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DEPENDENCE OF STRESS AND HEAT FLUX UPON TEMPERATURE GRADIENT
IN A HOMOGENEOUS IDEALLY ELASTIC SOLID

A THESIS
Presented to
The Faculty of the Graduate Division
by
José Villanueva y Guardia

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Doctor of Philosophy
in the School of Engineering Mechanics

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DEPENDENCE OF STRESS AND HEAT FLUX UPON TEMPERATURE GRADIENT
IN A HOMOGENEOUS IDEALLY ELASTIC SOLID.
Este trabajo está dedicado a mi esposa e hijos.
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SUMMARY

The constitutive equations of homogeneous ideally elastic materials possessing definite isotropic symmetries in the reference state are found after postulating that whenever such a material is simultaneously subjected to a deformation gradient \( D = [x_{\alpha \gamma}] \) and to a temperature gradient \( \text{grad } \theta = [\theta_{\alpha}] \) both the stress tensor \( t = [t_{\alpha \beta}] \) and the heat flux vector \( h = [h_{\alpha}] \) are functions of the deformation gradient, the temperature gradient, and the material parameters relevant to the formulation.

Particular attention is given to the case of isotropic materials with a center of symmetry (holohedral) and to isotropic materials without a center of symmetry (hemihedral).

The approach used in deriving these constitutive equations consists of a direct application of the principle of material indifference and the determination of the integrity basis for invariant functions of vectors and tensors corresponding to the group of transformations of the reference state which characterizes the symmetries of the material in each case. This method is based on Hilbert's basis theorem; it was first employed in problems of this type by Pipkin and Rivlin in their paper on the formulation of constitutive equations in continuum physics (4).

A thorough dimensional analysis is performed on these constitutive equations. This analysis shows the exact form in which the material parameters involved in the formulation enter in the equations.

The constitutive equations corresponding to the isotropic material
with a center of symmetry are

\[ h = k(\xi_0 I + \xi_1 c + \xi_2 c^2) \cdot \text{grad } \theta \quad (60) \]

where \( c = [c_{ij}] = [x_\alpha x_\beta, A] \), and

\[ t = \mu(\psi_0 I + \psi_1 c + \psi_2 c^2) + \lambda[\psi_3 \theta + \psi_4 (c\theta + \theta c) + \psi_5 (c^2 \theta + \theta c^2)] \quad (61) \]

where \( \theta = [\theta, \theta, \theta, \beta] \), \( \lambda = \frac{\alpha^2 k^2}{\rho c^2} \), and where the \( \xi_i (i = 0, 1, 2) \) and the \( \psi_i \)
\((i = 0, 1, \ldots, 5)\) are dimensionless functions of the nine dimensionless scalars \( c^{-1/2}, \text{tr}_c, \text{tr}_c^2, \text{tr}_c^3, \lambda \text{tr}_c^2, \lambda \text{tr}_c^3, \lambda \text{tr}_c^4, a(\theta - \theta_0) \), and \( T = \frac{\alpha u}{\rho c} \).

Similarly, the response equations for an isotropic material without a center of symmetry are found to be given by

\[ h = k(n_0 I + n_1 c + n_2 c^2) \cdot \text{grad } \theta + \Pi(n_3 b + n_4 c \cdot b + n_5 c^2 \cdot b) \cdot \text{grad } \theta \quad (63) \]

where \( \Pi = \frac{k^2}{\mu} \sqrt{\frac{\alpha}{c}} \), and

\[ t = \mu(\psi_0 I + \psi_1 c + \psi_2 c^2) + \lambda[\psi_3 \theta + \psi_4 (c\theta + \theta c) + \psi_5 (c^2 \theta + \theta c^2)] + \]
\[ + \Pi[\psi_6 (cb - bc) + \psi_7 (c^2 b - bc^2) + \psi_8 (cbc^2 - c^2 bc)] + \]
\[ + \frac{\lambda \Pi}{\mu} [\psi_9 (bc\theta - \theta bc) + \psi_{10} (bc^2 \theta - \theta bc^2)]. \quad (64) \]
where $\Omega = k \sqrt{\frac{\alpha}{c}}$, and where the $\eta_i (i = 0, 1, \ldots, 5)$ and the $\eta_i (i = 0, 1, \ldots, 10)$ are dimensionless functions of the dimensionless scalar groupings $c^{-1/2}, trc, trc^2, trc^3, \lambda trc, \lambda trc^2, \lambda trc^3, a(\theta - \theta_0)$ and $T = \frac{\mu}{\rho c}$, where $\tau = \frac{k}{\mu} \sqrt{\frac{\alpha}{c}}$.

From equation (61) one finds that for a two-dimensional temperature distribution given by $\theta = k_1 x + k_2 y$ the stress system required to prohibit deformation must satisfy the universal relationship $k_1 k_2 (t_{xx} - t_{yy}) = (k_1^2 - k_2^2) t_{xy}$. The exceptional case is a material for which $\lambda \theta_0 = 0$.

Considering next the case of isochoric shear in the presence of a two-dimensional temperature distribution one finds that the shear stress no longer depends solely on odd powers of the tangent of the shear angle, but on even powers as well, showing that, for a maintained temperature distribution, a reversal of the stress will not correspond to an exact reversal of the deformation. Again, the normal stress depends upon odd powers of the tangent of the shear angle in contrast to the result of classical finite elasticity with uniform temperature. These results are expressed by equations (85-88).

One also considers the case of a homogeneous cylindrical tube uniformly stretched in the presence of a radial temperature gradient, it is found in this case that the prescribed deformation will not (in general) be maintained by a uniform axial stress; indeed, one finds that the axial stress is a function of the radial distance from the geometric center of the tube. Furthermore, it is found that an end torque and lateral pressures are necessary to maintain the deformation in the presence of the temperature gradient. The system of forces necessary to maintain the
deformation is found to be a definite function of the temperature gradient as well as of the deformation.

Considering next the case of a homogeneous and isotropic (holohedral) cylindrical tube which is simultaneously stretched, twisted, and subjected to an axial temperature gradient it is found that the axial stresses are not uniform (in general), also that it is necessary to apply lateral pressures to the tube, as well as a torque in order to maintain the desired deformation. It is further found that in this particular case there is no heat flux across the lateral surfaces of the tube; however, a definite phenomenon is observed in the sense that there is a component of the heat flux in the tangential direction, which indicates that the heat flux travels in helices. It is emphasized here that this phenomenon is a consequence of the combined effect of the twisting, the stretching, and the axial temperature gradient, for if any of these is absent then the tangential component of the heat flux will be zero.

Some of the results of this work have been published in the September, 1964 issue of the Journal of Applied Mechanics in a joint paper by Marris and Villanueva.
CHAPTER I

INTRODUCTION

This work is motivated by the idea that stress in an elastic material can be developed through temperature gradients as well as deformation gradients. One develops the constitutive equations of an ideally elastic solid under the combined action of deformation gradients and temperature gradients. These constitutive equations are then applied to particular geometries.

The material is assumed to be homogeneous and possesses definite isotropic properties in its reference state of uniform temperature and zero stress.

Specifically one seeks the rational consequences of postulating that the stress tensor \( \tau \) and the heat flux vector \( h \) are both dependent upon the deformation gradient \( \frac{\delta x_\alpha}{\delta x_A} \) and the temperature gradient vector

\[
\text{grad } \theta = \nabla \theta = \left[ \frac{\delta \theta}{\delta x_\alpha} \right]
\]

where \( x_\alpha \) are the coordinates in the deformed state of a point whose coordinates in the reference state are given by \( x_A \).

This postulate is justified through the principle of equipresence. In general this principle asserts that "a variable present as an independent variable in one constitutive equation should be present in all." This principle has been invoked by Coleman and Noll (1) and by Coleman and Mizel (2) in their fundamental researches on the thermostatics and the thermodynamics of continuous media. Coleman and Mizel (2) propose, on the basis of this principle for a rigid material that not only the heat
flux vector \( \mathbf{h} \), but also the specific internal energy \( \varepsilon \) and the specific entropy \( \eta \) each be non-linear functions not only of temperature but also of the first \( n \) spatial gradients of temperature (\( n \) being a finite positive integer). Coleman and Mizel then establish on the basis of the second law of thermodynamics that the scalar expressions for \( \varepsilon \) and \( \eta \) are independent of the gradients of temperature of all orders, and furthermore show that the heat flux vector \( \mathbf{h} \) must be more sensitive to changes in the first order temperature gradient \( \theta, \alpha \) than to higher order temperature gradients. The first result requires modification for the case of an elastic material. Coleman and Mizel's second result, however, indicates that it is profitable to restrict considerations to first order temperature gradients in the elastic case. At this point, one may remark upon the so-called Curie's theorem generally invoked in researches in irreversible thermodynamics which asserts essentially that couplings between tensors of different order are not possible. Counter-examples to this theorem are found in the kinetic theory of gases (see for example reference 5). Furthermore, most of the results which are attributed to this theorem are obtainable from the principle of material indifference\(^1\) and the rest are a consequence of the linearization processes imposed on these researches.

The present research is concerned with materials whose isotropic properties are:

1. Isotropic materials with a center of symmetry (holohedral).

---

1. The principle of material indifference requires that all constitutive equations of continuum physics be form-invariant under arbitrary rotations of the physical system.
These are materials whose constitutive equations are form-invariant with respect to the full orthogonal group of transformations of the natural (reference) state. The full orthogonal group of transformations consists of all transformations whose determinant is either plus or minus one. Any homogeneous steel would be an example of this type of material.

2. Isotropic materials with no center of symmetry (hemihedral). This type of material has its constitutive equations form-invariant with respect to the proper orthogonal group of transformations of the reference state only (i.e., rotations are allowed but not reflections). The proper orthogonal group of transformations consists of all transformations whose determinant is plus one. Quartz, which has a spiral microstructure, may be pulverized and repacked to create such a material.

Thus, summarizing, one postulates that for an ideally elastic material which is homogeneous in its reference state and unstressed at the reference temperature, both the stress tensor $\sigma$ and the heat flux vector $h$ are functions of the deformation gradient ($D = \frac{\partial x}{\partial X}$), the temperature gradient vector ($\text{grad } \theta = \nabla \theta = [\theta_{,\alpha}]$), and the material parameters relevant to the formulation.
CHAPTER II
THE TENSORIAL FORM OF THE CONSTITUTIVE EQUATIONS

Consider a body that is deformed in such a way that a particle initially at \( X_\alpha \) in a rectangular Cartesian coordinate system \( x \) moves to \( x_\alpha \) in the same system. Suppose also that a static temperature distribution with non-zero gradient of components \( \theta , \alpha \) in the coordinate system \( x \) is acting on the body simultaneously with the deformation gradient. The basic postulates are expressed mathematically, thus,

\[
\mathbf{t} = \mathbf{f} (D, \text{grad } \theta) \tag{1}
\]

or \( t_{\alpha \beta} = f_{\alpha \beta} (x_\gamma, \theta, \gamma) \)

and

\[
\mathbf{h} = \mathbf{g} (D, \text{grad } \theta) \tag{2}
\]

or \( h_\alpha = g_\alpha (x_\gamma, \theta, \gamma) \)

where dependence of \( f \) and \( g \) upon material parameters is understood and will be considered in a subsequent chapter.

Applying a result that Pipkin and Rivlin (4, equation 2.27) established by using the principle of material indifference, it follows that the dependence of \( t_{\alpha \beta} \) on the arguments \( x_\gamma, \theta, \gamma \) must be of the form
\[ t_{\alpha\beta} = \frac{\delta x_\alpha}{\delta X_A} \frac{\delta x_\beta}{\delta X_B} F_{AB}(C_{DG}, \Theta, G, C^{-1/2}) \]  

(3)

where \( F_{AB} \) is a single-valued function of its arguments and

\[ C_{DG} = \frac{\delta x_\gamma}{\delta x_D} \frac{\delta x_\gamma}{\delta x_G}, \]

\[ \Theta, G = \frac{\delta x_\gamma}{\delta x_G} \frac{\delta \Theta}{\delta x_\gamma}, \text{ and} \]

\[ C = \det C_{DG} = |C_{DG}| \]  

(4)

Similarly one writes for the heat flux

\[ h_\alpha = \frac{\delta x_\alpha}{\delta X_A} G_{A}(C_{DG}, \Theta, G, C^{-1/2}) \]  

(5)

where \( C^{-1/2} \) is a pure scalar.

**Isotropy Considerations**

At this point it is necessary to consider the restrictions which are imposed on the constitutive equations of the form (1) and (2) or their equivalents (3) and (5) by any symmetry which the material may possess in its reference state, in which it is undeformed and at a constant temperature \( \Theta_0 \). The symmetry of the material is characterized with respect to a rectangular cartesian system \( x \) by a group of transformations \( \{s\} \) which is a sub-group of the full orthogonal group. Each member transformation of the group \( \{s\} \) transforms the coordinate system \( x \) into an equivalent system \( x^* \) in which the constitutive equation of the material takes the same form as in the system \( x \).
Let the systems $x$ and $x^*$ be related by a transformation of the group $\{s\}$, then

\[ x^*_\alpha = s_{\alpha\beta}x_\beta \]  

(6)

where

\[ s_{\gamma\alpha}s_{\gamma\beta} = s_{\alpha\gamma}s_{\beta\gamma} = \delta_{\alpha\beta} \]  

(7)

$\delta_{\alpha\beta}$ being the Kronecker delta.

One now defines $C^*$ and $\theta^*$ in a manner analogous to $C_{DG}$ and $\theta_{DG}$ as

\[ C^*_{DG} = \frac{\delta x^*}{\delta x^*_G} \frac{\delta x^*}{\delta x^*_G} \]  

(8)

and

\[ \theta^*_{DG} = \frac{\delta x^*}{\delta x^*_G} \frac{\delta \theta}{\delta x^*_G} \]  

(9)

Since the coordinate system $x^*$ is equivalent to the system $x$ one has from equations (3) and (5)

\[ t^*_{\alpha\beta} = \frac{\delta x^*}{\delta x^*_A} \frac{\delta x^*_B}{\delta x^*_A} F_{AB}(C^*_{DG}, \theta^*_{DG}, C^*_{-1/2}) \]  

(10)

and
\[ h_a = \frac{\delta x^*_a}{\delta x^*_A} \Gamma_A(x^*, \theta^*, C^{*_1/2}) \]  

(11)

since \( x^* \) and \( x \) are related through equations (6) and (7) it follows that

\[ \begin{align*}
x^*_A = S_{AB} x^*_B, \\
h_a \left( \frac{\delta x^*_A}{\delta x^*_a} \right) = S_{AB} h_a \left( \frac{\delta x^*_B}{\delta x^*_a} \right), \\
C_{AB} = S_{AM} S_{BN} C_{MN}, \\
\theta^*_A = S_{AM} \theta^*_M, \text{ and} \\
t^a_B \left( \frac{\delta x^*_A}{\delta x^*_a} \right) = S_{AM} S_{BN} t^a_B \left( \frac{\delta x^*_M}{\delta x^*_a} \right) \left( \frac{\delta x^*_N}{\delta x^*_a} \right).
\end{align*} \]

(12)

From (5), (6), (7), (11), and (12) one now obtains

\[ G_A(x^*, \theta^*, C^{*_1/2}) = S_{AB} G_B(x^*, \theta^*, C^{*_1/2}) \]  

(13)

Now, if \( \psi^*_A \) and \( \psi^*_A \) are the components in the coordinate systems \( x \) and \( x^* \) of an arbitrary vector \( \psi \), then

\[ \psi^*_A = S_{AB} \psi^*_B \]  

(14)

From (13), multiplying throughout by \( \psi^*_A \) and using (14)

\[ \psi^*_A G_A(x^*, \theta^*, C^{*_1/2}) = \psi^*_A G_A(x^*, \theta^*, C^{*_1/2}) = G. \]  

(15)
From equation (15) it follows that

$$G_A(C_{DG}, \theta G, C^{-1/2}) = \frac{\delta G}{\delta \psi_A}$$  \hspace{1cm} (16)$$

Equation (14) implies that $G$ is a scalar invariant, under the group of transformations $\{s\}$ which characterizes the symmetry of the material, of the vectors $\psi_A$ and $\theta G$, of the symmetric tensor $C_{DG}$, and of the scalar $C^{-1/2}$. Thus, $G$ must be expressible as a polynomial in $C^{-1/2}$ and the elements of an integrity basis (under the group of transformations $\{s\}$) for the vectors $\psi_A$ and $\theta G$ and the symmetric tensor $C_{DG}$. Since $G$ is linear in the components of the vector $\psi_A$, one may express it in the form

$$G = \sum_{R=1}^{Q} P_R A_R$$  \hspace{1cm} (17)$$

where $A_R (R = 1, 2, \ldots, Q)$ are the elements of the integrity basis which are linear in the vector $\psi_A$ and $P_R (R = 1, 2, \ldots, Q)$ are polynomials in $C^{-1/2}$ and the elements of the integrity basis which do not involve $\psi_A$ at all.

Similarly, for the stress tensor one obtains from equations (3),

1. Hilbert (5) has shown that for any group of transformations $\{s\}$ there exists a finite set of polynomial scalar invariants of $\mu$ vectors and $\nu$ tensors such that any scalar invariant (under the group of transformations $\{s\}$) of these vectors and tensors can be expressed as a polynomial in the members of the finite set. Such a set of polynomial invariants is called an invariant basis for polynomials in the $\mu$ vectors and $\nu$ tensors corresponding to the group of transformations $\{s\}$. If the invariant basis is such that none of its members is expressible as a polynomial in the others, then it is said to be an integrity basis. See also ref. (7).
(6), (7), (10) and (12):

\[
F_{AB}^{*} = \frac{1}{2}(C_{DG}, \theta, G, C^{-1/2}) =
\]

\[
S_{AM}^{*} S_{BN} F_{MN}^{*} (C_{DG}, \theta, G, C^{-1/2})
\]

(18)

Now let \( \psi_{A}^{*} \) and \( \phi_{A}^{*} \) be the components in the coordinate systems \( x \) and \( x^{*} \) of two arbitrary vectors \( \psi \) and \( \phi \). Then

\[
\psi_{A}^{*} = S_{AB} \psi_{B}^{*}
\]

and

\[
\phi_{A}^{*} = S_{AB} \phi_{B}^{*}
\]

(19)

and from (18) multiplying throughout by \( \psi_{A}^{*} \phi_{B}^{*} \) and using (19) one finds:

\[
\psi_{A}^{*} \phi_{B}^{*} F_{AB}^{*} (C_{DG}, \theta, G, C^{-1/2}) =
\]

\[
= \psi_{A} \phi_{B} F_{AB} (C_{DG}, \theta, G, C^{-1/2}) = F
\]

(20)

Hence, it follows from (20) that

\[
F_{AB} (C_{DG}, \theta, G, C^{-1/2}) = \frac{\delta^{2} F}{\delta \psi_{A} \delta \phi_{B}}
\]

(21)

Again, equation (20) shows that \( F \) is a scalar invariant, under the group of transformations \{s\} which characterizes the symmetry of
the material, of the vectors $\phi_A$, $\phi_B$ and $\theta_G$, of the symmetric tensor $C^{DG}$, and of the scalar $C^{-1/2}$. Thus $F$ must be expressible as a polynomial in $C^{-1/2}$ and the elements of an integrity basis, under the transformation group $\{s\}$, for the vectors $\psi_A$, $\phi_B$ and $\theta_G$, and the symmetric second order tensor $C^{DG}$. Since $F$ is multilinear\(^1\) in the components of the vectors $\psi_A$ and $\phi_B$, one may express $F$ in the form

$$F = \sum_{R=1}^{Q} T_R B_R$$

where $B_R (R = 1, 2, \ldots, Q)$ are the elements of the integrity basis which are multilinear in the vectors $\psi_A$ and $\phi_B$, and $T_R (R = 1, 2, \ldots, Q)$ are polynomials in $C^{-1/2}$ and the elements of the integrity basis which are independent of the vectors $\psi_A$ and $\phi_B$.

\(^1\) Isotropic Materials Possessing a Center of Symmetry (Holohedral)

If when the material is undeformed and the temperature gradient is zero, it is isotropic and possesses a center of symmetry, then the appropriate transformation group expressing the symmetries of the material is the full orthogonal group. In this case, one obtains.

The Heat Flux Equation. The elements of an integrity basis, corresponding to the full orthogonal group, for the vectors $\psi_A$ and $\theta_G$ and the symmetric tensor $C^{DG}$ which are linear in $\psi_A$ (see ref. 4) may be taken as

\[^1\] In this context multilinear is to be interpreted as simultaneously linear in each of the vectors $\psi$ and $\phi$.\]
Those elements which are independent of the vector $\psi_A$ may be taken as

$$\begin{align*}
&\theta_A^A, \quad \theta_A^C \theta_B^B, \quad \text{and} \\
&\psi_A^C \theta_{AB}^D \theta_B^B
\end{align*}$$

Replacing $A_R (R = 1, 2, \ldots, Q)$ in (17) by (23), one obtains

$$G = P_1 \psi_A \theta_A^A + P_2 \psi_C \theta_A^C \theta_B^B +$$

$$+ P_3 \psi_A^C \theta_{AB}^D \theta_B^B$$

where $P_1$, $P_2$, and $P_3$ are polynomials in the quantities (24), $C^{-1/2}$, and material parameters. But from equation (16)

$$G_A = \frac{\delta G}{\delta \psi_A},$$

hence

$$G_A = P_1 \theta_A^A + P_2 \psi_A^C \theta_B^B + P_3 \psi_A^C \theta_{AB}^D \theta_B^B$$

and from equation (5),
\[ h_a = \frac{\delta x}{\delta x_A} G_A(C_{DG}, \theta, \epsilon^{1/2}) \]

one obtains

\[ h_a = \frac{\delta x}{\delta x_A} (P_1 e_A + P_2 C_{AB}, B + P_3 C_{AD} C_{DB}, B) \]  

which from the definitions of \( C_{AB} \) and \( \theta, A \) may be shown to transform into

\[ h_a = (P_1 c_{a\beta} + P_2 c_{\alpha}\gamma c_{\gamma\beta} + P_3 c_{a\delta} c_{\delta\eta} c_{\eta\beta}) \theta, \beta \]

where \( c_{a\beta} \) is defined by

\[ c_{a\beta} = \frac{\delta x}{\delta x_A} \frac{\delta x}{\delta x_A} \]  

If one now uses the Cayley-Hamilton theorem for the matrix \( C = [c_{a\beta}] \),

This may be stated as follows

\[ c^3 - c^2 trC + 1/2 [(trC)^2 - trC^2] c \]
\[- 1/6 [(trC)^3 - 3(trC)(trC^2) + 2 trC^3] I = 0 \]

where \( I \) is the unit matrix. One notes that

\[ c^3 = [c_{a\gamma} c_{\gamma\delta} c_{\delta\beta}] \text{ and } \]
\[ c^2 = [c_{a\gamma} c_{\gamma\beta}] \]

and that
Applying equations (30) to (33) to equation (28) one sees that

\[ h_\alpha = (Q_0 \delta_{\alpha\beta} + Q_1 c_{\alpha\beta} + Q_2 c_{\alpha\gamma} c_{\gamma\beta}) \theta_{,\beta} \]  

where \( Q_0, Q_1 \) and \( Q_2 \) are polynomials in the quantities (24), \( C^{-1/2} \), and material parameters. Using equations (4) and (29), one has

\[ \theta_{,\alpha} \theta_{,\beta} = \theta_{,\alpha} c_{\alpha\beta}, \theta_{,\beta} \]  
\[ \theta_{,\alpha} A_{\gamma\delta} \theta_{,\beta} = \theta_{,\alpha} c_{\alpha\gamma} c_{\gamma\delta} \theta_{,\beta} \]  
\[ \theta_{,\alpha} C_{AB} \theta_{,\beta} = \theta_{,\alpha} c_{\alpha\gamma} c_{\gamma\delta} \delta_{,\gamma} \theta_{,\beta} \]  
\[ \theta_{,\alpha} C_{AB} \theta_{,\beta} = \theta_{,\alpha} c_{\alpha\gamma} c_{\gamma\delta} \delta_{,\gamma} \theta_{,\beta} \]  

thus, using equation (30) one notes that the terms \( Q_0, Q_1, \) and \( Q_2 \) of equation (34) are expressible as polynomials in

\[ \text{tr}_C, \text{tr}_C^2, \text{tr}_C^3, \]
\[ \text{tr}_\Theta, \text{tr}_\Theta^2, \text{and} \text{tr}_\Theta^3 \]  

where \( \Theta = [\theta_{,\alpha}, \theta_{,\beta}] \) is the temperature gradient product tensor, and in
and material parameters. Using vector notation, $h$ may be written as

$$h = (Q_0 I + Q_1 c + Q_2 c^2) \cdot \text{grad } \theta.$$  \hspace{1cm} (37)

The preceding derivation is a modification of the general results of Pipkin and Rivlin in reference (4). It has been included in order to insure continuity. The derivations that follow, although a consequence of the work of Pipkin and Rivlin, are both extensions and applications of their work.

The Stress Equation. The products, multilinear in the vectors $\psi_A$ and $\phi_B$, which can be formed from the elements of an integrity basis for the second order tensor $C_{AB}$ and the vectors $\psi_A$, $\phi_B$ and $\theta_A$, for this particular case of an isotropic material possessing a center of symmetry may be taken as

$$\psi_A \delta_{AB} \phi_B, \quad \psi_A C_{AB} \phi_B, \quad \psi_A C_{AD} C_{DB} \phi_B,$$

$$\psi_A (\theta, B C_{AD} \theta, D + \theta, A C_{BD} \theta, D) \phi_B,$$

$$\psi_A (\theta, B C_{AD} C_{DE} \theta, E + \theta, A C_{BD} C_{DE} \theta, E) \phi_B,$$

and $\psi_A \theta, A, \theta, B \phi_B$. \hspace{1cm} (38)

Those products which are independent of the vectors $\psi_A$ and $\phi_B$ are

$$\theta, A \theta, A, \quad \theta, A C_{AB} \theta, B, \quad \theta, A C_{AD} C_{DB} \theta, B.$$
C_{AA} = C_{AB} C_{BA}' and C_{AD} C_{DB} C_{BA}.

(39)

Hence, from equation (22) one finds,

\[ F = T_0 \phi_A \phi_B + T_1 \phi_A \phi_B + \]
\[ + T_2 \phi_A \phi_A \phi_B + T_3 \phi_A \phi_B + \]
\[ + T_4 \phi_A (\theta_A \phi_A \phi_B + \theta_B \phi_B \phi_B) + \]
\[ + T_5 \phi_A (\theta_A \phi_A \phi_B + \theta_B \phi_B \phi_B) \]

(40)

and equation (21) now yields

\[ F_{AB} = \frac{\delta F}{\delta \phi_A \phi_B} = T_0 \delta_{AB} + T_1 \delta_{AB} + \]
\[ + T_2 \delta_{AB} + T_3 \delta_{AB} + \]
\[ + T_4 (\theta_A \delta_{AB} + \theta_B \delta_{AB}) + \]
\[ + T_5 (\theta_A \delta_{AB} + \theta_B \delta_{AB}) \]

(41)

where \( T_0, T_1, T_2, T_3, T_4 \) and \( T_5 \) are polynomials in the quantities (39), \( C^{-1/2} \) and material parameters.

From equation (3), one now obtains

\[ ^\alpha B \delta = \frac{\delta x_A}{\delta x_A} \frac{\delta x_B}{\delta x_B} F_{AB} = \]
If one now uses equations (29) to (33), one sees that equation (42) may be written as

\[
\begin{align*}
\mathbf{t}_{\alpha\beta} &= R_0 \delta_{\alpha\beta} + R_1 \epsilon_{\alpha\beta} + R_2 \epsilon^\gamma \theta \epsilon_{\gamma\delta} + R_3 \theta \epsilon, \theta, \beta \\
&+ R_4 (c_0 \epsilon^\theta \epsilon, \theta, \alpha + c_2 \epsilon^\theta \epsilon, \theta, \alpha) \\
&+ R_5 (c_0 \epsilon^\theta \epsilon, \theta, \alpha + c_2 \epsilon^\theta \epsilon, \theta, \alpha) \\
&+ R_i (c_0 \epsilon^\theta \epsilon, \theta, \alpha + c_2 \epsilon^\theta \epsilon, \theta, \alpha)
\end{align*}
\]  

(43)

where the \(R_i(i = 0, 1, 2, 3, 4, 5)\) are polynomials in \(c^{-1/2}\), material parameters and in the quantities given in (36).

Equation (43) may now be expressed in vector notation as

\[
\mathbf{t} = R_0 \mathbf{I} + R_1 \mathbf{e} + R_2 \mathbf{c}^2 + R_3 \mathbf{\Theta} + \\
+ R_4 (c \mathbf{\Theta} + \mathbf{c} \mathbf{\Theta}) + R_5 (c^2 \mathbf{\Theta} + \mathbf{c}^2 \mathbf{\Theta})
\]  

(44)

where \(\mathbf{\Theta}^d = [\Theta, \alpha, \theta, \beta]\).

Isotropic Material Without a Center of Symmetry (Hemihedral)

The groups of transformations \(\{s\}\) characterizing isotropic
materials without a center of symmetry is the proper orthogonal group (rotation group) consisting of all orthogonal transformations with $|s_{AB}| = +1$.

The Heat Flux Equation. The elements of an integrity basis (corresponding to the proper orthogonal group of transformations) for the vectors $\psi_A$ and $\theta_A$ and the symmetric second order tensor $C_{AB}$ which are linear in $\psi_A$ (see ref. 4) may be taken as

$$
\begin{align*}
\psi_A^\theta, A, & \quad \psi_A^\theta C_{AB}, B, & \quad \psi_A^\theta C_{AD} C_{DB}, B, & \quad e_{ABC}^\theta, B, C, \\
e_{ABC}^\theta C_{BD}, D, C, & \quad e_{ABC}^\theta C_{BD} C_{DE}, E, C.
\end{align*}
$$

Those elements which are independent of the vector $\psi_A$ are

$$
C_{AA}, \quad C_{AB} C_{BA}, \quad C_{AD} C_{DB} C_{BA}, \quad \theta_A \theta_A, \quad \theta_A^\theta A, \quad \theta_A^\theta C_{AB}, B, \quad \theta_A^\theta C_{AD} C_{DB}, B,
$$

and $e_{ABC}^\theta, A, B, C.$

One may now construct the function $G$ from equations (17) and (45) as follows:

$$
G = P_0 \psi_A^\theta, A + P_1 \psi_A^\theta C_{AB}, B + P_2 \psi_A^\theta C_{AD} C_{DB}, B + P_3 e_{ABC}^\theta, B, C + P_4 e_{ABC}^\theta C_{BD} D, C + P_5 e_{ABC}^\theta C_{BD} C_{DE}, E, C.
$$

Hence, since $G_A = \frac{\delta G}{\delta \psi_A}$, one obtains
\[ G_A = P_0 \theta_A + P_1 \theta_{AB},_B + P_2 \theta_{AD},_B + \]
\[ + P_3 \theta_{ABC},_B + P_4 \theta_{ABCD},_B + \]
\[ + P_5 \theta_{ABCDE},_B \]  

(48)

where \( P_i (i = 0, 1, \ldots 5) \) are polynomials in the seven invariants (46), \( C^{-1/2} \) and material parameters.

Thus, from equation (5), it follows that

\[ h_\alpha = \frac{\delta x}{\delta X_A} G_A (C DG, \theta, C, C^{-1/2}) = \]
\[ = \frac{\delta x}{\delta X_A} (P_0 \theta_A + P_1 \theta_{AB},_B + P_2 \theta_{AD},_B + \]
\[ + P_3 \theta_{ABC},_B + P_4 \theta_{ABCD},_B + \]
\[ + P_5 \theta_{ABCDE},_B). \]  

(49)

From the definitions of \( C_{AB} \) and \( \theta_A \) and the Cayley-Hamilton theorem, equation (49) is readily transformed to

\[ h_\alpha = Q_0 \delta_\alpha \theta,_\beta + Q_1 \alpha \beta \theta, _\beta + Q_2 \alpha \gamma \gamma \beta \theta, _\beta + \]
\[ + Q_3 \alpha \beta \gamma \gamma \beta \theta, _\beta + Q_4 \alpha \beta \gamma \gamma \beta \theta, _\beta + \]
\[ + Q_5 \alpha \beta \gamma \gamma \beta \theta, _\beta \]  

(50)

where \( Q_i (i = 0, 1, 2, \ldots 5) \) are functions of the invariants

\[ \alpha \alpha, \alpha \beta \beta \alpha, \alpha \beta \gamma \gamma \alpha, \theta, _\alpha, _\alpha \]
and of $c^{-1/2}$ and of material parameters. Equation (50) may now be written as

$$h_\alpha = (Q_0 \delta_{\alpha \beta} + Q_1 c_{\alpha \beta} + Q_2 c_{\alpha \gamma} c_{\gamma \beta} +$$

$$+ Q_3 e_{\alpha \beta} \gamma_{\gamma} + Q_4 e_{\alpha \beta} c_{\gamma \delta} +$$

$$+ Q_5 e_{\alpha \beta} c_{\gamma \delta} c_{\delta \eta} c_{\eta \theta} \theta_{\beta})$$

The vector form of this equation is

$$h = [Q_0 I + Q_1 c + Q_2 c^2 + Q_3 b +$$

$$+ Q_4 c \cdot b + Q_5 c^2 \cdot b] \cdot \text{grad } \theta,$$

where $b = [e_{\alpha \beta \gamma}, \beta]$ is the temperature gradient alternating tensor.

The Stress Equation. The stress equation for hemihedral materials is determined following a procedure analogous to that used in the case of holohedral materials.

The elements, which are multilinear in the vectors $\psi_A$ and $\phi_B$, of an integrity basis corresponding to the proper orthogonal group for the vectors $\psi_A$, $\phi_B$ and $\theta_A$ and the tensor $C_{AB}$ may be taken as (see ref. 4)

$$\psi_A^{\delta_{AB}} \phi_B, \quad \psi_A^{C_{AB}} \phi_B, \quad \psi_A^{C_{AD}} C_{DB} \phi_B, \quad \psi_A^{\theta, A \theta, B \phi_B},$$
Those elements which are completely independent of the vectors \(\psi_A\) and \(\phi_B\) are given by expressions (46).

If one now uses expressions (54) in equations (22), (21) and (31), and then applies the definitions in (29) and equations (30) to (33) to the result, one finds that the stress equation may be written as

\[
\mathbf{t} = R_0 I + R_1 c + R_2 c^2 + R_3 \theta + \nonumber \\
+ R_4 (c\theta + \theta c) + R_5 (c^2\theta + \theta c^2) + \\
+ R_6 (cb - bc) + R_7 (c^2b - bc^2) + \\
+ R_8 (c^2b^2 - c^2bc) + R_9 (bc\theta - c\theta bc) + \\
+ R_{10} (bc^2\theta - \theta c^2b) .
\]

where \(R_i (i = 0, 1, \ldots, 10)\) are functions of \(c^{-1/2}\), the scalar invariants given by (51) and material parameters.
CHAPTER III

SCALAR PARAMETERS INVOLVED IN THE EQUATIONS
AND CONSIDERATIONS OF DIMENSIONAL INVARIANCE

The ideally elastic material remembers a single reference state to which it returns without hysteresis when released from stress. The material is assumed to possess definite isotropic properties and to be homogeneous in the reference state. As affirmed in the nomenclature, the reference state is defined by the coordinates \( X_A \). One assumes the material to be unstressed and at a uniform temperature in the reference state. The material parameters relevant to the formulation are a natural elasticity, a natural thermal conductivity, a natural coefficient of thermal expansion, a natural specific heat, a natural density and a natural temperature. One may without loss of generality take for these quantities their values in the reference state and designate them by \( \mu_o, k_o, \alpha_o, c_o, \rho_o, \) and \( \theta_o \), respectively. Their values at any other state, and in particular in the configuration (deformed state), are given through the equation of state of the material; thus

\[
\frac{\mu}{\mu_o} = f \left( \frac{\rho}{\rho_o}, \frac{\theta}{\theta_o} \right), \\
\frac{\alpha}{\alpha_o} = f \left( \frac{\rho}{\rho_o}, \frac{\theta}{\theta_o} \right), \\
\frac{c}{c_o} = f \left( \frac{\rho}{\rho_o}, \frac{\theta}{\theta_o} \right), \quad \text{and} \quad \frac{k}{k_o} = f_k \left( \frac{\rho}{\rho_o}, \frac{\theta}{\theta_o} \right). 
\]
One may now write equations (1) and (2) in forms which include the above material parameters as

\[ t = f(D, \text{grad } \theta, u, k, \alpha, c, \rho, (\theta - \theta_0)), \]  
\[ (57) \]

and

\[ h = g(D, \text{grad } \theta, u, k, \alpha, c, \rho, (\theta - \theta_0)) \]  
\[ (58) \]

where \( u, k, \alpha \) and \( c \) are given by expressions (56).

One notes that explicit dependence on \( X_A \) is eliminated by the assumption of homogeneity in the reference state. One may drop this assumption if one agrees to remember that \( u_0, k_0, \alpha_0, c_0, \) and \( \rho_0 \) are to be specified functions of \( X_A \).

The dimensions involved in the constitutive equations (37), (44), (53), (55), (57), and (58) expressed in terms of the basic dimensions of mass, length, time and temperature are as follows:

\[ \dim t = \frac{M}{LT^2}, \quad \dim c = 0, \quad \dim \theta = \frac{G^2}{L^2}, \]

\[ \dim (\text{grad } \theta) = \dim (b) = \frac{\Theta}{L}, \quad \dim h = \frac{M}{T^3}, \]

\[ \dim u = \frac{M}{LT^2}, \quad \dim k = \frac{ML}{T^3}, \]

\[ \dim \alpha = \frac{1}{\Theta}, \quad \dim c = \frac{L^2}{\Theta T^2}, \quad \dim \rho = \frac{M}{L^3}, \]

Thus, the response equations for the isotropic material with a
center of symmetry (equations (37) and (44)) may be written as follows:

\[ h = k(\xi_0 I + \xi_1 c + \xi_2 c^2) \cdot \text{grad } \theta \]  
(60)

and

\[ t = \mu(\psi_0 I + \psi_1 c + \psi_2 c^2) + \]
\[ + \lambda[\psi_3 \theta + \psi_4 (c \theta + \theta c) + \psi_5 (c^2 \theta + \theta c^2)] \]
(61)

where \( \lambda = \frac{2k^2}{pc^2} \), and where \( \xi_i (i = 0, 1, 2) \), and \( \psi_i (i = 0, 1, \ldots, 5) \) are now dimensionless functions of the 13 variables \( \mu, k, \alpha, c, \rho, (\theta - \theta_0), \text{trc}, \text{trc}^2, \text{trc}^3, \text{trc}_3, \text{trc}_2, \text{trc}_2 \theta, \) and \( c^{-1/2} \).

Since one has four fundamental dimensions \( M, L, T, \) and \( \Theta \), Buckingham's theorem requires that both the variable \( \xi_i \) and the \( \psi_i \) be functions of 13-4 or 9 dimensionless ratios, and one may thus write

\[ \xi_i = \xi_i (c^{-1/2}, \text{trc}, \text{trc}^2, \text{trc}^3, \lambda \text{trc} \theta, \lambda \text{trc}^2 \theta, \alpha(\theta - \theta_0), \text{and } T = \frac{\alpha \mu}{\rho c}). \]  
(62)

Writing \( T = \frac{\alpha \mu (\theta - \theta_0)}{\rho c (\theta - \theta_0)} \), one notes that the dimensionless grouping \( T \) represents the ratio of the elastic energy stored as a result of the deformation created by the temperature difference \( (\theta - \theta_0) \) to the heat energy stored due to the same temperature difference.

Similarly, one obtains the exact form of the response equations for the isotropic material without a center of symmetry as
\[ h = k (\eta_0 \mathbf{i} + \eta_1 \mathbf{c} + \eta_2 \mathbf{c}^2) \cdot \nabla \theta + 
\]
\[ + \pi (\eta_3 b + \eta_4 c \cdot b + \eta_5 \mathbf{c}^2 \cdot b) \cdot \nabla \theta \]

(63)

where \( \pi = \frac{k^2}{\mu} \sqrt{\frac{a}{c}} \), and

\[ t = \mu (\eta_0 \mathbf{i} + \eta_1 \mathbf{c} + \eta_2 \mathbf{c}^2) + 
\]
\[ + \lambda [\eta_3 \mathbf{i} + \eta_4 (\mathbf{c} \cdot \mathbf{c} + \mathbf{c}^2 \cdot \mathbf{c})] + 
\]
\[ + \Omega [\eta_5 (\mathbf{c} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{c}) + \eta_7 (\mathbf{c}^2 \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{c}^2) + \eta_8 (\mathbf{c} \cdot \mathbf{b}^2 - \mathbf{b} \cdot \mathbf{c}^2)] + 
\]
\[ + \frac{\lambda \Omega}{\mu} [\eta_9 (\mathbf{b} \cdot \mathbf{c} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{c} \cdot \mathbf{b}) + \eta_{10} (\mathbf{b} \cdot \mathbf{c}^2 \cdot \mathbf{b} - \mathbf{c} \cdot \mathbf{b}^2 \cdot \mathbf{b})]. \]

(64)

where \( \Omega = k \sqrt{\frac{a}{c}} \) and where the \( \eta_i (i = 0, 1, \ldots, 5) \), and the \( \eta_1 (i = 0, 1, \ldots, 10) \) are now dimensionless functions of the 14 variables \( \mu, k, a, c, p, (\theta - \theta_0), \text{trc}, \text{trc}^2, \text{trc}^3, \text{trc}^2 \cdot \text{trc}, \text{trc} \cdot \text{trc}^2, \text{trc} \cdot \text{trc}^2 \cdot \text{trc}, \text{trc} \cdot \text{trc}^2 \cdot \text{trc}^2, \text{trc} \cdot \text{trc}^2 \cdot \text{trc}^2 \cdot \text{trc}, \text{trc} \cdot \text{trc}^2 \cdot \text{trc}^2 \cdot \text{trc}^2 \), and \( c^{-1/2} \); hence, by Buckingham's theorem, one may write

\[ \eta_i = \eta_i (c^{-1/2}, \text{trc}, \text{trc}^2, \text{trc}^3, \lambda \text{trc}, \lambda \text{trc}^2, \lambda \text{trc}^2 \cdot \text{trc}, \alpha (\theta - \theta_0), \text{and } T = \frac{\alpha u}{\rho c}) \]

(65)

where \( \tau = \frac{k}{\mu} \sqrt{\frac{\alpha}{c}} \).

The argument \( c^{-1/2} \) of equations (62) and (65) may be eliminated since \( c = \frac{1}{a} \) may be expressed as

\[ c = \frac{1}{a} (2 \text{trc}^3 - 3 \text{trc} \cdot \text{trc}^2 + (\text{trc})^3) \]

(66)

and since the functions \( \xi_i, \psi_i, \eta_i \), and \( \eta_i \) are assumed to be single-valued functions.
CHAPTER IV

SOLUTIONS OF PARTICULAR PROBLEMS

One now considers the application of the theories developed above to the following particular cases:

1. An isotropic body subjected to a two-dimensional temperature distribution and simultaneously constrained so that it cannot deform,

2. An isotropic body subjected to a two-dimensional temperature distribution and simultaneously subjected to simple isochoric shear, and

3. Stresses and heat transmission in a stretched and twisted tube in the presence of temperature gradients.

Thermal Stresses in a Body Constrained to No-Deformation

Consider the stress system which is required to maintain the material in its undeformed (reference) state in the presence of a temperature gradient.

For this case \( \sigma = \sigma^2 = I \), and hence equation (61) reduces to

\[
\mathbf{t} = \mu (\psi_0 + \psi_1 + \psi_2)I + \lambda (\psi_3 + \psi_4 + \psi_5)\Theta
\]  

or

\[
\mathbf{t} = \mu A_0 I + \lambda B_0 \Theta
\]  

1. Throughout this chapter and whenever employed in relation to the problems the word isotropic is to be read isotropic and with a center of symmetry.
where \( A_o \) and \( B_o \) are functions of \( \text{tr}Q, \frac{\theta}{\theta_o}, \rho/\rho_o, \alpha(\theta - \theta_o), \) and \( T = \frac{\alpha u}{\rho c} \).

For a two-dimensional temperature distribution given by

\[
\theta = k_1 x + k_2 y
\]

one has

\[
\Theta = \begin{pmatrix} k_1^2 & k_1 k_2 & 0 \\ k_1 k_2 & k_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

from which

\[
\text{tr} \Theta = k_1^2 + k_2^2.
\]

The stresses are then given by

\[
\begin{align*}
t_{xx} &= \alpha A_o + \lambda B_o k_1^2, \\
t_{yy} &= \alpha A_o + \lambda B_o k_2^2, \quad \text{and} \\
t_{xy} &= \lambda B_o k_1 k_2
\end{align*}
\]

where \( A_o \) and \( B_o \) are functions of \( k_1^2 + k_2^2, \alpha(\theta - \theta_o), \) and the dimensionless number \( T = \frac{\alpha u}{\rho c} \).

Assuming that for the elastic material under consideration \( \lambda B_o \neq 0 \).
one sees immediately the universal relationship

\[ k_1 k_2 (t_{xx} - t_{yy}) = (k_1^2 - k_2^2) t_{xy} \]  \( (73) \)

The relationship (73) need not hold for materials for which \( \lambda B_0 = 0 \); however, subject to the exclusion of this particular class, the relationship is independent of the material.

A normal stress difference will result unless \( k_1 = \pm k_2 \); i.e., unless the temperature distribution is

\[ \theta = k_1 (x \pm y) \]  \( (74) \)

so that the isothermals are at 45 degrees to one of the normal stress directions.

Equations (72) show that a shear stress is required to maintain the undeformed state unless \( k_1 \) or \( k_2 \) is zero; i.e., unless the isothermals are parallel to one or other of the normal stress directions.

In the absence of deformation the heat flux, equation (60), reduces to the form

\[ h_\alpha = k (\xi_0 + \xi_1 + \xi_2) \theta_\alpha = k \Gamma_0 \theta_\alpha \]  \( (75) \)

where \( \Gamma_0 \) is a dimensionless function of \( tr_0, \frac{\theta}{\theta_0}, \rho/\rho_0, \alpha(\theta - \theta_0) \), and \( T = \frac{\alpha \mu}{\rho c} \), and where from thermodynamic considerations

\[ k \Gamma_0 < 0. \]  \( (76) \)
If one accepts Fourier's Law of heat conduction one may take

\[ \Gamma_0 = \xi_0 + \xi_1 + \xi_2 = -1. \]  \hspace{1cm} (77)

**Isochoric Shear in the Presence of Temperature Gradients**

Consider now the simple shear defined by the homogeneous linear transformation

\[ \begin{align*}
    x &= X + SY \\
y &= Y \\
z &= Z
\end{align*} \]  \hspace{1cm} (78)

with a temperature gradient in the configuration state given by

\[ \Theta = k_1 x + k_2 y. \]  \hspace{1cm} (79)

Planes X constant in the reference state turn through the finite angle S about their lines of intersection with the plane Y = 0. A rectangle in the reference state is transformed into a parallelogram of the same area. The prescribed deformation thus maintains the volume of an element and is isochoric.

It is required to determine the stresses required to maintain the deformation in the presence of the steady temperature distribution.

Application of the Cayley-Hamilton theorem allows one to write equation (61) in the form
\[ t = \mu[\psi_1 c^{-1} + \psi_0 \mathbb{I} + \psi_1 \mathbb{I}] + \\
+ \lambda[\psi_2 \Theta + \psi_3 (c \Theta + \Theta c) + \psi_4 (c^{-1} \Theta + \Theta c^{-1})] \quad (80) \]

where

\[ c^{-1} = [X_A, \alpha X_A, \beta]. \quad (81) \]

Thus, from equations (78) and (79) one obtains

\[ \Theta = \begin{pmatrix} k_1^2 & k_1 k_2 & 0 \\ k_1 k_2 & k_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (82) \]

\[ c = \begin{pmatrix} 1 + s^2 & s & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (83) \]

and

\[ c^{-1} = \begin{pmatrix} 1 & -s & 0 \\ -s & 1 + s^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (84) \]
The elementary invariants of $c$ are $I_c = II_c = 3 + S^2$, $III_c = 1$ with the last equation verifying that the deformation is isochoric.

Equation (80) gives

$$t_{xx} = \mu[(\psi_1 + \psi_0 + \psi_1) + \psi_1 S^2] +$$

$$+ \lambda[\{\psi_2 + 2(\psi_3 + \psi_4)\} k_1^2 + 2(\psi_3 - \psi_4) k_1 k_2 S + 2\psi_2 k_1^2 S^2]$$

$$t_{yy} = \mu[(\psi_1 + \psi_0 + \psi_1) + \psi_1 S^2] +$$

$$+ \lambda[\{\psi_2 + 2(\psi_3 + \psi_4)\} k_2^2 + 2(\psi_3 + \psi_4) k_1 k_2 S + 2\psi_2 k_2^2 S^2]$$

$$t_{zz} = \mu(\psi_1 + \psi_0 + \psi_1)$$

and

$$t_{xy} = \mu(\psi_1 - \psi_1) S + \lambda[\{\psi_2 + 2(\psi_3 + \psi_4)\} k_1 k_2$$

$$+ (\psi_3 - \psi_4)(k_1^2 + k_2^2) S + (\psi_3 + \psi_4) k_1 k_2 S^2]$$

where the $\psi_i$ are functions of $S^2$ and of

$$\text{tr}^2 = k_1^2 + k_2^2$$

$$\text{trc}^2 = (k_1 + Sk_2)^2 + k_2^2$$

$$\text{trc}^2 = (Sk_1 - (1 + S^2)k_2)^2 + (k_1 - Sk_2)^2$$

and the scalars $\alpha(\theta - \theta_0)$, and $T = \frac{au}{\rho c}$
From equations (85) one finds the normal stress difference

\[ t_{xx} - t_{yy} = u(\psi_1 - \psi_{-1})S^2 + \]
\[ + \lambda(\psi_2 + 2(\psi_3 + \psi_4))(k_1^2 - k_2^2) + \]
\[ + 2[\psi_2 k_1^2 - \psi_4 k_2^2]S^2 \]  

(88)

In the absence of temperature gradients (isothermal conditions) equations (86) and (88) reduce, respectively, to

\[ t_{xy} = u(\psi_1 - \psi_{-1})S \]

and

\[ t_{xx} - t_{yy} = u(\psi_1 - \psi_{-1})S^2 \]  

(89)

leading to the well-known relation between the normal stresses required to maintain isochoric shear, the shear stress and the angle of shear

\[ t_{xx} - t_{yy} = t_{xy} S \]  

(90)

which must be obeyed by all ideally elastic materials whether the deformation be finite or infinitesimal.

When a temperature gradient exists one sees that only if either \( k_1 \) or \( k_2 \) is zero, so that the temperature gradient is parallel to one of the directions of normal stress, will the required shear stress \( t_{xy} \)
depend solely upon odd powers of the shear $S$. In general, interaction between $k_1$ and $k_2$ will introduce a term in $S^2$ and higher powers of $S$ showing that $t_{xy}$ will not be exactly reversed with a reversal of $S$ in the presence of temperature gradients.

Again, the normal stresses and the normal stress differential will only involve even-powers of $S$ provided either $k_1$ or $k_2$ is zero. In addition to the strain-independent normal stresses introduced by the temperature gradients, interaction between $k_1$ and $k_2$ leads to terms in the first and third powers of the deformation $S$ in the required normal stress and in the third power of the deformation in the normal stress difference. One finds a non-linear growth in each of the normal stresses with deformation in the presence of temperature gradients.

One notes that the term in $t_{xx} - t_{yy}$ independent of $S$ vanishes when $k_1 = k_2$. This again is in agreement with the general results of the preceding section.

The results of this section and those of the previous section are published in a joint paper by Marris and Villanueva (see reference 8).

**Stresses and Heat Transmission in a Stretched and Twisted Tube in the Presence of Temperature Gradients**

Consider a homogeneous and isotropic cylindrical tube which is stretched and twisted in such a way that a point $P(X_1, X_2, X_3)$ in the reference state, whose coordinates are given by

$$X_1 = R\cos\phi, \quad X_2 = R\sin\phi, \quad \text{and} \quad X_3 = Z \quad (91)$$
where $R^2 = x_1^2 + x_2^2$, and $\phi = \arctan \frac{x_2}{x_1}$ is mapped onto a point $p(x_1, x_2, x_3)$ of the deformed state according to the relations

$$x_1 = r(R)\cos(\phi + KZ), \quad x_2 = r(R)\sin(\phi + KZ), \quad \text{and} \quad x_3 = \varepsilon Z \quad (92)$$

where $K$ is the angle of twist per unit length and $\varepsilon$ is the longitudinal strain.

The form of $r(R)$ in equations (92) depends on the rheological properties of the material and on the system of forces applied to the tube.

From equations (29) and (92) one may determine the matrix $[c_{ab}]$ for a point $Q$ located at $(r, \theta, x_j)$ in the deformed state, thus,

$$[c] = \begin{pmatrix} (r')^2 & 0 & 0 \\ 0 & \left(\frac{r}{R}\right)^2(1+K^2R^2) & r\varepsilon K \\ 0 & r\varepsilon K & \varepsilon^2 \end{pmatrix} \quad (93)$$

where $r' = \frac{dr}{dR}$.

The matrix (93) not only gives the components of $c$ in a local cartesian coordinate system at $Q$ with axes in the radial, circumferential and axial directions, but also at any other point in the solid.

From (93) one now finds
Now let the temperature gradient in the deformed state be given by

\[
[\text{grad } \theta] = \begin{bmatrix}
\frac{\delta \theta}{\delta r}, & 0, & \frac{\delta \theta}{\delta z} = [n_r, 0, n_z]
\end{bmatrix}
\]

(95)

where \(n_r = n_r(z)\), and \(n_z = n_z(r, z)\) in general. Thus

\[
[\Theta] = \begin{bmatrix}
0^2 & 0 & n_r n_z \\
0 & 0 & 0 \\
n_r n_z & 0 & n_z^2
\end{bmatrix}
\]

(96)

Expressions (93), (94), (95), and (96) when substituted in equation (60) yield the components of the heat flux vector as

\[
h_r = k\{\xi_0 + (r')^2 \xi_1 + (r')^4 \xi_2\} n_r ,
\]

\[
h_\phi = k(\varepsilon r K)\{\xi_1 + [(\frac{r}{R})^2(1 + R^2 k^2) + \varepsilon^2] \xi_2\} n_z ,
\]

(97)

\[
h_z = k\{\xi_0 + \varepsilon^2 \xi_1 + \varepsilon^2 (r^2 k^2 + 1) \xi_2\} n_z .
\]
The components of the stress tensor are found from equation (61) as

\[ t_{rr} = \mu \{ \psi_0 + (r')^2 \psi_1 + (r')^4 \psi_2 \} + \]
\[ + \lambda \{ \psi_3 + 2(r')^2 \psi_4 + 2(r')^4 \psi_5 \} n^2, \]
\[ t_{\phi\phi} = \mu \{ \psi_0 + \left( \frac{r}{R} \right)^2 (1 + R^2 K^2) \psi_1 + r^2 \left[ \frac{1}{R^2} (1 + R^2 K^2) + \epsilon^2 \right] \psi_2 \}, \]
\[ t_{zz} = \mu \{ \psi_0 + \epsilon^2 \psi_1 + \epsilon^2 (r^2 K^2 + \epsilon^2) \psi_2 \} + \]
\[ + \lambda \{ \psi_3 + 2 \epsilon^2 \psi_4 + 2 \epsilon^2 (r^2 K^2 + \epsilon^2) \psi_5 \} n^2, \]
\[ t_{r\phi} = \lambda (r^2 K) \{ \psi_4 + \left[ \left( \frac{r}{R} \right)^2 (1 + R^2 K^2) + \epsilon^2 \right] \psi_5 \} n_r n_z, \]
\[ t_{rz} = \lambda \{ \psi_3 + \left[ \epsilon^2 + (r')^2 \right] \psi_4 + \left[ (r')^4 + \epsilon^2 (r^2 K^2 + \epsilon^2) \right] \psi_5 \} n_r n_z, \]

and

\[ t_{\phi z} = \mu \{ r^2 K \psi_1 + \epsilon^2 (r^2 K^2 + \epsilon^2) \psi_2 \} + \]
\[ + \lambda \{ r^2 K \{ \psi_4 + \left( \frac{r}{R} \right)^2 (1 + R^2 K^2) + \epsilon^2 \} \psi_5 \} n^2, \] \hspace{1cm} (98)

The \( \xi_i (i = 0, 1, 2) \) and the \( \psi_i (i = 0, 1, 2, 3, 4, 5) \) of equations (97) and (98) are the dimensionless functions specified by equations (62). Thus, the \( \xi_i \) and the \( \psi_i \) are dimensionless functions of the dimensionless quantities

\[ \text{tr}_{-} = (r')^2 + \left( \frac{r}{R} \right)^2 (1 + R^2 K^2) + \epsilon^2, \]
\[ \text{tr}_{-}^2 = (r')^4 + \left( \frac{r}{R} \right)^4 (1 + R^2 K^2)^2 + 2 \left( \frac{r}{R} \right)^2 R^2 K^2 \epsilon^2 + \epsilon^4, \]
We have \( \text{tr}_{\Sigma}^3 = (r')^6 + \frac{(r')}{R} (1 + R^2 \kappa^2)^3 + 3 \frac{(r')^4}{R} (1 + R^2 \kappa^2)^2 R^2 \kappa^2 \varepsilon^2 + \)
\[ + 3 \frac{(r')^2}{R} R^2 \kappa^2 \varepsilon^4 + \varepsilon^6, \]
\[
\lambda \text{tr}_0 = \lambda (n_r^2 + n_z^2),
\]
\[
\lambda \text{tr}_{\phi} = \lambda [(r')^2 n_r^2 + \varepsilon^2 n_z^2],
\]
\[
\lambda \text{tr}_{\phi} = \lambda [(r')^4 n_r^2 + \varepsilon^2 (r^2 \kappa^2 + \varepsilon^2) n_z^2],
\]
\[
\alpha(\theta - \theta_0), \text{ and the dimensionless grouping } T = \frac{\alpha}{\rho c}.
\]

One is now ready to study in detail the particular case for which 
\( n_r = \text{constant}, \ n_z = 0, \) and \( K = 0 \) (i.e., the tube is stretched but 
not twisted and the temperature gradient is radial). Assume also that 
the end of the tube at \( z = 0 \) is fixed against rotation and motion in 
the \( z \)-direction. Under these assumptions equations (97) and (98) give 
for the components of the heat flux

\[
h_r = k(\xi_0 + (r')^2 \xi_1 + (r')^4 \xi_2) n_r, \text{ and}
\]
\[
h_\phi = h_z = 0.
\]

The components of the stress tensor are correspondingly given as

\[
t_{rr} = u(\psi_0 + (r')^2 \psi_1 + (r')^4 \psi_2) + \]
\[ + \lambda \{\psi_3 + 2(r')^2 \psi_4 + 2(r')^4 \psi_5 \} n_r^2, \]
\[
t_{\phi\phi} = \psi_0 + (r')^2 \psi_1 + (r')^4 \psi_2, \]
\[
\]
\[ t_{zz} = \xi(\psi_o + \epsilon^2 \psi_1 + \epsilon^4 \psi_2) , \]

\[ t_{rz} = t_{r\psi} = 0 , \text{ and} \]

\[ t_{z\psi} = \xi \epsilon^4 \psi_2 . \]

Thus, it appears that the combined action of uniformly stretching the tube and the radial temperature gradient produces a normal stress \( t_{zz} \) in the direction of the stretching which is not uniform, but definitely a function of \( r(R) \). In general, \( t_{zz} \) will also depend on \( z \), on the rheological properties of the material, on the forces applied to the tube and on the square of the temperature gradient \( \nabla n \).

The resultant axial force which must be applied to the tube in order to maintain the desired deformation in the presence of the temperature gradient may be found by integration as

\[
F_z = 2\pi \int_{r_o(R_0)}^{r_l(R_1)} r \, dr \left| t_{zz} \right| \left[ \right. \text{evaluated at } z=1=\varepsilon L \]

where \( r_o(R_0) \) and \( r_l(R_1) \) are the inner and outer radii of the tube in the deformed state and \( R_o \) and \( R_l \) are the corresponding radii in the reference state.

The presence of a normal stress in the \( \phi \) direction \( (t_{\phi\phi}) \) indicates a possibility of buckling of the lateral surfaces in the case of a thin-walled tube under the appropriate conditions.

One notices that there is a tendency of the tube to twist, which
is to be prevented by the application of an external torque

\[ T = 2\pi \int_{r_o(R_o)}^{r_1(R_1)} r \, dr \, |_{z=1=\varepsilon L} \]  \hspace{1cm} (103)

applied at the end of the tube.

It is also found that lateral pressures must be applied to the surfaces of the tube in order to maintain the prescribed deformation in the presence of the temperature gradient. The distribution of these pressures is found to be

\[ P_o = t_{rr} |_{r_o(R_o)} \] at the inner surface, and

\[ P_1 = t_{rr} |_{r_1(R_1)} \] at the outer surface.

The total heat flux may be evaluated by integration as follows

\[ q_r = 2\pi \int_{z=0}^{z=1=\varepsilon L} h_r \, dz = 2\pi \int_{r=r_o(R_o)}^{r=r_1(R_1)} hr \, dz \] \hspace{1cm} (105)

where \( h_r \) is given by equation (100). Since \( h_z = 0 \), one concludes that there is no heat flux across the ends of the tube.

Consider now the case in which the cylindrical tube is homogeneous and it is being simultaneously stretched, twisted and subjected to an axial temperature gradient \( h_z = \text{constant} \). One finds for this par-
ticular case that the heat flux vector has the components

\[ h_r = 0, \quad h_{\phi} = k(r^2)\{\xi_1 + [(r^2)\left(1 + R^2K^2\right) + \varepsilon^2]\xi_2}\eta_z, \]

and

\[ h_z = k\{\xi_0 + \varepsilon^2\xi_1 + \varepsilon_1 + \varepsilon^2(r^2K^2 + 1)\xi_2\}\eta_z. \quad (106) \]

The components of the stress tensor are

\[ t_{rr} = \mu\{\psi_0 + (r')^2\psi_1 + (r')^4\psi_2\}, \]

\[ t_{\phi\phi} = \mu\{\psi_0 + (r^2)\left(1 + K^2R^2\right)\psi_1 + r^2\left[\frac{1}{R^2}(1 + R^2K^2) + K^2\varepsilon^2\right]\psi_2\}, \]

\[ t_{zz} = \mu\{\psi_0 + \varepsilon^2\psi_1 + \varepsilon^2(r^2K^2 + \varepsilon^2)\psi_2\} + \lambda\{\psi_3 + 2\varepsilon^2\psi_4 + 2\varepsilon^2(r^2K^2 + \varepsilon^2)\psi_5\} \eta_z^2, \]

\[ t_{r\phi} = t_{r2} = 0, \quad \text{and} \]

\[ t_{z\phi} = \mu(r^2K\psi_1 + \varepsilon^2(r^2K^2 + \varepsilon^2)\psi_2) + \lambda(r^2K\psi_4 + [(r^2)\left(1 + R^2K^2\right) + \varepsilon^2]\psi_5) \eta_z^2. \quad (107) \]

Again it is found that the axial forces required to maintain the prescribed deformations are not necessarily uniform, but functions of \( r(R) \). In fact, this axial stress \( t_{zz} \) depends also on the axial temperature gradient as shown by equation (107). The resultant external forces required to maintain the deformation are again an axial force \( (F_z) \), a
torque (T) applied at the free end, and lateral pressures. These forces may be found from equations (102), (103) and (104), where \( t_{zz} \), \( t_{\phi z} \) and \( t_{rr} \) are now given by equations (107).

One notes that there is no heat flux across the lateral surfaces of the tube since \( n_r = 0 \); however, there is a definite phenomenon in the sense that there is a component of the heat flux in the \( \phi \) (tangential) direction, which indicates that the heat flux travels in helices. One emphasizes that this phenomenon is a consequence of the combined effect of the twisting, the stretching and the temperature gradient, since if \( \varepsilon = 0 \) or if \( K = 0 \) this component of the heat flux would not exist (see equation 106).

The total heat flux passing through the ends of the tubes is

\[
q = 2\pi \int_{r_0}^{r_1} h_z r dz \quad (108)
\]

If one specifies that the material is incompressible, one finds that \( \frac{\delta x_A}{\delta x_A} = 1 \), and it follows that since \( c_{\alpha\beta} = \begin{bmatrix} x_{\alpha\beta} & x_{\beta\alpha} \end{bmatrix} \), then

\[
c = |c_{\alpha\beta}| = 1; \text{ thus, from (93) one finds that}
\]

\[
\frac{e^2 - e}{\varepsilon} = R \quad (109)
\]

thus

\[
r^2 = r_o^2 + \frac{R^2 - R_o^2}{\varepsilon}, \quad (110)
\]
hence, only $r_0$ is left to be determined by the properties of the material and the system of forces applied. In any particular experiment $r_0$ may be measured, and $r(R)$ is thus determined without reference to any knowledge of the rheological properties of the material.
CHAPTER V

CONCLUSIONS

The response equations for the isotropic material with a center of symmetry were found to be

\[ h = k(\xi_0 I + \xi_1 \mathbf{c} + \xi_2 \mathbf{c}^2) \cdot \text{grad } \theta \] (60)

for the heat flux, and

\begin{align*}
\tau &= \mu(\psi_0 I + \psi_1 \mathbf{c} + \psi_2 \mathbf{c}^2) + \\
&\quad + \lambda[\psi_3 \mathbf{c}^2 + \psi_4 (\mathbf{c}^2 + \theta^2) + \psi_5 (\mathbf{c}^2 \theta + \theta^2)]
\end{align*}
(61)

for the stress tensor, where \( \lambda = \frac{\sigma^2 k^2}{\rho c^2} \), and where \( \xi_i (i = 0, 1, 2) \) and \( \psi_i (i = 0, 1, 2, \ldots, 5) \) are dimensionless functions of the nine dimensionless ratios \( c^{-1/2}, \text{tr}\mathbf{c}, \text{tr}\mathbf{c}^2, \text{tr}\mathbf{c}^3, \lambda\text{tr}\mathbf{c}, \lambda\text{tr}\mathbf{c}^2, \lambda\text{tr}\mathbf{c}^3, \alpha(\theta - \theta_0), \) and \( T = \frac{\alpha u}{\rho c} \).

The corresponding response equations for isotropic materials without a center of symmetry were found to be

\[ h = k(\eta_0 I + \eta_1 \mathbf{c} + \eta_2 \mathbf{c}^2) \cdot \text{grad } \theta + \\
+ \tau(\eta_3 \mathbf{b} + \eta_4 \mathbf{c} \cdot \mathbf{b} + \eta_5 \mathbf{c}^2 \cdot \mathbf{b}) \cdot \text{grad } \theta \] (63)
where \( \pi = \frac{k^2}{\mu} \sqrt{\frac{a}{c}} \), and

\[
\tau = u(i_o I + i_3 C + i_2 C^2) + \\
+ \lambda [i_3 \theta + i_4 (c_0 + \theta c) + i_5 (c_0^2 + 2 \theta c)] + \\
+ \Omega [i_6 (cb - bc) + i_7 (c^2 b - bc^2) + i_8 (c c_0^2 - c^2 bc)] + \\
+ \frac{\lambda \Omega}{\mu} [i_9 (bc \theta - \theta cb) + i_{10} (bc \theta - \theta cb)].
\]

(64)

where \( \Omega = k \sqrt{\frac{a}{c}} \), and where the \( \eta_i (i = 0, 1, 2, \ldots, 10) \) and the \( \iota_i (i = 0, 1, 2, \ldots, 10) \) are dimensionless functions of the dimensionless scalar groupings \( c^{-1/2}, trc, trc^2, trc^3, \lambda trc \), \( \lambda trc^2 \), \( \lambda trc^3 \), \( \alpha (\theta - \theta_0) \) and \( T = \frac{\mu}{\rho c} \), where \( \tau = \frac{k}{\mu} \sqrt{\frac{a}{c}} \).

It was found from equation (61) that for a two-dimensional temperature distribution \( \theta = k_1 x + k_2 y \) the stress system necessary to prohibit deformation should satisfy the universal relationship \( k_1 k_2 (t_{xx} - t_{yy}) = (k_1^2 - k_2^2) t_{xy} \). The exceptional case is a material for which \( \lambda B_0 = 0 \).

Considering next the case of isochoric shear in the presence of a two-dimensional temperature distribution, it was found that for a maintained temperature distribution a reversal of stress will not correspond to an exact reversal of the deformation. It was further found that in this case the normal stresses depend upon odd powers of the tangent of the shear angle in contrast to the results of classical finite elasticity with uniform temperature. These results are expressed by equations (85-88).

Considering next a homogeneous and isotropic cylindrical tube
which was uniformly stretched in the presence of a radial temperature gradient it was found that the axial stresses necessary to maintain the deformation were not constant in general, but a function of the radial distance from the geometric center of the tube; furthermore, it was a definite function of the temperature gradient. It was further shown that these axial stresses were not sufficient to maintain the prescribed deformation, but that there was need of applying a torque and lateral pressures to the tube in order to produce the prescribed deformation. These results are given by equations (101-104).

Finally, when one considered the case of a homogeneous and isotropic cylindrical tube which was simultaneously stretched and twisted in the presence of an axial temperature gradient it was found that there was no heat flux across the lateral surfaces of the tube; however, a definite phenomenon was observed in the sense that there was a tangential component of the heat flux, which indicated that the heat flux travels in helices. Expressions were found for the external forces required to maintain the deformation and for the total heat flux across the free end of the tube. These results are given by equations (102-107).

The foregoing results are, of course, a consequence of the postulates set forth in the Introduction of this work.

Experimental work in this problem as well as in related fields, such as electro-elasticity and magneto-elasticity, is to be encouraged. However, one emphasizes that more analyses of the type of this work (following the axiomatic method) are required in order to point out what to look for in the experiments and hence make worthwhile such a task.
NOMENCLATURE

\( X_A \)  
\((A = 1, 2, 3)\)  
= material coordinates, coordinates specifying the continuum in the reference state. In general, majuscule script with Latin majuscule index refers to the reference state.

\( x_\alpha \)  
\((\alpha = 1, 2, 3)\)  
= spatial coordinates, coordinates specifying the continuum in the deformed state. In general, minuscule script with Greek minuscule script refers to the deformed state.

\[ D = x_\alpha, A = \frac{\partial x_\alpha}{\partial X_A} \]  
= deformation gradient. The usual comma notation denotes partial differentiation.

\( \text{grad} \ \theta = \nabla \theta \)  
= temperature gradient vector.

\( \theta_\alpha = \frac{\partial \theta}{\partial x_\alpha} \)  
= scalar component of \( \text{grad} \ \theta \).

\( t \)  
= stress tensor.

\( t_{\alpha\beta} \)  
= scalar element of \( t \).

\( h \)  
= heat flux vector.

\( h_\alpha \)  
= scalar component of \( h \).

\( x \)  
= rectangular Cartesian coordinate system.

\( x^* \)  
= equivalent rectangular Cartesian coordinate system related to \( x \) by \( x^*_\alpha = s_{\alpha\beta} x_\beta \).

\( \{s\} \)  
= group of transformations which defines the symmetry of the material and is a sub-group of the full orthogonal group.
\( s_{\alpha\beta} \) = element of a transformation of the group \( \{ s \} \).

\( \delta_{\alpha\beta} \) = Kronecker delta.

\( C = [C_{DG}] = [x_{Y,D} x_{Y,G}] \) = deformation tensor in the reference state.

\( c = [c_{\alpha\beta}] = [x_{\alpha,\beta} x_{\alpha,\beta}] \) = deformation tensor in the deformed state.

\( \theta = [\theta_{\alpha\beta}] \) = temperature gradient product tensor.

\( b = [e_{\alpha\beta\gamma}\theta_{\gamma}] \) = temperature gradient alternating tensor.

\( \mu, k, \alpha, c, \rho \) = material elasticity, thermal conductivity, coefficient of thermal expansion, specific heat and density in configuration state. Appended suffix \( o \) indicates values in reference state.

\( \theta, \theta_o \) = temperature in configuration state and reference state, respectively.

\( \text{tr}A \) = trace of tensor \( A \).

\( \Gamma \) = dimensionless grouping \( \frac{\alpha \mu}{\rho c} \).

\( \lambda = \frac{\alpha^2 k^2}{\rho c^2} \).

\( \tau = \frac{k^2}{\mu} \sqrt{\frac{\alpha}{c}} \).

\( \Omega = k \sqrt{\frac{\alpha}{c}} \).

\( \tau = \frac{k}{\mu} \sqrt{\frac{\alpha}{c}} \).
\( p_i, q_i \) = scalar coefficient functions in the constitutive equations of heat flux.

\( t_i, r_i \) = scalar coefficient functions in the constitutive equations of stress.

\( \xi_i, \eta_i \) = dimensionless scalar coefficient functions in the constitutive equations of heat flux.

\( \psi_i, \zeta_i \) = dimensionless scalar coefficient functions in the constitutive equations of stress.

\( k_1, k_2, n_r, n_z \) = constant coefficients in prescribed two-dimensional temperature distributions.

\( S \) = tangent of angle of shear.

\( \epsilon \) = longitudinal strain.

\( K \) = angle of twist per unit length.
BIBLIOGRAPHY


Jose Villanueva y Guardia was born in Santiago de Cuba, Cuba on March 31, 1937. He attended elementary schools in Sagua de Tanamo, Cuba and in Kingston, Jamaica. He graduated from high school in 1954 from the Instituto Santiago. From 1954 to 1956 he attended the Universidad de Oriente.

He entered the Georgia Institute of Technology in September, 1957, where he was awarded the degrees of Bachelor of Mechanical Engineering (1959), and a Master of Science in Mechanical Engineering (1961). He married the former Thais Gomez in 1961. They have a son, Jose III (three years old), and a daughter, Ana Maria (two years old). He began his graduate studies in the School of Engineering Mechanics of the Georgia Institute of Technology in 1961, where he has also been a staff member since 1961.

He is a member of the Society of the Sigma Xi, Tau Beta Pi, and Pi Tau Sigma. His articles have appeared in the Journal of Solar Energy and the Journal of Applied Mechanics of the ASME.