THERMAL STRESSES IN A RECTANGULAR PLATE

A THESIS

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the Faculty of the Graduate Division
by
John H. Murphy

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of the Requirements for the Degree
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THERMAL STRESSES IN A RECTANGULAR PLATE

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SUMMARY

A numerical procedure is developed for solving general two-dimensional thermal stress problems in the elastic-plastic range. Details of the procedure are fully worked out for application to rectangular plates. Results of an analytical solution and experimental investigation of an Inconel plate are presented.
CHAPTER I

INTRODUCTION

The basic underlying theories of thermal stress analysis and plastic deformation of ductile materials are well developed.\(^1\),\(^2\) These areas have been of both theoretical and practical interest for some time; however, not until recently have a large number of practicing engineers needed the results of investigations on these topics. The time is rapidly approaching when it will be necessary that a large number of rather complicated problems in these two areas be readily analyzed and solved.

Without listing any particular applications, it is well known that high temperature considerations are very important in many areas of structural analysis. Furthermore, considerations of economy and weight make it necessary that more efficient use be made of materials. This latter consideration often requires the use of materials at stress levels above the elastic range. Therefore, methods of stress analysis are needed for situations where the material undergoes plastic flow. Plastic flow analysis is also important in the basic study of material behavior; for example, fatigue life is directly dependent on the amount of plastic flow.\(^4\)

Although there is considerable literature on the subject, there is a distinct void on practical methods of formulation and solution. It is believed that a series of articles by Manson\(^3\) and a more recent book by Lubahn and Felgar\(^7\) will do a great deal in rectifying this situation.
Practical formulations and methods of solution would probably not even be possible now if it were not for the widespread availability of high speed digital computers.

A method of analyzing thermal stresses in the elastic-plastic range has been developed by Mendelson and Manson. Their technique starts with an "elastic solution," that is, the problem is solved as if the material had actually remained elastic during the loading. The strain distribution from the "elastic solution" is taken as a good first approximation to the actual strain distribution, and an iteration procedure is then used to obtain the actual strain distribution. The method as developed by Mendelson and Manson applies to one-dimensional problems, and no one has generalized the procedure. The questions arise: Can the idea be used and applied to general two-dimensional problems? How is this done? Does the "elastic solution" provide a good starting point? How good is it? It is the objective of this study to answer these questions.

The general two-dimensional equations are developed in a suitable form and an iteration procedure for their solution is outlined. Application is made to the two-dimensional problem of thermal stress in a rectangular plate. An experimental investigation of strains in the rectangular plate is conducted for comparison with the analytical solution.

In selecting the problem for application of the method developed, two assumptions were made: (1) the plastic deformation is small and (2) proportional loading exists. These assumptions were made for convenience of application and can easily be removed. Small plastic deformation requires that the total strains be less than about 2 per cent. This is not much of a restriction when it is realized that ductile
materials begin to flow plastically at strain values on the order of 0.1 per cent. Large deformations introduce nonlinear terms into the definition of strain and are not considered here. Proportional loading is a term which will be defined in the text; it applies to the manner in which the load is applied to the material. The theory and procedure developed, however, are easily applied to any general type of loading and how this may be accomplished will be clearly shown.
ELASTIC-PLASTIC THEORY FOR SMALL DEFORMATION

The essential characteristics of the behavior of solid materials under load can be determined by a uniaxial tensile test in which a bar is subject to an increasing tensile stress. The nominal stress is the load divided by the original cross-sectional area and the nominal strain is the elongation divided by the original length. A typical plot of nominal stress vs. nominal strain for a ductile material is shown in Figure 1. The subscript \( \sigma \) will be used to denote the stress and strain resulting from this type of loading.

As the stress increases the strain will increase linearly with it up to some value of the stress at point \( P \). This part of the curve corresponds to a range in which the material behaves in an elastic manner. Elastic behavior is characterized by small strains (the slope of the curve being on the order of \( 3 \times 10^6 \) to \( 30 \times 10^6 \) psi), and the fact that if the load is removed the material will recover its original shape. As the stress is increased above point \( P \) the curve departs from linearity and the material enters the plastic range. Actually the point \( P \) is not well defined for most materials but for practical purposes can be taken as the point where the curve ceases to be linear. If the material is stressed into the plastic range to the point \( Q \) and the load removed, the material will not recover its original shape but will unload along the line \( QST \), which is roughly straight and parallel to the original elastic line.
Figure 1. Uniaxial Stress-Strain Curve for a Ductile Material.
Therefore, when the load is removed there will be a permanent set in the bar of an amount $e''_o$.

We see then that if the material is stressed into the plastic range to a point such as Q the total strain in the bar can be considered as being made up of two parts, an elastic or recoverable part $e'_o$, and a plastic or permanent part $e''_o$. That is:

$$e''_o = e'_o + e''_o \quad (1)$$

where

$$e'_o = \frac{d'o}{E}$$

If the material unloads along a straight line parallel to the elastic line, as in Figure 1, the plastic part of the strain is the same as the deviation of the point Q from the elastic line extended. Reloading the material after this unloading will result in tracing the line TSQ back up to point Q and then continuing along the original curve. That is to say, once a material has been stressed plastically the stress-strain curve is altered with the point Q replacing the point P. If the load had only been removed to the point S and then reloaded, the material would retrace the straight portion SQ and then continue along the original curve. Further unloading and reloading at any point such as U will proceed in the same manner as above. Also it is usually assumed that the material has the same stress-strain curve in compression as in tension, and that the alteration of the stress-strain curve due to plastic flow is the same in compression and tension.

Actually the above picture is somewhat simplified in that any material will exhibit some hysteresis. Other deviations from the picture
are also present in real materials particularly if the material is loaded in compression after plastic flow occurs. To include these effects would over-complicate an already difficult problem, so it is quite common practice to base calculations and experimental correlations on the picture presented.

The Condition for Yielding

Early investigators concerned themselves with the problem of predicting the condition for transition from elastic to plastic behavior, that is, in predicting the condition for yielding of the material. Of several theories which have been advanced, two have found wide acceptance and seem to adequately predict the condition for yielding of ductile materials. One of these was first formulated by Tresca and states that the initiation of plastic behavior is governed by the maximum shear stress in the material and that yielding occurs when the maximum shear stress reaches a certain critical value. If $\sigma_1, \sigma_2, \sigma_3$, are the principal normal stresses with $\sigma_1 > \sigma_2 > \sigma_3$ this condition may be expressed by:

$$\sigma_1 - \sigma_3 = \sigma_{op}, \quad (2)$$

where $\sigma_{op}$ is the yield point in the uniaxial tensile test, that is, point $P$ in Figure 1. The second theory is the energy of distortion condition. This theory was first presented by Mises and later given physical significance by Hencky. According to this second condition, yielding occurs when the distortion energy reaches a certain critical value. The distortion energy in the material is that part of the total strain energy which causes distortion of the material in contrast with the energy which causes a change of volume. In equation form this condition is expressed by:
\[
\frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sigma_{0p}.
\]  

The left-hand member of this equation is referred to as the significant stress.

Examination of equations (2) and (3) reveals that the maximum shear theory in general predicts yielding somewhat sooner than the distortion energy condition. The maximum difference is about 15 per cent. Due to the fact that it is somewhat conservative and easily applied for simple types of loading, the shear theory is widely used in machine design. However, it is generally agreed that the distortion energy condition predicts yielding more accurately and actually is considerably simpler to apply for general types of loading. The distortion energy or Mises condition is the most widely accepted yield condition.

Since the second decade of the twentieth century, attention has been directed to the formulations of the conditions and equations describing the deformation in the plastic range. The development of a relationship between the plastic strain and the stress is complicated by two apparent facts: first, that the relationship is nonlinear and second, that there is no unique correspondence between plastic strain and stress for a general loading history. The fact that there is no unique correspondence between stress and strain can be seen by examining Figure 1. We see that if the material is loaded into the plastic range and then unloaded, the unloading path follows an elastic line in which there is no further change in the plastic strain. Therefore, once the material has been stressed into the plastic range there can be many stress states for
the same plastic strain state. We see that the amount of plastic strain present in the material depends on the entire loading history. There exists a particular type of loading for which this problem of nonuniqueness does not arise, namely, proportional loading in which the ratios of the principal stresses remain constant during loading and in which no unloading occurs. For proportional loading the final stress state describes the entire loading history and there is a unique value of the plastic strain.

The theory which relates the total plastic strain to the stress is called the deformation theory and it is applicable for proportional loading. It has recently been shown by Budiansky\textsuperscript{12} that the restriction to constant principal stress ratios can be relaxed somewhat and still be correct if the variation in principal stress ratios is not too great. For the general case of loading, an incremental theory must be used. An incremental theory relates the increment or change in plastic strain to the stress. We first describe the deformation theory and then show how this is extended to the increment theory.

The Deformation Theory

Consider a material which is subject to a slowly increasing general state of stress in which the principal stress ratios remain constant. Initially the material behaves in an elastic manner and the relationship between the principal stresses and the principal strains is given by the generalized Hooke's Law.

\[
\varepsilon_1 = \frac{1}{E} \left[ \sigma_1 - \nu (\sigma_2 + \sigma_3) \right] \\
\varepsilon_2 = \frac{1}{E} \left[ \sigma_2 - \nu (\sigma_3 + \sigma_1) \right] \\
\]

(4)
When the state of stress reaches some condition, as given by equation (3) say, the material enters the plastic range. The relationships between the plastic strains and stresses are formulated based on four assumptions, which are based on experimental results. The first three are called rules of flow, Nadai, and the fourth is concerned with the strain hardening characteristics of the material. These assumptions are:

(1) The directions of the principal plastic strains coincide with the directions of the principal stresses.

(2) Any change in density of the mass is entirely associated with elastic deformation, the change in density associated with plastic deformation being zero.

(3) The Mohr circle representing the state of plastic strain remains continuously geometrically similar to that representing the stress state (Figure 2).

(4) There exists a universal stress-strain relationship for the material. That is, during plastic flow there exists a functional relation between the principal plastic strain components and the principal stresses which is independent of the mode of loading.

By examining Figure 2 we see that condition (3) can be expressed as follows:

\[ \varepsilon_3' = \frac{1}{E} [\sigma_3 - \nu (\sigma_1 + \sigma_2)] \]

\[ \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3} = \frac{\varepsilon_1'' - \varepsilon_2''}{\varepsilon_1'' - \varepsilon_3''} \]

and
Figure 2. Mohr Circles for Stress and Plastic Strain.
This condition states that the ratios of the principal shear stresses to the principal plastic shear strains are equal.

Now from condition (2), since the change in mass associated with plastic deformation is zero, and considering small deformations, we have

\[ \varepsilon''_1 + \varepsilon''_2 + \varepsilon''_3 = 0. \]  

Equations (5) and (6) are three independent equations which can be solved for \( \varepsilon''_1, \varepsilon''_2, \) and \( \varepsilon''_3 \) to obtain

\[ \varepsilon''_1 = \frac{2}{3} R \left[ \sigma_1 - \frac{1}{2} (\sigma_2 + \sigma_3) \right] \]  

\[ \varepsilon''_2 = \frac{2}{3} R \left[ \sigma_2 - \frac{1}{2} (\sigma_3 + \sigma_1) \right] \]  

\[ \varepsilon''_3 = \frac{2}{3} R \left[ \sigma_3 - \frac{1}{2} (\sigma_1 + \sigma_2) \right] \]  

The value of \( R \) can be taken as any one of the ratios (5), or can be expressed in terms of all the principal stresses and strains by writing equations (5) as:

\[ \varepsilon''_1 - \varepsilon''_2 = R (\sigma_1 - \sigma_2) \]  

\[ \varepsilon''_2 - \varepsilon''_3 = R (\sigma_2 - \sigma_3) \]  

\[ \varepsilon''_3 - \varepsilon''_1 = R (\sigma_3 - \sigma_1) \]  

Squaring both sides and adding,

\[ R = \frac{\sqrt{(\varepsilon_1^n - \varepsilon_2^n)^2 + (\varepsilon_2^n - \varepsilon_3^n)^2 + (\varepsilon_3^n - \varepsilon_1^n)^2}}{\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}} \]

or

\[ \frac{2}{3} R = \frac{\sqrt{2/3} \sqrt{(\varepsilon_1^n - \varepsilon_2^n)^2 + (\varepsilon_2^n - \varepsilon_3^n)^2 + (\varepsilon_3^n - \varepsilon_1^n)^2}}{\sqrt{2/3} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}} . \]

The denominator is recognized as the significant stress and will be denoted by \( \sigma \). The numerator is called the significant plastic strain and will be notated by \( \varepsilon^n \).

Therefore we have

\[ \varepsilon_1^n = \frac{\varepsilon^n}{\sigma} [\sigma_1 - \frac{1}{2} (\sigma_2 + \sigma_3)] \]  
\[ \varepsilon_2^n = \frac{\varepsilon^n}{\sigma} [\sigma_2 - \frac{1}{2} (\sigma_3 + \sigma_1)] \]  
\[ \varepsilon_3^n = \frac{\varepsilon^n}{\sigma} [\sigma_3 - \frac{1}{2} (\sigma_1 + \sigma_2)] . \]

Now the fourth basic assumption in the development of this theory is that there exists a universal stress-strain relationship. It is convenient to look for a relation of the form:

\[ \varepsilon^n = f(\sigma) \]  

Note that this form is compatible with the Mises condition of yielding, which states that yielding occurs when the significant stress
reaches some value. We know that once a material has been stressed into the plastic range the yield stress is changed, the value of the new yield stress depending on the amount of plastic strain induced. Equation (9) states exactly this as it gives the value of the significant stress required to produce further yielding which is a function of the amount of plastic strain which has already occurred.

The form of the function in equation (9) can be evaluated by any convenient test, such as a uniaxial tensile test. In fact, for a tensile test the significant stress and significant plastic strain are exactly the axial stress and strain respectively. Therefore, the tensile test curve supplies the desired functional relationship. That this form of the function is essentially correct has been verified by many investigators, who have shown that a plot of significant stress vs. significant plastic strain for complicated types of loadings results in the same curve as the uniaxial stress-strain curve.

Since the direction of the principal stresses and the principal strains coincide, a rotation of axis allows us to write the stress-strain equations for a general rectangular coordinate system as:

\[
\begin{align*}
\varepsilon_x'' &= \frac{\varepsilon''}{\sigma} [\sigma_x - \frac{1}{2} (\sigma_y + \sigma_z)] \\
\varepsilon_y'' &= \frac{\varepsilon''}{\sigma} [\sigma_y - \frac{1}{2} (\sigma_z + \sigma_x)] \\
\varepsilon_z'' &= \frac{\varepsilon''}{\sigma} [\sigma_z - \frac{1}{2} (\sigma_x + \sigma_y)] \\
\gamma_{xy}'' &= 3 \frac{\varepsilon''}{\sigma} \tau_{xy} \\
\gamma_{yz}'' &= 3 \frac{\varepsilon''}{\sigma} \tau_{yz} \\
\gamma_{zx}'' &= 3 \frac{\varepsilon''}{\sigma} \tau_{zx}
\end{align*}
\]
The Increment Theory

As previously stated, the deformation theory relates the total plastic strain to the stress. A more fundamental formulation of plastic flow theory relates the rate of change of the plastic strains to the stresses. In the solution of practical problems it is necessary to break the loading history up into a sequence of loadings, each occurring over some finite time interval. The loading during any time interval is considered as proportional loading and therefore the deformation theory can be applied over this interval. We can then write equations (10) if we think of the strain as the incremental change of strain during this time interval and the stress as the stress state at the end of the time interval.

\[
\Delta \varepsilon''_x = \frac{\Delta \varepsilon''}{\sigma} \left[ \sigma_x - \frac{1}{2} (\sigma_y + \sigma_z) \right]
\]

with similar equations for the other strain components where

\[
\Delta \varepsilon'' = \sqrt{\frac{2}{3}} \sqrt{(\Delta \varepsilon''_1 - \Delta \varepsilon''_2)^2 + (\Delta \varepsilon''_2 - \Delta \varepsilon''_3)^2 + (\Delta \varepsilon''_3 - \Delta \varepsilon''_1)^2}
\]

Finally, we can write equations for determining the total plastic strain in the material after a general loading history as:

\[
\varepsilon''_x = \frac{\Delta \varepsilon''}{\sigma} \left[ \sigma_x - \frac{1}{2} (\sigma_y + \sigma_z) \right] + \sum \Delta \varepsilon''_x
\]

where \( \sum \Delta \varepsilon''_x \) represents the summation of the x component of plastic strain which has occurred up to the beginning of the time interval under question,
CHAPTER III

FORMULATION OF THE TWO-DIMENSIONAL PROBLEM

As previously stated, the object of this work is to develop and apply a method for solving general two-dimensional thermal stress problems in the elastic-plastic region. The method presented is applicable to any two-dimensional plane stress problem; however, in order to make the presentation clear and for purposes of demonstration, a particular geometry has been selected for solution. In this chapter the particular problem selected will first be described and then the two-dimensional equations developed. It will be apparent that the equations presented are general and not restricted to any specific problem. Certain terms in the equations will then be evaluated or put into a form applicable to the particular problem selected. The method of solving the equations will be given in the next chapter.

One more significant point should be stated: the development in this chapter and in the next will be based on the deformation theory. However, the entire procedure is easily adapted to an incremental approach, and how this could be accomplished will be shown.

Description of Problem

The particular problem considered is that of two-dimensional plane stress in a rectangular plate of length $A$ and width $B$ (Figure 3).

All surfaces of the plate are free so that there is actually no external loading acting on the plate. Stresses in the plate are due to a
Figure 3. Rectangular Plate Configuration.
temperature variation in the XY plane. Actually, the temperature is a function of Y only. The temperature profile is maintained by adding heat uniformly over a narrow strip of width b about the longitudinal center line, and transferring heat out along the two edges parallel to this center line. All other surfaces of the plate are insulated. The temperature profile is shown in Figure 4.

**Basic Equations**

The equations describing the stress and strain distribution in solid bodies are obtained from three different physical considerations: the statics or equilibrium of the body, the geometry of deformation, and the stress-strain relationships for the material.

As shown in text on the theory of elasticity, for plane stress problems the following equations are obtained.

From equilibrium:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = 0.
\]

From the geometry of deformation (compatibility equation):

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.
\]

The significant difference between elasticity problems and elastic-plastic thermal stress problems lies in the stress-strain relationships. The total strain in the material is made up of strains due to the stress
Figure 4. Heat Transfer and Temperature Distribution in the Rectangular Plate.
plus strains due to thermal expansion. The strain due to stress can be considered as consisting of elastic strain plus plastic strain. Therefore we have for example:

\[ \varepsilon_x = \varepsilon_x' + \varepsilon_x'' + \alpha T, \]

where

\[ \alpha = \text{thermal coefficient of expansion} \]
\[ T = \text{temperature above some initial stress-free temperature}. \]

The temperature change affects the normal strains only, as can be seen by considering a volume element. If the volume element is subject to a temperature rise, each linear dimension changes the same amount with a resulting change in the volume without distortion.

In Chapter II it was shown that the elastic parts of the total strains are related to the stresses by Hooke's Law. The plastic parts are complicated functions of the stresses as expressed by equations (9) and (10). The total strains can thus be expressed in terms of the stresses and the temperature. However, for the method of solution used it is more convenient to make the substitution only for the elastic part. For plane stress problems, \( \sigma_z = \tau_{yz} = \tau_{zx} = 0 \), therefore the stress-strain relations are written:

\[ \varepsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y) + \varepsilon_x'' + \alpha T \]
\[ \varepsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x) + \varepsilon_y'' + \alpha T \]
\[ \gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy} + \gamma''_{xy} \]
Solution of the problem requires the determination of the stress and strain distribution which satisfies the following: equilibrium, equations (11); compatibility, equation (12); stress-strain relations, equations (9), (10), and (13); and the appropriate boundary conditions. This complicated set of equations is condensed by first substituting equations (13) into (12). Use is then made of equations (11) to eliminate the shear stress $\tau_{xy}$. By performing these substitutions, and doing some rearranging and cancellation, equation (12) can be replaced by

$$
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\sigma_x}{E} + \frac{\sigma_y}{E}\right) = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\varphi I)
$$

(14)

Next, equations (11) are eliminated by defining a "stress function." If we define a function $\phi^*$ by the following:

$$
\sigma_x = \frac{\partial^2 \phi^*}{\partial y^2},
$$

$$
\sigma_y = \frac{\partial^2 \phi^*}{\partial x^2},
$$

$$
\tau_{xy} = -\frac{\partial^2 \phi^*}{\partial x \partial y},
$$

it is easily verified that equations (11) are satisfied identically. In terms of $\phi^*$, equation (14) becomes

$$
\left(\frac{\sigma_x^2}{\partial x^2} + 2 \frac{\sigma_x \sigma_y}{\partial x \partial y} + \frac{\sigma_y^2}{\partial y^2}\right)\left(\frac{\phi^*}{E}\right) = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\varphi I)
$$

(16)

$$
-\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial y^2}\right)
$$
For the rectangular plate it was found convenient to express equation (16) in dimensionless form.

Let

\[ \xi = \frac{x}{A} \]
\[ \eta = \frac{y}{B} \]
\[ \kappa = \frac{B}{A} \]
\[ \varphi = \frac{\varphi^*}{B^2 E} \]

Equation (16) becomes

\[ \left( \kappa^4 \frac{\partial^4 \varphi}{\partial \xi^4} + 2\kappa^2 \frac{\partial^4 \varphi}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 \varphi}{\partial \eta^4} \right) \varphi = -\left( \kappa^2 \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \eta^2} \right) \varphi_{TT} \quad (17) \]

- \left( \kappa^2 \frac{\partial^2 \epsilon_{xy}}{\partial \xi^2} - \kappa \frac{\partial^2 \epsilon_{xy}}{\partial \xi \partial \eta} + \frac{\partial^2 \epsilon_{xy}}{\partial \eta^2} \right).

**Boundary Conditions**

It can be shown that for plane stress problems, \( \varphi^* \) must satisfy the following conditions at the boundary:

\[ \frac{\partial^2 \varphi^*}{\partial y^2} \frac{dy}{ds} + \frac{\partial^2 \varphi^*}{\partial x \partial y} \frac{dx}{ds} = -X \]
\[ \frac{\partial^2 \varphi^*}{\partial x^2} \frac{dx}{ds} + \frac{\partial^2 \varphi^*}{\partial x \partial y} \frac{dy}{ds} = -Y \]

where \( X \) and \( Y \) are the \( x \) and \( y \) components of boundary loading and \( s \) is arc length along the boundary.

For the rectangular plate with free edges, these conditions become
These conditions are satisfied if on the boundary \( \varphi \) is of the form:

\[
\varphi = ax + by + c ,
\]

where \( a, b, \) and \( c \) are arbitrary. For convenience we take \( a = b = c = 0 \).

Therefore, the boundary conditions are equivalent to

\[
\varphi = 0 \quad (18)
\]

\[
\frac{\partial \varphi}{\partial n} = 0 ,
\]

where \( \frac{\partial \varphi}{\partial n} \) represents the directional derivative normal to the boundary.

The same boundary conditions obviously apply to \( \varphi \) as well.

**Evaluation of the Laplacian**

It is seen that if the coefficient of thermal expansion, \( \alpha \), is constant, the basic equation to be solved, equation (16) or (17), contains the Laplacian of the temperature field on the right-hand side. The temperature distribution is given so that this becomes a known function. For the rectangular plate the temperature depends on \( y \) only and it is assumed, that steady-state exists. Under these conditions, the heat conduction equation reduces to

\[
\frac{d^2T}{dy^2} = -\frac{\varphi}{k} ,
\]

where
\[ q = \text{volume heat source term} \]

\[ k = \text{thermal conductivity}. \]

Neglecting losses from the surface, there is no heat energy added or lost throughout the plate except along the longitudinal center line. Therefore, the Laplacian is identically zero everywhere except along the center line where heat is transferred to the plate. The value of the Laplacian in this region can be determined by making an energy balance on a volume element of length dy and unit depth in the x direction (see Figure 4).

Let \( \bar{q} = \text{heat transfer per unit area at surface} \)

\[ q = \text{heat conducted in } y \text{ direction} \]

\[ \bar{q} = -kt \frac{dT}{dy} \text{ (Fourier's Law)} \]

Then the energy balance gives:

\[ 2(\bar{q}dy) + q = q + dq \]

\[ 2\bar{q}dy = dq \]

\[ 2\bar{q}dy = \frac{d}{dy}(-kt \frac{dT}{dy}) dy \]

\[ 2\bar{q} = -kt \frac{d^2T}{dy^2} \]

or

\[ \frac{d^2T}{dy^2} = -\frac{2\bar{q}}{kt} \]

Since the heat added at the surface of the plate is conducted in the y direction and out through the edge:
\[ 2q\left( \frac{b}{2} \right) = ( -kt \frac{dT}{dy} )_{y=B} \]

so

\[ -\frac{2q}{kt} = \frac{2}{b} \left( \frac{dT}{dy} \right)_{y=B} \]

for

\[ b < B, \quad \left( \frac{dT}{dy} \right)_{y=B} \approx -\frac{\Delta T}{B/2} \]

Therefore

\[ \frac{d^2 T}{dy^2} = -\frac{4\Delta T}{Bb} \]

The temperature term on the right-hand side of equation (16) then is given by the following function:

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\Delta T) = -\frac{4\alpha \Delta T}{Bb} \]

for

\[ \left( \frac{B - b}{2} \right) < y < \left( \frac{B + b}{2} \right), \]

\[ = 0 \text{ otherwise}. \]

For use in equation (17), in terms of the variables \( \xi \) and \( \eta \), this expression becomes

\[ \left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) (\Delta T) = -\frac{4B \Delta T}{b} \]

for

\[ \left( \frac{1}{2} - \frac{b}{2B} \right) < \eta < \left( \frac{1}{2} + \frac{b}{2B} \right), \]

\[ = 0 \text{ otherwise}. \]
CHAPTER IV

THE METHOD OF SOLUTION

The solution of two-dimensional thermal stress problems where both elastic and plastic deformations occur requires the solution of equation (17) for the stress function $\phi$ subject to boundary conditions, equation (18). Once a solution for $\phi$ is obtained, the stresses can be calculated from equations (15). The elastic strains are then determined from equations (4) and the plastic strains are determined by using equations (10) along with the stress-strain curve for the material.

Equation (17) resembles the nonhomogeneous biharmonic equation, but it must be remembered that the plastic terms appearing on the right-hand side are functions of the stresses and therefore of the stress function itself. The relationship between the plastic strains and the stresses depends on the strain-hardening characteristics of the material and in general is not expressible in any simple analytical form. For this reason as well as the fact that equation (17) is nonlinear, it seems almost hopeless to ever obtain any sort of general solution. An iteration procedure is therefore proposed and used to obtain the solution.

No consideration is given to the question as to whether or not the procedure is inherently convergent. Mendelson and Manson have shown that, for the one-dimensional problems they considered, the conditions for convergence are satisfied. The method starts from an "elastic solution" and investigations have shown 5 that under small plastic deformation the
strain distribution is very nearly the same as the strain distribution from an "elastic solution." Undoubtedly there are conditions under which the procedure would not converge; it is believed that this would occur in problems of large plastic strain. The question of convergence should not subtract from the usefulness of the method. The method can be applied and it seems clear that if the mathematical equations are satisfied along with the stress-strain properties of the material, the answer is correct.

The Iteration Procedure

Essentially, the method of solution will consist of a sequence of "elastic solutions," or in other words, a sequence of solutions to equation (17), each one of which is obtained by holding the right-hand side fixed. For each solution the value of the plastic strain terms on the right-hand side is obtained from the previous solution. Evaluation of the plastic strain terms is accomplished by assuming that the previous solution gives a good approximation to the strain distribution but not necessarily to the stress distribution. However, the stress distribution from the previous solution is also used in evaluating the plastic strain term, as will be shown.

First, a plot is made of significant stress vs. significant strain as shown in Figure 5. As was shown in Chapter II, the relationship between significant stress and significant plastic strain is exactly the same as the relationship between stress and plastic strain from the uniaxial tensile test. It can be shown that in the elastic range:

\[ \varepsilon' = \frac{2(1+v)}{3E} \sigma \]
Figure 5. Significant Stress vs. Significant Strain.
where \( \varepsilon' \) is a significant elastic strain defined in a manner analogous to the significant plastic strain, except using the elastic parts of the strain. Therefore Figure 5 looks exactly like Figure 1 except for a slightly different slope of the elastic line. However, the curve is used only to obtain values for the significant plastic strain, and so for convenience, the uniaxial stress-strain curve may be used. No error will be introduced in the final results.

Using the notation:

\[
F'' = \left( \frac{\partial^2 e''}{\partial y^2} - \frac{\partial^2 e''}{\partial x \partial y} + \frac{\partial^2 e''}{\partial x^2} \right)
\]

the procedure is formalized as follows:

1. Set \( F'' = 0 \) and solve equation (17) for \( \phi \). Call this solution \( \phi^{(0)} \). From this solution the stress and strain distribution can be determined (\( \sigma^{(0)} \) and \( \varepsilon^{(0)} \)). This is equivalent to assuming that the material has remained elastic during the loading. In reality the yield point has been exceeded at some points in the material and this solution gives values for \( \sigma^{(0)} \) and \( \varepsilon^{(0)} \) which lie at point P in Figure 5. Assuming that the strain from this solution is very nearly correct, it is seen from Figure 5 that for this total amount of strain, part is elastic and part plastic. Therefore, dropping from point P to point P', the significant plastic strain \( \varepsilon''^{(0)} \) is read from the curve. With \( \varepsilon''^{(0)} \) determined in this way and the stresses from the original solution, the plastic strain components are calculated from equations (10). From the plastic strain components, calculate \( F''^{(0)} \).

2. Use \( F''^{(0)} \) in equation (17) and solve for \( \phi^{(1)} \). Evaluate the stress \( \sigma^{(1)} \) corresponding to this value of \( \phi^{(1)} \). An approximation to the
stress-strain distribution is now taken as point Q, \((\sigma^{(1)}, \varepsilon^{(0)})\). It is seen that this distribution satisfies the required equations but does not satisfy the stress-strain properties of the material as point Q does not lie on the stress-strain curve. Again, assuming that the total strain is nearly correct, drop straight down to the curve at point Q! and read off the value of \(\varepsilon^{(1)}\). Use \(\varepsilon^{(1)}\) and \(\sigma^{(1)}\) along with the stress components \(\sigma_x^{(1)}, \sigma_y^{(1)}\) and \(\tau_{xy}^{(1)}\) in equations (10) to determine the plastic strain components. Calculate \(F^{(1)}\).

The process is repeated over and over again and in this way a sequence of points P, Q, S, ..., is generated. At the nth stage, one of these points corresponds to the stress-strain state \((\sigma^{(n)}, \varepsilon^{(n-1)})\), which satisfies all equations but does not satisfy the stress-strain characteristics of the material. The process is continued until we lie sufficiently close to the actual stress-strain curve at all points in the material. It is noticed that the method of calculation should generate a sequence of points, all of which lie above the actual stress-strain curve.

Since at each stage the plastic terms in equation (17) are known, it is only necessary to solve the nonhomogeneous biharmonic equation. Theoretically, any method of doing this is applicable; however, the plastic terms are known only in numerical value at points throughout the body and therefore it is natural to look for a numerical method of solution. In fact, several other methods of solution were tried with very little success and it appears that a numerical method of solution is the only practical one. In order to effect a numerical solution it is necessary to express the equation in finite difference form.
Finite Difference Formulation

To obtain a numerical solution to equation (17), the rectangular plate is replaced by a rectangular array of grid points, and equation (17) is replaced by a system of linear equations involving the values of the unknown function $\phi$ at these grid points. In fact, since equation (17) is expressed in dimensionless form, the rectangular plate is replaced by the unit square. The array of grid points breaks the unit square up into $M^2$ squares of size

$$h = \frac{1}{M}$$

Indices $i$ and $j$ are used to locate the grid points where:

$$i = 0, 1, 2, \ldots, M$$
$$j = 0, 1, 2, \ldots, M$$

See Figure 6. Using the notation:

$$\nabla_x^2 \phi_{i,j} = \phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}$$

and

$$\nabla_x^4 \phi_{i,j} = \phi_{i+2,j} - 4\phi_{i+1,j} + 6\phi_{i,j} - 4\phi_{i-1,j} + \phi_{i-2,j}$$

the finite difference formulation of equation (17) is:

$$k^4 \nabla_x^4 \phi_{i,j} + 2k^2 \nabla_x^2 \nabla_y^2 \phi_{i,j} + \nabla_y^4 \phi_{i,j} = h^4 g_{i,j}$$

$$i,j = 1, 2, \ldots, M-1$$
Figure 6. Grid Configuration for Finite Difference Formulation.

\[ \xi_i = \frac{X}{A} \]
where \( g_{ij} \) represents the temperature and plastic strain terms, the values of which are known at each grid point.

The boundary conditions are

\[
\begin{align*}
\varphi_{i,j} &= 0 & i &= 0, M \tag{21} \\
\varphi_{i+1,j} &= \varphi_{i-1,j} & j &= 1, 2, \ldots, M-1
\end{align*}
\]

and

\[
\begin{align*}
\varphi_{i,j} &= 0 & j &= 0, M \\
\varphi_{i,j+1} &= \varphi_{i,j-1} & i &= 1, 2, \ldots, M-1
\end{align*}
\]

The boundary condition \( \varphi_{i+1,j} = \varphi_{i-1,j} \), for example, represents the zero normal slope condition where use has been made of an imaginary grid point outside the boundary.

Equations (20) represent \((M-1)^2\) linear equations in \((M-1)^2\) unknowns. That is, there is a linear equation associated with each of the interior grid points which involves the value of \( \varphi \) at this point as well as at the 12 surrounding grid points. Of course the equations at points adjacent to the boundary are adjusted to account for the boundary conditions.

Several methods are available for solving such a system of equations. These methods start by making an initial guess for the solution and then iterating on this initial guess. Two such methods are the Southwell Relaxation method\(^{17}\) and the Gauss-Seidel method\(^{18}\). Write equations (20) in the form:

\[
K'_x \varphi_{x}^4 + 2K'_y \varphi_{y}^2 + \varphi_{ij} + \varphi_{ij}^4 - h^4 g_{ij} = R_{ij}.
\]
If the initial guess is correct the equations are satisfied identically and $R_{ij}$ equal zero for all $i$ and all $j$. In general the initial guess is not the correct solution and so not all of the $R_{ij}$'s are zero. The $R_{ij}$'s are called the residuals. In the Southwell method the equations are examined to find the largest residual. At the grid point corresponding to the equation with the largest residual, the value of $\phi_{ij}$ is adjusted to make the residual zero. This process is then repeated over and over until all residuals are brought to a sufficiently small value. In the Gauss-Seidel method the entire grid is swept over, reducing each residual in turn to zero irrespective of its magnitude. This process is then repeated over and over again.

For the present problem a 10 by 10 grid was set up over one quadrant of the plate (100 grid points). From the symmetry of the problem it is only necessary to use the one quadrant. An initial guess was made for the values of $\phi$ at each point and the residuals computed. The residuals ranged from value on the same order of magnitude as $\phi$ down to about 0.1 $\phi$. The Southwell Relaxation method was then programmed for computation on the Burroughs 220 digital computer. After a total running time of approximately 2 hours (several thousand relaxations) the values of the residuals ranged from about 0.5 to 0.1 $\phi$. The Gauss-Seidel method was then programmed and computations carried out for an additional 2 hours on the computer (600 sweeps over the grid or 60,000 relaxations). The value of the average residual was still about 0.1 $\phi$. Since in the overall scheme the biharmonic equation must be solved several times, it became apparent that unless a more efficient method could be found for obtaining a solution the whole procedure would be rather impractical. Such a method was
found. The Alternating Direction method of Conte and Dames\textsuperscript{19} can start with an initial guess of zero everywhere and produce average values of the residuals on the order of 0.001 $\varphi$ in 20 minutes of computer time. Further computations seem to be useless and it appears that this is about as small a value of the residual as can be obtained. Examination of the equations reveals that this should give solutions to the set of equations good to three significant figures. In fact, the method was applied to the calculation of the deflection of several rectangular plates under lateral loads (which requires solution of the biharmonic equation) and values were obtained which agreed with the accepted solutions\textsuperscript{21} to three significant figures.

The Method of Conte and Dames

In what follows the superscript indicates the number of the iteration as it applies to the Conte-Dames method and not to the overall iteration method of solving the elastic-plastic problem.

Consider the following equations.

\[
\phi_{ij}^{(n+\frac{1}{2})} = \phi_{ij}^{(n)} - r_{n+1} \left[ K_{ij} \phi_{ij}^{(n+\frac{1}{2})} + 2K_{ij} \phi_{ij}^{(n)} \right] + h_{ij}^{(n)}
\]

\[
\phi_{ij}^{(n+1)} = \phi_{ij}^{(n+\frac{1}{2})} - r_{n+1} \left[ \nabla_y \phi_{ij}^{(n+1)} - \nabla_y \phi_{ij}^{(n)} \right]
\]

\[1, j = 1, 2, \ldots, M-1\]

together with the required boundary conditions on $\varphi$. Where $r_{n+1}$ is an iteration parameter which is chosen to accelerate convergence.

The solution is obtained as follows. First an initial guess is made for the values of $\phi_{ij}^{(0)}$. The first of equations (22) is implicit
in \( X \) and represents \( M-1 \) equations in \( M-1 \) unknowns for each \( j \). Thus as \( j \) goes through the values 1, 2, \ldots, \( M-1 \) the equations are set up and solved along each row. This gives a value \( \varphi_{ij}^{(1/2)} \) at each grid point and constitutes one-half iteration. The second of equations (22) is implicit in \( y \) and thus represents \( M-1 \) equations in \( M-1 \) unknowns for each \( i \). Thus as \( i \) goes through the values 1, 2, \ldots, \( M-1 \) the equations are set up and solved for each column. This now gives a new value \( \varphi^{(1)} \) at each grid point and constitutes one iteration.

By examining any one of the set of \( M-1 \) equations in \( M-1 \) unknowns which have to be solved, it is seen that they are of the "quidiagonal" type. That is, the matrix of coefficients contains at most five non-zero elements (on the main diagonal and on the two adjacent diagonals). Conte and Dames give an elimination procedure, which they credit as an extension of a method due to L. H. Thomas, for very rapidly solving such a system.

To obtain a feeling for this method consider a 10 by 10 grid (100 grid points). One iteration will involve the solution of 10 equations in 10 unknowns—20 times: ten times as we in turn set up the equation for each row (for each \( j \)) and proceed up the grid, and 10 times as we in turn set up the equations for each column (for each \( l \)) and proceed over the grid from left to right. For the rectangular plate it was found that 21 iterations were required. This required solving 10 equations in 10 unknowns 420 times. In addition, certain computations and transfers were required in the computer program between each iteration. As previously stated, excellent results were obtained in about 20 minutes. As will be shown later, one problem was solved using a 20 by 20 grid, which required about one hour on the computer for convergence. It is believed that any
other method of solving the finite difference problem would be out of the
question and the writer regards the Conte-Dames method as most satisfactory.

The success of the whole procedure depends on the proper choice of
the iteration parameters \( r_{n+1} \). Conte and Dames considered the problem in
which the boundary condition required the second normal derivative to be
zero. For this problem they were able to show that a near optimum choice
of iteration parameters is given by

\[
r_k = \frac{1-k}{16}
\]

\[k = 1, 2, \ldots, t\]

where \( t \) is the number of iterations required. One first decides on the
desired reduction in the error, \( P_t(a) \). The value of \( a \) is determined from

\[
P_t(a) = \left[ \frac{1-a}{1+a} \right]^{1/2} \exp\left(-\frac{a^{3/2}}{1-a}\right)^4
\]

The number of iterations required is then

\[t \geq 1 + 4 \frac{\log \left[ \sin \frac{\pi h}{2} \right]}{\log a}\]

For large reductions in error it is better to keep the number of iterations
to a small value and repeat the whole process if necessary. All things
considered, it appears that a good choice is:

\[a = 0.2\]

this gives:
\[ P_t(a) = 0.01 \]

or a reduction in error by a factor of \(10^{-2}\). For a 10 by 10 grid this requires \(t = 7\), or 7 iterations. Best results were obtained by repeating the procedure 3 times for a total of 21 iterations.

For the rectangular plate in which the boundary condition requires the first normal derivative to be zero, Conte and Dames have shown in a later paper that the Alternating Direction method will converge, but have not been able to show how to choose the iteration parameter. Several different schemes for choosing the parameter were tried, but best results were obtained by using the same parameters as predicted above.

**Extension to Increment Theory**

Now, as previously stated, the entire foregoing discussion has been based on the deformation theory of plastic flow. The correct results are obtained by this theory if the loading has been proportional, that is, if the plate has been subject to an increasing temperature gradient in which the temperature profiles at each instant are all similar. Many practical problems, however, do not fall into the class of proportional loading; for example, it may be required to determine the residual stress distribution in the plate after it is allowed to cool. The solution to this problem would require the use of the increment theory and consideration will now be given as to how this would be done.

Suppose the plate has been heated and the stress and strain distribution have been determined by applying the iteration method of solution outlined. The plate is then allowed to cool, and it is assumed that the temperature profiles during cooling can be determined. As the plate first
begins to cool it would probably be accompanied by unloading at each point. This would be reflected in a decrease in the significant stress at each point. As long as the significant stress is decreasing, the material is unloading along an elastic line and no further plastic flow occurs. Therefore the stress-strain distribution can be determined by solving equation (17) with the plastic strain terms held at the same value as when cooling first began. This process is continued until it is found that at some points the significant stress has increased to a value higher than the value at the start of cooling. Additional plastic flow has then occurred at these points and the iteration method would be used to bring the value of stress and strain back into agreement with the required stress-strain curve. During this iteration solution, the plastic strain terms at those points where the significant stress is still less than its initial value would not be changed. The process is thus continued until the final equilibrium temperature is reached. At the equilibrium state, the temperature term in equation (17) would be zero but the plastic strain term would not. The plastic strain term would have a value which depends on the accumulation or summation of the plastic strain induced during cooling. Solution of equation (17) would then give the residual stress and strain distribution.
CHAPTER V

ANALYTICAL SOLUTION

Using the technique developed in the previous chapter, the stress and strain distribution in the rectangular plate have been determined. The results of this calculation are presented in this chapter.

The Inconel Plate

A rectangular plate 36 inches by 24 inches by 0.25 inch thick was obtained from the International Nickel Co. The plate was hot-rolled and annealed, the direction of rolling coinciding with the longitudinal (or X) direction. The International Nickel Co. also supplied a chemical analysis and a statement that the plate had been ultrasonically tested and met the requirements of MIL-STD-271. Inconel was selected as a material which would maintain fairly constant physical properties up to relatively high temperatures. Typical stress-strain curves for hot-rolled Inconel at temperatures up to 600° F were supplied by the International Nickel Co.; these curves showed a reduction in yield strength from room temperature to 600° F of only 8.5 per cent. This is within the range of variation which might be expected from different samples of material even when all are tested at the same temperature. An International Nickel Co. publication shows that creep effects for this material are negligible at temperatures less than 800° F.

A rectangular test plate was machined from the original plate and conformed to the following dimensions and physical properties.
\[ A = 30 \text{ inches} \]
\[ B = 20 \text{ inches} \]
\[ K = \frac{B}{A} = \frac{2}{3} \]
\[ \alpha = 8.3 \times 10^{-6} \text{ in/in-}^\circ \text{F} \]
\[ E = 30.0 \times 10^6 \text{ psi} \]
\[ \text{yield point} = 30.0 \times 10^3 \text{ psi.} \]

The value of the thermal coefficient of expansion is taken from the International Nickel Co. publication and represents an average value over the temperature range from room to 600° F. The values of the modulus of elasticity and the other stress-strain characteristics were obtained from several tensile tests performed on longitudinal specimens cut from the plate. The stress-strain curve is shown in Figure 7. Several tensile tests were also performed using specimens cut in the transverse direction. Transverse tensile properties were in general slightly higher than the longitudinal values. However, the difference was less than 10 per cent and since the predominant stresses are in the longitudinal direction, the longitudinal stress-strain curve was used.

**Definition of Problem and Solution**

Referring to Figure 3 in Chapter III, the plate is assumed to be heated over a strip of width \( b = 1 \) inch about the longitudinal center line. Heating is initiated from some stress free, uniform temperature condition. The temperature of the plate along the longitudinal center line is slowly increased until there is a temperature difference between the center and the edges of 480° F. It is assumed that the temperature profiles at each instant are similar to the profile shown in Figure 4; under
Figure 7. Stress-Strain Curve for Inconel.
these conditions proportional loading has existed. The problem then is to determine the stress and strain distribution existing in the plate under this maximum temperature condition. It will be found that this temperature difference is sufficient to have caused some plastic flow in the plate.

The solution is begun by first solving equation (17) with the plastic terms set equal to zero. The value of the Laplacian of the temperature field is

$$\frac{4}{b} \frac{\partial^2 T}{\partial \eta^2} \Delta \frac{dT}{b} = 4 \left( \frac{20}{(8.3 \times 10^{-6})(480)} \right)$$

(1)

$$= 0.3187$$

for

$$0.475 < \eta < 0.525$$

A 20 by 20 grid was placed on one quadrant of the plate and, using the above value of the Laplacian as $g_{ij}$ at grid points falling within the heated strip, equation (17) was solved by the Alternating Direction method. Values of the significant stress from this solution are shown in Figure 8. Values are only shown at every other grid point. This solution assumes that the material has remained elastic during the heating, but in reality the yield point has been exceeded and plastic flow has occurred at every point where the significant stress is greater than 30,000 psi. It is seen that plastic flow has occurred over areas of the plate at the center and at the midpoints of the sides.

A second approximation to the stress-strain distribution was calculated as follows. First, using Figure 7, a plot was made of significant plastic strain vs. total strain:
Figure 8. Significant Stress Distribution from "Elastic Solution" ($\approx 10^{-3}$ psi).
\[ \varepsilon'' = G(\varepsilon) . \]

For purposes of computation, this relation was approximated by a fifth degree polynomial. Now at every grid point where the significant stress is greater than 30,000 psi, the total strain is approximately given by

\[ \varepsilon'(o) = \frac{\sigma'(o)}{E} ; \]

therefore at these points the part of the total strain which is plastic is computed from:

\[ \varepsilon''(o) = G\left(\frac{\sigma'(o)}{E}\right) . \]

As outlined in the previous chapter, the values of \( \sigma'(o) \) and \( \varepsilon''(o) \) allow the determination of the plastic strain terms in equation (17). Equation (17) is then resolved to obtain a new significant stress distribution, \( \sigma'(1) \). So the second approximation to the stress-strain state is that corresponding to a significant stress of \( \sigma'(1) \) and a significant plastic strain \( \varepsilon''(o) \). Or, this can be stated as a stress-strain state where the significant stress is \( \sigma'(1) \) and the total significant strain is approximately

\[ \varepsilon'(1) = \frac{\sigma'(1)}{E} + \varepsilon''(o) . \]

A third approximation to the stress-strain state was then computed. At each grid point the total strain is

\[ \varepsilon'(1) = \frac{\sigma'(1)}{E} + \varepsilon''(o) ; \]
therefore the part of the total strain which is plastic is

\[ \varepsilon^{(1)} = \frac{e^{(1)}}{E} + \varepsilon^{(0)} \]

This allows re-evaluation of the plastic strain terms and a new solution to equation (17). The process is then continued in the same manner as above. After the nth iteration, the stress-strain state is approximated by the corresponding values \( \sigma^{(n)} \) and \( \varepsilon^{n(n-1)} \). Or, in other words, a stress-strain state where the total strain is given by

\[ \varepsilon^{(n)} = \frac{\sigma^{(n)}}{E} + \varepsilon^{n(n-1)} \]

Therefore the significant plastic strain for the next iteration is calculated from:

\[ \varepsilon^{(n)} = \frac{\sigma^{(n)}}{E} + \varepsilon^{n(n-1)} \]

After 4 iterations the significant stress at any point was only changing in the third figure, so the procedure was terminated. Final significant stress distribution in the plate is shown in Figure 9. Figures 10, 11, and 12 show the final values of \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \) at points in the plate. Figure 13 is a plot of \( \sigma_x \) variation over the transverse center line; this is given for comparing the final values obtained with the values from the initial "elastic solution." Additional results from this solution are given in Chapter VI where certain longitudinal strains are compared with strains measured in the experimental investigation.

At each stage of the iteration, the biharmonic equation was solved by the Alternating Direction method of Conte and Dames. This procedure
Figure 9. Significant Stress Distribution ($\sigma \times 10^{-3}$ psi).
Figure 10. x Stress Distribution ($\sigma_x \times 10^{-3}$ psi).
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\(\eta = 0\) & \hline
44.8 & 25.6 & 6.94 & -0.84 & -5.46 & -7.96 & -8.36 & -8.02 & -8.61 & -8.24 & -8.06 \\
34.6 & 21.8 & 6.99 & 0.0 & -4.44 & -6.91 & -7.38 & -7.85 & -7.87 & -7.30 & -7.22 \\
27.1 & 17.7 & 6.47 & 0.62 & -3.39 & -5.71 & -6.19 & -6.62 & -6.45 & -6.10 & -8.05 \\
19.1 & 13.5 & 3.51 & 0.91 & -2.43 & -4.46 & -4.82 & -5.22 & -5.04 & -4.78 & -4.60 \\
12.1 & 9.52 & 4.29 & 0.81 & -1.68 & -3.27 & -3.53 & -3.76 & -3.54 & -3.36 & -3.23 \\
6.08 & 6.12 & 2.05 & 0.20 & -1.07 & -2.17 & -2.31 & -2.35 & -2.33 & -2.13 & -1.91 \\
1.84 & 3.21 & 1.63 & 0.39 & -0.59 & -1.22 & -1.23 & -1.12 & -0.95 & -0.84 & -0.85 \\
0.12 & 1.63 & 0.51 & 0.11 & -0.31 & -0.45 & -0.29 & -0.27 & -0.24 & -0.21 & -0.22 \\
0.0 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
\hline
\end{tabular}
\caption{\(\gamma\) Stress Distribution (\(\sigma_\gamma \times 10^{-3}\) psi).}
\end{table}
Figure 12. Shear Stress Distribution ($-\tau_{xy} \times 10^{-3}$ psi).
Figure 13. Comparison of Final Value of $d_x$ with Value Obtained from "Elastic Solution" ($\xi = \frac{1}{2}$).
was programmed for solution on the Burroughs 220 computer using the Burroughs algebraic compiler, which is a hardware representation of Algol. Additional discussion of the method of solution along with the computer program itself is given in the Appendix.

The fifth degree polynomial approximating the relationship between $\varepsilon''$ and $\varepsilon$ was determined using a method due to Forsythe. This method is ideal for computer use but does not give the equation in a convenient form for publication, and so the resulting polynomial will not be exhibited here. However, it can be stated that 12 data points were used to fit the equation and the maximum deviation of the polynomial from any data point was less than 4 per cent.

At each stage of the iteration the Alternating Direction method gave a solution to the finite difference equations with the residual less than 0.001 times the value of the stress function at every point. An additional check on the accuracy of the solution to the finite difference equations was made by calculating the integrals

$$
\int_0^1 \sigma_x \, dy \quad \text{and} \quad \int_0^1 \sigma_y \, dx
$$

over each cross section (along grid lines). From equilibrium conditions the value of every integral should be zero. It was found that the values of the integrals were all less than 0.0005 times the value of the maximum stress on the cross section.
CHAPTER VI

EXPERIMENTAL PROGRAM

An experimental program was conducted using the Inconel plate. The plate conformed in both dimensions and physical properties with those used in the analytical solution. Both temperature and strain measurements were made.

Description of Apparatus

A picture of the experimental equipment is shown in Figure 14. The plate is located in a box of sufficient size to allow at least 6 inches of insulation on the top and bottom surfaces and on the two ends. The entire box was filled with vermiculite. Heat is supplied to the plate by means of two G. E. calrod heaters, Catalog Number 5-D 12, embedded in one inch wide copper bars. The bars are shown clamped to the top and bottom surfaces of the plate along the longitudinal center line. Clamping pressure was held to a minimum in order to minimize the effect of constraint. The two edges parallel to the heated center line are cooled by water running through 0.25 inch thick wall copper pipes. These pipes are machined flat on one side with a small lip so as to fit snugly against the plate edges. Alignment of the cooling pipes is accomplished by means of holes at each end of the box, the pipes are not clamped to the plate in any way and are free to move slightly with the plate. Each pipe has its own valve so that the water flow can be controlled separately.

Copper was used for both the heating and cooling surfaces in order to equalize the temperature over the length of the plate. Calculations
Figure 14. Experimental Equipment.
indicated that it should be possible to maintain the maximum temperature in the plate with only a 2° or 3° F rise in water temperature. Inlet and outlet water temperatures were not measured during the test, but even under maximum heat transfer conditions there was no detectable temperature increase.

At the right in Figure 14 is shown the temperature measuring equipment. Temperature measurements in the plate are made with Iron-Constantan thermocouples and measured with an L and N type 8692 temperature potentiometer. One thermocouple however, the one nearest the center of the plate, is connected to an L and N Speedmax G recorder which is used in conjunction with an L and N series 60 controller. The controller supplies the heater voltage, and thus accurate temperature control is possible. A small indentation 1/16 inch in diameter by 1/32 inch deep is drilled in the plate surface, the thermocouples are bent in such a way as to maintain good surface contact and fastened down with Sauereisen type 63 heater cement.

At the left in Figure 14 is shown the strain measuring equipment. This consists of a Baldwin switching-balancing unit, Serial Number 1047, and a Baldwin type L strain indicator. The strain gage selected was the BLH "universally temperature-compensated" gage type FNM-50-12E. Application of this gage presented several interesting problems and so a more detailed discussion is presented below.

The Temperature-Compensated Gage

Two problems need special consideration in the selection of resistance type strain gages for use at elevated temperatures. These are concerned with the choice of a bonding agent and with the problem of temperature compensation. Several organic type cements are available for use at
temperatures in the 200° to 300° F range and a few organic cements can be used up to 500° F. For applications over 500° F, ceramic cements must be employed. Temperature compensation up to about 300° F can be accomplished by the use of "selected melt" gages in which the gage material is carefully controlled so as to minimize resistance change with temperature when mounted on the material for which it is specified. Availability of compensating type gages for use at higher temperatures is limited. After investigation it appeared that the type FNH-50-12E gage, bonded with R. A. Allen Co. ceramic cement Alien P-1, was the only gage which could satisfy all requirements. This gage can be used for strain measurements up to 850° F.

The selection of the cement was based on a cement evaluation report published by BLH. Bonding of this gage (as are all high temperature gages) is extremely tedious. Proper mounting and curing of the cement required about 15 hours. Considerable effort was needed to develop a satisfactory technique. After the gage is installed, it is necessary to soak the entire test piece for several hours at a temperature above the maximum temperature expected in the actual test. Furthermore it is recommended that the test piece be cycled several times from room temperature. These requirements are needed to fully stabilize the cement and are necessary before any reproducible strain readings can be obtained. It was found that about 10 hours soaking at 700° F plus about six cycles from room temperature to 700° F resulted in good reproducibility in the calibration runs.

The FNH-50-12E gage is shown in Figure 15 along with a schematic drawing of its required circuitry. The gage is seen to consist of a Nichrome V foil element and a platinum wire element. The Nichrome is the
Figure 15. Type FNH-50-12E Strain Gage and Typical Installation Circuitry.
strain sensing element and the platinum is for temperature compensation. The principle of operation of a resistance type strain gage is that the gage resistance changes when it is subject to strain. Therefore, a measurement of this resistance change allows the strain to be determined.

Unfortunately, the resistance of the gage also changes with temperature; this is due to a change in resistance of the gage material plus an effect due to the difference in the thermal expansion of the gage and the material on which it is mounted. Ordinarily the gage is connected as one arm of a Wheatstone bridge and a "dummy" or "compensating" gage is connected in the opposite arm. The compensating gage is placed in the same vicinity as the active gage; therefore, any temperature variation effects both gages and there is no unbalancing of the bridge. The type FNH-50-12E eliminates the need for a separate compensating gage. As shown in Figure 15, the Nichrome element is connected as one arm of the bridge and the platinum element, along with a ballast resistor in series with it, is connected as the opposite arm. Both elements show fairly linear resistance change with temperature. By proper selection of the ballast resistor the percentage change in resistance of the platinum element, caused by temperature change, can be made to cancel the percentage change in the resistance of the Nichrome element, thereby minimizing bridge output. The value of the ballast resistor can be determined from the following equation:

$$R_B = \frac{A_T}{\Delta \sigma} \left[ R_T \left( 1 + \frac{R_G}{R_T} \right) \right] = R_T - R_L$$

where
\[ R_B = \text{value of ballast resistor} \]
\[ R_T = \text{resistance of platinum element} \]
\[ R_G = \text{resistance of Nichrome element} \]
\[ R_{LT} = \text{lead wire resistance in compensating arm} \]
\[ R_{LG} = \text{lead wire resistance in active arm} \]
\[ \Delta T = \text{temperature coefficient of resistance change for compensating element} \]
\[ \Delta G = \text{temperature coefficient of resistance change for active element} \]

The values of \( R_T \) and \( R_G \) are supplied by the manufacturer. Also the value of \( \Delta T/\Delta G \) is supplied for the gage mounted on 316 Stainless steel; for other materials the value is adjusted by multiplying by the ratio of the coefficient of expansion of 316 Stainless to the coefficient of expansion for the test material.

To test the temperature compensating quality of the gage, a gage was mounted on an Inconel test bar and heated uniformly in an oven from room temperature to 700° F. Calculations predicted a value of 178 ohms for the ballast resistor. The apparent strain vs. temperature for this value of the ballast resistor as well as for other values is shown in Figure 16. These results are in good agreement with typical curves shown by the manufacturer.

As previously mentioned, the strains were measured with a BLH type L strain indicator. The active Nichrome element has a resistance of about 120 ohms, and the platinum element along with the ballast resistor will run about 180 ohms. Such a wide difference in resistance in the two legs of the bridge is much more than can be accommodated by the type L indicator. Therefore, it is necessary to connect the gage into a complete bridge.
Figure 16. Apparent Strain vs. Temperature for Various Values of Ballast Resistor.
external to the indicator and to use the indicator to measure the output of this bridge. Figure 17 shows a complete wiring diagram of the circuit used, including the type 1047 switching unit. It is seen that the external bridge contains two variable resistors. One is the ballast resistor and the other is for initial balance of the bridge.

To conclude the discussion of this gage, some remarks are required regarding certain corrections which must be made to the indicated strains obtained in the test. First, a correction is needed to account for the apparent strain due to temperature variation. As previously shown, the effect of temperature can be greatly reduced by using the self-compensating gage but it is not completely eliminated and is too large to be ignored. Another correction is necessary because of gage factor variation. The gage factor is defined as

\[ f = \frac{\Delta R}{R} \]

and is an inherent property of the gage material. However, the gage factor does vary with temperature, and information supplied by the manufacturer showed that a reduction of approximately 10 per cent could be expected over the temperature range from room to 1000°F, this reduction was assumed to be linear. An additional correction on the gage factor is due to the use of an external bridge. Referring to the Wheatstone bridge below, it can be shown\(^{26}\) that the ratio of bridge output to bridge input is given by:

\[ \frac{\Delta V}{E} = \frac{R_G R_1}{(R_G + R_1)^2} \cdot f \]
To Type L Indicator

External Bridge

304.7 Ohms

0 to 1000

223.4 Ohms

195.4 Ohms

0 to 1000

1028

Type 1047 Switch

Specimen

R_G

R_T

Figure 17. Strain Gage Circuit (All Resistances in Ohms).
The type L indicator is calibrated for the condition \( R_G = R_1 \) which gives:

\[
\frac{\Delta V}{E} = 0.25 \text{ ft }.
\]

For the values used in the external bridge we have:

\[
\frac{\Delta V}{E} = 0.24 \text{ ft }.
\]

The net result is that the bridge sensitivity has been reduced by about 4 per cent and therefore the strain readings will be about 4 per cent low. Considering both effects, the strain readings will be from 4 to 14 per cent low over the temperature range from room to 1000° F.

Consideration was also given to the effect of gage transverse sensitivity. Calculations indicated that even for a standard wire gage located at the same positions as the test gages, the transverse strain would affect the gage reading by less than one per cent. Foil gages are in general less sensitive to transverse strain than are wire gages; therefore, the effect of transverse strain is negligible.
Test Procedure and Results

Locations of the thermocouples and strain gages are shown in Figure 18. Twelve thermocouples were placed on the plate in order to check the temperature profile. In addition, six other thermocouples were placed on the plate in order to check the symmetry of the profile. Thermocouple number one was connected to the recorder and used to maintain temperature control. Two strain gages are located on the transverse center line and are oriented to measure the longitudinal strain. The strain gage lead wires are number 26 nickel-clad copper with Fiberglas insulation and are spot welded to the gages.

Prior to test, the strain gages were calibrated by heating the plate uniformly in an oven up to 700°F. Apparent strain vs. temperature for the two gages is shown in Figure 19.

The test was conducted by initially heating the plate until a temperature difference of about 100°F was obtained. During this heating the instrumentation was checked and found to be working satisfactorily. The plate was then allowed to cool until a uniform temperature distribution was reached. This occurred at a temperature of approximately 125°F. Maintaining the edges at a constant temperature the plate was then slowly heated until a temperature difference between the center and the edges of ±80°F was obtained. Temperature and strain readings were taken at various intervals. Heating was conducted very slowly in order to maintain the temperature profile essentially linear. The total time required was 10.5 hours.

Temperature and strain values obtained during the test are presented in Tables 1 and 2. In addition, Figure 20 shows the temperature profile at various intervals during the heating. Figures 21 and 22 show the strain values recorded and compare the results with the analytical solution of Chapter V.
Figure 18. Strain Gage and Thermocouple Locations.
Figure 19. Calibration Curves, Apparent Strain vs. Temperature for Test Gages.
Figure 20. Test Temperature Profiles, Thermocouples 2, 6, 12, and 15.
Figure 21. Longitudinal Strain vs. Time, Gage 1.
Figure 22. Longitudinal Strain vs. Time, Gage 2.
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Table 1. Temperature Data (°F)
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A method for solving two-dimensional thermal stress problems in the elastic-plastic range has been presented and substantiated by experimental investigation. The method is practical and useful. It can be used to obtain fairly accurate values for stress and strain distribution, if one is willing to spend enough time and effort, or it can be used to very quickly obtain approximate or engineering values for the stresses and strains. For an accurate estimation of stress and strain values, the one disadvantage of the method is the amount of computation which is necessary. The present problem required about 6 hours on the digital computer and this time would increase considerably for more general problems involving complicated loading histories. The main time consumer is in the solving of the biharmonic equation. It is believed that the solution of the biharmonic equation is best accomplished numerically, as was done here, and that of the numerical methods available the Alternating Direction method is the best. Convergence of the Alternating Direction method, however, has been proven only for rectangular regions. Therefore, this method may not be available for more general geometries. For geometries other than the rectangle the computation time would be greater. In spite of the time required it is felt that the present method is simply applied and offers the only method available for solving a very difficult problem.
From examination of the analytical results, the following conclusions can be drawn.

1. The "elastic solution" offers an excellent starting point for the iteration procedure. In fact, for moderate amounts of strain such as existed in the present problem, the "elastic solution" would be sufficient for most engineering requirements. This is due to the fact that the strain distribution as obtained from the "elastic solution" is essentially correct, or that the strains are invariant. Even for problems involving fairly large amounts of plastic flow, one or two iterations would probably suffice. Thus for engineering calculations the problems of time could be largely eliminated.

2. It will be noticed that the elastic-plastic boundary in the material is slightly different in the final solution from that predicted in the "elastic solution." That is, the "elastic solution" has predicted yielding at points where yielding actually did not occur and vice versa. The conclusion is that for accurate results under large amounts of plastic flow, two-dimensional problems may require an incremental approach even under proportional loading.

Results of the experimental program verify the analytical procedure. As seen in Figures 21 and 22, in general the measured values of strain were slightly higher than the analytical curve. The primary reason for this is the difference between the analytical and actual temperature profiles. The analytical solution is based on steady-state temperature field with a temperature difference of 480°F. It was impossible, of course, to have steady-state conditions at all times during the heating. Examination of Figure 20 reveals that there was in reality some curvature to the
temperature profiles. Furthermore, heating of the plate was not uniform over its length. Examination of the temperature data reveals that the temperature difference along the transverse center line was greater than 480° F. Both these facts would make the strain greater than calculated by the analytical procedure.

**Recommendations**

1. It is believed that additional experience and understanding should be obtained by solving other problems. For example, the present method could be easily applied to the problem of a rectangular plate with a rectangular hole.

2. Additional work is needed on applying the method to other geometries besides the rectangle. This will require the development or discovery of an efficient method for solving the biharmonic equation.

3. Finally, it is felt that by using a more fundamental definition of strain, the method is capable of being extended for application to problems involving large amounts of plastic strain. This extension should be undertaken.
BIBLIOGRAPHY


APPENDIX

COMPUTER PROGRAM FOR SOLVING THE BIHARMONIC EQUATION

As shown in Chapter IV the Alternating Direction method is an
iteration scheme for solving the biharmonic equation. The iteration pro­
cedure is defined by equations (22). One iteration consists of starting
with an initial guess for the value of the function at each grid point,
$\psi^{(n)}$, and making a double sweep over the grid; first, by setting up and
solving linear equations involving the value of the function at points
in a row (constant $j$), and then by setting up and solving linear equa­
tions involving the values of the function at points in a column (constant
$i$).

Due to the symmetry in the rectangular plate, it is only necessary
to solve the biharmonic equation over one quadrant. Figure 23 defines the
notation for the grid used in this solution. Note that the boundary and
symmetry conditions are accounted for by making use of grid points out­
side of this quadrant. For example:

\[ \psi_{2,j} = \psi_{1,2} = 0, \]
\[ \psi_{1,j} = \psi_{3,j} \]
\[ \psi_{23,j} = \psi_{21,j} \]
\[ \psi_{24,j} = \psi_{20,j} \]

and so forth.

To make the procedure clearer, equations (22) are written in expanded
form and the unknowns collected on the left-hand side. The first of (22) is:
Figure 23. Grid Configuration for Computer Solution.
The second of (22) is:

\[ k^2 \phi_{i-2,j} + 4k^2 \phi_{i-1,j} + (6k^2 + \frac{1}{r_{n+1}}) \phi_{i,j} + 4k^2 \phi_{i+1,j} + k^4 \phi_{i+2,j} \]

\[ = \left( \frac{1}{r_{n+1}} - 8k^2 - 6 \right) \phi_{i,j} + 4(k^2 + 1) \phi_{i,j+1} + \phi_{i,j-1} \]

\[ + 4k^2 \left( \phi_{i+1,j} + \phi_{i-1,j} \right) - 2k^2 \left( \phi_{i+1,j+1} + \phi_{i-1,j+1} \right) \]

\[ + \phi_{i+1,j-1} + \phi_{i-1,j-1} - \left( \phi_{i,j+2} + \phi_{i,j-2} \right) + k^4 g_{i,j} \]

\[ i = 3, 4, \ldots, 22 \]

These equations could be more conveniently written in matrix notation. For example, the second of (22A) is

\[ \phi_{i,j+2} = 4\phi_{i,j+1} + (6 + \frac{1}{r_{n+1}}) \phi_{i,j} - 4\phi_{i,j-1} + \phi_{i,j-2} \]

\[ = \frac{1}{r_{n+1}} \phi_{i,j} + 6\phi_{i,j} + \phi_{i,j+2} + \phi_{i,j-2} - 4 \left( \phi_{i,j+1} + \phi_{i,j-1} \right) \]

\[ j = 3, 4, \ldots, 22 \]

where
The boundary and symmetry conditions have been taken into account in writing the expression for the matrix of coefficients, $\mathbf{Y}$. 

The computer program first defines a procedure called QUIDI for solving a set of linear equations of the quidiagonal type, such as (22A). In this procedure the input data are the number of equations, $N$, the matrix of coefficients, $\mathbf{X}(,)$, and the right-hand side of the equations, the vector $\mathbf{F}(,)$. The solutions to the set of equations is stored in the vector $\mathbf{U}(,)$.

An initial guess for the solution is read in as DATA, and the right-hand side of the biharmonic equation is read in as RSIDE. For each of nine different values of the iteration parameter, the program first sweeps up the grid setting up and solving the first of (22A) along each row (for each $j$), then sweeps across the grid setting up and solving the second of (22A). Each double sweep is one iteration and the nine iterations constituted one cycle. Three cycles are executed, or a total of 27 iterations.

At the end of each iteration the boundary and symmetry conditions are
re-evaluated. The final solution of the biharmonic equation is stored in either of the arrays $X(,)$ or $U(,)$. 
COMMENT THE ALTERNATING DIRECTION METHOD OF CONTE AND DAMES FOR SOLVING
THE BIHARMONIC EQUATION.

PROCEDURE QUIDI(N,X(),F(),A(),B(),C(),D(),E(),W(),BE(),G(),DL(),H(),U())

BEGIN  INTEGER I, N
FOR I=(3,1,N)  A(I)=X(I,I-2)
FOR I=(2,1,N)  B(I)=X(I,I-1)
FOR I=(1,1,N)  C(I)=X(I,I)
FOR I=(1,1,N-1)  D(I)=X(I,I+1)
FOR I=(1,1,N-2)  E(I)=X(I,I+2)
W(1)=C(1)
BE(1)=(D(1))/W(1)
BE(N)=0.0
G(1)=(E(1))/W(1)
G(N)=G(N-1)=0.0
H(1)=(F(1))/W(1)
DL(2)=B(2)
W(2)=C(2)-(DL(2))(BE(1))
BE(2)=(D(2)-(DL(2))(G(1)))/W(2)
G(2)=(E(2))/W(2)
H(2)=(F(2)-(DL(2))(H(1)))/W(2)
FOR I=(3,1,N-2)
BEGIN
DL(I)=B(I)-(A(I))(BE(I-2))
W(I)=C(I)-(A(I))(G(I-2))-(DL(I))(BE(I-1))
BE(I)=(D(I)-(DL(I))(G(I-1)))/W(I)
G(I)=(E(I))/W(I)
H(I)=(F(I)-(A(I))(H(I-2))-(DL(I))(H(I-1)))/W(I)
END
I=N-1
DL(I)=B(I)-(A(I))(BE(I-2))
W(I)=C(I)-(A(I))(G(I-2))-(DL(I))(BE(I-1))
BE(I)=(D(I)-(DL(I))(G(I-1)))/W(I)
\[ H(I) = \frac{(F(I) - (A(I))(H(I-2)) - (DL(I))(H(I-1)))}{(W(I))} \]

\[ I = N \]

\[ DL(I) = (A(I))(BE(I-2)) \]

\[ W(I) = C(I) - (A(I))(G(I-2)) - (DL(I))(BE(I-1)) \]

\[ H(I) = \frac{(F(I) - (A(I))(H(I-2)) - (DL(I))(H(I-1)))}{(W(I))} \]

\[ U(N+2) = H(N) \]

\[ U(N+1) = H(N-1) - (BE(N-1))(U(N+2)) \]

FOR \( I = (N-2, -1, 1) \)

\[ U(I+2) = H(I) - (BE(I))(U(I+3)) - (G(I))(U(I+4)) \]

RETURN

END QUIDI()

INTEGER \( I, J, M, N, P, Q, L \)

ARRAY \( X(24, 24), U(24, 24), RS(20, 20), YY(20, 20), F(20), A(20), B(20), C(20), D(20), E(20), V(20), BE(20), G(20), DL(20), H(20) \)

\( P = 0 \)

\( N = 24 \)

\( M = N - 4 \)

\( K = 666666667 \)

\( DEL = 0.5/M \)

\( Q = 2 \)

TRAN.. READ($$DATA)

READ($$RSIDE)

FOR \( I = (1, 1, M-1) \)

\[ YY(I, I+1) = -4.0 \]

FOR \( I = (2, 1, M-1) \)

\[ YY(I, I-1) = -4.0 \]

FOR \( I = (1, 1, M-2) \)

\[ YY(I, I+2) = 1.0 \]

FOR \( I = (3, 1, M-1) \)

\[ YY(I, I-2) = 1.0 \]

\[ YY(M, M-1) = -8.0 \]

\[ YY(M, M-2) = 2.0 \]

CYCLE.. \( P = P + 1 \)

FOR \( L = (0, 1, 8) \)

BEGIN \( R = ((1.0)/(1.0))((5.0)*L) \)

FOR \( I = (1, 1, M) \)

\[ FOR J = (1, 1, M) \]

\[ YY(I, J) = (K*4)(YY(I, J)) \]

FOR \( I = (2, 1, M-2) \)

\[ YY(I, I) = ((6.0)(K*4)) + ((1.0)/(R)) \]

\[ YY(I, I) = ((7.0)(K*4)) + ((1.0)/(R)) \]

\[ YY(M-1, M-1) = ((7.0)(K*4)) + ((1.0)/(R)) \]

FOR \( J = (3, 1, N-2) \)

BEGIN \( FOR I = (1, 1, M) \)
F(I) = \((1*0)/(R)\) - \((8*0)(K*2)\) - \((6*0)(U(I+2*J))\) + \((4*0)(K*2) + (1*0)) \((U(I+2*J+1)+U(I+2*J-1))\) + \((4*0)(K*2)\) \((U(I+3*J)*U(I+1*J))\)
\((2*0)(K*2)\) \((U(I+3*J+1)+U(I+3*J-1))\) \((U(I+3*J+2)+U(I+3*J-2))\) \((DEL)^4\) \((RS(I+J))\)

QUIDI(M,YY(J),F(I),A(),B(),C(),D(),E(),V(),BE(),G(),DL(),H(),X(J))

FOR I=(1,1,M-1) $ YY(I,I+1)=-4.0$
FOR I=(2,1,M-1) $ YY(I,I-1)=-4.0$
FOR I=(1,1,M-2) $ YY(I,I+2)=1.0$
FOR I=(3,1,M-1) $ YY(I,I-2)=1.0$
YY(M,M-1)=-8.0 $ YY(M,M-2)=2.0$
FOR J=(2,1,M-2), M
YY(J,J)=\((6*0)+((1*0)/(R))\)
YY(1,J)=\((7*0)+((1*0)/(R))\)
YY(M-1,M-1)\((7*0)+((1*0)/(R))\)
FOR I=(3,1,N-2)
BEGIN FOR J=(1,1,M) $ F(J)=\((1*0)/(R))\) \((X(I+2*J+2))\) + \((6*0)(U(I,J+2))\) + \((U(I,J+4))\) + \((U(I,J))\)
\((4*0)(U(I,J+3)+U(I,J+1))\)
QUIDI(M,YY(J),F(),A(),B(),C(),D(),E(),V(),BE(),G(),DL(),H(),X(J))

FOR I=(1,1,N) $ X(I,1)=X(I,3)$
FOR I=(1,1,N-4) $ X(I,N-1)=X(I,N-3)$
X(1,1)=X(3,1) $ X(N,1)=X(N-4,1)$
END $ X(N,1)=X(N-4,1)$
FOR I=(1,1,N) $ X(1,1)=X(I,1))$ $ U(I,J)=X(I,J)$
IF P LEQ Q GO TO CYCLE
WRITE ($$ANS,FMTJ$
GO TO TRANS
INPUT DATA FOR I=(1,1,N) $ FOR J=(1,1,N) $ U(I,J))
INPUT RSIDE FOR I=(1,1,M) $ FOR J=(1,1,M) $ RS(I,J))
OUTPUT ANS FOR I=(1,1,N) $ FOR J=(1,1,N) $ X(I,J))
FORMT FMT(*5*,B2*,F15*,8*,W0)$
FINISH
VITA

John Hope Murphy was born in Atlanta, Georgia on January 13, 1927. He attended public schools in Atlanta and was graduated from Boys' High School in 1945.

After spending two years in the United States Navy he returned to Atlanta in 1947 and entered Emory University. While at Emory he was initiated into the Kappa Alpha fraternity. He entered the Georgia Institute of Technology in 1948 and received the degree Bachelor of Mechanical Engineering in 1951.

From 1951 to 1953 he was employed by the Sandia Corporation of Albuquerque, New Mexico as a Mechanical Engineer. From 1953 to 1955 he was associated with the Engineering Experiment Station at Georgia Tech. While associated with the Engineering Experiment Station he did part time graduate work and received the degree Master of Science from Georgia Tech in 1955. In 1955 he was appointed Assistant Professor of Mechanical Engineering.

He married the former Jeannette Snee of Atlanta in 1948. They have five children, a son and four daughters.

He is a member of the Pi Tau Sigma Mechanical Engineering honor society, the Phi Kappa Phi honor society, and associate member of the Society of Sigma Xi.