The Design of Incentives for the Management of Supply and Demand

A Thesis
Presented to
The Academic Faculty

by

Matthew J. Drake

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

H. Milton Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology
December 2006

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The only thing that will redeem mankind is cooperation.

Bertrand Russell

I am not trying to relieve others by putting a burden on you; but since you have plenty at this time, it is only fair that you should help those who are in need. Then, when you are in need and they have plenty, they will help you. In this way both are treated equally.

2 Corinthians 8: 13–14
For my parents, Stanley and Christine Drake, for their love and encouragement

and

For Sydney Davis Magidson (1915–1987) for laying the groundwork.

The conversation goes on.
As I look at the calendar, I see that I am writing these acknowledgments almost four years to the day after I moved to Atlanta to attend Georgia Tech. I attended the graduate-student orientation along with fifty-one of my new colleagues a few days later, and I remember being told that fewer than half of us in that room would actually finish our dissertations and graduate with a Ph.D. from the School of Industrial and Systems Engineering. Boy, that’s great! I definitely didn’t need another reason to doubt myself at that point. I was a twenty-one-year-old, business-school graduate from a Pennsylvania university that no one here had ever heard of, and I was entering a program in which I couldn’t even understand the notation on the chalkboard in my first class. How was I going to hold my own in a group of people who were much more prepared for this journey than I was?

Thankfully, I had a group of people in my life who acted as beacons of light and encouragement during my darkest moments. Anyone who has attained a doctoral degree will tell you that there are countless moments of doubt, confusion, frustration, and even despair; but if you’re lucky, these moments don’t last too long and don’t compound on each other. This is where the angels in your life come in; they keep the low points from becoming unscalable escarpments. (In my case, however, they were also masters of ensuring that the highs would not get too high.) If I were to thank everyone who played a part in my successful completion of the work that follows, this section would be twice as long as the dissertation itself. It must suffice for me to acknowledge those who played the largest roles in my life during my time at Georgia Tech, but everyone else not specifically mentioned here should know that I value their presence in my life, and I hope I can play a role of similar significance in their lives in the future.

I cannot thank Julie Swann, my adviser, enough for taking a chance on me. She had only been on the job for eight months when we started working together; and instead of searching out students with a more traditional background, she was willing to integrate
me into her research program right away and always managed to accept my knowledge and abilities at their current levels at that point in time. Working with her was the best decision I’ve ever made. I don’t know of anyone else who would have met with me as often as I needed, even up to four or five times per week at some times. I don’t know what I would have done without that kind of close attention as I was gaining confidence in my researching ability. Regardless of how difficult the research became, she always kept a clear vision of where we wanted to go. If I ever have the opportunity to advise a student in the future, I feel confident that I can be successful because I’ve had a front-row seat to a master class in advising for the last four years.

For the last three years, Paul Griffin has been like a second adviser to me. His presence was invaluable in acting as another wall for me to bounce ideas off of. He always affirmed my abilities and never failed to express his confidence that I would succeed. Perhaps most valuable, however, were the conversations about baseball or books in his office or over a scotch. Regardless of how bad the day was going, I would inevitably feel more grounded and ready to face the challenges after only chatting with Paul for a few minutes. He has a calming effect that always helped me put things into perspective and realize that all of the hurdles I was facing were attainable.

I would be remiss if I didn’t also thank my family and friends who put up with my moving many hundreds of miles away in order to pursue this degree. Even though I was so far away, I always felt close to home after a phone call or a letter. They also understood that some times I needed to concentrate on my work and could not be as available to them as I had been in the past. They never seemed to get angry or annoyed because they understood that I was doing something that was very important to me. I hope now that I am moving closer that I can repay them by becoming a more active part of their lives again.

The following are several groups of people who bear special recognition in my accomplishing this goal:

- To Georgia Tech professors—Pinar Keskinocak, Mark Ferguson, Steve Hackman, Joel Sokol, Craig Tovey, and Beril Toktay—for imparting their wisdom.
To Dan Guide, John Mawhinney, Bill Presutti, and Tom Pollack for their advice and encouragement to pursue this degree.

To Fr. Ray French, C.S.Sp., for affirmation.

To Chris Walker for showing me the value of the virtue of happiness.

To Brian Fralix, Ray Hagtvedt, and Lori Houghtalen for their friendship and support through these four years. I can’t imagine a better group of people to walk with me on this journey.

To my students in ISyE 3103 in the Spring 2006 semester at Georgia Tech for confirming that I have chosen the correct vocation.

Again, I reiterate that there are so many people, both mentioned here by name and anonymous, who hold such a special place in my heart. This is for all of us.
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This dissertation analyzes the economic incentives involved in three distinct supply chain and revenue management decision environments. The first study examines the adoption of the percent deviation contract in a supply chain to induce the buyer to share some of the demand risk in an environment in which the buyer would typically place her order when she has full knowledge of the customer demand levels. The subgame-perfect Nash Equilibrium decisions are characterized, and the percent deviation is shown to achieve full supply chain channel coordination in cases where a simpler contract cannot. Pareto-improving examples based on industry demand data are presented and discussed. The second section considers a revenue management problem for sports and entertainment organizations. Given that the organization starts the selling season by offering ticket packages exclusively, the optimal time during the selling season for the organization to begin selling individual-event tickets is derived. Extensions of the base model are developed to include multiple ticket packages and heterogeneous ticket packages. The model is illustrated using empirical data sets obtained from the Georgia Tech Athletic Department and the Atlanta Symphony Orchestra. The third section develops a model of vendor-controlled category management in which vendors are in charge of the stocking and assortment decisions for a given amount of shelf space at a vendor when the retailer retains control over the retail price. The subgame-perfect Nash Equilibrium strategies for two vendors and a single retailer are analyzed, and a revenue-sharing contract is shown to coordinate the channel when the vendors can produce multiple brands in a given product category and shelf space is sufficiently large or small.
CHAPTER I

INTRODUCTION

The study of the economic incentives faced by decision makers rests at the intersection of several traditional industrial engineering and management science disciplines. Models of incentive structures often incorporate optimization in the context of game theory because the decision maker’s benefit or cost is not simply a function of her own actions but also depends on the actions of other entities in the decision environment. Game theory provides a framework to capture the interdependence between decisions and payoffs as well as the sequential nature of the decision making process. Optimization is also requisite in any incentive analysis; the decision maker must be able to determine the best course of action in order to maximize her benefit or minimize her cost.

Incentive problems appear in every discipline that is in any way affected by the actions of rational decision makers. This dissertation focuses on the specific economic incentives and decision environments that arise when organizations seek to improve the effectiveness of their supply and demand management practices. Many firms have been able to reap substantial benefits by improving their supply chain operations—such as procurement, inventory control, production, warehousing, and distribution—and by partnering with strategic suppliers and customers. In some scenarios, though, an organization may have a fixed quantity of perishable inventory that she wants to utilize in the most effective manner. Since the short-term supply is fixed, she can turn to actions and policies that affect the demand for these goods in order to maximize revenue. Models that establish these policies fall under the broad category of revenue management. Game theory and optimization models can provide guidance in designing the terms of these relationships in order to improve the performance of the entire supply chain, while ensuring that the gains are not realized at the expense of one of the channel members.
This dissertation is comprised of three main chapters, each of which develops a mathematical model for a specific decision environment. Each of these models isolates the tradeoffs that a decision maker must balance in order to obtain the maximum benefit while accounting for the strategic decisions of other stakeholders. The first main chapter studies the incentives generated by a new type of contract that originated in the truckload transportation industry. We analyze the structure and decisions of the percent deviation contract, a supply chain contract that encourages the sharing of information and demand risk between a single supplier and buyer. This contract induces a dynamic game of perfect information, and we characterize the subgame-perfect Nash Equilibria under various contract scenarios. We establish ways to set the contract parameters to coordinate the supply chain and show that the percent deviation contract is able to achieve channel coordination in some cases where the quantity flexibility contract fails. In order to aid the implementation of the percent deviation contract in practice, we develop ways to set the parameters to satisfy both parties’ individual-rationality constraints. We conclude with numerical examples based on industry data that highlight the important results by illustrating the channel gains in each case over a traditional contracting arrangement.

The second major chapter considers a revenue management problem faced by sports teams and entertainment venues. Like airlines and hotels, sports teams and entertainment venues can benefit from revenue management efforts for their ticket sales. Demand for these tickets is such that some consumers are willing to purchase a package of tickets for multiple events in advance instead of buying single tickets as they are needed. Teams and entertainment venues usually offer bundles of tickets early in their selling horizon and put single-event tickets on sale at a later date. We model the seller’s optimal a priori timing decision for offering individual tickets in order to maximize revenue when bundle and single-ticket customers each arrive according to a linear, Markovian death process. When the marginal benefit of selling tickets in the bundle is less than that obtained from selling them individually, we find that the seller should never practice mixed bundling. When the reverse is true, however, we show that it may be optimal to offer both bundles and individual tickets. We find that the timing decision is independent of the initial inventory level and establish
comparative statics for the optimal timing decision as model parameters such as bundle size, marginal revenues, and customer arrival rates vary. We extend our results to find the optimal time for offering multiple bundle sizes or nonhomogeneous products.

The final major chapter presents a decentralized model of category management in a retail environment. Category management is a popular retail practice in which pricing, assortment, and stocking decisions are coordinated across products within a particular retail category such as laundry detergent or toothpaste. Recently some retailers have been delegating this assortment decision to their vendors, but it is not known how well this approach performs. We analyze an industry-motivated model of category management in which the retailer allows each vendor to make stocking and assortment decisions for a given amount of shelf space. We determine the optimal pricing policies for the retailer and the best stocking and assortment policies for each vendor in a two-stage decentralized system. We find that when the vendors’ stocking incentives run contrary to the retailer’s preferences, the retailer can be considerably worse off (i.e., experience profit losses as high as 40%) by delegating these responsibilities to the vendors. We compare the delegated system to a retailer-controlled channel and a centralized supply chain and demonstrate how a minimum-profit constraint can induce the vendors to achieve the total profit of a retailer-controlled channel but it may not induce centralized performance. To improve the performance of the decentralized, vendor-controlled channel, we show that a revenue-sharing arrangement with a discounted wholesale price is guaranteed to achieve full supply chain coordination in a vendor-controlled channel when the vendors produce multiple substitutable products and shelf space is limited or ample. Revenue sharing may not coordinate a vendor-controlled channel with medium levels of shelf space, but we characterize the difference in the decisions compared with the centralized channel’s decisions, which is small for most parameter realizations. Through our analysis of revenue sharing, we establish that a revenue-sharing contract can coordinate stocking levels in a general, uncapacitated supply chain consisting of multiple vendors setting stocking levels for multiple substitutable products.
CHAPTER II

FACILITATING DEMAND RISK-SHARING WITH THE
PERCENT DEVIATION CONTRACT

2.1 Introduction

The proliferation of computerized information systems in the 1990s facilitated the establishment of supply chain partnerships in which demand information is shared between firms. The upstream firms can use this information to reduce the traditional demand distortion due to the bullwhip effect (Lee et al., 1997). Some firms have also incorporated this information into contracts that induce their supply chain partners to share demand risk, thereby improving supply chain efficiency. Many researchers and practitioners (e.g. Lee (2004) and Finley and Srikanth (2005)) have advocated demand risk-sharing as a necessary condition for supply chain collaboration efforts to be successful in practice. In this chapter we analyze one such contracting mechanism, which we denote as the percent deviation contract.

One industry that stands to benefit from application of the percent deviation contract is truckload transportation. While these carriers generally have standing weekly orders for loads with their bigger customers, most shippers call dispatch requesting a pickup in a few hours. Current legislation limiting driver hours has brought the efficiency of operations into the forefront of truckload carriers’ concerns. A survey by the American Trucking Association estimates that trucking companies can expect a 17 percent productivity drop as a result of the new restrictions (Strong, 2004). In July 2004 a U.S. federal appeals court overturned these new regulations, but it is likely that some of the restrictions on driver hours will remain (Machalaba, 2004).

Trucking companies have employed a variety of actions to account for the loss of efficiency. Many carriers have sharply increased their rates or accessorials charges to attain their profitability goals, but this increase has caused some shippers to switch to regional
less-than-truckload (LTL) carriers (Schulz, 2004). Other carriers have begun charging their shippers for excess loading time, since now these delays are counting against the time that drivers can be on the road (Strong, 2004). The percent deviation contract offers carriers an alternative method of improving efficiency by allocating some demand risk to the shippers. A large truckload carrier originally proposed the idea for this contract but did not know how to set the parameters or whether or not the contract would be beneficial.

The percent deviation mechanism is applicable in traditional manufacturer-retailer channels as well, especially those in which one party currently bears the weight of demand risk. One obvious application would be an industry where the retailer places an order only when he receives an order from his customer, such as home construction, equipment integrators, window replacement, or door-to-door sales.

We analyze the strategic properties of the percent deviation contract in which the buyer gives an initial order estimate and the supplier pre-acquires inventory. Once the buyer’s customer demand is realized, she places her final order, and the supplier fulfills all or a portion of the order, possibly by expediting. We characterize the subgame-perfect Nash Equilibria decisions when the supplier has a fixed expediting capacity and in the special case of infinite capacity. We discuss methods of channel coordination to optimize the performance of the entire system. Since the buyer assumes some demand risk under the percent deviation contract, her expected profit may be less than that under a traditional contracting structure; therefore, we develop two methods that the supplier can use to satisfy the buyer’s individual-rationality constraint. Numerical examples utilizing demand distributions estimated from industry shipping data illustrate how the percent deviation contract can be used to create a Pareto-improving system for each party.

2.2 Literature Review

The breadth of supply chain contracting literature has grown significantly over the last two decades as researchers and practitioners have examined strategic relationships between supply chain partners. (See Tsay et al. (1998) for a review of traditional contracting mechanisms.) One stream of supply chain contracting literature has proposed and analyzed
methods of coordinating decentralized decisions to attain the optimal supply chain profit. Examples of these studies include Weng (1997), Parlar and Weng (1997), Taylor (2002), and Huggins and Olsen (2003). We discuss below the most relevant contracting references, which model a system with multiple, sequential decisions.

Tsay (1999) analyzes a quantity flexibility contract in which the retailer commits to purchasing no less than a certain percentage of the initial forecast while the supplier agrees to fulfill up to a certain percentage above the forecast. He also evaluates the sharing of demand risk that produces the coordinated channel. Tsay and Lovejoy (1999) extend these results to a rolling horizon decision environment. In contrast to quantity flexibility, the percent deviation contract places no limits on the buyer’s final order, although it adds complexity to the decision environment by including additional contract parameters. We show in Section 2.4.4 that this added complexity can be justified because the percent deviation contract succeeds in coordinating the supply chain in several cases where the quantity flexibility contract is known to be unable to coordinate the channel.

Donohue (2000) and Cachon (2004) analyze contracts with two-tier pricing structure that induce early commitment from buyers. In both of these contracts the buyer is bound to her order in both periods, whereas in our contract the first order is only an estimate of demand and can be freely adjusted once demand is known. These two papers only consider the full compliance contract regime where the supplier must fulfill the entire order; whereas, we model the supplier’s compliance decision explicitly.

Several contracts employ an options framework where the buyer makes a firm order commitment and purchases options for additional goods to be exercised if demand is high. Cachon and Lariviere (2001) consider a single period model with options and forecast sharing. Since the buyer has an incentive to provide a biased forecast, they develop conditions that facilitate the credible sharing of forecasts under both full and voluntary compliance. They also analyze information asymmetry in which the manufacturer has some private information about demand. They conclude by showing that this options framework is a general model that encompasses other contract structures such as returns and additional sources of supply. However, in our case, we have an additional parameter (the deviation range),
no firm commitment, and no upper bound on the final order amount, so the contract we study cannot be reduced to their model. Barnes-Schuster et al. (2002) extend the options framework using a two-period model with correlated demand between periods.

A recent series of studies (see, for example, Jin and Wu (2001) and Erkoc and Wu (2005)) have analyzed reservation fee supply contracts in which the buyer pays a (usually) deductible fee to reserve capacity along with an exercise fee for the final order quantity. The manufacturer builds capacity based on the reservations made, but they can also build excess capacity to offer at a higher spot rate once demand is realized. The aforementioned studies only consider linear reservation fee contracts—where each unit ordered is charged the same prices. The percent deviation contract is a special case of a piecewise-linear reservation fee contract in which the reservation and exercise prices differ for various portions of the order.

In addition to contracting, several papers (e.g. Lee et al. (2000), Cachon and Fisher (2000), and Balakrishnan et al. (2004)) have examined various ways of reducing the bullwhip effect through information sharing in decentralized supply chains. Kulp et al. (2004) study the benefits the manufacturer gains under different degrees of information sharing and collaboration. They find that most of the manufacturers’ benefit from information integration comes from collaborative activities such as vendor-managed inventory and collaborative forecasting instead of simply sharing information. On the contrary, our results suggest that the information sharing induced by the percent deviation contract enable the supplier to attain a higher expected profit through increased efficiency and the sharing of demand risk with his customers.

Another stream of literature analyzes the effect of supply chain information asymmetry and forecast updating. Corbett et al. (2004) compare various contract structures under full and asymmetric information. Fisher and Raman (1996) and Iyer and Bergen (1997) evaluate the benefits of Quick Response manufacturing. Ferguson et al. (2005) develop a structure for analyzing supply chains under information updating where the final demand is the sum of two independent random variables; we introduce a similar form of information updating into an infinite capacity scenario in Section 2.4.3.

Our contribution includes analysis of a risk-sharing contract where decisions made by
the buyer and supplier explicitly depend on each other and are solvable in the framework of a dynamic, extensive form game. This dynamic game necessarily results in a more complex contract, but we also show that this contract can be strictly Pareto-improving for both parties. Our contract is most similar to the quantity flexibility contract, but ours does not enforce limits on the buyer’s final behavior, so we show that this contract can coordinate the supply chain in some cases where quantity flexibility cannot. Our analysis focuses on the buyer’s assumption of some demand risk unlike traditional relationships in this decision environment where she places an order when demand is known with certainty and experiences no loss when the realized demand is especially high or low. Because of this increased demand risk and because the purchase is often of a commodity good, we incorporate individual-rationality constraints to ensure buyer participation.

The next section develops the model for the general case where the supplier has an expediting capacity constraint as well as the special case of infinite capacity. Section 2.4 develops conditions on the contract parameters that satisfy each party’s participation constraints, details ways to coordinate the channel in each decentralized scenario, and compares the percent deviation contract with the well-known quantity flexibility contract. Section 2.5 provides the results of several numerical examples under various contract conditions based on demand distributions estimated from a major manufacturer’s weekly shipping activity. Conclusions and suggestions of future research are given in Section 2.6.

2.3 Models and Scenarios

The percent deviation contract accommodates the following sequence of decisions. The buyer provides an initial estimate of its final-order demand that will be placed at a later date. The seller can then use this information to acquire goods in advance (e.g., a truckload carrier can preposition trucks or coordinate backhauls to optimize his transportation network) at a low cost in anticipation of this demand. When the buyer’s demand is known with certainty, she places her actual order with the supplier. Depending on the contract parameters, the seller can choose to satisfy additional demand by expediting or subcontracting at a high cost or can choose to fulfill only the demand equal to the number of previously-acquired goods.
The percent deviation penalty is the mechanism that punishes the buyer for unrealistic estimates. If the buyer’s final order is within a certain percentage above or below her initial estimate, no penalty is charged. If the order exceeds the limits, the supplier charges a penalty on all goods ordered outside of the tolerable range.

2.3.1 Notation and Assumptions

We employ notation adapted from Donohue (2000). The buyer receives \( r \) dollars in revenue for each unit, and she pays a wholesale price, \( w \), to the supplier. We assume that the buyer earns a positive gross margin from these transactions (i.e., \( r > w \)). Consumer demand for a period is given by the random variable \( X \), which has a continuous, differentiable probability distribution function, \( f(x) \). If the buyer cannot satisfy her customers’ demand (due to lack of product availability), she incurs a customer penalty of \( \beta \) per unit. This \( \beta \) could also be viewed as the higher cost from using an alternative supplier not under long-term contract.

The seller faces a cost of \( c_1 \) dollars to acquire goods in anticipation of demand and must pay \( c_2 \) dollars to satisfy demand after the firm order has been placed. We assume that \( c_2 > c_1 \), so the \( c_2 \) can be thought of as an expediting or subcontracting cost. If the supplier has excess inventory at the end of the period, he receives a unit salvage value of \( v \). It is natural to assume that \( w > v \) and \( c_1 > v \), which ensure that the supplier does not receive too much of a benefit from selling goods for salvage. Since the seller may choose not to satisfy the buyer’s entire order, he must pay the buyer \( \alpha \) for each unit ordered but not delivered. We assume that \( \alpha < \beta \), which signifies that lost customers are more costly for the buyer than for the supplier.

The per-unit penalty that the buyer must pay the supplier for orders outside of the allowable deviation range is denoted by \( p \), while \( d \in [0, 1] \) is the percentage that defines the range. We assume that the buyer only pays the deviation penalty on units ordered and ultimately provided by the supplier. The buyer’s initial forecast of her order is given by \( q_1 \), and the actual order is \( q_2 \). The number of units the supplier acquires in advance of demand is \( t_1 \), and the additional goods expedited or subcontracted are denoted by \( t_2 \). The supplier has a maximum expediting capacity of \( M \) units. (In Section 2.3.2.3 we consider a special
This particular way of modeling the supplier’s capacity bears further consideration. It is important to note that the capacity for the supplier’s pre-acquisition decision is infinite. By setting the $t_1$ value, the supplier is *de facto* determining the capacity of the system as a whole, which is equal to $t_1 + M$. This structure is appropriate for buyer-supplier transactions in which the supplier has a lot of capacity in his system, but he must make the allocation decisions over many customers before production occurs. Therefore, if the supplier knows that a particular buyer will require many units in a given period, he can plan his production to satisfy the large order; closer to the purchase date, however, he can only provide a limited amount of excess capacity if necessary because the rest of his system is dedicated to fulfilling orders from other customers.

We make the following assumptions that improve tractability but are not likely to impede the application of the results. The first assumption is that all costs are linear per unit of demand for a single product line because we are interested in the structure of the incentives. Another assumption is the existence of complete, symmetric cost, capacity, and demand information between the two parties. (We relax this assumption in Section 2.4.3 by allowing the buyer to have a private demand forecast; the supplier constructs a conditional distribution of demand based on the buyer’s initial order estimate.) When the buyer places her final order, she knows the exact demand as is usual in truckload transportation and the other relevant channels discussed in Section 2.1.

If the actual demand amount exceeds the upper limit of the deviation range, the cost and penalty parameters determine whether or not the buyer’s order equals the full demand. In order for the buyer to order above the deviation threshold, the net cash flow from satisfying the demand must exceed the penalty that she must pay her customer for not satisfying demand. If the inequality

$$r - w - p > -\beta$$

holds, then $q_2 = X$, or the actual customer demand. If this inequality is not satisfied (for instance, if the penalty for ordering outside of the deviation range is too high), $q_2 = \min\{X, (1 + d)q_1\}$. The additional assumption $w > p$ assures that if the actual demand is
below the lower limit of the deviation range, \((1 - d)q_1\), the buyer orders the actual demand amount.

### 2.3.2 General Model with Finite Expediting Capacity

We begin our analysis with the decentralized structure in which each party makes decisions to optimize his individual expected profit. Even though the supplier has an expediting capacity of \(M\) units, the cost of expediting these units, \(c_2\), might be too high for him to choose to do so. In order for the supplier to use any of this expediting capacity, the cash flow from expediting must be higher than the cost of failing to expedite. These flows are dependent on whether or not the supplier will receive the deviation penalty on some or all of these units. The two buyer scenarios discussed above, which are dependent on whether or not the buyer is willing to place orders above the upper limit of the deviation range, generate different parameter conditions dictating the supplier’s expediting decision; therefore, we must consider each of these buyer scenarios independently.

#### 2.3.2.1 Buyer Orders Entire Demand

In the following two instances, the supplier’s expediting decision can be determined \textit{a priori}, without knowledge of how many units for which the buyer will pay the deviation penalty. If \(w - c_2 > -\alpha\), then the supplier finds it beneficial to expedite whether or not he receives the deviation penalty on any units; consequently, \(t_2^* = (\min\{q_2 - t_1, M\})^+\). We will denote this case as I.A. Similarly, if \(w - c_2 + p < -\alpha\), the supplier would not choose to expedite any units even if he were receiving the deviation penalty on all of the units; and thus, \(t_2^* = 0\). This will be scenario I.B.

We will use backward induction to solve for the subgame-perfect Nash Equilibria in each scenario. We now formulate the expected profit functions for the buyer and supplier in case I.A., where \(q_2^* = X\) and \(t_2^* = (\min\{q_2^* - t_1, 0\})^+\). The supplier chooses \(t_1\) to maximize his expected profit function:

\[
\Pi_{I.A.}^S = w \left[ \int_0^{t_1+M} xf(x)dx + (t_1 + M) (1 - F(t_1 + M)) \right] + v \int_0^{t_1} (t_1 - x) f(x)dx + p \int_0^{\min\{t_1 + M, (1 - d)q_1\}} (\min\{t_1 + M, (1 - d)q_1\} - x) f(x)dx +
\]
Figure 1: Regions of system capacity defining the form of the supplier’s expected profit function

\[
p \left[ \int_{(1+d)q_1}^{t_1+M} (x - (1 + d)q_1) f(x) dx + (t_1 + M - (1 + d)q_1)^+ (1 - F(t_1 + M)) \right] - c_1 t_1 - c_2 \left[ \int_{t_1}^{t_1+M} (x - t_1) f(x) dx + M (1 - F(t_1 + M)) \right] - \alpha \int_{t_1+M}^{\infty} (x - t_1 - M) f(x) dx. \tag{2}
\]

There are three separate functions that represent realizations of the expected profit function in (2) based on the relationship between the total system capacity, \( t_1 + M \), and the boundaries of the deviation range, \((1 - d)q_1\) and \((1 + d)q_1\). The expected profit regions are depicted in Figure 1. In region \( S_1 \) the supplier sets capacity so that he cannot even satisfy the lower limit of the deviation range. Region \( S_2 \) prescribes that the total system capacity lies somewhere in the deviation range. In these first two regions, the buyer will never pay the deviation penalty for orders above the upper limit of the range because these units will never be fulfilled. Region \( S_3 \) specifies that the system capacity exceeds the upper limit of the deviation range. In each region only one of the three separate expected profit realizations is feasible, regardless of which expected profit is higher in the region. The following observation establishes that the overall expected profit function is continuous.

**Observation 1** The individual expected profit function realizations that are active in two adjacent feasible regions are equal at the boundary (i.e., \( \Pi_{S.I.A.I.}^S((1-d)q_1-M) = \Pi_{S.I.A.II.}^S((1-d)q_1-M) \) and \( \Pi_{S.I.A.II.}^S((1+d)q_1-M) = \Pi_{S.I.A.III.}^S((1+d)q_1-M) \)).

**Lemma 1** If \( w + \alpha > p + c_2 \), the supplier’s expected profit function in (2) is piecewise-concave.\(^1\)

\(^1\)Here we define a piecewise-concave function as a continuous, piecewise function whose separate segments are individually concave.
Proofs of all lemmas, propositions, and theorems are presented in Appendix A. Each of the individual realizations of (2) has a corresponding maximizing value that is derived from the solution to the following equations,

\[ t_{I.A.I.}^1 \in \left\{ t : F(t + M) = \frac{w + \alpha - c_1 - (c_2 - v)F(t)}{w + \alpha - c_2 - p} \right\} \]

\[ t_{I.A.II.}^1 \in \left\{ t : F(t + M) = \frac{w + \alpha - c_1 - (c_2 - v)F(t)}{w + \alpha - c_2} \right\} \]

\[ t_{I.A.III.}^1 \in \left\{ t : F(t + M) = \frac{w + \alpha - c_1 + p - (c_2 - v)F(t)}{w + \alpha - c_2 + p} \right\} \]

which are all dependent on the supplier’s available expediting capacity. We can always find solutions to these equations by applying the Intermediate Value Theorem since the left-hand sides are all bounded between 0 and 1. The following lemma defines a relationship between two of these optimal values that helps us to simplify the supplier’s best response function.

**Lemma 2** If \( w + \alpha > p + c_2 \), then \( t_{I.A.I.}^1 \geq t_{I.A.II.}^1 \).

To understand the supplier’s best response, we can consider the individual functional maximizers and their relationship to each other and the boundaries of the feasible regions. We establish some additional properties of these functions and their maximizers in Appendix A. The following theorem characterizes the supplier’s best response, which is dependent on the specific value of the buyer’s decision, \( q_1 \), via the feasible region boundary conditions. Figure 16 in Appendix A provides a graphical depiction of the optimal response, given specific parameter values.
**Theorem 1** The supplier’s best response to a given value of $q_1$ when $w + \alpha > p + c_2$ is

\[
\begin{align*}
t_1^* (q_1) = & \begin{cases} 
    t_1^{I,A.I.}, & \text{if } t_1^{I,A.I.} \leq (1 - d)q_1 - M \ \& \ 
    t_1^{I,A.II.} \leq (1 + d)q_1 - M; \\
    t_1^{I,A.II.}, & \text{if } (1 - d)q_1 - M \leq t_1^{I,A.II.} \ \& \ 
    t_1^{I,A.III.} \leq (1 + d)q_1 - M; \\
    t_1^{I,A.III.}, & \text{if } t_1^{I,A.II.} \geq (1 + d)q_1 - M \\
    (1 - d)q_1 - M, & \text{if } t_1^{I,A.II.} \leq (1 - d)q_1 - M \leq t_1^{I,A.I.} \ \& \ 
    t_1^{I,A.III.} \leq (1 + d)q_1 - M; \\
    \arg \max_{t_1^{I,A.I}, t_1^{I,A.II}.} \Pi_{I,A}^S, & \text{if } t_1^{I,A.I} \leq (1 - d)q_1 - M \ \& \ 
    t_1^{I,A.III.} \geq (1 + d)q_1 - M; \\
    \arg \max_{(1 - d)q_1 - M, t_1^{I,A.III.}} \Pi_{I,A}^S, & \text{if } t_1^{I,A.II.} \leq (1 - d)q_1 - M \leq t_1^{I,A.I.} \ \& \ 
    t_1^{I,A.III.} \geq (1 + d)q_1 - M; \\
    \arg \max_{t_1^{I,A.II.}, t_1^{I,A.III.}} \Pi_{I,A}^S, & \text{if } (1 - d)q_1 - M \leq t_1^{I,A.II.} \ \& \ 
    t_1^{I,A.II.} \leq (1 + d)q_1 - M \ \& \ 
    (1 + d)q_1 - M \leq t_1^{I,A.III.}.
\end{cases}
\end{align*}
\]

The buyer must choose the $q_1$ that maximizes her expected profit while anticipating the supplier’s response to her given value. The buyer’s expected profit function is given by

\[
\Pi_{I,A}^B (r-w) \left[ t_1^* (q_1) + M \right] f(x)dx + (t_1^* (q_1) + M) \left( 1 - F (t_1^* (q_1) + M) \right) \\
- p \int_0^{\min \{ (1 - d)q_1, t_1^* (q_1) + M \} } \left( x \left( t_1^* (q_1) + M \right) - x \right) f(x)dx \\
- p \int_{(1 + d)q_1}^{t_1^* (q_1) + M} (x - (1 + d)q_1) f(x)dx \\
- p \left[ (t_1^* (q_1) + M - (1 + d)q_1)^+ \left( 1 - F (t_1^* (q_1) + M) \right) \right] \\
- (\alpha - \beta) \int_0^{t_1^* (q_1) - M} \left( x - t_1^* (q_1) - M \right) f(x)dx.
\]

When we consider the supplier’s best response $t_1$ values (4 in all), the first three values in (6) provide three realizations of the buyer’s expected profit function in (7) based on the values of the minimum and maximum operators. There is a fourth realization corresponding
Table 1: Possible SPNE decision pairs and feasibility conditions for case I.A.’s explicit $\bar{t}_1(q_1)$ decisions

<table>
<thead>
<tr>
<th>Decision Pair</th>
<th>Feasibility Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(q_1^{I.A.I.}, \bar{t}_1)$</td>
<td>$(q : q \geq \max \left{ \frac{t_1^{I.A.I.} + M}{1-d}, \frac{t_1^{I.A.I} + M}{1+d} \right}, \bar{t}_1^{I.A.I.})$</td>
</tr>
<tr>
<td>$(q_1^{I.A.II.}, \bar{t}_1)$</td>
<td>$\max \left{ \frac{F^{-1}(0)}{1-d}, \frac{\max{t_1^{I.A.II.}, t_1^{I.A.III.}} + M}{1+d} \right}, \bar{t}_1^{I.A.II.})$</td>
</tr>
<tr>
<td>$(q_1^{I.A.III.}, \bar{t}_1)$</td>
<td>$(q : \frac{(1-d)F((1-d)q)}{(1+d)(1-F((1+d)q))}, \frac{t_1^{I.A.III.}}{1+d})$</td>
</tr>
<tr>
<td>$(q_1^{I.A.IV.}, \bar{t}_1)$</td>
<td>$\max \left{ \frac{t_1^{I.A.II.} + M}{1-d}, \frac{t_1^{I.A.AAAA.} + M}{1+d} \right}, \frac{t_1^{I.A.IV.}}{1-d}$</td>
</tr>
</tbody>
</table>

We must consider this realization as well because the supplier’s response is explicitly dependent on the buyer’s initial decision unlike the other three possible responses. The following lemma establishes the piecewise-concavity of (7) with respect to these four realizations.

**Lemma 3** The buyer’s expected profit function realizations resulting from (7) are concave.

Since each realization is concave, we can apply the Karush-Kuhn-Tucker (KKT) conditions (c.f. Bazaraa et al. (1993: 151–55)) over each realization’s feasible range of decisions to determine the constrained optimal values of $q_1$. We characterize the following potential subgame-perfect Nash Equilibrium decision pairs by maximizing the four realizations of (7) over the region of $q_1$ where each corresponding explicit value of $t_1^*(q_1)$ is valid.

**Theorem 2** The possible subgame-perfect Nash Equilibrium (SPNE) decision pairs for explicit $t_1(q_1)$ decisions are given in Table 1.

We must also determine the buyer’s optimal decision over the regions where she knows that the supplier will be selecting the maximizing argument from a set of two values. From Theorem 1, we establish the following ranges of $q_1$ under which each of these situations is
Since all three situations involve the possible decision $t_{1.A.III.1}$, we define the difference function, $\Delta(t) \equiv \Pi_{I.A.}^S(t_{1.A.III.1}) - \Pi_{I.A.}^S(t)$, where $t$ is any other possible supplier pre-acquisition amount. While it is difficult to determine the exact feasible region for $q_1$ that induces each of the possible $t_1(q_1)$ values, we can use these difference functions to explain how a buyer would determine her optimal decision for a given set of parameters.

**Proposition 1** The structure of the three difference functions for the decisions in (8)–(10) enables us to determine the specific ranges of $q_1$ that induce each of the two possible supplier values for $t_1$.

For each instance there are at most seven decision pairs from Table 1 and obtained from the procedure in Proposition 1, but some of these decisions may not be mutually feasible given a set of problem parameters. Since this set contains a maximum of seven elements, the buyer can evaluate her expected profit in (7) with respect to each of the feasible pairs and select the initial order estimate that yields the highest expected profit to obtain the overall subgame-perfect Nash Equilibrium for this sequential supply chain game.

Each of the formulas used in computing the potential decision pairs has an economic interpretation. The buyer always sets her initial order estimate with a goal of minimizing the expected deviation penalty that she pays under each scenario. In the case where she can conceivably experience a lower and an upper deviation penalty, the quantity she chooses balances the expected lower and upper deviation penalties. The supplier also seeks to balance the expected revenue that he receives from pre-acquiring inventory with the cost of doing so as well as the expected expediting and shortage costs. Even though the resulting formulas are more complicated, the supplier follows the same rationale as in a traditional newsvendor contract.
We now consider case I.B., where the supplier chooses not to expedite any units after the buyer places her final order. This case is analogous to case I.A when $M = 0$ since the supplier can be viewed as having an effective expediting capacity of zero units if he chooses not to expedite; the results are simpler, though, because there is one fewer parameter to consider. Recall that this case occurs when $w - c_2 < -\alpha$, signifying a high expediting cost and a low $\alpha$ penalty paid to the buyer for not satisfying the full order.

The supplier’s expected profit function, denoted $\Pi_{I.B.}^S$, is the same as in (2) with $M = 0$. The profit function is piecewise-concave if $w + \alpha > p + v$; the derivation is identical to that of Lemma 1. Since the individual realizations of the overall expected profit function are each concave, we can solve for their maximum values by setting the first derivative equal to zero.

$$t_{I.B.}^1 \in \left\{ t : F(t) = \frac{w + \alpha - c_1}{w + \alpha - v - p} \right\}$$

(11)

$$t_{I.B.}^{II} \in \left\{ t : F(t) = \frac{w + \alpha - c_1}{w + \alpha - v} \right\}$$

(12)

$$t_{I.B.}^{III} \in \left\{ t : F(t) = \frac{w + \alpha - c_1 + p}{w + \alpha - v + p} \right\}$$

(13)

These decisions correspond with the equations in (3)–(5) again, with $M = 0$. (We expect this result because I.B. is a special case of I.A.) Note that these decisions have a familiar newsvendor form of a critical ratio of the underage costs to the sum of the underage and overage costs. The deviation penalty appears in (11) and (13) where the supplier’s capacity decision explicitly determines the expected lower and upper deviation penalty, respectively, that he will receive.

Using the relationships in Lemma 19 (stated in Appendix A) to simplify the feasibility conditions, the supplier’s best response in this scenario is characterized by the following theorem.
Theorem 3 The supplier’s best response to a given value of \( q_1 \) when \( w + \alpha > p + v \) is

\[
t^*_1(q_1) = \begin{cases} 
  t^{I,B.I.}_1, & \text{if } t^{I,B.I.}_1 \leq (1 - d)q_1 \ \& \ \not\exists \\
  t^{I,B.III.}_1, & \text{if } (1 - d)q_1 \leq t^{I,B.II.}_1 \ \& \ \not\exists \\
  \arg \max_{t^{I,B.I.}_1, t^{I,B.II.}_1} \Pi^S_{I,B.}, & \text{if } t^{I,B.I.}_1 \leq (1 - d)q_1 \ \& \ \exists \\
  \arg \max_{(1 - d)q_1, t^{I,B.III.}_1} \Pi^S_{I,B.}, & \text{if } t^{I,B.II.}_1 \leq (1 - d)q_1 \leq t^{I,B.I.}_1 \ \& \ \not\exists \\
  \arg \max_{t^{I,B.II.}_1, t^{I,B.III.}_1} \Pi^S_{I,B.}, & \text{if } (1 - d)q_1 \leq t^{I,B.II.}_1 \leq (1 + d)q_1 \ \& \ \not\exists \\
  (1 + d)q_1, & \text{if } t^{I,B.II.}_1 \leq (1 - d)q_1 \ \& \ t^{I,B.III.}_1 \geq (1 + d)q_1.
\end{cases}
\]

While it may not seem like it at first glance, the feasibility conditions for each of the decisions in (14) correspond to those in (6). They exhibit a simpler form because of the additional relationship between the \( t_1 \) values defined in Lemma 19.

The buyer’s expected profit function is the same as that given in (7) where \( M = 0 \); thus, the piecewise-concavity established in Lemma 3 applies to this case as well. We can again apply the KKT conditions to solve the buyer’s constrained optimization problem.

The possible subgame-perfect Nash Equilibrium decision pairs in case I.B. are the same as those for case I.A. (given in Table 1) except with the I.B. supplier decision values replacing the I.A. decisions. The buyer can apply the methods described in the proof of Proposition 1 to determine her optimal initial order estimate in the cases in which the supplier’s best response is the value among a set of maximizing arguments for two of the expected profit realizations. Once the feasible set of possible decision pairs is determined, the buyer can again substitute each of them into her expected profit function to find the maximizing

\[\]
decision pair, which is the SPNE.

If the contract’s parameters are such that neither of the above scenarios (I.A. or I.B.) apply, then the supplier’s expediting decision is dependent on the magnitude of the final order. In order for it to be Pareto optimal for the supplier to fulfill the order, the cash flow from satisfying must be greater than the cash flow from not satisfying, or \((w - c_2)(q_2 - t_1) + p(q_2 - (1 + d)q_1) > -\alpha(q_2 - t_1)\). Solving for \(q_2\), the seller satisfies the extra demand if

\[
q_2 > \frac{(w - c_2 + \alpha)t_1 + p(1 + d)q_1}{w - c_2 + \alpha + p} \equiv \Lambda.
\]

Formally, the supplier’s expediting decision is

\[
t_2^* = \begin{cases} 
(\min\{q_2 - t_1, M\})^+, & \text{if } q_2 > \Lambda \\
0, & \text{if } q_2 < \Lambda.
\end{cases}
\]

The dependence of supplier compliance on the magnitude of the final order may be problematic for both parties. The buyer must wait until the demand realization to know if the supplier is going to comply with the entire order, causing her added supply uncertainty. Knowing that the expediting decision rests with the magnitude of the order might induce the buyer to inflate her final order so that the supplier will fulfill the entire amount. The buyer’s strategic behavior is detrimental for the supplier because he could be induced to expedite when he would not otherwise.

To alleviate these difficulties, we recommend that the parties set the negotiated parameters—\(p\) and \(\alpha\)—such that the contract assumes another case. This could be accomplished by letting \(\alpha > c_2 - w\), shifting the contract to the I.A. case. Of course, shifts to other scenarios are possible through negotiations, depending on the relative market power of the parties. Since both parties have an incentive to set the contract parameters to move the contract to other cases, we omit this situation from the analysis.

2.3.2.2 Buyer Does Not Order Above Upper Deviation Limit

In this scenario the buyer chooses not to order above the upper limit of the deviation range (i.e., \(q_2^* = \min\{X, (1 + d)q_1\}\)) because the inequality in (1) is not satisfied; therefore, the supplier’s expediting decision is only dependent on the deviation penalty if the pre-acquisition amount is less than the lower limit of the deviation range. The supplier will
expedite \((t^*_2 = \min\{(q_2 - t_1)^+, M\})\) if \(w - c_2 > -\alpha\); we call this case II.A. When the supplier chooses not to expedite \((t^*_2 = 0)\) because \(w - c_2 < -\alpha\), we denote this situation as case II.B.

The supplier’s expected profit function for case II.A. is

\[
\Pi^S_{II.A.} = w \left[ \int_0^{\min\{t_1 + M, (1 + d)q_1\}} x f(x) dx \right] + w \left[ \min\{t_1 + M, (1 + d)q_1\} \right] (1 - F(\min\{t_1 + M, (1 + d)q_1\})) +
\]

\[
p \int_0^{\min\{t_1 + M, (1 + d)q_1\}} (\min\{t_1 + M, (1 - d)q_1\} - x) f(x) dx + v \int_0^{t_1} (t_1 - x) f(x) dx - c_1 t_1 -
\]

\[
c_2 \left[ \int_0^{\min\{t_1 + M, (1 + d)q_1\}} (x - t_1) f(x) dx \right] -
\]

\[
c_2 \left[ (\min\{t_1 + M, (1 + d)q_1\} - t_1) (1 - F(\min\{t_1 + M, (1 + d)q_1\})) \right] -
\]

\[
a \left[ \int_{t_1 + M}^{(1 + d)q_1} (x - t_1 - M) f(x) dx + ((1 + d)q_1 - t_1 - M)^+ (1 - F((1 + d)q_1)) \right].
\]

In this scenario, the maximum demand that will be satisfied is not simply determined by the supplier’s capacity but is also a function of the upper limit of the deviation range. The supplier’s expected profit function again has three realizations based on the values assumed by the minimums and maximum and is piecewise-convex when \(w + \alpha > p + c_2\).

Individually-optimizing the first two realizations of the supplier’s expected profit function yields \(t^*_{II.A.I.} = t^*_{I.A.I.}\) in (3) and \(t^*_{II.A.II.} = t^*_{I.A.II.}\) in (4), respectively. The third realization, in which \(t_1 \geq (1 + d)q_1 - M\), yields the optimal solution

\[
t^*_{II.A.III.} \in \left\{ t : F(t) = \frac{c_2 - c_1}{c_2 - v} \right\}.
\]

This solution is not dependent on the supplier’s expediting capacity, \(M\), because the supplier’s total capacity, \(t_1 + M\), is greater than the maximum buyer order, \((1 + d)q_1\). Thus, as long as the capacity is sufficiently high, it does not matter exactly how large it is because the supplier will always be able to satisfy the buyer’s order.

Since the first two II.A. decisions are the same as those in case I.A., the relationship between these decision values defined in Lemma 2 still applies. Combining this result with that of Lemma 20 (stated in Appendix A), yields the supplier’s best response function.
Theorem 4  The supplier’s best response to a given value of $q_1$ when $w + \alpha > p + c_2$ is

$$t_1^*(q_1) = \begin{cases} 
  t_1^{II.A.I.}, & \text{if } t_1^{II.A.I.} \leq (1-d)q_1 - M; \\
  t_1^{II.A.II.}, & \text{if } (1-d)q_1 - M \leq t_1^{II.A.II.} \leq (1+d)q_1 - M; \\
  t_1^{II.A.III.}, & \text{if } t_1^{II.A.II.} \geq (1+d)q_1 - M; \\
  (1-d)q_1 - M, & \text{if } t_1^{II.A.II.} \leq (1-d)q_1 - M \leq t_1^{II.A.I.}; \\
  (1+d)q_1 - M, & \text{if } t_1^{II.A.II.} \leq (1+d)q_1 - M \leq t_1^{II.A.II.}.
\end{cases} \tag{16}$$

Unlike the supplier’s best response function for case I.A. in (6), the best response in (16) establishes explicit $t_1$ decisions for every possible $q_1$ value that the buyer could choose. In this case we also have an additional supplier decision that was not possible in case I.A.

The buyer’s expected profit function in case II.A. is given by

$$\Pi_B^{II.A.} = (r - w) \left[ \int_0^{\min\{t_1^*(q_1) + M, (1+d)q_1\}} x f(x) dx \right] + (r - w) \left[ \min\{t_1^*(q_1) + M, (1+d)q_1\} (1 - F(\min\{t_1^*(q_1) + M, (1+d)q_1\})) \right] - p \int_0^{\min\{(1-d)q_1, t_1^*(q_1) + M\}} (\min\{1-d)q_1, t_1^*(q_1) + M\} - x) f(x) dx + \alpha \left[ \int_{t_1^*(q_1) + M}^{(1+d)q_1} (x - t_1^*(q_1) - M) f(x) dx + ((1+d)q_1 - t_1^*(q_1) - M)^+ (1 - F((1+d)q_1)) \right] - \beta \int_{\min\{t_1^*(q_1) + M, (1+d)q_1\}}^{\infty} (x - \min\{t_1^*(q_1) + M, (1+d)q_1\}) f(x) dx.$$

This function has five realizations depending on the value of $t_1^*(q_1)$. Since each expected profit function is concave from the same derivation as Lemma 3, we can apply the KKT conditions to each of the function realizations over the feasible regions defined in (16) to determine the possible SPNE solutions.

Theorem 5  The possible SPNE decision pairs for case II.A. are given in Table 2.

The buyer can again find the overall SPNE for this game by determining which of the decision pairs in Table 2 are feasible and then choosing the pair that maximizes her expected profit function, $\Pi_B^{II.A.}$. As in case I.A., both parties’ decisions seek to balance overage and underage costs.

The results for case II.B., where the buyer chooses not to expedite any units, are the same as those for case II.A. with $M = 0$. The supplier’s possible best responses are $t_1^{II.B.I} = t_1^{I.B.I}$. 

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### Table 2: Possible SPNE decision pairs and feasibility conditions for case II.A.

<table>
<thead>
<tr>
<th>((q_1^{II.A.I.}, t_1^{II.A.I.}))</th>
<th>Feasibility Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q_1^{II.A.I.} \equiv { q : F((1 + d)q) = 1 } ), (t_1^{II.A.I.})</td>
<td>(q_1^{II.A.I.} \geq \frac{t_1^{II.A.I.} + M}{1 - d})</td>
</tr>
<tr>
<td>((q_1^{II.A.II.}, t_1^{II.A.II.}) \equiv { q : p(1 - d)F((1 - d)q) = \alpha(1 + d)(1 - F((1 + d)q)) }, t_1^{II.A.II.})</td>
<td>(q_1^{II.A.II.} \leq \frac{t_1^{II.A.II.} + M}{1 + d})</td>
</tr>
<tr>
<td>((q_1^{II.A.III.}, t_1^{II.A.III.}) \equiv { q : p(1 - d)F((1 - d)q) = (r - w + \beta)(1 + d)(1 - F((1 + d)q)) }, t_1^{II.A.III.})</td>
<td>(q_1^{II.A.III.} \leq \frac{t_1^{II.A.III.} + M}{1 + d})</td>
</tr>
<tr>
<td>((q_1^{II.A.IV.}, t_1^{II.A.IV.}) \equiv { q : (r - w + \beta)(1 - d) + 2d\alpha = \alpha(1 + d)F((1 + d)q) + [(r - w + \beta + p)(1 - d) - \alpha(1 + d)]F((1 - d)q)), (1 - d)q_1^{II.A.IV.} - M))</td>
<td>(q_1^{II.A.IV.} \leq \frac{t_1^{II.A.IV.} + M}{1 - d})</td>
</tr>
</tbody>
</table>

in (11), \(t_1^{II.B.II.} = t_1^{II.B.II.}\) in (12), \(t_1^{II.B.III.} = t_1^{II.A.III.}\) in (15), and the two boundaries, \((1 - d)q_1 - M\) and \((1 + d)q_1 + M\). The possible SPNE are the same as those in Table 2 with the II.B. decisions substituted for the II.A. ones, and the buyer can again find the overall SPNE by identifying the feasible decision pair that maximizes her expected profit function.

#### 2.3.2.3 Infinite Expediting Capacity

We conclude our presentation of the general models by considering a special case in which the supplier’s expediting capacity is infinite (or especially large for practical purposes). These uncapacitated models have an especially simple structure that enables us to develop (quasi) closed-form optimal decisions. Since this case is based on the expediting capacity, we must only develop models for cases analogous to I.A. and II.A. above, in which the supplier chooses to expedite. It does not matter how much extra capacity the supplier has if he would not choose to use any of it.

We first consider the case where the buyer orders the actual demand in all cases and the supplier chooses to expedite (i.e., \(q_2^* = X\) and \(t_2^* = (\min \{ q_2 - t_1, M \})^+\)) since (1) holds and \(w - c_2 > -\alpha\). We denote this scenario as III.A. The supplier’s expected profit function is again the same as in (2) with \(M = \infty\), but this substitution results in the following simpler
\[ \Pi_{III.A}^S = \int_0^\infty x f(x)dx + \int_0^{(1-d)q_1} ((1-d)q_1 - x) f(x)dx + p \int_0^{(1+d)q_1} (x - (1+d)q_1) f(x)dx + v \int_0^{t_1} (t_1 - x) f(x)dx - c_1 t_1 - c_2 \int_0^\infty (x - t_1) f(x)dx, \] (17)

which is continuously differentiable (not piecewise-continuous like (2)). Consequently, we do not have to deal with multiple functional realizations as in Sections 2.3.2.1 and 2.3.2.2. (17) is concave by the same methodology used in the proof of Lemma 1, so we can solve for the optimal pre-acquisition amount using first order conditions. This yields

\[ t_{1,III.A}^I \in \left\{ t : F(t) = \frac{c_2 - c_1}{c_2 - v} \right\}. \] (18)

This is the same expression as \( t_{1,III.A}^I \) in (15) and the one that Donohue (2000) derives for the optimal production quantity in the first of two production periods and the optimal decision for the centralized channel (see Section 2.4.2.1). Since the system is uncapacitated regardless of the supplier’s \( t_1 \) decision, the supplier also chooses \( t_1^* \) according to (18) regardless of the buyer’s initial order estimate.

Similarly, the buyer’s expected profit function is the same as in (7), but infinite supplier capacity yields the simplified form:

\[ \Pi_{III.A}^B = (r - w) \int_0^\infty x f(x)dx - p \int_0^{(1-d)q_1} ((1-d)q_1 - x) f(x)dx - p \int_0^{(1+d)q_1} (x - (1+d)q_1) f(x)dx. \] (19)

Since the supplier always chooses to expedite in order to satisfy the buyer’s order regardless of its size, the buyer’s expected profit function is no longer dependent on the supplier’s \( t_1 \) decision. In this case, the \( t_1 \) decision only affects the supplier’s profitability and not her ability to fulfill the buyer’s order.

The buyer’s optimal initial order estimate is given by \( q_{1,III.A}^I \equiv \{ q : (1-d)F((1-d)q) = (1+d)(1 - F((1+d)q)) \} \), which corresponds with her optimal decision in case I.A.III. in which the supplier is able to satisfy orders above the upper limit of the deviation range. The decision \( q_{1,III.A}^I \) is the value of \( q_1 \) that equates the marginal expected deviation penalty for
demands below the lower limit of the range, \( p(1 - d)F((1 - d)q_1) \), to the marginal expected penalty for orders above the upper limit, \( p(1 + d)(1 - F((1 + d)q_1)) \). Since the nominal deviation penalty, \( p \), is the same regardless of whether the deviation was a lower deviation or an upper deviation, it is irrelevant to the buyer’s decision. Of course, if there were two deviation penalties, \( p_l \) and \( p_u \), they would affect the buyer’s decision.

Thus, the SPNE for the III.A. case is \((q_1^*, t_1^*(q_1^*)) = (q_1^{III.A.}, t_1^{III.A.})\) for all parameter sets such that \( w - c_2 > -\alpha \). It applies in situations where the supplier always has enough extra capacity in her network to satisfy the buyer’s order. It would be most reasonable when the buyer’s requirements are small compared with the supplier’s capabilities. Consequently, the supplier would only need to utilize the more complicated capacitated contracts for customers who require a large portion of his capacity. Since these buyers are larger, they are presumably more important to the supplier, so he would have more incentive to utilize a more complicated contract for these customers.

We now analyze the second infinite expediting capacity scenario, in which the buyer’s actual order will not exceed \((1 + d)q_1\), the upper limit of the deviation range, because inequality (1) is not satisfied. In this case, which corresponds to the capacitated case II.A., we have \( q_2^* = \min\{X, (1 + d)q_1\} \) and \( t_2^* = (q_2 - t_1)^+ \).

The supplier’s expected profit function must reflect the fact that the final order quantity will not exceed the upper limit of the deviation range. The supplier’s expected profit function becomes

\[
\Pi_{III.B.}^S = w \left[ \int_0^{(1+d)q_1} xf(x)dx + (1 + d)q_1(1 - F((1 + d)q_1)) \right] + \
\n\n\left[ v \int_0^{t_1} (t_1 - x)f(x)dx + p \int_0^{(1-d)q_1} ((1 - d)q_1 - x)f(x)dx - c_1 t_1 - \
\n\n\right. \
\left. - c_2 \int_{t_1}^{(1+d)q_1} (x - t_1)f(x)dx + ((1 + d)q_1 - t_1)(1 - F((1 + d)q_1)) \right],
\]

which he maximizes subject to \( 0 \leq t_1 \leq (1 + d)q_1 \). KKT optimality conditions yield the quantity \( t_1^{III.B.} = \min \left\{ t_1^{III.A.}, (1 + d)q_1 \right\} \). The supplier solves for the same pre-acquisition quantity as in case III.A., in which the buyer orders the full demand. If this pre-acquisition amount exceeds the upper limit of the deviation range, he simply adjusts the actual quantity to the upper limit because he knows that the buyer’s final order will never exceed this value.
The buyer’s expected profit function must be modified to

\[ \Pi_{III.B}^B = (r - w) \left[ \int_0^{(1+d)q_1} xf(x)dx + (1 + d)q_1(1 - F((1 + d)q_1)) \right] - p \int_0^{(1-d)q_1} ((1 - d)q_1 - x)f(x)dx - \beta \int_{(1+d)q_1}^{\infty} (x - (1 + d)q_1)f(x)dx. \]  

(20)

This results in an optimal initial order estimate of

\[ q_{III.B}^1 \in \{ q : p(1-d)F((1-d)q_1) = (r - w + \beta)(1 + d)(1 - F((1 + d)q_1)) \}. \]  

(21)

Notice that unlike the solution \( q_{III.A}^1 \), \( q_{III.B}^1 \) is dependent on the deviation penalty, since the deviation penalty will only be paid on orders below the lower limit of the range. The following lemma establishes a relationship between the two order estimate quantities.

**Lemma 4** If the buyer will not order above the upper limit of the deviation range (when \( p > r - w + \beta \)), then she deflates her initial order estimate compared with what she would have estimated if she were to order the entire demand (i.e., \( q_{III.A}^1 \geq q_{III.B}^1 \)).

This result is counterintuitive because we would expect the buyer to have an incentive to increase her order estimate since she knows that she will not order above the upper limit. By increasing the upper limit of the range, she ensures that she can satisfy additional customer demand. The contrary result in Lemma 4 holds because the deviation penalty is so large that the buyer would rather not satisfy some customer orders for large demand realizations than risk having to pay the lower deviation penalty on small demands. Thus, the initial order estimate is smaller to reduce the expected deviation penalty that she will incur.

### 2.4 Economic Analysis and Model Extensions

#### 2.4.1 Individual Rationality Constraints

The practical implementation of the percent deviation contract is necessarily impacted by the competitive power of the parties. If the buyer has a powerful market presence, she will likely be able to negotiate favorable contract terms by threatening to use another supplier who offers a more traditional agreement. (We assume that the contract is used
in a competitive industry, so the buyer can find another supplier with comparable service performance and quality.) The terms of the contract, therefore, must satisfy the buyer’s individual-rationality constraint, which says that under the percent deviation contract she must be able to attain an expected profit at least as great as she could under a traditional mechanism. See Tirole (1988) for a detailed discussion of individual-rationality constraints. If this constraint is not satisfied, she will switch to another supplier.

In this section we compare our percent deviation contract to the status quo of a traditional wholesale-price contract. In the cases where the supplier’s expediting capacity is limited, the percent deviation mechanism can induce the supplier to pre-acquire significantly more inventory than he would under the wholesale-price contract. This additional ability to meet demand is beneficial for both parties, resulting in higher expected profits without further contract modifications. In situations where the supplier does not increase his pre-acquisition quantity significantly (i.e., the deviation penalty is not high enough to induce him to pre-acquire much more inventory), it is clear that the buyer will earn less expected profit under the percent deviation contract because she shares some of the demand risk by paying the deviation penalty for orders outside of the allowable range.

There are several ways in which the parties can adjust the terms of the percent deviation contract to satisfy the buyer’s individual-rationality constraint. The supplier can offer the buyer a fixed transfer payment to share some of his gain. In some cases the supplier can offer a discounted wholesale price, \( w' \), that gives the buyer the same expected profit as she would attain under the traditional wholesale-price contract. We present numerical examples of each of these strategies in Section 2.5. The remainder of this section illustrates the methodology required to find the requisite discounted wholesale price.

The III.A. infinite capacity model is comparable to the traditional newsvendor, wholesale-price (NV) contract, since the supplier chooses and has the capacity to satisfy the entire order. The buyer’s expected profit function under the NV contract is given by \( \Pi_{NV} = (r - w) \int_0^\infty xf(x)dx \). Notice that this function is not dependent on any decision by the supplier, because under a traditional contract in this setting the buyer places orders for exactly the number of units needed with no demand risk. Comparing this expected profit to
that under the percent deviation contract in (17) with the $q^{III.A.}$ decision, the contracting parties wish to find $w'$ such that $\Pi^{B_{NV}}(w) \geq \Pi^{B_{III.A.}}(w')$, to ensure that the buyer earns at least as much expected profit under the percent deviation contract as she does under the original wholesale-price contract. We find that the discounted wholesale price given by

$$w' \leq \left( w - p \frac{\int_{0}^{(1-d)q^{III.A.}} ((1-d)q^{III.A.} - x)f(x)dx + \int_{(1+d)q^{III.A.}}^{\infty} (x - (1+d)q^{III.A.})f(x)dx}{\int_{0}^{\infty} xf(x)dx} \right)^+ \tag{22}$$

satisfies the buyer’s participation constraint. The term in brackets represents the percentage of time that the deviation penalty will be paid. Consequently, the supplier must provide an allowance for this expected penalty if the buyer is to realize the same expected profit as in the newsvendor contract. The positive indicator on the right side of (22) is necessary because the term in brackets could be negative. If the right side assumes the value of zero, there is no discounted wholesale price mechanism that can satisfy the buyer’s rationality constraint with the given contract parameters.

The supplier also has an individual-rationality constraint that can be satisfied. To induce the buyer to participate in the percent deviation contract in this case, the supplier must offer the discounted wholesale price discussed above. This reduces his expected profit from the high theoretical profit he could earn with the original newsvendor wholesale price. If the supplier can reduce his early acquisition cost to $c'_1$ using the advanced information provided by the percent deviation contract (as the truckload carrier that originally proposed the mechanism envisioned), he could earn the high theoretical profit with parameters $(w', c'_1)$. The seller’s individual-rationality constraint finds the $c'_1$ such that $\Pi^{S_{III.A.}}(w', c'_1) \geq \Pi^{S_{III.A.}}(w, c_1)$. This constraint is satisfied if

$$c'_1 t^*_1(c'_1) \leq \left( c_1 t^*_1(c_1) - p \left[ \int_{0}^{(1-d)q^*_1} ((1-d)q^*_1 - x)f(x)dx + \int_{(1+d)q^*_1}^{\infty} (x - (1+d)q^*_1)f(x)dx \right] \right)^+, \tag{23}$$

where $t^*_1(z) = F^{-1} \left( \frac{z - \frac{1}{2}}{\frac{1}{2} - \epsilon} \right)$. Since the expression for $t^*_1$ is dependent on $c_1$, this condition is algebraically complex even for simple demand functions; consequently, we omit the details, but the derivation is straightforward for given problem parameters.
2.4.2 Channel Coordination

Supply chain research has shown that the total supply chain profit is maximized by a centralized firm making decisions that are best for the system as a whole. One main objective of supply chain contracts is to align each entity’s own incentives to induce decentralized decisions that attain the maximal centralized supply chain profit. This achievement is commonly referred to as “channel coordination.” In addition to channel coordination, the buyer and/or the supplier may want to define the terms of the percent deviation contract to induce specific behavior (i.e., the supplier chooses to expedite or the buyer always orders the full demand). Therefore, we now examine methods of setting the parameters of the percent deviation contract to coordinate the channel or to generate other desired behaviors.

2.4.2.1 Centralized Channel Benchmark

In terms of a centralized channel, the buyer and the seller are viewed as a single entity trying to maximize its own expected profit. Hence, there is no wholesale price \((w)\) paid from the sales department (the buyer) to the manufacturing department (the seller), and the penalties levied by one party on the other under the percent deviation contract are not valid. The buyer’s decisions are not relevant either since the single company does not order from itself; the combined firm must only determine the number of units to acquire early and the number to expedite.

If the cost structure for the centralized channel is such that \(r - c_2 > -\beta\), the firm will satisfy additional demand beyond the number of pre-acquired goods up to capacity \(M\). In this case the number of units to be expedited is given by \(t_2 = (\min \{X - t_1, M\})^+\). The channel’s expected profit function is

\[
\Pi_{C.I} = r \left[ \int_0^{t_1+M} xf(x)dx + (t_1 + M)(1 - F(t_1 + M)) \right] + v \int_{t_1}^{t_1+M} (t_1 - x)f(x)dx - c_1t_1 - c_2 \left[ \int_{t_1}^{t_1+M} (x - t_1)f(x)dx + M(1 - F(t_1 + M)) \right] - \\
\beta \int_{t_1+M}^\infty (x - t_1 - M)f(x)dx.
\]

(24)

Since this profit function is concave (because it has a traditional newsvendor form), first
order optimality conditions show that the optimal solution for $t_1$ is

$$t_1^{C.I.} \in \left\{ t : F(t + M) = \frac{r + \beta - c_1 - (c_2 - v)F(t)}{r + \beta - c_2} \right\}. $$

If $r - c_2 < -\beta$, the loss from expediting or subcontracting to meet the marginal demand is larger than the cash outlay from the penalty paid to the customer for not satisfying its demand. Accordingly, the centralized channel will not expedite at the higher cost $c_2$; formally, we have $t_2 = 0$. The channel expected profit function now becomes

$$\Pi_{C.I} = r \int_{t_1}^{\infty} x f(x) dx + t_1 (1 - F(t_1)) + v \int_{t_1}^{t_1} (t_1 - x) f(x) dx - c_1 t_1 - \beta \int_{t_1}^{\infty} (x - t_1) f(x) dx. $$

The profit function in (25) is very similar to (24), except the expected revenue has been adjusted to reflect the fact that the centralized channel will not satisfy any demand more than $t_1$. The optimal number of units to acquire early is given by $t_1^{C.II.} \in \left\{ t : F(t_1) = \frac{r + \beta - c_1}{r + \beta - v} \right\}$.

### 2.4.2.2 Infinite Expediting Capacity Channel Coordination

We first consider the infinite expediting capacity scenario where the shipper orders the entire demand, denoted by III.A. above. The total supply chain profit, obtained by adding together the individual buyer and supplier profit functions, is

$$\Pi_{III.A.}^{SC} = r \int_{0}^{\infty} x f(x) dx + v \int_{0}^{t_1} (t_1 - x) f(x) dx - c_1 t_1 - c_2 \int_{t_1}^{\infty} (x - t_1) f(x) dx, $$

which is exactly the centralized profit function for case C.I. given in (24) when $M = \infty$.

The optimal pre-acquisition amount found by optimizing (24) is the same as $t_1^{III.A.}$ in (18). This shows that any parameter values, as long as they are consistent with the inequalities defining these two cases, will coordinate the channel and produce the maximum centralized profit. The percent deviation parameters ($p$ and $d$), therefore, are merely a mechanism for profit distribution between the two parties in this case.

The second infinite capacity scenario (III.B.) is more complex to coordinate directly because the buyer does not order the full demand if it is greater than the upper limit of the deviation range because of the relatively high penalty cost. This does not clearly correspond to either of the centralized scenarios, since they are differentiated by whether or not it is
beneficial for the centralized firm to meet the excess demand and since the centralized decision maker never has such a penalty applied to partial orders. In this case the parties can achieve the centralized profit by setting the deviation penalty such that \( p < r - w + \beta \). This causes the contract to shift from III.B. to III.A. and, therefore, to the centralized case C.I. as well.

2.4.2.3 Finite Expediting Capacity Channel Coordination

Since the subgame-perfect Nash Equilibria for the scenarios in which the supplier has finite (possibly zero) expediting capacity have a complicated form, it is difficult to develop a coordinating mechanism that covers all of the possible equilibrium decision pairs. Consequently, we examine a possible decision pair for a particular case, and then we show how the system can be coordinated given that particular decision. The procedure described below is applicable to all other possible decision pairs and case scenarios.

We consider scenario I.B., in which the buyer orders the entire demand but the supplier does not expedite to fulfill the entire order, where the corresponding decision pair is \( (q_{I.B.III.1}, t_{I.B.III.1}) \). The following lemma contains the channel coordinating condition for this case. (The same channel coordinating condition also applies in scenario I.A. with the same decision pair, but we omit the results here for brevity.)

Lemma 5 The decentralized channel in scenario I.B. (and I.A.) in which the SPNE decision pair is \( (q_{I.B.III.1}, t_{I.B.III.1}) \) will be coordinated if the contract parameters are set such that

\[
\alpha + p + w = r + \beta. \tag{26}
\]

The left-hand side of (26) comprises parameters that represent payments between the buyer and the supplier. These are set during contract negotiations as opposed to the right-hand side, which only contains parameters that we assumed were exogenous to the contract because they involve an outside party to the contract (the buyer’s customer). The parties can coordinate the channel by setting \( \alpha, p, \) and \( w \) according to (26).
2.4.3 Information Asymmetry Extension

In this section we relax the assumption of complete, symmetric information between the parties and consider the situation where the buyer obtains a private forecast of demand based on her particular business environment before giving her initial order estimate to the supplier. We employ the demand model for information asymmetry discussed in Ferguson et al. (2005).

Suppose that demand, denoted by $Z$, is given by the sum of two random variables, $X$ and $Y$. Let $X$ be the buyer’s forecast of demand and $Y$ be the forecast error, having a Normal$(0, \sigma^2)$ distribution. We also assume that $X$ and $Y$ are bounded\(^3\), such that $L_X \leq X \leq H_X$ and $L_Y \leq Y \leq H_Y$. To assure that demand is nonnegative, let $L_X \geq |L_Y|$.

The buyer knows $X$ with certainty, but she does not know the value of $Y$ when she makes the $q_1$ decision. Since the forecast errors are normally distributed, let $f(y)$ be the normal pdf with standard deviation $\sigma$. The supplier knows the buyer’s forecast accuracy (and thus, $f(y)$), but he does not know the forecast value with certainty. The seller has a conditional distribution of $X$ given the initial order estimate, which we denote by $g(x|q_1)$. This conditional distribution could have been constructed from previous transactions in the contract period.

We begin by analyzing the decision structure in the III.A. infinite capacity case under this form of information asymmetry. Given the forecast $x$, the buyer’s expected profit function is

$$
\Pi_{Buyer,F.I.Info}^{\text{F.I.Info}} = (r - w) \int_{L_Y}^{H_Y} (x + y) f(y) dy - p \int_{L_Y}^{(1-d)q_1-x} ((1-d)q_1 - x - y) f(y) dy - p \int_{(1+d)q_1-x}^{H_Y} (x + y - (1+d)q_1) f(y) dy.
$$

This profit function is concave, so first order optimality conditions imply that the optimal $q_1^*$ is found by solving

$$
(1 + d) = (1 - d) F((1 - d)q_1^* - x) + (1 + d) F((1 + d)q_1^* - x).
$$

(28)

Note the structural similarities between the equation in (28) and the symmetric information

\(^3\)Since $Y$ is normally distributed, it is reasonable to let $H_Y = 3\sigma$ and $L_Y = -3\sigma$ for practical applications.
optimal decision, $q_1^*$. The bounds of the deviation range is merely shifted by the demand forecast, and the cumulative density is evaluated with respect to the forecast error distribution. Note also that, without loss of generality, we can assume that 

$$H_Y + x \leq H_Y$$

because we must evaluate $F((1 + d)q_1^* - x)$ to find the optimal decision. If a value of $q_1$ solves the equation in (28), then $q_1^* = \frac{H_Y + x}{(1 + d)}$ also solves the equation and satisfies the inequality.

Given the buyer’s order estimate, the supplier’s expected profit function can be written as

$$\Pi_{Supplier}^{F.I.Info} = w \int_{L_X}^{H_X} \int_{L_Y}^{H_Y} (x + y)g(x|q_1^*)f(y)dydx +$$

$$p \int_{L_X}^{H_X} \int_{L_Y}^{H_Y} ((1 - d)q_1^* - x - y)g(x|q_1^*)f(y)dydx +$$

$$p \int_{L_X}^{H_X} \int_{(1 + d)q_1^* - x}^{H_Y} (x + y - (1 + d)q_1^*)g(x|q_1^*)f(y)dydx +$$

$$v \int_{L_X}^{H_X} \int_{L_Y}^{H_Y} (t_1 - x - y)g(x|q_1^*)f(y)dydx - c_1 t_1 -$$

$$c_2 \int_{L_X}^{H_X} \int_{L_Y}^{H_Y} (x + y - t_1)g(x|q_1^*)f(y)dydx,$$

The first order conditions imply that the optimal $t_1^*$ is the value that solves

$$\int_{L_X}^{H_X} (t_1 - x)g(x|q_1^*)dx = \frac{c_2 - c_1}{c_2 - v}. \quad (29)$$

The left-hand side of (29), which is equivalent to the expected value of $t_1 - X$ with respect to the conditional distribution of $X$, is monotonically increasing in $t_1$; thus, it exhibits the single crossing property with the critical ratio on the right side. This implies that the equation can be solved numerically for $t_1$.

It is interesting to observe that the profit functions and decisions under this form of information asymmetry have the same form as their symmetric information counterparts. This suggests that the assumption of symmetric information does not degrade the structure of the decisions. In fact, similar results hold for several capacitated cases as well; for brevity we omit these details.
2.4.4 Comparison to Quantity Flexibility Contracts

Since the percent deviation contract provides the buyer with order flexibility around an initial order estimate, it is constructive to compare its channel performance with the quantity flexibility contract, which affords the buyer similar flexibility. If the quantity flexibility contract yields the same supply chain results, it renders a new contracting scheme such as the percent deviation mechanism superfluous. Tsay (1999) establishes that the quantity flexibility contract cannot coordinate the supply chain when the buyer is not bound by a minimum purchase commitment. The percent deviation contract, on the other hand, does coordinate the channel without establishing a floor on the buyer’s order. We establish this result formally in this section.

Let us consider analysis for a particular case, e.g., the I.B. scenario in which the SPNE is \((q_1^{I.B.III}, t_1^{I.B.III})\). Recall that this scenario can be coordinated with respect to centralized case C.II. by setting \(\alpha + p + w = r + \beta\). To compare the quantity flexibility and percent deviation contracts, we need to analyze them in a similar framework. We apply the basic quantity flexibility contract structure but modify as follows to correspond to the percent deviation decision environment. We assume that the buyer’s actual order in the quantity flexibility contract is made after the customer demand has been realized, as in the percent deviation scenario. Consequently, the supplier commits to fulfilling a maximum of \(t_1\) units.

The buyer establishes a minimum purchase commitment of \((1 - \delta)q_1\) units when she provides the initial order estimate, \(q_1\). If the buyer ends up ordering more units than she ultimately requires to satisfy the realized demand (as a result of the minimum purchase quantity), she receives \(u\) dollars per unit as a salvage value.

We assume that leftover units of inventory are no more valuable to the buyer than they are to the supplier (i.e., \(u \leq v\)). This is practical for several reasons. While it is true that goods generally appreciate in value as they move downstream in a supply chain, the buyer is not physically performing additional functions to add value to the product; consequently, the actual sale price of the salvaged product should be no higher than that which the supplier could receive if he sold it in the secondary market. Leftover product should be more valuable to the supplier in terms of expected revenue since he could likely use the
product to fulfill demand from another buyer while the buyer may have limited outlets to offload the extra product. This is especially true in the market for truckload transportation, which was an inspiration for the percent deviation contract. Carriers would obviously place more value on an unassigned truck than any one particular shipper might.

We will solve for the subgame-perfect Nash equilibrium decisions under a quantity flexibility contract via backward induction. The parameters in the I.B. scenario are such that the buyer orders \( q_2^* = \max \{ X, (1 - \delta)q_1 \} \), where \( X \) denotes the realized customer demand. The supplier’s expected profit can thus be written as

\[
\Pi_{QF}^S = w \left[ \int_0^{(1-\delta)q_1} (1-\delta)q_1 f(x)dx + \int_{(1-\delta)q_1}^{t_1} xf(x)dx + t_1(1-F(t_1)) \right] + v \left[ \int_0^{(1-\delta)q_1} (t_1 - (1-\delta)q_1)f(x)dx + \int_{(1-\delta)q_1}^{t_1} (t_1-x)f(x)dx \right] - c_1 t_1 - \alpha \int_{t_1}^{\infty} (x-t_1)f(x)dx. \tag{30}
\]

Since the supplier’s expected profit function is concave, first-order optimality conditions imply that the supplier’s optimal decision is \( t_1 = F^{-1} \left( \frac{w+\alpha-c_1}{w+\alpha-v} \right) \). There is one additional consideration, though, since the buyer is guaranteed to order at least \((1-\delta)q_1\). The supplier should pre-acquire at least the minimum purchase quantity because he is guaranteed to sell it. Thus, the supplier’s optimal decision is \( t_1^{QF} = \max \{ (1-\delta)q_1, F^{-1} \left( \frac{w+\alpha-c_1}{w+\alpha-v} \right) \} \).

The buyer’s expected profit function is given by

\[
\Pi_{QF}^B = \rho \left[ \int_0^{t_1^{QF}} xf(x)dx + t_1^{QF}(1-F(t_1^{QF})) \right] - w \left[ \int_0^{(1-\delta)q_1} (1-\delta)q_1 f(x)dx + \int_{(1-\delta)q_1}^{t_1^{QF}} xf(x)dx + t_1^{QF}(1-F(t_1^{QF})) \right] + v \left[ \int_0^{(1-\delta)q_1} ((1-\delta)q_1-x)f(x)dx + (\alpha-\beta) \int_{t_1^{QF}}^{\infty} (x-t_1^{QF})f(x)dx \right]. \tag{31}
\]

We can solve for the buyer’s optimal decision, as before, by assuming that the supplier’s decision takes on each of the two possible values and then optimizing the buyer’s profit subject to the constraint that makes the supplier’s decision valid. The decision pair \( (q_1^{QF}, t_1^{QF}) = \left( F^{-1} \left( \frac{r-w+\beta-\alpha}{r-v+\beta-\alpha} \right), F^{-1} \left( \frac{r-w+\beta-\alpha}{r-v+\beta-\alpha} \right) \right) \) is optimal when \( F^{-1} \left( \frac{r-w+\beta-\alpha}{r-v+\beta-\alpha} \right) \geq F^{-1} \left( \frac{w+\alpha-c_1}{w+\alpha-v} \right) \), which reduces to \((c_1-v)(r+\beta-\alpha) + wv - c_1 u \geq (w+\alpha)(w-u)\). If this
inequality is reversed, \( (q_1^{QF}, t_1^{QF}) = \left( \frac{F^{-1}(\frac{w+\alpha-c_1}{w+\alpha})}{1-\delta}, F^{-1}(\frac{w+\alpha-c_1}{w+\alpha-v}) \right) \). In this case the supplier’s decision is fixed regardless of the value of \( q_1^{QF} \), so the buyer can reduce her demand risk by offering \( q_1^{QF} \in \{q|F((1-\delta)q) = 0\} \) such that there is no probability of customer demand below the minimum purchase amount. In summary, the SPNE decisions for the quantity flexibility contract are

\[
\left( q_1^{QF}, t_1^{QF} \right) = \begin{cases} 
\left( F^{-1}(\frac{r-w+\beta-\alpha}{r-v+\beta-\alpha}), F^{-1}(\frac{r-w+\beta-\alpha}{r-v+\beta-\alpha}) \right), & \text{if } (c_1-v)(r+\beta-\alpha) + wv - c_1u \geq (w+\alpha)(w-u) \\
\{q|F((1-\delta)q) = 0\}, F^{-1}(\frac{w+\alpha-c_1}{w+\alpha-v}) \), & \text{otherwise.} 
\end{cases}
\] (32)

Suppose the parameter values are such that the quantity flexibility equilibrium decisions are the first pair in (32). We can write the expected total supply chain profit as the sum of the agents’ individual expected profit functions, which reduces to

\[
\Pi_{QF}^{SC} = r \left( \int_0^{t_1^{QF}} xf(x)dx + t_1^{QF} \left( 1 - F \left( t_1^{QF} \right) \right) \right) + u \int_0^{t_1^{QF}} \left( t_1^{QF} - x \right) f(x)dx - c_1 t_1^{QF} - \beta \int_{t_1^{QF}}^{\infty} (x - t_1^{QF}) f(x)dx. \] (33)

Note that if \( u = v \), for any value of \( t_1 \) we have \( \Pi_{QF}^{SC}(t_1) = \Pi_{C.II.}^{SC}(t_1) \), where \( \Pi_{C.II.}^{SC}(t_1) \) is the centralized supply chain profit in (25).

**Theorem 6** The percent deviation contract coordinates the supply chain in the following cases where the quantity flexibility contract fails to coordinate:

i. When the salvage value is higher at the supplier \( (u < v) \), there are cases in which the centralized supply chain profit under the percent deviation contract always exceeds that attainable from the quantity flexibility contract.

ii. When the salvage values are equal for both parties \( (u = v) \), channel coordination efforts for quantity flexibility require either setting \( \alpha < 0 \) or \( w < c_1 \), both of which do not make sense in this decision environment.

In other supply chain contracting structures such as revenue-sharing agreements, it is possible for suppliers to benefit by selling goods for a wholesale price below their marginal

35
Table 3: Parameter declarations for numerical examples

<table>
<thead>
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<th>Parameter</th>
<th>Exp(.17297)</th>
<th>Unif(0,18)</th>
<th>Parameter</th>
<th>Exp(.17297)</th>
<th>Unif(0,18)</th>
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</tr>
<tr>
<td>( c_2 )</td>
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<td>22</td>
<td>( p )</td>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>3</td>
<td>1</td>
<td>( M )</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

cost of production as the second part of Theorem 6 requires. This strategy is successful because the supplier is receiving part of the buyer’s revenue in addition to the wholesale price that he charges the buyer. Looking at the supplier’s expected profit function under the quantity flexibility contract in (30), the supplier can either obtain \( w \) or \( v \) for each of the \( t_1 \) units he pre-acquires in advance of the buyer’s order. If each of these values are less than \( c_1 \), he cannot earn positive expected profit by selling below his marginal cost.

We have thus shown that there are cases in which the quantity flexibility contract cannot coordinate the supply chain, while the percent deviation contract is able to achieve coordinated performance. The main difficulty the quantity flexibility contract has in this decision environment is that it establishes a minimum purchase commitment for the buyer. The percent deviation contract provides the buyer more flexibility by allowing them to choose to pay the penalties associated with ordering outside of the deviation range. Of course, in order to gain this flexibility, the contract must be more complex; therefore, the percent deviation contract would likely be more costly to manage in practice.

2.5 Numerical Analysis

In this section we provide several numerical examples that illustrate the behavior of the percent deviation contract in various decision environments discussed above as well as how parameters can be set to satisfy individual-rationality constraints and to coordinate the channel. We estimated the demand distributions used below from weekly shipping data provided by a major U.S. manufacturer. The demand random variable represents the number of shipments per week required from the supplier to a retailer on a particular
origin-destination lane; we consider two such lanes. For one of the lanes, the exponential distribution gave the best fit, and for the other the uniform distribution was appropriate. We failed to reject chi-squared goodness of fit statistics at the 10% significance level for each of the two distributions. For the cost and contract parameters, we constructed values that make relative sense in this manufacturer’s business setting.

2.5.1 Exponential Demand Example (Case I.A.)

Consider weekly demand that follows an exponential distribution with $\lambda = 0.17297$ and the cost parameters listed in Table 3. These parameters define a contract in case I.A., since $r - w + \beta > p$ and $w + \alpha > c_2$; all of the supplier’s expected profit function realizations are concave because $w + \alpha > c_2 + p$. Thus, the buyer orders the exact demand, and the supplier chooses to expedite units (up to his capacity of 5). Under a traditional wholesale-price contract with inventory pre-acquisition and expediting, the supplier pre-acquires $t_{NV} \in \left\{ t : F(t + M) = \frac{w - c_1 - (c_2 - v)F(t)}{w - c_2} \right\}$, or $t_{NV} = 4.7667$ units. This results in an expected profit of 189.89 for the buyer and 29.21 for the supplier; thus, the total supply chain profit for the wholesale-price contract is 219.10. The centralized-channel pre-acquisition quantity is 10.4930, yielding a maximal channel expected profit of 243.45. The main problem with the wholesale-price contract is that the supplier does not have an incentive to pre-acquire enough inventory because the buyer is not sharing any of the demand risk. This low pre-acquisition amount restricts the total system’s ability to satisfy realized customer demand, which dampens the system’s profit potential.

For the same parameter set, the percent deviation contract with $(p, d) = (5, 0.2)$ is Pareto-improving for both parties as compared with the wholesale-price contract. Figures 2(a) and 2(b) depict the expected profit functions for the supplier and the buyer, respectively, as a function of the two main decision variables, $q_1$ and $t_1$. Note the piecewise form of these expected profit functions, which reflects the different profit function realizations with their distinct optimal solutions. Applying the solution procedure detailed in Section 2.3.2.1, the SPNE decision pair is $(q_1^*, t_1^*(q_1^*)) = (q_{I.A.III.}^*, t_{I.A.III.}) = (5.1810, 6.0391)$. These decisions yield an expected profit of 192.55 for the buyer and 37.37 for the supplier and a
Figure 2: Expected profit functions of (a) the supplier and (b) the buyer for the Exp(0.17297) example

Figure 2: Expected profit functions of (a) the supplier and (b) the buyer for the Exp(0.17297) example

We can design the percent deviation contract parameters to ensure that the channel coordination condition in (26) is met. Namely, we need $\alpha + w + p = r + \beta$, so we can satisfy this inequality by setting $p = 52$. This induces an equilibrium pair of $(q^*_1, t^*_1(q^*_1)) = (q^{I.A.III.}_1, t^{I.A.III.}_1 = t^{C.I.}_1) = (5.1810, 10.4930)$, which gives channel-optimal expected profits of 84.35 and 159.10 for the buyer and supplier, respectively, and a total expected supply chain profit of 243.45, as designed.

In order to induce the supplier to pre-acquire the channel-optimal inventory amount, the buyer has had to relinquish a substantial amount of profit to the supplier. The buyer’s coordinated expected profit does not satisfy her individual-rationality constraint, which requires that her expected profit be at least 189.89, the buyer’s expected profit from the wholesale-price contract. If the buyer received a fixed transfer payment, then she would be

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4 Since the deviation penalty is so large, the condition for concavity on the supplier’s first profit function realization is no longer satisfied. This does not matter, though, because the buyer would never choose an equilibrium in this realization, which requires that she pay the deviation penalty for every unit of demand satisfied.
willing to accept the percent deviation contract. In this case, the fixed transfer payment must be larger than 105.54. This payment, denoted $F$, should not be too high, though; or else the supplier would be better off under the original wholesale-price contract as well. Thus, for any fixed supplier-to-buyer transfer payment in the range $F \in (105.54, 129.89)$, the percent deviation contract is coordinated and strictly Pareto-improving for both parties as compared to the wholesale-price contract.

2.5.2 Uniform Demand Example (Case I.B.)

We now consider an example with uniform demand and parameters as defined in Table 3. Since $r - w + \beta > p$ and $w + \alpha < c_2$, a percent deviation contract in this case would fall in scenario I.B., where the buyer orders the full demand and the supplier chooses not to expedite because it is too expensive. Under a traditional wholesale-price contract, the supplier pre-acquires 12.7059 units of inventory. The buyer and supplier expected profits are 95.54 and 76.24, respectively, resulting in a total supply chain expected profit of 171.78. If the firms acted as a centralized channel, the pre-acquisition amount would be 15.2727 with a total expected profit of 177.82.

Figures 3(a) and 3(b) depict the expected profit functions for the supplier and the buyer under a percent deviation contract in this example. We can apply the solution procedure for case I.B. to determine the SPNE decision pair of $(q_1^*, t_1^*(q_1^*)) = (q_1^{I,B.III.}, t_1^{I,B.III.}) = (10.3846, 15.0968)$, which results in expected profits of 71.53 and 106.26 for the buyer and the supplier, respectively, and a total supply chain expected profit of 177.79. Note that this decentralized percent deviation contract produces a supply chain profit very close to that of the centralized channel; this is due to the fact that the supplier’s $t_1$ decision value is approximately equal to that of the centralized channel.

While the above percent deviation contract is close to coordinated as currently constructed, it does not satisfy the buyer’s individual-rationality constraint when compared

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5Note that, in this case, the channel coordinating condition is a function of the wholesale price, so we do not attempt to satisfy the buyer’s individual-rationality constraint using a wholesale price discount as discussed in Section 2.4.1.

6We see in Table 3 that the supplier has $M = 5$ units of available expediting capacity. This number is irrelevant here because regardless of how much extra capacity the supplier has, he will not use any of it because expediting is too costly.
Figure 3: Expected profit functions of (a) the supplier and (b) the buyer for the Unif(0,18) example

with the wholesale-price contract. Consequently, the percent deviation contract must be modified to give the buyer an incentive to accept it over the status quo. If the supplier offers a discounted wholesale price (as discussed in Section 2.4.1) of 15.2346, which represents an approximate discount of 15% off the original price of 18, the equilibrium decision pair becomes \((q_1^*, t_1^*(q_1^*)) = (q_{I.B.III.1}, t_{I.B.III.}) = (10.3846, 14.1812)\). This contract results in expected profits of 95.54 and 82.08 for the buyer and the supplier, respectively, and a total supply chain expected profit of 177.62, which is still close to the centralized optimum of 177.82. This percent deviation contract with a discounted wholesale price satisfies the buyer’s participation constraint and provides a higher profit for the supplier in relation to the traditional wholesale-price contract. Thus, for this example the individual-rationality constraint and the Pareto-improving condition are more important than channel coordination since the decentralized percent deviation contracts are close to being coordinated without any additional consideration.
2.6 Conclusions and Further Research

In this chapter we have characterized the subgame-perfect Nash Equilibria of a dynamic supply chain game induced by the percent deviation contract, a mechanism that was motivated by our discussions with a major firm in the transportation industry. Due to the sequential extensive form of this supply chain game, many of the decisions are functions of those decisions made in earlier stages of the game. We also showed mechanisms for coordinating the decentralized channel, and we illustrated two methods to ensure that both parties’ individual-rationality constraints are satisfied. The percent deviation contract can achieve channel coordination in some situations where the quantity flexibility contract fails to coordinate; this is mainly because the contract does not force the buyer to commit to a minimum purchase quantity. Several numerical examples based on demand distributions estimated from industry data show that a properly-designed percent deviation contract can be strictly Pareto-improving for both parties.

The main result we have shown is that the percent deviation contract is a viable, albeit somewhat complicated, mechanism whereby the supplier can transfer some of his demand risk to the buyer. The prospect of receiving a deviation penalty for large or small buyer orders induces the supplier to pre-acquire more inventory than he ordinarily would, which increases the total capacity of the system. This extra ability to satisfy end-user demand benefits the entire system, enabling Pareto improvements for both parties.

Several trajectories exist for future research in this area. The first direction includes relaxing some of the assumptions that we made in these models. A natural extension would be adding some information asymmetry by including one party’s proprietary information on costs or capacity. One could also extend the analysis by including nonlinear costs (reflecting production economies of scale) or some other pricing policy such as quantity discounts. More generally, future work incorporating dynamic decision environments could be useful, especially in multi-echelon supply chains. Comparison studies of various contracting mechanisms applied to the same scenario could lead to Pareto-improvements similar to the ones we found. Further analysis is also needed to incorporate the advanced demand information into operational production and transportation network models. Only then will the true
value of the percent deviation contract be estimated for the system as a whole.

**Acknowledgements**

This research was funded, in part, by The Logistics Institute Leaders in Logistics Grant from Lucent Technologies and NSF Grants DMI-0223364 and DMI-0348532.
CHAPTER III

OPTIMAL TIMING OF PACKAGE PROMOTIONS FOR SPORTS AND ENTERTAINMENT TICKETS

3.1 Introduction

Revenue management has its origins in the transportation industry, and extensive research in the field has been accomplished in the past two decades. Much of this research has concerned optimal and heuristic pricing policies for selling a fixed number of airline tickets for a specific flight to a stochastic stream of customers with widely varying reservation prices and utilities in order to maximize the expected revenue. Extensions have been developed for products with similar demand characteristics such as hotel reservations and car rentals. A decidedly smaller library of research exists concerning revenue management for non-travel industries. In particular, the market for sports and entertainment event tickets has not received a great deal of attention.

Tickets for sporting and entertainment events have many similar demand characteristics as airline tickets or hotel reservations; thus, these organizations are likely to benefit from developing revenue management techniques similar to those of the airlines. Like airline tickets, event tickets have value at a specific point in time (the designated start of a particular game or performance) and are worthless afterwards. Teams and entertainment venues offer many different classes of seats with corresponding unique prices determined by quality just as airlines offer different fare classes that have specific restrictions. Capacity is fixed in the short term for both industries, and the schedule of events is set months before they occur. The marginal cost of selling an extra ticket when there is available capacity for a particular event is negligible in each industry as well; this encourages firms to sell as many tickets as they can even at deeply-discounted prices.
There are some important differences between the two industries that necessitate separate modeling frameworks. Passengers must have a reason to travel to a specific location regardless of how undefined that reason might be; whereas, sporting and entertainment events are consumer ends in and of themselves and, therefore, are responsive to different kinds of promotions. A much greater proportion of event tickets are purchased at the box office window on the day of the performance than at the airport on the day of a flight. Airlines are also able to alter their prices dynamically in response to realized demand, while teams and entertainment venues typically publish a price schedule before the beginning of the season and must stick to it for the entire year. The only way to alter the price during the season is to offer discounts for specific events or customer groups.

Many spectators at sporting events are willing to consider purchasing tickets to multiple games at one time, whereas typical passengers for airlines only purchase tickets for one flight at a time. This enables the sports teams to offer their tickets in mixed bundles—selling goods both individually as well as in packages. Mixed bundling is a form of second-degree price discrimination because the same goods can be sold at different marginal prices to different consumers and the individual consumers self-select their bundle from the same menu of offerings. Teams can restrict the availability of tickets for especially popular games to buyers of ticket packages to extract extra revenue from consumers who value these events highly. Georgia Tech, for example, is able to induce fans to purchase half-season packages in order to secure coveted tickets to the bi-annual home football game with the rival University of Georgia.

All teams offer different packages of games designed to serve different customer demographics. Season ticket plans, for example, appeal to a particular segment of sports fans; not too many individuals are willing or able to make the commitment of time and money required by a season ticket. On the other end of the fan spectrum, a great many people prefer to purchase single game tickets on an ad hoc basis, often at the box office on the day of the game. There are many reasons for this: busy schedules, weather concerns, out-of-town fans, etc. For one reason or another, these customers are unlikely to make a commitment for more than one game at a time regardless of the team’s marketing campaigns and price.
A sizeable group of consumers, however, lies between these two extremes. They are open to the possibility of purchasing a multiple-game package, but not containing as many games as a full-season plan. Their purchase can be influenced by the team’s sales efforts. One Major League Baseball franchise, in our discussions, expressed exasperation about their ticket sales over the last seven years. Attendance has steadily decreased, but the effect has been dulled by increases in ticket prices and their share of local television revenue and national revenue generated by Major League Baseball. The percentage of ticket sales in the form of single-game tickets had increased drastically over this seven-year period of time. The team had kept its fan base strong, but many of these customers were only purchasing a few games as opposed to the partial-season ticket plans that they had bought in the past. Even though ticket packages may be sold at a discounted marginal price, all organizations with which we have spoken emphasized the importance of selling the packages to reduce the risk of having extra seats at the event time, to reduce sales transaction costs, and to increase customer loyalty. Of particular interest to this team were the marketing and sale of two partial-season plans: one consisting of twenty games and one for nine games. They were particularly interested in the best time to begin offering the nine-game packages so as to maximize revenue.

In this chapter we study the above optimal timing problem, which is one aspect of the complex revenue management environment faced by sports franchises and entertainment organizations. We model the seller’s decision of the time within the selling horizon at which she should make single tickets available or, equivalently, when she should promote the individual tickets. The seller faces conflicting incentives in this timing problem because he would like to sell as many ticket packages as possible, but if he waits too long to put the individual tickets on sale, he may not have enough time in the selling horizon to sell his all of his inventory.

We consider a static decision environment because many organizations publish their schedule of ticket availability early in the selling horizon. Fourteen of the 30 Major League Baseball teams announced their single-ticket availability for the 2006 season at least two
weeks before the tickets went on sale, and four of them (Cleveland, Kansas City, Philadelphia, and San Diego) made this announcement over a month ahead of time (MLB, 2006). Many minor league baseball teams (see, e.g., Lake County (OH) Captains (2006), Washington (PA) Wild Things (2006), and Mobile (AL) Bay Bears (2006)) followed a similar policy. The Jacksonville Jaguars (2006) of the National Football League also publicized the sale date of individual tickets several weeks early. Cultural organizations such as New York’s Metropolitan Opera (2005), Atlanta’s Fox Theatre (2006), and the Georgia Institute of Technology’s Robert Ferst Center for the Arts (2005) make the timing decision of offering single-event tickets very early in the selling season as well and announce the dates to the public far in advance. These cultural organizations also allow special “patrons” (significant donors) to purchase tickets in a fixed period of time before season tickets go on sale.

The \textit{a priori} nature of our timing problem provides an extra constraint when compared with most airline revenue management models because airlines can adjust their timing and pricing decisions in response to realized demand levels. Sports and entertainment organizations would undoubtedly benefit, in terms of increased ticket sales, by reserving this decision until later in the time horizon so that more of the demand history could be used to make the decision. The fact that many different organizations publish their ticket sales dates so far in advance suggests that they realize some benefit by doing so. Benefits would include the reduction of expedited printing costs for advertising materials, more time for workforce planning, and a smaller amount of worker uncertainty by settling the sale dates sooner. It is difficult, though, to believe that these benefits alone are significant enough to justify sacrificing their selling flexibility. The proliferation of this practice must be driven by customer behavior and expectations. An earlier commitment to the sales date allows the organization to advertise the dates for a longer period of time. Loyal customers may react negatively to an unexpected reduction in the sales period, and the organizations do not want to risk an adverse reaction because the package buyers are their most important customers.

In this chapter we model the seller’s optimal timing decision when customers arrive according to a pure, linear Markovian death process. Interestingly, the optimal policy is
independent of the initial inventory level due to the special structure of the linear death process. For many possible parameter values, the seller should not practice mixed bundling; instead, she should offer either only season ticket or individual tickets for the entire selling season. We determine how the optimal policy changes as a function of other model parameters such as the bundle size, marginal revenues, and customer arrival rates. We extend our results to the scenario where the seller makes multiple-size bundle offerings as well as the case where the event tickets have nonhomogeneous demand characteristics.

While we develop this model in the context of a sports ticket market, the framework can be extended to other industries where blocks of goods are sold in different quantities at different points in time. One possibility is the sale of hotel rooms first to groups and then to individual guests. Another application would be the allocation of fixed capacity by a contract manufacturer or third-party warehouse.

3.2 Literature Review

Optimal timing problems have been studied in many different research contexts. One significant problem concerns the optimal time to exercise a financial or real option. An additional stream of literature considers the optimal timing of pricing policies. Feng and Gallego (1995) model the timing of a single price change, either a markup or a markdown, and show that a time threshold policy that is dependent on the remaining inventory is optimal. This model is a stochastic extension of the deterministic solution provided in Gallego and van Ryzin (1994). Feng and Xiao (2000b) extend this setting to multiple monotonic (upwards or downwards) price changes, and Feng and Xiao (2000a) allow for multiple reversible price changes. When the assets to be sold are especially perishable, Feng and Xiao (1999) suggest to include a penalty factor to account for the business risk involved and solve for the closed-form solution to the continuous-time problem. Petruzzi and Monahan (2003) characterize the optimal time for a firm to terminate the primary selling season (in terms of a sales target) in a discrete time setting where demand in a given period is an increasing function of the cumulative demand until that point in time. Our model differs from all of these in that it considers bundles of products where demand is dependent on the remaining
inventory level at the time of the transaction.

The economics and marketing streams of literature contain several studies about ticket pricing for sports and other entertainment events. Leslie (2004) uses data from a Broadway production to estimate a discrete-choice random utility model of ticket sales with three different seat qualities under various pricing mechanisms. Rosen and Rosenfield (1997) develop an economic model with two seat qualities. They characterize revenue-maximizing prices given the quantities of each seat quality and then determine the quantity of seats to allocate to each quality based on this optimum pricing policy. Neither of these studies considers the bundling of tickets (selling season tickets or partial-season tickets), and they do not incorporate a temporal element in their pricing decisions.

Venkatesh and Mahajan (1993) determine the optimal pricing policies under the three types of bundling: pure commodities (selling only individual goods), pure bundling (selling only bundles of goods), and mixed bundling (selling goods both separately and within packages). Other variations on pricing goods sold in bundles under various competitive conditions are presented in Venkatesh and Kamakura (2003), Bakos and Brynjolfsson (2000), Salinger (1995), McAfee et al. (1989), and Dansby and Conrad (1984). Again, there is no consideration of offering different bundles at different points in time; these studies are concerned solely with the pricing decisions.

DeGraba and Mohammed (1999) develop a model of intertemporal mixed bundling with two types of consumers in a concert ticket environment. In their model bundles of tickets to two shows are sold first, and then single-performance tickets are sold at a later period of time. They show that the concert promoter can offer a price for the bundle of two shows that exceeds the price of the individual tickets if they can induce fear in the consumers that they will be sold out of seats by the time the single-event tickets go on sale. Their main concern is the pricing decision, while ours is the timing of the offering of the smaller bundle. (In their model the single-show tickets are offered at an exogenous point in time.)

A number of recent articles and conference presentations have highlighted the application of operations research and management science techniques to the problems faced by professional sports teams. Much of this research, however, has been in developing improved
game schedules that satisfy an extremely large set of preference constraints and (in some studies) minimize travel distance between matches through the use of large-scale integer programming models (see, e.g., Bean and Birge (1980) and Nemhauser and Trick (1998)) and predicting players’ and teams’ performance through statistical analysis (see, e.g., Kvam and Sokol (2005)). Fewer papers in operations research have focused on maximizing a team’s internal operations and policies.

To summarize our contributions to the existing literature, we consider the timing decision for offering different products (i.e. various-sized bundles) in contrast to the optimal timing studies that model price markdowns and markups for a single product. We are primarily concerned with the time at which to offer particular bundles unlike the commodity bundling literature that characterizes the optimal pricing decision for different bundles. Another contribution of our research is the employment of a demand model in which the quantity demanded is a decreasing function of the remaining inventory level. The rationale for this characteristic of demand is explained in Section 3.3.1. Our models address revenue management possibilities in a non-traditional market, that for sports and entertainment tickets. While we frame the problem in the context of the sale of event tickets, the models and insights derived are general enough to apply to the sale of contract manufacturing capacity in different-sized blocks of time.

The remainder of the chapter is organized as follows. Section 3.3 contains the general model for a single timing decision in which the seller offers either a bundle or individual units from a fixed quantity of homogeneous inventory. Section 3.4 contains extensions of the general model to the offering of multiple ticket packages in addition to individual tickets as well as the bundling of non-homogeneous products. Section 3.5 details empirical analysis of data sets comprised of ticket sales transactions for the 2003 Georgia Tech home football season and the 2005-2006 Atlanta Symphony Orchestra performance season. The conclusions of the study are provided in Section 3.6.
3.3 Single Ticket Package Model

Many entertainment venues advertise two types of ticket offerings: single-event tickets and season ticket packages. Season tickets are generally sold first in the selling season, and single-event tickets are made available at a later date. In this section we develop a model that a venue can use to determine the optimal time to start selling the single-event tickets.

3.3.1 Notation and Assumptions

For generality of application, we shall subsequently refer to all tickets as “units” and ticket packages as “bundles.” We consider a single seller who faces a selling season of \( T \) time units in length with a fixed \( k \) units of inventory to sell. There is no possible inventory replenishment during the selling season because that would entail adding capacity to the stadium, arena, or venue. These \( k \) units have homogenous demand characteristics except for the demand rate decreasing as the inventory level erodes. The seller can offer these goods as single units or bundles consisting of \( a \) units each, and we assume that \( k \) is an integer multiple of \( a \) to avoid complications from units that cannot be allocated to bundles. (This assumption becomes less restrictive as \( k \) increases.) As a result the firm can sell a maximum of \( k/a \) bundles or \( k \) individual units (or any combination thereof). Bundles of units yield a unit revenue of \( r_L \) (and a total revenue of \( r_L a \)), and individual units sell for \( r_S \) each.

Admittedly, many factors affect the demand for tickets to sporting events and entertainment performances, so it might seem simplistic to aggregate the tickets into a homogenous set of inventory. This assumption, though, enables us to control for all of the unique demand characteristics and isolate the effects of the independent demand processes and the prices on the optimal timing policy. We relax this assumption in Section 3.4.2 and consider two sets of fixed inventory with different demand characteristics.

The seller must determine the \textit{a priori} optimal time \( u \in [0, T] \) to begin selling individual units given that she began the horizon by selling bundles. Unlike other dynamic revenue

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1These “season” tickets need not necessarily comprise the entire season of performances. Each theater or hall uniquely determines the number of events that constitutes a season ticket. The criterion we employ for a season ticket package is that it contains admission to more than one event.
Table 4: Notation (in order of discussion)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>The end of the selling season</td>
</tr>
<tr>
<td>$k$</td>
<td>The number of units available to be sold before time $T$</td>
</tr>
<tr>
<td>$a$</td>
<td>The number of units in each bundle</td>
</tr>
<tr>
<td>$r_L$</td>
<td>Unit price of goods in the bundle (containing $a$ units)</td>
</tr>
<tr>
<td>$r_S$</td>
<td>Unit price of goods sold individually</td>
</tr>
<tr>
<td>$u$</td>
<td>The time to begin selling individual units (the decision variable)</td>
</tr>
<tr>
<td>$X_L(\tau)$</td>
<td>The pure death process corresponding to demand for bundles</td>
</tr>
<tr>
<td>$X_S(\tau)$</td>
<td>The pure death process corresponding to demand for single units</td>
</tr>
<tr>
<td>$\alpha_L$</td>
<td>The rate of demand decline from selling each bundle</td>
</tr>
<tr>
<td>$\alpha_S$</td>
<td>The rate of demand decline from selling a single unit</td>
</tr>
</tbody>
</table>

management models that establish protection levels or booking limits for each time period in the selling season, sports and entertainment organizations typically publish their ticket offering timeline at the beginning of the sales horizon; therefore, the timing decision of offering single-event tickets must be made before any sales have occurred. Those organizations that do not announce the date that individual tickets go on sale are still committed to a narrow time window due to the necessity of coordinating marketing campaigns for single tickets and their customers’ frequent desire to know the date far in advance.

As is the case for many consumer products, sports and entertainment tickets embody the maxim that all are not created equal. Tickets to different events obviously have different values in the minds of the consumers based on the popularity of the participants, but even tickets to the same event that are located within the same price class can be valued very differently. Most, if not all, venues sell the remaining seats that are closest to the field or stage whenever they make a transaction. Consequently, as more and more seats are sold in a price class, the ones that remain are less valuable because they are further away from the action. It is natural in this context to model demand as a decreasing function of the remaining inventory. The declining demand rate, thus, serves as a proxy for the quality degradation of the remaining inventory as goods are sold. To capture this interdependence between demand and inventory level, we model customer arrivals according to a pure, linear Markovian death process, a type of renewal process where the exponential inter-arrival rate decreases at each arrival.
The two pure, linear Markovian death processes corresponding to demand for bundles and individual tickets are denoted by \{X_L(\tau) : \tau \in [0,T]\} and \{X_S(\tau) : \tau \in [0,T]\}, respectively. \(X_L(\tau)\) denotes the number bundles remaining after selling bundled units from time period 0 to \(\tau\). By definition, \(X_L(0) = k/a\), and \(X_L(\tau) = 0\) means that the seller has sold out of inventory by time \(\tau\). The only values that \(X_L(\tau)\) can assume are \(k/a, k/a - 1, \ldots, a, 0\). \(X_S(\tau)\) denotes the number of units remaining after selling individual units for \(\tau\) time periods. \(X_S(\tau) = 0\) means that the firm has sold out. Since the seller begins offering single units at time \(u\), it follows that \(X_S(0) = aX_L(u)\), which is the remaining inventory after selling bundles for \(u\) units of time.

The characterization of customer arrivals as two pure, linear Markovian death processes leads to closed-form expressions (provided in Taylor and Karlin (1998)) for the probabilities of the death processes being in any state after a given amount of selling time. The death process for bundles, \(X_L(\tau)\), has customer arrival rates \(\mu^i_L = \alpha_L i, i = k/a, k/a - 1, \ldots, 1, 0\), where \(i\) is the number of bundles remaining for sale and \(\alpha_L\) is a constant linear rate. It follows that \(\mu_0 = 0\), which implies that no customers arrive when there is no remaining inventory.\(^2\) Similarly, the death process for single units, \(X_S(\tau)\), has customer arrival rates of \(\mu^j_S = \alpha_S j, j = n, n - 1, \ldots, 1, 0\), where \(j\) is the number of individual units remaining, \(n\) is the remaining inventory level when the seller puts individual units on sale, and \(\alpha_S\) is a constant linear rate. Table 4 summarizes the notation employed for the general model.

We assume that customer arrivals in these two demand processes are independent of each other and of price. One sports executive was quite adamant in telling us that his season ticket and individual ticket demand streams were independent. Furthermore, only one process is active at a given time. Only bundles are sold during the time interval \([0,u]\); and once the decision is made to sell single units, they are sold exclusively on \([u,T]\).

The first interval corresponds exactly to the way sports and entertainment organizations offer tickets—only season tickets are available at first. When venues put individual tickets

---

\(^2\)This is a necessary condition to ensure that the death process terminates at state 0 (i.e. zero units remaining). This assumption that the demand rate goes to zero with inventory is somewhat restrictive, but it may be justified by the fact that the only seats that typically remain at low levels of inventory are either very high or provide obstructive views. It is reasonable to expect that the demand rate for these tickets would be quite small.
on sale, they will still sell season tickets and larger packages; but the vast majority of transactions and the bulk of the marketing efforts are for single-event tickets. Consequently, it is reasonable to assume that only individual units are sold for the latter part of the interval; this assumption coincides with Georgia Tech and the Atlanta Symphony Orchestra’s policies that are analyzed in Section 3.5.

The final assumption we make is that any leftover goods at the end of the selling season have no value. Clearly, like airline seats and hotel rooms, tickets to events are worthless after the start of the game or performance. It is important to note, however, that we assume that the end of the selling horizon occurs before any events occur. This eliminates the difficult scenario that occurs when one game included in an unsold package is played but the tickets to other games comprising that package could still be sold. Thus, the following model is predicated on a team or organization’s desire to maximize the revenue it receives before the performance season starts. The organization benefits from this goal through a reduction of the necessity for ticket promotions during the course of the season, a lower likelihood of having many tickets left over on the actual event day, and the access to the funds from the sales several months before the event.

3.3.2 Optimal Timing Decision for Selling Individual Units

Given the above description of the seller’s decision environment, we can characterize her expected revenue function as follows. Let \( J(k, T|u) \) denote the seller’s expected revenue given that she has a selling season of length \( T \), a fixed inventory level of \( k \), and she decides to sell large bundles for the time interval \([0, u)\) and individual units over the interval \([u, T]\). The seller faces the following expected revenue maximization problem:

\[
\max_{u \in [0,T]} J(k, T|u) = r_L a E_L[k/a - X_L(u)] + r_S E_S[aX_L(u) - X_S(T - u)].
\]  

(34)

The first part of (34) is the expected revenue obtained from selling bundled units, where \( k/a - X_L(u) \) represents the number of bundles sold. Note that the \( L \) subscript denotes that the expectation is taken with respect to the bundle death process. The second part is the expected revenue from the single unit sales. The linear death processes result in the
following closed-form state probabilities (Taylor and Karlin (1998))

\[
P(X_L(u) = n) = \frac{(k/a)!}{n!(k/a - n)!} e^{-n\alpha_L u} (1 - e^{-\alpha_L u})^{k/a - n}
\]

\[
P(X_S(T - u) = l) = \frac{(an)!}{l!(an - l)!} e^{-\alpha_S(T-u)} (1 - e^{-\alpha_S(T-u)})^{an - l},
\]

where \(n\) is an index corresponding to the remaining number of bundles after offering bundles for \(u\) time periods and \(l\) is an index corresponding to the remaining inventory of individual tickets after offering single tickets for \(T - u\) periods of time. (Note that the single-unit death process begins with \(an\) units of inventory, where \(n\) is the state of the bundle death process at time \(u\).) These are binomial probabilities where the probabilities of a success on a trial are \(e^{-\alpha_L u}\) and \(e^{-\alpha_S(T-u)}\) for the bundle and single-unit demand processes, respectively. It is important to note that a “success” in the context of this problem corresponds to not selling the good because the state of the process is the remaining inventory level. The number of independent trials for each binomial probability, respectively, are \(k/a\) and \(an\).

The closed-form binomial state probabilities enable us to simplify equation (34) as follows.

**Lemma 6** If customer demand arrives according to two pure, linear Markovian death processes, the seller’s expected revenue function in (34) reduces to

\[
J(k, T|u) = k \left[ r_L (1 - e^{-\alpha_L u}) + r_s e^{-\alpha_L u} (1 - e^{-\alpha_S(T-u)}) \right].
\]

Proofs of all lemmas and theorems are provided in Appendix B. Further examination of the expected revenue function in (35) yields the result in the following lemma.

**Lemma 7** The seller’s expected revenue function is concave in the decision \(u\) if \(r_L \geq r_S\).

Since the expected revenue function is concave, the seller can solve her optimization problem \(\max_{u \in [0, T]} J(k, T|u)\) using traditional constrained concave optimization methods. If the unit revenue obtained from selling a good in a bundle is at least as large as that received if the unit is sold individually (i.e. \(r_L \geq r_S\)), then the following theorem completely characterizes the seller’s optimal timing decision for various combinations of the other parameters.
Theorem 7 The optimal time, $u^*$, for the seller to offer single units in the time interval $[0,T]$ when $r_L \geq r_S$ is given by

\[ u^* = \begin{cases} 
  T - \frac{\ln\left(\frac{r_S}{r_L-r_S}\frac{a_S-a_L}{a_S}\right)}{\alpha_S} & \text{if } \alpha_S > \alpha_L \\
  T & \text{if } \alpha_S \leq \alpha_L
\end{cases} \]

Note that in some cases, $u^* = 0$, implying that only individual tickets should be sold. Likewise, $u^* = T$ suggests that the tickets should only be offered as bundles. These policies of offering multiple products for different lengths of time run contrary to the practice of some firms (but not all). Examining the conditions provides insight into the cases where a singular offering is preferable.

The value $e^{-\alpha_S T}$ can be thought of as the probability that each unit will not be sold even if the goods are sold individually for the entire time horizon. The ratio $r_L - r_S$ is the percentage benefit in revenue obtained from offering a unit in a bundle as opposed to selling it individually, and $\frac{a_S-a_L}{a_L}$ is the percentage increase in customer arrivals from offering single units. If $\frac{r_L - r_S}{r_S} < e^{-\alpha_S T} \left(\frac{a_S-a_L}{a_L}\right)$, then the revenue benefit from bundling does not outweigh the loss in quantity demanded (i.e. customer arrivals) even in the worst case when the good would not even be sold if it were offered by itself for the entire time horizon; thus, $u^* = 0$, and the units should only be offered individually. Analogously, the inequality $\frac{r_L - r_S}{r_S} > \frac{a_S-a_L}{a_L}$ states that the revenue benefit from bundling outweighs the percentage reduction in customer arrivals; consequently, only bundles should be offered for the entire selling season and $u^* = T$. That is, bundles should be solely offered if the marginal revenue benefit is high and the accompanying loss in demand is not too severe.

We now establish some comparative statics for the optimal timing decision characterized by Theorem 7. While the bundle size, $a$, does not appear explicitly in the optimal value of $u^*$, it is reasonable to expect the bundle arrival rate, $\alpha_L$, to be a function of the bundle size. Suppose that $\alpha_L = \gamma/a$, which implies that the demand rate decreases as the bundle size increases. The following lemma describes the relationship between the bundle size and the optimal $u^*$. 

55
Figure 4: Expected revenue function when $k = 20$, $a = 2$, $T = 30$, $r_S = 10$, $\alpha_L = 0.1$, $\alpha_S = 0.8$

**Lemma 8** As the bundle size increases, the individual tickets should be offered earlier in the selling horizon (i.e., $u^*$ decreases).

Since an increase in the bundle size is assumed to generate a decrease in the bundle demand rate parameter, $\alpha_L$, Lemma 8 implies that the optimal $u^*$ value decreases (increases) as $\alpha_L$ decreases (increases). This is the behavior we would expect because a decrease in $\alpha_L$ means that selling bundles is less attractive for the seller, so she should find it beneficial to offer them for a shorter period of time.

Examining the behavior of the $u^*$ function as the marginal revenue parameters vary, we can establish the following lemma:

**Lemma 9** As the marginal revenue $r_L$ ($r_S$) increases, the individual tickets should be offered later (earlier) in the selling horizon.

Figure 4 shows the expected revenue function for different values of $r_L$ that are all greater than $r_S$. The optimal timing decision increases with $r_L$; as the unit revenue from selling bundles increases, the firm would like to sell more and more bundles. They find it beneficial to offer bundles for a longer portion of the time horizon. Note that in the first
three curves the optimal timing decision lies in the interior of the selling horizon, meaning that the firm offers both bundles and individual units at different points in time. The final curve \((r_L = 52)\) increases monotonically; therefore, \(u^* = T = 30\) and the firm only offers bundles.

The impact of a change in the individual ticket demand rate on the optimal timing decision is a function of the model parameters as detailed in the following lemma.

**Lemma 10** As the single ticket demand rate \(\alpha_S\) increases, the individual tickets should be offered earlier if \(\frac{1}{\alpha_S - \alpha_L} > \frac{\ln \left( \frac{r_S}{r_L} \left( \frac{\alpha_S - \alpha_L}{\alpha_S} \right) \right)}{\alpha_S}\). Otherwise, individual tickets should be offered later in the time horizon.

The parameter conditions required in Theorem 7 bear further consideration. One might argue that the assumption that \(r_L \geq r_S\) is unrealistic because most sports teams and entertainment venues offer unit price discounts for tickets purchased in a bundle. While this observation is sometimes true, the definition of the unit revenue parameters can be extended to the value of selling the good either in a bundle or by itself. Selling goods in bundles reduces the transaction costs and the required marketing costs compared with an individual sale because these costs are spread over multiple goods. Many sports teams experience a very high renewal rate on their season tickets and partial-season plans. By selling a bundle of tickets, it is possible that the team could be establishing a long-term stream of demand with a given customer. The discounted expected present value of future ticket purchases could also be incorporated into the definition of \(r_L\), helping to satisfy the condition for concavity. The Atlanta Symphony also told us that package subscribers are also more likely to make donations to the symphony, which is an important consideration for all non-profit organizations. All the professional and college sports teams and cultural organizations with which we have had discussions have emphasized the importance of selling ticket packages as opposed to single-game tickets; this universal response suggests that these organizations receive additional benefit from selling ticket packages as opposed to individual tickets.

There are, however, markets in which goods are typically sold at no discount or together.
at a higher price than the sum of their individual prices. The Georgia Tech Athletic Department charges the same unit price for season tickets to its football games as it does for individual sales; New York’s Metropolitan Opera follows a similar pricing policy. DeGraba and Mohammed (1999) develop a model where concert venues can charge this higher bundle price if some consumers feel that there is a chance that their desired event will be sold out before single-event tickets go on sale. The venue can charge the concert-goers extra for the time utility of securing their seats well in advance of the actual event. In markets for collectibles such as antiques or sports cards, sellers routinely charge more (and often substantially more) for a complete set of items than they would for the goods individually.

We now consider the optimal timing decision when \( r_S > r_L \) by first characterizing the shape of the expected revenue function in this scenario.

**Lemma 11** When \( r_L < r_S \) and \( \alpha_L > \alpha_S \), the expected revenue function given in (35) is

i. monotonically increasing in \( u \) if \( \frac{r_S - r_L}{r_S} \leq e^{-\alpha_S T} \left( \frac{\alpha_L - \alpha_S}{\alpha_L} \right) \),

ii. monotonically decreasing in \( u \) if \( \frac{r_S - r_L}{r_S} \geq \frac{\alpha_L - \alpha_S}{\alpha_L} \),

iii. decreasing and then increasing in \( u \) within the selling horizon if

\[
e^{-\alpha_S T} \left( \frac{\alpha_L - \alpha_S}{\alpha_L} \right) < \frac{r_S - r_L}{r_S} < \frac{\alpha_L - \alpha_S}{\alpha_L}.
\]

**Theorem 8** The optimal time, \( u^* \), for the seller to offer single units in the time interval \([0, T]\) when \( r_L < r_S \) is given by

i. \( u^* = \arg \max_{\{0, T\}} \{J(k, T|0), J(k, T|T)\} \), if \( \alpha_L > \alpha_S \).

ii. \( u^* = 0 \), if \( \alpha_L \leq \alpha_S \).

Figure 5 depicts the expected revenue function for different values of \( \alpha_S \) when \( r_L < r_S \). For small values of \( \alpha_S \), the expected revenue function increases monotonically, so \( u^* = T \). Conversely, for larger values of \( \alpha_S \), the curve decreases monotonically and \( u^* = 0 \).

Theorem 8 shows that if the seller offers a unit discount for the bundled goods (or maintains a constant unit price), it is never optimal for her to offer bundles for one portion of the selling season and then single units for the balance. The proliferation of discounted ticket
packages of various sizes along with single tickets suggests that the teams and organizations consider the entire value of offering the good in a bundle compared with the value of offering it singly. Even though we as consumers see a discounted price, \( r_L \geq r_S \) for decision-making purposes.\(^3\)

**Corollary 1**  *The optimal value for \( u^* \) is independent from the initial inventory level.*

This result runs contrary to our intuition that the initial inventory of goods available for sale should have some impact on the timing decision. Figure 6 depicts an example that highlights the observation in Corollary 1, which follows directly from Theorems 7 and 8. The independence stems from the fact that each state probability is binomial, signifying that each bundle and/or unit has the same independent probability of being sold in the

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\(^3\)Another possible explanation for the behavior is, of course, either that the sellers have not performed the analysis required to determine the revenue-maximizing behavior or that consumers are so used to the option of buying various-sized ticket packages at discounted prices that the sellers cannot change their policy without incurring a large negative backlash from their customers.
time horizon. The first part of (35) represents the expected value of selling each unit in a bundle, and the second is the expected value of selling the unit individually given that the unit has not been sold in a bundle. The seller, therefore, seeks to find the time at which the marginal value of selling a unit in a bundle is equal to its marginal individual sale value, and this time is independent of the initial inventory level. However, we have found other situations (in particular, when demand is characterized by a nonlinear, Markovian death process) where the optimal timing decision is dependent on the initial inventory level.

3.4 General Model Extensions

While the general model developed in the previous section results in a simple, easy-to-implement decision rule, it was built under several assumptions that may not hold for all sports teams and entertainment venues. This section discusses several extensions to the original model. The first model allows for multiple ticket package offerings in addition to individual tickets, and the second considers the optimal timing decision for offering bundles and individual tickets when the events have nonhomogeneous demand characteristics.
3.4.1 Multiple Ticket Package Model

Many sports teams and entertainment venues offer multiple ticket packages designed to satisfy a wide range of customer needs and preferences (as opposed to just a season ticket and single tickets). In order to characterize the optimal timing decisions for offering multiple packages, we extend the single package model developed in Section 3.3 to one that accommodates two bundle offerings—large and small—in addition to individual tickets.

We maintain all of the assumptions of the previous model. Customer arrivals are still characterized by pure, linear Markovian death processes. The parameters associated with the small bundles have a subscript “M” and are analogous to their large bundle and single unit counterparts. The seller now has timing decisions $u_1$ and $u_2$, where $u_1$ is the time she begins selling small bundles and $u_2$ is the time that individual units go on sale. Only one type of bundle is offered at any point in time; that is, large bundles are sold in $[0, u_1)$, small bundles are sold in $[u_1, u_2)$, and individual units are sold in $[u_2, T]$.

The seller’s optimization problem is given by

$$\max_{u_1, u_2} \ r_La_L E_L \left[ \frac{k}{a_L} - X_L(u_1) \right] + r_M a_M E_M \left[ \frac{a_L X_L(u_1)}{a_M} - X_M(u_2 - u_1) \right] + r_S E_S \left[ a_M X_M(u_2 - u_1) - X_S(T - u_2) \right]$$

s.t. 0 ≤ $u_1$ ≤ $T$

$u_1$ ≤ $u_2$ ≤ $T$.

Since the demand processes for the three offerings are again pure, linear Markovian death processes, we can derive a reduced-form expected revenue function using an analogous method as in Lemma 6.

**Lemma 12** If customer demand arrives according to independent pure, linear Markovian death processes, the seller’s expected revenue function, given in (36), reduces to

$$J(k, T | u_1, u_2) \equiv k [r_L (1 - e^{-\alpha_L u_1}) + r_M e^{-\alpha_L u_1} (1 - e^{-\alpha_M (u_2 - u_1)}) + r_S e^{-\alpha_L u_1 - \alpha_M (u_2 - u_1)} (1 - e^{-\alpha_S (T - u_2)})].$$

\(^4\)Most, if not all, Major League Baseball teams offer many different bundles ranging from full season tickets (81 games) to nine-game or even six-game packages.
Table 5: Possible optimal solutions for $u_1$ and $u_2$

<table>
<thead>
<tr>
<th>Case</th>
<th>$u_1^*$</th>
<th>$u_2^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.1</td>
<td>$u_1^*=0$</td>
<td>$u_2^*=0$</td>
</tr>
<tr>
<td>I.2</td>
<td>$u_1^*=0$</td>
<td>$u_2^* \in (0,T)$</td>
</tr>
<tr>
<td>I.3</td>
<td>$u_1^*=0$</td>
<td>$u_2^*=T$</td>
</tr>
<tr>
<td>II.1</td>
<td>$u_1^* \in (0,T)$</td>
<td>$u_2^<em>=u_1^</em>$</td>
</tr>
<tr>
<td>II.2</td>
<td>$u_1^* \in (0,T)$</td>
<td>$u_2^* \in (u_1^*,T)$</td>
</tr>
<tr>
<td>II.3</td>
<td>$u_1^* \in (0,T)$</td>
<td>$u_2^*=T$</td>
</tr>
<tr>
<td>III.</td>
<td>$u_1^*=T$</td>
<td>$u_2^*=T$</td>
</tr>
</tbody>
</table>

In order to apply traditional constrained concave optimization techniques to solve for the optimal timing policy, the objective function (in this case, the expected revenue function) must be jointly concave in $u_1$ and $u_2$. The following lemma establishes a sufficient condition similar to that of the single decision model for the expected revenue function to be concave.

**Lemma 13** The expected revenue function in (37) is jointly concave in $u_1$ and $u_2$ if $r_L \geq r_M \geq r_S$.

Since the expected revenue function is concave and the constraints are linear, the seller can use the Karush-Kuhn-Tucker (KKT) optimality conditions (c.f. Bazaraa et al. (1993: 151–55)) for constrained optimization to solve her optimization problem:

$$
\max_{u_1,u_2} \ J(k,T|u_1,u_2) \tag{38}
$$

subject to

$$
0 \leq u_1 \leq T
$$

$$
u_1 \leq u_2 \leq T.
$$

Examination of the optimization problem in (38) reveals seven possible solution combinations for the timing decisions. The first case is that $u_1^* = 0$, signifying that no large bundles should be sold. This case divides into three subcases depending on whether $u_2^*$ equals $u_1^*$, $T$, or lies in between the two. Similarly, it is possible that $u_1^* \in (0,T)$, which means that large bundles are sold for a certain length of time. This scenario has the same three subcases as the previous one. The final possibility is that $u_1^* = T = u_2^*$, or only large bundles should be sold. These seven cases are summarized in Table 5.
Table 6: Optimal solutions for $u_1$ and $u_2$ in each case

<table>
<thead>
<tr>
<th>Case</th>
<th>$u_1^*$</th>
<th>$u_2^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>I.2</td>
<td>0</td>
<td>$T - \ln\left(\frac{r_M-r_S}{r_l-r_S}\right)\frac{\alpha_S-\alpha_M}{\alpha_M}$</td>
</tr>
<tr>
<td>I.3</td>
<td>$T - \ln\left(\frac{r_M-r_S}{r_l-r_S}\right)\frac{\alpha_S-\alpha_L}{\alpha_L}$</td>
<td>$T$</td>
</tr>
<tr>
<td>II.1</td>
<td>$T - \ln\left(\frac{r_M-r_S}{r_l-r_S}\right)\frac{\alpha_S-\alpha_L}{\alpha_L}$</td>
<td>$T - \ln\left(\frac{r_S}{r_l-r_S}\right)\frac{\alpha_S-\alpha_M}{\alpha_M}$</td>
</tr>
<tr>
<td>II.2</td>
<td>$T - \ln\left(\frac{r_M-r_S}{r_l-r_M}\right)\frac{\alpha_M-\alpha_L}{\alpha_L}$</td>
<td>$T - \ln\left(\frac{r_S}{r_l-r_M}\right)\frac{\alpha_S-\alpha_M}{\alpha_M}$</td>
</tr>
<tr>
<td>III.</td>
<td>$T$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

Theorem 9 The optimal solutions to the seller’s two-decision problem in (38) when $r_L \geq r_M \geq r_S$ are provided in Table 6. The parameter conditions that characterize each case are given in Table 7.

Since we are now considering an optimization problem with two decision variables, it is possible to graph the expected revenue as a function of the two variables to gain a better understanding of the shape of the function. The expected revenue function in figure 7(a) has optimal decision pair II.1 from Table 5 because $\alpha_S > \alpha_L > \alpha_M$ and $e^{-\alpha_S T} \left(\frac{\alpha_S-\alpha_L}{\alpha_L}\right) < \frac{r_L-r_S}{r_S} < \frac{\alpha_S-\alpha_L}{\alpha_L}$. The appropriate row in Table 7 shows that $u_1^* = u_2^* \approx 18.84$, which means that large bundles are sold in the interval $[0, 18.84)$ and individual tickets are sold in $[18.17, 20]$. The expected revenue function in figure 7(b) has optimal decision pair II.3 since $\alpha_M > \max\{\alpha_L, \alpha_S\}$ and $e^{-\alpha_M T} \left(\frac{\alpha_M-\alpha_L}{\alpha_L}\right) < \frac{r_L-r_M}{r_M} < \frac{\alpha_M-\alpha_L}{\alpha_L}$. Table 7 gives us that $u_1^* \approx 18.17$ and $u_2^* = 20$, signifying that the firm should only offer large and small bundles and no individual units.

The main implication of Theorem 9 is that in order to obtain an optimal timing policy that offers all three sizes of products, the arrival rate parameters exhibit the opposite ordering as the unit revenues, just as in the single decision model. That is, we must have $r_L \geq r_M \geq r_S$ and $\alpha_S > \alpha_M > \alpha_L$. 

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Table 7: Optimal cases for all combinations of parameter conditions

<table>
<thead>
<tr>
<th>Arrival Rate Conditions</th>
<th>Other Parameter Conditions</th>
<th>Opt. Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_L \geq \max{\alpha_M, \alpha_S} )</td>
<td>None</td>
<td>III.</td>
</tr>
<tr>
<td>( \alpha_M \geq \max{\alpha_L, \alpha_S} )</td>
<td>( \frac{r_L-r_M}{r_M} \leq e^{-\alpha_M T} \left( \frac{\alpha_M-\alpha_L}{\alpha_L} \right) )</td>
<td>I.3</td>
</tr>
<tr>
<td>( \alpha_M \geq \max{\alpha_L, \alpha_S} )</td>
<td>( e^{-\alpha_M T} \left( \frac{\alpha_M-\alpha_L}{\alpha_L} \right) &lt; \frac{r_L-r_M}{r_M} &lt; \frac{\alpha_M-\alpha_L}{\alpha_L} )</td>
<td>II.3</td>
</tr>
<tr>
<td>( \alpha_M \geq \max{\alpha_L, \alpha_S} )</td>
<td>( \frac{r_L-r_M}{r_M} \geq \frac{\alpha_M-\alpha_L}{\alpha_L} )</td>
<td>III.</td>
</tr>
<tr>
<td>( \alpha_S &gt; \alpha_L &gt; \alpha_M )</td>
<td>( \frac{r_L-r_S}{r_S} \leq e^{-\alpha_S T} \left( \frac{\alpha_S-\alpha_M}{\alpha_L} \right) )</td>
<td>I.1</td>
</tr>
<tr>
<td>( \alpha_S &gt; \alpha_L &gt; \alpha_M )</td>
<td>( e^{-\alpha_S T} \left( \frac{\alpha_S-\alpha_M}{\alpha_M} \right) &lt; \frac{r_L-r_S}{r_S} &lt; \frac{\alpha_S-\alpha_M}{\alpha_L} )</td>
<td>I.1</td>
</tr>
<tr>
<td>( \alpha_S &gt; \alpha_L &gt; \alpha_M )</td>
<td>( \frac{r_L-r_S}{r_S} \geq \frac{\alpha_S-\alpha_M}{\alpha_L} )</td>
<td>III.</td>
</tr>
<tr>
<td>( \alpha_S &gt; \alpha_M &gt; \alpha_L )</td>
<td>( \frac{r_M-r_S}{r_S} \leq e^{-\alpha_S T} \left( \frac{\alpha_S-\alpha_M}{\alpha_L} \right) ; \frac{r_M-r_S}{r_S} \leq e^{-\alpha_S T} \left( \frac{\alpha_S-\alpha_M}{\alpha_L} \right) )</td>
<td>I.1</td>
</tr>
<tr>
<td>( \alpha_S &gt; \alpha_M &gt; \alpha_L )</td>
<td>( e^{-\alpha_M T} \geq \frac{\alpha_L (r_L-r_M)(\alpha_S-\alpha_M)}{\alpha_M (r_M-r_S)(\alpha_M-\alpha_L)} \left( \frac{\alpha_M-\alpha_L}{\alpha_L} \right) ) ( \frac{1}{\alpha_S} )</td>
<td>II.2</td>
</tr>
<tr>
<td>( \alpha_S &gt; \alpha_M &gt; \alpha_L )</td>
<td>( \frac{r_M-r_S}{r_S} \geq \frac{\alpha_L (\alpha_S-\alpha_M)}{\alpha_M (\alpha_S-\alpha_L)} )</td>
<td>I.3</td>
</tr>
<tr>
<td>( \alpha_S &gt; \alpha_M &gt; \alpha_L )</td>
<td>( e^{-\alpha_S T} \left( \frac{\alpha_S-\alpha_M}{\alpha_M} \right) &lt; \frac{r_M-r_S}{r_S} &lt; \frac{\alpha_S-\alpha_M}{\alpha_L} )</td>
<td>II.1</td>
</tr>
<tr>
<td>( \alpha_S &gt; \alpha_M &gt; \alpha_L )</td>
<td>( T - \ln \left( \frac{\alpha_L (\alpha_S-\alpha_M)}{\alpha_M (\alpha_S-\alpha_L)} \right) \frac{1}{\alpha_M} &gt; \right) )</td>
<td>II.2</td>
</tr>
<tr>
<td>( \alpha_S &gt; \alpha_M &gt; \alpha_L )</td>
<td>( e^{-\alpha_M T} \left( \frac{\alpha_M-\alpha_L}{\alpha_L} \right) &lt; \frac{r_M-r_S}{r_S} &lt; \frac{\alpha_M-\alpha_L}{\alpha_L} )</td>
<td>II.3</td>
</tr>
<tr>
<td>( \alpha_S &gt; \alpha_M &gt; \alpha_L )</td>
<td>( \frac{r_L-r_M}{r_M} \geq \frac{\alpha_M-\alpha_L}{\alpha_L} ; \frac{r_L-r_S}{r_S} \geq \frac{\alpha_S-\alpha_M}{\alpha_L} )</td>
<td>III.</td>
</tr>
</tbody>
</table>
3.4.2 Multiple Commodity Bundling Model

In this section we extend our analysis on the optimal timing decision to account for non-homogeneous products. Many varied factors affect a person’s desire to attend a particular sporting event. *Ceteris paribus*, a given team’s attendance typically increases as its performs at a higher level on the field. Fans generally have fewer schedule restrictions on weekends, so weekend games often fetch larger crowds than weekday games. Attendance at outdoor sporting events such as baseball can be adversely affected by inclement weather as well. The opposing team is also a major attendance determinant because of the number of that team’s local fans, a pre-established rivalry between the teams, and the popularity of the other teams’ players. Teams routinely bundle games from each category together in order to leverage the appeal of the popular games into extra sales of tickets for the weaker games. Each game possesses individual demand characteristics that may affect its efficacy in generating bundle sales.

We assume that a team is able to divide its home games into two groups: one containing games with favorable demand characteristics (weekend games, popular opponents, etc.) and one having the less desirable games. We also assume that demand for each ticket

---

**Figure 7:** (a) Expected revenue function when \( k = 20, a = 2, T = 20, r_L = 20, r_m = 18, r_s = 15, \alpha_L = 0.4, \alpha_M = 0.1, \alpha_S = 0.7 \) and (b) Expected revenue function when \( k = 20, a = 2, T = 20, r_L = 20, r_m = 18, r_s = 15, \alpha_L = 0.5, \alpha_M = 0.7, \alpha_S = 0.2 \)
category when the tickets are sold individually arrives according to independent, linear, Markovian death processes, denoted $X_H(\tau)$ for the high demand tickets and $X_S(\tau)$ for the sparse demand tickets. Since we are designating the two categories as “high” and “sparse” demand, it is natural to expect that the associated death rate parameters are such that $\alpha_H > \alpha_S$; that is, for a given level of remaining inventory, customers arrive for the high demand products faster than for the sparse demand products. When the tickets are packaged together, demand arrives according to another independent, linear, Markovian death process, $X_L(\tau)$. Again we consider the optimal time, $u^* \in [0, T]$, at which the team should begin offering both kinds of tickets individually, given that bundles are sold at the beginning of the selling season.

Consider a team whose schedule and seating capacity constitute a total, fixed, initial inventory of $k_H$ high demand tickets and $k_S$ sparse demand tickets across all games of these types. For example, $k_H$ is equal to the total number of games that are designated as “high” demand multiplied by the capacity of the stadium. The team wants to create ticket packages containing $a_H$ and $a_S$ games from each category, respectively. As before, the marginal revenue that the team receives for each ticket sold in a bundle is denoted by $r_L$, and the marginal revenue obtained from an individual sale\(^5\) is $r_H$ for high demand tickets and $r_S$ for sparse demand tickets. We assume that all of the tickets in a given demand category are homogeneous; that is, when combining tickets to form bundles, the team can choose any $a_H$ and $a_S$ of the tickets from each category. As a result, the team can offer $\min\{k_H/a_H, k_S/a_S\}$ ticket packages, and the expected revenue maximization problem is given by

$$\max_{u \in [0,T]} J(k_H, k_S, a_H, a_S, T | u) = r_L(a_H + a_S)E_L[\min\{k_H/a_H, k_S/a_S\} - X_L(u)] + r_H E_H[a_H X_L(u) - X_H(T - u)] + r_S E_S[a_S X_L(u) - X_S(T - u)].$$

(39)

The team assigns the games in its schedule before the selling season starts and, thus,

\(^5\)While many teams change a single price for all games, an increasing number of sports teams is utilizing differential pricing by identifying “premium games” for which they can charge a higher ticket price than the other games. Of course, the following results apply for teams that use unitary pricing by letting $r_H = r_S$. 

knows which quantity attains the minimization in (39), so we continue the analysis by assuming that $k_H/a_H \leq k_S/a_S$. This means that the team has extra sparse demand tickets that were included in bundles. We will establish results *ex post facto* for the converse situation.

**Lemma 14** The seller’s expected revenue function in (39) reduces to

\[
J(k_H, k_S, a_H, a_S, T | u) = r_L k_H \frac{a_H + a_S}{a_H} (1 - e^{-\alpha_L u}) + r_H k_H e^{-\alpha_L u} (1 - e^{-\alpha_H (T-u)}) + r_S k_H \frac{a_S}{a_H} e^{-\alpha_L u} (1 - e^{-\alpha_S (T-u)}) + r_S (1 - e^{-\alpha_S (T-u)}) \left( k_S - k_H \frac{a_S}{a_H} \right). \tag{40}
\]

We can establish the following parameter sufficiency condition for the concavity of the expected revenue function.

**Lemma 15** The seller’s reduced-form expected revenue function is concave in $u$ if $r_L \geq \max\{r_H, r_S\}$.

This concavity condition is similar to that derived for the single ticket package decision model developed in Section 3.3.2. The seller must be able to obtain a higher marginal benefit from offering the goods in a bundle as opposed to selling them individually. Since the expected revenue function is concave, we can compute the optimal timing decision using the KKT conditions.

**Theorem 10** The optimal time, $u^*$, for the seller to begin offering the goods individually when $r_L \geq \max\{r_H, r_S\}$ is given by

\[
u^* = \begin{cases} 
0, & \text{if } r_L e^{-\alpha_H (T_0 - \alpha_L)} + r_S \frac{a_S}{a_H} e^{-\alpha_S (T_0 - \alpha_L)} + r_S e^{-\alpha_S T} \left( k_S - k_H \frac{a_S}{a_H} \right) \geq 0, \\
T, & \text{if } e^{-\alpha_L T} (r_L \alpha_L - r_H \alpha_H) + r_S e^{-\alpha_S T} (r_L \alpha_L - r_S \alpha_S) \geq \frac{r_S}{k_H} \left( k_S - k_H \frac{a_S}{a_H} \right).
\end{cases} \tag{41}
\]

If neither of the above conditions holds, the optimal time, $u^*$, is the solution to

\[
r_L \alpha_L \frac{a_H + a_S}{a_H} e^{-\alpha_L u} - r_H \alpha_H e^{-\alpha_L u - \alpha_H (T-u)} - r_H \alpha_L e^{-\alpha_L u} (1 - e^{-\alpha_H (T-u)}) = r_S \alpha_S \frac{a_S}{a_H} e^{-\alpha_L u - \alpha_S (T-u)} + r_S e^{-\alpha_S u} (1 - e^{-\alpha_S (T-u)}) + r_S \alpha_S \left( \frac{k_S}{k_H} - \frac{a_S}{a_H} \right) e^{-\alpha_S (T-u)}. \tag{42}
\]
In contrast to the single decision model, inspection of the results in Theorem 10 shows that the optimal timing decision for two nonhomogeneous products is dependent on the initial inventories and explicitly on the bundle size. The inventory levels enter the optimal decision equation in (42) in the final term, which represents the extra sparse demand units that are not used in forming bundles. The results for the converse situation where $k_H/a_H > k_S/a_S$ are symmetric with the previous case and are summarized in the following corollary.

**Corollary 2** The optimal time, $u^*$, for the seller to begin offering the goods individually when $k_H/a_H > k_S/a_S$ and $r_L \geq \max\{r_H, r_S\}$ is given by

$$u^* = \begin{cases} 
0, & \text{if } r_H \frac{a_H}{a_S} e^{-\alpha_H T} (\alpha_H - \alpha_L) + r_S e^{-\alpha_S T} (\alpha_S - \alpha_L) + 
\quad r_H \alpha_H e^{-\alpha_H T} \left( \frac{k_H}{k_S} - \frac{a_H}{a_S} \right) \geq \frac{a_H}{a_S} \alpha_L (r_L - r_H) + \alpha_L (r_L - r_S) 
T, & \text{if } r_H \alpha_H \left( \frac{k_H}{k_S} - \frac{a_H}{a_S} \right). 
\end{cases}$$

If neither of the above conditions holds, the optimal time, $u^*$, is the solution to

$$r_L \alpha_L \frac{a_H + a_S}{a_S} e^{-\alpha_L T} u - r_S \alpha_S e^{-\alpha_L u - \alpha_S (T-u)} - r_S \alpha_L e^{-\alpha_L u} (1 - e^{-\alpha_S (T-u)}) =$$

$$r_H \alpha_H \frac{a_H}{a_S} e^{-\alpha_L u - \alpha_H (T-u)} + r_H \alpha_L \frac{a_H}{a_S} e^{-\alpha_L u} (1 - e^{-\alpha_H (T-u)}) +$$

$$r_H \alpha_H \left( \frac{k_H}{k_S} - \frac{a_H}{a_S} \right) e^{-\alpha_H (T-u)}.$$ (44)

In order to illustrate the effect that considering nonhomogeneous products can have on the seller’s optimal timing decision, we present a numerical example. Consider a team that has 200 tickets to offer during a selling season that lasts for 20 units of time. She receives a marginal benefit of $10 per ticket sold in packages containing ten tickets and $8 when the ticket is sold individually. (She charges the same price for high and sparse demand individual tickets.) Customers arrive according to linear, Markovian death processes, and the arrival rate parameter for bundles is $\alpha_L = 0.1$. We consider the following four possible scenarios that vary with respect to demand aggregation and bundle composition.

1. All of the tickets are homogeneous, and the individual tickets have an arrival rate parameter $\alpha_S = 1$. This case has the expected revenue function given in (35), which was developed in Section 3.3.2.
2. The seller divides the tickets into two demand categories. There are $k_H = 80$ high demand tickets and $k_S = 120$ sparse demand tickets. The high demand tickets have the arrival rate parameter $\alpha_H = 1.1$, while the sparse demand tickets have the parameter $\alpha_S = 0.9$. The bundles of ten tickets include five high demand and five sparse demand tickets.

3. This case is the same as case 2 except that the discrepancy between the individual ticket arrival rate parameters is larger. In this case $\alpha_H = 1.6$ and $\alpha_S = 0.4$.

4. This case is the same as case 3 except that the bundles now include four high demand tickets and six sparse demand tickets.

Figure 8 displays the expected revenue functions and the optimal $u^*$ decisions for the above four scenarios. Comparing the nonhomogeneous goods cases (2 and 3) with the homogeneous case (1), we see that the optimal decision does not change much from the homogeneous case when the demands are similar (case 2), but it varies widely when the demand rates are far apart (case 3). Comparing the third case, which has even bundles, with the fourth case, in which the bundles are uneven, we see that the team should offer
the uneven bundles for a longer period of time. The reason is that by selling more bundles, she will have fewer sparse demand tickets left to sell during the single-ticket sales period than high demand tickets. These sparse demand tickets have a low demand rate, so it is beneficial for her to have fewer of these tickets to sell individually.

### 3.5 Empirical Analysis

Our main objectives in using empirical data were to verify the plausibility of the demand assumptions we made and also to determine the optimal timing policy for a specific organization. We obtained transactional data sets from the Georgia Tech Athletic Department and the Atlanta Symphony Orchestra. The first set contained the history of individual customer ticket sales transactions for the eight months preceding the 2003 Georgia Tech home football season. Georgia Tech’s Bobby Dodd Stadium has a capacity of 55,000 seats. The average attendance at home games during the 2003 football season was 48,813, but this includes an average of 7,959 student tickets per game that were distributed free of charge.

Figure 9 depicts the sale of season tickets (six games), mini-packages (three games), and individual tickets per week as a function of the remaining inventory of tickets at the beginning of the week. Note that the peaks in demand correspond to the timing of the
university’s ticket promotions. We are most interested in the sales pattern following the promotional peak of a ticket product but before the promotion of the next product, since this is likely to represent the natural demand rate as a function of the remaining inventory. For the period of time after a product’s promotional peak, sales of that product do, in fact, decline with the remaining inventory level. The assumption that this decline is linear is satisfied with varying degrees of success by the three products’ demand processes, and demand is clearly affected by the timing of the promotions.

We estimate the arrival rate parameters for each type of ticket package using linear regression, although the interpretation of these estimates is limited because the demand data is a function of Georgia Tech’s current timing decision. The arrival rate for the season tickets ($\alpha_L = 0.1161$) is much larger than that of mini-packages ($\alpha_M = 0.0122$) and individual tickets ($\alpha_S = 0.0206$). Since the team offers all of its packages at the same unit price per ticket regardless of their size, we can apply Theorem 9 and the conditions given in Table 7 to find that a team facing these arrival parameters should offer season tickets as long as possible. This is slightly different from Georgia Tech’s 2003 selling policy which put mini-packages on sale with approximately five weeks remaining in the selling horizon and single tickets on sale with one week left. We estimate that the team would have benefited by selling more season tickets at the end of the horizon without detracting too much from the mini-package and individual ticket sales, but it is not meaningful for us to quantify the optimality gap in expected revenue between the team’s current policy and the optimal policy since the parameter estimates really depend on the university’s current promotions.

The second data set we studied contained both package ticket sales and individual-event ticket sales from the 18-week, 2005-2006 Atlanta Symphony Orchestra concert season. The symphony typically performs the same piece of music for three shows over a weekend. Atlanta Symphony Hall seats 1,762 per performance, and the seats are spread over three levels (Orchestra, Loge/Dress Circle, and Balcony). The symphony offers many different types of subscription (season) packages containing different quantities of tickets, including an option where patrons build their own 8-concert series. The average price of a ticket sold in packages for this season was $31.21, while the average price of an individual ticket was
$27.58.

The Atlanta Symphony package and individual ticket sales by week as a function of the remaining inventory level are shown in Figure 10. As with the Georgia Tech ticket sales, the first peaks in each time series correspond to the week during which the particular ticket type was initially promoted. Both kinds of tickets exhibit sales patterns that generally decrease with the remaining inventory from their initial demand spikes. It is clear here, however, that the symphony realizes a much higher percentage of its sales from individual tickets than Georgia Tech does for its football tickets. Even though they would prefer to sell packages, it seems as if the symphony would benefit from offering individual tickets earlier in the selling season because there does not appear to be much demand for packages. These suspicions were confirmed by estimating the model parameters as $\alpha_L = 0.0124$ for ticket packages and $\alpha_S = 0.0366$ for the individual tickets. This results in an optimal time of $u^* = 0$, which means that the symphony should sell only single tickets for the entire selling season. Of course, since the model does not fit the data perfectly, the symphony should still offer packages at the beginning of the season to satisfy its loyal patrons; but it seems
clear that single tickets should be offered earlier in the selling horizon.

### 3.6 Conclusions and Future Extensions

In this chapter we have characterized the optimal time for a sports franchise or entertainment venue to begin offering individual tickets when customers arrive according to two independent pure, linear death processes. We found that in many cases the firm should only make one offering—either bundles or season tickets—and, more surprisingly, that the optimal timing policy was independent of the initial inventory level. Comparative statics for the optimal decision coincided with our expectations for the variation of the timing decision with bundle size, revenue, and demand rate parameters. We extended the single-decision model to the scenario when the seller offers two types of ticket packages as well as individual tickets. We also extended the model to allow for two independent sets of tickets with unique demand characteristics. Examination of ticket sales data from Georgia Tech’s 2003 football season and the Atlanta Symphony Orchestra’s 2005-2006 concert season supported the assumption of the pure, linear Markovian death process that demand is a decreasing function of the remaining inventory level. We estimated the model’s arrival parameters from the data set and found that the team could have benefited from selling season tickets for a few weeks longer.

This research is an important step in the application of management science techniques to the sports and entertainment ticket industries. It characterizes the optimal policy for the timing of mixed-bundle offerings. It is also an important extension of revenue management methods by incorporating bundling and timing decisions instead of the usual pricing modifications. We hope that other researchers begin to investigate the vast array of interesting problems that affect the businesses of sports and entertainment.

There are several paths in which to build on the work contained herein. Even though many organizations follow a static timing policy, others do not; thus, it would be interesting to model the dynamic timing decision environment for offering bundles versus individual tickets. Another interesting extension to the general model is one that allows for sales of both bundles and single tickets once individual tickets go on sale. If multiple offerings
occur concurrently, the models should account for customer substitution between different package offerings. One team with which we spoke claimed that its ticket package and individual ticket demands were independent; thus, they offer all of their packages and single tickets over the entire selling horizon. Many other teams, though, stagger their offerings and acknowledge the existence of customer substitution between packages. While the complexity of these models will likely limit the potential for closed-form analytical results, additional methods are needed to provide insight into the effect of customer substitution on the optimal policy.

**Acknowledgments**

This research was supported by NSF grant DMI-0348532. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.
CHAPTER IV

MECHANISMS TO IMPROVE DECENTRALIZED CATEGORY MANAGEMENT WITH VENDOR-SPECIFIC PRODUCT PORTFOLIOS

4.1 Introduction

Faced with ever-shrinking margins, increased competition from brick-and-mortar stores and online e-tailers, and increasing demands from customers, most retailers are engaged in a constant search for ways to reduce their costs while at the same time increasing their level of customer service. This journey has led many retailers, especially grocers, to the potential savior known as category management. Although the term has become an umbrella moniker that can include many different practices and goals, all definitions of category management emphasize the joint management and planning for an entire category of products (e.g., soft drinks, laundry detergent, or socks) instead of individual stock-keeping units (SKUs) with the goal of improving the availability, pricing, product assortment, and timing of promotions with respect to customer demand (Dupre and Gruen, 2004). A report by Cannondale Associates estimates that retailers have experienced a 14-percent growth in sales and manufacturers a corresponding 8-percent increase in sales as a result of category management practices (Raskin, 2003).

Category management efforts may differ with respect to which firm is responsible for performing the analysis and making the recommendations. Some retailers such as Giant Eagle (Tortola, 2004), The County Grocer (Felix, 2006), and Spartan Stores (Garry, 2005) employ their own personnel to manage their product categories. Other retailers, though, have allocated category management responsibilities to their vendors. The main rationale for allowing the vendors to make the category recommendations is that individual retailers cannot be experts in all product segments. According to Pat Garvey, manager of retail
floor space for apparel supplier VF, “No matter how expert [retailers] are in their area, they can’t be an expert on every single brand a lot of the time” (Ryan, 2004). The retailer also saves the cost and effort involved in managing a category, which can be considerable, by allocating this responsibility to her vendors. As a former category manager put it, “As a retailer, you get the best minds in the business working for you free of charge. Why not take advantage of that?” (Raskin, 2003).

One form of vendor-controlled category management is category captainship, in which one vendor in a category is designated as the “captain” who is responsible for making the category decisions for his products as well as any competitors’ products. There are many examples in practice of this type of arrangement working well, but it is obvious that there is an inherent potential in this kind of structure for the captain vendor to recommend his products to the detriment of his major competing vendor. Category captainship arrangements have also come under scrutiny as a potential violation of antitrust legislation. The Federal Trade Commission (2001) has identified three types of antitrust violation that can be facilitated by category management: (1) obtaining confidential information about competitors’ plans, (2) competitive exclusion of other vendors’ products, (3) fostering collusion between retailers or vendors. Category captainship arrangements typically violate antitrust law if they restrict competition or reduce consumer utility (Desrochers et al., 2003). Several high-profile court decisions in the smokeless tobacco, cigarette, and tortilla markets have deemed particular category captain programs in violation of antitrust statutes; there are cases currently pending involving the sale of cranberries and soft drinks as well (Desrochers et al., 2003; Raskin, 2003). The governments of the United States and Canada have each issued reports that identify category captainship as a business practice of anticompetitive concern, and Israel has required governmental approval for category management initiatives undertaken by its three largest retailers (Desrochers et al., 2003).

One way to mitigate antitrust concerns is to allow each vendor to make decisions about its own products, a practice that some retailers have used with great success. Therefore, we analyze an alternative form of category management called vendor-specific category management (VSCM), in which each vendor has the responsibility of managing the stocking and
assortment decisions for its own shelf space provided by the retailer. VSCM-type arrange-
ments have been used successfully by apparel companies such as VF and Gold Toe (Ryan,
2004; DesMarteau, 2004). The retailer is still able to avoid the cost and effort required
by category management, and she benefits from having the vendors, who are the experts
in their particular product category, make the shelf-space management recommendations.
The potential agency problem of having a captain vendor make recommendations about
other vendors’ products is eliminated; thus, the anticompetitive concerns about the prac-
tice should be reduced, if not removed completely. It is not clear, however, what effects
these types of arrangements have on profits and consumer utility in competitive markets.

The major contribution of our study is the analysis of a decentralized category man-
agement mechanism in which each vendor controls his own space and stocking levels at
the retailer while the retailer sets the retail prices. We characterize the subgame-perfect
Nash Equilibrium decisions made by each party in a two-stage supply chain game with two
vendors and a single retailer. Our focus is on analyzing these decisions to capture insights
about when this decentralized approach is beneficial and what strategies are effective in
improving the channel’s performance. When the parties’ preferences as to which products
to stock under circumstances of extremely limited shelf space are aligned, VSCM naturally
performs close to the optimal retailer-controlled channel; however, when the incentives are
misaligned, the retailer can be considerably worse off (profit loss as high as 40%) than if
he retained the assortment decision. We compare the VSCM mechanism with a traditional
retailer-controlled channel as well as a centralized supply chain. We show that a minimum-
profit constraint can achieve the performance of a retailer-controlled channel, but it can be
costly to set up and may not achieve full channel coordination. In response, we demon-
strate that a revenue-sharing contract can be effective in coordinating the supply chain in
many cases. We add to the current research on revenue-sharing contracts by showing that
revenue sharing coordinates the channel when shelf space is ample and when vendors can
offer multiple products under deterministic demand.
4.2 Literature Review

Category management has its origins in the study of methods for shelf-space allocation. Corstjens and Doyle (1981) develop a model where the main driver of the assortment decision is the relationship between the amount of shelf space assigned to each product and the store’s profitability by incorporating direct shelf-space elasticities and cross-product space elasticities. More recent studies (e.g., van Ryzin and Mahajan (1999) and Cachon et al. (2002)) have utilized a multinomial logit (MNL) model of consumer choice in determining a retailer’s optimal assortment strategy. Chong et al. (2001) use an MNL consumer choice model and develop three brand-width measures that incorporate the homogeneity and heterogeneity of products within a manufacturer’s product portfolio and across manufacturers into the assortment decision. Hall et al. (2003) study a fixed-horizon dynamic pricing and ordering model of the retailer’s category management decision. Category-managing decision makers often require forecasts of multiple data sets including substitution effects; in response to the difficulty of producing these forecasts, Curry et al. (1995) show how Bayesian Vector Autoregressive models can be used effectively in these environments, and Jiang et al. (2005) extend this model to incorporate nonstationary time-series elements, thereby improving the category management mechanics. In each of the studies above, the retailer retains the assortment decision, and none of them consider the supply chain implications for the vendors of using these models for shelf-space management.

Another relevant group of studies model the supply chain effect of centralized brand management at the retailer. Zenor (1994) develops a model in which vendors can choose to create a coalition with their products when they offer wholesale prices to retailers with no space considerations, and Martín-Herrán et al. (2006) consider the effect of the vendors’ wholesale prices on the retailer’s resulting shelf-space allocation. Basuroy et al. (2001) develop a competitive market with two vendors each producing a single product and two retailers. They examine the equilibria characteristics that arise when the retailers practice traditional, individual brand management or centralized category management. Moorthy (2005) determines the effect of retailer competition and individual-brand management versus category management strategies at two competing retailers on how different cost changes.
are allocated through the channel. In all of these studies, the assortment decision still rests with the retailer; whereas, these responsibilities are delegated to the vendors in the VSCM model we study.

The papers that are most related to our study analyze scenarios of category captainship, in which the retail assortment decision rests with one of the vendors. In all of these models, the retailer adopts the captain’s recommendation in all circumstances; in practice, however, some retailers do not necessarily implement all of the captain’s suggestions (Raskin, 2003). Kaipia and Tanskanen (2003) present a case study in which the shelf-space assortment is kept close to optimal when the vendor manages the store shelves. Kurtuluş and Toktay (2005) consider category captainship with two vendors each producing a single product and one retailer. One of the vendors is chosen as the captain and is responsible for managing a given amount of retailer shelf space subject to a minimum profit constraint dictated by the retailer. They find that in some situations the non-captain vendor’s product is excluded from the assortment under category management when it would not be if the retailer retained the assortment decision. Kurtuluş and Toktay (2006) examine the effect that category captainship has on the variety experienced by consumers and compare the effect of three types of retailer targets (profit, sales, and variety). Wang et al. (2003) focus mainly on the appropriate choice of a category captain vendor who will control the pricing and allocation decisions for the entire category. Our VSCM model allows each vendor to manage his own retailer-allocated shelf space (instead of a single captain vendor making decisions for the entire category) while the retailer retains price control, and we develop methods for coordinating the supply chain.

Since the retailer transfers control of its shelf space to its vendors, our model can be thought of as an extension of traditional vendor-managed inventory (VMI) methods. VMI is a well-studied technique for reducing the bullwhip effect in serial supply chain in which the retailer gives control over his stocking levels to the vendors to mitigate the effects of order batching for the vendors, to reduce his inventory control costs, and to allow the vendor to coordinate production and distribution across multiple retailers. See Dong and Xu (2002), Mishra and Raghunathan (2004), Choi et al. (2004), and Bernstein et al. (2007)
for examples of the latest research on VMI methods. Our VSCM model takes traditional VMI one step farther by allowing the vendors to select the product assortment at the retailer in addition to the stocking level.

Many methods exist for achieving the centralized supply chain profit in a decentralized channel by eliminating the double-marginalization effect of individual decision makers. Wang and Gerchak (2001) present a mechanism that coordinates by using an inventory subsidy payment for a market in which when customer demand is dependent on the shelf space allocated to each product. The revenue-sharing contract, however, is a well-studied mechanism for channel coordination. Perhaps the most famous industry application of revenue sharing is in the video rental industry (see Dana and Spier (2001) for details). Giannoccaro and Pontrandolfo (2002), Wang et al. (2004), Cachon and Lariviere (2005), and Bernstein and Federgruen (2005) all develop supply-chain-coordinating models utilizing revenue sharing with a single supplier and a single or multiple retailers. Luo and Çakanyildirim (2005) establish that a revenue-sharing contract can be Pareto-improving for both parties in a VMI system. In this chapter we show that when the vendors make decisions on product assortment and stocking levels rather than setting wholesale prices, revenue sharing can improve the performance of our category management channel in many cases. In all of the above revenue-sharing models, the manufacturer is assumed to produce only one product; we add to the existing literature on revenue sharing by showing that revenue-sharing contracts can also coordinate the supply chain when multiple vendors offer substitutable products in a market segment.

The remainder of the chapter is organized as follows. The next section develops the vendor-specific category management model and establishes the retailer-control and centralized benchmark models. Section 4.4 discusses two benchmark cases and compares the performance of the vendor-specific category management system with these benchmarks. Several methods of aligning the parties’ incentives are analyzed in Section 4.5, and Section 4.6 provides concluding remarks and suggestions for future research in this area.
4.3 Vendor-Specific Category Management (VSCM) Model

Let us consider a two-stage supply chain for a single product category consisting of two vendors and one retailer. There are many examples of markets in which firms produce many substitutable products within a single product category including toothpaste, laundry detergent, beer, and athletic shoes. Substitution is a key factor in stocking decisions within a category. To focus on providing insights for this issue, we consider a market in which Vendor 1 manufacturers two products for the category and Vendor 2 produces a third; all three products are substitutes for each other within the particular category. We expect some of our main insights to extend naturally to larger category markets with many vendors and products. We analyze a single-period system in order to isolate the decision drivers without confounding the results with the effects of multi-period inventory control. The system we analyze can be thought of as the steady state of a typical infinite-horizon inventory system.

The serial decision structure under VSCM is as follows. The retailer determines the shelf spaces, $S_1$ and $S_2$ (measured in units of product), she will make available to Vendor 1 and Vendor 2, respectively. Many retailers such as Dollar General, Foot Locker, and CVS operate small stores in which the shelf space allocated to each product category is limited; consequently, the effective use of this space is of paramount concern to the retailer and her vendors. In our model we assume that the vendor space-allocation decision has been made *a priori*, but it is also possible to consider this decision as an additional decision stage within our framework. This assumption is consistent with some current retail practices in which a retailer “sells” the space to the vendor either explicitly through slotting allowances or indirectly by providing some level of promotional effort to induce a certain level of sales (Klein and Wright, 2004).

Each vendor simultaneously determines the quantity of its good(s) to stock at the retailer subject to the allocated space. The retailer then simultaneously determines the prices at which to sell her vendor-prescribed inventory to maximize her profit. Some retailers such as Dollar General retain pricing control in their interactions with their vendors. The retailer maintaining the pricing decision can also be viewed as a method for validating the vendors’ stocking decisions. Even if the vendors make recommendations about the retail prices in
practice, the retailer will only adopt these prices if they confirm their appropriateness. By allowing the retailer to set the prices in our VSCM model, we explicitly include the fact that she must approve of whatever prices are ultimately charged. Note that this is an important difference in our model compared to models of category captainship in which the retailer adopts the captain’s pricing recommendations completely. In order to prevent the vendors from dumping unwanted inventory on the retailer, we make the following assumption which is consistent with the models of consignment-type VMI discussed in Bernstein et al. (2007).

**Assumption 1** The retailer only pays the vendor for units that she ultimately sells.

We will denote the two different products manufactured by Vendor 1 as Products 1 and 2. The product made by Vendor 2 will be known as Product 3. The wholesale prices, production costs, and retailer shelf space allocations are exogenous to our decision environment. In our model one could think of the wholesale price being determined by a fixed markup over cost or by a long-term contract; retail prices are often much more malleable in short-term, day-to-day operations in response to shifting consumer preferences. In addition, if the vendors set the wholesale prices as well as determine the shelf-space allocation, they could take advantage of the retailer and capture the entire supply chain profit; so under such circumstances, the retailer would be unlikely to cede full control of its shelf space to her vendors. Fixing the wholesale prices also allows us to study the vendors’ stocking preferences without the confounding influence of the wholesale pricing decision. The vendors select the $q_i$ amounts that they will stock at the retailer, and the retailer chooses the vector of prices, $\mathbf{p} = (p_1, p_2, p_3)$, that she will charge the end consumers. The notation is summarized in Table 8.

Following the tradition in the marketing and economics literature (see, e.g., Levitan and Shubik (1971), McGuire and Staelin (1983), and Kim and Staelin (1999)), we utilize linear demand functions with price substitution effects in our analysis. This type of demand function results from an underlying consumer utility maximization problem and captures the substitution characteristics of traditional downward-sloping demand; namely, the quantity demanded of each product is decreasing in its own price and increasing in the prices of the
Table 8: Summary of model notation

<table>
<thead>
<tr>
<th>w_i</th>
<th>Wholesale price (per unit) paid to the vendor producing product i</th>
</tr>
</thead>
<tbody>
<tr>
<td>c_i</td>
<td>Production cost (per unit) paid by the vendor producing product i</td>
</tr>
<tr>
<td>q_i</td>
<td>Number of units of product i supplied by its manufacturing vendor</td>
</tr>
<tr>
<td>p_i</td>
<td>Retail price (per unit) charged by the retailer for product i</td>
</tr>
<tr>
<td>S_j</td>
<td>Shelf space (measured in units) allocated by the retailer to vendor j, j = 1, 2</td>
</tr>
<tr>
<td>D_i(p)</td>
<td>Consumer demand for product i as a function of retail prices, p</td>
</tr>
</tbody>
</table>

other goods. See Kurtuluş and Toktay (2005) for a derivation of this demand function from the representative consumer utility model in Shubik and Levitan (1980). This deterministic demand structure allows us to isolate the effect of the supply chain incentives created by the vendor-specific category management strategy without having the results confounded with the effects of demand variability. The end-user demand functions for each of the three products are as follows.

\[
D_1(p) = a_1 - b_1p_1 + \beta_{12}p_2 + \beta_{13}p_3 \\
D_2(p) = a_2 - b_2p_2 + \beta_{21}p_1 + \beta_{23}p_3 \\
D_3(p) = a_3 - b_3p_3 + \beta_{31}p_1 + \beta_{32}p_2
\]

All of the parameters comprising the demand functions are assumed to be non-negative, which ensures that the goods are substitutes. The substitution effects result in complex equilibrium decisions; so for readability, we use compact notation for these decisions in discussion of the results and provide the full decisions in Appendix C. Complete information exists between all of the parties, so each player knows the others’ costs, space allocations, and the system of demand functions.

The following assumption will allow us to establish concavity of the retailer’s profit function. It requires that each good’s own-price effect is twice-as-large as any of the cross-price effects; price increases for competing products will cause some consumers to substitute other products, but each of the other product brands will only experience a fraction of the total substitution purchases because they are in competition with each other.

**Assumption 2** Each of the \( b_i \) terms is greater than the sum of product \( i \)’s cross-price
effects with one of the other two products. More succinctly, we have $b_i \geq \beta_{i,j} + \beta_{j,i}$ for all $i = 1, 2, 3$ and $j \neq i$.

The final assumption we make is that Vendor 1’s products have profit margins that are sufficiently “close.” We want to consider situations where Vendor 1 would choose to stock both products for medium levels of shelf space rather than stock the maximum amount of his high-margin product that he could sell and leave empty shelf space. If this assumption did not hold, then the reduction in sales of Product 1 required to offer Product 2 would outweigh the gain from selling Product 2.

**Assumption 3** Without loss of generality, Product 1 attains the highest unit profit margin for Vendor 1. The profit margins are such that $w_2 - c_2 \geq \rho(w_1 - c_1)$, where

$$
\rho = \frac{b_3(\beta_{12} + \beta_{21}) + \beta_{31}\beta_{23} + \beta_{13}\beta_{32}}{2(b_2b_3 - \beta_{23}\beta_{32})}.
$$

**4.3.1 Retailer’s Pricing Decision**

Since we have a two-stage dynamic game of perfect information, we will solve for the equilibrium strategies using backward induction. We begin, therefore, by examining the retailer’s optimal pricing best response given that each of the vendors has provided $q_i$ of their respective products. The retailer is faced with the following profit maximization problem.

$$
\max_{p_1, p_2, p_3} \sum_{i=1}^{3} (p_i - w_i)D_i(p)
$$

s.t. $\quad 0 \leq D_i(p) \leq q_i \quad i = 1, 2, 3$

$$
p_1, p_2, p_3 \geq 0
$$

**Lemma 16** The retailer’s objective function in her profit maximization problem is concave in the retail prices.\(^1\)

Since the objective function is concave, and we have constraints that are linear in the decision variables (retail prices), we can solve the retailer’s problem using the Karush-Kuhn-Tucker (KKT) conditions (c.f. Bazaraa et al. (1993: 151–55)). Because of the structure of

\(^1\)Since the demand functions are deterministic, we could equivalently solve for the retailer’s optimal selling quantities subject to the supply quantities received from each vendor.

\(^2\)The proofs of all lemmas and theorems are provided in Appendix C.
the demand functions induced by Assumption 2, the retailer would never choose to sell a negative quantity or offer a product at a negative price even if the non-negativity constraints were not present in her problem. This leaves us with four possible retailer scenarios related to the retailer’s available level of supply, which we discuss below.

1. **None of the products are constrained.** The retailer has enough units to supply her simultaneously-unconstrained profit-maximizing quantities. We solve for the prices that make the gradient of the profit function equal to zero to obtain the set of optimal prices, \( \hat{p}_i \) for \( i = 1, 2, 3 \), which is given in equations (73)–(75) in Appendix C. We will denote the retailer’s corresponding unconstrained profit-maximizing sales quantities as \( \hat{D}_i = D_i(\hat{p}_1, \hat{p}_2, \hat{p}_3) \) for \( i = 1, 2, 3 \).

2. **One of the products is constrained.** The retailer sells out of one product and has enough supply to sell her resulting desired amount of the other two products. For example, suppose that the retailer sells out of Product 2, which means that \( D_2(p) = q_2 \). Scenario 2 has two analogous scenarios in which Product 1 and Product 3 are the only ones sold out, respectively. The prices can be found by switching all of the “2” subscripts with either “1” or “3,” depending on which product is sold out. This enables us to solve for one of the prices as a function of the other prices and \( q_2 \). We can then equate the gradient of the Lagrangian with zero by solving for the other two prices and the KKT multiplier for the supply constraint for Product 2 to obtain \( p_i(q_2) \) for \( i = 1, 2, 3 \) given in (76)–(78) in Appendix C. The resulting customer demands, \( Q_i(q_2) \) for \( i = 1, 2, 3 \) are given in (79)–(81).

3. **Two of the products are constrained.** The retailer sells out of two products and has enough supply to sell her resulting desired amount of the third product. For example, suppose that the retailer sells out of Products 2 and 3; thus, \( D_2(p) = q_2 \) and \( D_3(p) = q_3 \). Scenario 3 also has two analogous scenarios in which each of the other two products replaces Product 1 as the unconstrained good. As in Scenario 2 the prices and sales quantities for these cases are found by replacing all “1” subscripts with “2” and “3,” respectively. We can solve these two equations for two of the retail prices and then
solve for the third price and the two KKT supply constraint multipliers that make
the gradient of the Lagrangian equal to zero to obtain $p_i(q_2, q_3)$ for $i = 1, 2, 3$ given in
(82)–(84) in Appendix C. The resulting customer demands are given in (85)–(87).

4. **All of the products are constrained.** The retailer sells out of all three products. The
optimal retail prices are obtained by solving the three $D_i(p) = q_i$ equations, which
yields (88)–(90) in Appendix C. The resulting customer demands for each product
are the $q_i$ values we used in determining the corresponding prices.

The notation we use to represent the unconstrained customer demands shows explicitly
that they are a function of the constrained supply of various products. For example, $Q_1(q_2)$
is meant to denote the unconstrained demand for Product 1 when exactly $q_2$ units of Product
2 will be sold.

In order to determine which of the four scenarios is valid for a particular vector of supply
quantities, $q = (q_1, q_2, q_3)$, one could look at the values of the KKT multipliers and see if
they are non-negative. The structure of the optimal multipliers in our problem, however, is
prohibitively complex for meaningful analysis. Fortunately, we can equivalently determine
which scenario is applicable by computing the optimal customer sales quantities in each of
the four scenarios and determining which set of quantities satisfies all of the constraints in
the retailer’s profit maximization problem. We will use this observation to establish the
constraints for the vendors’ decision problems discussed in the next section.

4.3.2 **Vendors’ Shelf-Space Stocking Decisions**

Now that we have characterized the retailer’s best response to any set of supply quantities,
we can incorporate this into the vendors’ shelf-space optimization problems. Note that since
each vendor has control over his own shelf space, we have an optimization problem for each
vendor that the two vendors solve simultaneously. The fact that Vendor 2 only produces
one good simplifies this simultaneous analysis so that we can provide insights about Vendor
1’s stocking incentives.

Each of the vendors seeks to maximize his profit subject to the shelf space that the
retailer has allocated him. His profit function changes, however, according to the pricing
actions taken by the retailer in response to the quantities of products he supplies. Consequently, there are several different vendor cases, each of which induces a separate retailer scenario from the set described in the previous section. We provide the vendors’ (simultaneous) profit maximization problems for each of these scenarios in Table 9.\textsuperscript{3}

The optimization problems in Table 9 differ in the expressions for the quantity of each product for which the vendors will receive revenue from the retailer as well as the shelf space constraints that determine the ranges of supply quantities in which this case will be applicable. For example, in Case IV the retailer is constrained in the supply of Product 2, and she is able to sell her corresponding unconstrained amounts of Products 1 and 3. (Note that this corresponds to Retailer Scenario 2 discussed in Section 4.3.1.) As a result, Vendor 1 receives revenue for all $q_2$ units of Product 2 that he supplies since the retailer will sell the entire amount, but he will only earn revenue on a maximum of $Q_1(q_2)$ units of Product 1. Vendor 1 prefers to stock Product 1 over Product 2, so he will always supply a quantity of Product 1 equal to $Q_1(q_2)$ in this case and we have adjusted his profit function to reflect this policy. Likewise, in this case Vendor 2 earns revenue on a maximum of $Q_3(q_2)$ units of Product 3, and his profit function reflects this. Each of the profit-maximization problems for the other vendor cases can be interpreted analogously where the demand quantities determine the constraints. Examination of the optimization problems in the six vendor cases yields the following observation.

**Observation 2** In any case where the vendor’s profit function contains the retailer’s sales quantity for his revenue and his supply amount ($q_i$) for his costs, the vendor should only supply the amount that the retailer will sell.

Observation 2 is an easily-seen result of Assumption 1. Each vendor’s revenue is based on what the retailer ultimately sells, so it does him no good to supply more units that the retailer would choose to sell. This is one of the built-in incentive alignment mechanisms in our vendor-specific category management model compared with the models of category

\textsuperscript{3}The optimization problems for Vendor 1 in Cases III and IV are written to reflect that Product 1 is the more profitable product for Vendor 1 to sell. Each of these cases has an analogous scenario (denoted III.B and IV.B) in which Product 2 has the higher margin. These problems can be written by switching all of the “1” subscripts in the original problems to “2” and vice versa.
Table 9: Vendors’ profit maximization problems in each vendor scenario

Case I: All products constrained

<table>
<thead>
<tr>
<th>Vendor 1</th>
<th>Vendor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{max}_{q_1,q_2} (w_1 - c_1)q_1 + (w_2 - c_2)q_2$</td>
<td>$\text{max}_{q_3} (w_3 - c_3)q_3$</td>
</tr>
<tr>
<td>s.t. $q_1 + q_2 \leq S_1$</td>
<td>s.t. $q_3 \leq S_2$</td>
</tr>
<tr>
<td>$q_1 \leq Q_1(q_2, q_3)$</td>
<td>$q_3 \leq Q_3(q_1, q_2)$</td>
</tr>
<tr>
<td>$q_2 \leq Q_2(q_1, q_3)$</td>
<td></td>
</tr>
</tbody>
</table>

Case II: Vendor 2 unconstrained

<table>
<thead>
<tr>
<th>Vendor 1</th>
<th>Vendor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{max}_{q_1,q_2} (w_1 - c_1)q_1 + (w_2 - c_2)q_2$</td>
<td>$\text{max}_{q_3} (w_3Q_3(q_1, q_2) - c_3q_3$</td>
</tr>
<tr>
<td>s.t. $q_1 + q_2 \leq S_1$</td>
<td>s.t. $q_3 \leq S_2$</td>
</tr>
<tr>
<td>$q_1 \leq Q_1(q_2)$</td>
<td>$q_3 \geq Q_3(q_1, q_2)$</td>
</tr>
<tr>
<td>$q_2 \leq Q_2(q_1)$</td>
<td></td>
</tr>
</tbody>
</table>

Case III: Product 1 unconstrained and Products 2 and 3 constrained

<table>
<thead>
<tr>
<th>Vendor 1</th>
<th>Vendor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{max}_{q_1,q_2} w_1Q_1(q_2, q_3) - c_1q_1 + (w_2 - c_2)q_2$</td>
<td>$\text{max}_{q_3} (w_3 - c_3)q_3$</td>
</tr>
<tr>
<td>s.t. $q_1 + q_2 \leq S_1$</td>
<td>s.t. $q_3 \leq S_2$</td>
</tr>
<tr>
<td>$q_1 \geq Q_1(q_2, q_3)$</td>
<td>$q_3 \leq Q_3(q_1, q_2)$</td>
</tr>
<tr>
<td>$q_2 \leq Q_2(q_1, q_3)$</td>
<td></td>
</tr>
</tbody>
</table>

Case IV: Products 1 and 3 unconstrained and Product 2 constrained

<table>
<thead>
<tr>
<th>Vendor 1</th>
<th>Vendor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{max}_{q_1,q_2} w_1Q_1(q_2) - c_1q_1 + (w_2 - c_2)q_2$</td>
<td>$\text{max}_{q_3} (w_3Q_3(q_2) - c_3q_3$</td>
</tr>
<tr>
<td>s.t. $q_1 + q_2 \leq S_1$</td>
<td>s.t. $q_3 \leq S_2$</td>
</tr>
<tr>
<td>$q_1 \geq Q_1(q_2)$</td>
<td>$q_3 \geq Q_3(q_2)$</td>
</tr>
<tr>
<td>$q_2 \leq D_2$</td>
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</table>

Case V: Products 1 and 2 unconstrained and Product 3 constrained

<table>
<thead>
<tr>
<th>Vendor 1</th>
<th>Vendor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{max}_{q_1,q_2} w_1Q_1(q_3) - c_1q_1 + w_2Q_2(q_3) - c_2q_2$</td>
<td>$\text{max}_{q_3} (w_3 - c_3)q_3$</td>
</tr>
<tr>
<td>s.t. $q_1 + q_2 \leq S_1$</td>
<td>s.t. $q_3 \leq S_2$</td>
</tr>
<tr>
<td>$q_1 \geq Q_1(q_3)$</td>
<td>$q_3 \leq D_3$</td>
</tr>
<tr>
<td>$q_2 \geq Q_2(q_3)$</td>
<td></td>
</tr>
</tbody>
</table>

Case VI: All products unconstrained

<table>
<thead>
<tr>
<th>Vendor 1</th>
<th>Vendor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{max}_{q_1,q_2} w_1\hat{D}_1 - c_1q_1 + w_2\hat{D}_2 - c_2q_2$</td>
<td>$\text{max}_{q_3} (w_3\hat{D}_3 - c_3q_3$</td>
</tr>
<tr>
<td>s.t. $q_1 + q_2 \leq S_1$</td>
<td>s.t. $q_3 \leq S_2$</td>
</tr>
<tr>
<td>$q_1 \geq \hat{D}_1$</td>
<td>$q_3 \geq \hat{D}_3$</td>
</tr>
<tr>
<td>$q_2 \geq \hat{D}_2$</td>
<td></td>
</tr>
</tbody>
</table>

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Observation 3. In all of the cases, the vendors’ profit functions are linear in the supply quantity decision variables; thus, the profits are strictly increasing or decreasing with the decision variables. Consequently, the equilibrium solution is determined by identifying the set of constraints that are tight.

Observation 3 suggests a methodology that we will use in determining the vendors’ equilibrium strategies. Since we know that at least one of the constraints will be tight, we can substitute the values into vendor 1’s profit function and determine from the sign of the derivative if he wants \( q_1 \) or \( q_2 \) to be large or small. Then we can solve explicitly for the shelf-space conditions that ensure that the vendors’ decision values are valid in the current scenario. The solution approach is presented in Appendix C, and the results, assuming Product 1 generates the higher margin for the vendor, are summarized in Table 10. (See Appendix C for the exact equations for stocking quantities and the definitions of any notation not discussed thus far.)

Observation 4. Since Vendor 1’s profit function is linear in his two supply quantities in all cases, he prefers to stock solely the highest margin product for as long as possible.

Observation 4 implies that stocking Product 1 is always preferable to stocking Product 2 for Vendor 1 as long as the retailer chooses to sell the quantity he supplies. Any shelf-space allocations that have a given proportion of Product 1 will always generate a higher profit for Vendor 1 than any allocation that includes fewer units of Product 1. This has major implications if the vendor is responsible for selecting the allocation.

The vendor cases in Table 9 exhibit a natural ordering that allows us to develop the spectrum of applicable cases depicted in Figure 11 as a function of the vendors’ shelf-space allocations. Consider the cases where Vendor 2’s shelf space is small and \( q_3^* = S_2 \). Since Vendor 1 is making the assortment decision, he wants to sell as many units of Product 1 (his
Table 10: Equilibrium decision sets and validity conditions for each vendor scenario

<table>
<thead>
<tr>
<th>Case</th>
<th>Decisions</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>[ q_1^* = S_1 ] &lt;br&gt; [ q_2^* = 0 ] &lt;br&gt; [ q_3^* = S_2 ]</td>
<td>[ S_1 \leq Q_1(q_2 = 0, q_3 = S_2) ] &lt;br&gt; [ S_2 \leq Q_3(q_1 = S_1, q_2 = 0) ]</td>
</tr>
<tr>
<td>II.</td>
<td>[ q_1^* = S_1 ] &lt;br&gt; [ q_2^* = 0 ] &lt;br&gt; [ q_3^* = Q_3(q_1 = S_1, q_2 = 0) ]</td>
<td>[ S_1 \leq Q_1(q_2 = 0) ] &lt;br&gt; [ S_2 \geq Q_3(q_1 = S_1, q_2 = 0) ]</td>
</tr>
<tr>
<td>III.</td>
<td>[ q_1^* = S_1 - q_2^{III} ] &lt;br&gt; [ q_2^* = q_2^{III} ] &lt;br&gt; [ q_3^* = S_2 ]</td>
<td>[ S_1 \geq Q_1(q_2 = 0, q_3 = S_2) ] &lt;br&gt; [ S_1 \leq Q_1(q_3 = S_2) + Q_2(q_3 = S_2) ] &lt;br&gt; [ S_2 \leq Q_3(q_1 = S_1 - q_2^{III}, q_2 = q_2^{III}) ]</td>
</tr>
<tr>
<td>IV.</td>
<td>[ q_1^* = S_1 - q_2^{IV} ] &lt;br&gt; [ q_2^* = q_2^{IV} ] &lt;br&gt; [ q_3^* = Q_3(q_2 = q_2^{IV}) ]</td>
<td>[ S_1 \geq Q_1(q_2 = 0) ] &lt;br&gt; [ S_1 \leq D_1 + D_2 ] &lt;br&gt; [ S_2 \geq Q_3(q_1 = S_1 - q_2^{IV}, q_2 = q_2^{IV}) ]</td>
</tr>
<tr>
<td>V.</td>
<td>[ q_1 = Q_1(q_3 = S_2) ] &lt;br&gt; [ q_2 = Q_2(q_3 = S_2) ] &lt;br&gt; [ q_3 = S_2 ]</td>
<td>[ S_1 \geq Q_1(q_3 = S_2) + Q_2(q_3 = S_2) ] &lt;br&gt; [ S_2 \leq D_3 ]</td>
</tr>
<tr>
<td>VI.</td>
<td>[ q_1 = D_1 ] &lt;br&gt; [ q_2 = D_2 ] &lt;br&gt; [ q_3 = D_3 ]</td>
<td>[ S_1 \geq D_1 + D_2 ] &lt;br&gt; [ S_2 \geq D_3 ]</td>
</tr>
</tbody>
</table>

highest-margin product) as he can. Consequently, he will fill his shelf space with Product 1 if space is limited (\[ S_1 \leq Q_1(q_2 = 0, q_3 = S_2) \]). This corresponds with Vendor Case I. Once his space exceeds this boundary, he can either start stocking some of Product 2 or can stock only Product 1 and leave some space empty. Assumption 3 ensures that the marginal profit gain from stocking some of Product 2 is larger than the marginal loss by selling fewer units of Product 1, so he will start stocking some of Product 2. This amount, though, will only be enough so that Product 1’s unconstrained sales amount exactly fills up the space (i.e., \[ q_2 + Q_1(q_2, q_3 = S_2) = S_1 \]); thus, the decision environment moves to Case III. When his space is very large, then Vendor 1 supplies the resulting unconstrained sales amount when \[ q_3 = S_2 \], which represents a move to Case V. When Vendor 2 has a large amount of space, Vendor 1 faces a similar spatial decision spectrum only with Vendor 2 supplying unconstrained quantities of Product 3. Now that we have characterized the equilibrium decisions, we can use them to analyze the performance of the system as a whole.
4.4 Vendor-Specific Category Management’s Performance Relative to Benchmark Systems

In this section we investigate the performance characteristics of the VSCM system compared to a traditional, retailer-controlled (RCM) channel as well as the efficient, centralized supply chain. In the RCM channel, the retailer determines both the stocking levels and the product assortment on her shelves. To ensure that we compare analogous systems, we assume that the retailer has already assigned $S_1$ units of space to Vendor 1’s products and $S_2$ units to Vendor 2’s single product.

4.4.1 Analysis of the Retailer-Controlled Channel

In a RCM channel, the retailer seeks to determine her profit-maximizing quantities of each product to purchase from the vendors subject to the shelf-space constraints. It is clear that the retailer will purchase only such quantities that she is able to sell since we are considering a market with deterministic demand. Consequently, the retail prices will be those from Retailer Scenario 4 (discussed in Section 4.3.1), where the retailer sells out of her supply of all of the products. Recall that these prices are denoted $p_i(q_1, q_2, q_3)$. We can
now write the retailer’s profit-maximization problem in a RCM channel as

\[
\max_{q_1, q_2, q_3} \sum_{i=1}^{3} (p_i(q_1, q_2, q_3) - w_i)q_i
\]

\[\text{s.t.} \quad q_1 + q_2 \leq S_1
\]

\[q_3 \leq S_2
\]

\[q_1, q_2, q_3 \geq 0.
\]

The retailer’s problem has the same decision structure as in the VSCM channel. That is, there are six decision regions corresponding the vendor cases depicted in Figure 11. Since the retailer is concerned with maximizing her profit (and not the vendors’), it is clear that the optimal decision values will be different from those in the VSCM system, which are still feasible in the retailer-controlled channel. This means that the retailer must be worse off by allowing the vendors to manage their own product assortments. The issue, though, is capturing the magnitude of the retailer’s loss. The retailer may still be willing to adopt a VSCM system if her profit loss is small enough to be offset by the gain from not having to exert the effort (and cost) of managing the category assortment and stocking levels. As we will show with several numerical examples, the degree of profit loss for the retailer is dependent on the natural alignment of each party’s incentives concerning the preferential product to stock when Vendor 1’s space is extremely limited as well as the initial allocation of shelf space to each vendor. Misaligned stocking incentives can make VSCM very costly for the retailer.

The optimal stocking strategy for the RCM channel also has a range of space values in which only one of Vendor 1’s products is stocked. Assuming that consumers value the option to purchase different brands, we can compare these single-product ranges under the VSCM and RCM schemes. In many consumer products’ industries, consumers possess strong brand allegiance to particular products. Kim et al. (2002) suggest that each brand be thought of as an imperfect substitute for the other brands in order to represent the fact that some households would suffer an extreme loss in utility from the exclusion of a particular brand. In addition, retailers may be the only source of particular household goods (e.g., toothpaste or laundry detergent) in economically-depressed areas. (See Lavin (2005) for a case study

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of the establishment of a Pathmark supermarket store in the Harlem section of New York City.) If these retailers do not stock particular brands, then the consumers who live in these areas will have a limited amount of brand choice, thus decreasing their utility.

The following theorem shows that the consumers can be worse off with respect to their product variety availability under VSCM than under RCM when the parties’ single-product stocking preferences are aligned.\(^4\) The main implication of the theorem is that the allocation of category management decisions can have an impact not only on profit, but also on customer service and consumer welfare.

**Theorem 11** *In cases where Vendor 2’s shelf space is small \(q^*_2 = S_2\) and when the retailer and Vendor 1 prefer to stock the same initial product offered by Vendor 1, the space level under retailer-controlled category management where both goods begin to be stocked does not exceed the corresponding space level under vendor-specific category management.*

The result in Theorem 11 that the retailer prefers to increase the product assortment at lower levels of space runs contrary to a result of the category captainship model presented in Kurtuluş and Toktay (2005). They find that the category captain makes decisions that can reduce the product variety available to the customers and that the retailer prefers to have more product homogeneity than the captain when she is deciding to add another product to the assortment. In our case both vendors make an assortment decision, and the retailer prefers more variety than the vendors since she chooses to include an additional product to the category assortment at a lower level of shelf space. The fact that consumers can be worse off under VSCM makes the design of incentive-coordinating mechanisms even more important to consider in instances where Vendor 1 prefers to stock only one product.

### 4.4.2 Performance of Vendor-Specific Category Management with Misaligned Incentives

In this section we provide a numerical example in which the single-product stocking preferences of Vendor 1 and the retailer are mismatched to compare the VSCM system to a

\(^4\)A similar result was evident in all the numerical examples we have seen when the parties’ preferences for single-product stocking are different, but we do not expect the result to hold in general.
Table 11: Parameter declarations for numerical example with misaligned incentives

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>30</td>
<td>$\beta_{12}$</td>
<td>0.9</td>
<td>$w_1$</td>
<td>4</td>
</tr>
<tr>
<td>$a_2$</td>
<td>15</td>
<td>$\beta_{13}$</td>
<td>0.8</td>
<td>$w_2$</td>
<td>2</td>
</tr>
<tr>
<td>$a_3$</td>
<td>20</td>
<td>$\beta_{21}$</td>
<td>0.85</td>
<td>$w_3$</td>
<td>3</td>
</tr>
<tr>
<td>$b_1$</td>
<td>5</td>
<td>$\beta_{23}$</td>
<td>0.75</td>
<td>$c_1$</td>
<td>2</td>
</tr>
<tr>
<td>$b_2$</td>
<td>3</td>
<td>$\beta_{31}$</td>
<td>0.7</td>
<td>$c_2$</td>
<td>1</td>
</tr>
<tr>
<td>$b_3$</td>
<td>4</td>
<td>$\beta_{32}$</td>
<td>0.95</td>
<td>$c_3$</td>
<td>1.25</td>
</tr>
</tbody>
</table>

RCM channel and a centralized supply chain. In particular, Vendor 1 would like to stock as much as Product 1 as possible in small shelf-space areas, while the retailer prefers stocking Product 2. This is obviously the case in which the retailer stands to lose the most from allowing the vendors to control her shelf space, and we find that indeed the retailer’s profit loss can be quite severe.

The model parameters utilized in this example are presented in Table 11. (The graph of the six vendor cases in Figure 11 was generated from this example.) It is clear that Vendor 1 prefers stocking Product 1, but the retailer’s single-product stocking preference is more difficult to establish. To determine which product the retailer prefers to stock first, one looks at the margin she receives at minimal stocking levels (where zero units of both products are stocked). For any value of $q_3$, the optimal retail prices are $p_1(q_1 = 0, q_2 = 0, q_3) = 9.2039 - 0.0608q_3$ and $p_2(q_1 = 0, q_2 = 0, q_3) = 9.8450 - 0.0876q_3$. The maximum amount of product 3 that the retailer can sell when neither of the other two products are stocked is $Q_3(q_1 = 0, q_2 = 0) = 10.5686$, and at this maximum value for $q_3$, the retail price of Product 2 is larger than Product 1. Coupled with the fact that Product 1 has a higher wholesale price, this confirms that the retailer prefers to stock Product 2 first. Since the production costs exhibit the same ordering as the wholesale prices, the centralized firm also prefers to stock Product 2 solely.

The retailer’s profit under VSCM as a function of the shelf space allocated to the two vendors is provided in Figure 12(a). The shelf-space boundaries at which the slope of the profit function changes drastically correspond to the space conditions at which the VSCM channel switches cases. For large values of $S_1$ and $S_2$, the profit curve plateaus, representing
Figure 12: (a) Retailer’s profit function under VSCM and (b) Efficiency of retailer’s profit under VSCM compared to a RCM channel

the fact that the retailer prefers to leave extra space at these high levels. Note also that the retailer’s profit is non-decreasing in each shelf-space value since the retailer should never be worse off when he has more space (because she could always choose not to sell the entire supply she receives as a result of retaining retail pricing power).

The retailer’s net benefit from adopting the VSCM channel structure comprises gains due to delegation of the cost and effort required to manage the category that are offset by the profit loss from accepting the vendors’ stocking and assortment decisions. Since the retailer would only initiate a VSCM structure if she received a positive net benefit, we examine the performance of the VSCM channel relative to the (presumed) status quo of RCM in Figure 12(b) to determine the potential magnitude of the profit loss. We see that the retailer can earn up to 40% less profit under the VSCM system. The profit loss is most severe when Vendor 2 has little shelf space and Vendor 1 has a “medium” level of space because Vendor 1 chooses to fill this relatively-large amount of space with Product 1 when the retailer would rather have more of Product 2. When Vendor 1’s space is small, the effect of the incentive misalignment is mitigated somewhat by the fact that Vendor 1 only has a little bit of space to stock in the first place. When Vendor 1 has no space or a
very large amount of space, the VSCM approaches or reaches full efficiency either because Vendor 1’s decision set reduces or the parties’ interests naturally align when space is not a constraining factor.

Figures 13(a) and 13(b) depict the vendors’ profits as a function of the two shelf-space values. As we would expect, each vendor’s profit is non-decreasing in his own shelf space and is non-increasing in the other vendor’s space. Each vendor’s profit function has its steepest slope when the other vendor has a small amount of space. This stems from the fact that the vendor can take advantage of the customers’ willingness to purchase substitutes if their preferred product is not available by increasing the availability of his product(s) when the other vendor has limited space at the retailer.

In Figure 12(b) we saw that the retailer can be significantly worse off under a VSCM channel. Some of this lost profit, though, is captured by the vendors through their ability to select product stocking levels and assortments. Consequently, the comparison of the total supply chain profit under VSCM and the optimal centralized profit in Figure 14 shows that the maximum efficiency loss in this example is approximately 20%, which is still quite significant. The two efficiency graphs illustrate that incentive realignment is especially

Figure 13: (a) Vendor 1 and (b) Vendor 2’s profit function under a VSCM system
4.4.3 Performance of Vendor-Specific Category Management with Naturally-Aligned Incentives

In this section we consider a numerical example in which the retailer and Vendor 1 both prefer the same single product in small shelf-space areas. To make the example as close as possible to the previous example, we use all of the same parameter values in Table 11 except that $a_1 = 60$ (instead of 30). This adjusts the retailer’s initial marginal profit for Product 1 to $p_1(q_1 = 0, q_2 = 0, q_3) - w_1 = 11.8989 - 0.0608q_3$ and to $p_2(q_1 = 0, q_2 = 0, q_3) - w_2 = 10.1730 - 0.0876q_3$ for Product 2. The margin for Product 1 is larger than that of Product 2 for any value of $q_3$, so the retailer (and the centralized channel) now prefers to stock Product 1 in areas of small shelf space. Since the rest of the parameters are the same as in the previous example, the vendor’s product preference is still for Product 1.

Our main interest is in the effect of this natural incentive alignment on the retailer’s efficiency with respect to a RCM channel (shown in Figure 15(a)) and the total supply chain performance.
Figure 15: (a) Efficiency of retailer’s profit under VSCM compared to RCM channel and (b) Total supply chain efficiency of a VSCM channel relative to the centralized supply chain performance.

chain efficiency relative to the centralized channel (provided in Figure 15(b)). We see that the retailer’s efficiency maintains the same graphical form, but her maximum profit loss is approximately 10%, compared with 40% in the previous case. Thus, formal incentive alignment is less important from the retailer’s perspective if the vendor chooses the same product to stock solely; the retailer be willing to accept such profit losses if she can avoid the administrative and analytical costs of category management. The total supply chain profit is close to optimal as well, with the maximum efficiency loss being less than 6%. Interestingly, the decentralized VSCM system behaves like a de facto centralized system when both vendors have small space allocations since Vendor 1 fills his entire shelf space with the same product that the centralized system would choose. In the example with mis-aligned stocking incentives (Figure 14), the VSCM system does not achieve total efficiency in these shelf spaces because Vendor 1 is stocking the “wrong” product from a supply chain perspective.
4.5 Methods of Incentive Coordination

As we saw in Section 4.4.2, the retailer can experience a significant loss in profit from allowing the vendors to control their shelf space compared with what she could earn from retaining that decision in a RCM channel. Consequently, the retailer would likely want to introduce either performance standards or an incentive-realignment mechanism to adjust the vendors’ decentralized performance to a level in accordance with her interests. Following our discussions with industry about a mechanism currently in place in one VSCM system, we first investigate the imposition of a minimum-performance threshold to control the vendors’ actions.

**Theorem 12** The retailer can induce Vendor 1 to stock her optimal RCM sales quantities in a VSCM channel by imposing a minimum-profit-per-unit-of-space or total-minimum-profit requirement.

At first glance, it seems as if this constraint does an adequate job in aligning the parties’ decisions. Technically, it works fine, but it is somewhat unsatisfying for practice because by dictating the profit that she must earn, the retailer is actually determining the stocking quantities that she will receive. She may be nominally allocating the assortment decision to the vendors, but the vendors do not really have control over the decision since they have to stock the retailer-controlled optimal quantities in order to satisfy the performance standard. One way to determine the profit requirements is to use the profit the retailer was achieving when she managed the stocking decisions, but this may be ineffective when new products are introduced to the category. The retailer is still incurring the costs required to calculate the appropriate profit requirements for the constraint, and she does not receive the benefit from the vendors’ industry expertise in making the category decisions. Further, this mechanism may not achieve centralized channel performance because the retailer’s pricing decision is based on product margins that do not correspond with those of the centralized channel. In light of these shortcomings of the minimum-performance-standard mechanisms, we investigate the adoption of a revenue-sharing contract structure in this environment to allow the vendors to retain responsibility for the stocking and assortment decisions both
nominally and practically.

**Theorem 13** A revenue-sharing contract in which the retailer shares a \((1 - \alpha)\) fraction of its revenue from each product to the product’s manufacturer and the vendors charge wholesale prices, \(w_i = \alpha c_i\) for \(i = 1, 2, 3\), results in a VSCM channel that attains the same profit as the optimal centralized supply chain when Vendor 1’s shelf space is extremely limited or suitably ample. When vendor shelf space is middling, revenue sharing may be Pareto improving (especially if Vendor 1’s single-product stocking incentives differ from the centralized channel’s preferences), but it is not guaranteed to be effective in all cases.

Theorem 13 establishes an extension of the classic revenue-sharing contract to guarantee coordination of a channel with multiple substitutable products and multiple vendors, albeit with deterministic demand. Unlike the minimum-performance requirement options above, a revenue-sharing mechanism does not subject the vendors to additional constraints. It adjusts the profit functions that they optimize in determining their stocking quantities. For a specific value of \(\alpha\), one of the vendors or the retailer can be worse off under VSCM than he would be under a traditional RCM channel since the \(\alpha\) parameter determines how to split the optimal supply chain profit. In order to develop a strictly Pareto-improving solution for all parties, a particular \(\alpha\) could be chosen or the party that benefits the most could share some of their gain with the adversely-affected party via a lump-sum side payment.

The revenue-sharing mechanism is not necessarily able to achieve channel coordination for medium-sized levels of shelf space. This deficiency stems from the fact that in this case the retailer will sell all of the units of each product that the Vendor 1 supplies. She cannot use her pricing decision as a credible threat to induce the vendor to provide the centralized quantities as she can when shelf space is ample because the capacity is limited enough that she is still willing to sell the non-coordinated amounts. When capacity is very limited, the centralized channel stocks one product in Vendor 1’s space, and Vendor 1 follows suit because he would prefer the vendor’s profit function under medium-sized shelf space, however, is sufficiently “close” to the centralized channel so that his stocking quantities will not vary too much from the optimal centralized quantities (as demonstrated
in the following theorem).

**Theorem 14** In the cases where the revenue-sharing contract fails to coordinate the VSCM channel and Vendor 2 provides \( q_3 \) units of Product 3, Vendor 1’s optimal supply quantities differ from the optimal centralized quantities by

\[
\pm q_3 \left( \frac{b_2 \beta_{31} + \beta_{21} \beta_{32}}{b_1 b_3 - \beta_{13} \beta_{31}} - \frac{b_1 \beta_{32} + \beta_{12} \beta_{31}}{2(b_3 \beta_{12} + \beta_{13} \beta_{32})} \right).
\]

(46)

Note that the difference in the optimal stocking levels in Theorem 14 is not very large for most parameter realizations satisfying Assumption 2. It is especially small if the products have similar substitution characteristics with each other and similar own-price elasticities. The difference is a result of the profit function of the retailer (and the centralized channel) including the price of Product 3, which is also affected by the stocking levels of Products 1 and 2. Vendor 1’s profit function does not have this extra term; thus, it does not appear in his optimal quantities. In these cases Vendor 2 may not supply the centralized quantity of Product 3, but he will provide the centralized channel’s best response to the stocking levels provided by Vendor 1 since his incentives are aligned with those of the centralized channel.

For illustration of Theorem 14, we apply a revenue-sharing mechanism to the example in Section 4.4.2 with misaligned single-product stocking incentives. Consider a scenario in which Vendor 1 has four units of shelf space and Vendor 2 has five units of space. In a VSCM channel without revenue sharing, Vendor 1 fills his entire shelf space with Product 1, yielding a total supply chain profit of 54.34. The optimal RCM decisions are \( q_{RCM} = (0.1768, 3.8234, 5) \), which yields a supply chain profit of 57.00 (44.07 for the retailer, 4.18 for Vendor 1, and 8.75 for Vendor 2). If a revenue-sharing contract with \( \alpha = 0.84 \) is employed, the optimal VSCM stocking quantities are \( q_{VSCM} = (1.0282, 2.9718, 5) \), which results in a supply chain profit of 57.53. The retailer now earns a profit of 48.33, Vendor 1 realizes 4.47, and Vendor 2 obtains 4.73. The retailer can provide a lump-sum payment to Vendor 2 in order to increase his profit above 8.75 while still retaining profit higher than 44.07; this results in a strictly-Pareto improving solution over RCM for all parties. Other values of \( \alpha \) will also achieve the same supply chain profit, but one of the parties may have to provide a lump-sum payment to achieve a strictly-Pareto improving solution. The maximal
centralized channel profit is 57.56 (obtained from optimal stocking quantities $q^{Central} = (1.3036, 2.6964, 5)$), which shows that revenue sharing attains performance very close to the centralized channel in this example. The revenue-sharing VSCM channel generates total supply chain profit within 0.05% of the centralized channel while earning 5.87% more supply chain profit than a VSCM channel with no incentive alignment mechanism.

The rationale used in the proof of Theorem 13 for environments with large shelf space says that the first vendor has an incentive to overstock each of his products relative to the quantities that the retailer prefers to sell when the other vendor supplies the centralized quantities. Consequently, the vendor’s best response is to stock the centralized quantities as well because he will not be paid for any units that the retailer fails to sell. Nothing in this rationale leads us to believe that an analogous argument would fail in a market with multiple vendors each producing multiple products. This leads us to conjecture that a revenue-sharing agreement similar to that discussed in Theorem 13 would also coordinate the behavior of a VSCM channel in markets with more vendors and products and no shelf-space considerations.

**Corollary 3** A revenue-sharing contract is effective in coordinating the supply chain when vendors choose the stocking levels for multiple substitutable products and retailer shelf space is not constrained.

### 4.6 Conclusions and Future Research

In this chapter we have analyzed the performance of a type of decentralized category management channel called vendor-specific category management (VSCM) that is currently being used in the apparel industry. The VSCM structure is different from models of category captainship that allocate the stocking and assortment decisions at the retailer to a single, “captain” vendor; under VSCM each vendor is responsible for maintaining the stocking and assortment of its own products in a designated area of shelf space at the retailer. Since each vendor controls his own space, the VSCM structure eliminates (or dampens) many of the risks of competitive exclusion and antitrust violations inherent under category captainship.
We apply backwards induction to a two-stage supply chain game to find the optimal retailer pricing decisions that define scenarios within which the vendor optimizes. When the wholesale prices and vendor shelf-space allocations are set in advance, we find that the vendor who manufactures two goods chooses to stock one product exclusively in small areas of shelf space and then gradually increases the stock of the second product as he gains more space at the retailer. In some cases the decentralized channel will stock fewer types of products under VSCM as compared to a retailer-controlled (RCM) system; thus, the consumers’ utilities suffer because they have fewer product options to choose from. The retailer’s profit under VSCM can be as much as 40% lower than her profit in RCM; although, this can be mitigated by requiring the vendors to satisfy a minimum-profit constraint. This profit constraint, however, can be costly to quantify and basically dictates the decisions that the vendor must make; and it may not be able to attain the centralized supply chain performance. In response to these shortcomings, we show that when shelf space is sufficiently small or large, a revenue-sharing arrangement is effective in coordinating the channel with respect to a centralized supply chain. Under medium levels of shelf space, revenue sharing may not fully coordinate the system, but the gap in the equilibrium decisions may be small, especially when the own-price effects and substitution parameters for the products are similar. Our findings also imply that revenue sharing is guaranteed to coordinate a general decentralized system in which the vendors choose stocking levels for multiple substitutable products with deterministic demand.

The fact that many retailers are utilizing vendor-controlled category management practices and the comparatively few academic studies of the effects of these arrangements on supply chain performance suggest that this area provides many opportunities for future research. Since the vendors are controlling the stocking decisions, they should be able to realize efficiencies from coordinating production and distribution across products and customers. These efficiencies could lead the vendors to adjust their stocking decisions in particular periods. In our model we assumed that the retailer’s initial shelf-space allocations for each vendor had been determined exogenously. It would be interesting to study this allocation decision in light of the VSCM model and how it could potentially change over time.
with all of the vendors’ assortment decisions. It is also important to investigate the use of category management arrangements in larger-scale systems with many vendors producing many substitutable products as well as the effects of information sharing or information asymmetry on the parties’ decisions.

An overwhelming consensus in the trade literature about category management is that a necessary requirement for successful implementation of category management is the existence of collaboration and trust between the vendor(s) and the retailer. Too often, though, the vendors and the retailer have disparate, if not conflicting, goals for their relationship (Orgel, 2004). According to Tom Fox, a category management consultant, “A big part of category management is collaboration, and I don’t think many retailers do that very well” (Russo, 2004). Further empirical research on facilitating category management relationships in the vein of Lindblom and Olkkonen’s (2006) study of the effect of vendors’ market power on their ability to make category decisions would help eliminate these practical impediments to the true implementation of aligned category management mechanisms that always have the end-customer’s interests at their core.

**Acknowledgments**

This research was supported by NSF grants DMI-0348532 and DMI-0245352. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.
CHAPTER V

CONCLUDING REMARKS

As businesses continue to collaborate with their supply chain partners in order to improve their competitive position with their rivals, it is important to understand the incentives faced by decision makers in a dynamic supply chain environment. Supply chain optimality is often difficult to achieve, partly because each entity typically makes decisions with his own interests in mind. Once the parties’ economic behaviors have been characterized, managers can design mechanisms to align each firm’s interests with those that are optimal for the system as a whole. These mechanisms allow the total channel to earn a higher profit than it could without them, and the system gains can be split among the individual members to create a strictly Pareto-improving environment.

In this dissertation we studied the economic incentives evident three distinct supply chain and revenue management decision environments. The first major chapter focused on a new form of supply chain contract, known as the percent deviation contract, that we applied to an environment in which the buyer places her order with the supplier only after determining exactly what her customer demand is for that particular selling period. The supplier has the option of producing or allocating inventory in advance of the buyer’s order at a reduced cost; in doing so, the supplier assumes all of the demand risk, and this limits the amount of capacity he is willing to establish in advance. By requiring the buyer to provide an initial order estimate ahead of her actual demand realization, the percent deviation contract eases the amount of risk borne by the supplier, which induce him to build extra capacity. This enables the entire channel to satisfy a larger proportion of the end-customer demand and increases the total supply chain profit. We illustrated the application of the percent deviation contract to obtain strictly-Pareto improving solutions with several numerical examples based on industry demand data. Although the percent deviation contract assumes a complex form, this complexity is justified because it is effective in coordinating a
capacitated supply chain under circumstances in which similar established mechanisms such as the quantity flexibility contract fail to achieve full channel coordination. With the library of work on supply chain coordination steadily increasing in size, more comparison studies (both analytical and behavioral) including various contracting mechanisms are required to establish the effects of using one type of contract over another in various business scenarios.

The second main chapter of the dissertation presented a model of revenue management for the sale of season tickets and individual-event tickets by sports teams and entertainment venues. Sports and entertainment organizations face many of the same issues as airlines and hotels, thus making this industry a strong candidate to benefit from revenue management practices; one major difference, though, is that consumers are more willing to purchase tickets for multiple sports or entertainment events in ticket packages at one time. These organizations universally value the sale of ticket packages over single-event tickets; consequently, they typically offer the packages first in the selling season before individual tickets go on sale. We characterized the optimal time at which the team or venue should start selling the individual tickets (or smaller ticket packages) in cases where the demand rate decreases as the level of remaining tickets falls due to the common practice of selling the best seats in a section first. We also showed that the timing decision is affected by the degree of product heterogeneity in each ticket package.

This area provides many opportunities for future study, as there is currently little research on the pricing and bundling of sports and entertainment tickets. It is natural to extend our results with models that include a dynamic timing decision or allow for ticket packages to be sold once the individual tickets go on sale. Any models that allow multiple types of tickets to be sold at the same time should include a provision for customer substitution between offerings. The composition of ticket packages (i.e., which events should be packaged together) is another important issue that requires attention. Many organizations are even allowing their customers to create their own packages; these organizations would benefit from knowing which, if any, restrictions on these package compositions would be most effective in generating revenue while at the same time possibly reserving capacity for the most popular events.
The final major chapter presented a model of a supply chain structure in which each vendor is responsible for making the product stocking and assortment decisions for its own products at the retailer’s facility. By delegating this responsibility to the vendors, the retailer saves the often significant administrative expense of managing a product category while tapping the expertise of the vendors who are the experts in the markets for their products. Since the parties make their decisions based on different unit profit margins due to the double marginalization inherent in decentralized supply chains, the retailer can experience a significant loss in profit (as high as 40%) under a vendor-controlled system if she has a single-product stocking preference that differs from the vendor. The customers can also be worse off since they will likely face a reduced product variety compared with their choices when the retailer makes the assortment decision. A revenue-sharing arrangement with a discounted wholesale price can fully coordinate this decentralized channel to perform as an efficient, centralized supply chain when vendor shelf space is sufficiently small or ample. Our results are significant because they extend the revenue-sharing framework to an environment in which coordination is achieved across multiple products being supplied by multiple vendors with no shelf-space considerations and they explicitly consider channel coordination within a structure of category management. It is important to extend these results in the future to a larger-scale system with multiple vendors producing many products in a particular product category to verify that similar results hold. Further research is also required on the behavioral characteristics that build the trust essential for managing category relationships in practice.
Proof of Lemma 1. We will define the three realizations of (2) as follows:

\[
\Pi_{I.A.I.}^S = w \left[ \int_0^{t_1+M} xf(x)dx + (t_1 + M) (1 - F(t_1 + M)) \right] + \\
p \int_0^{t_1+M} (t_1 + M - x) f(x)dx + v \int_0^{t_1} (t_1 - x)f(x)dx - c_1t_1 - \\
c_2 \int_{t_1}^{t_1+M} (x - t_1)f(x)dx + M (1 - F(t_1 + M)) \right] - \\
\alpha \int_{t_1+M}^{\infty} (x - t_1 - M)f(x)dx 
\]

(47)

\[
\Pi_{I.A.II.}^S = w \left[ \int_0^{t_1+M} xf(x)dx + (t_1 + M) (1 - F(t_1 + M)) \right] + \\
p \int_0^{(1-d)q_1} ((1-d)q_1 - x) f(x)dx + v \int_0^{t_1} (t_1 - x)f(x)dx - c_1t_1 - \\
c_2 \int_{t_1}^{t_1+M} (x - t_1)f(x)dx + M (1 - F(t_1 + M)) \right] - \\
\alpha \int_{t_1+M}^{\infty} (x - t_1 - M)f(x)dx 
\]

(48)

\[
\Pi_{I.A.III.}^S = w \left[ \int_0^{t_1+M} xf(x)dx + (t_1 + M) (1 - F(t_1 + M)) \right] + \\
p \int_0^{(1-d)q_1} (1-d)q_1 - x) f(x)dx + \\
p \left[ \int_{(1-d)q_1}^{t_1+M} (x - (1+d)q_1)f(x)dx + (t_1 + M - (1+d)q_1) (1 - F(t_1 + M)) \right] + \\
v \int_0^{t_1} (t_1 - x)f(x)dx - c_1t_1 - \alpha \int_{t_1+M}^{\infty} (x - t_1 - M)f(x)dx - \\
c_2 \int_{t_1}^{t_1+M} (x - t_1)f(x)dx + M (1 - F(t_1 + M)) \right). 
\]

(49)

The second derivative of (47) taken with respect to \( t_1 \) is negative for all values of \( t_1 \) if \( w + \alpha > p + c_2 \). The second derivatives of (48) and (49) are negative for all values of \( t_1 \) without the extra condition. ■

Proof of Lemma 2. Suppose, on the contrary, \( t_{I.A.I.}^{t_i} < t_{I.A.II.}^{t_i} \), which implies that
Figure 16: Supplier’s best response depiction for Scenario I.A.

\[ F(t_{1,A.I.}^I + M) \leq F(t_{1,A.II.}^I + M) \]. Substituting the values given in (3) and (4), we have

\[
\frac{w + \alpha - c_1 - (c_2 - v)F(t_{1,A.I.}^I)}{w + \alpha - c_2 - p} \leq \frac{w + \alpha - c_1 - (c_2 - v)F(t_{1,A.II.}^I)}{w + \alpha - c_2}
\]

\[
(c_2 - v)(w + \alpha - c_2) \left[ F(t_{1,A.II.}^I) - F(t_{1,A.I.}^I) \right] \leq -p \left( w + \alpha - c_1 - (c_2 - v)F(t_{1,A.II.}^I) \right).
\]

The left side of (50) is positive, and the right side is negative since the numerator in (4) must be positive at \( t_{1,A.II.}^I \) (The denominator is positive due to the parameter relationship defining case I.A.) This leads to a contradiction.

Proof of Theorem 1. There are five possible values for the supplier’s best response. Each of the three realizations of the supplier’s expected profit function has an individual maximizer, shown in (3), (4), and (5). In addition, the two points where the pieces of the profit function converge \( (t_1 = (1 - d)q_1 - M \) and \( t_1 = (1 + d)q_1 - M \) are possible solutions. These solutions would occur when the maximizing \( t_1 \) values do not lie in their corresponding feasible regions. In order to establish the result in Theorem 1, we must first make some preliminary observations about the expected profit function that will help us later in the main proof.

Observation 5 For all values of \( t_1 \) less than the lower boundary of the deviation range \( ((1 - d)q_1 - M) \), \( \Pi_{I,A.II.}^S(t_1) > \Pi_{I,A.I.}^S(t_1) \) because the term representing the expected value
of the lower deviation penalty paid is larger in \( \Pi_{I,A,II}^S \). For values of \( t_1 \) greater than the lower boundary, \( \Pi_{I,A,II}^S(t_1) < \Pi_{I,A,I}^S(t_1) \).

**Observation 6** For all values of \( t_1 \) less than the upper boundary of the deviation range \((1 + d)q_1 - M\), \( \Pi_{I,A,II}^S(t_1) > \Pi_{I,A,III}^S(t_1) \) because the term representing the expected value of the upper deviation penalty paid in \( \Pi_{I,A,III}^S \) is negative. For values of \( t_1 \) greater than the upper boundary, \( \Pi_{I,A,II}^S(t_1) < \Pi_{I,A,III}^S(t_1) \).

The supplier’s best response function depends on the values of the three maximizers relative to the feasible boundaries. There are 27 possible cases because each of the three decisions can potentially lie in three regions; however, the following results show that several of these cases are not possible.

**Lemma 17** *It is not possible to have* \( t_{1,A,I}^I < (1 - d)q_1 - M < t_{1,A,II}^I \).*

**Proof.** This result follows directly from Lemma 2.

**Lemma 18** *It is not possible to have* \( t_{1,A,III}^I < (1 + d)q_1 - M < t_{1,A,II}^I \).*

**Proof.** Assume that this relationship is true. Since the piecewise functions are concave from Lemma 1, \( t_{1,A,III}^I \) is the single maximum of \( \Pi_{I,A,III}^S \), and \( \Pi_{I,A,III}^S(t_1) \) is decreasing for values of \( t_1 > t_{1,A,III}^I \). Consequently, \( \Pi_{I,A,III}^S(t_{1,A,III}^I) > \Pi_{I,A,III}^S((1 + d)q_1 - M) > \Pi_{I,A,III}^S(t_{1,A,II}^I) \). Since \( \Pi_{I,A,III}^S((1 + d)q_1 - M) = \Pi_{I,A,III}^S((1 + d)q_1 - M) \) from Observation 1, we have \( \Pi_{I,A,III}^S(t_{1,A,III}^I) > \Pi_{I,A,II}^S((1 + d)q_1 - M) > \Pi_{I,A,III}^S(t_{1,A,II}^I) \). Observation 6 states that \( \Pi_{I,A,III}^S(t_{1,A,III}^I) > \Pi_{I,A,II}^S(t_{1,A,II}^I) \), which implies \( \Pi_{I,A,III}^S(t_{1,A,III}^I) > \Pi_{I,A,II}^S(t_{1,A,II}^I) \). The statement \( \Pi_{I,A,II}^S((1 + d)q_1 - M) > \Pi_{I,A,II}^S(t_{1,A,II}^I) \) contradicts the result that \( t_{1,A,II}^I \) maximizes \( \Pi_{I,A,II}^S \).

After using Lemmas 17 and 18 to reduce the number of possible cases, we can determine the overall best response for each given the values of the individual maximizers by applying Observations 1, 5, and 6. Summarizing all of the scenarios, we obtain the solution given in (6) and depicted in Figure 16.
The second derivative of (51) is zero, so this component function is concave. The second
each function over the region of
are concave from Lemma 3, we can use the KKT conditions to solve for the optimal
thus, this function is concave. Similarly, the second derivative of (53) is

\[ \Pi^B_{I.A.II.} = (r - w) \left[ \int_0^{t_1^*(q_1)} x f(x)dx + (t_1^*(q_1) + M) (1 - F(t_1^*(q_1))) \right] - \\
p \int_0^{t_1^*(q_1)} (t_1^*(q_1) + M - x) f(x)dx + \\
(\alpha - \beta) \int_{t_1^*(q_1)}^\infty (x - t_1^*(q_1) - M) f(x)dx \]  

(51)

The second derivative of (51) is zero, so this component function is concave. The second
derivative of (52) equals \(-p(1 - d)^2 f((1 - d)q_1)\), which is negative for all values of \(q_1\); thus, this function is concave. Similarly, the second derivative of (53) is \(-p(1 - d)^2 f((1 - d)q_1) - p(1 + d)^2 f((1 + d)q_1)\), which is negative for all \(q_1\). The second derivative of (54) is

\( -(r - w - \alpha + \beta)(1 - d)^2 f((1 - d)q_1) - p(1 - d)^2 f((1 - d)q_1) \), which is also negative for all \(q_1\), making this function concave as well.

Proof of Theorem 2. Since the buyer’s four individual profit function realizations
are concave from Lemma 3, we can use the KKT conditions to solve for the optimal \(q_1\) for
each function over the region of \(q_1\) values where the function is valid, as defined in 1.

We first maximize (51) over the region \(q_1 \geq \frac{t_1^{A.I.} + M}{1 - d}\). Since (51) is not dependent on

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\( q_1 \), any value \( q_1^{I.A.I.} \equiv \left\{ q : q \geq \frac{t_1^{I.A.I.} + M}{1-d} \right\} \) is optimal.

**Proof of Proposition 1.** We use the difference function to determine when the supplier chooses each \( t_1 \) for given values of \( q_1 \) in a range where his best response is known to be the maximizer of his expected profit from a set of two values. We then use this information to characterize the ranges of \( q_1 \) that induce each value of \( t_1 \). If the difference function is positive for a value of \( q_1 \), then the supplier will choose \( t_1^{I.A.I.} \); he will select the other possible decision if the function is negative. The number of ranges for \( q_1 \) is determined by the number of changes of sign in the difference function. The difference function related to the decision in (8) is given by

\[
\Delta \left( t_1^{I.A.I.} \right) = w \left[ \int_{t_1^{I.A.I.} + M}^{t_1^{I.A.I.} + M + M} x f(x)dx \right] + \\
w \left[ \left( t_1^{I.A.I.} + M \right) \left( 1 - F \left( t_1^{I.A.I.} + M \right) \right) - \left( t_1^{I.A.I.} + M \right) \left( 1 - F \left( t_1^{I.A.I.} + M \right) \right) \right] + \\
p \left[ f_0^{1-d} \left( (1-d)q_1 - x \right) f(x)dx - \int_{0}^{t_1^{I.A.I.} + M} \left( t_1^{I.A.I.} + M - x \right) f(x)dx \right] + \\
p \left[ f_0^{1-d} \left( x - (1+d)q_1 \right) f(x)dx \right] + \\
p \left[ \left( t_1^{I.A.I.} + M - (1+d)q_1 \right) \left( 1 - F \left( t_1^{I.A.I.} + M \right) \right) \right] + \\
\left[ t_1^{I.A.I.} - \int_{0}^{\infty} \left( t_1^{I.A.I.} - x \right) f(x)dx \right] - c_1 \left( t_1^{I.A.I.} - t_1^{I.A.I.} \right) - \\
\alpha \left[ \int_{0}^{\infty} \left( x - t_1^{I.A.I.} - M \right) f(x)dx - \int_{0}^{t_1^{I.A.I.} + M} \left( x - t_1^{I.A.I.} - M \right) f(x)dx \right] - \\
e_2 \left[ \int_{t_1^{I.A.I.} + M}^{\infty} \left( x - t_1^{I.A.I.} - M \right) f(x)dx - \int_{t_1^{I.A.I.} + M}^{t_1^{I.A.I.} + M} \left( x - t_1^{I.A.I.} \right) f(x)dx \right] + \\
e_2 \left[ M \left( F \left( t_1^{I.A.I.} + M \right) - F \left( t_1^{I.A.I.} + M \right) \right) \right].
\]

This difference function is convex in \( q_1 \) since \( \frac{\partial^2 \Delta}{\partial q_1^2} = p(1-d)^2 f((1-d)q_1) + p(1+d)^2 f((1+d)q_1) > 0 \). We can begin by evaluating the difference function at the two endpoints of the region defined in (8); that is, \( q_1 = \frac{t_1^{I.A.I.} + M}{1-d} \) and \( q_1 = \frac{t_1^{I.A.I.} + M}{1+d} \). If the difference function is positive for one value and negative for the other, then convexity implies that there exists a single threshold value of \( q_1 \) in the interval where the difference function changes sign. The buyer can use these supplier decision values to evaluate her best selection of \( q_1 \) in this region with respect to her expected profit function.

If the difference function is positive for both endpoint values of \( q_1 \), then it is possible that there are zero, one, or two points where the function switches sign. If there are zero or one switching points, then the supplier will choose \( t_1 = t_1^{I.A.I.} \) for all values of \( q_1 \) in the
region. If there are two switching points, then for values of \( q_1 \) between these two values, the supplier will choose \( t_1 = t_{11}^{A.I} \). He will choose \( t_1 = t_{11}^{A.III} \) for all other values of \( q_1 \).

If the difference function is negative for both endpoint values, then convexity implies that it will be negative for all values of \( q_1 \); thus, the supplier will always choose \( t_1 = t_{11}^{A.I} \).

The difference function related to the decision in (10) is given by

\[
\begin{align*}
\Delta \left( t_{11}^{A.III} \right) &= w \left[ \int_{t_{11}^{A.III} + M}^{t_{11}^{A.III} + M} x f(x) dx \right] + \\
&\quad w \left[ \left( t_{11}^{A.III} + M \right) \left( 1 - F \left( t_{11}^{A.III} + M \right) \right) - \left( t_{11}^{A.III} + M \right) \left( 1 - F \left( t_{11}^{A.III} + M \right) \right) \right] + \\
&\quad p \left[ \int_{(1+d)q_1}^{t_{11}^{A.III} + M} (x - (1 + d)q_1) f(x) dx \right] + \\
&\quad p \left[ \left( t_{11}^{A.III} + M - (1 + d)q_1 \right) \left( 1 - F \left( t_{11}^{A.III} + M \right) \right) \right] + \\
&\quad v \left[ \int_{t_{11}^{A.III} - q_1}^{t_{11}^{A.III} - q_1} (1 + t_{11}^{A.III} - x) f(x) dx - \int_{t_{11}^{A.III} - q_1}^{t_{11}^{A.III} - q_1} (1 - t_{11}^{A.III} - x) f(x) dx \right] - \\
&\quad c_1 \left( t_{11}^{A.III} - t_{11}^{A.III} \right) - \\
&\quad \alpha \left[ \int_{t_{11}^{A.III} + M}^{t_{11}^{A.III} + M} \left( x - t_{11}^{A.III} + M \right) f(x) dx - \int_{t_{11}^{A.III} + M}^{t_{11}^{A.III} + M} \left( x - t_{11}^{A.III} - M \right) f(x) dx \right] - \\
&\quad c_2 \left[ \int_{t_{11}^{A.III} + M}^{t_{11}^{A.III} + M} \left( x - t_{11}^{A.III} + M \right) f(x) dx - \int_{t_{11}^{A.III} + M}^{t_{11}^{A.III} + M} \left( x - t_{11}^{A.III} - M \right) f(x) dx \right] + \\
&\quad c_2 \left[ (F \left( t_{11}^{A.III} + M \right) - F \left( t_{11}^{A.III} + M \right) \right) \right].
\end{align*}
\]

This difference function is convex in \( q_1 \) since \( \frac{\partial^2 \Delta}{\partial q_1^2} = p(1 + d)^2 f((1 + d)q_1) > 0 \), and it is also decreasing in \( q_1 \) because \( \frac{\partial \Delta}{\partial q_1} = -p(1 + d)(1 - F((1 + d)q_1)) < 0 \). This means that if the difference function is negative when \( q_1 = \frac{t_{11}^{A.III} + M}{1 + d} \), which is the lower limit of the range defined in (10), then the supplier will always choose \( t_1 = t_{11}^{A.III} \). Likewise, if the difference function is positive at the upper endpoint of the range \( \left( q_1 = \min \left\{ \frac{t_{11}^{A.III} + M}{1 - d}, \frac{t_{11}^{A.III} + M}{1 + d} \right\} \right) \), then the supplier will always select \( t_1 = t_{11}^{A.III} \). If the difference function is positive for the lower endpoint and negative for the upper endpoint, then there exists exactly one point where the difference function changes sign, and we have two distinct ranges of \( q_1 \) values where the two \( t_1 \) decisions are chosen.
The difference function related to the decision in (9) is given by

\[ \Delta ((1 - d)q_1 - M) = w \left[ \int_{t^{I,A.III.} \cdot (1 - d)q_1}^{t^{I,A.III.} + M} x f(x) dx \right] + 
\]
\[ w \left[ \left( t^{I,A.III.} + M \right) (1 - F \left( t^{I,A.III.} + M \right)) - (1 - d)q_1 (1 - F ((1 - d)q_1)) \right] + 
\]
\[ p \left[ f_{(1 + d)q_1}^{t^{I,A.III.} + M} (x - (1 + d)q_1) f(x) dx \right] + 
\]
\[ p \left[ \left( t^{I,A.III.} + M - (1 + d)q_1 \right) \left( 1 - F \left( t^{I,A.III.} + M \right) \right) \right] + 
\]
\[ v \left[ \int_{0}^{t^{I,A.III.}} (f(x) dx - \int_{0}^{(1 - d)q_1} M ((1 - d)q_1 - M - x) f(x) dx \right] - 
\]
\[ c_1 \left( t^{I,A.III.} - (1 - d)q_1 + M \right) - 
\]
\[ \alpha \left[ \int_{t^{I,A.III.} - M}^{\infty} x f(x) dx - \int_{(1 - d)q_1}^{\infty} (x - (1 - d)q_1) f(x) dx \right] - 
\]
\[ c_2 \left[ \int_{t^{I,A.III.} + M}^{\infty} F((1 - d)q_1) f(x) dx \right] + 
\]
\[ c_2 \left[ M \left( F((1 - d)q_1) - F \left( t^{I,A.III.} + M \right) \right) \right]. \]

Here one of the potential supplier decisions is an explicit function of the buyer’s \( q_1 \) decision, so the difference function is more complex. Specifically, the function is not necessarily convex or concave. For a given set of parameters, then the exact switching points can be determined by simple numerical search methods. In many realizations the difference function will be well-behaved; thus, a similar analysis to that performed for the previous two cases above would suffice for these situations.

Proof of Theorem 3. The proof of this result follows the same logic as that of Theorem 1, utilizing the results from Lemma 19, which is stated below.

**Lemma 19** If \( w + \alpha > p + v \), then \( t^{I,B.II.} \leq \min \left\{ t^{I,B.I.}, t^{I,B.III.} \right\} \).

**Proof.** We will follow the same contradiction procedure as in the proof of Lemma 2. First, assume that \( t^{I,B.II.} > t^{I,B.I.} \), which implies that \( F \left( t^{I,B.II.} \right) \geq F \left( t^{I,B.I.} \right) \). Further substitution yields

\[ \frac{w + \alpha - c_1}{w + \alpha - v} \geq \frac{w + \alpha - c_1}{w + \alpha - v - p} \]
\[ -p(w + \alpha - c_1) \geq 0, \]

which leads to a contradiction since \( w + \alpha - c_1 \geq 0 \).
Similarly, we assume that $t_{1}^{I.B.II.} > t_{1}^{I.B.III.}$; consequently, $F(t_{1}^{I.B.II.}) \geq F(t_{1}^{I.B.III.})$.

We obtain the following contradiction through substitution.

\[
\frac{w + \alpha - c_{1}}{w + \alpha - v} \geq \frac{w + \alpha - c_{1} + p}{w + \alpha - v + p} \Rightarrow v \geq c_{1},
\]

which is a contradiction since $c_{1} > v$. ■

**Lemma 20** $t_{1}^{II.A.II.} \geq t_{1}^{II.A.III.}$.

**Proof.** We will establish this result by contradiction, so we assume that $t_{1}^{II.A.II.} < t_{1}^{II.A.III.}$.

This implies that $F(t_{1}^{II.A.II.}) \leq F(t_{1}^{II.A.III.}) = \frac{c_{2} - c_{1}}{c_{2} - v}$. Using this result, we have

\[
F(t_{1}^{II.A.II.} + M) = \frac{w + \alpha - c_{1} - (c_{2} - v)F(t_{1}^{II.A.II.})}{w + \alpha - c_{2}} \\
\geq \frac{w + \alpha - c_{1} - (c_{2} - v)F(t_{1}^{II.A.III.})}{w + \alpha - c_{2}} \\
= \frac{w + \alpha - c_{1} - (c_{2} - v)\left(\frac{c_{2} - c_{1}}{c_{2} - v}\right)}{w + \alpha - c_{2}} = 1.
\]

The only way that this can hold is if the solution for (4) is not unique. Further, the inequality above is strict if $F(\cdot)$ is strictly increasing; in this case the relationship can never hold. Thus, we have a contradiction. ■

**Proof of Theorem 4.** The feasibility conditions for each of the possible $t_{1}$ values follow from Lemmas 2 and 20 as well as Observations 1, 5, and 7. (The latter of these is stated below).

**Observation 7** For all values of $t_{1}$ less than the upper boundary of the deviation range $((1 + d)q_{1} - M, \Pi_{II.A.III.}(t_{1}) > \Pi_{II.A.II.}(t_{1})$ because the difference, $\Delta$, between the two functions is

\[
\Delta = w \left[ \int_{t_{1} + M}^{(1 + d)q_{1}} x f(x) dx + (1 + d)q_{1}(1 - F((1 + d)q_{1}) - (t_{1} + M)(1 - F(t_{1} + M)) \right] - \\
c_{2} \left[ \int_{t_{1} + M}^{(1 + d)q_{1}} (x - t_{1}) f(x) dx \right] + \\
c_{2} \left[ ((1 + d)q_{1} - t_{1})(1 - F((1 + d)q_{1}) - M(1 - F(t_{1} + M)) \right] + \\
\alpha \left[ \int_{t_{1} + M}^{(1 + d)q_{1}} (x - t_{1} - M) f(x) dx + ((1 + d)q_{1} - t_{1} - M)(1 - F((1 + d)q_{1})) \right],
\]
which is greater than zero because \( w + \alpha > c_2 \). For values of \( t_1 \) greater than the upper boundary, \( \Pi_{II.A.III.}^{S}(t_1) < \Pi_{II.A.II.}^{S}(t_1) \).

Note that this result is the reverse of that established in Observation 6 for case I.A. This is due to the fact that the buyer’s final order will not exceed the upper limit of the deviation range.

From Lemmas 2 and 20 we establish that \( t_{II.A.I.}^{I} \geq t_{II.A.II.}^{I} \geq t_{II.A.III.}^{I} \). Using this relationship between the decisions and the relationships between the expected profit realizations defined in the observations, we obtain the feasible ranges and corresponding optimal \( t_1(q_1) \) values in (16).

**Proof of Theorem 5.** This results by applying the same procedure used in the proof of Theorem 2.

**Proof of Lemma 4.** We are given that \( p > r - w + \beta \) because we are in case III.B. Suppose that \( q_{II.A.I}^{I} \leq q_{II.A.II.}^{I} \), which means that \( p(1-d)F \left( (1-d)q_{II.A.I}^{I} \right) \leq p(1-d)F \left( (1-d)q_{II.A.II.}^{I} \right) \). From the definitions of \( q_{II.A.I}^{I} \) and \( q_{II.A.II.}^{I} \), this implies that \( p(1+d) \left( 1 - F \left( (1+d)q_{II.A.I}^{I} \right) \right) \leq (r-w+\beta)(1+d) \left( 1 - F \left( (1+d)q_{II.A.II.}^{I} \right) \right) \). We know, though, that \( p > r - w + \beta \) and \( 1 - F \left( (1+d)q_{II.A.I}^{I} \right) \geq 1 - F \left( (1+d)q_{II.A.II.}^{I} \right) \), since \( q_{II.A.I}^{I} \leq q_{II.A.II.}^{I} \). Thus, the above inequality cannot hold, and we have a contradiction.

**Proof of Lemma 5.** This case can easily be compared with the centralized case C.II. in which the centralized firm also does not expedite. The total expected supply chain profit for the voluntary compliance case is

\[
\Pi_{SC.IB.} = r \left[ \int_{0}^{t_1} xf(x)dx + t_1(1 - F(t_1)) \right] + v \int_{0}^{t_1} (t_1 - x)f(x)dx - c_1t_1 - \beta \int_{t_1}^{\infty} (x-t_1)f(x)dx.
\]  

Comparing (55) with the centralized supply chain profit in (25), it is easily seen that the two profits will be equal if the \( t_1 \) decisions are equal, which is accomplished if \( \frac{w + \alpha - c_1 + p}{w + \alpha - v + p} = \frac{r + \beta - c_1}{r + \beta - v} \). Simplifying this equality yields the channel coordinating condition.

**Proof of Theorem 6.** If \( u < v \), then clearly \( \Pi_{SC.QF}^{SC} \leq \Pi_{SC.C.II.}^{SC} \) for every value of \( t_1 \), and there exist some values of \( t_1 \) where the inequality is strict. Consequently, coordination is
not possible in these cases because leftover goods are less valuable in the buyer’s possession, which is where they reside under quantity flexibility.

Now let \( u = v \). If \( t_1^{QF} = t_1^{C.II} \), then \( \Pi^{SC}_{QF} = \Pi^{SC}_{C.II} \), and we would have a coordinated supply chain. Thus, we want to have \( \frac{r-w+\beta-\alpha}{r-v+\beta-\alpha} = \frac{r+\beta-c_1}{r+\beta-v} \). Since \( \alpha \geq 0 \), the penalty the supplier pays the buyer for not satisfying units she orders, is the one parameter over which the contracting parties are assumed to have control under quantity flexibility, we solve for the coordinating condition

\[
\alpha = \frac{(r + \beta - v) (c_1 - w)}{c_1 - v}.
\] (56)

Examining the components of (56) individually, we see that the first term in the numerator is greater than zero because \( r > v \) and \( \beta \geq 0 \), as is the denominator. So if \( w > c_1 \) by our initial assumption, then this would require a negative \( \alpha \). We could have a positive coordinating \( \alpha \) if we allowed the supplier to sell the goods below cost.

We consider (52) over the region \( q_1 \leq t_{1A.II}^{11} + M \cap q_1 \geq t_{1A.II}^{11} + M \cap q_1 \geq t_{1A.III}^{11} + M \). Taking the partial derivative and setting it equal to zero yields \( p(1-d)F((1-d)q_1) = 0 \). Since only the lower deviation penalty exists in this profit function realization, the buyer wants to make her initial order estimate as small as possible to avoid paying the penalty. Consequently, \( q_1^* = q_{1A.II}^1 \equiv \max \left\{ F^{-1}(0), \frac{\max \{ t_{1A.II}^{11}, t_{1A.III}^{11} \} + M}{1+d} \right\} \).

We want to maximize (53) over the region \( q_1 \leq t_{1A.II}^{11} + M \cap q_1 \geq t_{1A.II}^{11} + M \cap q_1 \geq t_{1A.III}^{11} + M \). First order optimality conditions yield \( q_1^* = q_{1A.III}^1 \equiv \{ q : (1+d) = (1-d)F((1-d)q) + (1+d)F((1+d)q) \} \), which is feasible if it is smaller than \( t_{1A.III}^{11} + M \).

Finally, we maximize (54) over the region \( q_1 \geq t_{1A.III}^{11} + M \cap q_1 \leq t_{1A.II}^{11} + M \cap q_1 \geq t_{1A.III}^{11} + M \). The first order conditions yield \( q_1^* = q_{1A.IV}^1 \equiv \left\{ q : F((1-d)q_1) = \frac{r-w-\alpha-\beta}{r-w-\alpha+\beta+p} \right\} \), which is feasible if \( \max \left\{ \frac{t_{1A.II}^{11} + M}{1-d}, \frac{t_{1A.III}^{11} + M}{1+d} \right\} \) ≤ \( q_1^* \leq \frac{t_{1A.II}^{11} + M}{1-d} \). ■
APPENDIX B

APPENDIX FOR CHAPTER 3

Proof of Lemma 6. Conditioning on the state of the bundle death process at time $u$, we obtain

$$J(k, T|u) = r_L a \sum_{n=0}^{k/a} \left( \frac{k}{a} - n \right) P(X_L(u) = n) + r_s \sum_{n=0}^{k/a} \left( an - X_S(T - u) \right) P(X_L(u) = n).$$

Now we must condition on the state of the single-unit death process after $T - u$ time units in order to simplify the expected value of the random variable, yielding

$$J(k, T|u) = r_L a \sum_{n=0}^{k/a} \left( \frac{k}{a} - n \right) P(X_L(u) = n) + r_s \sum_{n=0}^{k/a} \left( an - \sum_{l=0}^{an} l P(X_S(T - u) = l) \right) P(X_L(u) = n).$$

Algebraic simplification results in the following expression for the expected revenue function:

$$J(k, T|u) = kr_L - r_L a \sum_{n=0}^{k/a} n P(X_L(u) = n) + r_s \sum_{n=0}^{k/a} \left( an - \sum_{l=0}^{an} l P(X_S(T - u) = l) \right) P(X_L(u) = n).$$

(57)

The sum in the parentheses in (57) is simply the expected value of a binomial random variable with parameters $e^{-\alpha S(T-u)}$ and $an$. Substituting the product of these parameters into (57) for the sum gives us

$$J(k, T|u) = kr_L - r_L a \sum_{n=0}^{k/a} n P(X_L(u) = n) + r_s a (1 - e^{-\alpha S(T-u)}) \sum_{n=0}^{k/a} n P(X_L(u) = n).$$

(58)

The remaining sums represent the expected value of a binomial random variable having parameters $e^{-\alpha T_u}$ and $k/a$. Substituting the product of these parameters into (58) and simplifying gives the result. $\blacksquare$
Proof of Lemma 7. The second derivative of $J(k, T|u)$ is given by

$$\frac{\partial^2 J(k, T|u)}{\partial u^2} \equiv J'' = k \left[ -\alpha_L^2 (r_L - r_S)e^{-\alpha_L u} - r_S (\alpha_S - \alpha_L)^2 e^{-\alpha_L u - \alpha_S (T-u)} \right].$$

(59)

The first term in (59) is non-positive because $r_L \geq r_S$ and $e^{-x} > 0 \forall x \geq 0$. The second term is negative as well, regardless of the values of the alpha parameters. Thus, $J(k, T|u)$ is concave in $u$ if $r_L \geq r_S$. 

Proof of Lemma 13. In order to establish the joint concavity of the expected revenue function in (37), we must show that the Hessian is negative-semidefinite. That is, the first element of the Hessian must be non-positive, and the determinant of the Hessian should be non-negative. (For details, see any introductory book on nonlinear optimization such as Rardin (1998: 747).) We shall consider the requirement for the determinant of the Hessian, denoted $|H|$, first.

$$|H| = k^2 e^{-2\alpha_L u_1 - \alpha_M (u_2 - u_1)} \alpha_L^2 \alpha_M^2 (r_L (r_M - r_S) - r_M (r_M - r_S)) +$$

$$k^2 e^{-2\alpha_L u_1 - \alpha_M (u_2 - u_1) - \alpha_S (T-u_2)} (r_L - r_M) r_S \alpha_M^2 (\alpha_M - \alpha_S)^2 +$$

$$k^2 e^{-2\alpha_L u_1 - 2\alpha_M (u_2 - u_1) - \alpha_S (T-u_2)} (r_M - r_S) r_S \alpha_S^2 (\alpha_L - \alpha_M)^2$$

Each of these terms is non-negative provided that $r_L \geq r_M \geq r_S$. Now we turn our attention to the first element of the Hessian (also considered the first principal submatrix). This term corresponds to $\frac{\partial^2 J(k,T|u_1,u_2)}{\partial u_1^2} \equiv J''_1$, and we need it to be non-positive in order to establish concavity.

$$J''_1 = k [e^{-\alpha_L u_1} \alpha_L^2 (r_M - r_L) + e^{-\alpha_L u_1 - \alpha_M (u_2 - u_1)} (\alpha_M - \alpha_L)^2 (r_S - r_M) -$$

$$e^{-\alpha_L u_1 - \alpha_M (u_2 - u_1) - \alpha_S (T-u_2)} (\alpha_M - \alpha_L)^2 r_S]$$

All of the exponential and squared terms are non-negative, so if we have the same ordering of the unit revenues ($r_L \geq r_M \geq r_S$), then the first principal submatrix is non-positive. Thus, the Hessian is negative semidefinite, and the expected revenue function is jointly concave in $u_1$ and $u_2$. 

Proof of Theorem 7. We shall first prove (i.), so we assume that $\alpha_S > \alpha_L$. The concavity of $J$ established in Lemma 7 allows us to solve for the optimal value of $u$ by
setting the first derivative equal to zero.

\[
\frac{\partial J(k, T|u)}{\partial u} \equiv J' = k \left[ \alpha_L (r_L - r_S) e^{-\alpha_L u} - r_S (\alpha_S - \alpha_L) e^{-\alpha_S u - \alpha_S (T - u)} \right]
\]

(60)

Solving \( J' = 0 \) for \( u \) yields

\[
u^* = T - \frac{\ln \left[ \left( \frac{r_S}{r_L - r_S} \right) \left( \frac{\alpha_S - \alpha_L}{\alpha_L} \right) \right]}{\alpha_S},
\]

(61)

which we constrain to be within \([0, T]\). Combining the conditions under which \( u^* \geq 0 \) and \( u^* \leq T \) yields that the seller’s optimal decision is given by (61) if \( e^{-\alpha_S T} \left( \frac{r_L - r_S}{r_S} \right) \leq e^{-\alpha_S T} \left( \frac{\alpha_S - \alpha_L}{\alpha_L} \right) \).

If \( \frac{r_L - r_S}{r_S} < e^{-\alpha_S T} \left( \frac{\alpha_S - \alpha_L}{\alpha_L} \right) \), then \( J' < 0 \), which means that the expected revenue is decreasing in \( u \); consequently \( u^* = 0 \). Similarly, \( \frac{r_L - r_S}{r_S} > \frac{\alpha_S - \alpha_L}{\alpha_L} \) implies that \( J' > 0 \) and the expected revenue is increasing in \( u \); thus, \( u^* = T \). These three optimal values and conditions can be aggregated into the expression in item (i.) of the theorem statement.

Intuitively, (ii.) follows directly from the fact that when bundled goods obtain at least as high a unit price as single units and customers desiring bundles arrive at least as often as individual buyers, the firm has no incentive to sell the goods individually. If \( \alpha_S \leq \alpha_L \), then \( J' \) is non-negative over the entire selling horizon and \( J(k, T|u) \) increases monotonically in \( u \). Therefore, \( u^* = T \).

**Proof of Lemma 8.** We first establish the result for the case where \( \alpha_S > \alpha_L \equiv \gamma/a \).

Suppose that \( u^* \in (0, T) \) and therefore \( u^* = T - \frac{\ln \left[ \left( \frac{r_S}{\gamma/a - r_S} \right) \left( \frac{\alpha_S - \gamma/a}{\gamma/a} \right) \right]}{\alpha_S} \); then by taking the partial derivative of \( u^* \) with respect to \( a \), we obtain \( \frac{\partial u^*}{\partial a} = \frac{-1}{a(\alpha_S - \gamma/a)} \). This ratio is always negative since \( \alpha_S > \gamma/a \); thus, \( u^* \) is decreasing in \( a \).

The condition for \( u^* = 0 \) is \( \frac{r_L - r_S}{r_S} < e^{-\alpha_S T} \left( \frac{\alpha_S - \gamma/a}{\gamma/a} \right) \). As \( a \) increases, the ratio on the left remains constant while the ratio on the right increases. This means that it is easier for this condition to be satisfied; as a result, the decision remains at \( u^* = 0 \) as \( a \) increases.

The condition for \( u^* = T \) is \( \frac{r_L - r_S}{r_S} > \frac{\alpha_S - \gamma/a}{\gamma/a} \). As \( a \) increases, the ratio on the left remains the same while the ratio on the right increases. This implies that the optimal \( u^* \) moves away from \( T \) because it is more difficult for this relationship to hold as \( a \) increases.

If \( \alpha_S \leq \gamma/a \), then \( u^* = T \) as \( a \) increases initially and until \( a \) is large enough that \( \alpha_S > \gamma/a \). After that point, the policy transitions to the above case.
Proof of Lemma 9. Suppose that $u^* \in (0, T)$; therefore, we have $u^* = T - \ln \left( \left. \frac{rS}{\gamma/a - rS} \right| \alpha_S \right) / \alpha_S$; and $\frac{\partial u^*}{\partial r_L} = \frac{1}{\alpha_S (r_L - rS)}$. This ratio is always positive since we are still assuming $r_L \geq r_S$; thus, $u^*$ is increasing in $r_L$. Looking at the conditions under which $u^* = 0$ and $u^* = T$, we see that the optimal decision is moving toward $T$ and away from 0 since the condition for $T$ to be optimal is easier to satisfy as $r_L$ increases.

Similarly, $\frac{\partial u^*}{\partial r_S} = \frac{-rL}{rS \alpha_S} \left( \frac{rL - rS}{\alpha_L - \alpha_S} \right)$, which is always negative, so $u^*$ is decreasing as $r_S$ increases.

Examination of the optimal decisions at the endpoints of the horizon again yields the result that the decision moves toward 0 and away from $T$.

Proof of Lemma 10. Suppose that $u^* \in (0, T)$; then we have $\frac{\partial u^*}{\partial \alpha_S} = \frac{1}{\alpha_S (\alpha_S - \alpha_L)} + \ln \frac{\left. \frac{rS}{\gamma/a - rS} \right| \alpha_S \right)}{\alpha_S}$ Setting $\frac{\partial u^*}{\partial \alpha_S} < 0$, we obtain the condition under which $u^*$ is decreasing in $\alpha_S$. Setting the partial derivative greater than zero establishes the converse result.

Proof of Lemma 11. The second derivative of the expected revenue function, which is given in (59), can be positive for some parameter values within the selling horizon if $r_L < r_S$; thus, the function is not necessarily concave. That said, we can determine the shape of the expected revenue function when $\alpha_L > \alpha_S$.

The expected revenue function is monotonically decreasing if $J'$, given in (60), is less than or equal to zero for all $u \in [0, T]$. This requirement simplifies to $e^{-\alpha_S (T - u)} \leq \left( \frac{rS - rL}{rS} \right) \left( \frac{\alpha_L}{\alpha_L - \alpha_S} \right)$, $\forall u \in [0, T]$. The left side of this inequality is greatest when $u = T$, so if the inequality holds for that value of $u$, it holds for all other values. Substituting $u = T$ above yields the condition $\frac{rS - rL}{rS} \geq \frac{\alpha_L - \alpha_S}{\alpha_L}$ for $J(k, T | u)$ to be monotonically decreasing.

The expected revenue is monotonically increasing if $J' \geq 0$ over the entire time horizon. This requirement simplifies to $e^{-\alpha_S (T - u)} \geq \left( \frac{rS - rL}{rS} \right) \left( \frac{\alpha_L}{\alpha_L - \alpha_S} \right)$, $\forall u \in [0, T]$. The left side of the inequality is smallest when $u = 0$, so the inequality will hold for all other values of $u$ if it is satisfied at $u = 0$. Substitution yields the condition $\frac{rS - rL}{rS} \leq e^{-\alpha_S T} \left( \frac{\alpha_L - \alpha_S}{\alpha_L} \right)$ for the expected revenue function to be monotonically increasing.

When $e^{-\alpha_S T} \left( \frac{\alpha_L - \alpha_S}{\alpha_L} \right) < \frac{rS - rL}{rS} < \frac{\alpha_L - \alpha_S}{\alpha_L}$, straight computation shows that $J'(0) < 0$ and $J'(T) > 0$. This tells us that the expected revenue function is decreasing at the
beginning of the horizon and increasing at the end. Setting $J' = 0$ and solving for $u$, we obtain a unique solution. Thus, $J$ has one critical point, and $J'$ can change its sign only once. This implies that $J$ must decrease as $u$ increases from zero; once it reaches its minimum, $J$ increases until $u = T$. ■

**Proof of Lemma 13.** Lemma 11 establishes that the expected revenue function is either monotonic or it decreases and then increases over the entire selling horizon when $\alpha_L > \alpha_S$. In the case where the expected revenue function is monotonically decreasing, this results in an optimal $u^* = 0$. When the expected revenue function increases monotonically, the optimal timing decision is $u^* = T$. The case when $J$ decreases and then increases suggests that the optimal decision is at one of the endpoints of the time horizon; the seller must simply compare expected profits at $u = 0$ and $u = T$ and choose the time with the higher profit.

These three optimal solutions are summarized by item (i.) in the theorem statement. The seller can always evaluate the expected revenue function under strict bundles or strict individual tickets and choose the pure policy that maximizes her revenue.

As in the proof of Theorem 7, the second result comes from the observation that if units sold individually obtained a higher unit revenue than goods sold in bundles and generated customer arrivals as least as quickly, it is optimal for the seller to offer only single units for the entire time horizon. When $\alpha_L \leq \alpha_S$, $J'(u) < 0$ for all $u \in [0, T]$. Consequently the expected revenue function is monotonically decreasing and $u^* = 0$. ■

**Proof of Theorem 9.** We first observe that if the unit revenue and the arrival rate for a particular product offering (i.e. large bundles, small bundles, or individual units) are both less than or equal to the corresponding parameters for another offering, the seller has no incentive to offer that product. This observation is crucial to the rest of the proof.

The KKT optimality conditions are applicable only if the expected revenue function is concave, so Lemma 13 implies that we must have $r_L \geq r_M \geq r_S$. The resulting optimality equations are the following two partial derivative equations and the four complementary slackness conditions. In addition to finding a solution to this system, we must ensure that
the multipliers (the λs and µs) are non-negative.

\[ k \left[ e^{-\alpha_L u_1} \alpha_L (r_L - r_M) + e^{-\alpha_L u_1 - \alpha_M (u_2 - u_1)} (\alpha_L - \alpha_M) \left( r_M - r_S (1 - e^{-\alpha_S (T - u_2)}) \right) \right] + \lambda_1 - \lambda_2 - \mu_1 = 0 \] (62)

\[ k \left[ e^{-\alpha_L u_1 - \alpha_M (u_2 - u_1)} \alpha_M (r_M - r_S) + e^{-\alpha_L u_1 - \alpha_M (u_2 - u_1) - \alpha_S (T - u_2)} (\alpha_M - \alpha_S) r_S \right] + \mu_1 - \mu_2 = 0 \] (63)

\[ \lambda_1 u_1 = 0 \] (64)

\[ \lambda_2 (T - u_1) = 0 \] (65)

\[ \mu_1 (u_2 - u_1) = 0 \] (66)

\[ \mu_2 (T - u_2) = 0 \] (67)

We can first divide the parameter space by the relationship between the arrival rates and determine the optimal decisions for each of these orderings. If \( \alpha_L \geq \max \{ \alpha_M, \alpha_S \} \), then there is no incentive for the seller to offer anything other than large bundles since they obtain the highest revenue and result in the highest customer arrival rate. Thus, \( u_1^* = T = u_2^* \).

If \( \alpha_M \geq \max \{ \alpha_L, \alpha_S \} \), then there is no reason for the firm to offer any single tickets. Consequently, \( u_2^* = T \), and the possible decision sets (from Table 5) are I.3, II.3, and III. In case I.3, \( u_1^* = 0 \), so we must also have \( \lambda_2^* = \mu_1^* = 0 \) to satisfy equations (65) and (66). That leaves us with the two partial derivative equations ((62) and (63)) to solve with decision variables \( \lambda_1 \) and \( \mu_2 \). To ensure that these solutions are non-negative, we need

\[ \frac{r_L - r_M}{r_M} \leq e^{-\alpha_M T} \left( \frac{\alpha_M - \alpha_L}{\alpha_L} \right) \].

In case II.3, \( u_1^* \in (0, T) \) so \( \lambda_1^* = \lambda_2^* = \mu_1^* = 0 \) to satisfy the complementary slackness conditions. Solving the two partial derivative equations for \( u_1 \) and \( \mu_2 \) yields \( u_1^* = T - \frac{\ln \left( \frac{r_S}{r_L - r_S} \left( \frac{\alpha_S - \alpha_L}{\alpha_S} \right) \right)}{\ln \left( \frac{r_S}{r_L - r_S} \right)} \), which lies within \( (0, T) \) if \( e^{-\alpha_M T} \left( \frac{\alpha_M - \alpha_L}{\alpha_L} \right) \) < \( \frac{r_L - r_M}{r_M} \) < \( \frac{\alpha_M - \alpha_L}{\alpha_L} \). \( \mu_2^* > 0 \) because \( \alpha_M > \alpha_S \). If case III. is optimal, then \( u_1^* = u_2^* = T \), \( \lambda_1^* = 0 \), and \( \lambda_2^* = \mu_1^* \) to satisfy the complementary slackness conditions. After solving the partial derivative equations for \( \mu_1 \) and \( \mu_2 \), we require \( \frac{r_L - r_M}{r_M} \geq \frac{\alpha_M - \alpha_L}{\alpha_L} \) in order for both of the multipliers to be non-negative.

If \( \alpha_S > \alpha_L > \alpha_M \), then there is no incentive for the seller to offer small bundles and \( u_2^* = u_1^* \). It follows that the only possible decision cases are I.1 (when only single units
are sold), II.1 (when both large bundles and single units are sold), and III. (when only large bundles are sold). In case I.1 both timing decisions equal zero, and $\lambda_2^* = \mu_2^* = 0$ to satisfy the complementary slackness conditions. Solving the partial derivative equations for $\lambda_1$ and $\mu_1$, we must have both $r_L - r_S \leq e^{-\alpha_S T} \left( \frac{\alpha_S - \alpha_L}{\alpha_L} \right)$ and $r_M - r_S \leq e^{-\alpha_S T} \left( \frac{\alpha_S - \alpha_M}{\alpha_M} \right)$ to ensure that the multipliers are non-negative. The second inequality is satisfied any time the first inequality holds because $r_L \geq r_M$ and $\alpha_L > \alpha_M$, so only the first inequality is required. In case II.1 $u_1^* = u_2^*$, and these values lie in the interval $(0, T)$. The complementary slackness conditions require that $\lambda_1^* = \lambda_2^* = \mu_2^* = 0$, so we can solve the partial derivative equations for $u_1$ and $\mu_1$. This yields $u_1^* = T - \ln \left( \frac{r_S}{r_L - r_S} \right) \left( \frac{\alpha_S - \alpha_L}{\alpha_L} \right)$, which lies in $(0, T)$ if $e^{-\alpha_S T} \left( \frac{\alpha_S - \alpha_L}{\alpha_L} \right) < \frac{r_L - r_S}{r_S} < \frac{\alpha_S - \alpha_L}{\alpha_L}$. The expression derived for $\mu_1^*$ is non-negative if this inequality holds. Finally, if case III. is optimal, then $u_1^* = u_2^* = T$, and we need $\lambda_1^* = 0$ and $\lambda_2^* = \mu_1^*$. Solving the partial derivative equations for $\mu_1$ and $\mu_2$, we obtain non-negative expressions if $\frac{r_L - r_S}{r_S} \geq \frac{\alpha_S - \alpha_L}{\alpha_L}$.

If $\alpha_S > \alpha_M > \alpha_L$, all seven cases from Table 5 are possible. In order to determine the necessary conditions for each solution to be optimal, we hypothesize that the solution is optimal and then solve for the conditions that ensure that the Lagrangian multipliers are non-negative and the original model constraints are satisfied.

1. If case I.1 is optimal, then $u_1^* = u_2^* = 0$. The complementary slackness conditions imply that $\lambda_2^* = \mu_2^* = 0$. Solving the partial derivative equations for $\lambda_1^*$ and $\mu_1^*$, we require $\frac{r_M - r_S}{r_S} \leq e^{-\alpha_S T} \left( \frac{\alpha_S - \alpha_M}{\alpha_M} \right)$ and $\frac{r_L - r_S}{r_S} \leq e^{-\alpha_S T} \left( \frac{\alpha_S - \alpha_L}{\alpha_L} \right)$.

2. If case I.2 is optimal, then $u_1^* = 0$ and complementary slackness conditions imply that $\lambda_2^* = \mu_1^* = \mu_2^* = 0$. Solving the partial derivative equations for $u_2^*$ and $\lambda_1^*$, we obtain $u_2^* = T - \frac{\ln \left( \frac{r_S}{r_M - r_S} \right) \left( \frac{\alpha_S - \alpha_M}{\alpha_M} \right)}{\alpha_S}$. We require $e^{-\alpha_M T} \left( \frac{\alpha_S - \alpha_M}{\alpha_M} \right) < \frac{r_M - r_S}{r_S} < \frac{\alpha_S - \alpha_M}{\alpha_M}$ and $e^{-\alpha_M T} \geq \frac{\alpha_L}{\alpha_S} \left( \frac{r_L - r_M}{r_M - r_S} (\alpha_M - \alpha_L) \left[ \frac{r_M - r_S}{r_S (\alpha_S - \alpha_M)} \right]^{1/\alpha_S} \right)$ to ensure that $u_2^* \in (0, T)$ and that $\lambda_1^* \geq 0$.

3. If case I.3 is optimal, then $u_1^* = 0$ and $u_2^* = T$. The complementary slackness conditions imply that $\lambda_2^* = \mu_1^* = 0$. Solving the partial derivative equations for $\lambda_1^*$ and $\mu_2^*$ yields the conditions $r_L - r_M \leq e^{-\alpha_M T} \left( \frac{\alpha_M - \alpha_L}{\alpha_L} \right)$ and $\frac{r_M - r_S}{r_S} \geq \frac{\alpha_S - \alpha_M}{\alpha_M}$ which ensure
4. If case II.1 is optimal, then $u_1^* = u_2^*$ and complementary slackness conditions imply that $\lambda_1^* = \lambda_2^* = \mu_2^* = 0$. Solving the partial derivative equations for $u_1^*$ and $\mu_1^*$ yields $u_1^* = u_2^* = T - \ln\left(\frac{\alpha_S}{\alpha_L}\right)\left(\frac{a_S - \alpha_L}{a_L}\right)$. We require $e^{-\alpha_ST}\left(\frac{a_S - \alpha_L}{a_L}\right) < \frac{r_L - rs}{r_S} < \frac{a_S - \alpha_L}{a_L}$ to ensure that $u_1^* \in (0, T)$ and $\frac{r_M - rs}{rs} < \frac{a_L(a_S - a_M)}{a_M(a_S - a_L)}$ to guarantee that $\mu_1^* \geq 0$.

5. If case II.2 is optimal, then $u_1^*$ and $u_2^*$ both lie in the interior of the selling interval. All of the multipliers are equal to zero since the constraints are satisfied by solving for the unconstrained optimum. The partial derivative equations imply that $u_1^* = T - \ln\left(\frac{\alpha_S}{\alpha_L}\right)\left(\frac{a_S - \alpha_L}{a_L}\right)$ and $u_2^* = T - \ln\left(\frac{\alpha_S}{\alpha_L}\right)\left(\frac{a_S - \alpha_L}{a_L}\right)$. In order to ensure that $u_1^* \in (0, T)$ and $u_2^* \in (u_1^*, T)$, we require $e^{-\alpha_ST}\left(\frac{a_S - \alpha_M}{a_M}\right) < \frac{r_M - rs}{rs} < \frac{a_L(a_S - a_M)}{a_M(a_S - a_L)}$, as well as $T - \ln\left(\frac{\alpha_S}{\alpha_L}\right)\left(\frac{a_S - \alpha_M}{a_M}\right) \geq \frac{\alpha_L(a_S - a_M)}{a_M(a_S - a_L)}$.

6. If case II.3 is optimal, then $u_1^* \in (0, T)$ and $u_2^* = T$. The complementary slackness conditions imply that $\lambda_1^* = \lambda_2^* = \mu_2^* = 0$. Solving the partial derivative equations for $u_1^*$ and $\mu_2^*$ yields $u_1^* = T - \ln\left(\frac{\alpha_S}{\alpha_L}\right)\left(\frac{a_M - \alpha_L}{a_L}\right)$. We need $e^{-\alpha_MT}\left(\frac{a_M - \alpha_L}{a_L}\right) < \frac{r_L - r_M}{r_M} < \frac{a_M - \alpha_L}{a_L}$ to ensure that $u_1^* \in (0, T)$ and $\frac{r_M - rs}{rs} \geq \frac{a_S - a_M}{a_M}$ so that $\mu_2^* \geq 0$.

7. If case III. is optimal, then $u_1^* = u_2^* = T$. The complementary slackness conditions imply that $\lambda_1^* = 0$. This leaves three free multipliers to be solved for using the two partial derivative equations. To ensure that the equations can be solved with non-negative multiplier values, we must have $\frac{r_L - r_M}{r_M} \geq \frac{a_M - \alpha_L}{a_L}$ and $\frac{r_L - rs}{rs} \geq \frac{a_S - \alpha_L}{a_L}$.

\section*{Proof of Lemma 14.} Since we are deriving the results for the situation where $k_H/a_H \leq k_S/a_S$, we have \(k_S - k_H\frac{a_H}{a_H}\) additional sparse demand tickets that cannot be formed into bundles. We begin by conditioning on the state of $X_L(\tau)$ at time $n$ in the expected revenue function in (39).

\[
J = r_L(a_H + a_S) \sum_{n=0}^{k_H/a_H} \left(\frac{k_H}{a_H} - n\right) P(X_L(u) = n) + \frac{k_H}{a_H} \sum_{n=0}^{k_H/a_H} \left(\frac{k_H}{a_H} - n\right) P(X_L(u) = n).
\]
Using the fact that the state probabilities for the linear, Markovian death process are binomial as in the proof of Lemma 6, we can simplify (68) as follows.

\[
J = \left. r_L k_H \frac{a_H}{a_H} + a_S \right|_{a_H} (1 - e^{-\alpha_L u}) + r_H k_H e^{-\alpha_L u} - r_H \sum_{n=0}^{k_H} X_H(T-u) P(X_L(u) = n) + \\
\left. r_S k_H \frac{a_S}{a_H} e^{-\alpha_L u} + r_S \left( k_S - k_H \frac{a_S}{a_H} \right) \right|_{a_S} - r_S \sum_{n=0}^{k_H} X_S(T-u) P(X_L(u) = n). 
\]

Conditioning on both of the individual ticket demand processes yields

\[
J = \left. r_L k_H \frac{a_H}{a_H} + a_S \right|_{a_H} (1 - e^{-\alpha_L u}) + r_H k_H e^{-\alpha_L u} - r_H \sum_{n=0}^{k_H} \left( \sum_{m=0}^{a_H n} m P(X_H(T-u) = m) \right) P(X_L(u) = n) + \\
\left. r_S k_H \frac{a_S}{a_H} e^{-\alpha_L u} + r_S \left( k_S - k_H \frac{a_S}{a_H} \right) \right|_{a_S} - r_S \sum_{n=0}^{k_H} \left( \sum_{l=0}^{a_S n + k_S - k_H \frac{a_S}{a_H}} l P(X_S(T-u) = l) \right) P(X_L(u) = n). 
\]

Substituting the binomial state probabilities and simplifying yields the reduced form expected revenue function given in (40).

**Proof of Lemma 15.** The second derivative of the reduced-form expected revenue function in (40) is given by

\[
\frac{\partial^2 J}{\partial u^2} = k_H \alpha_L^2 e^{-\alpha_L u} \left[ (r_H - r_L) + (r_S - r_L) \frac{a_S}{a_H} \right] - r_H k_H e^{-\alpha_L u - \alpha_H (T-u)} (\alpha_H - \alpha_L)^2 - \\
r_S \frac{a_S}{a_H} e^{-\alpha_L u - \alpha_S (T-u)} (\alpha_S - \alpha_L)^2. 
\]
Proof of Theorem 10. Since the expected revenue function in (40) is concave in \( u \) if \( r_L \geq \max\{r_H, r_S\} \), we can use the KKT optimality conditions to solve the seller’s optimization problem in (39). We have the following system of optimality equations consisting of the partial derivative equation and two complementary slackness conditions; plus, we must also ensure that the two multipliers (\( \lambda_1 \) and \( \lambda_2 \)) are non-negative at optimality.

\[
\begin{align*}
    r_L k_H \frac{a_H + a_S}{a_H} & \alpha_L e^{-\alpha_L u} - r_H k_H \alpha_H e^{-\alpha_L u - \alpha_H(T-u)} - r_H k_H \alpha_L e^{-\alpha_L u}(1 - e^{-\alpha_H(T-u)}) - \\
    r_S k_H \frac{a_S}{a_H} & \alpha_S e^{-\alpha_L u - \alpha_S(T-u)} - r_S k_H \frac{a_S}{a_H} \alpha_L e^{-\alpha_L u}(1 - e^{-\alpha_S(T-u)}) - \\
    r_S (k_S - k_H \frac{a_S}{a_H}) & \alpha_S e^{-\alpha_S(T-u)} + \lambda_1 - \lambda_2 = 0 \\
    \lambda_1 u &= 0 \\
    \lambda_2 (T - u) &= 0
\end{align*}
\]

(70)

We will establish the optimal solutions by positing the three possibilities for \( u^* \) and then determining the parameter requirements to satisfy the KKT conditions.

1. Assume that \( u^* = 0 \). The slackness condition in (72) implies that \( \lambda_2^* = 0 \). Substituting these two values into the partial derivative equation in (70) enables us to solve that equation for \( \lambda_1^* \). Setting \( \lambda_1^* \geq 0 \) yields the condition for \( u^* = 0 \) in the theorem statement.

2. Assume that \( u^* = T \). The slackness condition in (71) implies that \( \lambda_1^* = 0 \). Substituting these two values into the partial derivative equation in (70) enables us to solve that equation for \( \lambda_2^* \). Setting \( \lambda_2^* \geq 0 \) yields the condition for \( u^* = T \) in the theorem statement.

3. Assume that \( u^* \in (0, T) \). The slackness conditions imply that \( \lambda_1^* = \lambda_2^* = 0 \). Substituting these two values into the partial derivative equation, we can solve for \( u^* \).

■

Proof of Corollary 2. The results for the scenario when \( k_H/a_H > k_S/a_S \) are symmetric to those developed above for the converse case. Consequently, we can derive the reduced-form expected revenue function as in Lemma 14. Then we can establish the
concavity of the expected revenue function if $r_L \geq \max\{r_H, r_S\}$ as in Lemma 15. We conclude by characterizing the optimal decision policy analogously to Theorem 10. ■
APPENDIX C

APPENDIX FOR CHAPTER 4

C.1 Retailer’s optimal prices and quantities for the four retailer scenarios

Retailer Scenario 1: None of the products are constrained

\[ \hat{p}_1 = \frac{H_1(4b_2b_3 - (\beta_{23} + \beta_{32})^2) + H_2(2b_3(\beta_{12} + \beta_{21}) + (\beta_{13} + \beta_{31})(\beta_{23} + \beta_{32}))}{2[4b_1b_2b_3 - b_1(\beta_{23} + \beta_{32})^2 - b_2(\beta_{13} + \beta_{31})^2 - b_3(\beta_{12} + \beta_{21})^2 - (\beta_{12} + \beta_{21})(\beta_{13} + \beta_{31})(\beta_{23} + \beta_{32})]} \]

\[ H_1 = a_1 + b_1w_1 - \beta_{21}w_2 - \beta_{31}w_3, \]

\[ H_2 = a_2 + b_2w_2 - \beta_{12}w_1 - \beta_{32}w_3, \]

\[ H_3 = a_3 + b_3w_3 - \beta_{13}w_1 - \beta_{23}w_2. \]
Retailer Scenario 2: One of the products is constrained

\[ p_1(q_2) = \frac{K_1(2b_2(b_2b_3 - \beta_{23}b_{22})) + K_3b_2(b_2(\beta_{13}\beta_{31}) + \beta_{21}\beta_{32} + \beta_{23}\beta_{12})}{2(b_2b_3 - \beta_{23}\beta_{32})(2b_1b_2 - \beta_{12}\beta_{21}) - (b_2\beta_{13} + \beta_{12}\beta_{23})^2} - \]

\[ 2b_2\beta_{12}(b_3\beta_{21} + \beta_{31}\beta_{23}) - (b_2\beta_{31} + \beta_{21}\beta_{32})(b_2\beta_{31} + 2b_2\beta_{13} + \beta_{21}\beta_{32}) \]

\[ p_2(q_2) = \frac{a_2 - q_2}{b_2} + \frac{K_3(2b_1b_2\beta_{23} + b_2\beta_{21}(\beta_{13} + \beta_{31}) + \beta_{23}^2\beta_{12} - \beta_{21}\beta_{23}\beta_{32})}{2(b_2b_3 - \beta_{23}\beta_{32})(2b_1b_2 - \beta_{12}\beta_{21}) - (b_2\beta_{13} + \beta_{12}\beta_{23})^2} - \]

\[ 2b_2\beta_{12}(b_3\beta_{21} + \beta_{31}\beta_{23}) - (b_2\beta_{31} + \beta_{21}\beta_{32})(b_2\beta_{31} + 2b_2\beta_{13} + \beta_{21}\beta_{32}) \]

where \( K_1 = H_1 + \frac{(a_2-q_2)(\beta_{12}+\beta_{21})}{b_2} \), \( K_2 = H_2 - 2(a_2 - q_2) \), and \( K_3 = H_3 + \frac{(a_2-q_2)(\beta_{23}+\beta_{12})}{b_2} \).

The resulting customer demands are

\[ Q_1(q_2) = \frac{a_1b_2 + \beta_{12}(a_2-q_2)}{b_2} + \]

\[ K_1[b_2^2(\beta_{13}(\beta_{13} + \beta_{31}) - 2b_1b_3) + b_2(\beta_{23}(2b_1\beta_{32} + 2b_2\beta_{13} + \beta_{12}\beta_{31})) + \beta_{21}(2b_2\beta_{12} + \beta_{13}\beta_{32}) + \beta_{12}\beta_{23}(\beta_{12}\beta_{23} - \beta_{21}\beta_{32})] + \]

\[ K_2[b_2(\beta_{13}(b_1\beta_{23} + \beta_{13}\beta_{21} + \beta_{21}\beta_{31}) - b_1(2b_2\beta_{21} + \beta_{31}\beta_{23}) + \beta_{13}\beta_{21}\beta_{32} + b_1\beta_{23}(\beta_{21}\beta_{32} + \beta_{12}\beta_{23}) + \beta_{12}\beta_{21}(2b_1\beta_{21} + 2\beta_{31}\beta_{23} + \beta_{13}\beta_{23}) + K_3[\beta_{12}\beta_{21}(2b_1\beta_{21} - \beta_{21}\beta_{32} - \beta_{12}\beta_{23}) + b_2(\beta_{13}\beta_{23} - \beta_{21}\beta_{32}) + (\beta_{13} - \beta_{31}(b_1b_2 - \beta_{12}\beta_{21}))]\]

\[ 2(b_2b_3 - \beta_{23}\beta_{32})(2b_1b_2 - \beta_{12}\beta_{21}) - (b_2\beta_{13} + \beta_{12}\beta_{23})^2 - \]

\[ 2b_2\beta_{12}(b_3\beta_{21} + \beta_{31}\beta_{23}) - (b_2\beta_{31} + \beta_{21}\beta_{32})(b_2\beta_{31} + 2b_2\beta_{13} + \beta_{21}\beta_{32}) \]
Retailer Scenario 3: Two of the products are constrained

\[ Q_2(q_2) = q_2 \]

\[ Q_3(q_2) = \frac{a_3b_2 + \beta_32(a_2 - q_2)}{b_2} + \]

\[ K_1[\beta_23\beta_32(\beta_12\beta_23 - \beta_21\beta_32) + b_2(b_3(\beta_21\beta_32 - \beta_12\beta_23) + \]

\[ (\beta_31 - \beta_13)(b_2b_3 - \beta_23\beta_32))] + K_2[b_2(\beta_31(b_3\beta_21 + \beta_23\beta_31 + \beta_13\beta_23) - \]

\[ b_3(b_1\beta_23 + \beta_13\beta_21))] + b_3\beta_21(\beta_12\beta_23 + \beta_21\beta_32) + \beta_12\beta_23\beta_31 + \]

\[ \beta_23\beta_32(2b_1\beta_23 + 2\beta_13\beta_21 + \beta_21\beta_31)] + \]

\[ K_3[b_2^2(\beta_31(\beta_13 + \beta_31) - 2b_1b_3) + b_2(2b_3\beta_12 + 2\beta_31\beta_32 + \beta_13\beta_32) + \]

\[ \beta_23(2b_1\beta_32 + \beta_12\beta_31)) + b_2\beta_23(2\beta_21\beta_32 - \beta_12\beta_23)] \]

\[ 2(b_2b_3 - \beta_23\beta_32)(2b_1b_2 - \beta_12\beta_21) - (b_2\beta_31 + \beta_12\beta_23)^2 - \]

\[ 2b_2\beta_12(b_3\beta_21 + \beta_31\beta_23) - (b_2\beta_31 + \beta_21\beta_32)(b_2\beta_31 + 2b_2\beta_13 + \beta_21\beta_32) \]

\[ H_1(b_2b_3 - \beta_23\beta_32) + H_2(b_3\beta_21 + \beta_31\beta_23) + H_3(b_2\beta_31 + \beta_21\beta_32) + \]

\[ (a_2 - q_2)(b_3(\beta_12 - \beta_21) + \beta_13\beta_32 - \beta_31\beta_23) + \]

\[ (a_3 - q_3)(b_2(\beta_13 - \beta_31) + \beta_12\beta_23 - \beta_21\beta_32) \]

\[ 2(b_1(b_2b_3 - \beta_23\beta_32) - \beta_12(b_3\beta_21 + \beta_31\beta_23) - \beta_31(b_2\beta_31 + \beta_21\beta_32)) \]

\[ (b_3\beta_21 + \beta_31\beta_23)[H_1(b_2b_3 - \beta_23\beta_32) + H_2(b_3\beta_21 + \beta_31\beta_23) + \]

\[ H_3(b_2\beta_31 + \beta_21\beta_32)] + \]

\[ (a_2 - q_2)[2b_1b_3(b_2b_3 - \beta_23\beta_32) - 2b_3\beta_13(b_2\beta_31 + \beta_21\beta_32) - \]

\[ (b_3\beta_21 + \beta_31\beta_23)((b_3(\beta_12 + \beta_21) + \beta_31\beta_23 - \beta_13\beta_23))] + \]

\[ (a_3 - q_3)[2b_1\beta_23(b_2b_3 - \beta_23\beta_32) - 2\beta_13\beta_23(b_2\beta_31 + \beta_21\beta_32) - \]

\[ (b_3\beta_21 + \beta_31\beta_23)(b_2(\beta_31 - \beta_13) + \beta_12\beta_23 + \beta_21\beta_32)) \]

\[ 2(b_2b_3 - \beta_23\beta_32)(b_1(b_2b_3 - \beta_23\beta_32) - \beta_12(b_3\beta_21 + \beta_31\beta_23) - \]

\[ \beta_13(b_2\beta_31 + \beta_21\beta_32)) \]

\[ p_1(q_2, q_3) = \frac{(a_2 - q_2)(b_3(\beta_12 - \beta_21) + \beta_13\beta_32 - \beta_31\beta_23) + \]

\[ (a_3 - q_3)(b_2(\beta_13 - \beta_31) + \beta_12\beta_23 - \beta_21\beta_32) \]

\[ 2(b_1(b_2b_3 - \beta_23\beta_32) - \beta_12(b_3\beta_21 + \beta_31\beta_23) - \beta_31(b_2\beta_31 + \beta_21\beta_32)) \]

\[ (b_3\beta_21 + \beta_31\beta_23)[H_1(b_2b_3 - \beta_23\beta_32) + H_2(b_3\beta_21 + \beta_31\beta_23) + \]

\[ H_3(b_2\beta_31 + \beta_21\beta_32)] + \]

\[ (a_2 - q_2)[2b_1b_3(b_2b_3 - \beta_23\beta_32) - 2b_3\beta_13(b_2\beta_31 + \beta_21\beta_32) - \]

\[ (b_3\beta_21 + \beta_31\beta_23)((b_3(\beta_12 + \beta_21) + \beta_31\beta_23 - \beta_13\beta_23))] + \]

\[ (a_3 - q_3)[2b_1\beta_23(b_2b_3 - \beta_23\beta_32) - 2\beta_13\beta_23(b_2\beta_31 + \beta_21\beta_32) - \]

\[ (b_3\beta_21 + \beta_31\beta_23)(b_2(\beta_31 - \beta_13) + \beta_12\beta_23 + \beta_21\beta_32)) \]

\[ 2(b_2b_3 - \beta_23\beta_32)(b_1(b_2b_3 - \beta_23\beta_32) - \beta_12(b_3\beta_21 + \beta_31\beta_23) - \]

\[ \beta_13(b_2\beta_31 + \beta_21\beta_32)) \]
The resulting customer demands are

\[
p_3(q_2, q_3) = \frac{(b_2\beta_{31} + \beta_{21}\beta_{32})[H_1(b_2b_3 - \beta_{23}\beta_{32}) + H_2(b_3\beta_{21} + \beta_{31}\beta_{23}) + H_3(b_2\beta_{31} + \beta_{21}\beta_{32})] + (a_2 - q_2)[2b_1\beta_{32}(b_2b_3 - \beta_{23}\beta_{32}) - 2\beta_{12}\beta_{32}(b_3\beta_{21} + \beta_{31}\beta_{23}) - (b_2\beta_{31} + \beta_{21}\beta_{32})(b_3(\beta_{21} - \beta_{12}) + \beta_{31}\beta_{23} + \beta_{13}\beta_{23})] + (a_3 - q_3)[2b_1b_2(b_2b_3 - \beta_{23}\beta_{32}) - 2b_2\beta_{12}(b_3\beta_{21} + \beta_{31}\beta_{23}) - \beta_{13}(b_2\beta_{31} + \beta_{21}\beta_{32})]}{2(b_2b_3 - \beta_{23}\beta_{32})(b_1(b_2b_3 - \beta_{23}\beta_{32}) - \beta_{12}(b_3\beta_{21} + \beta_{31}\beta_{23}) - \beta_{13}(b_2\beta_{31} + \beta_{21}\beta_{32})}.
\]

(84)

The resulting customer demands are

\[
Q_1(q_2, q_3) = \frac{q_3(b_2(\beta_{13} + \beta_{31}) + \beta_{12}\beta_{23} + \beta_{21}\beta_{32})}{2(b_2b_3 - \beta_{23}\beta_{32})}
\]

(85)

\[
Q_2(q_2, q_3) = q_2
\]

(86)

\[
Q_3(q_2, q_3) = q_3.
\]

(87)

**Retailer Scenario 4: All of the products are constrained**

\[
p_1(q_1, q_2, q_3) = \frac{(a_1 - q_1)(b_2b_3 - \beta_{23}\beta_{32}) + (a_2 - q_2)(b_3\beta_{12} + \beta_{13}\beta_{32}) + (a_3 - q_3)(b_1\beta_{23} + \beta_{13}\beta_{21})}{b_1(b_2b_3 - \beta_{23}\beta_{32}) - \beta_{12}(b_3\beta_{21} + \beta_{31}\beta_{23}) - \beta_{13}(b_2\beta_{31} + \beta_{21}\beta_{32})}
\]

(88)

\[
p_2(q_1, q_2, q_3) = \frac{(a_1 - q_1)(b_3\beta_{21} + \beta_{31}\beta_{23}) + (a_2 - q_2)(b_1b_3 + \beta_{13}\beta_{31}) + (a_3 - q_3)(b_1\beta_{23} + \beta_{13}\beta_{21})}{b_1(b_2b_3 - \beta_{23}\beta_{32}) - \beta_{12}(b_3\beta_{21} + \beta_{31}\beta_{23}) - \beta_{13}(b_2\beta_{31} + \beta_{21}\beta_{32})}
\]

(89)

\[
p_3(q_1, q_2, q_3) = \frac{(a_1 - q_1)(b_2\beta_{31} + \beta_{21}\beta_{32}) + (a_2 - q_2)(b_1\beta_{32} + \beta_{12}\beta_{31}) + (a_3 - q_3)(b_1b_2 - \beta_{12}\beta_{21})}{b_1(b_2b_3 - \beta_{23}\beta_{32}) - \beta_{12}(b_3\beta_{21} + \beta_{31}\beta_{23}) - \beta_{13}(b_2\beta_{31} + \beta_{21}\beta_{32})}
\]

(90)
C.2 Derivation of the vendors’ equilibrium strategies

We will consider the optimization problems for each of the cases in Table 9, starting with Case I. In this case both vendors receive limited shelf space, so we know that the shelf space constraints will be tight (i.e., $q_1^* + q_2^* = S_1$ and $q_3^* = S_2$). Substituting $q_1^* = S_1 - q_2^*$ into Vendor 1’s profit function, we have $(w_1 - c_1)(S_1 - q_2) + (w_2 - c_2)q_2$. The derivative of the profit function is $-(w_1 - c_1) + (w_2 - c_2)$, and since Product 1 has the higher profit margin, Vendor 1 wants $q_2$ as small as possible. Therefore, $q_2 = 0$ and $q_1 = S_1$. To obtain the shelf-space conditions to remain in this case where the vendors are constrained, we must ensure that the quantities chosen by the vendors are less than that which the retailer would sell if space were not considered.

In Case II, Vendor 2 has a (relatively) larger amount of shelf space, but Vendor 1 still has a small amount. We know that vendor 1’s shelf-space constraint will be tight, so we follow the same procedure as in Case I to obtain $q_1^* = S_1$ and $q_2^* = 0$. Observation 2 implies that $q_3^* = Q_3(q_1 = S_1, q_2 = 0)$. To obtain the shelf-space conditions to remain in this case, we must ensure that Vendor 1 is constrained in what he would like to supply and Vendor 2 has enough space to supply the retailer’s desired amount given the other two quantities.

In Case III, Vendor 2 has a small amount of space, and Vendor 1 has enough space that he wants to supply both of his products and fill the entire shelf space. Thus, both of the shelf space constraints will be tight, meaning that $q_3^* = S_2$. Observation 2 implies that $q_1^* = Q_1(q_2, q_3 = S_2)$. Substituting this value into Vendor 1’s profit function and differentiating with respect to $q_2$ yields the requirement of the profit margins given in Assumption 3 and the value of $\rho$ in order for Vendor 1 to choose to supply a non-zero amount of Product 2. Since the derivative with respect to $q_2$ is positive, we solve the space constraint $Q_1(q_2, q_3 = S_2) + q_2 = S_1$ for $q_2$ to get

$$q_{2ll}^{III} = \frac{(2S_1 - a_1 + b_1w_1)(b_2b_3 - \beta_{23}\beta_{32}) - (a_2 + \beta_{21}w_1)(b_3\beta_{12} + \beta_{13}\beta_{32})}{2(b_2b_3 - \beta_{23}\beta_{32}) - b_3(\beta_{12} + \beta_{21}) - \beta_{31}\beta_{23} - \beta_{13}\beta_{32}}.$$

The shelf-space conditions imply that Vendor 1 has enough space to want to supply both products, but not enough to provide the unconstrained selling amount; Vendor 2 must be
constrained in what he would like to supply given Vendor 1’s quantities.

Case IV is the same as Case III except that Vendor 2 has ample capacity to stock the maximum amount that the retailer will sell. Vendor 1 determines his allocation by solving $Q_1(q_2) + q_2 = S_1$ for $q_2$ to obtain $q_2^*$. We will denote the solution to this equation as $q_2^{IV}$. The optimal quantity of Product 1 to supply is $q_1 = S_1 - q_2^{IV}$. We need to verify that the derivative of the profit function, $(w_1 - c_1)\frac{\partial Q_1(q_2)}{\partial q_2} + (w_2 - c_2)$, is positive so that Vendor 1 wants to supply a nonzero amount of Product 2. Assumption 3 already ensures that the derivative is positive because $Q_1(q_2) \leq Q_1(q_2, q_3 = S_2)$ since Vendor 1 wants to supply more of Product 1 when there is less supply of Product 3. Observation 2 tells us that $q_3^* = Q_3(q_2 = q_2^{IV})$. The shelf-space conditions show that Vendor 1 supplies both products and fills up the entire space and that Vendor 2 leaves some space empty because he has enough shelf space to supply his desired amount.

In Case V, Vendor 2 has limited shelf space, so $q_3^* = S_2$. Vendor 1 has enough space to supply his resulting unconstrained amounts of Products 1 and 2, so $q_1^* = Q_1(q_3 = S_2)$ and $q_2 = Q_2(q_3 = S_2)$. In Case VI each vendor has enough shelf space, so they supply the retailer’s unconstrained amounts, $q_i = \hat{D}_i$.

\section*{C.3 Proofs of Lemmas and Theorems}

\textbf{Proof of Lemma 16.} Substituting the customer demands into the retailer’s profit function, we obtain

$$\Pi_R(p) = (p_1 - w_1)(a_1 - b_1p_1 + \beta_{12}p_2 + \beta_{13}p_3) + (p_2 - w_2)(a_2 - b_2p_2 + \beta_{21}p_1 + \beta_{23}p_3) + (p_3 - w_3)(a_3 - b_3p_3 + \beta_{31}p_1 + \beta_{32}p_2).$$

To establish that the retailer’s profit function in (91) is concave, we must show that its Hessian is negative (semi-)definite. This means that the determinants of the Hessian’s principal submatrices must be alternating negative (or non-positive) and positive (or non-negative). (For details, see any introductory book on nonlinear optimization such as Rardin...
The Hessian of the retailer’s profit function is

\[
H(p_1, p_2, p_3) = \begin{bmatrix}
-2b_1 & \beta_{12} + \beta_{21} & \beta_{13} + \beta_{31} \\
\beta_{12} + \beta_{21} & -2b_2 & \beta_{23} + \beta_{32} \\
\beta_{13} + \beta_{31} & \beta_{23} + \beta_{32} & -2b_3
\end{bmatrix}.
\] (92)

The determinant of the first \((1 \times 1)\) principal submatrix of (92) is \(-2b_1\), which is less than zero. The determinant of the second \((2 \times 2)\) principal submatrix is \(4b_1b_2 - (\beta_{12} + \beta_{21})^2\). Assumption 2 says that \(b_1 \geq \beta_{12} + \beta_{21}\) and \(b_2 \geq \beta_{12} + \beta_{21}\), so this determinant is positive. The determinant of the entire Hessian in (92) is

\[
2b_2(\beta_{13} + \beta_{31})^2 + 2b_1(\beta_{23} + \beta_{32}) + 2b_3(\beta_{12} + \beta_{21}) + 2(\beta_{12} + \beta_{21})(\beta_{13} + \beta_{31})(\beta_{23} + \beta_{32}) - 8b_1b_2b_3.
\] (93)

We can simplify the expression in (93) to

\[
2b_1[(\beta_{23} + \beta_{32})^2 - b_2b_3] + 2b_2[(\beta_{13} + \beta_{31})^2 - b_1b_3] + 2b_3[(\beta_{12} + \beta_{21})^2 - b_1b_2] + 2[(\beta_{12} + \beta_{21})(\beta_{13} + \beta_{31})(\beta_{23} + \beta_{32}) - b_1b_2b_3].
\] (94)

Assumption 2 implies that each of the terms of (94) in brackets is negative, which makes the entire determinant of \(H(p_1, p_2, p_3)\) negative. The alternating (starting with negative) signs of the determinants of the principal submatrices establishes that Assumption 2 is sufficient for concavity of the retailer’s profit function.

**Proof of Theorem 11.** Suppose that Product 1 is both parties’ preferred good. We will establish the result in this case by contradiction. Under VSCM Vendor 1 stocks only Product 1 when his allocated shelf space is less than or equal to \(Q_1(q_2 = 0, q_3 = S_2)\). Let us assume that the retailer would order \(q_1^* = S_1\) for shelf-space values exceeding \(Q_1(q_2 = 0, q_3 = S_2)\). This implies that she would choose to sell the quantities \((q_1, q_2, q_3) = (q_1^*, 0, S_2)\). If this were the case, Vendor 1 could increase his profit by stocking only Product 1 at the same shelf space since he makes a unit margin of \(w_1 - c_1 > w_2 - c_2\) for every unit the retailer sells. This contradicts the fact that vendor 1 maximizes his profit by switching to selling both products when \(S_1 \geq Q_1(q_2 = 0, q_3 = S_2)\). Consequently, the space level at
which Vendor 1 begins stocking both products cannot be lower than the level at which the retailer would choose to stock both goods. ■

**Proof of Theorem 12.** The retailer considers both space constraints simultaneously when making her stocking decisions in a RCM channel. Each of the vendors only considers his own shelf-space constraint when making the stocking decisions for a VSCM channel. Since the retailer’s constraint set is tighter than each of the vendor’s constraint sets, the retailer-optimal stocking quantities are feasible decisions for the vendors.

Suppose the retailer sets a performance criterion such that the vendors satisfy a minimum profit-per-unit-of-space (or total minimum profit) condition equal to the profit per unit of space that she earns from that vendor’s products from the retailer-optimal stocking quantities. Since the vendors’ profit margins are assumed to be positive (i.e., \( w_i > c_i \)), each vendor’s profit is strictly increasing in the quantity of each product he stocks up until the point at which he stocks more than the retailer will sell. Consequently, he will not benefit from stocking less than the retailer-controlled optimal quantities because he could achieve higher profit by stocking more. He will not stock more than the retailer-optimal quantities because the retailer will choose not to sell the extra units, and the vendor will not be paid for them. It follows that each vendor’s optimal response to this performance criterion is to stock the exact retailer-optimal quantities. ■

**Proof of Theorem 13.** The centralized channel’s optimization problem is similar to the retailer-controlled channel’s problem in (45), but the production costs replace the wholesale prices.

\[
\max_{q_1, q_2, q_3} \sum_{i=1}^{3} (p_i(q_1, q_2, q_3) - c_i)q_i \\
\text{s.t.} \quad q_1 + q_2 \leq S_1 \\
\quad q_3 \leq S_2 \\
\quad q_1, q_2, q_3 \geq 0.
\]

We denote the optimal solution to this problem as \( \mathbf{q}_{Central} = (q_{1 Central}, q_{2 Central}, q_{3 Central}) \).

Now consider a VSCM channel with wholesale prices \( w'_i = \alpha c_i \) for \( i = 1, 2, 3 \) and in which the retailer shares a \( (1 - \alpha) \) fraction of the revenue she receives from each product.
with that product’s manufacturer. The retailer’s optimization problem for selecting the quantities of each product to sell is

\[
\max_{q_1, q_2, q_3} \alpha \left[ \sum_{i=1}^{3} p_i(q_1, q_2, q_3)q_i \right] - w'_1q_1 - w'_2q_2 - w'_3q_3
\]

\[(96)\]

subject to \(0 \leq q_i \leq q_{Vendor}^i\) for \(i = 1, 2, 3\),

where \(q_{Vendor}^i\) denotes the quantity of product \(i\) supplied by the manufacturer of that product. Through parameter substitution, we can rewrite the retailer’s profit function in (96) as \(\Pi_{Retailer}^{RS} = \alpha \left[ \sum_{i=1}^{3} (p_i(q_1, q_2, q_3) - c_i)q_i \right]\), which is equivalent to the centralized supply chain’s profit function in (95) multiplied by \(\alpha\). Consequently, the retailer faces the same sales incentives as the centralized channel, so if the vendors stock enough of each product to ensure feasibility, the retailer’s optimal selling quantities will be equal to the centralized solution, \(q^{Central}\). If the vendors stock too much, then the retailer will correct for this by only selling the centralized quantities.

In establishing the vendors’ Nash Equilibrium strategies, we show that each vendor’s best response to the situation where the other vendor plays a given strategy is to play the given strategy as well. We consider Vendor 2’s optimization problem first, in which he stocks \(q_3\) units of Product 3. Vendor 2’s profit function under revenue sharing (after algebraic simplification and assuming Vendor 1 supplies quantities \(q_1\) and \(q_2\) of Products 1 and 2, respectively) is \(\Pi_{V2}^{RS} = p_3(q_1, q_2, q_3)D_3(p) - c_3q_3 - \alpha Q_3^C(p)(p_3(q_1, q_2, q_3) - c_3)\), where \(p_3(q_1, q_2, q_3)\) is the retail price for Product 3 chosen by the retailer and \(Q_3^C(p)\) is the quantity of Product 3 the retailer and the centralized channel choose to sell. From Observation 2 we know that the vendor will never supply more than the retailer is willing to sell, so we have \(q_3^* = Q_3^C(p)\) and \(\Pi_{V2}^{RS} = (1 - \alpha)(p_3(q_1, q_2, q_3^*) - c_3)q_3^*\). Setting the derivative of vendor 2’s profit function equal to zero yields the vendor’s optimal quantity

\[
q_3^* = \frac{\tilde{p}_3 - c_3}{b_1(b_2b_3 - \beta_{23}\beta_{21}) - \beta_{12}(b_1\beta_{24} + \beta_{23}\beta_{24}) - \beta_{13}(b_3\beta_{24} + \beta_{21}\beta_{24})},
\]

\[(97)\]

where \(\tilde{p}_3\) includes all of the terms of \(p_3(q_1, q_2, q_3)\) that are not dependent on \(q_3\). (That is,
\( \tilde{p}_3 = p_3(q_1, q_2, q_3) + \frac{\partial p_3(q_1, q_2, q_3)}{\partial q_3} q_3. \) If we fix the quantities of Products 1 and 2 and optimize the retailer’s profit in (96) over the \( q_3 \) decision, we obtain

\[
q_3^* = \frac{\tilde{p}_3 - c_3 + \frac{\partial p_1(q_1, q_2, q_3)}{\partial q_3} + \frac{\partial p_2(q_1, q_2, q_3)}{\partial q_3}}{\frac{1}{b_1(b_2b_3-\beta_{23}/\beta_{21})} - \frac{1}{\beta_{12}(b_3b_{21}+\beta_{11}/\beta_{21})} - \frac{1}{\beta_{13}(b_2b_{31}+\beta_{11}/\beta_{21})}}.
\]

which is equivalent to the best response quantity of Product 3 the centralized channel. Since each of the partial derivatives in the numerator of (98) are negative, we see that Vendor 2’s optimal stocking quantity in (97) under revenue sharing is greater than the retailer’s optimal stocking quantity of Product 3. By retaining the retail pricing decision, the retailer will choose not to sell any units supplied over the centralized quantity, so Vendor 2’s best response is to provide exactly the quantity of Product 3 preferred by the centralized channel in all cases.

We apply a similar argument in showing that Vendor 1’s best response if Vendor 2 provides the centralized quantity of Product 3 is to stock the centralized quantities of its own goods as well in Cases I, II, V, and VI. Vendor 1’s best response in Cases III and IV will not necessarily be to provide the centralized quantities; we establish the difference between the equilibrium quantities in the proof of Theorem 14. The revenue-sharing profit function for Vendor 1 will end up being the same in all six vendor cases (regardless of whether revenue sharing coordinates or not), so we will show the derivation for Case III only.

In Case III the vendor supplies the amount of Product 1 that is desired by the retailer given quantities \( q_2 \) and \( q_3 \), which in this case is the same as the centralized channel’s preferred quantity. Given that Vendor 2 stocks \( q_3 \) units of Product 3, Vendor 1’s profit function under revenue sharing in Case III is

\[
\Pi^{RS}_{V_1} = Q^C_1(q_2, q_3) - c_1 q_1 - \alpha [p_1(Q^C_1(q_2, q_3), q_2, q_3) - c_1] Q^C_1(q_2, q_3) + (1 - \alpha) [p_2(Q^C_1(q_2, q_3), q_2, q_3) - c_2] q_2,
\]

where \( Q^C_1(q_2, q_3) \) is the centralized channel’s best response sales amount of Product 1 given the supplies of the other goods. This value is found by replacing \( w_1 \) in \( Q_1(q_2, q_3) \), provided in (85), with \( c_1 \) since the centralized channel optimizes with respect to production cost instead of wholesale price.
The second term in (99) is the only term strictly dependent on \( q_1 \), however; and to increase the supply of Product 1 would increase the vendor’s cost. Thus, Vendor 1 wants to set \( q_1^* \) as small as possible. In order to recognize revenue on \( Q_1^C(q_2, q_3) \), he must supply at least this many units of Product 1; thus, \( q_1^* = Q_1^C(q_2, q_3) \). This is the same result as in Observation 2 that the vendor will never supply more units than the retailer will sell in equilibrium. We can rewrite Vendor 1’s profit function as

\[
\Pi_{V1}^{RS} = (1 - \alpha) \left[ (p_1(q_1, q_2, q_3) - c_1) q_1 + (p_2(q_1, q_2, q_3) - c_2) q_2 \right],
\]

subject to the constraints \( q_1 \leq Q_1^C(q_2, q_3) \) and \( q_2 \leq Q_2^C(q_1, q_3) \), which assure that the vendor does not realize revenue for more units than the retailer is willing to sell.

In Centralized Cases I and II, the centralized channel (and, consequently, the retailer) prefer to stock one of Vendor 1’s products in the entire \( S_1 \) space.\(^2\) Applying a similar methodology as used above for Vendor 2 in Equations (97) and (98), we see that Vendor 1 prefers to stock the same single product at higher levels of shelf space than the centralized channel does; thus, any time the centralized channel prefers to stock only one of Vendor 1’s products, Vendor 1 will follow suit under revenue sharing.

In Centralized Cases V and VI, Vendor 1’s shelf space is so large that the centralized channel can stock its unconstrained quantities of both of the vendor’s products. Consequently, we do not have to consider the shelf-space constraint in solving for the vendor’s optimal strategy in these cases. Setting each of the partial derivatives of (100) equal to zero and solving for the quantities yields

\[
q_1^{V1} = \frac{2(b_1 b_3 - \beta_{13} \beta_{31})(\tilde{p}_1 - c_1) - [b_3(\beta_{12} + \beta_{21}) + \beta_{13} \beta_{32} + \beta_{31} \beta_{23}](\tilde{p}_2 - c_2)}{4(b_1 b_3 - \beta_{13} \beta_{31})(b_2 b_3 - \beta_{23} \beta_{32}) - (b_3(\beta_{12} + \beta_{21}) + \beta_{13} \beta_{32} + \beta_{31} \beta_{23})^2} \quad (101)
\]

\[
q_2^{V1} = \frac{2(b_2 b_3 - \beta_{23} \beta_{32})(\tilde{p}_2 - c_2) - [b_3(\beta_{12} + \beta_{21}) + \beta_{13} \beta_{32} + \beta_{31} \beta_{23}](\tilde{p}_1 - c_1)}{4(b_1 b_3 - \beta_{13} \beta_{31})(b_2 b_3 - \beta_{23} \beta_{32}) - (b_3(\beta_{12} + \beta_{21}) + \beta_{13} \beta_{32} + \beta_{31} \beta_{23})^2} \quad (102)
\]

where \( \tilde{p}_1 \) and \( \tilde{p}_2 \) are defined analogously to \( \tilde{p}_3 \) above. The quantities that maximize the

\(^2\)Note that the dividing line for the shelf space values under which revenue sharing is guaranteed to work is determined by the centralized cases. The overall results is still the same—revenue sharing coordinates the channel when vendor shelf space is limited or ample.
retailer’s profit (and the centralized channel’s profit) in (96) for fixed values of \( q_3 \) are

\[
q_{Retailer,1} = \frac{2(b_1 b_3 - \beta_{13}\beta_{31})(\bar{p}_1 - c_1 + \frac{\partial p_1(q_1,q_2,q_3)}{\partial q_1})}{4(b_1 b_3 - \beta_{13}\beta_{31})(b_2 b_3 - \beta_{23}\beta_{32})-(b_3(b_1 + \beta_{21})+\beta_{13}\beta_{32}+\beta_{31}\beta_{23})^2} - \frac{b_1(b_3 b_2 + \beta_{31}\beta_{23})-\beta_{13}(b_2 b_3 + \beta_{21}\beta_{32})}{b_1(b_2 b_3 - \beta_{23}\beta_{32})-\beta_{12}(b_3 b_2 + \beta_{31}\beta_{23})-\beta_{13}(b_2 b_3 + \beta_{21}\beta_{32})}.
\]

\[
q_{Retailer,2} = \frac{2(b_2 b_3 - \beta_{23}\beta_{32})(\bar{p}_2 - c_2 + \frac{\partial p_2(q_1,q_2,q_3)}{\partial q_2})}{4(b_1 b_3 - \beta_{13}\beta_{31})(b_2 b_3 - \beta_{23}\beta_{32})-(b_3(b_1 + \beta_{21})+\beta_{13}\beta_{32}+\beta_{31}\beta_{23})^2} - \frac{b_2(b_3 b_1 + \beta_{31}\beta_{23})-\beta_{12}(b_2 b_3 + \beta_{21}\beta_{32})}{b_1(b_2 b_3 - \beta_{23}\beta_{32})-\beta_{12}(b_3 b_2 + \beta_{31}\beta_{23})-\beta_{13}(b_2 b_3 + \beta_{21}\beta_{32})}.
\]

We want to show that the vendor’s optimal quantities are each larger than the retailer’s (and centralized channel’s) quantities; thus, we want to have \( q_1^{V1} - q_{1,\text{Retailer}} \geq 0 \) and \( q_2^{V1} - q_{2,\text{Retailer}} \geq 0 \). These conditions simplify to

\[
b_1 b_3 \geq \beta_{13}\beta_{31} + \frac{(b_3(b_1 + \beta_{21})+\beta_{13}\beta_{32}+\beta_{31}\beta_{23})(b_1\beta_{32} + \beta_{12}\beta_{31})}{2(b_2\beta_{31} + \beta_{21}\beta_{32})}.
\]

\[
b_2 b_3 \geq \beta_{23}\beta_{32} + \frac{(b_3(b_1 + \beta_{21})+\beta_{13}\beta_{32}+\beta_{31}\beta_{23})(b_2\beta_{31} + \beta_{21}\beta_{32})}{2(b_1\beta_{32} + \beta_{12}\beta_{31})}.
\]

The parameter set when the two above conditions may not hold occurs when the \( b_i \) terms are as small as possible and the \( \beta_{j,k} \) terms as large as possible. In accordance with Assumption 2, we set \( \beta_{j,k} = \delta \) for \( j = 1, 2, 3 \) and \( j \neq k \). Each of the \( b_i \) terms is equal to \( 2\delta \) in order to make them as small as possible. Using these parameter values, (105) becomes

\[
4\delta^2 \geq \delta^2 + \frac{(4\delta^2 + \delta^2 + \delta^2)(2\delta^2 + \delta^2)}{2(2\delta^2 + \delta^2)} = 4\delta^2.
\]

Since (105) holds for this most-constrained parameter set, it also holds when some of the \( \beta_{j,k} \) terms are smaller and/or the \( b_i \) terms are larger. The same result holds for (106) by an analogous derivation.

We have now shown that Vendor 1 has the incentive to overstock both products in Cases V and VI under revenue sharing compared with the quantities that the retailer chooses to sell. We must still show that the vendor’s best response is to stock quantities of each product equal to the retailer’s preferred amounts, which are equal to the centralized quantities. It is clear that Vendor 1’s stocking levels should not exceed the retailer’s preferred amounts for both products, because the retailer would still sell her preferred amounts in this case and
the vendor would incur the production cost for the unsold units. It is possible, however, that Vendor 1 could earn a higher profit by stocking fewer than the retailer’s optimal quantity of one product and more of the other in Cases V and VI. We conclude the proof for these cases by showing that Vendor 1 attains higher profit by supplying the centralized quantities of each product instead of by undersupplying one and oversupplying the other with respect to the centralized equilibrium.

Consider first that Vendor 1 stocks $q_2 \leq q_2^{Retailer}$ of Product 2. The retailer is willing to sell a maximum of $q_1 = Q_1^C(q_2, q_3)$ units of Product 1 in response. Since Vendor 1’s profit function given $q_2 \leq q_2^{Vendor}$ is concave (by a similar derivation as that of Lemma 16), Vendor 1 achieves maximal profit by supplying $q_1 \geq q_1^{Vendor}(\geq Q_1^C(q_2, q_3))$ if the retailer were to sell the entire supply she receives. Since the retailer will not pay for units that she does not sell, Vendor 1 should provide the maximum amount of Product 1 that the retailer will sell; thus, $q_1 = Q_1^C(q_2, q_3)$. Substituting the best response sales amount of Product 1 into (100) yields

$$\Pi_{RS}^{V_1} = (1 - \alpha) \left[ (p_1(q_2, q_3) - c_1) Q_1^C(q_2, q_3) + (p_2(q_2, q_3) - c_2) q_2 \right]. \quad (107)$$

We know that (107) attains its unconstrained maximum at $q_2^* = q_2^{Vendor}$ in (102), since we are still optimizing the same profit function. If we can show that (107) is concave, we can prove that the optimal quantity for Product 2 with respect to the feasible set of $q_2 \leq Q_2^C(Q_1^C(q_2, q_3), q_3) = q_2^{Retailer}$ is $q_2^* = q_2^{Retailer}$ since the entire feasible set lies to the left of the unconstrained maximum.

The second derivative of (107) is

$$\frac{\partial^2 \Pi_{RS}^{V_1}}{\partial q_2^2} = 2(1 - \alpha) \left[ \frac{\partial p_1(q_2, q_3)}{\partial q_2} \frac{\partial Q_1^C(q_2, q_3)}{\partial q_2} + \frac{\partial p_2(q_2, q_3)}{\partial q_2} \right],$$

since all of the second partial derivatives of the price or quantity terms are zero because the prices and quantities are all linear in $q_2$. Substituting the actual values for the partial derivatives and simplifying yields the following concavity condition

$$2[2b_1b_3(b_2b_3 - \beta_{23} \beta_{32}) - 2b_3b_13(b_2 \beta_{31} + \beta_{21} \beta_{32}) - (b_3 \beta_{21} + \beta_{31} \beta_{23})(b_3(\beta_{12} + \beta_{21}) + \beta_{31} \beta_{23} - \beta_{13} \beta_{23})] \geq (b_3(\beta_{12} + \beta_{21}) + \beta_{13} \beta_{32} - \beta_{31} \beta_{23})(b_3(\beta_{12} + \beta_{21}) + \beta_{13} \beta_{32} + \beta_{31} \beta_{23}). \quad (108)$$
As before, this inequality has the greatest chance of being violated when \( \beta_{j,k} = \delta \) for \( j = 1, 2, 3 \) and \( j \neq k \) and \( b_i = 2\delta \) for \( i = 1, 2, 3 \). Substituting these values in (108) yields 0 = 0. For any other parameter values in accordance with Assumption 2, the left side of (108) is larger than the right side; thus, the vendor’s profit function is concave. As a result, Vendor 1 maximizes his profit by supplying the optimal retailer and centralized quantities of each product. The proof is analogous for all values of \( q_1 \leq q_1^{Retailer} \). Consequently, Vendor 1 stocks the centralized quantities in Cases V and VI (which are determined by the centralized channel’s decision problem); this results in a coordinated supply chain. Cases III and IV, in which coordination is not guaranteed under revenue sharing, are addressed in the proof of Theorem 14. ■

**Proof of Theorem 14.** In Case III, Vendor 1’s shelf-space constraint will be tight in both the vendor’s optimization problem as well as the centralized channel’s. We can substitute \( q_1^* = S_1 - q_2 \) into Vendor 1’s profit function in (100) to yield

\[
\Pi_{V1}^{RS} = (1 - \alpha) \left[ (p_1(S_1 - q_2, q_2, q_3) - c_1)(S_1 - q_2) + (p_2(S_1 - q_2, q_2, q_3) - c_2)q_2 \right].
\]  

(109)

Setting the derivative of (109) equal to zero gives us the optimal vendor quantity

\[
q_{Vendor}^{2} = \frac{(\tilde{p}_2 - c_2) - (\tilde{p}_1 - c_1) + \frac{\partial p_1(S_1 - q_2, q_2, q_3)}{\partial q_2} S_1 + \frac{\partial p_2(S_1 - q_2, q_2, q_3)}{\partial q_2} q_2}{2\frac{\partial p_1(S_1 - q_2, q_2, q_3)}{\partial q_2} - \frac{\partial p_2(S_1 - q_2, q_2, q_3)}{\partial q_2}},
\]

(110)

where \( \tilde{p}_1 \) and \( \tilde{p}_2 \) are defined, as before, to contain all of the terms of each price that are not dependent on \( q_2 \). The centralized channel’s optimal quantity is found by substituting \( q_1 = S_1 - q_2 \) into (95) and optimizing to yield

\[
q_{Central}^{2} = \frac{(\tilde{p}_2 - c_2) - (\tilde{p}_1 - c_1) + \frac{\partial p_1(S_1 - q_2, q_2, q_3)}{\partial q_2} S_1 + \frac{\partial p_2(S_1 - q_2, q_2, q_3)}{\partial q_2} q_3}{2\frac{\partial p_1(S_1 - q_2, q_2, q_3)}{\partial q_2} - \frac{\partial p_2(S_1 - q_2, q_2, q_3)}{\partial q_2}}.
\]

(111)

It is easily seen that the difference between the two quantities, \( q_{Vendor}^{2} \) and \( q_{Central}^{2} \), is

\[
\frac{\partial p_2(S_1 - q_2, q_2, q_3)}{\partial q_2} q_3 - \frac{\partial p_1(S_1 - q_2, q_2, q_3)}{\partial q_2} q_3 \frac{\partial p_2(S_1 - q_2, q_2, q_3)}{\partial q_2},
\]

which is equal to the parameterized fraction given in (46). Since \( q_1^* = S_1 - q_2^* \) in each case, the optimal stocking quantities for Product 1 will differ by the same amount. ■
REFERENCES


[35] Fox Theatre, Phone conversation held on March 6, Atlanta, GA, 2006.


