Appendix for Relicensing Fees as a Secondary Market Strategy

Nektarios Oraiopoulos*
Mark E. Ferguson†
L. Beril Toktay‡

February 2007

* Nektarios Oraiopoulos, College of Management, Georgia Institute of Technology, Atlanta, GA 30332. Phone: (404) -385-4887 , e-mail: nektarios.oraiopoulos@mgt.gatech.edu.
† Mark E. Ferguson, College of Management, Georgia Institute of Technology, Atlanta, GA 30332. Phone: (404) 894-4330, e-mail: mark.ferguson@mgt.gatech.edu.
‡ Beril Toktay, College of Management, Georgia Institute of Technology, Atlanta, GA 30332. Phone: (404) 385-0104, e-mail: beril.toktay@mgt.gatech.edu.

Appendix

Proof of Lemma 1

The utilities for the new and used product in the second period will be:

$$U_2 = \theta - p_2$$

and

$$U_u = \delta\theta - p_u - h$$

where h is the license fee

Solving for the "indifferent" consumer, we get the following inverse demand equations

$$\theta_1 = \frac{p_2 - p_u - h}{1-\delta}$$

$$\theta_2 = \frac{p_u + h}{\delta}$$

Which lead us to the following inverse demand functions:

price of new products : $p_2 = -q_2 + 1 - q_u\delta$

price of used products : $p_u = -q_u\delta + \delta - q_2\delta \ - h$

the respective profits will be :

$$Max_{q_2} \quad \Pi_2(q_2|q_u) = (p_2 - c)q_2 + hq_u = (1 - q_2 - q_u\delta - c)\, q_2 + hq_u$$

$$s.t \quad q_2 \geq 0$$

$$Max_{q_u,s}(q_u, s|q_2) \quad \Pi_u = (p_u - s)q_u$$

$$q_u \leq q_1$$

$$q_u \leq s(\frac{1}{\gamma} + 1) - p_1^* \tag{1}$$

$$q_u \geq 0$$

The Lagrangian for the RM's problem is

$L(q_u, s, \lambda_1, \lambda_2) = \Pi_u(q_u, s) + \lambda_1(s(\frac{1}{\gamma} + 1) - p_1^* - q_u) + \lambda_2(q_1 - q_u) + \mu_1 q_u$

The conditions for optimality are :

$\frac{\partial L}{\partial q_u} = 0,$

$\frac{\partial L}{\partial s} = 0,$

$\lambda_1(s(\frac{1}{\gamma} + 1) - p_1^* - q_u) = 0$

$\lambda_2(q_1 - q_u) = 0$

$\mu_1 q_u = 0$

$\lambda_1 \geq 0, \lambda_2 \geq 0, \mu_1 \geq 0,$

Assume $s(\frac{1}{\gamma} + 1) - p_1 - q_u > 0$,then $\lambda_1 = 0$. In this case, $\frac{\partial L}{\partial s} = -q_u \leqslant 0$.

However, if $q_u = 0$, there is no secondary market and thus, $s = 0$.Apparently, $s(\frac{1}{\gamma}+1)-p_1-q_u > 0$ will be a contradiction. Therefore, $\frac{\partial L}{\partial s} = -q_u < 0$.

Since this case can not meet the FOC ($\frac{\partial L}{\partial s} = 0$), from hereafter we assume that inequality (1) holds as an equality. Intuitively, the entrant would not be willing to buy more used units than the quantity she would launch in the secondary market.

Proof of Proposition 1

From Lemma 1 we know that, $q_u = s(\frac{1}{\gamma} + 1) - p_1^* \Longrightarrow$

$$s = \frac{\gamma(q_u + p_1^*)}{1+\gamma} \tag{2}$$

Therefore, we can rewrite the entrant's problem as:

$$Max_{q_u} \qquad \Pi_u = (p_u - \frac{\gamma(q_u + p_1^*)}{1+\gamma})q_u$$

$$q_u \leq q_1^* \tag{3}$$

$$q_u \geq 0$$

while the OEM's objective is given by:

$$Max_{q_2} \qquad \Pi_2 = (-q_2 + 1 - q_u\,\delta - c)\,q_2 + h^* q_u$$

$$q_2 \geq 0$$

We solve the following problem by assuming that constraint (3) is not binding at the optimal solution. After determining the optimal first period price (p_1^*) and the corresponding optimal quantity (q_1^*) we verify our assumption by substituting this value to constraint (3) and showing that it always holds as a strict inequality.

Solving simultaneously for the quantities of the two competitors, we obtain the following Nash Equilibria (N.E) :

$$q_2^* = -\frac{2\,\delta + 2\,\delta\,\gamma + 2\,\gamma - \delta^2 - \delta^2\gamma + \delta\,h^* + \delta\,h^*\gamma + \delta\,\gamma p_1^* - 2\,\delta\,c - 2\,\delta\,\gamma c - 2\,\gamma c}{-4\,\delta - 4\,\delta\,\gamma - 4\,\gamma + \delta^2 + \delta^2\gamma} \tag{4}$$

$$q_u^* = \frac{-\delta\,\gamma + 2\,h^* + 2\,h^*\gamma - \delta - \delta\,c + 2\,\gamma p_1^* - \delta\,\gamma c}{-4\,\delta - 4\,\delta\,\gamma - 4\,\gamma + \delta^2 + \delta^2\gamma} \tag{5}$$

substituting q_u^* from (5),

(2) can be rewritten as :

$$s = \frac{\gamma(-\delta\,\gamma + 2\,h^* + 2\,h^*\gamma - \delta - \delta\,c - 2\,\gamma p_1^* - \delta\,\gamma c - 4\,\delta\,p_1^* - 4\,\delta\,\gamma p_1^* + \delta^2 p_1^* + \delta^2\gamma p_1^*)}{(-4\,\delta - 4\,\delta\,\gamma - 4\,\gamma + \delta^2 + \delta^2\gamma)(1+\gamma)}$$

First Period Analysis

The price in the first period, is given by

$$p_1 \;=\; 1 - q_1 + s \tag{6}$$

solving simultaneously (**??**) and (6) we get

$$p_1 \;=\; -\frac{(-q_1\,\delta^2\gamma + 4\,q_1\,\gamma + 4\,q_1\,\delta\,\gamma + \delta^2\gamma - 4\,\gamma + 2\,h\gamma - 5\,\delta\,\gamma - \delta\,\gamma c + 4\,q_1\,\delta - q_1\delta^2 - 4\,\delta + \delta^2)(1+\gamma)}{4\,\delta + 4\,\delta\,\gamma + 4\,\gamma + 2\,\gamma^2 - \delta^2 - \delta^2\gamma}$$

First period profits are consist of only the sales from new products :

$$\begin{aligned} \Pi_1(q_1, h) \;&=\; (p_1 - c)q_1 \\ st \qquad & q_1 \geq 0, h \geq 0 \end{aligned}$$

and the cumulative profits over the two periods will be $\Pi = \Pi_1 + \Pi_2$

The OEM's overall objective will be :

$$\begin{aligned} Max_{q_1,h} \qquad & \Pi(q_1, h) \\ \text{st} \qquad & q_1 \geq 0 \\ & q_2^* \geq 0 \\ & q_u^* \geq 0 \\ & h \geq 0 \end{aligned}$$

Given, the second period equilibrium equations, an equivalent form is:

$$Max_{q_1,h} \qquad \Pi(q_1, h)$$

s.t

$$q_1 \geq 0 \tag{7}$$

$$h \geq \gamma q_1 + A \tag{8}$$

$$h \leq \gamma q_1 + B \tag{9}$$

$$h \geq 0$$

where $A = \frac{-\delta\,\gamma^2 - \gamma^2 + \gamma^2 c + \delta^2 \gamma + 2\,\delta\,\gamma c - 3\,\delta\,\gamma - 2\,\gamma + 2\,\gamma c + \delta^2 - 2\,\delta + 2\,\delta\,c}{\delta\,(1+\gamma)}$ and $B = \frac{\delta(1+c)}{2} - \gamma$

Note that, $B - A = \frac{(4\delta + 4\delta\gamma + 4\gamma + 2\gamma^2 - \delta^2 - \delta^2\gamma)(1-c)}{2\delta(1+\gamma)} > 0$ for $c < 1$,so $A < B$.

The Lagrangean is:

$L(q_1, h, \lambda_1, \lambda_2) = \Pi(q_1, h) + \lambda_1(\gamma q_1 + B - h) + \lambda_2(h - \gamma q_1 - A) + \mu_1 q_1 + \mu_2 h$

with the following conditions for optimality :

$$\frac{\partial L}{\partial q_1} = 0 \tag{10}$$

$$\frac{\partial L}{\partial h} = 0 \tag{11}$$

$$\lambda_1(\gamma q_1 + B - h) = 0 \tag{12}$$

$$\lambda_2(h - \gamma q_1 - A) = 0 \tag{13}$$

$$\mu_1 q_1 = 0 \tag{14}$$

$$\mu_2 h = 0 \tag{15}$$

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \mu_1 \geq 0, \mu_2 \geq 0$$

Case 1. $\gamma q_1 + B - h = 0$ and $q_1 > 0$, then $\lambda_2 = 0, \mu_1 = 0, \mu_2 = 0$.

Solving (10) and (11) gives

$\lambda_1 = -\frac{2c(\delta\gamma - \gamma + \delta)}{4\delta + 4\delta\gamma + 4\gamma + 2\gamma^2 - \delta^2(1+\gamma)}$

and $h = -\frac{1}{2}\frac{-\delta\gamma c + c\gamma - \delta c + \gamma^2 - \delta - \delta\gamma + \gamma}{1+\gamma}$

We need $\lambda_1 \geq 0$ which holds only for $\delta < \frac{\gamma}{1+\gamma}$

since $4\delta + 4\delta\gamma + 4\gamma + 2\gamma^2 - \delta^2(1+\gamma) > 0$

Also, $h \geq 0$ holds only for $c < \frac{\delta(1+\gamma)-\gamma-\gamma^2}{\gamma-\delta-\gamma\delta}$.

However, when $\delta < \frac{\gamma}{1+\gamma}$, then$\frac{\delta(1+\gamma)-\gamma-\gamma^2}{\gamma-\delta-\gamma\delta} < 0$ and since $c < \frac{\delta(1+\gamma)-\gamma-\gamma^2}{\gamma-\delta-\gamma\delta}$., we conclude that $c < 0$ which is false.

Since, $\lambda_1 \geq 0$ and $h \geq 0$ can not be satisfied simultaneously, this case can not meet the conditions for optimality.

In other words, constraint (9), is not binding at the optimal solution, and thus, $q_u^* > 0$

Case 2. $(h - \gamma q_1 - A) = 0$,and $q_1 > 0$ then $\lambda_1 = 0, \mu_1 = 0, \mu_2 = 0$.

Solving (10) and (11) gives

$$\lambda_2 = \frac{-8\delta c + 8\gamma - 8c\gamma + \delta^2\gamma c - 3\delta^2\gamma - 6\delta\gamma c + \delta^2 c - 3\delta^2 + 8\delta(1+\gamma)}{\delta(-4\delta - 4\delta\gamma - 4\gamma - 2\gamma^2 + \delta^2(1+\gamma))} \tag{16}$$

$$h = \frac{1}{2}\frac{4\delta c - 4\gamma + 4\gamma c + 2\delta^2\gamma + 3\delta\gamma c + 2\delta^2 - 4\delta - 5\delta\gamma - \delta\gamma^2}{\delta(1+\gamma)} \tag{17}$$

We need $\lambda_2 \geq 0$ or equivalently

$$-8\delta c + 8\gamma - 8c\gamma + \delta^2\gamma c - 3\delta^2\gamma - 6\delta\gamma c + \delta^2 c - 3\delta^2 + 8\delta(1+\gamma) \leq 0 \tag{18}$$

since $\delta(-4\delta - 4\delta\gamma - 4\gamma - 2\gamma^2 + \delta^2(1+\gamma)) < 0$.

(18) is equivalent to $c \geq \frac{(1+\gamma)(3\delta^2-8\delta)-8\gamma}{-8\gamma+\delta^2(1+\gamma)-8\delta-6\delta\gamma}$

for $\delta > \frac{\gamma}{1+\gamma}$,$\frac{(1+\gamma)(3\delta^2-8\delta)-8\gamma}{-8\gamma+\delta^2(1+\gamma)-8\delta-6\delta\gamma} < 1$ and thus (18) is possible.

We also need $h \geq 0 \Rightarrow 4\delta c - 4\gamma + 4\gamma c + 2\delta^2\gamma + 3\delta\gamma c + 2\delta^2 - 4\delta - 5\delta\gamma - \delta\gamma^2 \geq 0 \Rightarrow$
$c \geq \frac{\delta\gamma^2-2\delta^2+4\delta+5\delta\gamma+4\gamma-2\delta^2\gamma}{(4\delta+4\gamma+3\delta\gamma)}$

Again for $\delta > \frac{\gamma(\gamma+2)}{2(1+\gamma)}$, $\frac{\delta\gamma^2-2\delta^2+4\delta+5\delta\gamma+4\gamma-2\delta^2\gamma}{(4\delta+4\gamma+3\delta\gamma)} < 1$ and the condition is possible.

However, since $\frac{(1+\gamma)(3\delta^2-8\delta)-8\gamma}{-8\gamma+\delta^2(1+\gamma)-8\delta-6\delta\gamma} < \frac{\delta\gamma^2-2\delta^2+4\delta+5\delta\gamma+4\gamma-2\delta^2\gamma}{(4\delta+4\gamma+3\delta\gamma)}$, it is sufficient to have
$c \geq \frac{(1+\gamma)(3\delta^2-8\delta)-8\gamma}{-8\gamma+\delta^2(1+\gamma)-8\delta-6\delta\gamma}$ in which case $h > 0$

Case 3. $h - \gamma q_1 - A \neq 0$ and $\gamma q_1 + B - h \neq 0$ then $\lambda_1 = 0, \lambda_2 = 0, \mu_1 = 0, \mu_2 = 0$ (since $A \neq B$)

Solving (10) and (11) gives

$$\lambda_2 = \frac{3\delta^2+6\delta^2\gamma+8c\gamma+8\delta c+4\delta\gamma c-8\gamma-16\delta\gamma-8\delta-8\gamma^2-8\delta\gamma^2+4c\gamma^2-3\delta^2\gamma c-4\delta\gamma^2 c-3\delta^2 c+3\delta^2\gamma^2}{(1+\gamma)(3\delta^2\gamma+3\delta^2-8\delta\gamma-8\,\delta-8\gamma)}$$
$$h = \frac{1}{2}\frac{\delta^3\gamma c+3\delta^3\gamma+\delta^3 c+3\delta^3-3\delta^2\gamma^2-\delta^2\gamma c-11\delta^2\gamma-8\delta^2+8\gamma^2+8\delta\gamma^2}{3\delta^2\gamma+3\delta^2-8\delta\gamma-8\delta-8\gamma}$$

this case can also be feasible (at the optimality) as long as
for $c \leq -(\frac{3\delta^2+6\delta^2\gamma-16\delta\gamma-8\delta-8\gamma^2-8\gamma+3\delta^2\gamma^2-8\delta\gamma^2}{8\delta+4\delta\gamma+8\gamma-3\delta^2\gamma-4\delta\gamma^2-3\delta^2+4\gamma^2})$
and $c \leq \frac{3\delta^2\gamma^2-3\delta^3\gamma+11\delta^2\gamma-3\delta^3-8\gamma^2-8\delta\gamma^2+8\delta^2}{\delta^3+\delta^3\gamma-\delta^2\gamma}$

Case 4. $h-\gamma q_1-A=0$ and $\gamma q_1+B-h=0$,then $A=B$ which is false since $A<B$ so this case can not occur .

Case 5. $h=0$ and $q_1\neq 0$,then $\lambda_1=0,\lambda_2=0,\mu_1=0$

since $\delta>\gamma>\frac{\gamma}{1+\gamma}$ the case becomes feasible for $c\geq\frac{3\delta^2\gamma^2-3\delta^3\gamma+11\delta^2\gamma-3\delta^3-8\gamma^2-8\delta\gamma^2+8\delta^2}{\delta^3+\delta^3\gamma-\delta^2\gamma}$

To summarize the above Lagrangean analysis, we have the following conditions :

for $c\leq\frac{3\delta^2\gamma^2-3\delta^3\gamma+11\delta^2\gamma-3\delta^3-8\gamma^2-8\delta\gamma^2+8\delta^2}{\delta^3+\delta^3\gamma-\delta^2\gamma}$ we fall into case 3 ($q_2>0$ and $h>0$), while for $c\geq\frac{3\delta^2\gamma^2-3\delta^3\gamma+11\delta^2\gamma-3\delta^3-8\gamma^2-8\delta\gamma^2+8\delta^2}{\delta^3+\delta^3\gamma-\delta^2\gamma}$ we fall into case 5.

Finally, for very high values of $c\geq\frac{(1+\gamma)(3\delta^2-8\delta)-8\gamma}{-8\gamma+\delta^2(1+\gamma)-8\delta-6\delta\gamma}$ case 2 ($q_2=0$) becomes also feasible.

Proof of Remark 1

We start by case 2. From our previous analysis,

$$h^*=\frac{1}{2}\frac{4\delta c-4\gamma+4\gamma c+2\delta^2\gamma+3\delta\gamma c+2\delta^2-4\delta-5\delta\gamma-\delta\gamma^2}{\delta(1+\gamma)} \tag{19}$$

$$\frac{\partial}{\partial\delta}\frac{1}{2}\frac{4\delta c-4\gamma+4\gamma c+2\delta^2\gamma+3\delta\gamma c+2\delta^2-4\delta-5\delta\gamma-\delta\gamma^2}{\delta(1+\gamma)} \tag{20}$$

Thus, $\dfrac{\partial h^*}{\partial\delta}=\frac{2\delta^2+2\gamma\delta^2+4\gamma+-4c\gamma}{2\delta^2+2\gamma\delta^2}>0$ and $h^*$is decreasing in δ.

To prove that $h^*$is decreasing in γ it is sufficient to show that the numerator of (19) is decreasing in γ.

By taking the first derivative, we get $\dfrac{\partial}{\partial\gamma}(4\delta c-4\gamma+4\gamma c+2\delta^2\gamma+3\delta\gamma c+2\delta^2-4\delta-5\delta\gamma-\delta\gamma^2)=$ $=4c-5\delta+3c\delta-2\gamma\delta+2\delta^2-4=4(c-1)+3\delta(c-1)-2\gamma\delta+2\delta(\delta-1)$ which is obviously negative for $c<1$.

Thus,$h^*$is decreasing in γ.

Finally,$\dfrac{\partial h^*}{\partial c}=\frac{1}{2\delta+2\gamma\delta}(4\gamma+4\delta+3\gamma\delta)>0$

Similarly, we prove the properties for case 3, where

$h^*=\frac{1}{2}\frac{\delta^3\gamma c+3\,\delta^3\gamma+\delta^3c+3\,\delta^3-3\,\delta^2\gamma^2-\delta^2\gamma c-11\,\delta^2\gamma-8\,\delta^2+8\,\gamma^2+8\,\delta\,\gamma^2}{3\,\delta^2\gamma+3\,\delta^2-8\,\delta\,\gamma-8\,\delta-8\,\gamma}$

Note, however, that in this case $\dfrac{\partial h^*}{\partial c}=\frac{\delta^2(\delta-\gamma+\gamma\delta)}{6\delta^2-16\delta-16\gamma\delta-16\gamma+6\gamma\delta^2}<0$

Proof of Proposition 2

The utilities for the new and used product in the second period will be:

$U_2=\theta-p_2$ and $U_r=\delta\theta-p_u-h$ where h is the license fee

Solving for the "indifferent" consumer, we get the following inverse demand equations

$$\theta_1=\frac{p_2-p_u-h}{1-\delta}$$

$$\theta_2=\frac{p_u+h}{\delta}$$

price of new products : $p_2 = -q_2 + 1 - \delta \sum_{i=1}^{N} q_u^i$

price of used products : $p_u = \delta - q_2\delta \ - h - \delta \sum_{i=1}^{N} q_u^i$

the respective profits will be :

$$\begin{aligned} &\text{OEM}: \ Max_{q_2} \quad \Pi_2 = \left(-q_2 + 1 - \delta \sum_{i=1}^{N} q_u^i \ - c\right) q_2 + h \sum_{i=1}^{N} q_u^i \\ &\text{s.t} \quad q_2 \geq 0 \\ &\text{and for the } i^{th} \text{ RM}: \ Max_{q_u^i} \quad \Pi_r = (p_u - s) q_u^i \\ &\text{s.t} \quad \sum_{i=1}^{N} q_u^i \leq s(\tfrac{1}{\gamma} + 1) - p_1 \\ &\qquad \sum_{i=1}^{N} q_u^i \leq q_1 \\ &\qquad q_u \geq 0 \end{aligned}$$

Let $Q_u^{-i} = \sum_{j=1, j\neq i}^{N} q_u^j$

then we can rewrite the OEM's and RM's problems as :

$$\begin{aligned} &\text{OEM}: \ Max_{q_2} \quad \Pi_2 = \left(-q_2 + 1 - \delta q_u^i - \delta Q_u^{-i} \ - c\right) q_2 + h(q_u^i + Q_u^{-i}) \\ &\text{s.t} \quad q_2 \geq 0 \\ &\text{and for the } i^{th} \text{ RM}: \ Max_{q_u^i} \quad \Pi_u = (\delta - q_2\delta \ - h - \delta q_u^i - \delta Q_u^{-i} - s) q_u^i \\ &\text{s.t} \quad q_u^i + Q_u^{-i} \leq s(\tfrac{1}{\gamma} + 1) - p_1 \\ &\qquad q_u^i + Q_u^{-i} \leq q_1 \\ &\qquad q_u \geq 0 \end{aligned}$$

The FOC with respect to q_2 and q_r^i

$$-2q_2 + 1 - \delta q_u^i - \delta Q_u^{-i} \ - c = 0$$

$$\delta - q_2\delta \ - h - 2\delta q_u^i - \delta Q_u^{-i} - s = 0$$

however, since we assume N symmetric remanufacturers $Q_u^{-i} \ = (N-1) q_u^i$

and the FOC can be rewritten as

$$-2q_2 + 1 - \delta N q_u^i - c = 0 \tag{21}$$

$$\delta - q_2\delta \ - h - \delta(N+1) q_u^i - s = 0 \tag{22}$$

In addition, the market clearing price s will satisfy

$$N q_u^i = s(\frac{1}{\gamma} + 1) - p_1 \tag{23}$$

solving simultaneously (21) ,(22) ,and (23)

we get the equilibrium quantities

$q_2 = -\frac{\gamma N+\delta+\delta\gamma+\delta N+\gamma N\delta-\delta^2 N-\delta^2 N\gamma+\delta Nh+\delta N\gamma h+\delta N\gamma p_1-c\gamma N-\delta c-\gamma\delta c-N\delta c-\gamma N\delta c}{-2\gamma N-2\delta-2\delta\gamma-2\delta N-2\gamma N\delta+\delta^2 N+\delta^2 N\gamma}$
$q_u = \frac{-\delta-\delta\gamma+2h+2\gamma h-\delta c-\gamma\delta c+2\gamma p_1}{-2\gamma N-2\delta-2\delta\gamma-2\delta N-2\gamma N\delta+\delta^2 N+\delta^2 N\gamma}$
$s = \frac{\gamma\left(-\delta N+2Nh-2\delta p_1-N\delta c+\delta^2 Np_1-2\delta Np_1\right)}{-2\gamma N-2\delta-2\delta\gamma-2\delta N-2\gamma N\delta+\delta^2 N+\delta^2 N\gamma}$
From hereafter we proceed as in the case of one RM:

Since the OEM maximizes over q_1 and h, first we take the first derivative $\frac{\partial \Pi(q_1,h)}{\partial q_1}$ and set it equal to zero so that we derive the $q_1^*(h)$.

We can now substitute this value to the expression for the OEM's profits and apply the FOC with respect to h (since the function is concave in h)

The FOC yield :

$$h^* = \frac{(\delta^3 Nc+3\delta^3 N+3\delta^3 N\gamma+\delta^3 N\gamma c-4\delta^2 N-7\delta^2 N\gamma-3\delta^2\gamma^2 N-\delta^2 N\gamma c-4\delta^2\gamma-4\delta^2+4\gamma^2 N\delta+4\delta\gamma+4\delta\gamma^2+4\gamma^2 N)}{2(3\delta^2 N+3\delta^2 N\gamma-4\gamma N\delta-4\delta N-4\delta\gamma-4\delta-4\gamma N)}$$

From, the last expression :

$$\frac{\partial h^*}{\partial N} = -2\,\frac{\delta^3 c\left(-\gamma^2-\gamma+\delta+2\delta\gamma+\delta\gamma^2\right)}{\left(3\delta^2 N+3\delta^2 N\gamma-4\gamma N\delta-4\delta N-4\delta\gamma-4\delta-4\gamma N\right)^2}$$

$$\frac{\partial^2 h^*}{\partial N^2} = 4\,\frac{\delta^3 c\left(-\gamma^2-\gamma+\delta+2\delta\gamma+\delta\gamma^2\right)\left(3\delta^2+3\delta^2\gamma-4\delta\gamma-4\delta-4\gamma\right)}{\left(3\delta^2 N+3\delta^2 N\gamma-4\gamma N\delta-4\delta N-4\delta\gamma-4\delta-4\gamma N\right)^3}$$

Since we are in the case of positive license fees $\delta > \frac{\gamma}{1+\gamma} \Leftrightarrow \delta(1+\gamma) > \gamma \Leftrightarrow$
$\delta(1+\gamma)^2 > \gamma(1+\gamma) \Leftrightarrow \left(-\gamma^2-\gamma+\delta+2\delta\gamma+\delta\gamma^2\right) > 0 \Rightarrow \frac{\partial^2 h^*}{\partial N^2} < 0$
Also, we can readily see that
$\left(3\delta^2+3\delta^2\gamma-4\delta\gamma-4\delta-4\gamma\right) < 0$
$\left(3\delta^2 N+3\delta^2 N\gamma-4\gamma N\delta-4\delta N-4\delta\gamma-4\delta-4\gamma N\right) < 0$
and therefore $\frac{\partial^2 h^*}{\partial N^2} > 0$. Thus, we proved that h^* is convex decreasing in N.
Also,

$$\frac{\partial \Pi^*}{\partial N} = 4\,\frac{\delta c^2\left(\delta^2\gamma^2+2\delta^2\gamma+\delta^2-2\delta\gamma^2-2\delta\gamma+\gamma^2\right)}{\left(3\delta^2 N+3\delta^2 N\gamma-4\gamma N\delta-4\delta N-4\delta\gamma-4\delta-4\gamma N\right)^2}$$

$\frac{\partial^2 \Pi^*}{\partial N^2} = -8\,\frac{\delta c^2\left(\delta^2\gamma^2+2\delta^2\gamma+\delta^2-2\delta\gamma^2-2\delta\gamma+\gamma^2\right)\left(3\delta^2+3\delta^2\gamma-4\delta\gamma-4\delta-4\gamma\right)}{\left(3\delta^2 N+3\delta^2 N\gamma-4\gamma N\delta-4\delta N-4\delta\gamma-4\delta-4\gamma N\right)^3}$
Similarly,
$\left(\delta^2\gamma^2+2\delta^2\gamma+\delta^2-2\delta\gamma^2-2\delta\gamma+\gamma^2\right) = \delta(1+\gamma)[\delta(1+\gamma)-2\delta\gamma]+\gamma^2 > 0$
$\left(3\delta^2+3\delta^2\gamma-4\delta\gamma-4\delta-4\gamma\right) < 0$
$\left(3\delta^2 N+3\delta^2 N\gamma-4\gamma N\delta-4\delta N-4\delta\gamma-4\delta-4\gamma N\right) < 0$
so, $\frac{\partial^2 \Pi^*}{\partial N^2} < 0$

By substituting the optimal license fee, to the second period equation we derive the corresponding expressions and the proofs regarding the properties are similar .

Along the same lines, we can prove that $\frac{\partial q_r^*}{\partial N} < 0$ with $\frac{\partial^2 q_r^*}{\partial N^2} > 0, \frac{\partial Q_r^*}{\partial N} > 0$ with $\frac{\partial^2 Q_r^*}{\partial N^2} < 0$, and $\frac{\partial s}{\partial N} > 0$ and $\frac{\partial^2 s}{\partial N^2} < 0$

$$\frac{\partial q_r^*}{\partial N} = 2\,\frac{\left(3\,\delta^3\gamma^2 + 6\,\delta^3\gamma + 3\,\delta^3 - 7\,\delta^2\gamma^2 - 11\,\delta^2\gamma - 4\,\delta^2 + 4\,\gamma^2\right) c}{\left(3\,\delta^2 N + 3\,\delta^2 N\gamma - 4\,\gamma N\delta - 4\,\delta\,N - 4\,\delta\,\gamma - 4\,\delta - 4\,\gamma N\right)^2}$$

$$\frac{\partial^2 q_r^*}{\partial N^2} = -4\,\frac{\left(3\,\delta^3\gamma^2 + 6\,\delta^3\gamma + 3\,\delta^3 - 7\,\delta^2\gamma^2 - 11\,\delta^2\gamma - 4\,\delta^2 + 4\,\gamma^2\right) c\left(3\,\delta^2 + 3\,\delta^2\gamma - 4\,\delta\,\gamma - 4\,\delta - 4\,\gamma\right)}{\left(3\,\delta^2 N + 3\,\delta^2 N\gamma - 4\,\gamma N\delta - 4\,\delta\,N - 4\,\delta\,\gamma - 4\,\delta - 4\,\gamma N\right)^3}$$

$$\frac{\partial Q_r^*}{\partial N} = 8\,\frac{\left(-\gamma^2 - \gamma + \delta + 2\,\delta\,\gamma + \delta\,\gamma^2\right)\delta\,c}{\left(3\,\delta^2 N + 3\,\delta^2 N\gamma - 4\,\gamma N\delta - 4\,\delta\,N - 4\,\delta\,\gamma - 4\,\delta - 4\,\gamma N\right)^2}$$

$$\frac{\partial^2 Q_r^*}{\partial N^2} = -16\,\frac{\left(-\gamma^2 - \gamma + \delta + 2\,\delta\,\gamma + \delta\,\gamma^2\right)\delta\,c\left(3\,\delta^2 + 3\,\delta^2\gamma - 4\,\delta\,\gamma - 4\,\delta - 4\,\gamma\right)}{\left(3\,\delta^2 N + 3\,\delta^2\gamma - 4\,\gamma N\delta - 4\,\delta\,N - 4\,\delta\,\gamma - 4\,\delta - 4\,\gamma N\right)^3}$$

$$\frac{\partial s}{\partial N} = 8\,\frac{\left(\delta + \delta\,\gamma - \gamma\right) g\delta\,c}{\left(3\,\delta^2 N + 3\,\delta^2 N\gamma - 4\,\gamma N\delta - 4\,\delta\,N - 4\,\delta\,\gamma - 4\,\delta - 4\,\gamma N\right)^2}$$

$$\frac{\partial^2 s}{\partial N^2} = -16\,\frac{\left(\delta + \delta\,\gamma - \gamma\right)\gamma\delta\,c\left(3\,\delta^2 + 3\,\delta^2\gamma - 4\,\delta\,\gamma - 4\,\delta - 4\,\gamma\right)}{\left(3\,\delta^2 N + 3\,\delta^2 N\gamma - 4\,\gamma N\delta - 4\,\delta\,N - 4\,\delta\,\gamma - 4\,\delta - 4\,\gamma N\right)^3}$$