot{d}) \dot{\delta} + (c_n \delta + c_{nd} \delta d) \delta = 0 \]

These equations contain the partial derivatives with respect to rudder control deflection and rate of deflection as well as the rudder hinge moment equation. Using the same assumed solution substitution method as used in the control locked case and expanding the resulting

\[ ^6 \text{Ibid., pp. 458-460.} \]
determinant of the coefficients, a cubic equation in $\lambda$ is obtained:

$$A \lambda^3 + B \lambda^2 + C \lambda + D = 0$$

where

$$A = -J_z C_{hd}$$
$$B = C_{hd} \frac{c_{n\beta}}{2} - J_z C_{h} - \mu c_{c_{d\beta} r} C_{hd} \psi$$
$$C = \frac{c_{n\beta}}{2} C_{h} \psi - \mu c_{n\beta} c_{n\beta} c_{hd} + \mu C_{hd} c_{n\beta}$$
$$D = \mu (C_{n\beta} c_{n\beta} - C_{n\beta} C_{h} \psi)$$

In solving these equations for the coefficients, typical derivatives are used for the same airplane configuration as the controls locked case. The moment-of-inertia terms are varied in the same manner, i.e., the "J" values are varied in direct proportion to the variance in moments of inertia. The real roots of the cubic equation are determined by Horner's method and the complex roots are determined by the quadratic equation. The sign and the magnitude of the roots indicate the nature of the stability and the degree of convergence or divergence.

Dynamic lateral stability, ailerons free (rudder locked). -- To complete the study of the effects of increased inertia on the airplane's lateral dynamics, the modes of motion introduced through freeing the ailerons
are investigated. Using the general equations of motion to characterize the complete motion results in six roots. For a simpler mathematical treatment, the yawing and sideslipping motion can be neglected, eliminating the spiral mode and the control-fixed lateral oscillation mode which have only small influence on the rolling modes. Further, assuming the aileron's inertia to be negligible and that aileron mass balance shows little possibility of producing any undamped oscillation, the equations of motion may be reduced to the following pair:

\[
\left( \frac{C_{1p}}{2} - J_X \beta \right) \ddot{\phi} + \mu \left( C_{1d} \alpha + C_{1d} \beta \right) \ddot{\alpha} = 0
\]

\[
C_{hd} \dot{\phi} + \left( C_{h \delta} + C_{nd} \beta \right) \delta \alpha = 0
\]

Using the same procedure for the solution of these equations, the characteristic motion is defined by a quadratic in \( \lambda \), the roots of which indicate the nature of the modes. The quadratic in general form is:

\[
A \lambda^2 + B \lambda + C = 0
\]

where \( A = -C_{hd} \delta J_X \)

\[
B = C_{hd} \delta \frac{C_{1p}}{2} - J_X C_{h \delta} - \mu C_{1d} \alpha C_{hd} \phi
\]

\[
C = \mu C_{1d} \alpha C_{hd} \phi
\]

\(^9\text{Ibid., pp. 465-466.}\)
Using typical values of the derivatives based on the same airplane configuration and the usual variation in "J", the numerical values of the roots are determined.

Results

The numerical values of the coefficients of the characteristic equations have been determined using typical values for the derivatives and varying the moment-of-inertia terms. Table 1 shows the variations in the coefficients as the moment-of-inertia is increased.

After the coefficients are known, the roots of the characteristic equations are extracted by using Horner's method and the quadratic formula. The variations in the roots are tabulated in Table 2.

The variations in the period of oscillation and the time to damp to one-half amplitude with increased inertia are presented in Table 3. The values are presented in terms of $\tau$ which, for a given value of the airplane density factor, $\mu$, depends on the airplane's span and velocity.
<table>
<thead>
<tr>
<th>Flight Condition</th>
<th>Coefficient</th>
<th>True Moment-of-Inertia</th>
<th>5 Per cent Increase Moment-of-Inertia</th>
<th>25 Per cent Increase Moment-of-Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Controls</td>
<td>A</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Locked</td>
<td>B</td>
<td>13.4</td>
<td>12.76</td>
<td>10.74</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>67.4</td>
<td>62.67</td>
<td>48.185</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>394.0</td>
<td>358.04</td>
<td>253.68</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>-73.8</td>
<td>-65.9</td>
<td>-47.1</td>
</tr>
<tr>
<td>Rudder</td>
<td>Free</td>
<td>A</td>
<td>0.0003</td>
<td>0.000315</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
<td>0.01488</td>
<td>0.01558</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C</td>
<td>0.0282</td>
<td>0.0282</td>
</tr>
<tr>
<td></td>
<td></td>
<td>D</td>
<td>0.268</td>
<td>0.268</td>
</tr>
<tr>
<td>Aileron</td>
<td>Free</td>
<td>A</td>
<td>0.00028</td>
<td>0.000294</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
<td>0.012356</td>
<td>0.013086</td>
</tr>
<tr>
<td></td>
<td></td>
<td>C</td>
<td>0.09408</td>
<td>0.09408</td>
</tr>
</tbody>
</table>
### Table 2. Roots of the Characteristic Equations

<table>
<thead>
<tr>
<th>Flight Condition</th>
<th>Roots</th>
<th>True Moment-of-Inertia</th>
<th>5 Per cent Increase</th>
<th>25 Per cent Increase</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda$</td>
<td>$I$</td>
<td>$I_5$</td>
<td>$I_{25}$</td>
</tr>
<tr>
<td><strong>Locked</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Controls</td>
<td>$\lambda_1$</td>
<td>0.1815</td>
<td>0.1809</td>
<td>0.1793</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2$</td>
<td>-10.61</td>
<td>-10.12</td>
<td>8.63</td>
</tr>
<tr>
<td></td>
<td>$\lambda_3,4$</td>
<td>-1.48/6.011</td>
<td>-1.41/5.861</td>
<td>-1.14/5.3951</td>
</tr>
<tr>
<td><strong>Free</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rudder</td>
<td>$\lambda_1$</td>
<td>-48.1</td>
<td>-47.94</td>
<td>-47.90</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2,3$</td>
<td>-75/4.61</td>
<td>-73/4.34</td>
<td>-65/3.541</td>
</tr>
<tr>
<td>Aileron</td>
<td>$\lambda_1$</td>
<td>-9.8</td>
<td>-9.03</td>
<td>-8.7</td>
</tr>
<tr>
<td></td>
<td>$\lambda_2$</td>
<td>-35.8</td>
<td>-35.375</td>
<td>-32.0</td>
</tr>
</tbody>
</table>
Table 3. Variations In Period and Time to Damp to One-Half Amplitude for Variations in Moment-of-Inertia (Controls Locked)

<table>
<thead>
<tr>
<th></th>
<th>True Moment-of-Inertia</th>
<th>5 Per cent Increase</th>
<th>25 Per cent Increase</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
<td>I₀</td>
<td>I₂⁵</td>
</tr>
<tr>
<td><strong>Period (Sec.)</strong></td>
<td>1.05 τ</td>
<td>1.07 τ</td>
<td>1.16 τ</td>
</tr>
<tr>
<td><strong>Time to Damp To One-Half Amplitude (Sec.)</strong></td>
<td>0.468 τ</td>
<td>0.492 τ</td>
<td>0.605 τ</td>
</tr>
</tbody>
</table>
CHAPTER IV

CONCLUSIONS

The formulas which have been developed for estimating the additional mass corrections to the moments of inertia of airplanes apply only to the experimental moments of inertia determined by the pendulum method. The formulas provide the values of the additional moments of inertia of the airplane about the swinging axes, or the axes of rotation of the pendulum. When an airplane or dynamically scaled model is swung as an integral part of a pendulum, the measured moment of inertia is a virtual moment of inertia. It has been shown in Chapter II that the virtual moment of inertia of the airplane or model about the body axes is made up of the true moment of inertia about the body axes plus the additional moment of inertia about the swinging axes of the pendulum. Thus the measured moment of inertia determined by the pendulum method can be corrected to the true moment of inertia by use of the formulas developed for rapid calculation of the additional moments of inertia about the swinging axes.

Effects of Additional Moments of Inertia on Airplane Stability

The stability investigations for the purpose of this paper are limited to three cases of lateral stability. It is difficult to reach simple conclusions of significance in the mathematical treatment of the
stability against lateral disturbances because of the great number of poorly known force and moment coefficients and their derivatives encountered. Therefore, it is necessary to deal in trends and the nature of the motion rather than in numerical answers. In the stability investigations involving the effects of additional moments of inertia on lateral stability, one can merely indicate the sensitivity of these effects and the trends in a theoretical discussion of their influences. Controls Locked.—In the divergent mode of motion known as the spiral divergence, the additional moments of inertia tend to decrease the divergence. However, the decrease has negligible effect on the airplane's characteristics since most airplanes are spirally unstable to prevent unstable lateral oscillations. The increased inertia effects are small compared to increasing the dihedral, and thus $C_{l \beta}$, the dihedral effect. Increasing $C_{l \beta}$ can, in effect, produce spiral stability while reducing the damping of the lateral oscillations. Also the spiral divergence is considerably influenced by the lift coefficient, increasing as $C_L$ is increased. The additional moment-of-inertia effects could never change spiral instability into spiral stability. Therefore, neglecting the corrections for additional moments of inertia would have only slight influence on spiral stability compared to other forces acting on the airplane.

The heavy convergent mode of motion is not seriously affected by additional moments of inertia. Although increased inertia tends to reduce the degree of convergence, the lateral oscillation mode of motion is not very important in the controls locked case because of the normally heavy damping. The directional stability derivative, $C_{n \beta}$, is always
made large to keep the airplane far from the oscillatory boundary. The effects of excessive dihedral can bring the airplane sufficiently close to the oscillatory boundary to make lateral oscillations very objectionable to the pilot. Thus the relatively small effects of the additional moments of inertia has little influence on the usually heavily-damped lateral oscillation mode.

The additional moments of inertia tend to lengthen the period of oscillation while reducing the damping. Neglecting the moment-of-inertia corrections will show indications of a shorter period although increased damping, which is a more objectionable condition to the pilot.

Rudder Free.—The roots of the characteristic equation of the typical airplane in the rudder free condition indicate an extremely heavy convergence and a damped oscillation with light damping. The additional moment of inertia tends to reduce the heavy convergence and decrease the damping. However, these effects are small compared to the hinge moment parameters or rudder balance. For example, balancing out the rudder reduces the heavy convergence an order of magnitude compared to the increased inertia effects and can make the damping of the oscillatory, or "snaking", mode go negative. The rudder restoring tendency, \( C_{h g} \), and the hinge moment parameter, \( C_{h\alpha} \), are the sensitive variables in the "snaking" mode of lateral oscillations. Therefore, the simplifying assumption that the influence of additional moments of inertia can be neglected is valid.

Aileron Free.—The roots of the characteristic equation of the typical airplane for the aileron free case indicate two very heavy convergences
which can be thought of as the heavy rolling convergence and the snapping back of the aileron to the normal floating angle under the influence of the hinge moment parameter. As in the rudder free condition, the influence of the additional moment of inertia is very small compared to the aerodynamic balance variables. Even extreme values of the aileron hinge moment parameters, $C_{h\alpha}$ and $C_{h\phi}$, would be required to introduce oscillatory instability. Thus no noticeable differences could be detected in the characteristics of this mode with neglect of the moment-of-inertia corrections.
FIGURE 1 - SKETCH OF F6F-3 AIRPLANE
APPENDIX
APPENDIX I

METHODS OF CALCULATING THE ADDITIONAL MOMENTS-OF-INERTIA ABOUT THE REFERENCE SWINGING AXES OF THE GRUMMAN F6F-3 AIRPLANE

Pertinent Data.—Figure 1 shows a sketch of the F6F-3 in position for swinging. Pertinent coefficients and dimensional data are as follows:

Wing:

Area, $S$, sq ft $334$
Span, $b$, ft $42.833$
Aspect ratio, $A$ $5.5$
Mean chord, $c$, ft ($S/b$) $7.8$

Component of distance in the $X-Y$ plane from centroid of area of wing to $Y$-axis of rotation $W_y$, ft $1.72$
Taper ratio, $\lambda$ $2.1$
Dihedral, $\alpha$, deg $7.5$

Additional mass coefficient, $k$ (from ref. 3) .94
Additional moment-of-inertia coefficient, $k'$, for the $X$ swinging (from ref. 3) .86

For the $Y$ swinging, the reciprocal of the aspect ratio is used to give an additional moment-of-inertia coefficient $k''$ (from ref. 3) .13

Taper ratio correction, $D_x$ (from ref. 5) .82
Dihedral correction, $D_\alpha$ (from ref. 3) .7
Fuselage:

Length, \( L_f \), ft 33.833

Average width, \( w \), ft 3.62

Average depth, \( d \), ft 5.54

Distance from centroid of side area of fuselage to X-axis of rotation, \( l_{fx} \), ft 7.5

Component of distance in X-Y plane from centroid of top area of fuselage to Y-axis of rotation, \( l_{fy} \), ft 4.17

Distance from centroid of side area of fuselage to Z-axis of rotation, \( l_{fz} \), ft 25.58

Additional moment-of-inertia coefficients obtained from the width-depth ratios and fineness ratio of the fuselage (from ref. 3)

\[ k_{fy} = 1.34 \]

\[ k_{fz} = 0.68 \]

Horizontal tail:

Area, \( S \), sq ft 77.84

Span, \( b \), ft 18.5

Aspect ratio, \( A \) 4.4
Mean chord \( c \), ft \((s/b)\) 

Component of distance in the X-Y plane from centroid of area of horizontal tail to Y-axis of rotation, \( l_{xy} \) 

Additional mass coefficient, \( k \) (from ref. 3) 

Additional moment-of-inertia coefficient, \( k \), for the X swinging (from ref. 3) 

For the Y swinging the reciprocal of the aspect ratio is used to give an additional moment-of-inertia coefficient, \( k \) (from ref. 3) 

Taper ratio, \( \lambda \) 

Taper ratio correction, \( D_\lambda \) (from ref. 3) 

Vertical Tail: 

Area, \( S \), sq ft 

Span, \( b \), ft 

Aspect ratio, \( A \) 

Mean chord, \( c \), ft \((s/b)\) 

Distance from centroid of area to X-axis of rotation, \( l_{tx} \), ft 

Distance from centroid of area to Z-axis of rotation, \( l_{tz} \), ft 

Additional mass coefficient, \( k \) (from ref. 3) 

Additional moment-of-inertia coefficient, \( k \), for the X swinging (from ref. 3)
For the Z swinging the reciprocal of the aspect ratio is used to give additional moment-of-inertia coefficient, $k$ (from ref. 3).

Taper ratio, $\lambda$  
2.67

Taper ratio correction, $B_{\lambda}$ (from ref. 3)  
.75

Calculations. The additional moments of inertia about the swinging axes as shown in Figure 1 are calculated according to the approximate equations (from ref. 3).

**X-axis:**

\[
I_{X_a} = \frac{\rho}{48} \left( k' \, D_{\lambda} \, D_{\lambda} \, S \, b^2 \right)_w + \rho \left( k_{f} L_{w} \, w \, d \, L_{Xw} \right)_{\text{fus}}
\]

Substituting the proper values gives

\[
I_{X_a} = \frac{3.14}{48} \left( 0.002378 \right) \left[ (0.86)(0.82)(0.7)(334)^2(42.833) \right]_w + \left[ (0.002378)(1.34)(33.853)(3.62)(5.54)(7.5)^2 \right]_{\text{fus}}
\]

\[
= (3.66)_w + (121)_{\text{fus}}
\]

\[
= 487 \text{ slug ft}^2
\]

**Y-axis:**

\[
I_{Y_a} = \left[ \frac{\rho}{5} \, k_{f} L_{w} \, w \, d \, \left( \frac{\rho h}{k_{f}} \right) + \frac{3 \, d^2}{2 \, \pi^2} \right]_{\text{fus}} + \rho \left( k_{f} L_{w} \, w \, d \, L_{Yw} \right)_{\text{fus}} + \frac{\pi \rho}{4} \left( k \, S^2 \, L_{Yw} \right)_{\text{ht}}
\]
Substituting the proper values gives

\[ I_{y_a} = \left\{ \frac{0.002378}{5} (0.56)(33.833)(3.62)(5.54) \left[ \frac{33.833}{4} \right]^2 + \frac{3}{6.28} (5.54)^2 \right\}_{\text{fus}} \]

\[ + \left\{ (0.002378)(0.68)(33.833)(3.62)(5.54)(4.17)^2 \right\}_{\text{fus}} \]

\[ + \left\{ \frac{3.14(0.002378)}{4} \left[ \frac{0.92(77.84)}{18.5} \right]^2 \right\}_{\text{ht}} \]

\[ = (54.363)_{\text{fus}} + (19.079)_{\text{fus}} + (240.313)_{\text{ht}} \]

\[ = 313.755 \text{ slug ft}^2 \]

Z-axis:

\[ I_{z_a}' = \left\{ \frac{2}{5} k' L^2 \rho v d \left( \frac{L^2}{4} + \frac{3}{2} \frac{v^2}{\pi} \right) \right\}_{\text{fus}} \]

\[ + \rho \left( k' L^2 \rho v d L_{z}^2 \right)_{\text{fus}} + \frac{\pi \rho}{4} \left( K \frac{s^2}{b} L_{z}^2 \right)_{\text{vt}} \]

Substituting the proper values gives

\[ I_{z_a} = \left\{ \frac{0.002378}{5} (1.07)(33.833)(3.62)(5.54) \left[ \frac{33.833}{4} \right]^2 + \frac{3}{8.28} (3.62)^2 \right\}_{\text{fus}} \]

\[ + \left\{ 0.002378 (1.34)(33.833)(3.62)(5.54)(25.58)^2 \right\}_{\text{fus}} \]

\[ + \left\{ \frac{3.14(0.002378)}{4} \left[ \frac{0.51(23.4)}{18.25} \right]^2 \right\}_{\text{ht}} \]

\[ = (101.2)_{\text{fus}} + (1410)_{\text{fus}} + (207.5)_{\text{ht}} \]

\[ = 1718.7 \text{ slug ft}^2 \]
APPENDIX II

METHODS OF CALCULATING THE CHARACTERISTIC MOTIONS OF A TYPICAL AIRPLANE FOR VARIATIONS IN MOMENT-OF-INERTIA

Pertinent Data. — The following conditions and stability derivatives for a typical airplane are assumed:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_L$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>10</td>
</tr>
<tr>
<td>$C_{y\beta}$</td>
<td>0.28</td>
</tr>
<tr>
<td>$C_{n\beta}$</td>
<td>0.09</td>
</tr>
<tr>
<td>$C_{1\beta}$</td>
<td>0.04</td>
</tr>
<tr>
<td>$C_{1p}$</td>
<td>0.45</td>
</tr>
<tr>
<td>$C_{pr}$</td>
<td>0.12</td>
</tr>
<tr>
<td>$C_{1r}$</td>
<td>0.25</td>
</tr>
<tr>
<td>$C_{np}$</td>
<td>-0.125</td>
</tr>
<tr>
<td>$C_{n9 r}$</td>
<td>-0.06</td>
</tr>
<tr>
<td>$C_{nd 9 r}$</td>
<td>-0.061</td>
</tr>
<tr>
<td>$C_{b9}$</td>
<td>-0.24</td>
</tr>
<tr>
<td>$C_{b9}$</td>
<td>-0.48</td>
</tr>
<tr>
<td>$C_{b\psi}$</td>
<td>-0.012</td>
</tr>
<tr>
<td>$C_{bd 9}$</td>
<td>-0.01</td>
</tr>
<tr>
<td>$C_{15 \alpha}$</td>
<td>0.15</td>
</tr>
<tr>
<td>$C_{h\alpha}$</td>
<td>-0.014</td>
</tr>
<tr>
<td>$C_{1d 5 \alpha}$</td>
<td>0.0015</td>
</tr>
<tr>
<td>$C_{h\phi}$</td>
<td>0.0096</td>
</tr>
</tbody>
</table>
Sample Calculations. -- Characteristic motions of the airplane with controls locked. The equations of motion for level flight of an airplane with controls locked are:

\[
\begin{align*}
\frac{c_y}{\beta} - 2 \frac{c_l}{\beta} \frac{d}{d\beta} \psi + c_l \phi &= 0 \\
\mu c_{1l} \beta + \frac{c_{1l}}{2} \frac{d}{d\beta} \psi + \left(\frac{c_{1l}}{2} \alpha - J_x \alpha^2\right) \phi &= 0 \\
\mu c_{\alpha l} \beta + \left(\frac{c_{1l}}{2} - J_z \alpha\right) \frac{d}{d\beta} \psi + \frac{c_{1p}}{2} \phi &= 0
\end{align*}
\]

Substituting the assumed solutions

\[\beta = \beta e^{\lambda t/\lambda}; \frac{d}{d\beta} \psi = \lambda \frac{d}{d\beta} \psi, e^{\lambda t/\lambda}; \text{etc.}\]

in the equations of motion and dividing out \(e^{\lambda t/\lambda}\) results in three simultaneous equations in \(\lambda\). To determine the value of \(\lambda\), the determinant of the coefficients of \(\lambda\) is equated to zero and expanded. This determinant becomes:
Expanding this determinant, a quartic in $\lambda$ is obtained:

$$A \lambda^4 + B \lambda^3 + C \lambda^2 + D \lambda + E = 0$$

where

$$A = 1$$

$$B = -\frac{1}{2} \left( \frac{c_{y/\beta}}{JZ} + \frac{c_{nF}}{JX} \right)$$

$$C = \frac{1}{4} \frac{JZ}{JX} \left( \frac{c_{nF}}{JZ} - \frac{c_{nT}}{JZ} \right) + \frac{c_{y/\beta}}{4} \left( \frac{c_{nT}}{JZ} + \frac{c_{1P}}{JX} \right) + \frac{\mu c_{n/\beta}}{JZ}$$

$$D = -\frac{\mu}{2} \frac{JZ}{JX} \left( \frac{c_{n/\beta}}{JZ} - \frac{c_{1P}}{JZ} \right) + \frac{\mu}{2} c_{L} c_{1/\beta}$$

$$E = \frac{\mu}{4} \frac{JZ}{JX} \left( \frac{JZ}{JZ} c_{1/\beta} - c_{n/\beta} - c_{n/\beta} c_{1T} \right)$$

The above coefficients are then determined by substituting the proper values of the derivatives. The coefficients are determined for three different values of $JX$ and $LZ$ according to the true moment-of-inertia, a five per cent increase in moment-of-inertia and a twenty-five per cent increase in moment-of-inertia.

For the sample calculation, the coefficients of the quartic
will be solved for the case of a five per cent increase in moment-of-
 inertia,

\[ A \lambda^4 + B \lambda^3 + C \lambda^2 + D \lambda + E = 0 \]

\[ A = 1 \]

\[ B = -\frac{1}{2} \left( \frac{c_{y/\beta}}{J_Z} + \frac{c_{n r}}{J_Z} + \frac{c_{1 p}}{J_X} \right) \]

\[ = -\frac{1}{2} \left( -0.28 + \frac{(-0.12)}{(0.0315)} + \frac{(-0.45)}{(0.021)} \right) = 12.76 \]

\[ C = \frac{1}{4 J_x J_Z} \left( c_{1 p} \ c_{n r} - c_{1 r} \ c_{n p} \right) + \frac{c_{y/\beta}}{4} \left( \frac{c_{n r}}{J_Z} + \frac{c_{1 p}}{J_X} \right) + \frac{c_{n/\beta}}{J_z} \]

\[ = \frac{1}{4(0.021)(0.0315)} \left[ (-0.45)(-0.12) - (0.25)(-0.125) \right] + \frac{(-0.28)}{4} \left[ (-0.12) \ 0.0315 + (-0.45) \ 0.021 \right] + \frac{10(0.09)}{0.0315} \]

\[ = 32.3 + 1.77 + 28.6 = 62.67 \]

\[ D = -\frac{\lambda}{2 J_x J_Z} \left( c_{n/\beta} \ c_{1 p} - c_{1/\beta} \ c_{n p} \right) - \frac{\lambda}{2 J_x} \ c_{n/\beta} \ c_{1/\beta} \]

\[ - \frac{c_{y/\beta}}{8 J_x J_Z} \left( c_{1 p} \ c_{n r} - c_{1 r} \ c_{n p} \right) \]

\[ = -\frac{10}{2(0.021)(0.0315)} \left[ (0.09)(-0.45) - (-0.04)(-0.125) \right] \]

\[ - \frac{10}{2(0.021)} \ (1.0)(-0.04) \]

\[ - \frac{(-0.28)}{8(0.021)(0.0315)} \left[ (-0.45)(-0.12) - (0.25)(-0.125) \right] \]

\[ = 344 + 9.54 + 4.5 = 358.04 \]

\[ E = \frac{\lambda}{4 J_x J_Z} \left( c_{1/\beta} \ c_{n r} - c_{n/\beta} \ c_{1 r} \right) \]

\[ = \frac{10(1.0)}{4(0.021)(0.0315)} \left[ (-0.04)(-0.12) - (0.09)(0.25) \right] \]

\[ = 3770 \left[ (0.0048) - (0.0225) \right] = -66.9 \]
The quartic in $\lambda$ with the computed values of the coefficients becomes:

$$f(\lambda) = \lambda^4 + 12.76 \lambda^3 + 62.67 \lambda^2 + 358.04 \lambda - 66.9 = 0$$

$$f(-\lambda) = \lambda^4 - 12.76 \lambda^3 + 62.67 \lambda^2 - 358.04 \lambda - 66.9 = 0$$

Horner's method is used to find the real roots of this equation. By Descartes' rule of signs, the equation is seen to have one positive root and at most three negative roots. Using the method of trial divisor and testing with synthetic division, the positive root is found to lie between one-tenth and two-tenths. Transforming the equation in order of the diminishing roots, the positive real root is obtained.

\[
\begin{array}{cccccc}
1 & + & 12.76 & + & 62.67 & + & 358.04 & - & 66.9 \\
+ & .1 & & & & & & & .1 \\
\hline 1 & + & 12.86 & + & 63.956 & + & 354.4456 & - & 36.4436 \\
+ & .1 & & & & & & & .1 \\
\hline 1 & + & 12.96 & + & 65.3292 & + & 360.9688 & & \\
+ & .1 & & & & & & & .1 \\
\hline 1 & + & 13.06 & + & 66.558 & + & 367.9608 & - & 30.1996 \\
+ & .1 & & & & & & & .08 \\
\hline 1 & + & 13.16 & + & 66.5858 & + & 370.9609 & - & 30.1996 \\
+ & .08 & & & & & & & .3468 \\
\hline 1 & + & 13.24 & + & 67.6272 & + & 376.3762 & - & .3468 \\
+ .08 & & & & & & & .0000 \\
\hline 1 & + & 13.32 & + & 68.6777 & + & 381.8644 & - & .3468 \\
+ .08 & & & & & & & .0000 \\
\hline 1 & + & 13.40 & + & 69.7497 & + & 381.8644 & - & .3468 \\
+ .08 & & & & & & & .0000 \\
\hline
\end{array}
\]

$$\lambda_1 = 0.1809$$
To find the negative root the equation is written as
\[ f(-\lambda) = 0 \] and the approximate root is found by the trial division method and tested by synthetic division.

\[
f(-\lambda) = \lambda^4 - 12.76 \lambda^3 + 62.67 \lambda^2 - 358.04 \lambda - 66.9 = 0
\]

\[
\begin{array}{cccccc}
1 & -12.76 & + & 62.67 & - & 358.04 & - & 66.9 & | & 10 \\
+ 10.0 & - & 27.60 & + & 380.70 & - & 73.4 & \\
\hline
1 & -2.76 & + & 35.07 & - & 7.34 & - & 140.3 & \\
+ 10.0 & - & 72.40 & + & 1974.70 & \\
\hline
1 & + & 7.24 & + & 187.47 & + & 1587.56 & \\
+ 10.0 & + & 172.40 & \\
\hline
1 & + & 17.24 & + & 279.87 & \\
+ 10.0 & \\
\hline
1 & + & 27.24 & + & 279.87 & + & 1067.56 & - & 140.3, & .1 \\
+ .1 & + & 2.754 & + & 28.2604 & + & 199.56 & \\
\hline
1 & + & 27.34 & + & 282.604 & + & 1095.6294 & - & 36.74 & \\
+ .1 & + & 2.744 & + & 28.5348 & \\
\hline
1 & + & 27.44 & + & 285.348 & + & 1124.1592 & \\
+ .1 & + & 2.754 & \\
\hline
1 & + & 27.54 & + & 288.102 & \\
+ .1 & \\
\hline
1 & + & 27.64 & + & 288.102 & + & 1124.1552 & - & 30.74 & .02
\end{array}
\]

\[ \lambda_2 = -10.12 \]

The two real roots having been determined, the quartic equation is reduced to a quadratic by dividing by each of the two factors.

\[ \lambda_1 = 0.1809 \]
\[ \lambda_2 = -10.12 \]
\[ \lambda = 0.1809 \]
\[ \lambda = 10.12 \]

\[
\begin{array}{cccccccc}
\lambda^4 & + & 12.76 \lambda^3 & + & 62.67 \lambda^2 & + & 358.04 \lambda & - & 66.9 & \lambda & - & 0.1809 \\
\lambda^4 & - & 1809 \lambda^3 & + & 12.9409 \lambda^2 & + & 65.011 \lambda & + & 369.8 & \lambda^3 & + & 12.9409 \lambda^2 & - & 65.011 \lambda & - & 369.8 & - & 66.9
\end{array}
\]
This quartic is now reduced to the quadratic

\[ \lambda^2 + 2.8209 \lambda + 36.474 = 0 \]

Using the quadratic formula to solve the equation

\[
\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2b}
\]

where \( a \lambda^2 + b \lambda + c = 0 \)

we obtain \( \lambda_{3,4} \), the two imaginary roots

\[
\lambda_{3,4} = \frac{-2.8209 \pm \sqrt{(2.8209)^2 - 4(36.474)}}{2}
\]

\[
= -2.8209 \pm 11.72 \quad \text{or} \quad -1.41 \pm 5.861
\]
BIBLIOGRAPHY


