In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institute shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the Dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

7/25/68
A MOMENT TECHNIQUE FOR
SUBOPTIMAL ADAPTIVE NONLINEAR FILTERING

Approved:

Chairman:

Date approved by Chairman: Dec 15, 1969
ACKNOWLEDGMENTS

I wish to express my sincere appreciation to Dr. James R. Rowland, my thesis advisor, for his guidance, assistance, and encouragement during the development of this dissertation. Appreciation is also extended to Drs. Joseph L. Hammond and Cecil O. Alford for their services as members of the reading committee. In addition, I wish to acknowledge the assistance of Dr. Kendall L. Su, my program advisor, who gave me much invaluable advice and encouragement during the period of my doctoral studies.

Special appreciation is given to Professor Chao Shu, Head of the Department of Electrical Engineering, National Taiwan University, for initiating my study program in the United States and encouraging me to complete my doctoral program at the Georgia Institute of Technology.

Finally, to my parents and my wife goes my deepest appreciation for their understanding and patience. Without their support and encouragement this dissertation would not have been possible.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Acknowledgments</th>
<th>ii</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Tables</td>
<td>v</td>
</tr>
<tr>
<td>List of Illustrations</td>
<td>vi</td>
</tr>
<tr>
<td>Summary</td>
<td>viii</td>
</tr>
<tr>
<td><strong>CHAPTER</strong></td>
<td></td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>Background</td>
<td></td>
</tr>
<tr>
<td>Development of the General Filtering Equation</td>
<td></td>
</tr>
<tr>
<td>Specific Applications for Nonlinear Filtering</td>
<td></td>
</tr>
<tr>
<td>Approach to the Filtering Problem</td>
<td></td>
</tr>
<tr>
<td>Outline of the Thesis</td>
<td></td>
</tr>
<tr>
<td>II. THE METHOD OF MOMENTS</td>
<td>17</td>
</tr>
<tr>
<td>Statement of the Problem</td>
<td></td>
</tr>
<tr>
<td>Use of Polynomial Approximations</td>
<td></td>
</tr>
<tr>
<td>The Updating Algorithm</td>
<td></td>
</tr>
<tr>
<td>Summary and Conclusions</td>
<td></td>
</tr>
<tr>
<td>III. THE ZERO-ORDER PROBLEM</td>
<td>33</td>
</tr>
<tr>
<td>Solution Via the Moment Technique</td>
<td></td>
</tr>
<tr>
<td>An Example</td>
<td></td>
</tr>
<tr>
<td>Execution Time Versus Number of Moments</td>
<td></td>
</tr>
<tr>
<td>Comparison with the Linear Filter</td>
<td></td>
</tr>
<tr>
<td>The Effect of Correlation Between x and v</td>
<td></td>
</tr>
<tr>
<td>The Non-Additive Noise Case</td>
<td></td>
</tr>
<tr>
<td>Conclusions</td>
<td></td>
</tr>
<tr>
<td>IV. THE FIRST-ORDER DYNAMICAL PROBLEM</td>
<td>62</td>
</tr>
<tr>
<td>Solution of the Problem</td>
<td></td>
</tr>
<tr>
<td>An Example</td>
<td></td>
</tr>
<tr>
<td>Comparison with the Kalman Filter</td>
<td></td>
</tr>
<tr>
<td>Application to a Nonlinear System</td>
<td></td>
</tr>
<tr>
<td>Application to a Slowly Time-Varying System</td>
<td></td>
</tr>
<tr>
<td>Conclusions</td>
<td></td>
</tr>
<tr>
<td>CHAPTER</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>V. THE SECOND-ORDER DYNAMICAL PROBLEM</td>
<td>31</td>
</tr>
<tr>
<td>Approximation of a Joint Density Function by a Polynomial</td>
<td></td>
</tr>
<tr>
<td>Solution of the Second-Order Problem</td>
<td></td>
</tr>
<tr>
<td>An Example</td>
<td></td>
</tr>
<tr>
<td>Conclusions</td>
<td></td>
</tr>
<tr>
<td>VI. CONCLUSIONS AND RECOMMENDATIONS</td>
<td>110</td>
</tr>
<tr>
<td>Conclusions</td>
<td></td>
</tr>
<tr>
<td>Recommendations for Further Work</td>
<td></td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>114</td>
</tr>
<tr>
<td>VITA</td>
<td>116</td>
</tr>
<tr>
<td>Table</td>
<td>Description</td>
</tr>
<tr>
<td>-------</td>
<td>----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>1.</td>
<td>The Ensemble Average of $e^2$ for the Zero-Order Example</td>
</tr>
<tr>
<td>2.</td>
<td>Number of Operations Required for Data Accumulation and Signal Estimation</td>
</tr>
<tr>
<td>3.</td>
<td>Number of Operations for the Updating Scheme</td>
</tr>
<tr>
<td>4.</td>
<td>Typical Machine Time/Step in $\mu$sec</td>
</tr>
</tbody>
</table>
# LIST OF ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Vertical Position Estimation from Angular Measurements</td>
<td>13</td>
</tr>
<tr>
<td>2.</td>
<td>Transformation of $y_k$ into $\theta_k$ by (1.38)</td>
<td>13</td>
</tr>
<tr>
<td>3.</td>
<td>The Nonlinear Filter to be Designed</td>
<td>19</td>
</tr>
<tr>
<td>4.</td>
<td>Square Error Versus Order of Highest Moment Used for Approximating (2.23) by Legendre Polynomials</td>
<td>27</td>
</tr>
<tr>
<td>5.</td>
<td>A Flow Chart Showing the Updating of the Moments of $z$</td>
<td>30</td>
</tr>
<tr>
<td>6.</td>
<td>Distribution Region of $x_k, y_k$ for $x \in (x_{\text{min}}, x_{\text{max}})$ and $v \in (-1, 1)$</td>
<td>36</td>
</tr>
<tr>
<td>7.</td>
<td>Flow Chart for the Zero-Order Problem</td>
<td>40</td>
</tr>
<tr>
<td>8.</td>
<td>The Ensemble Average of $e^2$ for the Zero-Order Example</td>
<td>49</td>
</tr>
<tr>
<td>9.</td>
<td>Time Average of $e_0^2$ for the Zero-Order Example</td>
<td>47</td>
</tr>
<tr>
<td>10.</td>
<td>The Plot of $e^2$ vs $n_s$ for (a) $J = 10$, (b) $J = 0$</td>
<td>49, 50</td>
</tr>
<tr>
<td>11.</td>
<td>True Values and Updated Values of the Even Order Moments of $x$</td>
<td>56</td>
</tr>
<tr>
<td>12.</td>
<td>Plots of $e_n^2$ Versus $n_s$ Showing the Effects of Correlations</td>
<td>56</td>
</tr>
<tr>
<td>13.</td>
<td>Distribution Region of $x, z$ for the Non-Additive Noise Example</td>
<td>58</td>
</tr>
<tr>
<td>14.</td>
<td>A Plot of $e_n^2$ Versus $n_s$ for the Non-Additive Noise Example</td>
<td>58</td>
</tr>
<tr>
<td>15.</td>
<td>The Plot of $e_n^2$ Versus $n_s$ for the First-Order Example Without Updating</td>
<td>72</td>
</tr>
<tr>
<td>16.</td>
<td>The Plot of $e_n^2$ Versus $n_s$ for First-Order Example</td>
<td>73</td>
</tr>
<tr>
<td>17.</td>
<td>Piecewise Approximation of $h$ Given by (4.34)</td>
<td>78</td>
</tr>
<tr>
<td>18.</td>
<td>The Plot of $e_n^2$ Versus $n_s$ for the Time-Varying System</td>
<td>79</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>19.</td>
<td>Minimum Rectangle Containing the Distribution Region of $x$</td>
<td>89</td>
</tr>
<tr>
<td>20.</td>
<td>Minimum Rectangle Containing the Distribution Region of $x_1</td>
<td>Z_1$</td>
</tr>
<tr>
<td>21.</td>
<td>Reachable Set of $x_2</td>
<td>Z_1$ and the Associated Rectangle</td>
</tr>
<tr>
<td>22.</td>
<td>The Plot of $e^2$ Versus $n_s$ for the Second-Order Example</td>
<td>107</td>
</tr>
</tbody>
</table>
SUMMARY

This dissertation describes a moment algorithm for estimating the states of noisy linear, time-invariant discrete systems having nongaussian disturbances. It is assumed that the input signal is ergodic white noise with known finite bounds, but otherwise unknown. The measurement noise on the output is ergodic white noise having a known probability density function over a finite range. Both signal and measurement noises are permitted to be nongaussian. The system is assumed to be operating in the steady-state condition.

The moment technique is combined with the reachable set concept in applying the Bayesian decision rule and the least mean-square error criterion. To obtain a computationally feasible suboptimal solution, all density functions are either expressed directly in terms of their moments or approximated by polynomials whose coefficients are functions of the moments. Formulas relating the moments of the observed data, the states, and the input signal are established. Thus, information for the density function of the input signal is obtained as the moments of the measured data are formed. An arbitrary initial guess for the input signal density function, and thus for the moments of the input signal, is necessary. This density function is then updated sequentially as additional data are obtained. Following the updating of the input signal, the moments of the conditional states are automatically updated to yield an adaptive filtering algorithm.
The suboptimal adaptive nonlinear filter is applied to zero-order, first-order, and second-order problems. After the basic algorithm has been established for the zero-order problem, it is shown that the result is not seriously affected by a significant amount of correlation between the input signal and the measurement noise. However, the amount of correlation does influence the proper number of moments to be used in the new algorithm. The filter is then applied to a class of zero-order non-additive noise problems. In addition, the algorithm is applied to a nonlinear first-order system by linearizing over the range of operation and to a time-varying first-order system by averaging the time-varying parameter over several sampling intervals. These examples demonstrate that the new filtering algorithm performs satisfactorily even in those cases where the basic assumptions are only approximately satisfied. The reachable set concept is utilized for estimating the states in both first-order and second-order dynamical systems. Examples show that the new filtering algorithm developed in this dissertation compares favorably with the Kalman filter, especially in those systems where the input signal is much different from gaussian. In such cases the suboptimal adaptive nonlinear filter yields a much lower mean-square estimation error than the Kalman filter.
CHAPTER I

INTRODUCTION

An important class of problems in communications and control systems is the statistical estimation of the states of noisy dynamical systems. Such problems arise, for example, in determining the optimum controllers for many practical systems. In much of the recent work in control theory, it is necessary to assume that the states of the system are either exactly measurable or may be calculated instantaneously from other exactly known quantities. Because of external disturbances, instrument noise, and the intrinsic difficulty of physically performing the unbounded operation of differentiation, the assumption that the states of the system are known exactly is almost never justified.

During the 1960’s linear filtering theory, featuring the celebrated Kalman filter, has been used with some success. For example, the Kalman filter is a part of the navigational control system on Lockheed’s C-5A. A general assumption in the linear filtering problem is that both the signal and noise are gaussian. In this dissertation the gaussian assumption is relaxed, and the resulting optimal filter is no longer linear. Some previous results were reported on nonlinear filtering, but the basic computational problem was unsolved.

The nonlinear filtering problem is a subset of the general nonlinear estimation problem. Suppose the signal and noise at $t = kT$, where $k$ is some positive integer and $T$ is the sampling period, are
given by $x_k$ and $v_k$, respectively, and only the sum $z_k = x_k + v_k$ can be observed. The estimation problem is to determine the best value of $x_k$ after observing the values $z_1, z_2, \ldots, z_n$. The problem is classified as a data-smoothing problem if $k < n$, as a prediction problem if $k > n$, and as a filtering problem if $k = n$. The objective of this dissertation is to develop a new technique based on a method of moments to yield a feasible suboptimal solution for the nonlinear filtering problem.

**Background**

The general filtering problem has been a subject of research for several years. In the late 1940's, Wiener (1) posed a solution to the linear filtering problem in the form of an integral equation referred to as the Wiener-Hopf equation. He also gave a method involving spectral factorization for solving this equation in the important special case of stationary statistics and rational spectra. Following Wiener's basic work, Zadeh and Ragazzini (2) obtained the Wiener-Hopf equation for the finite memory case, and Booton (3) discussed the applicability of the integral equation for the nonstationary case. Thus, the linear filtering problem was reduced to the problem of solving the appropriate Wiener-Hopf integral equation.

With little exception, the nonstationary Wiener-Hopf integral equation has withstood attempts to solve it. Recently, Kalman and Bucy (4,5) developed a procedure which applies without modification to stationary or nonstationary statistics and finite or infinite smoothing intervals. The solutions by Wiener and by Kalman and Bucy are equivalent, and both are based on the assumptions that the signal to be
estimated is the output of a linear system driven by white noise and that the signal is corrupted by additive white noise. When the signal and noise are both gaussian, the Wiener-Kalman filter is indeed the optimal filter. However, its application (6,7) is limited due to the restrictions imposed by the assumptions mentioned above.

In recent years the nonlinear filtering problem has been receiving an increasing amount of attention in the engineering literature. Using the calculus of variations approach, Bryson and Frazier (8) derived a maximum likelihood filter with the assumption that the input and the corrupting noise in the measurement are both white noise. Cox (9) derived a continuous minimum mean-square error filter by the dynamic programming approach with a white gaussian noise input and measurement. Cox (10) also derived a discrete version. As in most nonlinear filtering schemes, the differences in these derivations are the criteria of optimality and the approximations used.

Schwartz and Stear (11) made some interesting computational comparisons and pointed out that no particular approximate nonlinear filter is consistently better than any other nonlinear filter, although nonlinear filters are in general better than a strictly linear filter, e.g. the Kalman-Bucy filter. Ho and Lee (12,13) approached the problem from the viewpoint of Bayesian estimation theory, which requires the fewest assumptions and restrictions on the statistics of the signal and noise as well as on the dynamical system. They gave an estimation algorithm for discrete systems and showed the Wiener-Kalman filter for linear estimation with gaussian signal and noise as a special example.
Schweppe (14) initiated a completely new concept. He assumed that the input to the dynamical system and the observation errors were completely unknown except for bounds on their magnitude or energy. Making use of the concept of reachable states, Schweppe gave a recursive algorithm to calculate a time-varying ellipsoid that always contains the system's actual state.

Although many variations of nonlinear filtering have been developed, the computational problems associated with nonlinear estimation are far from being solved. Generally the solutions are so complicated that they cannot be instrumented as on-line algorithms.

**Development of the General Filtering Equation**

As mentioned in the previous section, Ho and Lee gave the most general solution to the filtering problem. Since their work is significant for the development of the suboptimal filter in later chapters, a simplified version of the Ho and Lee result tailored to the problem of this thesis will be outlined in this section. In addition, the special case of the Kalman filter is included for later use in comparing with the proposed suboptimal adaptive scheme.

**Problem Statement**

It is assumed that at any stage \( k + 1 \), the following data are given as a result of the previous computation or as part of the problem statement:

1. The system equations governing the evaluation of states are given by
\[ x_{k+1} = g(x_k, w_k) \]  
\[ z_{k+1} = h(x_{k+1}, v_{k+1}) \]

where at stage \( k+1 \), \( x_{k+1} \) is an \( n \)-vector representing the states of the system, and \( v_{k+1} \) and \( z_{k+1} \) are \( m \)-vectors representing measurement noise and measurement data, respectively. Moreover, \( w_k \) is a \( p \)-vector representing the input signal at stage \( k \). The letters \( g \) and \( h \) represent, respectively, \( n \)- and \( m \)-vector functions of the indicated arguments.

(2) The complete set of measurements is given and denoted by

\[ Z_{k+1} = (z_1, z_2, \ldots, z_{k+1}) \]

(3) The following density functions are known

\[ f_{x_k|z_k}(x_k|z_k) \]
\[ f_{w_k}(w_k) \]
\[ f_{v_{k+1}}(v_{k+1}) \]

where \( w \) and \( v \) are independent white random discrete time series. In addition, it is assumed that \( w_k \) and \( v_{k+1} \) are independent of the states for all \( k \).

The problem is to estimate the states \( x_{k+1} \) based on the complete set of measurement data \( Z_{k+1} \).
The Bayesian Solution

It is well known (15) that the best estimate of $x_{k+1}$, given $Z_{k+1}$, in the sense of least mean-square error is

$$\hat{x}_{k+1} = E\left[ x_{k+1} \mid Z_{k+1} \right]$$

$$= \int_{-\infty}^{\infty} x_{k+1} \cdot f_{x_{k+1} \mid Z_{k+1}} \left( x_{k+1} \mid Z_{k+1} \right) \, dx_{k+1}$$

where $f_{x_{k+1} \mid Z_{k+1}} \left( x_{k+1} \mid Z_{k+1} \right)$ may also be expressed as

$$f_{x_{k+1} \mid Z_{k+1}} \left( x_{k+1} \mid Z_{k+1} \right) = \frac{f_{Z_{k+1} \mid x_{k+1}, Z_k} \left( x_{k+1} \mid Z_{k+1}, Z_k \right) \cdot f_{x_{k+1} \mid Z_k} \left( x_{k+1} \mid Z_k \right)}{f_{Z_{k+1} \mid x_{k+1}, Z_k} \left( Z_{k+1} \mid x_{k+1}, Z_k \right)}$$

(1.8)

Due to assumption (3) and (1.2), it is seen that with $x_{k+1}$ given $Z_{k+1}$ is independent of $Z_k$. Hence, (1.8) becomes

$$f_{x_{k+1} \mid Z_{k+1}} \left( x_{k+1} \mid Z_{k+1} \right) = \frac{f_{Z_{k+1} \mid x_{k+1}, Z_k} \left( x_{k+1} \mid Z_{k+1}, Z_k \right) \cdot f_{x_{k+1} \mid Z_k} \left( x_{k+1} \mid Z_k \right)}{f_{Z_{k+1} \mid x_{k+1}, Z_k} \left( Z_{k+1} \mid x_{k+1}, Z_k \right)}$$

(1.9)

The problem is to evaluate each of the three density functions on the right hand side of (1.9) in terms of known quantities.

Assuming that (1.2) can be inverted to give $v_{k+1}$, i.e.,
\[ v_{k+1} = h^{-1}(z_{k+1}, x_{k+1}) \]  
(1.10)

where \( x_{k+1} \) is considered to be a fixed vector in this calculation, then

\[ f_{z_{k+1} | x_{k+1}}(z_{k+1} | x_{k+1}) = f_{v_{k+1}}(v_{k+1} = h^{-1}(z_{k+1}, x_{k+1}) | x_{k+1}) |\text{det}(\frac{\partial v_{k+1}}{\partial z_{k+1}})| \]  
(1.11)

The second density function \( f_{x_{k+1} | z_{k}}(x_{k+1} | z_{k}) \) in (1.9) can be obtained from \( f_{w_{k+1} | x_{k}}(w_{k+1} | x_{k} | z_{k}) \) by the transformation given by (1.1), i.e.,

\[ x_{k+1} | z_{k} = g(x_{k} | z_{k}, v_{k}) \]  
(1.12)

with

\[ f_{w_{k+1} | z_{k}}(w_{k+1} | z_{k}) = f_{w_{k}}(w_{k}) \cdot f_{x_{k} | z_{k}}(x_{k} | z_{k}) \]  
(1.13)

which is valid because \( w_{k}, v_{k}, \) and \( x_{k} \) are mutually independent for all \( k \). Thus, only the denominator in (1.9) is yet to be found. Since the total probability of any random variable, e.g. \( x_{k+1} | z_{k} \), is unity, one has

\[ \int_{\mathbb{R}} f_{v_{k+1} | z_{k}}(v_{k+1} | z_{k}) \, dv_{k+1} = 1 \]  
(1.14)

Therefore, using (1.14) in (1.9), one may write

\[ f_{z_{k+1} | z_{k}}(z_{k+1} | z_{k}) = \int_{\mathbb{R}} f_{z_{k+1} | x_{k+1}}(z_{k+1} | x_{k+1}) \cdot f_{x_{k+1} | z_{k}}(x_{k+1} | z_{k}) \, dx_{k+1} \]  
(1.15)
Alternately, \( f_{x_{k+1} | Z_{k}} (z_{k+1} | Z_{k}) \) can also be obtained from

\[
f_{x_{k+1} v_{k+1} | Z_{k}} (x_{k+1} , v_{k+1} \mid Z_{k})
\]

by the transformation given by (1.2), i.e.,

\[
f_{x_{k+1} v_{k+1} | Z_{k}} (x_{k+1} , v_{k+1} \mid Z_{k}) = f_{x_{k+1} | Z_{k}} (x_{k+1} \mid Z_{k}) f(v_{k+1})
\]

with

\[
f_{x_{k+1} v_{k+1} | Z_{k}} (x_{k+1} , v_{k+1} \mid Z_{k}) = h(x_{k+1} \mid Z_{k}, v_{k+1})
\]

since \( x_k , v_k \) are independent for all \( k \), and \( v \) is white noise.

After these three density functions have been evaluated in the above manner, \( f_{x_{k+1} | Z_{k+1}} (x_{k+1} \mid Z_{k+1}) \) may be obtained from (1.9), and \( \dot{x}_{k+1} \) may be computed from (1.7).

The Bayesian rule in (1.9) will be used extensively in Chapters IV and V, where the filtering algorithm of Chapter III will be applied to dynamical systems.

The Kalman Filter

The results presented earlier in this section may also be used to derive the Kalman filter, which applies when the physical model in (1.1)-(1.2) is given by

\[
x_{k+1} = \Phi x_k + \Gamma w_k
\]

\[
z_{k+1} = H x_{k+1} + v_{k+1}
\]

where \( w \) and \( v \) are independent, white gaussian random sequences, \( \Phi \) is an \( n \) by \( n \) matrix, \( \Gamma \) is an \( n \) by \( p \) matrix, and \( H \) is an \( m \) by \( n \) matrix. Cor-
responding to (1.4), it is assumed that the density function

$$f_{x_k|Z_k}(x_k|Z_k)$$  \hfill (1.20)

is gaussian. The mean value and the covariance matrix of $x_k|Z_k$ are denoted by $\hat{x}_k$ and $P_k$, respectively, i.e.

$$E(x_k|Z_k) = \hat{x}_k$$  \hfill (1.21)

$$E[(x_k-\hat{x}_k)^2|Z_k] = Cov(x_k|Z_k) = P_k$$  \hfill (1.22)

The mean value and covariance matrices of $w_k$ and $v_{k+1}$ are given by

$$E(w_k) = E(v_{k+1}) = 0$$  \hfill (1.23)

$$Cov(w_k) = Q; \quad Cov(v_{k+1}) = R$$  \hfill (1.24)

where $Q$ is a $p$ by $p$ matrix and $R$ is an $m$ by $m$ matrix.

The desired result may be obtained by applying (1.9) directly to the system given by (1.18) and (1.19). The evaluation of the three density functions involved in the solution of (1.9) yields the Kalman recursive filtering equations. From the given conditions (1.20)-(1.24) and the linearity of the system (1.18), it is noted that $f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)$ is gaussian with

$$E(x_{k+1}|Z_k) = F_{k} \hat{x}_k$$  \hfill (1.25)

$$Cov(x_{k+1}|Z_k) = F_k P_k F_k^T + Q_{k+1}$$  \hfill (1.26)
Similarly, \( f_{\tilde{z}_{k+1} | z_k} (z_{k+1} | z_k) \) is gaussian with

\[
E(z_{k+1} | z_k) = H \hat{x}_k
\]

\[
\text{Cov}(z_{k+1} | z_k) = H M_{k+1} H^T + R
\]

(1.27)

(1.28)

Finally, by the same reasoning, \( f_{\tilde{x}_{k+1} | x_{k+1}} (x_{k+1} | x_{k+1}) \) is also gaussian with

\[
E(x_{k+1} | x_{k+1}) = H x_{k+1}
\]

\[
\text{Cov}(x_{k+1} | x_{k+1}) = R
\]

(1.29)

(1.30)

Inserting (1.25)-(1.30) into (1.29), one obtains

\[
f_{\tilde{x}_{k+1} | z_k} (x_{k+1} | z_k) = \frac{1}{(2\pi)^{n/2}|R|^{1/2}|M_{k+1}|^{1/2}}
\]

\[
\cdot \exp \left\{ -\frac{1}{2} \left[ (x_{k+1} - \hat{x}_k) M_{k+1}^{-1} (x_{k+1} - \hat{x}_k) + (z_{k+1} - H \hat{x}_k) (H M_{k+1} H^T + R)^{-1} (z_{k+1} - H \hat{x}_k) \right] \right\}
\]

(1.31)

Completing the square in the exponent of (1.31) gives
\[
\mathcal{P}_{x_k}^+ = f(x_k^+ | z_{k+1}) = \exp \left\{ -\frac{1}{2} (x_{k+1} - \hat{x}_{k+1}) P_{k+1}^{-1} (x_{k+1} - \hat{x}_{k+1}) \right\}
\]

where

\[
P_{k+1}^{-1} = M_{k+1}^{-1} + H^T R^{-1} H
\]

Using the matrix inversion lemma, one may rewrite (1.34) as

\[
P_{k+1} = M_{k+1} - M_{k+1} H^T (H M_{k+1} H^T + R)^{-1} H M_{k+1}
\]

Therefore, the recursive formula for the best estimate in (1.21) is

\[
\hat{x}_{k+1} = \hat{x}_k + M_{k+1} H^T (H M_{k+1} H^T + R)^{-1} (z_{k+1} - H \hat{x}_k)
\]

Equations (1.33)-(1.36) are referred to as the discrete Kalman filtering equations.

Specific Applications for Nonlinear Filtering

The optimal filter for a noisy system either containing a nonlinearity or having nongaussian disturbances is generally nonlinear. It is well known in optimal control theory that for the linear-quadratic
problem with additive input gaussian disturbances and gaussian measurement noise, the Separation Theorem can be applied to obtain the overall optimum system (16). This theorem, which is sometimes referred to as the Certainty-Equivalence Principle, states that for the linear-quadratic problem an optimum system can be designed by combining the optimal linear filter with the optimum deterministic feedback law. Analogous statements can not be made, in general, for nonlinear systems or systems with nongaussian disturbances. From the viewpoint of applications, however, the optimal estimates of the state variables represents a first step in any stochastic optimal control problem. The design of the optimal feedback control law in general cases remains an extremely difficult, and as yet unsolved, problem.

In the following paragraphs, two examples will be described to show the applications of nonlinear filtering. The first example concerns position estimation from angular measurements. As shown in Figure 1, the height $y_k$ of an object A is to be estimated by the angular measurement $z_k$ from Point 0. Measurements made at discrete points in time are given by

$$z_k = \theta_k + v_k$$  \hspace{1cm} (1.37)

where $v_k$ is the measurement noise. It is assumed that $d$ is fixed and known and that $y$ is stationary and distributed uniformly on some interval $(y_u, y_u)$. Since

$$\theta_k = \tan^{-1} \left( \frac{y_k}{d} \right)$$  \hspace{1cm} (1.38)
Figure 1. Vertical Position Estimation from Angular Measurements.

Figure 2. Transformation of $y_k$ into $\theta_k$ by (1.38).
the density function of \( \varphi_k \) is much different from gaussian as shown in Figure 2. Hence, it is expected that nonlinear estimation would be somewhat better than linear estimation. This assertion is verified for a particular example in Chapter III.

A second example is the estimation of the output shaft position \( \theta \) and velocity \( \dot{\theta} \) for a positional servomechanism which has a field-controlled dc motor. The presence of a nongaussian randomly varying disturbance torque \( L(t) \) requires a nonlinear filtering scheme. The dynamics of the motor are given by

\[
J_m \ddot{\theta} + \tau_m \dot{\theta} = L(t) + u
\]  

where \( J_m \) represents the moment of inertia of the rotating parts, \( \tau_m \) represents the viscous friction, and \( u \) is the input reference torque.

Suppose the output shaft position is monitored at discrete time intervals and has a measurement noise \( v_k \), i.e.

\[
z_k = \theta_k + v_k
\]  

Furthermore, the measurement noise is assumed to be bounded with a known probability density function. This class of problems is examined in Chapter V.

There are numerous other such applications for nonlinear filtering in control system design. The foregoing examples are intended only as representative of the types of problems which may be considered.

**Approach to the Filtering Problem**

The problem to be investigated in this thesis is the estimation
of the states in noisy linear discrete systems. The classes of systems which will be considered are those in which the dynamical behavior is described by first-order or second-order difference equations. The input is ergodic white noise with known bounds, but otherwise unknown. The measurement noise on the output is ergodic, independent, white noise which is uniformly distributed within some known bounds.

For estimation purposes the Bayesian decision rule and the least mean-square error criterion will be used. All the density functions of concern will be expressed in terms of their moments or will be approximated by polynomials where the coefficients are functions of the moments. Formulas relating the moments of the observed data, the states, and the input will be established. Thus, information for the density functions of the states, as well as the input, will be obtained as the moments of the measured data are found. An arbitrary initial guess for the density function, and thus for the moments of the input signal, will be made. This density function will be updated sequentially as additional data are obtained.

Several algorithms will be formulated, and the value of each will be verified by computer simulations. The results will be compared with those of the Kalman filter to illustrate the superior features of the new filtering procedure.

Outline of the Thesis

Following this introductory chapter, the problem is specified fully in Chapter II. The polynomial approximation of density functions is treated in detail, and the updating algorithm is shown by a computer
flow chart. In Chapter III the zero-order formulation is presented to illustrate the particular utility of the method of moments. The computer simulation of a particular example shows a favorable comparison with the linear filter. The effects of correlation between input and measurement noise as well as a non-additive noise problem are investigated. In Chapter IV the filtering algorithm for the first-order problem is presented. The same algorithm is tested for a specific nonlinear system and a slowly time-varying system. In Chapter V the filtering scheme is applied to second-order systems. The reachable state concept of Schweppe is fully utilized in this application. In Chapter VI some conclusions as well as some recommendations for further research are presented.
CHAPTER II

THE METHOD OF MOMENTS

The moment technique is applied in this dissertation to approximate and update density functions pertinent to the filtering scheme and to avoid the usual problems in evaluating the convolution integral for two density functions. According to the Bayesian decision rule, the optimal estimation in the sense of least mean-square error is given in the form of the integral (1.7) with the conditional density function given by (1.8). In most cases this integral is evaluated by the Monte Carlo technique or by numerical integration. Because a large amount of computer time is required to determine the optimal estimate of the states by either of those two techniques, neither can be instrumented as on-line algorithms. To overcome this difficulty, the density functions appearing in (1.8) are approximated by polynomials. Thus, the integration of (1.7) can be performed analytically and the resulting optimal estimation is in a computationally feasible form. Since in practical cases the polynomials are of finite degree, the estimation is suboptimal. Increasing the degrees of the polynomials increases the accuracy of the estimations, but also increases the required computing time. When two independent variables are added to form a new random variable, the convolution integral is usually unavoidable. The few exceptions occur in those cases where the sum and original two random variables have density functions of the same functional form or where
the summation can be performed easily by using characteristic functions. The moment technique discussed in this chapter is applied to overcome the disadvantages of the usual convolution method.

As shown in Chapter I, the calculation of the best estimate for the current states of a system requires the knowledge of the density functions of the input signal, the previous states, the noise, and the measurement data. However, the density function of the input signal is often unknown, and no information about the states of the driven system can be derived from it. Using the Bayesian rule, Cunningham and Breipohl (17) developed a mathematically simple and intuitively satisfying technique to estimate the distribution function of a random variable, in which prior knowledge is included in the empirical function. An equivalent result emphasizing the moment aspect was developed concurrently and independently during the course of this thesis research. The concept is applied in this chapter to estimate and update the moments of the measurement data and is extended in Chapters III, IV, and V to handle the moments of random variables involved in the estimation of the states.

In the following sections the problem is first formulated, and then the approximation of density functions by polynomials is treated in detail. Finally, the moment updating scheme is described with the aid of a computer flow chart.

Statement of the Problem

A model of the problem under consideration is shown in Figure 3. The quantities $w_k$, $x_k$, $v_{k+1}$, and $\hat{x}_{k+1}$ were defined in Chapter I in the development of the general filtering equation. The letter $d$ represents
a scalar in the zero-order problem and an $n$-vector in the $n$th order
dynamical problem, and $y_{k+1}$ is the scalar output of the system at $t = kT$
with

$$y_{k+1} = d^T x_{k+1}$$  \hfill (2.1)

Figure 3. The Nonlinear Filter to be Designed.

The system to be investigated can be placed in one of the
following three categories:

1. The zero-order system is simply given by

   $$x_k = w_k$$  \hfill (2.2)

   $$z_k = x_k + v_k$$  \hfill (2.3)

2. The first-order system has the following model

   $$x_{k+1} = h x_k + w_k$$  \hfill (2.4)
where \( h \) is a real scalar and \(|h| < 1\).

(3) The second-order system is described by

\[
\begin{align*}
x_{1,k+1} &= h_1 x_{1,k} + d_1 w_k \\
x_{2,k+1} &= h_2 x_{2,k} + d_2 w_k \\
y_{k+1} &= d_1 x_{1,k+1} + d_2 x_{2,k+1} \\
z_{k+1} &= y_{k+1} + v_{k+1}
\end{align*}
\]

where \( h_1, h_2, d_1, \) and \( d_2 \) are real scalars, and \(|h_1| < 1, |h_2| < 1\).

**Assumptions**

The following assumptions are made:

1. The input signal \( w \) and the measurement noise \( v \) are ergodic, discrete time series composed of mutually independent random variables.

2. Both \( w \) and \( v \) have known bounds.

3. The density function of \( v \), i.e. \( f_v(v) \), is known. For simplicity, it is assumed to be uniformly distributed on \((-1, 1)\).

4. The density function of \( w \), i.e. \( f_w(w) \), is unknown. However, \( f_w(w) \) must belong to a class of functions such that \( f_w(x) \) can be approximated by polynomials.

5. The noise \( v \) is independent of \( x \) and \( w \).
The system is operating in the steady-state condition. The problem is to find the suboptimal estimate of $x_{k+1}$ based on the measurement set $Z_{k+1}$ in the sense of least mean-square error.

Use of Polynomial Approximations

The orthogonality of Legendre polynomials makes it possible to expand certain functions, which satisfy Dirichlet's conditions, in a series of Legendre polynomials. Let the value of $x$ be scaled in such a way that $x$ is bounded on $(-1, 1)$, and the corresponding density function satisfies Dirichlet's condition on $(-1, 1)$. Thus, one has

$$f_x(x) = \sum_{i=0}^{\infty} A_i P_i(x) \text{ for } x \in (-1, 1)$$

(2.9)

where $A_i$'s are the coefficients, and $P_i(x)$'s are the Legendre polynomials. The general expression for the $i$th Legendre polynomial is given by

$$P_i(x) = \frac{1}{2^i} \sum_{j=0}^{m} \frac{(-1)^j}{j!} \frac{(2i-2)!}{(i-2j)!(i-j)!} x^{i-2j}$$

(2.10)

where

$$m = \begin{cases} \frac{i}{2} & \text{when } i \text{ is even} \\ \frac{i-1}{2} & \text{when } i \text{ is odd} \end{cases}$$

(2.11)

For example, the first few Legendre polynomials are

$$P_0(x) = 1$$

$$P_1(x) = x$$

(2.12)
\begin{align*}
P_2(x) &= \frac{1}{2} (3x^2 - 1) \\
P_3(x) &= \frac{1}{2} (5x^3 - 3x) \\
P_4(x) &= \frac{1}{8} (35x^4 - 30x^2 + 3) \\
P_5(x) &= \frac{1}{8} (63x^5 - 70x^3 + 15x) \\
P_6(x) &= \frac{1}{16} (231x^6 - 315x^4 + 105x^2 - 5) \\
\vdots
\end{align*}

In a practical case, (2.9) must be truncated to a finite number of terms. Therefore,

\[ f_x(x) = \sum_{i=0}^{N} A_i P_i(x) \quad x \in (-1, 1) \quad (2.13) \]

The square error introduced by this approximation is

\[ e^2 = \int_{-1}^{1} \left( f_x(x) - \sum_{i=0}^{N} A_i P_i(x) \right)^2 \, dx \quad (2.14) \]

Setting the derivative with respect to \( A_j \) of \( e^2 \) in (2.14) to zero, one obtains

\[ \int_{-1}^{1} \left( f_x(x) - \sum_{i=0}^{N} A_i P_i(x) \right) P_j(x) \, dx = 0 \quad (2.15) \]

\[ j = 0, 1, 2 \ldots N \]
Note that

\[
\int_{-1}^{1} P_i(x) P_j(x) \, dx = \begin{cases} 
\frac{2}{2j+1} & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]  
(2.16)

Hence,

\[
A_i = \frac{2i+1}{2} \int_{-1}^{1} f(x) P_i(x) \, dx
\]  
(2.17)

Using (2.17), the first seven Legendre coefficients may be computed as

\[
A_0 = \frac{1}{2} \int_{-1}^{1} f(x) \, dx = \frac{1}{2}
\]  
(2.18)

\[
A_1 = \frac{3}{2} \int_{-1}^{1} f(x) x \, dx = \frac{3}{2} m_x^{(1)}
\]

\[
A_2 = \frac{5}{2} \int_{-1}^{1} f(x) \frac{1}{2} \left(3x^2-1\right) \, dx = \frac{5}{4} \left(3m_x^{(2)} - 1\right)
\]

\[
A_3 = \frac{7}{2} \int_{-1}^{1} f(x) \frac{1}{8} \left(5x^3-3x\right) \, dx = \frac{7}{4} \left(5m_x^{(3)} - 3m_x^{(1)}\right)
\]

\[
A_4 = \frac{9}{2} \int_{-1}^{1} f(x) \frac{1}{8} \left(35x^4-30x^2+3\right) \, dx = \frac{9}{16} \left(35m_x^{(4)} - 30m_x^{(2)} + 3\right)
\]

\[
A_5 = \frac{11}{2} \int_{-1}^{1} f(x) \frac{1}{8} \left(63x^5-70x^3+15x\right) \, dx = \frac{11}{16} \left(63m_x^{(5)} - 70m_x^{(3)} + 15m_x^{(1)}\right)
\]
\[ A_6 = \frac{13}{2} \int_{-1}^{1} f_x(x) \cdot \frac{1}{16} (231x^6 - 315x^4 + 105x^2 - 5) \, dx \]

\[ = \frac{13}{32} \left( 231m_x^{(6)} - 315m_x^{(4)} + 105m_x^{(2)} - 5 \right) \]

where \( m_x^{(i)} \), with \( i = 1, 2, \ldots, 6 \), is the \( i \)th moment of \( x \). Substituting (2.18) into (2.13) and rearranging the terms, one may express the resulting equation in the form of an ordinary polynomial, i.e.

\[ f_x(x) \approx \sum_{i=0}^{N} a_i x^i \quad (2.19) \]

For \( N = 2 \), the coefficients in (2.19) are

\[ a_0 = -1.875m_x^{(2)} + 1.125 \quad (2.20) \]

\[ a_1 = 1.5m_x^{(1)} \]

\[ a_2 = 5.625m_x^{(2)} - 1.875 \]

Moreover, for \( N = 4 \), the coefficients in (2.19) become

\[ a_0 = 7.3828125m_x^{(4)} - 8.203125m_x^{(2)} + 1.7578125 \quad (2.21) \]

\[ a_1 = -13.125m_x^{(3)} + 9.375m_x^{(1)} \]

\[ a_2 = -73.828125m_x^{(4)} + 68.90625m_x^{(2)} - 8.203125 \]
\[ a_3 = 21.875m_x^{(3)} - 13.125m_x^{(1)} \]

\[ a_4 = 86.1328125m_x^{(4)} - 73.828125m_x^{(2)} + 7.3828125 \]

Finally, for \( N = 6 \), one obtains

\[ a_0 = -29.326172m_x^{(6)} + 47.373017m_x^{(4)} - 21.533203m_x^{(2)} + 2.392578 \]

\[ a_1 = 81.2109375m_x^{(5)} - 103.359375m_x^{(3)} + 28.7109375m_x^{(1)} \]

\[ a_2 = 615.849609m_x^{(6)} - 913.623075m_x^{(4)} + 348.837892m_x^{(2)} - 21.533203 \]

\[ a_3 = -378.984375m_x^{(5)} + 2605.517578m_x^{(4)} - 913.623075m_x^{(2)} + 47.373017 \]

\[ a_4 = -1847.548828m_x^{(6)} + 341.0859375m_x^{(3)} + 81.2109375m_x^{(1)} \]

\[ a_5 = 1354.869111m_x^{(6)} - 1847.548828m_x^{(4)} + 615.849609m_x^{(2)} \]
It was found that for a large class of density functions the approximations are quite good when moments of $x$ up to the sixth-order are used. Consider the density function given by

$$f_x(x) = \begin{cases} 
1 - |x| & \text{for } |x| < 1 \\
0 & \text{elsewhere}
\end{cases} \quad (2.23)$$

The square error versus the order of the highest moments used for (2.23) is shown in Figure 4. For the case of a truncated gaussian density function bounded on $(-1, 1)$, it was shown that the square error was less than $10^{-10}$ when only moments up to the sixth-order were used.

It is more straightforward to derive (2.20)-(2.22) by the following method in which the use of Legendre polynomials is not required. The density function $f_x(x)$ is approximated directly by a polynomial in the desired form (2.19). The square error in this approximation is

$$e^2 = \int_{-1}^{1} \left[ f_x(x) - \sum_{i=0}^{N} a_i x_i \right]^2 dx \quad (2.24)$$

Setting $\frac{\partial e^2}{\partial a_j} = 0$, with $j = 0, 1, 2, \ldots N$, one has

$$\int_{-1}^{1} \left[ f_x(x) - \sum_{i=0}^{N} a_i x_i \right] x^j dx = 0 \quad (2.25)$$

or equivalently
Figure 4. Square Error Versus Order of Highest Moment Used for Approximating (2.23) by Legendre Polynomials.
The equation (2.26) represents a set of \( N+1 \) simultaneous linear equations. If this set is solved for \( a_0, a_1, \ldots, a_N \), where \( N = 2, 4, \) and 6, the solutions will be identical to (2.20), (2.21), and (2.22), respectively.

The equation in (2.26) is referred to in numerical analysis as the normal equation. Ralston (18) warns that the coefficient matrix obtained in solving (2.26) for the unknown coefficients \( a_0, a_1, \ldots, a_N \) is often ill-conditioned. For that reason the use of orthogonal polynomials, such as the Legendre polynomials, is usually recommended in such cases. However, due to the high precision of the B-5500 digital computer utilized in this research, the solutions of (2.26) for \( N \) up to 6 were not affected by this numerical consideration, and either method described above could be used to obtain the correct results. The normal equation approach will be extended in Chapter V to the second-order problem.

The Updating Algorithm

Due to the basic assumptions made in the previous section, the measurement \( z \) is an ergodic, discrete random process. Therefore, one may use the concept of the sample mean to determine the moments of \( z \), i.e.,

\[
\frac{1}{N-j+1} \sum_{i=0}^{N} (1 - (-1)^{i+j+1}) a_i = m_X(j)
\]

for \( j = 0, 1, 2, \ldots, N \)
\[ \bar{m}^{(i)}_z = \frac{z^i_1 + z^i_2 + \ldots + z^i_{n_s}}{n_s} \quad i = 1, 2, \ldots, N \]  

(2.27)

where \( n_s \) is the number of sample points observed. At the beginning of the estimation procedure, the moments of \( z \) are determined from an initial guess of the density function of the input signal \( w \). As values of measured data \( z \) are obtained, the moments of \( z \) are updated sequentially.

In the following chapters the formulas relating the moments of \( w, x, \nu, \) and \( z \) will be derived for particular systems. Thus, the moments of \( x \) and \( w \) may be updated as a result of the updating of the moments of \( z \).

To update the moments at every sampling period was found to be inadvisable. Not only did this frequent updating require a large amount of computer time, but it made the updated moments fluctuate rapidly. After observing the results of several simulations it was found proper to update the moments only once every twenty intervals. The computational algorithm utilized for updating the moments of \( z \) is shown in Figure 5.

The algorithm may be described in seven steps as indicated in Figure 5. The initial moments of \( z \), denoted by \( m^{(i)}_{z_0} \) for \( i = 1, 2, \ldots, N \), are assigned with arbitrary values in Step 1. In Step 2 three storages are set to zero, where \( n_s \) is a counter indicating the total number of sample points observed, \( m^{(i)}_{zs} \) stores the accumulation of \( z^i_k \), i.e.

\[ m^{(i)}_{zs} = \sum_{k=1}^{n_s} z^i_k \]  

(2.28)
Begin

1. $m_{z_0}^{(1)} \leftarrow$ initial guess of moments of $z$

2. $n_{zs}^{(i)} \leftarrow 0$

3. $n_s \leftarrow n_c \leftarrow 0$

4. $z \leftarrow$ random number generator

5. $n_c = N_{up}$

6. $n_{z_0}^{(i)} \leftarrow m_{z_0}^{(i)} + z^i$

7. $n_{z_0}^{(i)} \leftarrow n_{z_0}^{(i)} + n_{z_0}^{(i)}$

Figure 5. A Flow Chart Showing the Updating of the Moments of $z$. 
and \( n_c \) is another counter indicating the number of sample points observed since the updating procedure. In Step 3 a random number is generated. In Step 4 \( z^i \) is added to \( \bar{m}_{z_s}^{(i)} \) and both counters \( n_s \) and \( n_c \) are increased by 1. The counter \( n_c \) is checked in Step 5 to determine whether a preset number \( N_{up} \) of sample points have been observed. If this number has not been observed, then the algorithm returns to Step 3, where another random number will be generated. If the counter \( n_c \) shows that \( N_{up} \) sample points have already been observed, then the algorithm will proceed further into Steps 6 and 7, where the counter \( n_c \) is set to zero again. Then the estimated moments of \( z \), denoted by \( \bar{m}_z^{(i)} \), are computed from (2.27). Finally, the updated moments of \( z \) are obtained as

\[
\bar{m}_z^{(i)} = \frac{n_o}{n_s + n_o} m_z^{(i)} + \frac{n_s}{n_s + n_o} \bar{m}_z^{(i)}
\]

where \( n_o \) indicates that \( m_z^{(i)} \)'s are equivalent to estimated moments obtained from \( n_o \) prior measurements. The value of \( n_o \) indicates the relative confidence in the initial guess.

**Summary and Conclusions**

The problem was divided into three categories and completely defined as the zero-order, first-order, and second-order problems. The magnitude of the moments involved was shown to indicate their relative importance in the complete filtering algorithm. The formulas expressing the coefficients of the approximating polynomials as functions of the moments of the corresponding random variables were derived for those cases where up to second, fourth, and sixth moments were used.
Finally, the algorithm for updating the moments was introduced by a programming flow chart, which will be applied in Chapters III, IV, and V.
CHAPTER III

THE ZERO-ORDER PROBLEM

The zero-order formulation is presented in this chapter primarily to illustrate the particular utility of the method of moments developed in Chapter II. After stating the problem briefly, a series of steps are investigated to result in the implementation of the solution by the Bayesian decision rule. Following those steps, a computer flow chart is provided to describe the complete filtering algorithm. Then a particular example which was simulated on the digital computer is presented. The suboptimal nonlinear filtering result is compared with the solutions obtained by the optimal nonlinear filter and by a linear filter. Furthermore, the required computer time versus the number of moments used is expressed in term of the number of operations involved, and a typical machine time per step of estimation is given. The effect of the correlation between input signal and measurement noise is also investigated. Finally, a non-additive noise example is presented to show another feature of the new technique.

Solution Via the Moment Technique

The zero-order model given in (2.2)-(2.3) may be expressed as

\[ z_k = x_k + v_k \]  

where the problem is to find the suboptimal estimate of \( x_k \) given \( z_k \).
in the sense of least mean-square error. Corresponding to (1.7)-(1.8), one has

$$\hat{x}_k = E[x_k | z_k]$$

$$= \int_{l_k}^{u_k} x_k \cdot \frac{f_{z_k | x_k}(z_k | x_k) \cdot f_{x_k}(x_k)}{f_{z_k}(z_k)} \, dx_k$$

(3.2)

where $l_k$ and $u_k$, which are to be determined later, form the interval $(l_k, u_k)$ in which $x_k | z_k$ is distributed. Due to the basic assumption that $v$ is uniformly distributed on $(-1, 1)$, one may use (3.1) to obtain

$$f_{z_k | x_k}(z_k | x_k) = f_{v_k}(v_k = z_k - x_k)$$

(3.3)

$$= \begin{cases} 0.5 & \text{when } |z_k - x_k| < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Corresponding to (1.15), $f_{z_k}(z_k)$ is found as

$$f_{z_k}(z_k) = \int_{l_k}^{u_k} f_{z_k | x_k}(z_k | x_k) \cdot f_{x_k}(x_k) \, dx_k$$

(3.4)

$$= \int_{l_k}^{u_k} 0.5 \cdot f_{x_k}(x_k) \, dx_k$$

Inserting (3.3) and (3.4) into (3.2), one obtains
\[ \hat{x}_k = \frac{\int_{t_k}^{u_k} x_k \cdot f_{x_k}(x_k) \, dx_k}{\int_{t_k}^{a_k} f_{x_k}(x_k) \, dx_k} \quad (3.5) \]

Due to the stationarity of \( x \), one has, for all \( k \)

\[ f_{x_k}(x_k) = f_x(x) \quad (3.6) \]

Approximating \( f_x(x) \) by a polynomial as shown in (2.19) and inserting the approximation into (3.5), the suboptimal estimate is

\[ \hat{x}_{kN} = \frac{\sum_{i=0}^{N} a_i (u_k^{i+2} - t_k^{i+2})/(i+2)}{\sum_{i=0}^{N} a_i (u_k^{i+1} - t_k^{i+1})/(i+1)} \quad (3.7) \]

where the letter \( N \) indicates the order of the highest moment used and the \( a_i \)'s are, as given by (2.20)-(2.22), functions of the moments of \( x \). The updating of the moments of \( x \) will be developed later after the determination of \( (t_k, u_k) \) has been considered.

**Bounds of \( x_k | z_k \)**

Since it is assumed that \( x \) is bounded on the interval \( (x_{\text{min}}, x_{\text{max}}) \), then the joint random variable \( x_k, z_k \) is distributed in a parallelogram as shown in Figure 6. Observing the right ends of Lines I and II, it is seen that
Figure 6. Distribution Region of $x, z_k$ for $x \in (x_{\min}, x_{\max})$ and $v \in (-1, 1)$.
\[ u_k = \begin{cases} x_{\text{max}} \quad & \text{when } z_k \geq x_{\text{max}} - 1 \\ z_k + 1 \quad & \text{when } z_k < x_{\text{max}} - 1 \end{cases} \tag{3.8} \]

Similarly from the left ends of Lines I and II, one has

\[ l_k = \begin{cases} z_k - 1 \quad & \text{when } z_k > x_{\text{min}} + 1 \\ x_{\text{min}} \quad & \text{when } z_k \leq x_{\text{min}} + 1 \end{cases} \tag{3.9} \]

For example, in the case where \( x \in (-1, 1) \), (3.8) and (3.9) become

\[ u_k = \begin{cases} 1 \quad & \text{if } z_k > 0 \\ z_k + 1 \quad & \text{if } z_k < 0 \end{cases} \tag{3.10} \]

\[ l_k = \begin{cases} z_k - 1 \quad & \text{if } z_k > 0 \\ -1 \quad & \text{if } z_k < 0 \end{cases} \tag{3.11} \]

Hence the method of determining bounds of \( x_k | z_k \) is quite simple, and this method will also be applied to the first-order and second-order problems in the following chapters.

**Updating of the Moments of x**

Since the density function of \( x \) is unknown, the prior information must be an initial guess. Hence, the \( a_l \)’s appearing in (3.7) will not
be very reliable in general. To update the filter given by (3.7), the procedure for updating the moments of \( x \) must be developed in detail.

Taking expected value of the \( i \)th power of both sides of (3.1), one has

\[
E[z_k^i] = E[(x_k + v_k)^i]
\]

\[
= E\left[ \sum_{j=0}^{i} \binom{i}{j} x_k^{i-j} v_k^j \right] 
\]

(3.12)

Since \( x \) and \( v \) are stationary and mutually independent, the relationship between the moments of \( z, x, \) and \( v \) may be obtained from (3.12) as

\[
m_z^{(i)} = \sum_{j=0}^{i} \binom{i}{j} m_x^{(i-j)} m_v^{(j)}
\]

(3.13)

Note that the \( m_v^{(j)} \)'s are assumed to be known and the estimated moments of \( z \) may be obtained from (2.27). One can estimate the moments of \( x \) by inserting \( \tilde{m}_z^{(i)} \) into (3.13) and by shifting the terms into proper positions such that

\[
\tilde{m}_x^{(i)} = \tilde{m}_z^{(i)} - \sum_{j=1}^{i} \binom{i}{j} \tilde{m}_x^{(i-j)} m_v^{(j)}
\]

(3.14)

where \( m_x^{(i)} \)'s are the estimated moments of \( x \), which should be computed sequentially for \( i = 1, 2, \ldots, N \). Thus, in addition to the estimated \( \tilde{m}_z^{(i)} \) and known \( m_v^{(j)} \)'s, the right hand side of (3.14) consists only of those \( m_x^{(j)} \)'s which have been computed previously. Finally, the moments of \( x \) are updated by applying a formula equivalent to (2.29), i.e.
\[ m_x^{(1)} = \frac{n_0}{n_s + n_c} m_x^{(1)} + \frac{n_s}{n_s + n_c} m_x^{(1)} \quad i = 1, 2, \ldots, N \] (3.15)

where \( n_s \) and \( n_c \) were defined in Chapter II as, respectively, the number of sample points observed and the equivalent number of sample points from which the initial guess of the moments of \( x \) are obtained. As before, the \( m_x^{(1)} \)'s represent the initial guess of the moments of \( x \).

The \( a_i \)'s in (3.7) are updated sequentially, since they are functions of \( m_x^{(1)} \)'s, which are updated sequentially according to (3.15). Hence, the adaptiveness of the filter given by (3.7) is accomplished through the utilization of various moments.

**Computer Simulation Flow Chart**

A flow chart is presented to explain the detailed steps of the complete filtering algorithm. Moreover, the flow chart is coded for computer simulation for the particular example given in the following section. The complete simulation procedure may be described in sixteen steps as shown in Figure 7. In Step 1 initially guessed values are assigned to \( m_x^{(1)} \) for \( i = 1, 2, \ldots, N \), and \( a_i \)'s are computed from the appropriate set of equation in (2.20)-(2.22) depending on the value of \( N \). In Step 2, five storages are set to zero, where \( k \) is a counter indicating the current sampling period, \( n_e \) has already been defined in Chapter II as a counter indicating the number of sampling points observed since the last updating procedure, and \( n_c \) is another counter indicating the number of sampling points observed since the last mean-square error was evaluated and printed. The storage \( m_{zs}^{(1)} \) is given by
Figure 7. Flow Chart for the Zero-Order Problem.
(2.28), and erasure stores the accumulation of square error. In Step 3, the time counter \( k \) counts 1 and the random numbers \( x_k \) and \( v_k \) are each generated by a random number generator. Then \( x_k \) and \( v_k \) are summed together to form the measurement data \( z_k \). In Step 4, \( z_1 \) is accumulated in \( m_{z_1}^{(1)} \) for \( i = 1, 2, \ldots, N \), and the counters \( n_c \) and \( n_e \) are each increased by 1. The counter \( n_c \) is checked in Step 5 to decide whether the updating scheme in Step 6 should be performed or be omitted until a later sampling interval. In Step 6, the counter \( n_c \) is set to zero again, and \( n_e \) is given the value of \( k \), which is the number of sampling points observed at that time. Then the updating scheme as shown in the block is performed in the manner described in the last section.

The filtering scheme is started in Step 7, where the bounds of \( x_k | z_k \), i.e. \( t_k \) and \( u_k \), are found. In Step 8, the denominator of the filter given by (3.7) is computed and denoted by \( d_e \). The value of \( d_e \) may be negative due to the errors in the \( a_i \)'s, which in turn are due to the inaccuracy in the moments of \( x \). The absolute value of \( d_e \) may also be too small to make a correct division. Hence it is checked in Step 9 to see whether \( d_e \) is smaller than a small positive number \( \varepsilon \), which is assumed to be equivalent to zero in a practical computation. If \( d_e \) is smaller than \( \varepsilon \), then the filter given by (3.7) is considered to be invalid, and the estimate is given in Step 13 as

\[
\hat{x}_{kN} = \frac{1}{2} (t_k + u_k)
\]

(3.16)

which is obtained by arbitrarily assuming that \( x \) is uniformly distributed on \( (x_{\min}, x_{\max}) \). In Steps 10 and 11, the numerator of (3.7) and the
estimate of $x_k$ is computed. If $\hat{x}_k$ falls outside of the interval $(u_k, u_k)$, then again the filter given by (3.16) is applied as shown by Steps 12 and 13. In Step 14 the square error is accumulated in the storage $\text{ers}$. The counter $n_e$ is checked in Step 15 to determine the printing of the mean-square error $e^2_N$ shown in Step 16 with

$$e^2_N = \frac{\text{ers}}{n_s} = \frac{\sum_{k=1}^{n_s} (\hat{x}_{kN} - x_k)^2}{n_s}$$  \hspace{1cm} (3.17)

Finally, the algorithm indicates a return to Step 3 for beginning the computations for a new sampling period.

An Example

In this section the zero-order algorithm is thoroughly tested on a specific example. The mean-square error introduced by the optimal filter is computed first. The optimal error is used to justify the quality of the suboptimal adaptive filter developed in the previous sections. The mean-square error versus the highest order of moments used in the approximating polynomial is also examined.

Statement of the Example

For the model given by (3.1)

$$z_k = x_k + v_k$$  \hspace{1cm} (3.1)

let the state $x$ have the following stationary density function

$$f_x(x) = \begin{cases} \frac{J+1}{2} x^J & \text{when } |x| < 1 \\ 0 & \text{elsewhere} \end{cases}$$  \hspace{1cm} (3.18)
where $J$ is some positive even number. Moreover, the noise $v$ is uniformly distributed on $(-1, 1)$. The density function given in (3.19) is such that when $J = 0$ both the suboptimal filter and linear filter are optimal. As $J$ becomes larger, and consequently the density function becomes much different from gaussian, the linear estimation is further from the optimal. On the other hand, the suboptimal estimation can be made very close to optimal by using higher order moments. This feature is explored in the following sections.

### The Optimal Filter

Let the optimal estimate of $x_k$ given $z_k$ be denoted by $x_{kop}$.

Note that from (3.10)-(3.11) one has

$$
(t_k, u_k) = \begin{cases} 
(z_{k-1}, 1) & \text{when } z > 0 \\
(-1, z_{k+1}) & \text{when } z < 0
\end{cases}
$$

(3.19)

The optimal estimate $x_{kop}$ may be obtained by inserting (3.18) into (3.5) and performing the integration to yield

$$
\hat{x}_{kop} = \begin{cases} 
\frac{J+1}{J+2} \cdot \frac{1 - (z_{k-1})^{J+2}}{1 - (z_{k-1})^{J+1}} & \text{when } z > 0 \\
\frac{J+1}{J+2} \cdot \frac{(z+1)^{J+2} - 1}{(z+1)^{J+1} + 1} & \text{when } z < 0
\end{cases}
$$

(3.20)

The corresponding mean-square estimation error is
\[ e_{op}^2 = \mathbb{E}[(x_k - \hat{x}_{kop})^2] \]

\[ = \int_{-Z}^{Z} \int_{-1}^{1} (x_k - \hat{x}_{kop})^2 f_{x_k z_k}(x_k, z_k) \, dx_k \, dz_k \]

\[ + \int_{-1}^{1} \int_{-1}^{1} (x - \hat{x}_{kop})^2 f_{x_k z_k}(x_k, z_k) \, dx_k \, dz_k \]

\[ = 2 \int_{-Z}^{Z} \int_{-1}^{1} (x_k - \hat{x}_{kop})^2 \, dz_k \int_{-1}^{1} f_{x_k z_k}(x_k, z_k) \, dx_k \, dz_k \]

The numerical values of \( e_{op}^2 \) are listed in the second column of Table 1 and plotted in Figure 8.

**The Suboptimal Filter**

It is assumed that the density function of \( x \) is known and given by (3.18). Thus, the moments of \( x \) may be computed as

\[ m_x(i) = \int_{-1}^{1} x^i \frac{J+1}{2} x^J \, dx \]

\[ = \begin{cases} 
\frac{J+1}{J+1} & \text{when } i \text{ is even} \\
0 & \text{when } i \text{ is odd}
\end{cases} \]
Table 1. The Ensemble Average of $e^2$ for the Zero-Order Example.

<table>
<thead>
<tr>
<th>J</th>
<th>$e^2_{op}$</th>
<th>$e^2_2$</th>
<th>$e^2_4$</th>
<th>$e^2_6$</th>
<th>$e^2_{linear}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.166667</td>
<td>0.166667</td>
<td>0.166667</td>
<td>0.166667</td>
<td>0.166667</td>
</tr>
<tr>
<td>2</td>
<td>0.176796</td>
<td>0.176796</td>
<td>0.176796</td>
<td>0.176796</td>
<td>0.214286</td>
</tr>
<tr>
<td>4</td>
<td>0.148112</td>
<td>0.151443</td>
<td>0.148112</td>
<td>0.148112</td>
<td>0.227273</td>
</tr>
<tr>
<td>6</td>
<td>0.124269</td>
<td>0.190682</td>
<td>0.124269</td>
<td>0.124269</td>
<td>0.233333</td>
</tr>
<tr>
<td>8</td>
<td>0.106289</td>
<td>0.201550</td>
<td>0.107424</td>
<td>0.106301</td>
<td>0.236842</td>
</tr>
<tr>
<td>10</td>
<td>0.092598</td>
<td>0.210309</td>
<td>0.12257</td>
<td>0.092705</td>
<td>0.239130</td>
</tr>
</tbody>
</table>

Figure 8. The Ensemble Average of $e^2$ for the Zero-Order Example.
Therefore, the $a_i$'s in the suboptimal filter given by (3.7) may be obtained from (2.20)-(2.22). The corresponding mean-square estimation error is computed from (3.21) with $\hat{x}_k^{op}$ being replaced by $\hat{x}_k^N$. The numerical values of $\epsilon_N^2$ for $N = 2, 4,$ and 6 are also listed in Table 1 and plotted in Figure 8. As expected, the following properties may be observed:

1. $\epsilon_2^2 = \epsilon_4^2 = \epsilon_6^2 = \epsilon_{op}^2$ when $\lambda = 0, 2$
2. $\epsilon_2^2 > \epsilon_4^2 > \epsilon_6^2 = \epsilon_{op}^2$ when $\lambda = 4$
3. $\epsilon_2^2 > \epsilon_4^2 > \epsilon_6^2 = \epsilon_{op}^2$ when $\lambda = 6$
4. $\epsilon_2^2 > \epsilon_4^2 > \epsilon_6^2 > \epsilon_{op}^2$ when $\lambda = 8, 10$

It is seen that $\epsilon_6^2$ is quite close to $\epsilon_{op}^2$ for $\lambda = 8$ and $\lambda = 10$. This implies that the suboptimal filter is a good approximation for the optimal filter in this example.

A computer simulation was performed to compute the time average of $\epsilon_6^2$ by taking the sample mean of $(x_k - \hat{x}_k^N)^2$ as shown by (3.17) with $N = 6$. The results are plotted in Figure 9, which, when compared with corresponding ensemble-averaged values listed in Table 1, shows that the mean-square error is ergodic as expected. Thus, the time-averaging technique yielded acceptable results.

**The Suboptimal Adaptive Filter**

The complete filtering algorithm described by the flow chart in Figure 7 was simulated for this particular example. It was initially guessed that the input signal was uniformly distributed on $(-1, 1)$, and $n_o$ was arbitrarily chosen to be 100. The mean-square errors were com-
Figure 9. Time Average of $e_6^2$ for the Zero-Order Example.
puted in the manner given by (3.17). Computer simulation results for $J = 10$ and $J = 0$ are shown in Figures 10(a) and 10(b), respectively. In Figure 10(a), comparing the values of $e_N^2$'s at $n_s = 3000$, it is seen that $e_2^2 > e_4^2 > e_6^2$. However, when $n_s < 2200$, $e_4^2$ was smaller than $e_6^2$. The inaccuracy of $m_x^{(6)}$ for the first 2200 samples had a degrading effect on the results, which tended to offset the advantages normally realized by using higher-order moments.

In Figure 10(b), which shows the case of $J = 0$, i.e. when $x$ is uniformly distributed on (-1, 1), it is seen that $e_2^2 < e_4^2 < e_6^2$. Since $f_x(x)$ is constant on (-1, 1), a zero-degree polynomial may be used as an exact expression for $f_x(x)$. Therefore, the use of a higher-order polynomial decreases the accuracy due to errors in the updated higher-order moments.

If the $e_N^2$ vs $n_s$ curves in Figures 10(a) and 10(b) are compared closely and if the computer time is also considered, it would be quite reasonable to select $N = 4$ as an appropriate tradeoff value, i.e. to use the moments up to fourth-order in the complete filtering procedure for this particular example.

Execution Time Versus Number of Moments

For on-line operation, the sampling rate is limited by the execution time of the filtering algorithm. In this section the execution time is separated into two parts and expressed in terms of the number of operations required.

Data Accumulation and Signal Estimation

The first consideration in determining the execution time is to
Figure 10(a). The plot of $e^2$ vs $n_s$ for $J = 10$. 
Figure 10(b). The Plot of $e^2$ vs $n_s$ for $J = 0$.

Note - Curves for $e^2_{\text{linear}}$ and $e^2_2$ coincide.
examine the computer time required to accumulate the measurement data and estimate the state of the system. In the flow chart of Figure 7, these operations are performed in Steps 4, 5, 7, 8, 9, 10, 11, 12, and 13, which are executed in almost every sampling period. The required number of operations is listed in Table 2, where the symbol \( \pm \) represents replacement, \( > \) represents comparison, and the other symbols are self-explanatory. It is seen that the execution time is approximately proportional to \( N \), the order of highest moment used in the algorithm.

Table 2. Number of Operations Required for Data Accumulation and Signal Estimation

<table>
<thead>
<tr>
<th></th>
<th>( N )</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm )</td>
<td>5( N + 5 )</td>
<td>15</td>
<td>25</td>
<td>35</td>
</tr>
<tr>
<td>( \times )</td>
<td>7( N + 5 )</td>
<td>19</td>
<td>33</td>
<td>47</td>
</tr>
<tr>
<td>( \div )</td>
<td>( 1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \geq )</td>
<td>3( N + 8 )</td>
<td>14</td>
<td>20</td>
<td>26</td>
</tr>
<tr>
<td>( \rightarrow )</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 3. Number of Operations Required for the Updating Scheme

<table>
<thead>
<tr>
<th></th>
<th>( N )</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm )</td>
<td>( \frac{N^2}{2} + \frac{3}{2} N + 2 )</td>
<td>9</td>
<td>24</td>
<td>47</td>
</tr>
<tr>
<td>( \times )</td>
<td>( \frac{3}{2} N^2 + \frac{N}{2} )</td>
<td>7</td>
<td>26</td>
<td>57</td>
</tr>
<tr>
<td>( \div )</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>( \rightarrow )</td>
<td>4( N + 5 )</td>
<td>13</td>
<td>21</td>
<td>29</td>
</tr>
</tbody>
</table>
Updating Scheme

Another consideration in determining the total computer time required is the time required in the updating scheme, which is performed in Step 6 of the flow chart given in Figure 7. This step is executed only once every twenty sampling periods. The required number of operations is shown in Table 3. It is seen that the execution time required for this part is almost proportional to $N^2$. A typical machine time based on the speed of the B-5500 is given in Table 4.

If the updating procedure, which is executed only once in every 20 sampling periods, can be handled by another fixed program computer, then the maximum permitted sampling rate will be greatly increased. One may also use part of the sampling period for the execution of the updating scheme to increase the permitted sampling rate. In such a case, the updated moments will not be available before a certain number

<table>
<thead>
<tr>
<th>N</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data Accumulation and Signal Estimation</td>
<td>1460</td>
<td>2330</td>
<td>3290</td>
</tr>
<tr>
<td>Updating Scheme</td>
<td>775</td>
<td>2030</td>
<td>4005</td>
</tr>
<tr>
<td>Total</td>
<td>2235</td>
<td>4360</td>
<td>7295</td>
</tr>
</tbody>
</table>
Comparison with the Linear Filter

The optimal filter will be linear if the input signal and measurement noise are both gaussian, or if they have some special density functions, e.g. \( J = 0 \) in the particular example of the last section. Applying the orthogonality principle to the zero-order problem, the linear estimator is

\[
\hat{x}_{k, \text{linear}} = \frac{m_x^{(2)}}{m_x^{(2)} + m_v^{(2)}} z_k
\]

(3.23)

Specifically, for the example given in the last section, \( m_x^{(2)} \) is given in (3.22) with \( i = 2 \), and \( m_v^{(2)} = \frac{1}{3} \). Thus, one has

\[
\hat{x}_{k, \text{linear}} = \frac{(J+1)/(J+3)}{(J+1)/(J+3) + \frac{1}{3}} z_k = \frac{3J + 3}{4J + 6} z_k
\]

(3.24)

and the corresponding mean-square error is given in (3.21) with \( \hat{x}_{k, \text{Hop}} \) replaced by \( \hat{x}_{k, \text{linear}} \). The numerical values of this error, denoted by \( \sigma_{\text{linear}}^2 \), are listed in Column 6 of Table 1 and plotted in Figure 8. Note that when \( J = 0 \) the coefficients \( a_j \) in the suboptimal filter given by (3.7) are all zero, except \( a_0 = 0.5 \), regardless of the value of \( N \). Making use of (3.19) in (3.7), one has

\[
\hat{x}_{k, \text{N}} = \frac{1}{2} z_k
\]

(3.25)
Also, if one inserts $J = 0$ into (3.20) and (3.24), one obtains

$$\hat{x}_{k_0} = \hat{x}_{k_{\text{linear}}} = \frac{1}{2} z_k$$  for  $J = 0$  \hfill (3.26)

Hence, both the linear filter and the suboptimal filter are optimal. This fact is also shown in Table 1 and in Figure 8.

Simulations for the linear filter were performed under the assumption that the second moment of the input signal $x$ was known exactly. The mean-square errors are plotted in Figures 10(a) and 10(b) for comparison. In Figure 10(a), which is for $J = 10$, it is seen that the suboptimal filters gave better results than the linear filter. However, in Figure 10(b), which is for $J = 0$, the curves shows that

$$e_2^2 > e_4^2 > e_6^2$$

where $e_{linear}^2$ is also the optimal error which can be obtained from simulation. Thus, the use of higher-order moments degraded the estimates $\hat{x}_k$ and $\hat{x}_{k_0}$ due to the inaccuracy in the updated higher-order moments.

It is obvious that there are fewer problems in using the linear filter for on-line operation. Hence, one may conclude that in those cases where either the sampling rate is too high to be handled by the suboptimal adaptive filter or the density function of the state is known and is not far from gaussian, then the linear filter is preferred. Otherwise, the suboptimal adaptive filter is more suitable for the estimation of the noise corrupted state.
The Effect of Correlation Between $x$ and $v$

In the updating scheme (3.14) is based on the assumption that $x$ and $v$ are mutually independent. If this is not true, then updated moments of $x$ will not be accurate, and the mean-square estimation error may be increased. In a simulation for $J = 10$ in the example of the previous section, the estimated correlation coefficient between $x$ and $v$ was found to be around 0.05 to 0.10, and the actual values and updated values of the even order moments of $x$ are plotted in Figure 11. In the figure the true values of $m^{(i)}_x$ were computed during the simulation as:

$$m^{(i)}_x = \frac{1}{n_s} \sum_{k=1}^{n_s} x_k^i$$

The mean-square estimation error $e^2$ is plotted in Figure 12, in which the curves of $e^2_6$ and $e^2_{\text{linear}}$ from Figure 10(a) are also plotted. From the values given in the figure, it is seen that even when correlation was presented the updated moments still had acceptable accuracy. A comparison of the curves in Figure 12 shows that the mean-square error was not seriously increased because of correlation, and it was still lower than $e^2_{\text{linear}}$. Further investigations with the aid of computer simulation showed that although the updated moments might be invalid due to high correlation between $x$ and $v$, the applicability of the suboptimal adaptive filter was not seriously degraded, because of the techniques used in Steps 9, 12, and 13 of the algorithm given in Figure 7. Hence, the suggested filtering algorithm is apparently not
Figure 11. True Values and Updated Values of the Even Order Moments of $x$.

Figure 12. Plots of $e^2$ Versus $n_s$ Showing the Effects of Correlations.
very sensitive to the correlation between the state and the measurement noise.

The Non-Additive Noise Case

The non-additive noise case may be treated essentially in the same manner as the additive noise case. Consider a zero-order system in which the measurement data is corrupted by multiplicative noise, e.g.

\[ z_k = x_k v_k \quad (3.29) \]

Furthermore, let

\[ f_x(x) = \begin{cases} 
1+x & \text{when } -1 < x < 0 \\
1-x & \text{when } 0 < x < 1 \\
0 & \text{elsewhere} 
\end{cases} \quad (3.30) \]

\[ f_v(v) = \begin{cases} 
1 & \text{when } 0 < v < 1 \\
0 & \text{elsewhere} 
\end{cases} \quad (3.31) \]

Otherwise, the basic assumptions listed in Chapter II remain the same.

The distribution region of the joint random variable \( x_k z_k \) is shown in Figure 13. After some manipulations, the optimal and suboptimal filters were obtained, respectively, as
Figure 13. Distribution Region of $x_k z_k$ for the Non-Additive Noise Example.

Figure 14. A Plot of $e_6^2$ Versus $n_s$ for the Non-Additive Noise Example.
\[
\frac{\text{sign}(z_k) \cdot \left( 1 - \frac{1}{2} \left( 1 - \frac{1}{|z_k|^2} \right) \right)}{\frac{1}{|z_k|} + \frac{1}{|z_k|^2}} \quad 0 < |z_k| < 1
\]
\[
z_{k, \text{op}} = \begin{cases} 
0 & 
0 < |z_k| < 1 \\
\frac{6}{\sum_{i=0}^{\infty} a_i (1-|z_k|^{i+1})/(i+1)} & 
1 > |z_k| > 0 \\
0 & 
z_k = 0 
\end{cases}
\]

where the \( a_i \)'s are functions of the moments of \( x \) given in (2.22). The optimal mean-square estimation error was computed by numerical integration as

\[
e_{\text{op}}^2 = 0.027
\]

The suboptimal adaptive filter was simulated with the initial guess that \( x \) was uniformly distributed on (-1, 1) and \( n_o = 100 \). The simulation algorithm was exactly the same as given by the flow chart in Figure 7, except that (3.14) was replaced by

\[
\frac{\bar{m}(i)}{x} = \frac{\bar{m}(i)}{\bar{m}(i)}
\]

and (3.7) was replaced by (3.33). The mean-square error \( e_{6}^2 \) resulting
from the simulation is plotted in Figure 14. It is seen that $e_0^2$ converged to the optimal value in a few hundred samples.

**Conclusions**

If the density function of the signal, or equivalently the moments of the signal, is known exactly, then the mean-square estimation error is decreased as the order of the highest moment used is increased. However, if the updated moments of the input signal deviate from the true values, e.g. due to correlation, the mean-square error may be increased with an increase in the order of highest moment used, since some additional error may be contributed by the inaccurate higher-order moments. On the other hand, the execution time for the complete filtering algorithm increases rapidly with the order of highest moments involved. Both of these factors must be considered in determining the number of moments to be used in the algorithm.

There are other important features of the suboptimal adaptive filter. In a particular example, the estimation merit was not affected very much by correlation between the input signal and the measurement noise, although the updating scheme theoretically might become invalid when a sufficient amount of correlation is present. Furthermore, the new technique was found to be quite suitable for applying to the non-additive noise problem. Finally, the most important feature is that in those cases where the input signal was much different from gaussian, the suboptimal filter gave a much lower mean-square error than the linear filter.
The zero-order problem investigated in this chapter was presented to illustrate the significant features of the method of moments. The technique will be extended in the next chapter to yield a suboptimal adaptive filter for the first-order dynamical system.
CHAPTER IV

THE FIRST-ORDER DYNAMICAL PROBLEM

In this chapter the method of moments is applied to the first-order discrete time-invariant linear system. The relationship between the moments of the input signal and those of the measured data is established to permit the updating of the input signal moments. The characteristics of the system are fully utilized in computing the conditional moments and bounds of the state from stage to stage. A specific example was simulated, and the suboptimal result compared with that of the Kalman filter. Finally, the scheme was applied to a particular nonlinear system by linearizing the nonlinear element and to a slowly time-varying system by taking the average of the time-varying parameter over several sampling intervals.

Solution of the Problem

The problem was defined in Chapter II by the model given in (2.4) and (2.5). Based on the optimal estimate given in (1.7) with the conditional density function given in (1.9), the required algorithms will be developed in a step-by-step manner.

The Relationship Between the Moments of \( w \) and \( x \)

Since it has been assumed that the system has reached its steady-state, then one has
\[ E[x_k^{(i)}] = m_x^{(i)} \quad \text{for all } k \text{ and } i \quad (4.1) \]

Taking the expected value of the \( i \)th power of \((2.h)\), one obtains

\[ E[x_{k+1}^{i}] = E[(hx_k + w_k)^i] \quad (4.2) \]

Making use of \((4.1)\) and the mutual independence of \( x \) and \( w \), \((4.2)\) becomes

\[ m_x^{(i)} = h^i m_x^{(i)} + \sum_{j=1}^{i} (i\choose j) h^{i-j} m_x^{(i-j)} m_w^{(j)} \]

or equivalently

\[ m_x^{(i)} = \frac{1}{1-h^i} \sum_{j=1}^{i} (i\choose j) h^{i-j} m_x^{(i-j)} m_w^{(j)} \quad (4.3) \]

and

\[ m_w^{(i)} = (1-h)^i m_x^{(i)} - \sum_{j=1}^{i-1} (i\choose j) h^{i-j} m_x^{(i-j)} m_w^{(j)} \quad (4.4) \]

Hence, after estimating the moments of \( z \) by \((2.27)\) and computing the estimated moments of \( x \) from \((3.14)\), one may obtain the estimated moments of \( w \), i.e. \( m_w^{(i)} \), from \((4.4)\) and update these moments by \((2.29)\) with the subscript \( z \) being replaced by \( w \).

The Bounds and the Moments of \( x \)

Before any measurement is made, the unconditional moments of \( x \)
are computed from (4.3), where the \( m^{(i)} \)'s are either known a priori or initially guessed. With the repeated application of (2.4), one obtains

\[ x_{k+1} = h^{k+1} x_0 + w_k + h w_{k-1} + \ldots + h^{k-1} w_1 + h^k w_0 \]  

\( (4.5) \)

where \( k \), instead of indicating the sampling period after the filtering algorithm is started, indicates the sampling period before the system reaches its steady-state. Note that as \( k \to \infty \) the system approaches its steady-state operation and \( h^{k+1} \to 0 \). The upper bound of \( x \) denoted by \( u_0 \) is obtained by inserting into (4.5) the appropriate bounds for \( w_0 \), \( w_1 \), \ldots, and \( w_k \) as

\[ u_0 = \begin{cases} 
  w_{\text{max}} + h w_{\text{max}} + h^2 w_{\text{max}} + \ldots & \text{when } h > 0 \\
  w_{\text{max}} + h w_{\text{min}} + h^2 w_{\text{min}} + \ldots & \text{when } h < 0 
\end{cases} \]

\( (4.6) \)

\[ u_0 = \begin{cases} 
  \frac{1}{1-h} w_{\text{max}} & \text{when } h > 0 \\
  \frac{1}{1-h^2} (w_{\text{max}} + h w_{\text{min}}) & \text{when } h < 0 
\end{cases} \]

Similarly the lower bound of \( x \) denoted by \( l_0 \) is obtained as

\[ l_0 = \begin{cases} 
  \frac{1}{1-h} w_{\text{min}} & \text{when } h > 0 \\
  \frac{1}{1-h^2} (w_{\text{min}} + h w_{\text{max}}) & \text{when } h < 0 
\end{cases} \]

\( (4.7) \)
One may also intuitively assign \( x_{k+1} = u_0, x_k = u_0 \) and \( w_k = w_{\text{max}} \) in
(2.4), whereupon the first part of (4.6) is obtained immediately.
Similarly (4.7) and the second part of (4.6) may be obtained in an
intuitive manner. Starting with the unconditional moments and the
bounds of \( x \) and making use of the system equation and measurement
data, the conditional moments and bounds of \( x_{k+1|Z_k} \) and \( x_{k+1|Z_{k+1}} \)
will be evaluated in the remainder of this section.

Bounds and Moments of \( x_{k+1|Z_k} \)

Assuming the bounds and moments of \( x_k|Z_k \) have been found in the
last sampling period, one may evaluate the bounds of \( x_{k+1|Z_k} \) as

\[
\begin{align*}
\ell_{k+1,0} &= hu_{k} + w_{\min} \\
u_{k+1,0} &= hu_{k} + w_{\max}
\end{align*}
\]

where it is assumed that \( h > 0 \) without loss of generality, and the
notations are given as:

- \( \ell_k \) : lower bound of \( x_k|Z_k \)
- \( u_k \) : upper bound of \( x_k|Z_k \)
- \( \ell_{k+1,0} \) : lower bound of \( x_{k+1|Z_k} \)
- \( u_{k+1,0} \) : upper bound of \( x_{k+1|Z_k} \)

One may also obtain the moments of \( x_{k+1|Z_k} \) by taking the expected value of
\[
(x_{k+1} | Z_k)^i = (nx_k | Z_k + w_k)^i
\]

for \( i = 1, 2, \ldots, N \)

which is obviously the characteristics of the system with the given measurement set \( Z_k \). Since \( w \) is stationary and \( w_k \) is independent of \( x_k | Z_k \) due to the assumption that \( w_k \) is independent of \( x_k \) and \( v_k \), (4.9) becomes

\[
m_{x_{k+1} | Z_k}^{(i)} = \sum_{j=0}^{i} \binom{i}{j} m_{x_k | Z_k}^{(i-j)} m_{w}^{(j)}
\]

(4.10)

for \( i = 1, 2, \ldots, N \)

where \( m_{x_{k+1} | Z_k}^{(i)} \) and \( m_{x_k | Z_k}^{(i)} \) are the \( i \)th moments of \( x_{k+1} | Z_k \) and of \( x_k | Z_k \) respectively.

With the knowledge of the bounds and the moments of \( x_{k+1} | Z_k \), it is possible to approximate the density function of \( x_{k+1} | Z_k \) by a polynomial as described in the next subsection.

**Approximation of \( f_{x_{k+1} | Z_k} \)**

To approximate \( f_{x_{k+1} | Z_k} \) by a polynomial as given by (2.19), a transformation must be made such that the new variable will be distributed on \((-1, 1)\). Such a transformation is given as

\[
s_{k+1} = \frac{2}{u_{k+1, 0} - d_{k+1, 0}} (x_{k+1} - \frac{u_{k+1, 0} + v_{k+1, 0}}{2})
\]

(6.11)
where $s_{k+1}$ is the new random variable. Let

$$a = \frac{2}{v_{k+1,0} - t_{k+1,0}}$$

$$d = \frac{v_{k+1,0} + t_{k+1,0}}{v_{k+1,0} - t_{k+1,0}}$$

then (4.11) becomes

$$s_{k+1} = cx_{k+1} + d$$  \hspace{1cm} (4.13)

From (4.13), the moments of $s_{k+1}|Z_k$ are found as

$$\sigma^{(i)}_{s_{k+1}|Z_k} = \sum_{j=0}^{i} \binom{i}{j} d^{i-j} \sigma^{(j)}_{x_{k+1}|Z_k}$$  \hspace{1cm} (4.14)

Thus, the density function of $s_{k+1}|Z_k$ can be approximated by a polynomial as

$$f_{s_{k+1}|Z_k}(s_{k+1}|Z_k) \approx \sum_{i=0}^{N} b_i s_{k+1}^i$$  \hspace{1cm} (4.15)

with $b_i$'s computed from one of (2.20) - (2.22) as functions of $\sigma^{(i)}_{s_{k+1}|Z_k}$'s. Finally, making use of (4.13) again, one has

$$f_{x_{k+1}|Z_k}(x_{k+1}|Z_k) = \frac{\frac{ds_{k+1}}{dx_{k+1}}}{f_{s_{k+1}|Z_k}(s_{k+1}|Z_k = cx_{k+1} + d|Z_k)$$  \hspace{1cm} (4.16)

$$= \sum_{i=0}^{N} \frac{a_k x_i}{i}$$
where

\[ a_i = c^{i+1} \sum_{j=1}^{N} \binom{i}{j} b^j d^{j-1} \]  \hspace{1cm} (4.17)

The next step is to find the bounds and the density function of \( x_{k+1} | Z_{k+1} \) from the results obtained in the previous subsection and in this subsection.

**The Bounds and Density Function of \( x_{k+1} | Z_{k+1} \)**

The bounds of \( x_{k+1} | Z_{k+1} \) denoted by \( \ell_{k+1,0} \) and \( u_{k+1,0} \) are equivalent to \( x_{\text{min}} \) in (3.9) and \( x_{\text{max}} \) in (3.8), respectively. Hence, the bounds of \( x_{k+1} | Z_{k+1} \), i.e. \( \ell_{k+1} \) and \( u_{k+1} \), may be obtained from (3.9) and (3.8) as

\[
\begin{align*}
  u_{k+1} &= \begin{cases} 
    u_{k+1,0} & \text{when } z_{k+1} \geq u_{k+1,0} - 1 \\
    z_{k+1} + 1 & \text{when } z_{k+1} < u_{k+1,0} - 1 
  \end{cases} \\
  \ell_{k+1} &= \begin{cases} 
    z_{k+1} - 1 & \text{when } z_{k+1} > \ell_{k+1,0} + 1 \\
    \ell_{k+1,0} & \text{when } z_{k+1} \leq \ell_{k+1,0} + 1 
  \end{cases}
\end{align*}
\]  \hspace{1cm} (4.18)

Making use of (1.9), (1.15), the assumption that \( v_{k+1} \) is uniformly distributed on \((-1, 1)\), and the measurement equation (2.5), one has

\[
K_{k} = \frac{\int_{\ell_{k+1}}^{u_{k+1}} f_{x_{k+1} | Z_{k+1}}(x_{k+1} | Z_{k}) \, dx_{k+1}}{\int_{\ell_{k+1}}^{u_{k+1}} f_{x_{k+1} | Z_{k+1}}(x_{k+1} | Z_{k}) \, dx_{k+1}}
\]  \hspace{1cm} (4.20)
Inserting (4.16) into (4.20), one obtains

\[
\ell_{x_{k+1} | Z_{k+1}}(x_{k+1} | Z_{k+1}) = \frac{1}{\sum_{i=0}^{N} a_i} \sum_{i=0}^{N} a_i (x_{k+1}^i - \ell_{x_{k+1}}^i) / (i+1)
\]

where \(a_i'\) is equal to \(a_i\) divided by the denominator of (4.21). Finally, the moments of \(x_{k+1} | Z_{k+1}\) are given for the computation in the next sampling period as

\[
\mu_{x_{k+1} | Z_{k+1}}^j = \int_{x_{k+1}}^{x_{k+1}^j} \sum_{i=0}^{N} a_i' x_{k+1}^i dx_{k+1}
\]

\[
= \frac{1}{\sum_{i=0}^{N} a_i} \sum_{i=0}^{N} a_i (u_{k+1}^i - \ell_{x_{k+1}}^i) / (i+1)
\]

for \(j = 1, 2, \ldots, N\).

The suboptimal estimate is the conditional mean value, i.e. the first moment of \(x_{k+1} | Z_{k+1}\), and may be obtained from (4.22) by setting \(j = 1\). Hence, one has
Up to this point the complete filtering scheme has been described in detail. However, the updating of the suboptimal filter given by (4.23) needs to be considered further.

### Automatic Updating of the Moments $x_{k+1} | Z_k$

In (4.10) the moments of $x_{k+1} | Z_k$ are given as functions of $m^{(i)}_{x_k | Z_k}$'s and $m^{(d)}_{x_k | Z_k}$'s. The latter are updated only once every twenty sampling intervals, but the moments $m^{(i)}_{x_{k+1} | Z_k}$ have an increased accuracy because of automatic updating. In the updating scheme, there is the possibility that the bounds of $x_k | Z_k$, i.e. $\ell_k$ and $u_k$, are very near to each other at some of the sampling intervals. When this occurs, the density function of $x_k | Z_k$ may be approximated either as being uniformly distributed on $(\ell_k, u_k)$ or as an impulse at $\frac{1}{2} (\ell_k + u_k)$. Then the moments of $x_k | Z_k$ computed from such a density function will have high accuracy, and, consequently, the inaccuracy due to the initial guess and the approximations from stage to stage will not be significant. As the moments of $x_{k+1} | Z_k$ are updated automatically, so is the suboptimal filter given by (4.23).

### An Example

For the first-order system in (2.4), consider an input signal...
with a density function given by
\[
\begin{align*}
f_w(w) = \begin{cases} 
\frac{11}{2} w^10 & \text{when } |w| < 1 \\
0 & \text{elsewhere}
\end{cases}
\end{align*}
\]

Initially, the input signal was assumed to be known exactly and the suboptimal filtering was performed with the updating scheme omitted. The order of highest moment used was four, and the corresponding mean-square estimation errors $\sigma_i^2$ versus $n$ for $h = 0.1$ and $0.6$ were determined. These results, which are plotted in Figure 15, will be used later to check the quality of the updating scheme.

Computer simulations for $h = 0.1$ and $0.6$ were also performed with updating where the input signal was initially assumed to be uniformly distributed on $(-1, 1)$, and $n_0 = 100$. The mean-square errors obtained are plotted in Figure 16. That the values of $\sigma_i^2$ are a little larger than those plotted in Figure 15 can be attributed to a small degree of correlation between the input signal and the measurement noise in the computer simulation.

**Comparison with the Kalman Filter**

The Kalman filter given in (1.33)-(1.36) was applied to the specific example of the previous section, where
\[
\begin{align*}
\Phi &= h \\
\Gamma &= 1 \\
H &= 1
\end{align*}
\]
Figure 15. The Plot of $e^2_{ik}$ Versus $n_s$ for the First-Order Example Without Updating.
Figure 16. The Plot of $e_{4}^2$ Versus $n_s$ for First-Order Example.
and

$$Q = \text{Cov}(w) = m_w^{(2)} = \frac{11}{13}$$  \hfill (4.26)

$$R = \text{Cov}(v) = \frac{1}{3}$$

$$P_0 = \text{Cov}(x_0 | z_0) = \text{Cov}(x) = m_x^{(2)}$$

$$= \frac{1}{1-h^2} \left( 2hm_w^{(1)} m_x^{(1)} + m_w^{(2)} \right)$$

$$= \frac{1}{1-h^2} \frac{11}{13}$$

The mean-square estimation errors for $h = .1$ and $.6$ computed in the simulations are also plotted in Figure 16. A comparison of those curves in Figure 16 shows that the suboptimal adaptive filter was much better for this specific case. On the other hand, there are certain disadvantages in using the new filter. The new filtering algorithm is more complicated and requires more computer time, e.g., about 10 msecs are required to perform each step in the complete filtering algorithm on the B-5500 computer.

**Application to a Nonlinear System**

In developing the suboptimal adaptive filter, the whole concept was based on the stationarity of the states, which in turn depended upon the system being linear and time-invariant. Therefore, to arbitrarily apply this filter to a nonlinear system, it is necessary to linearize
the nonlinear element. Consider the system
\[ x_{k+1} = h \, \text{sign}(x_k) \left| x_k \right|^b + w_k \]  
(4.27)
where \( b \) is a real number which varies on the interval \((0.5, 3)\). The density function of the input signal \( w \) is given by (4.24).

The nonlinear element is to be linearized within the bounds of the state as
\[ \text{sign}(x_k) \left| x_k \right|^b = C x_k \]  
(4.26)
where \( C \) is a constant to be determined later. Due to the symmetry of the input signal density function, one may identify the lower and upper bounds of the state as \(-u\) and \(+u\), respectively, where \( u \) may be obtained by solving
\[ u = hu^b + 1 \]  
(4.29)
since \( |w| < 1 \). Minimizing the square error in a curve fitting procedure for (4.26), one obtains
\[ C = \frac{3}{b+2} \, u^{b-1} \]  
(4.30)
Thus, in the filtering and updating scheme the system becomes
\[ x_{k+1} = h' x_k + w_k \]  
(4.31)
where
The computer simulations for this nonlinear system gave good results when \( h = 0.1 \) and \( b \) was varied on the interval \((0.5, 3)\). For example, in the case where \( b = 3 \) the updated moments of \( w \) after 3000 measurements were given by

\[
\left\{ m_w^{(1)}, m_w^{(2)}, m_w^{(3)}, m_w^{(4)} \right\} = \{0.005, 0.842, 0.004, 0.715\} \tag{4.33}
\]

which were very close to the actual values \( \{0.005, 0.845, 0.004, 0.731\} \). Moreover, the mean-square error was found to be 0.113, which was somewhat smaller than that shown in Figure 16. The difference in mean-square error was partly due to the intrinsic property of the density function of the state. Another simulation with \( b = 0.5 \) and \( h = 0.6 \) was also performed. In this case the nonlinear term played a more important role in the system due to the larger value of \( h \). The data equivalent to (4.33) were \( \{0.005, 0.836, 0.003, 0.605\} \), which compared with those given in (4.33) were a little farther away from the actual values. The mean-square estimation error was found to be 0.186, which is somewhat greater than the corresponding value given in Figure 16. Hence, the applicability of the new technique to nonlinear system depends on how accurately the system can be approximated by a linear system. Similar conclusions are also valid for time-varying systems as shown in the following section.
Application to a Slowly Time-Varying System

In this section it is assumed that the multiplication factor $h$ in (2.4) is slowly time-varying according to

$$h = 0.4 + B \cdot \sin \left( \frac{k\pi}{2000} \right)$$

where $B$ is a real number used to adjust the changing rate of $h$. In the updating scheme $h$ may be approximated by piecewise constant values as shown in Figure 17. The analytical expression for the piecewise approximation in Figure 17 is

$$h = 0.4 + B \cdot \frac{1}{2} \left[ \sin(500I\pi/2000) + \sin(500(I+1)\pi/2000) \right]$$

for $500I < k < 500(I+1)$

where $I = 0, 1, 2, \ldots$. The algorithm used in updating the moments is modified accordingly. Before the starting of the $(500I+1)$th sampling period, the initially guessed moments $m^{(1)}$'s are replaced by the updated moments which are obtained after $500I$ measurements have been made, and $n^o$ is replaced by $n^o + 500I$. At this point the updating scheme is started again in the same manner as for $k = 1$.

Simulations were performed with the input signal given in (4.24) with $B = 0.01, 0.05, 0.1, 0.2$, and $0.4$. The mean-square error $e^2_4$ was computed for each case and is plotted in Figure 18, in which $e^2_4$ from the time-invariant case is also plotted and identified as $B = 0$. The curves indicate that the suboptimal adaptive filtering is applicable to time-varying systems which have sufficiently slow time variations.
Figure 17. Piecewise Approximation of $h$ Given by (4.34).
Figure 18. The Plot of $e_{14}^2$ Versus $n_8$ for the Time-Varying System.
Conclusions

The relationships between the measurement noise and the state and between the state and the input signal were established under the assumption of stationarity. Hence, these relationships may not be applied to update the moments of the conditional state $x_{k+1} | z_k$ for which the density function varies. However, the known bounds of the input signal, the dynamical characteristics of the system, and the updated moments of the input signal ensure that the conditional moments of the state are updated automatically.

The complete filtering algorithm, which was extended from the zero-order case to the first-order case by utilizing the dynamical characteristics of the system, was verified for representative systems by computer simulation. Furthermore, the results of the simulations showed that the suboptimal adaptive filter compares favorably with the Kalman filter.

Application of the new technique to the first-order problem indicated that computation time may prohibit the use of the algorithm for higher-order systems. In Chapter V the technique is extended to a class of second-order systems and the attendant computational problems are examined in detail.
CHAPTER V

THE SECOND-ORDER DYNAMICAL PROBLEM

The system to be investigated in this chapter has a single input and a single output, and the state equations can be expressed in the special form given in (2.6)-(2.7) with the measurement function given in (2.8). The existence of two states requires that the filtering algorithm be performed geometrically on a plane. The joint density function is approximated by a two variable polynomial with its coefficients expressed as functions of the joint moments of the states. A minimum rectangle containing the reachable set of the states is determined in a recursive manner in each sampling period. As before, relationships are established between the moments of the input signal, the moments of the measurement data, and the joint moments of the states. Through the use of a linear transformation, corresponding to a rotation and scaling of the axes of the phase variable plane, the suboptimal adaptive filtering technique is extended to this second-order system. A specific example was simulated to show the advantages of the new filter over the Kalman filter.

Approximation of a Joint Density Function by a Polynomial

The joint density function of two random variables may be approximated by a two variable polynomial. Let the joint random variables \( q_1 \) and \( q_2 \) be distributed on a square with \( q_1 \in (-1, 1) \) and \( q_2 \in (-1, 1) \),
\( q_2 \in (-1, 1) \). The joint density function of these two random variables, i.e., \( f_q(q) \) with \( q \) representing a 2-vector composed of \( q_1 \) and \( q_2 \), may be approximated as

\[
f_q(q) \approx \sum_{n_1=0}^{N} \sum_{j=0}^{n_1} a_{ij} q_1^i q_2^j
\]

where \( i = n_1 - j \) and \( N \) is the degree of the polynomial as well as the order of highest joint moment to be used. To minimize the square error in this approximation, one has

\[
s_{ij} = \frac{1}{N} \sum_{n_1=0}^{N} \frac{1}{(1+n_1+1)(j+n_2+1)} (1 - (-1)^{i+n_1+1}) (1 - (-1)^{j+n_2+1}) a_{ij}
\]

for \((n_1, n_2) = (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \ldots, (N-1, 1), (N-2, 2), \ldots, (1, N-1), (0, N)\)

where

\[
m_q^{(n_1, n_2)} = [q_1^{n_1} q_2^{n_2}]
\]

Solving (5.2), one obtains the \( a_{ij} \)'s as functions of \( m_q^{(n_1, n_2)} \)'s. In particular for \( N = 4 \), one has
\[ a_{00} = 1.89843750 - 5.27343750m_q(20) - 5.27343750m_q(02) \]
\[ + 3.69140625m_q(10) + 3.51562500m_q(22) + 3.69140625m_q(04) \]
\[ a_{10} = 5.62500000m_q(10) - 6.56250000m_q(30) - 2.81250000m_q(12) \]
\[ a_{01} = 5.62500000m_q(01) - 2.81250000m_q(21) - 6.56250000m_q(03) \]
\[ a_{20} = -5.27343750 + 37.96875000m_q(20) + 3.51562500m_q(02) \]
\[ - 36.91406250m_q(40) - 10.54687500m_q(22) \]
\[ a_{11} = 25.87500000m_q(11) - 19.68750000m_q(31) + m_q(13) \]
\[ a_{02} = -5.27343750 + 3.51562500m_q(20) + 37.96875000m_q(02) \]
\[ - 10.54687500m_q(22) - 36.91406250m_q(04) \]
\[ a_{30} = -6.56250000m_q(10) + 10.93750000m_q(30) \]
\[ a_{21} = -2.81250000m_q(01) + 8.43750000m_q(21) \]
\[ a_{12} = -2.81250000m_q(10) + 8.43750000m_q(12) \]
\[ a_{03} = -6.56250000m_q(01) + 10.93750000m_q(03) \]
\[ a_{40} = 3.69140625 - 36.91406250m_q(20) + 13.96940625m_q(10) \]
The coefficients in (5.4) were used in simulating the example to be presented later in this chapter.

**Solution of the Second-Order Problem**

For the second-order model given in (2.6)-(2.8), the problem is to find the suboptimal estimate of the states \( x_{k+1} \), which is a 2-vector in the second-order problem, based on the measurement set \( Z_{k+1} \) in the sense of least mean-square error. The updating scheme is investigated and then a complete algorithm is developed to obtain the conditional mean of \( x_{k+1} \) given \( Z_{k+1} \).

The Relationships Between the Moments of \( w, x, \) and \( z \)

As mentioned in the previous chapters, the adaptiveness of the filter is a result of the automatic updating of the moments of \( x_{k+1} | Z_k \) following the updating of the moments of \( w \). In this subsection the relationships between the moments of \( w \) and \( x \) and between those of \( x \) and \( z \) are developed as a major part of the updating scheme.

From (2.6), one may write
\[ E \left[ x_{1,k+1}^i x_{2,k+1}^j \right] = E \left[ (n_1 x_{1,k+1} w_{1})^i (n_2 x_{2,k+1} w_{2})^j \right] \]

\[ = E \left[ n_2 \sum_{i=0}^{n} n_4 \sum_{j=0}^{n} \left( \frac{i}{n_2} \right) \left( \frac{j}{n_4} \right) \left( n_2 \right)^{n_3} \left( n_4 \right)^{n_5} n_1^{n_6} n_4^{n_7} \left( n_2 + n_4 \right)^{n_8} \left( n_2 + n_4 \right)^{n_9} \right] \]

where \( n = i - n \) and \( n_5 = j - n \). Making use of the assumptions of stationarity and mutual independence of \( x \) and \( w \), one obtains

\[ m_x^{(i,j)} = \frac{1}{1 - n_4} \sum_{i=0}^{n} \left( \frac{i}{n_2} \right) \left( \frac{j}{n_4} \right) \left( n_2 \right)^{n_3} \left( n_4 \right)^{n_5} n_1^{n_6} n_4^{n_7} \left( n_2 + n_4 \right)^{n_8} \left( n_2 + n_4 \right)^{n_9} \]

Rearranging (5.6) yields

\[ m_x^{(i,j)} = \frac{1}{1 - n_4} \sum_{i=0}^{n} \left( \frac{i}{n_2} \right) \left( \frac{j}{n_4} \right) \left( n_2 \right)^{n_3} \left( n_4 \right)^{n_5} n_1^{n_6} n_4^{n_7} \left( n_2 + n_4 \right)^{n_8} \left( n_2 + n_4 \right)^{n_9} \]

Thus, if \( m_w^{(i,j)} \) are known, then the moments of \( x \) can be computed from (5.7) sequentially for increasing orders of the moments of \( x \).

Similarly from (2.8), one may write

\[ E \left[ x_{1,k+1}^i x_{2,k+1}^j \right] = E \left[ (d_1 x_{1,k+1} + d_2 x_{2,k+1} + d_3 x_{3,k+1})^i \right] \]

\[ = E \left[ \sum_{i=0}^{n_2} \left( \frac{i}{n_2} \right) \left( \frac{i}{n_2} \right) \left( n_2 \right)^{n_3} \left( n_4 \right)^{n_5} n_1^{n_6} n_4^{n_7} \left( n_2 + n_4 \right)^{n_8} \left( n_2 + n_4 \right)^{n_9} \right] \]
Due to the assumption of stationarity and the mutual independence of $x$ and $v$, (5.8) becomes

$$m_{z}^{(1)} = \sum_{n_1=0}^{i-n_1} \sum_{n_2=0}^{i-n_2} \binom{i-n_1}{n_1} \binom{i-n_2}{n_2} d_1 \cdot d_2 \cdot m_{x}^{(n_1)} m_{v}^{(n_2)}$$

(5.9)

To estimate the moments of $w$ from the estimated moments of $z$, one may use (5.7) and (5.9). From (5.9), or directly from (2.8), one has

$$\tilde{m}_z^{(1)} = d_1 \tilde{m}_x^{(10)} + d_2 \tilde{m}_x^{(01)} + \tilde{m}_v^{(1)}$$

(5.10)

Making use of (5.7) for $(i, j) = (1, 0)$ and $(0, 1)$, $\tilde{m}_x^{(10)}$ and $\tilde{m}_x^{(01)}$ are given as

$$\tilde{m}_x^{(10)} = \frac{d_1}{1-n_1} \tilde{m}_w^{(1)}$$

(5.11)

$$\tilde{m}_x^{(01)} = \frac{d_2}{1-n_2} \tilde{m}_w^{(1)}$$

Inserting (5.11) into (5.10) and solving for $\tilde{m}_w^{(1)}$ in terms of the estimated first moment of $z$ and the known moment of $v$, one obtains

$$\tilde{m}_w^{(1)} = \frac{\tilde{m}_z^{(1)} - \tilde{m}_v^{(1)}}{\left(\frac{d_1}{1-n_1} - \frac{d_2}{1-n_2}\right)}$$

(5.12)
Therefore, \( \tilde{m}_x^{(10)} \) and \( \tilde{m}_x^{(01)} \) may be computed from (5.11).

Similarly, from (5.9), or directly from (2.8), one has

\[
m_z^{(2)} = \frac{2}{1} m_x^{(10)} + 2d_1d_2m_x^{(11)} + 2(d_x^{(02)} + d_x^{(01)}) m_y^{(1)} \]

Making use of (5.7) for \((i, j) = (2, 0), (1, 1), \) and \((0, 2), \tilde{m}_x^{(20)}, \tilde{m}_x^{(11)}, \) and \( m_x^{(02)} \) are given as

\[
\tilde{m}_x^{(20)} = \frac{1}{1-h_1^2} \left( 2h_1d_1m_x^{(10)} \tilde{m}_w^{(1)} + d_1^2 \tilde{m}_w^{(2)} \right) 
\]

\[
\tilde{m}_x^{(11)} = \frac{1}{1-h_1^2} \left( h_1d_1m_x^{(10)} \tilde{m}_w^{(1)} + h_2d_1m_x^{(01)} \tilde{m}_w^{(1)} + d_1d_2 \tilde{m}_w^{(2)} \right) 
\]

\[
\tilde{m}_x^{(02)} = \frac{1}{1-h_2^2} \left( 2h_2d_2m_x^{(01)} \tilde{m}_w^{(1)} + d_2^2 \tilde{m}_w^{(2)} \right) 
\]

Inserting (5.14) into (5.13) and solving for \( \tilde{m}_w^{(2)} \) in terms of the estimated second moment of \( z, \) the estimated first moments of \( x, \) and the known moments of \( v, \) one obtains
\[
\tilde{m}_w(2) = \left[ \tilde{m}_z(2) - 2(d_1 \tilde{m}_x^{(10)} + d_2 \tilde{m}_x^{(01)}) \tilde{m}_w^{(1)} \right] - \tilde{m}_v(2) - \frac{2h_1 d_1^2}{1-h_2^2} \tilde{m}_x^{(10)} \tilde{m}_w^{(1)} \tag{5.15}
\]

\[
\left/ \left( \frac{2d_1 d_2^2}{1-h_1^2 h_2^2} + \frac{h_2 d_1}{1-h_2^2} \right) \right.
\]

Then \(\tilde{m}_x^{(20)}\), \(\tilde{m}_x^{(11)}\), and \(\tilde{m}_x^{(02)}\) can be computed from (5.14) for estimating the higher-order moments of \(w\). Repeating the above procedure, one may obtain all the required estimated moments of \(w\) from \(\tilde{m}_z^{(i)}\)'s and apply the updating formula equivalent to (2.29) with all the subscripts \(z\) replaced by \(w\) to obtain the updated moments of \(w\).

**The Bounds of \(x\)**

Simulations revealed that the states were distributed such that a minimum rectangle containing the region of the states could be obtained as in Figure 19. The subscript 0 in Figure 19 is used to indicate that no measurement has been made yet. The boundaries of the rectangle will be determined after a rotation of the coordinates has been made. One axis of the rectangle makes an angle \(\theta\) with the \(x_{1,0}\)-axis with

\[
\theta = \tan^{-1} \left( \frac{d_2}{d_1} \right) \tag{5.16}
\]
Figure 19. Minimum Rectangle Containing the Distribution Region of $x$. 
Rotating the $x_{1,0}$- and $x_{2,0}$-axes by the angle $\theta$ and denoting the new coordinates by $r_{1,0}$ and $r_{2,0}$ as shown in Figure 19, one has

$$r_{1,0} = x_{1,0} \cos \theta + x_{2,0} \sin \theta = \frac{1}{\sqrt{d_1^2+d_2^2}} (d_1 x_{1,0} + d_2 x_{2,0})$$  \hspace{1cm} (5.17)

$$r_{2,0} = x_{2,0} \cos \theta - x_{1,0} \sin \theta = \frac{1}{\sqrt{d_1^2+d_2^2}} (d_1 x_{2,0} - d_2 x_{1,0})$$

The transformation given in (5.17) also yields two new random variables $r_{1,0}$ and $r_{2,0}$, and the rectangle in Figure 19 containing the joint random variable $r_{1,0} r_{2,0}$ can be expressed as

$$l_{1,0} < r_{1,0} < u_{1,0}$$

$$l_{2,0} < r_{2,0} < u_{2,0}$$

where $l_{1,0}$, $u_{1,0}$, $l_{2,0}$ and $u_{2,0}$ are the bounds of the rectangle, which may be obtained in the same manner as (4.6) and (4.7) were derived.

Without loss of generality, one may assume that

$$h_1 > 0 ; h_2 > 0 ; h_2 > h_1 ; d_1 > 0 ; d_2 > 0$$  \hspace{1cm} (5.19)

Therefore,

$$u_{1,0} = \operatorname{Max}_{x_{1,0}} \frac{1}{\sqrt{d_1^2+d_2^2}} (d_1 x_{1,0} + d_2 x_{2,0}) = \frac{1}{\sqrt{d_1^2+d_2^2}} \left( \frac{d_1^2}{1-h_1} + \frac{d_2^2}{1-h_2} \right)$$  \hspace{1cm} (5.20)
\[ t_{1,0} = \min_{x_0} \frac{1}{\sqrt{d_1^2 + d_2^2}} \begin{bmatrix} d_1 x_{1,0} + d_2 x_{2,0} \end{bmatrix} = \frac{1}{\sqrt{d_1^2 + d_2^2}} \left( \frac{d_1^2}{1-h_1} + \frac{d_2^2}{1-h_2} \right) w_{\text{min}} \]

\[ u_{2,0} = \max_{x_0} \frac{1}{\sqrt{d_1^2 + d_2^2}} \begin{bmatrix} d_1 x_{2,0} - d_2 x_{1,0} \end{bmatrix} = \frac{1}{\sqrt{d_1^2 + d_2^2}} \left( \frac{d_1 d_2}{(1-h_2) - \frac{d_1 d_2}{1-h_1}} \right) w_{\text{max}} \]  \hspace{1cm} (5.21)

\[ t_{2,0} = \min_{x_0} \frac{1}{\sqrt{d_1^2 + d_2^2}} \begin{bmatrix} d_1 x_{2,0} - d_2 x_{1,0} \end{bmatrix} = \frac{1}{\sqrt{d_1^2 + d_2^2}} \left( \frac{d_1 d_2}{(1-h_2) - \frac{d_1 d_2}{1-h_1}} \right) w_{\text{min}} \]

From (2.7) and (5.17), it is seen that

\[ y_0 = \sqrt{d_1^2 + d_2^2} r_{1,0} \]  \hspace{1cm} (5.22)

Hence, the upper bound and lower bound of \( y_0 \), denoted by \( u y_0 \) and \( l y_0 \), respectively, are

\[ u y_0 = \sqrt{d_1^2 + d_2^2} u_{1,0} \]  \hspace{1cm} (5.23)

\[ l y_0 = \sqrt{d_1^2 + d_2^2} l_{1,0} \]

A minimum rectangle containing the reachable set of \( x_{k+1} | z_k \) must be investigated. Starting from the information given in (5.20) and (5.21), the bounds of this minimum rectangle may be computed from stage to stage.
It will be shown later that the minimum rectangle containing the distribution region of $x_{k+1} | Z_k$ always has one of its axes making the same angle $\theta$ with the $x_{1,k+1}$-axis on the phase plane. Hence, the same rotation given in (5.17) is made at each sampling period, and the new variables obtained from this transformation are denoted by $r_{1,k+1}$ and $r_{2,k+1}$. For indicating the bounds of the conditional states based on the new variables, the following notations are defined for $k = 0, 1, 2, \ldots$, as

- $u_{1,k+1,0}$ : upper bound of $r_{1,k+1}$ given $Z_k$
- $l_{1,k+1,0}$ : lower bound of $r_{1,k+1}$ given $Z_k$
- $u_{2,k+1,0}$ : upper bound of $r_{2,k+1}$ given $Z_k$
- $l_{2,k+1,0}$ : lower bound of $r_{2,k+1}$ given $Z_k$
- $u_{1,k+1}$ : upper bound of $r_{1,k+1}$ given $Z_{k+1}$
- $l_{1,k+1}$ : lower bound of $r_{1,k+1}$ given $Z_{k+1}$
- $u_{2,k+1}$ : upper bound of $r_{2,k+1}$ given $Z_{k+1}$
- $l_{2,k+1}$ : lower bound of $r_{2,k+1}$ given $Z_{k+1}$

The notations for indicating the bounds of the output $y$ are defined as
The notations defined above will be used throughout the remainder of this section.

Minimum Rectangle Containing the Reachable Set of $x_{k+1} \mid Z_k$

Since the states are stationary, due to the assumption that the system is in the steady-state, the bounds related to $x_1$ before any measurement is made are identical to those given in (5.20), (5.21), and (5.23), i.e.

$$u_{1,1,0} = u_{1,0} ; \quad l_{1,1,0} = l_{1,0}$$

$$u_{2,1,0} = u_{2,0} ; \quad l_{2,1,0} = l_{2,0}$$

$$\omega_{y,1,0} = \omega_{y_0} ; \quad \omega_{\beta,1,0} = \omega_{\beta_0}$$

Figure 19 is redrawn in Figure 20, where the second subscript 1 attached to each coordinate is used to indicate that the state under consideration is at the end of the first sampling period, and the minimum rectangle is identified as $P_{11}P_{12}P_{15}P_{16}$. 
Figure 20. Minimum Rectangle Containing the Distribution Region of $x_{1|Z_1}$.

Figure 21. Reachable Set of $x_{2|Z_1}$ and the Associated Minimum Rectangle.
Measuring $z_1$ and using (3.8)-(3.9), the bounds of $y_1|z_1$ are determined as

$$u_{y_1} = \begin{cases} 
uy_{1,0} & \text{when } z_1 \geq uy_{1,0}-1 \\
|z_1| & \text{when } z_1 < uy_{1,0}-1
\end{cases} \tag{5.25}$$

$$l_{y_1} = \begin{cases} 
|z_1|-1 & \text{when } z_1 > l_{y_1,0}+1 \\
l_{y_1,0} & \text{when } z_1 \leq l_{y_1,0}+1
\end{cases}$$

From (5.23), the bounds of $r_{1,1}|z_1$ are

$$u_{2,1} = \frac{1}{\sqrt{d_1^2+d_2^2}} uy_1$$

$$l_{2,1} = \frac{1}{\sqrt{d_1^2+d_2^2}} l_{y_1} \tag{5.26}$$

The measurement $z_1$ gives no further information for $r_{2,1}$; therefore, the bounds of $r_{2,1}|z_1$ are identical to the bounds of $r_{2,1}$ before any measurement is made, i.e.

$$u_{2,1,0} = u_{2,1,0} ; \quad l_{2,1,0} = l_{2,1,0} \tag{5.27}$$

The distribution region of $x_1|z_1$ is the rectangle $P_{11}P_{12}P_{13}P_{14}$ as shown by the shaded area in Figure 20. Letting the $r_{1,1},r_{2,1}$-coordinates of $P_{1i}$, for $i = 1, 2, 3, 4$, be denoted by $(r_{1,1i}, r_{2,1i})$, one has
\[(r_{1,1}', r_{2,1}') = (u_{1,1}', u_{2,1}') \text{ for } P_{11}\] (5.28)

\[(r_{1,2}', r_{2,2}') = (u_{1,1}', t_{2,1}') \text{ for } P_{12}\]

\[(r_{1,3}', r_{2,3}') = (t_{1,1}', t_{2,1}') \text{ for } P_{13}\]

\[(r_{1,4}', r_{2,4}') = (t_{1,1}', u_{2,1}') \text{ for } P_{14}\]

with \(u_{1,1}', t_{1,1}', u_{2,1}', t_{2,1}'\) given in (5.26) and (5.27). Solving (5.17) for \(x_{1,0}\) and \(x_{2,0}\), one obtains

\[x_{1,0} = r_{1,0} \cos \theta - r_{2,0} \sin \theta\] (5.29)

\[x_{2,0} = r_{1,0} \sin \theta + r_{2,0} \cos \theta\]

Denoting the \(x_{1,1}, x_{2,1}\)-coordinates of \(P_{11}\) as \((x_{1,1}', x_{2,1}')\), and making use of (5.29), with every second subscript 0 being replaced by 1 the coordinate of \(P_{11}\), for \(i = 1, 2, 3, 4\), may be written as

\[P_{11} : (x_{1,11}', x_{2,11}') = (r_{1,11} \cos \theta - r_{2,11} \sin \theta, r_{1,11} \sin \theta + r_{2,11} \cos \theta)\]

for \(i = 1, 2, 3, 4\) (5.30)

with \(r_{1,11}\) and \(r_{2,11}\) given in (5.28).

From the knowledge of the distribution region of \(x_{\perp}|Z_{\perp}\), the reachable set of \(x_{\perp}|Z_{\perp}\) may be determined using the assumption of (5.19).
The rectangle \( P_{11}, P_{12}, P_{13}, P_{14} \) in Figure 20 is redrawn in Figure 21.

The transformation

\[
\begin{align*}
x_{1,2} &= h_1 x_{1,1} \\
x_{2,2} &= h_2 x_{2,1}
\end{align*}
\]  

(5.31)

transforms \( P_{11}, P_{12}, P_{13}, \) and \( P_{14} \) into \( Q_1, Q_2, Q_3, \) and \( Q_4, \) respectively. Adding the vector \((d_{1\text{max}}, d_{2\text{max}})\) to the position vectors of \( Q_1, Q_2, \) and \( Q_3 \) yields the points \( P_{21}, Q_5, \) and \( Q_6, \) respectively, and adding another vector \((d_{1\text{min}}, d_{2\text{min}})\) to the position vectors of \( Q_1, Q_4, \) and \( Q_3 \) yields \( Q_8, Q_7, \) and \( P_{25}, \) respectively. Using these points the shaded area, which is the reachable set of \( x_{21}, \) is formed. Finally, a minimum rectangle containing the shaded area is found as \( P_{21}, P_{22}, P_{25}, \) \( P_{26}. \) Again the axis of this rectangle makes the same angle \( \theta \) with the \( x_{1,2} \)-axis, due to the fact that the vector \((d_{1w}, d_{2w})\) makes an angle \( \theta \) (or \( \theta + 180^\circ \)) with the \( x_{1,2} \)-axis. From Figure 21, the bounds of the rectangle are determined by the coordinates of \( P_{21} \) and \( P_{25}. \) From (5.31) and the procedure just described, the \( x_{1,2} \)-coordinates of \( P_{21} \) and \( P_{25} \) are

\[
\begin{align*}
P_{21} : (x_{1,21}, x_{2,21}) &= (h_1 x_{1,11} + d_{1\text{max}}, h_2 x_{2,11} + d_{2\text{max}}) \\
P_{25} : (x_{1,25}, x_{2,25}) &= (h_1 x_{1,13} + d_{1\text{min}}, h_2 x_{2,13} + d_{2\text{min}})
\end{align*}
\]  

(5.32)

with \((x_{1,11}, x_{2,11})\) given in (5.30). Making use of (5.17), the \( x_{1,2} \)-coordinates of \( P_{21} \) and \( P_{25} \) are found as
\[ P_{21} : (x_{1,21} \cos \theta + x_{2,21} \sin \theta, x_{2,21} \cos \theta - x_{1,21} \sin \theta) \]  

\[ P_{25} : (x_{1,25} \cos \theta + x_{2,25} \sin \theta, x_{2,25} \cos \theta - x_{1,25} \sin \theta) \]

Hence, the bounds of the rectangle \( P_{21} P_{22} P_{25} P_{26} \) are

\[ u_{1,2,0} = x_{1,21} \cos \theta + x_{2,21} \sin \theta \]  

\[ l_{1,2,0} = x_{1,25} \cos \theta + x_{2,25} \sin \theta \]  

\[ u_{2,2,0} = x_{2,21} \cos \theta - x_{1,21} \sin \theta \]  

\[ l_{2,2,0} = x_{2,25} \cos \theta - x_{1,25} \sin \theta \]

with \( x_{1,21} \) and \( x_{2,21} \) given in (5.32). The bounds of \( y_2 | Z_1 \) may be obtained from (5.34) and (5.22) as

\[ w_{y_2,0} = \sqrt{d_1^2 + d_2^2} u_{1,2,0} \]  

\[ l_{y_2,0} = \sqrt{d_1^2 + d_2^2} l_{1,2,0} \]

The entire procedure may be used recursively to obtain the minimum rectangle containing the reachable set of \( x_{k+1} | Z_k \), where the corresponding bounds are computed from one sampling period to the next. Since one axis of the rectangle makes the same angle \( \theta \) with the \( x_{1,k} \) axis for all \( k \), the estimation algorithm will be performed in a new
coordinate system in which one axis makes the angle $\theta$ with $x_{1,k}$-axis.

Furthermore, the rectangle is scaled into a square for the purpose of polynomial approximation.

The Moments of $x_{k+1} \mid Z_k$

Assuming that the moments of $x_k \mid Z_k$ have been found in the previous sampling period and using the characteristics given in (2.6), one has

$$m^{(ij)}_{x_{k+1} \mid Z_k} = E[(x_{1,k+1}^i x_{2,k+1}^j) \mid Z_k]$$

$$= E[(h_{1} x_{1,k} + d_{1} w_{k})^i (h_{2} x_{2,k} + d_{2} w_{k})^j \mid Z_k]$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \binom{n}{n_1, n_2} m_{n_1, n_2, n_3, n_4}$$

for all combinations of $i$ and $j$, where $n_{1} = i - n_{2}$ and $n_{3} = j - n_{4}$.

These conditional moments are used to obtain the moments of the new random variables corresponding to the new coordinates, $s_{1,k+1}$ and $s_{2,k+1}$ described in the next subsection.

Scaling of the Minimum Rectangle

Let $s_{k+1}$ be a 2-vector representing a new joint random variable obtained by applying a linear transformation to $x$ such that

$$s_{1,k+1} \mid Z_k \in (-1, 1)$$

(5.37)
This linear transformation may be divided into two steps. In the first step, (5.17) is applied with all subscripts 0 being replaced by \( k+1 \). In the second step, the following scaling procedure is performed.

Inserting (5.17) with appropriate subscripts into (5.38), one obtains

\[
\begin{align*}
{s_1}_{k+1} &= \frac{2}{u_{1,k+1,0} - \ell_{1,k+1,0}} (r_{1,k+1} - \frac{u_{1,k+1,0} + \ell_{1,k+1,0}}{2}) \\
{s_2}_{k+1} &= \frac{2}{u_{2,k+1,0} - \ell_{2,k+1,0}} (r_{2,k+1} - \frac{u_{2,k+1,0} + \ell_{2,k+1,0}}{2})
\end{align*}
\]  

(5.38)

where

\[
\begin{align*}
c_3 &= \frac{2}{u_{1,k+1,0} - \ell_{1,k+1,0}} \cos \theta \\
c_4 &= \frac{2}{u_{1,k+1,0} - \ell_{1,k+1,0}} \sin \theta
\end{align*}
\]  

(5.40)
\[ c_5 = - \frac{u_{1,k+1,0} + \ell_{1,k+1,0}}{u_{1,k+1,0} - \ell_{1,k+1,0}} \]

\[ d_3 = - \frac{2}{u_{2,k+1,0} - \ell_{2,k+1,0}} \sin \theta \]

\[ d_4 = \frac{2}{u_{2,k+1,0} - \ell_{2,k+1,0}} \cos \theta \]

\[ d_5 = - \frac{u_{2,k+1,0} + \ell_{2,k+1,0}}{u_{2,k+1,0} - \ell_{2,k+1,0}} \]

Thus, the conditional moments of \( s_{k+1} \) may be computed as

\[
\begin{align*}
\mathbb{E}[s_{1,k+1} | Z_k] &= E \left[ (d_3 x_{1,k+1} + c_4 x_{2,k+1} + c_5) \right] \\
\mathbb{E}[s_{2,k+1} | Z_k] &= E \left[ (d_3 x_{2,k+1} + c_4 x_{2,k+1} + d_5) \right] \\
\mathbb{E}[s_{3,k+1} | Z_k] &= E \left[ (d_3 x_{2,k+1} + c_4 x_{2,k+1} + d_5) \right]
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}[s_{4,k+1} | Z_k] &= E \left[ (d_3 x_{2,k+1} + c_4 x_{2,k+1} + d_5) \right] \\
\mathbb{E}[s_{5,k+1} | Z_k] &= E \left[ (d_3 x_{2,k+1} + c_4 x_{2,k+1} + d_5) \right]
\end{align*}
\]

Thus, the conditional moments of \( s_{k+1} \) may be computed as

\[
m_{ij}^{(k+1)} | Z_k = E[s_{1,k+1} s_{2,k+1} | Z_k]
\]

\[
= E\left[ (c_3 x_{1,k+1} + c_4 x_{2,k+1} + c_5) \right] (d_3 x_{2,k+1} + d_4 x_{2,k+1} + d_5) | Z_k]
\]

where \( n_1 = i - n_2 - n_3 \), and \( n_4 = j - n_5 - n_6 \). With the moments of \( s_{k+1} | Z_k \) computed, and under the condition given in (5.37), the density function of \( s_{k+1} | Z_k \) can be approximated as
\begin{equation}
\frac{f_{s_{k+1}|Z_k}(s_{k+1}|Z_k)}{f_{s_{k+1}|Z_k}(s_{k+1}|Z_k)} = \frac{N}{n_i} \sum_{j=0}^{n_i} \Sigma a_{ij} s_{1,k+1} s_{j}^{2,k+1}
\end{equation}

where \(i = n_i - j\) and \(a_{ij}\)'s are computed from (5.4). From \(f_{s_{k+1}|Z_k}(x_{k+1}|Z_k)\) the density function of \(s_{k+1}|Z_{k+1}\) may be obtained by using the Bayesian rule as shown in the next subsection.

**Moments of \(s_{k+1}|Z_{k+1}\)**

To find the estimate of \(x_{k+1}\) and the moments of \(x_{k+1}|Z_{k+1}\), it is necessary to compute the moments of \(s_{k+1}|Z_{k+1}\). By the same reasoning as in deriving (4.20), the density function of \(s_{k+1}|Z_{k+1}\) is

\begin{equation}
f_{s_{k+1}|Z_k}(s_{k+1}|Z_k) = \frac{f_{s_{k+1}|Z_k}(s_{k+1}|Z_k)}{\int_{s_{1,k+1}}^{s_{2,k+1}} f_{s_{k+1}|Z_k}(s_{k+1}|Z_k) ds_{1,k+1}}
\end{equation}

where the integration limits for \(s_{2,k+1}\) are given in (5.37), since the measurement \(z_{k+1}\) gives no further information on the bounds of \(r_{2,k+1}\) as shown in (5.27). The integration limits for \(s_{1,k+1}\) are obtained by inserting the bounds of \(r_{1,k+1}\) into the first equation of (5.38), i.e.

\begin{equation}
u_{s_{1,k+1}} = \frac{2}{u_{1,k+1,0}^2} \left( u_{1,k+1} - \frac{u_{1,k+1,0} e_{1,k+1,0}}{2} \right)
\end{equation}
\[ t_{s, l+1, k+1} = \frac{2}{u_{1, l+1, 0} - t_{l+1, k+1, 0}} \left( t_{l+1, k+1} - \frac{u_{l+1, k+1, 0} + t_{l+1, k+1, 0}}{2} \right) \]

with \( u_{l+1, k+1} \) and \( t_{l+1, k+1} \) determined in the manner shown by (5.25) and (5.26). Substituting (5.42) into (5.43), one obtains

\[
f_{s_{k+1}} | Z_{k+1} = \left( \begin{array}{c} s_{k+1} \\ Z_{k+1} \end{array} \right) \]

\[
= \frac{\prod_{n=0}^{N} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{i,j}^s s_{l+1, k+1} s_{l+2, k+1}}{\int_{s_{l+1, k+1}}^{s_{l+2, k+1}}} \frac{1}{u_{l+1, k+1, 0} - t_{l+1, k+1, 0}} \left( t_{l+1, k+1} - \frac{u_{l+1, k+1, 0} + t_{l+1, k+1, 0}}{2} \right) \int_{s_{l+1, k+1}}^{s_{l+2, k+1}} ds_{l+1, k+1} ds_{l+2, k+1} \]

\[
= \sum_{n=0}^{N} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{i,j}^s s_{l+1, k+1} s_{l+2, k+1} \]

where \( i = n_1 - j \) and \( a_{i,j}^s \) is equal to \( a_{i,j} \) divided by the integration term in (5.45). The moments of \( s_{k+1} | Z_{k+1} \) are given as

\[
\langle n_1 n_2 | Z_{k+1} = \int_{s_{l+1, k+1}}^{s_{l+2, k+1}} \sum_{n=0}^{N} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{i,j}^s s_{l+1, k+1} s_{l+2, k+1} \sum_{n=0}^{N} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{i,j}^s s_{l+1, k+1} s_{l+2, k+1} ds_{l+1, k+1} ds_{l+2, k+1} \]

Thus, it is now possible to compute the moments of \( x_{k+1} | Z_{k+1} \).
Moments of $x_{k+1} | Z_{k+1}$

The first moments of $x_{k+1} | Z_{k+1}$ is the estimate of $x_{k+1} | Z_{k+1}$ for the current sampling period, and all the moments of $x_{k+1} | Z_{k+1}$ up to the $N$th order must be computed for use in the filtering algorithm for the next sampling period.

Rearranging (5.38) to express $r_{1,k+1}$ and $r_{2,k+1}$ in terms of $s_{1,k+1}$ and $s_{2,k+1}$ and substituting the result into (5.29) where all the subscripts 0 are replaced by $k+1$, one may express $x_{k+1}$ in terms of $s_{k+1}$ as

$$x_{1,k+1} = c_6 s_{1,k+1} + c_7 s_{2,k+1} + c_8$$  \[5.47\]

$$x_{2,k+1} = d_6 s_{1,k+1} + d_7 s_{2,k+1} + d_8$$

where

$$c_6 = \frac{u_{1,k+1,0} - l_{1,k+1,0}}{2} \cos \theta$$

$$c_7 = -\frac{u_{2,k+1,0} + l_{2,k+1,0}}{2} \sin \theta$$

$$c_8 = \frac{u_{1,k+1,0} + l_{1,k+1,0}}{2} \cos \theta - \frac{u_{2,k+1,0} - l_{2,k+1,0}}{2} \sin \theta$$

$$d_6 = \frac{u_{1,k+1,0} - l_{1,k+1,0}}{2} \sin \theta$$
\[ d_7 = \frac{u_{2,k+1,0} + u_{2,k+1,0}}{2} \cos \theta \]
\[ d_8 = \frac{u_{1,k+1,0} + u_{1,k+1,0}}{2} \sin \theta + \frac{u_{2,k+1,0} + u_{2,k+1,0}}{2} \cos \theta \]

Similar to (5.41), one obtains

\[ m_{k+1}^{(1)} | z_{k+1} = \frac{1}{n_1} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_6} \frac{i!}{n_1! n_2! n_3!} \frac{j!}{n_4! n_5! n_6!} \left[ \frac{n_1}{c_6} c_7 c_8 d_6 d_7 d_8 m_{k+1}^{(1)} \right] \]

where \( n_1 = n_2 = n_3 \), \( n_4 = n_5 = n_6 \) and the suboptimal estimates of \( x_{1,k+1} | z_{k+1} \) and \( x_{2,k+1} | z_{k+1} \), i.e., \( \hat{x}_{1,k+1} \) and \( \hat{x}_{2,k+1} \), are \( m_{k+1}^{(1)} \) and \( m_{k+1}^{(01)} \), respectively.

The complete filtering algorithm has been fully established. Examples have shown that the algorithm requires about 0.5 sec real computer time per sampling interval on the B-5500 computer when the moments up to fourth-order are utilized.

An Example

The second-order algorithm was applied to a particular example. The mean-square errors obtained from computer simulations are compared with those obtained from the Kalman filter. The model used in this example is given as
\[ x_{1,k+1} = 0.1 x_{1,k} + v_{k} \]  
\[ x_{2,k+1} = 0.2 x_{2,k} + v_{k} \]  
\[ y_{k+1} = x_{1,k+1} + x_{2,k+1} \]  
\[ z_{k+1} = y_{k+1} + v_{k+1} \]  

The input signal has the density function

\[ f_{w}(w) = \begin{cases} \frac{J+1}{2} w^{J+1} & \text{when } |w| < 1 \\ 0 & \text{elsewhere} \end{cases} \]  

for \( J = 0, 10 \)

Computer simulations were performed by using moments up to fourth-order in the algorithm. The mean-square estimation errors were computed by

\[ e_{l}^{2} = \frac{1}{n} \sum_{k=1}^{n} \left[ (x_{1,l,k} - \hat{x}_{1,l,k})^2 + (x_{2,l,k} - \hat{x}_{2,l,k})^2 \right] \]  

and plotted in Figure 22.

Comparison with the Kalman Filter

The Kalman filter given in (1.33)-(1.36) was applied to (5.50)-(5.53) with
Figure 22. The Plot of $a^2$ Versus $n_s$ for the Second-Order Example.
\[
\hat{\Phi} = \begin{bmatrix}
0.1 & 0 \\
0 & 0.2 \\
\end{bmatrix}
\]

(5.55)

\[
\Gamma = [1 \quad 1]^T
\]

\[
\Lambda = \begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

and

\[
Q = \text{Cov}(w) = \frac{J+1}{J+3}
\]

(5.56)

\[
R = \text{Cov}(v) = \frac{1}{3}
\]

\[
\mathbf{P}_0 = \text{Cov}(x) = \begin{bmatrix}
\frac{1}{1-h_1^2} m_x^{(2)} & m_x^{(11)} \\
0 & \frac{1}{1-h_2^2} m_x^{(11)}
\end{bmatrix}
\]

(11) (2)

\[
\begin{bmatrix}
0 \\
\frac{1}{1-h_2^2} m_x^{(2)}
\end{bmatrix}
\]

(11) (02)
The mean-square errors $e^2_{\text{Kalman}}$ obtained from simulations were computed in the same manner as given by (5.54) and also plotted in Figure 22 for comparison. The curves show that the suboptimal adaptive filter was only a little better when $J = 0$, but demonstrated a considerable improvement for $J = 10$. Therefore, when the Kalman assumptions are almost satisfied, e.g. all noise sources are not far from gaussian, the Kalman filter is recommended because of its relative simplicity for on-line operation. However, when the noise is far from gaussian, the suboptimal adaptive nonlinear filter developed in this thesis is recommended.

Conclusions

The algorithm of the suboptimal adaptive filter for the second-order problem was straightforward to develop but cumbersome to use in real-time calculation. Because the main disadvantage of the new technique is the large amount of computer time required, applications are limited to systems with relatively low sampling rates. However, there are also some very attractive features of the new technique as shown in the increased filtering accuracy for the specific example. The mean-square error resulting from the new filter was only one-sixth of that obtained from the Kalman filter in the case considered where the input signal was much different from gaussian.
CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

This dissertation has considered the problem of suboptimally estimating the states of a linear, time-invariant, discrete noisy system. Using a moment technique, adaptive algorithms based on the Bayesian decision rule were developed for the static case, first-order systems, and a class of second-order systems.

Conclusions

The suboptimal adaptive filter was shown to be adaptive by the decrease in mean-square estimation error as the number of the data observed was increased. This adaptive characteristic may be directly attributed to the updating scheme. Through relationships between the moments of the input signal, the moments of the states, and the moments of measurement data, the initially unknown moments of the input signal were estimated and updated sequentially as additional measurements were obtained. Following the updating of the input signal, the moments of the conditional states were automatically updated to yield a complete nonlinear filtering algorithm.

If the moments of the input signal were known exactly, then the mean-square estimation error was decreased as the order of the moments used in the estimation algorithm was increased. However, if the updated moments deviated from their true values, e.g. due to correlation, the
mean-square error increased with an increase in the order of the highest moments used, since some additional error could be contributed to the inaccurate higher-order moments. On the other hand, the execution time for the complete filtering algorithm increased rapidly with the order of the highest moments involved. Both of these factors must be considered in determining the number of moments to be used in the algorithm. For those examples considered in this thesis research, acceptable results were obtained by using up to fourth-order moments.

The minimum set containing the reachable states of the system was determined for first-order systems and a class of second-order systems. However, there are certain theoretical difficulties in making an extension to large-scale systems. In addition, the computing speed limitations of the digital computer tend to discourage the extension to higher-order systems.

There are other important features of the suboptimal adaptive filter which should be emphasized. Simulations showed that the suggested filtering algorithm was rather insensitive in specific practical situations to correlation between the input signal and the measurement noise, although the updating scheme itself theoretically became invalid when a sufficient amount of correlation was present. The new technique was arbitrarily applied to some non-additive noise problems, slowly time-varying systems, and certain nonlinear systems with satisfactory results. Finally, the single most attractive feature of the new algorithm was that in those cases where the input signal was much different from gaussian, the suboptimal filter gave a much
lower mean-square estimation error than the Kalman filter. Furthermore, the adaptive filter was able to estimate and update the density function of the input signal, which was unknown a priori.

Recommendations for Further Work

Three problems related to this dissertation research are suggested for further study. The first recommendation is that the method be extended to higher-order systems. Secondly, to avoid the cumbersome calculations of the present method, a specific form of suboptimal nonlinear filter is suggested. Finally, it is recommended that the performance of the suboptimal adaptive nonlinear filter developed in this thesis research be investigated as part of a closed-loop optimal control system.

Even though the associated computational problems for higher-order systems are yet to be resolved in general, it may be worthwhile to extend the present algorithm to handle particular classes of higher-order systems. Theoretically, the extension for the updating scheme is quite straightforward. However, the reachable set of the states and the associated minimum rectangle are difficult to determine when the system order is higher than two. One reason for these problems in extending the existing method to higher-order systems is that a geometrical viewpoint was used exclusively in this thesis. An analytical method coupled with the present geometrical plane or state-space approach may very well make such an extension possible.

A possible approach to the problem considered in this dissertation is to utilize, instead of polynomial approximations, certain special
functions to form an approximate specific nonlinear filter. These functions must have approximately the same form from stage to stage, and the filter parameters must be determined by several lower-order moments. If these parameters may be computed from stage to stage in a recursive manner with a relatively small amount of computer time, and may also be estimated and updated from the measurement data, then the computational problems to be encountered in extending the technique of this thesis to higher-order systems may be solved.

Finally, an important application for the suggested algorithm is the estimation of states in optimal control systems. One may design a suboptimal adaptive controller by using the nonlinear suboptimal adaptive filter followed by an optimal deterministic controller. It may be very difficult to analyze the performance of such a system theoretically. However, exhaustive testing of such a system by computer simulation should be feasible.

This thesis has presented a new technique for nonlinear filtering which is both adaptive and suboptimal. The resulting filter has been compared favorably with the Kalman filter, especially in those cases where the disturbance signals were far from being gaussian. The problems outlined in this section are recommended as fruitful areas for further work.


Te-son Kuo, son of Hsu-tun Kuo and Chow-may Kuo, was born in Taoyuan, Taiwan, the Republic of China, on January 8, 1938. He attended the National Taiwan University and received his B.S.E.E. in June, 1960. During the period from September, 1963 to September, 1964, he attended the Philips International Institute of Technological Studies in Eindhoven, Netherlands. In 1964 and 1965, he worked as a teaching assistant and instructor at the National Taiwan University. In September 1966, he came to the United States to begin graduate study at the Georgia Institute of Technology under the sponsorship of a Fulbright-Hayes Grant. During the second year of his graduate program, he accepted an assistantship which supported him until December, 1969.

In October, 1966, Mr. Kuo married the former An-hsueh Chang, daughter of Tsuan Chang and Young-mei Chang, Taoyuan, Taiwan, the Republic of China. He will complete the requirements for his Ph. D. in Electrical Engineering in the field of automatic control systems in December, 1969.