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VIBRATION OF LONGITUDINALLY STIFFENED CYLINDRICAL SHELLS

A THESIS
Presented to
The Faculty of the Graduate Division
by
Stephen Anthony Rinehart

In Partial Fulfillment
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VIBRATION OF LONGITUDINALLY STIFFENED CYLINDRICAL SHELLS

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Chairman

Date approved by Chairman: June 27, 1969
This thesis is dedicated to Cindy and Sherry.
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TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>ACKNOWLEDGMENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF TABLES</td>
<td>v</td>
</tr>
<tr>
<td>LIST OF ILLUSTRATIONS</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF PRINCIPAL SYMBOLS</td>
<td>ix</td>
</tr>
<tr>
<td>SUMMARY</td>
<td>xii</td>
</tr>
</tbody>
</table>

Chapter

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>1</td>
</tr>
</tbody>
</table>

II. TOTAL ENERGY FOR SIMPLY SUPPORTED LONGITUDINALLY STIFFENED CYLINDRICAL SHELLS. 6

III. FREE VIBRATIONS OF SIMPLY SUPPORTED STIFFENED CYLINDER ACCORDING TO RITZ METHOD. 25

IV. GOVERNING DIFFERENTIAL EQUATIONS OF MOTION AND ANALYSIS OF STRINGER STIFFENED CYLINDRICAL SHELLS. 71

V. SUMMARY OF CONCLUSIONS AND RECOMMENDATIONS. 156

APPENDIX

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.</td>
<td>159</td>
</tr>
<tr>
<td>B.</td>
<td>163</td>
</tr>
<tr>
<td>C.</td>
<td>171</td>
</tr>
<tr>
<td>D.</td>
<td>180</td>
</tr>
<tr>
<td>E.</td>
<td>186</td>
</tr>
<tr>
<td>F.</td>
<td>191</td>
</tr>
<tr>
<td>G.</td>
<td>196</td>
</tr>
</tbody>
</table>

LITERATURE CITED 202

VITA 204
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Properties of Stringer Stiffened Shells</td>
<td>43</td>
</tr>
<tr>
<td>2. Comparison of Theoretical and Experimental Radial Frequencies for Four Equally Spaced Stringers</td>
<td>44</td>
</tr>
<tr>
<td>3. Eigenvectors for Radial Modes Showing the Extent of Modal Coupling for the Stiffened Shell with Four Stringers (Symmetric)</td>
<td>45</td>
</tr>
<tr>
<td>4. Eigenvectors for Radial Modes Showing the Extent of Modal Coupling for the Stiffened Shell with Four Stringers (Antisymmetric)</td>
<td>46</td>
</tr>
<tr>
<td>5. Theoretical and Experimental Radial Frequencies for Eight Equally Spaced Internal Stringers on a Simply Supported Cylinder</td>
<td>48</td>
</tr>
<tr>
<td>6. Frequencies of Vibration for Radial Modes (Four Stringers)</td>
<td>49</td>
</tr>
<tr>
<td>7. Frequencies of Vibration for Radial Modes (Eight Stringers)</td>
<td>49</td>
</tr>
<tr>
<td>8. Frequencies of Vibration for Longitudinal Modes (Four Stringers)</td>
<td>50</td>
</tr>
<tr>
<td>9. Frequencies of Vibration for Circumferential Modes (Four Stringers)</td>
<td>50</td>
</tr>
<tr>
<td>10. Comparison of Flügge's Theory versus Donnell's Theory for Four Equally Spaced Stringers on Simply Supported Cylinder</td>
<td>51</td>
</tr>
<tr>
<td>11. Comparison of Flügge's Theory versus Donnell's Theory for Eight Equally Spaced Stringers on Simply Supported Cylinder (m=1)</td>
<td>52</td>
</tr>
<tr>
<td>12. Comparison of Flügge's Theory versus Donnell's Theory for Eight Equally Spaced Stringers on Simply Supported Cylinder (m=2)</td>
<td>52</td>
</tr>
<tr>
<td>Table</td>
<td>Page</td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>13. Comparison of Flügge's Theory versus Donnell's Theory for Eight Equally Spaced Stringers on Simply Supported Cylinder (m=3)</td>
<td>53</td>
</tr>
<tr>
<td>14. Comparison of Donnell's Theory versus &quot;Orthotropic Theory&quot; for Four Equally Spaced Stringers on Simply Supported Cylinder</td>
<td>54</td>
</tr>
<tr>
<td>15. Comparison of Theoretical Radial Frequencies for Four Equally Spaced Stringers by Neglecting Rotaty Inertia of Stringers</td>
<td>55</td>
</tr>
<tr>
<td>16. Comparison of Theoretical Longitudinal Frequencies for Four Equally Spaced Stringers by Neglecting Rotatory Inertia of Stringers</td>
<td>56</td>
</tr>
<tr>
<td>17. Comparison of Theoretical Circumferential Frequencies for Four Equally Stringers by Neglecting Rotatory Inertia of Stringers</td>
<td>57</td>
</tr>
<tr>
<td>18. Comparison of Theoretical Radial Frequencies of the Present Analysis versus the Analysis of Reference [6] for 60 Stringers</td>
<td>58</td>
</tr>
<tr>
<td>19. Comparison of Theoretical Radial Frequencies for Eight Equally Spaced Stringers by Neglecting Rotatory Inertia of Stringers</td>
<td>59</td>
</tr>
<tr>
<td>20. Comparison of Theoretical Radial Frequencies for Eight Equally Spaced Stringers by Neglecting Rotatory Inertia of Stringers</td>
<td>60</td>
</tr>
<tr>
<td>21. Comparison of Present Theory versus Experimental Results of Hoppman (Ref. [1]) for 16 Equally Spaced Stringers</td>
<td>61</td>
</tr>
<tr>
<td>22. Comparison of Theoretical Frequencies of General Theory versus Ritz Analysis for Four Equally Spaced Stringers</td>
<td>147</td>
</tr>
<tr>
<td>23. Comparison of Theoretical Frequencies of General Theory versus Ritz Analysis for Eight Equally Spaced Stringers</td>
<td>148</td>
</tr>
<tr>
<td>24. Theoretical Frequencies of General Theory for a Clamped-Clamped Shell with Four Equally Spaced Stringers</td>
<td>149</td>
</tr>
</tbody>
</table>
**LIST OF ILLUSTRATIONS**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Stiffened Shell Coordinate System</td>
<td>9</td>
</tr>
<tr>
<td>2.</td>
<td>Stringer Coordinate Systems and Notation</td>
<td>13</td>
</tr>
<tr>
<td>3.</td>
<td>Radial Frequencies of Stiffened Cylinder with Four Stringers</td>
<td>62</td>
</tr>
<tr>
<td>4.</td>
<td>Radial Frequencies of Stiffened Cylinder with Eight Stringers</td>
<td>63</td>
</tr>
<tr>
<td>5.</td>
<td>Theoretical Mode Shapes for Stiffened Shell with Four Stringers, m=1</td>
<td>64</td>
</tr>
<tr>
<td>6.</td>
<td>Theoretical Mode Shapes for Stiffened Shell with Four Stringers, m=2 (Symmetric)</td>
<td>67</td>
</tr>
<tr>
<td>7.</td>
<td>Theoretical Mode Shapes for Stiffened Shell with Eight Stringers, m=1 (Symmetric)</td>
<td>69</td>
</tr>
<tr>
<td>8.</td>
<td>Schematic Sketch of Normal Loading</td>
<td>77</td>
</tr>
<tr>
<td>9.</td>
<td>Schematic Sketch of Twisting Moment</td>
<td>77</td>
</tr>
<tr>
<td>10.</td>
<td>Schematic Sketch of Tangential Shear Loading</td>
<td>77</td>
</tr>
<tr>
<td>11.</td>
<td>Schematic Sketch of Longitudinal Shear Loading</td>
<td>78</td>
</tr>
<tr>
<td>12.</td>
<td>Shell and Stringer Coordinate Systems and Loading on Stringers</td>
<td>106</td>
</tr>
<tr>
<td>13.</td>
<td>Theoretical Mode Shapes for Four Stringers (Simply Supported Case)</td>
<td>150</td>
</tr>
<tr>
<td>14.</td>
<td>Theoretical Circumferential Mode Shapes for Eight Stringers (Simply Supported Case)</td>
<td>153</td>
</tr>
<tr>
<td>15.</td>
<td>Theoretical Axial and Circumferential Modes for a Clamped-Clamped Cylinder with Four Stringers</td>
<td>154</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>16.</td>
<td>Stringer Displacements and Notation</td>
<td>162</td>
</tr>
<tr>
<td>17.</td>
<td>Shell Element Showing Forces and Moments</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(a) Shell Element Showing Stress Resultants</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(b) Shell Element Showing Stress Couples</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(c) Shell Element Showing Coordinate System</td>
<td>164</td>
</tr>
</tbody>
</table>
**LIST OF PRINCIPAL SYMBOLS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Radius of shell</td>
</tr>
<tr>
<td>$A_{mn}, B_{mn}, C_{mn}$</td>
<td>Symmetric Fourier coefficients</td>
</tr>
<tr>
<td>$A'<em>{mn}, B'</em>{mn}, C'_{mn}$</td>
<td>Antisymmetric Fourier coefficients</td>
</tr>
<tr>
<td>$A_r$</td>
<td>Cross-sectional area of the $r$th stringer</td>
</tr>
<tr>
<td>$(a_{yr}, a_{zr})$</td>
<td>Coordinates of the shear center with respect to the centroid</td>
</tr>
<tr>
<td>$C$</td>
<td>Centroid of stringer</td>
</tr>
<tr>
<td>$C_{sn}, C'_sn$</td>
<td>Arbitrary constants in complementary solution for the shell</td>
</tr>
<tr>
<td>$C_{wr}$</td>
<td>Warping constant for the $r$th stringer</td>
</tr>
<tr>
<td>$[D]$</td>
<td>Coefficient matrix</td>
</tr>
<tr>
<td>$e_{xx}, e_{yy}, e_{zz}$</td>
<td>Middle surface strain components in shell</td>
</tr>
<tr>
<td>$E$</td>
<td>Young's modulus for shell</td>
</tr>
<tr>
<td>$E^*_{r}$</td>
<td>Reduced Young's modulus for $r$th stringer</td>
</tr>
<tr>
<td>$(GJ)_r$</td>
<td>Torsional stiffness of the $r$th stringer</td>
</tr>
<tr>
<td>$h$</td>
<td>Shell thickness</td>
</tr>
</tbody>
</table>
Products of inertia with respect to the centroid of the rth stringer

Polar moment of inertia of rth stringer

St. Venant constant for uniform torsion

Submatrices of stiffness matrix

Total number of stringers

Change of curvatures of middle surface of the shell

Length of shell or stringers

Number of axial half-waves

Maximum value taken on by m

Submatrices of mass matrix

Number of circumferential waves

Maximum value taken on by n

Poisson's ratio for shell and stringer, respectively

Shear center of stringer

Surface loads on shell in the x, $\phi$, z directions, respectively

Column matrix of generalized coordinates

Time

Kinetic energy
$u, v, w$  Longitudinal, tangential, and radial middle surface displacements of the shell

$V$  Potential energy

$x, y, z$  Stringer fixed coordinates

$X, Y, Z$  Shell fixed coordinates

$\xi_r, \xi_r', \eta_r$  Longitudinal, tangential, and radial displacements of the $r$th stringer

$\Theta_{sr}$  Angle of twist of the $r$th stringer

$\phi$  Angular coordinate of the shell

$\phi_r$  Angular coordinate of the $r$th stringer

$\rho, \rho_r$  Shell and stringer density, respectively

$\omega$  Circular frequency of composite shell

$\Omega$  Frequency parameter

$\beta$  Shell stiffness parameter

$\delta_{ij}$  Kronecker delta
SUMMARY

This study is concerned with a theoretical analysis of the free vibration characteristics of a thin cylindrical shell reinforced by longitudinal ribs for any set of admissible boundary conditions. The usual assumption that the stringers are closely spaced is not made and hence the stiffened shell is not treated as an orthotropic continuum but rather as an isotropic cylinder interacting with a set of discrete thin-walled beams.

A new method of analyzing this type of structure is presented in which the shell-stringer interaction loads are represented by double Fourier series with unknown coefficients. Once the analysis of the shell and stringer problems has been completed, kinematical conditions at the line of attachment that require compatible deformations between the shell and stringers are enforced. This yields a linear eigenvalue problem in the unknown loading coefficients for determination of the eigenfrequencies of the composite structure.

The mathematical model which is developed is quite flexible. Since the number of stringers, and their spacing is left arbitrary, there are no restrictions due to smearing or averaging the stringer effects over the surface of the shell. Furthermore, the governing differential equations of the composite structure are satisfied as are all of the boundary conditions.
The Ritz method is employed in the analysis to compare the simplified Donnell's shell theory with Flügge's more exact theory for a stringer stiffened shell with simply supported boundary conditions. It was found that these theories were in good agreement with each other and with the experimental results found in the literature. It was also found that neglecting rotatory inertia of the shell and stringers was a justifiable assumption for the lower axial modes \((m<4)\). However, the assumption of neglecting in-plane inertias for the shell in calculating the radial frequencies can lead to results in serious disagreement with theoretical calculations based on retaining these terms. The analysis also shows that the presence of even a few stringers on a shell may substantially couple the circumferential modes when one considers the stringers as discrete structural elements. Also, the frequencies associated with symmetric and antisymmetric circumferential modes can have different values, which is not true for the unstiffened shell.

A general linear theory is developed based upon solving the coupled governing equations of motion for the shell and stringers. Donnell's equations are employed for the shell and V. Z. Vlasov's equations are used for the stringers. A particular numerical example is considered to illustrate the application of the general analysis to a realistic stiffened shell. The boundary conditions are chosen to be clamped-clamped. Also, results of the general analysis are compared with the Ritz analysis for the simply supported case and with some experimental data in the literature. Good agreement between mode shapes and frequencies was noted for this case.
CHAPTER I

INTRODUCTION

Since reinforced cylindrical shells are used extensively in the aerospace, nuclear, marine, and petrochemical industries as structural components in missiles, aircraft, submarines, nuclear reactor vessels, and refinery equipment, investigations on the free vibration characteristics of these shells continues to be of interest to researchers in the field of solid mechanics. The stiffening of these shells is accomplished by attaching longitudinal ribs and/or ring frames to the shell surface for the purpose of increasing the strength and stability of the shell while still retaining a relatively lightweight composite structure.

In the past, the usual method of analyzing reinforced cylindrical shells was to treat the structure as an orthotropic continuum with effective extensional and flexural stiffness by considering the stiffeners to be closely spaced and to average the stiffening effects over the surface of the shell as was done by Hoppman [1] and Miller [2]. This method was used by McElman, Mickulas, and Stein [3] to investigate the effects of stringer eccentricity (whether the stiffener center of gravity is inside or outside of the shell wall) on stiffened flat plates and cylindrical shells. The conclusion reached from this

*The numbers enclosed in brackets denote the correspondingly numbered reference in the Literature Cited.
investigation was that stiffener eccentricity effects are important and should be considered in the analysis of stiffened plates and cylindrical shells. However, they found that since eccentricity effects depend on the configuration and physical properties of both the cylinder and the stiffeners, it was rather difficult to make any concrete statements regarding stringer eccentricity.

An alternative approach in analyzing reinforced shells is to treat the stiffeners as discrete structural components. The advantage of this approach is that the stiffeners need not be closely spaced, identical in size, and may be few in number. The discrete approach has recently been adopted by several authors in investigating the vibration of stiffened cylindrical shells. Of the investigations using the discrete approach, references [4,5] deal only with longitudinal ribs and reference [6] considers both rings and stringers. Reference [4] considers the free vibration of stringer stiffened cylinders by using the orthotropic theory and the method of transfer matrices and compares these theories with experimental results for simply supported cylinders with four or eight stringers. The Rayleigh-Ritz energy method is employed in the other given references and with the exception of Egle and Sewall [6] only simply supported boundary conditions are considered. Ojalvo and Newman [5] were concerned with the effect of an attached mass point on the frequencies of a simply supported cylindrical shell with many equally spaced stringers. Egle and Sewall [6] use the energy method but leave the axial mode unspecified in their expression for the assumed displacement functions. The
authors suggest that beam-type axial modes be used to allow for various end conditions. Numerical results are presented for a simply supported, stringer stiffened cylindrical shell using Donnell's shell theory.

One of the disadvantages of using an approximate energy method is in the selection of the approximating functions. This depends on the user's intuition and experience and there is no systematic way to arrive at a best choice for an approximating sequence. The application of the Ritz method requires that the approximating functions be members of a complete system of functions and this is one of the basic difficulties connected with the method. Often it is difficult to find a complete system of coordinate functions satisfying the geometric boundary conditions of a problem. Then, too, since the coordinate functions do not in general satisfy the governing differential equations of motion of a given continuum it is sometimes difficult to assess the rate at which an approximate solution is converging to the exact solution. Often one must look for reassurances in making successive approximations to lend confidence to the results and where possible to compare the results with established information.

To investigate the problem of free vibration of a stiffened cylinder with arbitrary boundary conditions, an approach is adopted by which the coupled equations of motion are solved. This avoids one of the major disadvantages of the energy method in that one is not faced with the problem of having to select some set of approximating functions. Furthermore, one can satisfy the governing equations of motion of the composite structure and all of the boundary conditions exactly.
However, one is required to make a representation of the unknown forces of constraint which is avoided in an energy approach.

In proposing a general linear theory, it is very desirable to use as simplified a shell theory as possible in order to minimize the mathematical difficulties attendant with such an approach. To this end, an energy approach using a convenient set of coordinate functions can be useful in comparing the simplified Donnell's shell theory to the more exact Flügge's theory to see if they are in reasonable agreement for stiffened shells.

Three qualifications that are desirable in a mathematical model of an elastic system are (1) the model should adequately represent the important characteristics of the system, (2) the resulting mathematical equations describing the model's behavior should not be so unwieldly as to preclude a reasonable attack upon the problem, and (3) the actual results of the model should be reasonably accurate.

The energy method satisfies the first and second requirements for a good model. The results of the model are compared with existing experimental data to ascertain if the third criterion is satisfied. Furthermore, the energy method has been used in investigating the effect of the shear center not coinciding with the centroid, and the nonsymmetrical property (cross product of inertia) which is important for nonsymmetrical sections. From a differential equations point of view, the partial differential equations governing the motion of the stringer will be uncoupled if the shear center coincides with the centroid (or if such an effect can be considered negligible) and if the
cross product of inertia is zero (or can be neglected). Therefore, the energy method when used with convenient coordinate functions can be useful in furnishing one with a qualitative feel as well as explicit justification for various assumptions which one may postulate in attempting to formulate a tractable set of partial differential equations which govern the behavior of a stiffened cylindrical shell. Scruggs and Pierce [7] employed the energy method to investigate the free vibration of a ring and stringer stiffened shell for simply supported boundary conditions and obtained good agreement for the modes and frequencies investigated. The author, however, did not employ series solutions to represent the circumferential mode shape.

The purpose of this work is to develop a mathematical model for investigating the free vibration characteristics of a thin cylindrical shell reinforced by discrete stringers with arbitrary admissible boundary conditions by considering a thin monocoque cylinder interacting with a system of longitudinal ribs and using the energy method as a springboard to develop such a model.

Numerical examples are presented for both the Ritz analysis and the analysis by solving the coupled governing differential equations of motion.
CHAPTER II

TOTAL ENERGY FOR SIMPLY SUPPORTED LONGITUDINALLY STIFFENED CYLINDRICAL SHELLS

The Ritz method is employed in the present analysis to determine the natural frequencies and mode shapes of thin circular cylindrical shells stiffened by discrete elastic stringers with simply supported boundary conditions. Simply supported boundary conditions are chosen because they enable one to express the shell displacements in terms of a linearly independent, complete set of functions which satisfies all of the boundary conditions for the shell and stringers. Also, the existing experimental data for free vibration of stringer stiffened cylinders is only for simply supported boundary conditions. Therefore, this choice of boundary conditions enables one to adopt the Ritz method under extremely favorable conditions.

The numerical results of this method will be used later for comparison purposes with the results of the general analysis as given in Chapter IV. This will furnish an explicit comparison of two different approaches to the problem of free vibration of stiffened cylinders. If the results of these two methods are in good agreement, then this will furnish a confirmation of the derivation and programming of the general analysis.

The analysis incorporates rotatory inertia, bending, extension, and torsion of the stringers and does not neglect warping, transverse
stiffening effects, rotatory inertia of the shell, nor is it assumed that the shear center of the stringers coincides with the centroid nor does it neglect in-plane inertias in the numerical solution of the problem as is done in reference [6]. The general procedure for the Ritz method is given in the following steps.

The kinetic and potential energy functionals are constructed for the cylindrical shell and the stringers. These expressions are combined to give the Lagrangian for the stiffened cylinder in terms of the middle surface displacements of the shell by relating the displacements of the stringers to the middle surface displacements of the shell. Next, the displacements of the cylinder are assumed in the form of finite Fourier series with undetermined coefficients where each term of the series satisfies simply supported boundary conditions. The assumed displacements are substituted into the energy functionals and the Euler-Lagrange equations are utilized to yield a linear eigenvalue problem in the unknown amplitudes. The eigenvalue problem is then solved numerically for the approximate frequencies and mode shapes of the stiffened cylinder.

**Potential Energy of System**

The following assumptions concerning the shell are made:

(i) The shell material is isotropic and homogeneous.
(ii) The linear strain-displacement relations are employed.
(iii) The thickness of the shell is small compared with the smallest radius of curvature of its middle surface.
(iv) The stress components normal to the middle surface are small compared with the other stress components and may be neglected.

(v) The normals to the unstrained middle surface deform into the normals of the strained middle surface.

(vi) The normals to the strained middle surface undergo no elongation.

The assumptions (v) and (vi) together are commonly called the Kirchhoff-Love hypotheses.

The displacements $u$, $v$, and $w$ of the middle surface of the shell are assumed to be given by the following finite Fourier series in $\phi$ and $x$ (see Figure 1):

\[
\begin{align*}
  u(x,\phi,t) &= \sum_{m=0}^{M} \sum_{n=0}^{N} \left[ A_{mn}(t)\cos(n\phi) + A'_{mn}(t)\sin(n\phi) \right] \cos \frac{mn\pi x}{L} ,
  \\
  v(x,\phi,t) &= \sum_{m=1}^{M} \sum_{n=0}^{N} \left[ B_{mn}(t)\sin(n\phi) - B'_{mn}(t)\cos(n\phi) \right] \sin \frac{mn\pi x}{L} ,
  \\
  w(x,\phi,t) &= \sum_{m=1}^{M} \sum_{n=0}^{N} \left[ C_{mn}(t)\cos(n\phi) + C'_{mn}(t)\sin(n\phi) \right] \sin \frac{mn\pi x}{L} .
\end{align*}
\]

Each term of the series in Equations (2.1) satisfies the boundary conditions:

\[
  v = w = N_x = M_x = 0 \text{ at } x = 0, L .
\]
where $N_x$ is the longitudinal stress resultant and $M_x$ is the bending moment (see Appendix B). These are simply supported boundary conditions without axial constraint and are sometimes called "freely supported" in the literature. The unprimed coefficients $(A_{mn}, B_{mn}, C_{mn})$ are associated with the symmetric circumferential modes (i.e., modes which are symmetric with respect to the plane $\phi=0$) and the prime coefficients $(A'_{mn}, B'_{mn}, C'_{mn})$ are associated with the antisymmetric circumferential modes (i.e., modes which are antisymmetric with respect to the plane $\phi = 0$). It should be noted that the $mn$th term for $u$, $v$, and $w$ represents the modal functions of an unstiffened thin cylindrical shell with simply supported boundary conditions.

Figure 1. Stiffened Shell Coordinate System

The strain-displacement relations of the middle surface of a cylindrical shell for the coordinate system depicted in Figure 1 are given by Flügge [8] as:
The changes of curvature for a thin cylindrical shell are given by Flügge [8] as:

\[
\kappa_x = \frac{3^2 w}{a^2}, \quad \kappa_\phi = \frac{3^2 w}{a^2} + \frac{\mu}{a}, \\
\kappa_{x\phi} = \frac{1}{2} \frac{3u}{a^2} \frac{3v}{a^2} + \frac{1}{a} \frac{3^2 w}{3x^2}. 
\]

Substituting the expressions for the assumed displacements into Equations (2.3) and (2.4), the strains and changes of curvature of the shell become:

\[
e_{xx} = \frac{3u}{3x}, \quad (2.3)
\]

\[
e_{\phi\phi} = \frac{1}{a} \frac{3v}{3\phi} + \frac{w}{a},
\]

\[
\gamma_{x\phi} = \frac{1}{a} \frac{3u}{3\phi} + \frac{3v}{3x}. 
\]

\[
e_{xx} = -\sum_{m=1}^{M} \sum_{n=0}^{N} \left( A_{mn} \cos(n\phi) + A_{mn}' \sin(n\phi) \right) a \sin(\alpha x), \quad (2.5)
\]

\[
e_{\phi\phi} = \frac{1}{2} \sum_{m=1}^{M} \sum_{n=0}^{N} \left[ (B_{mn}(n) + C_{mn}) \cos(n\phi) + (B_{mn}'(n) + C_{mn}') \sin(n\phi) \right] \sin(\alpha x),
\]
\[ \gamma_{x\phi} = \sum_{m=1}^{M} \sum_{n=0}^{N} \left[ (B_{mn} - (n)A_{mn}) \sin(n\phi) + ((n)A_{mn}' - B_{mn}') \alpha_m \cos(n\phi) \right] \cos(\alpha_m x), \]

\[ \kappa_x = - \sum_{m=1}^{M} \sum_{n=0}^{N} \left( C_{mn} \cos(n\phi) + C_{mn}' \sin(n\phi) \right) a_m^2 \sin(\alpha_m x), \]

\[ \kappa_\phi = \frac{1}{a^2} \sum_{m=1}^{M} \sum_{n=0}^{N} \left( C_{mn} (1-n^2) \cos(n\phi) + C_{mn}' (1-n^2) \sin(n\phi) \right) \sin(\alpha_m x), \]

\[ \kappa_{x\phi} = \frac{1}{a^2} \sum_{m=1}^{M} \sum_{n=0}^{N} \left\{ \left( \frac{n}{2a} A_{mn} + \frac{\alpha_m}{2a} B_{mn} + \frac{(\alpha_m)(n)}{a} C_{mn} \right) \cos(n\phi) \right. \]

\[ \left. - \left( \frac{n}{2a} A_{mn} + \frac{\alpha_m}{2a} B_{mn} + \frac{(\alpha_m)(n)}{a} C_{mn} \right) \sin(n\phi) \} \cos(\alpha_m x) \right\} \]

where

\[ \alpha_m = \frac{m\pi}{L}. \]

The strain energy of a thin cylindrical shell is given by Love [9] as:

\[ V_{\text{SHELL}} = \frac{Eh^3a}{2(1-\nu^2)} \int_0^L \int_0^{2\pi} \left[ (e_{xx} + e_{\phi\phi})^2 - 2(1-\nu)(e_{xx} e_{\phi\phi} - \frac{1}{4} \nu^2) \right] dxd\phi + \]

\[ \frac{Eh^3a}{24(1-\nu^2)} \int_0^L \int_0^{2\pi} \left[ (\kappa_{xx} + \kappa_{\phi\phi})^2 - 2(1-\nu)(\kappa_{xx} \kappa_{\phi\phi} - \kappa_{xx\phi\phi}^2) \right] dxd\phi \] (2.6)

when the first term represents the membrane energy associated with the stretching of the shell middle surface, and the second term is the strain energy of bending.
Inserting Equations (2.5) into (2.6) and integrating yields:

\[
V_{\text{SHELL}} = \frac{\pi E h a l}{4(1-\nu^2)} \sum_{m=1}^{M} \sum_{n=0}^{N} \left\{ \left( -\alpha_m A + \frac{n}{a} B + \frac{1}{a} C \right)^2 + \left( -\alpha_m A' + \frac{n}{a} B' + \frac{1}{a} C' \right)^2 \right. \\
+ 2(1-\nu)\left[ \alpha_m A \left( \frac{n}{a} B + \frac{1}{a} C \right) + \frac{1}{4} \left( \alpha_m B - \frac{n}{a} A \right)^2 \right] \\
+ 2(1-\nu)\left[ \alpha_m A' \left( \frac{n}{a} B' + \frac{1}{a} C' \right) + \frac{1}{4} \left( \alpha_m B' - \frac{n}{a} A' \right)^2 \right] \}
\]

\[
+ \frac{\pi E h^3 a l}{48(1-\nu^2)} \sum_{m=1}^{M} \sum_{n=0}^{N} \left\{ \left[ \left( \frac{1}{a^2} (1-n^2) - \alpha_m^2 \right) c_{mn} \right]^2 \\
+ \left[ \frac{1}{a^2} (1-n^2) - \alpha_m^2 \right] c_{mn} \right\}^2 \\
+ 2(1-\nu)\left[ \alpha_m C \left( \frac{n}{a^2} \right) \left( \frac{1}{a^2} - \alpha_m^2 \right) + \left( -\frac{n}{2a^2} A + \frac{\alpha_m}{2a} B + \frac{\alpha_m(n)}{a} C \right)^2 \right] \\
+ 2(1-\nu)\left[ \alpha_m C' \left( \frac{n}{a^2} \right) \left( \frac{1}{a^2} - \alpha_m^2 \right) + \left( -\frac{n}{2a^2} A' + \frac{\alpha_m}{2a} B' + \frac{\alpha_m(n)}{a} C' \right)^2 \right] \quad (2.7)
\]

The strain energy of a stringer at the location \( \phi = \phi_r \) (see Appendix A) is:

\[
V_{sr} = \frac{1}{2} \int_0^l \left( E r_A r (\partial \zeta_r / \partial x)^2 + E r I y r (\partial^2 \xi_r / \partial x^2)^2 + E r I y r (\partial^2 \eta_r / \partial x^2)^2 \right) + \left( \partial \theta_r / \partial x \right)^2 \right) dx \\
+ 2E r I y r (\partial^2 \xi_r / \partial x^2)(\partial^2 \eta_r / \partial x^2) + E r C \omega r (\partial^2 \theta_r / \partial x^2)^2 + (GJ) \left( \frac{\partial \theta_r}{\partial x} \right)^2
\]

where \((\zeta_r, \xi_r, \eta_r, \theta_r)\) denote the displacements of the shear center of the \( r \)th
stringer (see Figure 2). It is assumed that the stringer is rigidly attached to the shell along a single line (attachment line) at the middle surface of the shell. Usually, the attachment line is assumed to be at the middle of the flange (see reference [5]).

![Stringer Coordinate Systems and Notation](image)

Figure 2. Stringer Coordinate Systems and Notation

Having determined an attachment line, we denote the displacements of this line by \((u_A, v_A, w_A)\). By using conventional thin-walled beam theory the displacements of any point in the stringer \((u_{sr}, v_{sr}, w_{sr})\) are related to the displacements of the shear center \((\xi_r, \eta_r, \eta_r)\) by the following linear transformation assuming \(a_y = a_z = 0\) and no warping:

\[
\begin{align*}
\frac{\partial \xi_r}{\partial x} - \frac{\partial \eta_r}{\partial x} \end{align*}
\]

\(2.9\)
\[ v_{sr} = \xi_r - z \theta_{sr}, \]
\[ w_{sr} = \eta_r + y \theta_{sr} \]

where \( \theta_{sr} \) denotes the angle of twist of the \( r \)th stringer. Therefore, the displacements of the attachment line are related to the displacements of the coordinate origin and shear center by:

\[ \xi_r = u_A + y_r \frac{\partial \xi_r}{\partial x} + z_r \frac{\partial \eta_r}{\partial x}, \quad (2.10) \]
\[ \xi_r = v_A + z_r \theta_{sr}, \]
\[ \eta_r = w_A - y_r \theta_{sr}. \]

From the second of Equations (2.10), we have

\[ \frac{\partial \xi_r}{\partial x} = \frac{\partial v_A}{\partial x} + z_r \frac{\partial \theta_{sr}}{\partial x}, \quad (2.11) \]

and from the third of Equations (2.10), we have

\[ \frac{\partial \eta_r}{\partial x} = \frac{\partial w_A}{\partial x} - y_r \frac{\partial \theta_{sr}}{\partial x} \quad (2.12) \]

Inserting Equations (2.11) and (2.12) into the first of equations (2.10) yields:
\[ \xi_r = u_A + y_r \frac{\partial v_A}{\partial x} + z_r \frac{\partial w_A}{\partial x} \] \quad (2.13)

and therefore the displacements of the shear center are given by:

(assuming \( a_y = a_z = 0 \)):

\[ \xi_r = u_A + y_r \frac{\partial v_A}{\partial x} + z_r \frac{\partial w_A}{\partial x}, \]

\[ \eta_r = v_A - z_r \theta_{sr}, \] \quad (2.14)

\[ \eta_r = w_A - y_r \theta_{sr}. \]

The stringer angle of twist is approximated by:

\[ \theta_{sr} = \frac{1}{a} \frac{\partial w_A}{\partial \phi} \] \quad (2.15)

If \( u, v, \) and \( w \) denote the middle surface displacement of the shell, then the mid-surface displacements along the attachment line are given by:

\[ u_A = u(x, \phi, t), \] \quad (2.16)

\[ v_A = v(x, \phi, t), \]

\[ w_A = w(x, \phi, t). \]

Inserting Equations (2.16) and (2.15) into Equations (2.14) gives the
following expressions for the displacements of the shear center:

\[
\zeta_r = u + y_r \frac{\partial v}{\partial x} + z_r \frac{\partial w}{\partial x}, \quad (2.17)
\]

\[
\xi_r = v + \frac{z_r}{a} \frac{\partial w}{\partial \phi},
\]

\[
\eta_r = w - \frac{y_r}{a} \frac{\partial w}{\partial \phi}.
\]

Substituting Equations (2.17) into Equation (2.8) yields:

\[
V_{sr} = \frac{1}{2} \int_0^l \left[ \frac{E I}{r} \left( \frac{\partial v}{\partial x} \right)^2 + \frac{E I}{r} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{G J}{r} \left( \frac{\partial w}{\partial x} \right)^2 \right] \phi - \phi_r \, dx +
\]

\[
\frac{1}{2} \int_0^l \left[ \frac{2 E I}{r} y' r \left( \frac{\partial v}{\partial x} \right)^2 + \frac{2 E I}{r} z' r \left( \frac{\partial w}{\partial x} \right)^2 + \frac{2 E I}{r} \left( \frac{\partial w}{\partial x} \right)^2 \right] \phi - \phi_r \, dx +
\]

\[
\frac{1}{2} \int_0^l \left[ \frac{y_r}{a} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{z_r}{a} \left( \frac{\partial w}{\partial x} \right)^2 \right] \phi - \phi_r \, dx (2.18)
\]
where the first integral represents the strain energy due to extension, bending and torsion of the stringers (including the effect of eccentricity of the stringer) and the second integral represents the strain energy due to warping and coupling between the twisting and bending.

Substituting Equations (2.1) into (2.18) and integrating yields the following expression for the strain energy at the location $\phi = \phi_r$:

$$V_{sr} = \frac{E^* A^* \ell}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left\{ -\alpha_m A_{mi} \cos(i\phi_r) - y_r \alpha_m^2 \right\}$$

$$B_m \sin(i\phi_r) - z_r^2 C_m \cos(i\phi_r) \right\}$$

$$\cos(j\phi_r) - y_r^2 B_m \sin(j\phi_r) - z_r^2 C_m \cos(j\phi_r) \right\} +$$

$$2\left\{ -\alpha_m A_{mi} \cos(i\phi_r) - y_r^2 B_m \sin(i\phi_r) - z_r^2 C_m \cos(i\phi_r) \right\}$$

$$\cos(i\phi_r) \right\} - \alpha_m A_{mi} \sin(i\phi_r) + \alpha_m^2 y_r B_m \cos(j\phi_r) -$$

$$\alpha_m^2 z_m C_{m} \sin(j\phi_r) \right\} + \alpha_m A_{mi} \sin(i\phi_r) + \alpha_m^2 y_r B_m \cos(j\phi_r) -$$

$$\cos(i\phi_r) - \alpha_m^2 z_m C_{m} \sin(i\phi_r) \right\} - \alpha_m A_{mi} \sin(j\phi_r) + \alpha_m^2 y_r B_m \cos(j\phi_r) -$$

$$B_m^2 \cos(j\phi_r) - \alpha_m^2 z_m C_{m} \sin(j\phi_r) \right\} + \frac{E^* A^* \ell}{4}$$

$$\sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left\{ [\alpha_m^2 C_{m}] \cos(i\phi_r) \cos(j\phi_r) +$$
\[
2[\alpha_m^2 c_m^{{\prime}}][\alpha_m^2 c_m^{{\prime}}] \cos(i \phi_r) \sin(j \phi_r) + [\alpha_m^2 c_m^{{\prime}}] \\
[\alpha_m^2 c_m^{{\prime}}] \sin(i \phi_r) \sin(j \phi_r) \}
\]

\[
\frac{E_{\rho'} \rho' \varphi'}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left[ -\alpha_m^2 b_m^{{\prime}} + \frac{z_{m}}{a} (\alpha_m^2)(i)c_m^{{\prime}} \right] \\
[\alpha_m^2 b_m^{{\prime}}] \sin(i \phi_r) \sin(j \phi_r) + \\
2[\alpha_m^2 b_m^{{\prime}} + \frac{z_{m}}{a} (\alpha_m^2)(i)c_m^{{\prime}}] \left[ \alpha_m^2 b_m^{{\prime}} - \frac{z_{m}}{a} (\alpha_m^2)(j)c_m^{{\prime}} \right] \\
\sin(i \phi_r) \cos(j \phi_r) + [\alpha_m^2 b_m^{{\prime}} - \frac{z_{m}}{a} (\alpha_m^2)(j)c_m^{{\prime}}] \cos(i \phi_r) \cos(j \phi_r) + \\
\frac{E_{\rho'} \rho' \varphi'}{2} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left[ -\alpha_m^2 c_m^{{\prime}} + \frac{z_{m}}{a} (\alpha_m^2)(j)c_m^{{\prime}} \right] \\
\cos(i \phi_r) \sin(j \phi_r) + [\alpha_m^2 c_m^{{\prime}}] \left[ \alpha_m^2 b_m^{{\prime}} \right] - \\
\left[ \frac{z_{r}}{a} \right] \left[ (\alpha_m^2)(j)c_m^{{\prime}} \right] \cos(i \phi_r) \cos(j \phi_r) + [\alpha_m^2 c_m^{{\prime}}] \\
[\alpha_m^2 b_m^{{\prime}} - \frac{z_{m}}{a} (\alpha_m^2)(j)c_m^{{\prime}}] \sin(i \phi_r) \sin(j \phi_r) - \\
\left[ \frac{z_{r}}{a} \right] \left[ (\alpha_m^2)(j)c_m^{{\prime}} \right] \sin(i \phi_r) \cos(j \phi_r) + \\
[\alpha_m^2 c_m^{{\prime}}] \left[ \alpha_m^2 b_m^{{\prime}} - \frac{z_{m}}{a} (\alpha_m^2)(j)c_m^{{\prime}} \right] \sin(i \phi_r) \cos(j \phi_r) +
\[
\frac{(GJ)\epsilon_{rs}}{4} \sum_{m=1}^{N} \sum_{i=0}^{N} \sum_{j=0}^{N} \left\{ \left[-\frac{\alpha}{a} c_{m\ell} \right] \left[-\frac{\alpha}{a} c_{m\ell} \right] \right\} + 2 \left[ \frac{\alpha}{a} c_{m\ell} \right] \left[ \frac{\alpha}{a} c_{m\ell} \right] \sin(i\phi_r) \cos(j\phi_r) + \\
\left[ \frac{\alpha}{a} c_{m\ell} \right] \left[ \frac{\alpha}{a} c_{m\ell} \right] \cos(i\phi_r) \cos(j\phi_r) + \\
\frac{E_{\text{wr}}^2}{4} \sum_{m=1}^{N} \sum_{i=0}^{N} \sum_{j=0}^{N} \left\{ \left[ \frac{\alpha}{a} c_{m\ell} \sin(i\phi_r) \right] \left[ \frac{\alpha}{a} c_{m\ell} \sin(i\phi_r) \right] \right\} - 2 \left[ \frac{\alpha}{a} c_{m\ell} \sin(i\phi_r) \right] \\
\left[ \frac{\alpha}{a} c_{m\ell} \cos(j\phi_r) \right] + \left[ \frac{\alpha}{a} c_{m\ell} \cos(i\phi_r) \right] \\
\left[ \frac{\alpha}{a} c_{m\ell} \cos(j\phi_r) \right] + \left[ \frac{\alpha}{a} c_{m\ell} \cos(i\phi_r) \right] \right\}
\]

Therefore, the total strain energy of the system is:

\[ V = V_{\text{shell}} + \sum_{r=1}^{K} V_{sr} \]  

where \( K \) is the total number of stringers attached to the shell's surface.

**Kinetic Energy of System**

The kinetic energy of a cylindrical shell (including rotatory inertia) is given by Ojalvo and Newman [5] as the following:
\[ T_{\text{SHELL}} = \frac{\rho A}{2} \int_0^\ell \int_0^{2\pi} \left( \dot{u}^2 + \dot{v}^2 + \dot{w}^2 + \right. \\
+ \beta a^2 \left[ \left( \frac{\partial \dot{w}}{\partial x} \right)^2 + \frac{1}{a^2} \left( \frac{\partial \ddot{w}}{\partial \phi} - \ddot{v} \right)^2 \right] \right) \, dx \, d\phi \]  
\tag{2.21}

where \( \rho \) is the mass density of the shell, \( \beta \) is the shell bending-stiffness parameter, and the dot indicates differentiation with respect to time. Inserting the expressions for the assumed displacements into Equation (2.21) and integrating gives:

\[ T_{\text{SHELL}} = \frac{\pi \rho A \ell}{4} \sum_{m=1}^{M} \sum_{n=0}^{N} \left\{ \left( \hat{A}_{mn} \right)^2 + \left( \hat{A}_{mn}^t \right)^2 + \left( \hat{B}_{mn} \right)^2 (1 + \beta) + \right. \\
2\beta (n) \hat{B}_{mn} + \left( \hat{B}_{mn}^t \right)^2 (1 + \beta) + 2\beta (n) \hat{C}_{mn} + \left( \hat{C}_{mn} \right)^2 \\
\left[ 1 + \beta (a^2 + n^2) \right] + \left( \hat{C}_{mn}^t \right)^2 [1 + \beta (a^2 + n^2)] \right\} \]  
\tag{2.22}

The kinetic energy of a typical stringer (including rotatory inertia) in terms of the middle surface displacements of the shell is given by (see Appendix A):

\[ T_{sr} = \frac{\rho A \ell}{2} \int_0^\ell \left\{ \ddot{u} + \frac{\partial}{\partial x} \left( \frac{\partial \dot{w}}{\partial x} + \frac{\partial \dot{w}}{\partial \phi} \right) \right\}^2 + \left\{ \dot{v} + \frac{\partial}{\partial x} \left( \frac{\partial \dot{w}}{\partial \phi} \right) \right\}^2 + \left\{ \dot{w} - \frac{\partial}{\partial \phi} \left( \frac{\partial \dot{w}}{\partial x} \right) \right\}^2 + \\
\frac{I_{pr}}{A} \left[ \frac{1}{a^2} \left( \frac{\partial \dot{w}}{\partial \phi} \right) \right]^2 - 2\frac{a}{\partial x} \left( \frac{\partial \dot{w}}{\partial \phi} \right) + 2 \frac{\partial}{\partial x} \left( \frac{\partial \dot{w}}{\partial \phi} \right) \right\} \, dx + \]  
\[ _\phi \int_{\phi_r} \]
\[ \frac{\rho I}{2} \int_0^L \left( \frac{\partial \omega}{\partial x} \right) \phi = \phi_r \, dx + \frac{\rho I}{2} \int_0^L \left( \frac{\partial \psi}{\partial x} \right) \phi = \phi_r \, dx \]

\[ \rho I y'z' \int_0^L \left( \frac{\partial \psi}{\partial x} \right) \frac{\partial \omega}{\partial x} \, dx \]  

(2.23)

Inserting Equations (2.1) into (2.23) and performing the integrations yield:

\[ T_{sr} = \frac{\rho A_i}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left\{ [\dot{A}_{mi} \cos(i \phi_r) + \alpha_y B_{mi} \sin(i \phi_r)] + [\dot{A}_{mj} \cos(j \phi_r) + \alpha_z C_{mj} \sin(j \phi_r)] + \right. \]

\[ \alpha_y B_{mj} \sin(j \phi_r) + \alpha_z C_{mj} \cos(j \phi_r) \right\} + \]

\[ \alpha_z C_{mj} \sin(j \phi_r) + 2\dot{A}_{mj} \cos(j \phi_r) + \alpha_y B_{mj} \sin(i \phi_r) \]  

\[ + \alpha_z C_{mj} \cos(i \phi_r)] + \alpha_y B_{mj} \sin(j \phi_r) \]  

\[ \alpha_z C_{mj} \sin(j \phi_r) \]  

\[ + \alpha_y B_{mj} \sin(j \phi_r) \]  

\[ + \frac{\rho A_i}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left\{ [\dot{A}_{mi} \cos(i \phi_r) + \alpha_y B_{mi} \sin(i \phi_r)] + \right. \]

\[ \alpha_z C_{mj} \cos(i \phi_r)] + \alpha_y B_{mj} \sin(j \phi_r) \]  

\[ + \frac{\rho A_i}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left\{ [\dot{A}_{mi} \cos(i \phi_r) + \alpha_y B_{mi} \sin(i \phi_r)] + \right. \]
\[ \dot{\mathbf{m}}_{m} = \left[ \frac{z_r}{a} \right] (i) \mathbf{c}_{m} \mathbf{I} \mathbf{m}_{j} \left[ \frac{z_r}{a} \right] (j) \dot{\mathbf{c}}_{m} \sin(i\phi_r) \sin(j\phi_r) + 2 \left[ \frac{z_r}{a} \right] (i) \mathbf{c}_{m} \mathbf{I} \mathbf{m}_{j} \left[ \frac{z_r}{a} \right] (j) \dot{\mathbf{c}}_{m} \sin(i\phi_r) \cos(j\phi_r) \]

\[ \left[ \frac{z_r}{a} \right] (i) \mathbf{c}_{m} \mathbf{I} \mathbf{m}_{j} \left[ \frac{z_r}{a} \right] (j) \dot{\mathbf{c}}_{m} \cos(i\phi_r) \cos(j\phi_r) \right] + \frac{z_r a^2}{4} \sum_{m=1}^{N} \sum_{i=0}^{N} \sum_{j=0}^{N} \left[ \mathbf{c}_{m} \cos(i\phi_r) + \frac{y_r(i)}{a} \dot{\mathbf{c}}_{m} \sin(i\phi_r) \right] \left[ \mathbf{c}_{m} \cos(j\phi_r) - \frac{y_r(j)}{a} \dot{\mathbf{c}}_{m} \sin(j\phi_r) \right] + 2 \left[ \dot{\mathbf{c}}_{m} \cos(i\phi_r) + \frac{y_r(i)}{a} \dot{\mathbf{c}}_{m} \sin(i\phi_r) \right] \left[ \dot{\mathbf{c}}_{m} \cos(j\phi_r) - \frac{y_r(j)}{a} \dot{\mathbf{c}}_{m} \sin(j\phi_r) \right] + \frac{z_r a^2}{4} \sum_{m=1}^{N} \sum_{i=0}^{N} \sum_{j=0}^{N} \left[ \left[ \begin{array}{c} \frac{y_r(i)}{a} \dot{\mathbf{c}}_{m} \cos(i\phi_r) \\ \frac{y_r(j)}{a} \dot{\mathbf{c}}_{m} \cos(j\phi_r) \end{array} \right] + \frac{z_r a^2}{4} \sum_{m=1}^{N} \sum_{i=0}^{N} \sum_{j=0}^{N} \left[ \begin{array}{c} \frac{y_r(i)}{a} \dot{\mathbf{c}}_{m} \sin(i\phi_r) \\ \frac{y_r(j)}{a} \dot{\mathbf{c}}_{m} \sin(j\phi_r) \end{array} \right] \right] + \left[ (-i/a) \dot{\mathbf{c}}_{m} \right] \left[ (-j/a) \mathbf{c}_{m} \right] \sin(i\phi_r) \sin(j\phi_r) + \]
\[ 2((-a)^{i}_{m_{i}})[(j+a)^{i}_{m_{j}}] \sin(i_{r}) \cos(j_{r}) + \]

\[ [(i+a)^{i}_{m_{i}}][(j+a)^{i}_{m_{j}}] \cos(i_{r}) \cos(j_{r}) - \]

\[ \rho_{r} a_{r} A_{r} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left( [\dot{c}_{m_{i}} \cos(i_{r})] \right) + \frac{\partial}{\partial_{r}} \frac{1}{2} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left( [\dot{c}_{m_{i}} \cos(i_{r})] \right) - \]

\[ \sin(j_{r}) + [\dot{c}_{m_{i}} \cos(i_{r})][j_{m_{j}} \cos(j_{r})] + \]

\[ [\dot{c}_{m_{i}} \sin(i_{r})][j_{m_{j}} \sin(j_{r})] + [\dot{c}_{m_{i}} \sin(i_{r})] - \]

\[ [j_{m_{j}} \cos(j_{r})] + \frac{\rho_{r} a_{r} A_{r}}{2a} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left( [\dot{c}_{m_{i}} \cos(i_{r})] \right) + \]

\[ \hat{\dot{b}}_{m_{i}} \sin(i_{r}) \right) \left( [j_{m_{j}} \sin(j_{r})] + [\hat{b}_{m_{i}} \sin(i_{r})][j_{m_{j}} \cos(j_{r})] - \]

\[ [\hat{b}_{m_{i}} \cos(i_{r})][j_{m_{j}} \cos(j_{r})] + \frac{\rho_{r} a_{r} A_{r}}{2a} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left( [\dot{c}_{m_{i}} \cos(i_{r})] \right) - \]

\[ \hat{b}_{m_{i}} \cos(i_{r}) \right) \left( [j_{m_{j}} \cos(j_{r})] + [\hat{b}_{m_{i}} \cos(i_{r})][j_{m_{j}} \cos(j_{r})] - \]

\[ [\dot{c}_{m_{i}} \sin(j_{r})] + [\dot{c}_{m_{i}} \sin(j_{r})][c_{m_{j}} \sin(j_{r})] + 2[\dot{c}_{m_{i}} \sin(j_{r})][c_{m_{j}} \sin(j_{r})] - \]

\[ \hat{b}_{m_{i}} \sin(i_{r}) \right) \left( [\dot{c}_{m_{i}} \sin(j_{r})] + [\dot{c}_{m_{i}} \sin(j_{r})][j_{m_{j}} \sin(j_{r})] - \]

\[ + \frac{\rho_{r} a_{r} A_{r}}{2a} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \left( [\dot{b}_{m_{i}} \alpha_{m} \sin(i_{r})][\hat{b}_{m_{j}} \alpha_{m} \sin(j_{r})] \right) - \]
\[
2[\dot{B}_{m_i}^{'}(\alpha_m)\sin(i \phi_r)][\dot{B}_{m_j}^{'}(\alpha_m)\cos(j \phi_r)] + [\dot{B}_{m_l}^{'}(\alpha_m)\cos(i \phi_r)][\dot{B}_{m_j}^{'}(\alpha_m)\cos(j \phi_r)] + \frac{\partial I_y z r}{4} \sum_{m=1}^{M} \sum_{i=0}^{N} \sum_{j=0}^{N} \{[\
\dot{S}_{m_i}^{'}(\alpha_m)\sin(i \phi_r)][\dot{S}_{m_j}^{'}(\alpha_m)\cos(j \phi_r)] - [\dot{B}_{m_l}^{'}(\alpha_m)\cos(j \phi_r)]\
[\dot{C}_{m_j}^{'}(\alpha_m)\cos(j \phi_r)] + [\dot{B}_{m_i}^{'}(\alpha_m)\sin(i \phi_r)][\dot{C}_{m_j}^{'}(\alpha_m)\cos(j \phi_r)], \\
\dot{S}_{m_i}^{'}(\alpha_m)\sin(i \phi_r)][\dot{C}_{m_j}^{'}(\alpha_m)\sin(j \phi_r)] - [\dot{B}_{m_l}^{'}(\alpha_m)\cos(i \phi_r)][\dot{C}_{m_j}^{'}(\alpha_m)\sin(j \phi_r)] \} (2.24)
\]

Therefore, the total kinetic energy of the system is:

\[
T = T_{\text{SHELL}} + \sum_{r=1}^{K} T_{sr} (2.25)
\]
CHAPTER III

FREE VIBRATIONS OF SIMPLY SUPPORTED STIFFENED CYLINDER ACCORDING TO RITZ METHOD

The equations of motion for free vibration are derived from Lagrange's equations:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_{mn}} \right) - \frac{\partial T}{\partial q_{mn}} + \frac{\partial V}{\partial q_{mn}} = 0
\]  

(3.1)

where \( T \) and \( V \) are the kinetic and potential energy of the system, respectively and \( q_{mn} \) is \( A_{mn}, A'_{mn}, B_{mn}, B'_{mn}, C_{mn} \) or \( C'_{mn} \). For free vibrations the generalized coordinates are harmonic functions of time; that is, we have:

\[
\begin{bmatrix}
A_{mn} \\
A'_{mn} \\
B_{mn} \\
B'_{mn} \\
C_{mn} \\
C'_{mn}
\end{bmatrix}
=
\begin{bmatrix}
a_{mn} \\
a'_{mn} \\
b_{mn} \\
b'_{mn} \\
c_{mn} \\
c'_{mn}
\end{bmatrix}
\cos \omega t
\]  

(3.2)

where \( \omega \) represents the natural frequency of the composite shell. If one introduces the following notation
then (3.1) can be written in the characteristic form of the Ritz method:

\[ \omega^2 \frac{\partial^2 T^*}{\partial q_{mn}^2} - \frac{\partial^2 V^*}{\partial q_{mn}^2} = 0 \quad (3.3) \]

where now \( q_{mn} \) is \( a_{mn}, a'_{mn}, b_{mn}, b'_{mn}, c_{mn} \) or \( c'_{mn} \). The expression for \( T^* \) is obtained from Equation (2.25) by replacing \( A_{mn}, A'_{mn}, B_{mn}, B'_{mn}, C_{mn} \) or \( C'_{mn} \) with \( a_{mn}, a'_{mn}, b_{mn}, b'_{mn}, c_{mn} \) or \( c'_{mn} \), respectively. Similarly, the expression for \( V^* \) is obtained from Equation (2.20) by replacing \( A_{mn}, A'_{mn}, B_{mn}, B'_{mn}, C_{mn} \) or \( C'_{mn} \) with \( a_{mn}, a'_{mn}, b_{mn}, b'_{mn}, c_{mn} \) or \( c'_{mn} \).

Inserting expressions for the derivatives of \( T^* \) and \( V^* \) into Equation (3.3), a set of 6N equations for each axial mode (wave number \( m \)) is generated because the axial modes are not coupled for the simply supported case. These equations are linear in the amplitudes of the displacement series and form an eigenvalue problem. In matrix form, these equations are:

\[ [K]q - \Omega [M]q = 0 \quad (3.4) \]

where
\([K]\) is the stiffness matrix, \(6N \times 6N\),
\([M]\) is the mass matrix, \(6N \times 6N\),
\(\{q\}\) is the matrix of generalized coordinates,

and

\[\Omega = (1-v^2)pa^2 \omega^2 / E\] is the frequency parameter.

Equations (3.4) can be reduced to the standard form \([C]\{\gamma\} = \lambda\{\gamma\}\),
by factoring \([K]\) into the product of a lower triangular matrix \([L]\) times
its transpose \([L]^T\), as follows:

\[ [K] = [L][L]^T \]  \hspace{1cm} (3.5)

this is permissible since \([K]\) is positive definite and symmetric. Pre-
multiplying (3.4) by \([L]^{-1}\) yields:

\[ [L]^{-1}[M]\{q\} = \lambda[L]^{-1}[L][L]^T\{q\}, \]  \hspace{1cm} (3.6)

or

\[ [L]^{-1}[M]\{q\} = \lambda[L]^T\{q\} \]  \hspace{1cm} (3.7)

where

\[ \lambda = 1/\Omega. \]
Rewriting this equation as

\[ [L]^{-1}[M] ([L]^{-1})^T [L]^T \{q\} = \lambda[L]^T \{q\} \]  \hspace{1cm} (3.8)

since

\[ ([L]^{-1})^T [L]^T = [I] \]

and letting

\[ [C] = [L]^{-1}[M][L]^{-1} \] and \[ \{Y\} = [L]^T \{q\} \]

we obtain the standard form

\[ [C] \{Y\} = \lambda \{Y\} \]  \hspace{1cm} (3.9)

where \([C]\) is a symmetric 6N\times 6N matrix.

The stiffness matrix \([K]\) is given by its submatrices as:

\[
[K] =
\begin{bmatrix}
  [k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\
  (k_{12})^T & [k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\
  (k_{13})^T & (k_{23})^T & [k_{33} & k_{34} & k_{35} & k_{36} \\
  (k_{14})^T & (k_{24})^T & (k_{34})^T & (k_{44})^T & (k_{54})^T & (k_{64})^T \\
  (k_{15})^T & (k_{25})^T & (k_{35})^T & (k_{45})^T & [k_{55} & k_{56} \\
  (k_{16})^T & (k_{26})^T & (k_{36})^T & (k_{46})^T & (k_{56})^T & [k_{66}]
\end{bmatrix}
\]  \hspace{1cm} (3.10)
The mass matrix \([M]\) is given in terms of its submatrices as:

\[
[M] = \begin{bmatrix}
    m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\
    (m_{12})^T & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\
    (m_{13})^T & (m_{23})^T & m_{33} & m_{34} & m_{35} & m_{36} \\
    (m_{14})^T & (m_{24})^T & (m_{34})^T & m_{44} & m_{45} & m_{46} \\
    (m_{15})^T & (m_{25})^T & (m_{35})^T & (m_{45})^T & m_{55} & m_{56} \\
    (m_{16})^T & (m_{26})^T & (m_{36})^T & (m_{46})^T & (m_{56})^T & m_{66}
\end{bmatrix}
\]  

(3.11)

The elements in the matrices \([K]\) and \([M]\) are given in Appendix C. The column matrix \(\{q\}\) of generalized coordinates is written as:

\[
\{q\} = \begin{bmatrix}
    \ddot{a} \\
    \ddot{b} \\
    \ddot{c} \\
    \ddot{a}' \\
    \ddot{b}' \\
    \ddot{c}'
\end{bmatrix}
\]  

(3.12)

where the elements \(\ddot{a}, \ddot{b}, \ddot{c}, \ddot{a}', \ddot{b}', \ddot{c}'\) are column vectors whose components are given by:

\[
\ddot{a}_j = a_{mj}
\]  

(3.13)
where \( m \) is a fixed axial mode number and \( j = 1, 2, \ldots, N \).

The addition of stringers on a cylindrical shell has a significant effect on the circumferential modes of the structure. The circumferential modes with different wave numbers \( n_1 \) and \( n_2 \) are coupled and the symmetric modes \( (a'_{mn}, b'_{mn}, c'_{mn}) \) are also coupled to the antisymmetric modes \( (a''_{mn}, b''_{mn}, c''_{mn}) \). However, it is possible to simplify Equations (3.4) by considering certain types of stringer configurations.

If the stringers have an axis of symmetry \( (I_{yzi} = 0) \) and are identical and equally spaced around the circumference of the shell then it is possible to uncouple the symmetric and antisymmetric modes. If the stringers are equally spaced, then the stringer distribution is given by:

\[
\phi_j = \frac{2\pi j}{K} \quad j = 1, 2, \ldots, k
\]

where \( K \) is the total number of stringers. This distribution is symmetric with respect to the x-z plane. For equally spaced, identical stringers, the terms which couple the circumferential modes of different wave numbers contain the following sums:
\[ X = \sum_{j=1}^{K} \sin \left( \frac{2\pi j}{K} n_1 \right) \sin \left( \frac{2\pi j}{K} n_2 \right) \]  

(3.14)

\[ Z = \sum_{j=1}^{K} \sin \left( \frac{2\pi j}{K} n_1 \right) \cos \left( \frac{2\pi j}{K} n_2 \right) \]

Miller [2] showed that these sums are given by:

\[
X = \begin{cases} 
0 & \text{if } n_1 + n_2 \neq 0, \pm k, \pm 2k, \ldots \\
\frac{k}{2} & \text{if } (n_1 + n_2) \text{ or } (n_1 - n_2) = 0, \pm k, \pm 2k, \ldots \\
0 & \text{if } (n_1 + n_2) \text{ and } (n_1 - n_2) = 0, \pm k, \pm 2k, \ldots 
\end{cases}
\]

(3.15)

\[
Y = \begin{cases} 
0, n_1 + n_2 \neq 0, \pm k, \pm 2k, \ldots \\
\frac{k}{2}, (n_1 + n_2) \text{ or } (n_1 - n_2) = 0, \pm k, \pm 2k, \ldots \\
k, (n_1 + n_2) \text{ and } (n_1 - n_2) = 0, \pm k, \pm 2k, \ldots 
\end{cases}
\]

\[ Z = \{0 \text{ for all } n_1 \text{ and } n_2\} \]

It may be noted that \(X = Y = 0\) if \(n_1 = n_2\) and \(n_1 + n_2 < K\) which states that there is no coupling between circumferential modes with wave numbers \(n_1\) and \(n_2\) if \(n_1 \neq n_2\) and \(n_1 + n_2 < K\). Therefore, the larger the number of stringers, the less number of coupled modes for the same number of terms in an assumed displacement series. However, if one has a few stringers, say 8, then there is no coupling between the first circumferential mode and modes 1, 2, 3, 4, 5, 6 but the first and seventh modes are coupled. Miller [2] and Ojalvo and Newman [5] used
the fact that if \( K > (n_1 + n_2) \) (i.e., if the stiffener spacing is small compared to the distance between successive circumferential nodal lines) then it is permissible to drop the sum on \( n \) in Equations (2.1) and assume that:

\[
\begin{align*}
    u(x, \phi) &= (A \cos n\phi + A' \sin n\phi) \cos \frac{m \pi x}{l} \\
    v(x, \phi) &= (B \sin n\phi - B' \cos n\phi) \sin \frac{m \pi x}{l} \\
    w(x, \phi) &= (C \cos n\phi + C' \sin n\phi) \sin \frac{m \pi x}{l}
\end{align*}
\]

Thus, the behavior of a shell with many stringers is that of a uniform orthotropic cylinder where the circumferential mode is a pure harmonic. This assumption is in very good agreement with experimental results for shells with a large number of stringers.

In the literature cited, one finds different expressions for the strain and kinetic energies of a shell and stringers. For example, Miller [2] considers only the strain energy of the stringers arising from the terms \( \frac{EA}{2} \int_0^l \left( \frac{\partial u}{\partial x} \right)^2 dx \) and \( \frac{EI_{zz}}{2} \int_0^l \left( \frac{\partial^2 w}{\partial x^2} \right)^2 dx \) as important but Ojalvo and Newman [5] include all the strain energy terms but warping and also neglect rotatory inertia of the stringers in the kinetic energy expression. Egle and Sewall [6] assume the shear center coincides with the centroid of the stringers and neglect the \( I_y \) and \( I_{yz} \) terms in the strain energy of the stringers. However, they account for transverse stiffening effects by incorporating "stiffening ratios" in the
bending energy expression for the shell which then takes the form:

$$\frac{D}{2} \int_0^{2\pi} \int_0^L \left[ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + \nu \left( 1 + \frac{D y}{D x} \right) \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \right] \text{adxdy}$$

As a result of this approximation, the fact that $I_{yz} = 0$ in order to uncouple the symmetric and antisymmetric modes does not explicitly come up. Further, it should be noted that $E_\text{r}$ in the present theory is $E_\text{s}/(1-\nu^2)$ and not Young's modulus of the thin-walled beam as is used in other references. This is a consequence of the assumption that in Vlasov's thin-walled beam theory one assumes that the section contour is inextensible which yields the result that the relation between longitudinal extensions $\varepsilon_x$ and normal stresses $\sigma_x$ is

$$\sigma_x = \frac{E_\text{s}}{1-\nu^2} \varepsilon_x \quad (3.17)$$

and not

$$\sigma_x = E_\text{s} \varepsilon_x \quad (3.18)$$

as in elementary beam theory. However, if one makes the assumption that the quantity $\nu^2$ is negligible compared with unity as was done by Vlasov [10] then we get the usual stress-strain relation of beam theory that:
Numerical Results and Discussion

Computer programs were written to solve the eigenvalue problem (i.e., Equation 3.4) in the ALGOL for use with the Burrough's B5500 digital computer. Because of the relatively large size of the eigenvalue problem, the equally spaced stringer distribution was assumed. For symmetric stringers this enables one to calculate the symmetric and antisymmetric modes separately. The eigenvalues and eigenvectors of (3.4) were calculated by the Householder-Givens method (see references [11] through [14]). This is a very fast technique which was found to be much faster and as accurate as the well-known Jacobi method. Even so, the subroutines employed required 30 minutes for the solution of a typical (90x90) size problem.

Free Vibration

Table 2 and Table 5 compare the frequencies of a simply supported cylindrical shell with four and eight internal stringers, respectively, as calculated by the present analysis with the experimental results of Schnell [4]. The material properties and geometry are given by case 1 in Table 1. The eigenvalues are compared for a fixed value of the axial mode number with a number of circumferential modes associated with it. Since the circumferential modes are coupled and therefore not harmonic as for the case of the unstiffened shell, there is a problem in identifying these modes. Usually, the term in the series with the largest coefficient is the predominant term in the mode shape. If this
is the nth term in the series then the mode will be called the nth circumferential mode. Thus, \( n=3 \) refers to a wave form around the circumference with three waves but the waves are simply not harmonic. From Table 2, it is seen that there are two natural frequencies, one is associated with a symmetric mode and the other is associated with an antisymmetric mode. These frequencies are not generally the same.

The symmetric and antisymmetric frequencies for the shell with four stringers are the same for odd values of \( n \), however. The reason for this is that for the odd symmetric modes, the stringers at \( \phi=0, \pi \) are at antinodal lines and the ones at \( \phi=\pi/2, 3\pi/2 \) are at nodes. For the case of the odd antisymmetric modes the stringers at \( \phi=0, \pi \) are at node lines, but the ones at \( \phi=\pi/2, 3\pi/2 \) are at antinodal lines. Therefore, the two circumferential modes are the same and one obtains the same frequencies. However, for the even symmetric modes all of the stringers are in bending (see Figure 5(a) for the symmetric modes), whereas for the even antisymmetric modes all of the stringers are in torsion (see Figure 5(b) for the antisymmetric modes). Since the strain energy of the stringers in bending is not generally the same as for torsion, the frequencies for the even symmetric modes will not be the same as the frequencies for the antisymmetric modes. Good agreement is noted for the theoretical values as compared with the experimental results. It may be noted that the experimental results give only a single frequency for a given circumferential mode. The reason for this is not known but it appears that a symmetric mode may have been excited in some cases (e.g., \( m=1, n=8 \)) and an antisymmetric mode
in others (e.g., m=1, n=12). The case for eight stringers is analogous. The symmetric and antisymmetric frequencies are the same for the odd values of n. The frequencies for the symmetric and antisymmetric modes are the same for n=2,6,10 since four stringers are at nodal lines and four stringers are at antinodal lines.

Figures 3-4 illustrate the fact that the frequency curves for the stiffened shell are analogous to those of the unstiffened shell (see reference [15]). For a given axial mode the frequencies decrease with increasing n down to a minimum and then increase again. The results also indicate that the value of n at which the minimum frequency occurs depends upon the axial wavelength. As m increases, the value of n corresponding to the minimum frequency also increases or remains the same. The eigenvalues were calculated using Donnell's shell theory and neglecting rotatory inertia of the shell wall. It was also found for the hat-shaped stringer ( ) that the frequencies and mode shapes were unchanged if one mode the further assumption that the shear center coincides with the centroid (i.e., . Since for this type of stringer, this result could be expected since the term contributes only a negligible amount of inertia to the structure and one has already accounted for the primary effects of such stringer eccentricity effects in the terms.

Tables 3-4 give the relative magnitudes of the components of some selected eigenvectors for the radial frequencies. These tables show the effect of circumferential modal coupling on the structure for four stringers. In the tables, the components are
normalized relative to the predominant radial component. Consider, for instance, the mode $m=2$, $n=5$ in Table 4. The dominant radial amplitude is $c_{25}$. All of the components have been normalized with respect to $c_{25}$. If the shell were not stiffened then $c_{25}$ would be the only non-zero component in this particular mode. The effect of attaching longitudinal ribs has been to couple these circumferential modes with different wave numbers together. It may be noted that the coefficients in the mode $m=2$, $n=1$ are not decreasing as rapidly as for the other modes. This is reflected in the mode shapes (see Figure 5(b)) in which the mode $(m=2, n=1)$ is not as smooth as the other mode shapes. The absolute value of the ratio of the largest secondary component to the dominant component is a measure of the strength of the coupling. In the mode $(m=2, n=5)$, this ratio is the largest for coupling between the ninth and fifth modes (ratio = .06). However, for the mode $(m=2, n=7)$, the ratio is the largest for coupling between the seventh and fifth modes (ratio = .13). In Table 3, considering the mode $(m=4, n=1)$, the dominant amplitudes for the complete set of eigenvectors are $a_{41}$, $b_{41}$, and $c_{41}$, but for the mode $(m=4, n=2)$ the dominant amplitudes are $a_{42}$, $b_{42}$, and $c_{42}$.

In Figures 5 and 6 some selected theoretical mode shapes are presented for the case of four internal stringers. These figures show the normalized radial deflection ($w$) plotted against a nondimensional circumferential coordinate. In addition Figure 5 also shows the normalized longitudinal deflection ($u$) and circumferential deflection ($v$) for the location $x = l/4$. The positions of the stringers on the shell are also given.
The eigenvalues for 4 axial modes and 25 symmetric circumferential modes are given in Tables 6-9 for the case of the internally stiffened cylinders with four or eight equally spaced stringers. It may be observed from these tables that for the radial frequencies, the frequencies of the shell reaches higher values with \( m \) increasing than with \( n \) increasing. This is because the contribution of the stringer to the potential energy increases with \( m \) while its kinetic energy is independent of \( m \) and \( n \). Thus, one has a stiffer system with higher frequencies. Arnold and Warburton [15] showed that for the unstiffened shell, that the displacement associated with the lowest frequency is predominantly radial. The intermediate frequency corresponds to mainly a longitudinal motion for small values of \( m \) and the circumferential component is associated with the highest frequencies. The intermediate and highest frequencies are associated almost entirely with extensional vibrations, in which the strain energy is large relative to that of the lowest frequencies. These characteristics are also true of the stringer stiffened shell as may be seen from examination of Tables 6-9. Usually the radial frequencies are the ones that are most easily excited and are the ones which are observed experimentally.

Yu [16] used the dynamic counterpart of Donnell's equations to investigate the free vibration characteristics of an unstiffened cylindrical shell. He obtained results in good agreement with Flügge's results for the more exact shell theory. Smith [17] presented a method for obtaining a closed form solution to Donnell's equations without
first making any simplifications. He obtained good agreement between experimental and theoretical results for simply supported and clamped-clamped shells. Tables 10-13 show a comparison of Flügge's shell theory versus Donnell's shell theory for the stiffened cylindrical shell treating the stringers as discrete elements and neglecting rotatory inertia of the shell. These results are in very good agreement for the numerical example considered.

In Table 14, the results of Donnell's shell theory are compared with the orthotropic results presented in reference [4] for the case of a cylindrical shell stiffened by means of four internal stringers. It may be immediately noticed that no results are given for n=1 for the orthotropic case. This is because the orthotropic theory yields an equivalent shell with too great a thickness and hence the shell is much stiffer than the actual stiffened cylinder. As a result, the frequencies are too high and no meaning can be attached to the results. It may be further noticed that the integer m for the orthotropic case corresponds to a harmonic wave. Therefore, the symmetric and antisymmetric frequencies are the same and only one frequency is given. This is, of course, in disagreement with the results of the Ritz energy method. The comparison of results shows reasonable agreement for most modes. However, there are some considerable discrepancies. For example, the antisymmetric frequency in the mode m=4, n=4 differs from the orthotropic result by about 23 per cent. Schnell and Heinrichsbauer [4] did not present any results stating as to when it was possible to use the orthotropic theory for a shell with only a few stringers to
predict frequencies. This does not in general seem feasible because of the large parameters involved.

Table 15 compares the theoretical frequencies for four internal stringers \(_{\text{J-J}}\) by neglecting rotatory inertia of the stringers and assuming the shear center coincides with the centroid \((a_z = 0)\). The assumption \(a_z = 0\) had no significant effect on the frequencies of the stiffened shell (less than 1/10 of 1 per cent change) and also yielded the same mode shapes. The results of these tables show excellent agreement. The primary rotatory inertia term was the contribution from the \(I_{zz}\) term. This comes from the \(m_{22}\) sub-matrix (see Appendix C, Equation C-22). Hence, if \(\alpha_m^2 I_{zz}/A_i \ll 1\), then this term has a negligible effect. The term becomes more important at high axial wave numbers, however.

The effect of neglecting rotatory inertia on the theoretical longitudinal and circumferential frequencies for four internal stringers is noted in Tables 16 and 17. A close examination of these tables reveals that making such an assumption also has little effect on the natural frequencies. Since the strain energy of the shell becomes very high for these modes, it seems reasonable to expect this result, although no experimental data is available.

A comparison of the results of the present theory retaining in-plane inertia versus the results of reference [6] which employed stiffening ratio's and neglected in-plane inertias for a shell stiffened by 60 stringers is presented in Table 18. The geometric properties are given by case 3 in Table 1. For this stringer configuration mode shapes
are almost pure harmonics. Serious disagreement can be noted for the results shown with the discrepancy increasing for higher circumferential waves. For example, in the mode \( m=2, n=15 \) the difference is about 22 per cent. The results for the analysis of reference [6] are higher since the in-plane inertias were neglected.

The assumption of neglecting rotatory inertia of the stringers for eight stringers is compared with the experimental results in Tables 19 and 20. Again, excellent agreement is noted for 4 axial waves and 25 circumferential waves (less than 2 per cent difference in most cases).

Table 21 compares the results of the present theory with the experimental results of Hoppman [1] for a cylindrical shell with 16 external stringers (type \( \mathbb{Z} \)) which are equally spaced around the circumference. The stringer properties are given by case 2 in Table 1. The orthotropic results are also compared. The results of the present analysis are given for the symmetric modes.

It may be noted that no orthotropic calculations are made for the \( n=1 \) mode since the "smear" analysis predicts frequencies which are too high. The results of the present analysis using Donnell's shell theory and retaining in-plane inertia terms are in very good agreement with the orthotropic results for the \( n \geq 3 \) modes. The worse agreement comes for the \( n=2 \) mode (i.e., \( m=4, n=2 \)) where the results differ about 20 per cent. The orthotropic theory predicts frequencies higher than the present analysis in this mode. One reason for this is that the orthotropic theory probably overstiffens the shell for this case, also.
The results between the present theory and the experiments are in good agreement except for the case when $m=4$ where the experimental results are higher than the results of the present analysis. This trend may be noted in the other modes also (i.e., $m=3$, $n=4,5$ and $m=2$, $n=4,5$). One possible reason for this discrepancy is that the boundary conditions may not have been exactly duplicated. Perhaps a longitudinal and/or rotational restraint may have been introduced.

The assumption of neglecting "rotatory" inertia of the stringers was investigated for this case, also. It was found that the difference between theoretical frequencies by employing this assumption was less than 1 per cent. The assumption of neglecting in-plane inertias of the shell was also investigated. It was found that this assumption caused much higher theoretical frequencies in the higher axial modes ($m>2$) because of the decrease in the inertia of the structure. It is concluded that the assumptions of neglecting in-plane inertias of the shell is not valid and it is recommended that this assumption not be employed.
<table>
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<td>(b_{\text{in}})</td>
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<td>(10^7)</td>
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<td>(2.59 \times 10^{-4})</td>
<td>(2.56 \times 10^{-4})</td>
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<td>(v)</td>
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<td>0.30</td>
<td>0.30</td>
</tr>
<tr>
<td>(k, \text{in})</td>
<td>38.85</td>
<td>15.53</td>
<td>23.75</td>
</tr>
<tr>
<td>((GJ)_s, \text{lb-in}^2)</td>
<td>10.76</td>
<td>1482</td>
<td>360</td>
</tr>
<tr>
<td>(A_s, \text{in}^2)</td>
<td>(1.589 \times 10^{-2})</td>
<td>(2.625 \times 10^{-2})</td>
<td>(2.899 \times 10^{-2})</td>
</tr>
<tr>
<td>(Z_s, \text{in})</td>
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<td>(+0.137)</td>
<td>(+0.152)</td>
</tr>
<tr>
<td>(a_{\text{y}, \text{in}})</td>
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</tr>
<tr>
<td>(a_{\text{z}, \text{in}})</td>
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</tr>
<tr>
<td>(I_{zz}, \text{in}^4)</td>
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<td>(3.42 \times 10^{-5})</td>
<td>(2.227 \times 10^{-5})</td>
</tr>
<tr>
<td>(I_{yy}, \text{in}^4)</td>
<td>(3.364 \times 10^{-4})</td>
<td>(9.651 \times 10^{-5})</td>
<td>(2.203 \times 10^{-3})</td>
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<tr>
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<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>(y_s, \text{in})</td>
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<td>0.0</td>
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<tr>
<td>(E_s, \text{psi})</td>
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<td>(10^7)</td>
<td>(10.5 \times 10^6)</td>
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<tr>
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<tr>
<td>(h/a)</td>
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<td>(t/a)</td>
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<td>8.07</td>
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*These data together with frequency tables were received in private communication from Professor F. J. Heinrichsbauer, Institut Für Festigkeit, West Germany, on April 10, 1967. The data for case (2) was obtained from reference [1] and for case (3) from reference [6].
Table 2. Comparison of Theoretical and Experimental Radial Frequencies for Four Equally Spaced Stringers

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<td>b</td>
<td>c</td>
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<td>302</td>
<td>*</td>
</tr>
<tr>
<td>3</td>
<td>157</td>
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<td>279</td>
<td>282</td>
</tr>
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<td>330</td>
<td>330</td>
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</tr>
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<td>405</td>
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<table>
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<td>2648</td>
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<td>*</td>
</tr>
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<td>1039</td>
<td>*</td>
</tr>
<tr>
<td>4</td>
<td>721</td>
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<td>*</td>
</tr>
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<td>460</td>
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<td>8</td>
<td>302</td>
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</tr>
<tr>
<td>12</td>
<td>418</td>
<td>418</td>
<td>422</td>
</tr>
</tbody>
</table>

Key:
- **a** = Symmetric modes, stiffened (Donnell's theory).
- **b** = Antisymmetric modes, stiffened (Donnell's Theory).
- **c** = Unstiffened, experimental results, Schnell and Heinrichsauer [6].
- **d** = Stiffened, experimental results, Schnell and Heinrichsauer.

Type of Stringer: TL.

* = No experimental results for this case.
Table 3. Eigenvectors for Radial Modes Showing the Extent of Modal Coupling for the Stiffened Shell with Four Stringers (Symmetric)

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</tr>
<tr>
<td>a₄₀</td>
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<td>8.6×10⁻³</td>
<td>0</td>
<td>1.0×10⁻²</td>
</tr>
<tr>
<td>a₄₁</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a₄₂</td>
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<td>0</td>
<td>-7.5×10⁻²</td>
<td>0</td>
<td>9.8×10⁻²</td>
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<td>0</td>
<td>5.75×10⁻²</td>
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<tr>
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<td>4.2×10⁻³</td>
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<td>-5.5×10⁻²</td>
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<td>7.1×10⁻³</td>
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<td>0</td>
</tr>
<tr>
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<td>2.0×10⁻²</td>
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Table 4. Eigenvectors for Radial Modes Showing the Extent of Modal Coupling for the Stiffened Shell with Four Stringers (Antisymmetric)

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<td>-2.36x10^{-3}</td>
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<td>0</td>
<td>0</td>
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Table 4. Eigenvectors for Radial Modes Showing the Extent of Modal Coupling for the Stiffened Shell with Four Stringers (Antisymmetric) (Continued)

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Key

a = Symmetric modes, stiffened (Donnell's theory).
b = Antisymmetric modes, stiffened (Donnell's theory).
c = Unstiffened, experimental results (Schnell and Heinrichsbauer).
d = Stiffened experimental results (Schnell and Heinrichsbauer).

Type of Stringer: \( \perp \) (Hat-Shaped).
### Table 6. Frequencies of Vibration for Radial Modes
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### Table 7. Frequencies of Vibration for Radial Modes
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Table 9. Frequencies of Vibration for Circumferential Modes
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Table 10. Comparison of Flügge's Theory versus Donnell's Theory for Four Equally Spaced Stringers on Simply Supported Cylinder

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Key:
- a = Symmetric modes (Flügge's theory).
- b = Symmetric modes (Donnell's theory).
- c = Unstiffened experimental.
- d = Experimental results for stiffened shell [4].

Type of Stringer: \( \mathcal{L} \) (Hat-shaped).

* = No experimental results for this case.
### Table 11. Comparison of Flügge's Theory versus Donnell's Theory for Eight Equally Spaced Stringers on Simply Supported Cylinder

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### Table 12. Comparison of Flügge's Theory versus Donnell's Theory for Eight Equally Spaced Stringers on Simply Supported Cylinder

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a = Symmetric modes (Flügge's theory).
b = Symmetric modes (Donnell's theory).
c = Antisymmetric modes (Flügge's theory).
d = Antisymmetric modes (Donnell's theory).
Table 13. Comparison of Flügge's Theory versus Donnell's Theory for Eight Equally Spaced Stringers on Simply Supported Cylinder

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a = Symmetric modes (Flügge's theory).
b = Symmetric modes (Donnell's theory).
c = Antisymmetric modes (Flügge's theory).
d = Antisymmetric modes (Donnell's theory).
Table 14. Comparison of Donnell's Theory versus "Orthotropic Theory" for Four Equally-Spaced Stringers on Simply-Supported Cylinder

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Key:
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b = Antisymmetric modes, stiffened.
c = Orthotropic results of Schnell and Heinrichsbaue
Type of Stringer: JL (Hat-Shaped).
* = No experimental results for this case.
Table 15. Comparison of Theoretical Radial Frequencies for Four Equally-Spaced Stringers by Neglecting Rotatory Inertia of Stringers

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- \( a \) = Symmetric modes (Donnell's theory).
- \( b \) = Symmetric modes (Donnell's theory but neglecting rotatory inertia of stringers).
Table 16. Comparison of Theoretical Longitudinal Frequencies for Four Equally-Spaced Stringers by Neglecting Rotatory Inertia of Stringers

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a = Symmetric modes (Donnell's theory).
b = Symmetric modes (Donnell's theory but neglecting rotatory inertia).
Table 17. Comparison of Theoretical Circumferential Frequencies for Four Equally-Spaced Stringers Neglecting Rotatory Inertia of Stringers

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Table 18. Comparison of Theoretical Radial Frequencies of the Present Analysis versus the Analysis of Reference [6] for 60 Stringers

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a = Results of present analysis (Donnell's theory).
b = Results of Reference [6].

Type of Stringer: M.
Table 19. Comparison of Theoretical Radial Frequencies for Eight Equally-Spaced Stringers by Neglecting Rotatory Inertia of Stringers

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a = Symmetric modes (Donnell's theory).
b = Symmetric modes (Donnell's theory but neglecting rotatory inertia of stringers).
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a = Symmetric modes (Donnell's theory).

b = Symmetric modes (Donnell's theory but neglecting rotatory inertia of stringers).
Table 21. Comparison of Present Theory versus Experimental Results of Hoppman [1] for 16 Equally-Spaced Stringers

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* = No experimental data given.

a = Donnell's theory (stiffened shell).
b = Experimental results (stiffened shell).
c = Orthotropic results (stiffened shell).
d = Experimental results (unstiffened shell).
e = Donnell's theory but neglecting rotatory inertia of the stringers.
Figure 3. Radial Frequencies of Stiffened Cylinder with Four Stringers
Figure 4. Radial Frequencies of Stiffened Cylinder with Eight Stringers
Figure 5(a). Theoretical Mode Shapes for Stiffened Shell with Four Stringers (Symmetric)
Figure 5(a). Theoretical Mode Shapes for Stiffened Shell with Four Stringers (Symmetric) (Continued)
Figure 5(b). Theoretical Mode Shapes for Stiffened Shell with Four Stringers (Antisymmetric)
Figure 6. Theoretical Mode Shapes for Stiffened Shell with Four Stringers (Symmetric)
Figure 6. Theoretical Mode Shapes for Stiffened Shell with Four Stringers (Symmetric) (Continued)
Figure 7. Theoretical Mode Shapes for Stiffened Shell with Eight Stringers (Symmetric)
Figure 7. Theoretical Mode Shapes for Stiffened Shell with Eight Stringers (Symmetric) (Continued)
CHAPTER IV

GOVERNING DIFFERENTIAL EQUATIONS OF MOTION AND ANALYSIS
OF STRINGER STIFFENED CYLINDRICAL SHELLS

In Chapter III the free vibration of a stringer stiffened cylindrical shell with simple supports was analyzed by using the Ritz method. The results of that analysis showed that the use of Donnell's shell theory led to results in good agreement with the more general Flügge's shell theory and with the experimental results cited in the literature but that neglecting the in-plane inertias in calculating the radial frequencies is not a completely justifiable assumption. Furthermore, neglecting rotatory inertia of the shell had no effect on the frequencies and mode shapes of the shell for the axial \((m<4)\) and circumferential \((n<30)\) modes investigated. Actually, this is to be expected since for an unstiffened thin cylindrical shell when one neglects shear distortion and rotatory inertia of the shell wall the results apply when \(\frac{h}{m} > 10h, \frac{n}{n} > \frac{10h}{a}\) where \(m\) is the number of axial half-waves and \(n\) is the number of full circumferential waves.

Therefore, Donnell's equations including in-plane inertias but neglecting rotatory inertia and shear distortion of the shell wall are now employed in order to investigate the free vibration of stringer stiffened, cylindrical shells with arbitrary admissible boundary conditions. The present analysis considers the thin cylindrical shell and stringers as discrete structural components.
The stringer-shell interaction loads are represented by double Fourier series with unknown coefficients. The nonhomogenous equations of motion of the cylindrical shell are used to determine the shell solutions for arbitrary admissible boundary conditions. One then obtains the displacements of the shell as functions of the unknown loading coefficients. Similarly, V. Z. Vlasov's equations of motion for the stringers are solved for the stringer displacements in terms of the Fourier loading coefficients for arbitrary admissible boundary conditions.

Once the analysis of the stringer and shell problems have been completed, the kinematical conditions at the line of attachment that require compatible deformations are imposed. This yields an infinite number of equations in the unknown Fourier coefficients for the determination of the eigenfrequencies of the composite structure. Once the frequencies are found, then calculations can be performed for the mode shapes of free vibration.

Equations of Motion for the Shell

The partial differential equations of motion of a thin cylindrical shell including surface forces according to Donnell's approximation for the coordinate system shown in Figure 12, p. 106, are:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{(1-\nu)}{2a^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{(1+\nu)}{2a} \frac{\partial^2 v}{\partial x \partial \phi} - \frac{v}{a} \frac{\partial w}{\partial x} + \frac{(1-\nu^2)}{Eh} p_x - \frac{(1-\nu^2)E}{E} \frac{\partial^2 u}{\partial t^2} = 0 \tag{4.1}
\]
\[
\frac{(1+v)}{2a} \frac{\partial^2 u}{\partial x^2} + \frac{(1-v)}{2} \frac{\partial^2 v}{\partial x^2} + \frac{1}{a^2} \frac{\partial^2 v}{\partial \phi^2} - \frac{1}{a^2} \frac{\partial^2 w}{\partial \phi^2} + \frac{(1-v^2)}{E \rho} \frac{\partial^2 v}{\partial t^2} = \frac{(1-v^2) \rho}{E} \frac{\partial^2 v}{\partial t^2} \tag{4.2}
\]

\[
\frac{v}{a} \frac{\partial u}{\partial x} + \frac{1}{a^2} \frac{\partial^2 w}{\partial \phi^2} - \frac{w}{a^2} - \frac{h^2}{12} \frac{\partial^2 u}{\partial x^2} + \frac{(1-v^2)}{E \rho} \frac{\partial^2 w}{\partial t^2} = \frac{(1-v^2) \rho}{E} \frac{\partial^2 w}{\partial t^2} \tag{4.3}
\]

where \( p_x, p_\phi, \) and \( p_z \) are the applied surface forces. Examination of Equations (4.1), (4.2), and (4.3) shows that each equation is a coupled partial differential equation in the middle surface displacements \( u, v, \) and \( w. \) In a similar manner as Donnell, these equations can be uncoupled by differentiation and elimination (see Appendix B). The equations that are derived from this uncoupling process are as follows:

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 w}{\partial \phi^2} = \frac{2(1+v) \rho}{E} \frac{\partial^2 u}{\partial t^2} + \frac{v}{a} \frac{\partial w}{\partial x} \tag{4.4}
\]

\[
- \frac{(1-v^2)}{E} \frac{\partial^2 u}{\partial x^2} + \frac{2(1-v^2)}{E} \frac{\partial^2 p_x}{\partial t^2} + \frac{2(1+v)}{E \rho} \frac{\partial^2 p_x}{\partial \phi^2} + \frac{(1-v^2)}{E} \frac{\partial^2 p_\phi}{\partial x^2} \tag{4.4}
\]

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 w}{\partial \phi^2} = \frac{2(1+v) \rho}{E} \frac{\partial^2 u}{\partial t^2} + \frac{v}{a} \frac{\partial w}{\partial x} \tag{4.4}
\]

\[
- \frac{(1-v^2)}{E} \frac{\partial^2 u}{\partial x^2} + \frac{2(1-v^2)}{E} \frac{\partial^2 p_x}{\partial t^2} + \frac{2(1+v)}{E \rho} \frac{\partial^2 p_x}{\partial \phi^2} + \frac{(1-v^2)}{E} \frac{\partial^2 p_\phi}{\partial x^2} \tag{4.4}
\]

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 w}{\partial \phi^2} = \frac{2(1+v) \rho}{E} \frac{\partial^2 u}{\partial t^2} + \frac{v}{a} \frac{\partial w}{\partial x} \tag{4.4}
\]

\[
- \frac{(1-v^2)}{E} \frac{\partial^2 u}{\partial x^2} + \frac{2(1-v^2)}{E} \frac{\partial^2 p_x}{\partial t^2} + \frac{2(1+v)}{E \rho} \frac{\partial^2 p_x}{\partial \phi^2} + \frac{(1-v^2)}{E} \frac{\partial^2 p_\phi}{\partial x^2} \tag{4.4}
\]
\[ \frac{(3-v)}{2} w^2 + \frac{1}{a^2} \frac{\partial w}{\partial \phi} - \frac{(1-v^2)}{Eh} p_{\phi} \right] - \frac{(1-v^2)}{Eha^2} \frac{\partial^2 p_{\phi}}{\partial \phi^2} - \]

\[ \frac{2(1-v^2)}{Eh(1-v)} \frac{\partial^2 p_{\phi}}{\partial x^2} + \frac{(1-v^2)(1+v)}{Eha(1-v)} \frac{\partial^2 p_x}{\partial x \partial \phi}, \quad (4.5) \]

\[ \frac{h^2}{12} w^8 + \frac{(1-v^2)}{a^2} \frac{\partial w}{\partial x} - \frac{(1-v^2)}{Eh} v^4 p_z = - \frac{2(1+v)p}{E} \frac{\partial^2}{\partial t^2} \left[ \right] \]

\[ \frac{(1-v^2)p_2}{E} \frac{\partial^2}{\partial t^2} - \frac{(3-v)}{2} v^2 \right] \left\{ \frac{p(1-v^2)}{E} \frac{\partial^2 w}{\partial t^2} + \frac{w}{a^2} + \frac{h^2}{12} v^4 w \]

\[ - \frac{(1-v^2)}{Eh} p_2 \right] + \frac{(1-v)}{2} v^4 w + \frac{v^2}{a^2} \frac{\partial^2 w}{\partial x^2} + \frac{1}{a^4} \frac{\partial^2 w}{\partial \phi^2} \right] - \]

\[ - \frac{2(1+v)p}{E} \frac{\partial^2}{\partial t^2} \left[ \frac{v(1-v^2)}{aEh} \frac{\partial p_x}{\partial x} - \frac{(1-v^2)}{Eha^2} \frac{\partial^2 p_{\phi}}{\partial \phi^2} \right] - \]

\[ \frac{v^2}{aEh} \frac{\partial^3 p_x}{\partial x^3} + \frac{(1-v^2)}{a^3 Eh} \frac{\partial^3 p_x}{\partial x \partial \phi^2} - \]

\[ \frac{1}{a^4} \frac{(1-v^2)}{Eh} \frac{\partial^3 p_{\phi}}{\partial \phi^3} - \frac{(2+v)(1-v^2)}{a^2(1-v)Eh} \frac{\partial^2 p_{\phi}}{\partial x^2 \partial \phi}, \quad (4.6) \]
where

\[ v^4 = v^2 v^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \phi^2} \right). \]

For the case where \( p_x = p_\phi = p_z = 0 \), then the above equations reduce to those obtained by Yu [16].

When the stringer is deformed, the interaction loads of the \( j \)th stringer may be summed into resultant vertical, longitudinal and tangential forces acting through the shear center and a resultant torque about the shear center. This approach was used by Lin [19] to investigate the random vibration of stringer-stiffened, simply-supported flat plates (i.e., fuselage panels). The resultant forces acting on the \( j \)th stringer are replaced by statically equivalent surface loads acting on the shell. The surface loads are then represented by double Fourier series in the circumferential and longitudinal directions. Mathematically, these loadings are given by:

(i) Normal load \( p^{(1)}_{2j}(x,\phi) \) (see Figure 8):

\[
P^{(1)}_{2j}(x,\phi) = \begin{cases} 
-\frac{z_j(x)}{2\epsilon}, & x \in [0,a] , \phi \in \left[ \frac{\epsilon l_j}{a}, \frac{\epsilon l_j}{a} + \frac{\epsilon l_j}{a} \right] \\
0, & \text{otherwise}
\end{cases}
\]  

(4.7)
(ii) Moment $p^{(2)}_{z_j}(x,\phi)$ (see Figure 9):

$$p^{(2)}_{z_j}(x,\phi) = \begin{cases} \frac{Q_j(x)}{\epsilon_{2j}}, & x\in[0,L], \phi\in\left[\phi_j - \frac{\epsilon_{2j}}{a}, \phi_j + \frac{\epsilon_{2j}}{a}\right] \\ 0, & \text{otherwise} \end{cases} \quad (4.8)$$

(iii) Tangential load $p_{\phi_j}(x,\phi)$ (see Figure 10):

$$p_{\phi_j}(x,\phi) = \begin{cases} \frac{Y_j(x)}{2\epsilon_{3j}}, & x\in[0,L], \phi\in\left[\phi_j - \frac{\epsilon_{3j}}{a}, \phi_j + \frac{\epsilon_{3j}}{a}\right] \\ 0, & \text{otherwise} \end{cases} \quad (4.9)$$

(iv) Longitudinal load $p_{x_j}(x,\phi)$ (see Figure 11):

$$p_{x_j}(x,\phi) = \begin{cases} \frac{X_j(x)}{2\epsilon_{a_j}}, & x\in[0,L], \phi\in\left[\phi_j - \frac{\epsilon_{a_j}}{a}, \phi_j + \frac{\epsilon_{a_j}}{a}\right] \\ 0, & \text{otherwise} \end{cases} \quad (4.10)$$

where $\phi_j$ is the location of the $j$th stringer.
(a) Longitudinal Direction   (b) Circumferential Direction

Figure 8. Schematic Sketch of Normal Loading

Circumferential Direction

Figure 9. Schematic Sketch of Twisting Moment

(a) Longitudinal Direction   (b) Circumferential Direction

Figure 10. Schematic Sketch of Tangential Shear Loading
We assume that the distributed loads which act upon the shell can be expanded into double Fourier series as follows:

\[ p_{zj}^{(1)}(x, \phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (Z_{jmn} \cos n\phi + Z_{jmn}' \sin n\phi)\cos \alpha_m x, \quad (4.11) \]

\[ p_{zj}^{(2)}(x, \phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (T_{jmn} \cos n\phi + T_{jmn}' \sin n\phi)\cos \alpha_m x, \quad (4.12) \]

\[ p_{\phi j}(x, \phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (Y_{jmn} \sin n\phi + Y_{jmn}' \cos n\phi)\cos \alpha_m x, \quad (4.13) \]

\[ p_{xj}(x, \phi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (X_{jmn} \cos n\phi + X_{jmn}' \sin n\phi)\sin \alpha_m x, \quad (4.14) \]

where the symmetric Fourier coefficients \((Z_{jmn}, T_{jmn}, Y_{jmn}, X_{jmn})\) are given by:
and the antisymmetric Fourier coefficients \( Z'_{jmn}, T'_{jmn}, Y'_{jmn}, X'_{jmn} \) are given by:

\[
Z'_{jmn} = \left( \frac{1}{\pi a} \right) \left( \frac{2}{\ell} \right) \int_0^\ell Z_j(x) \cos \alpha_m x \, dx,
\]

\[
T'_{jmn} = \left( \frac{1}{\pi a} \right) \cos n\phi_j \sin \frac{n\phi_{1j}}{a} \left( \frac{2}{\ell} \right) \int_0^\ell Z_j(x) \cos \alpha_m x \, dx,
\]

\[
Y'_{jmn} = \left( \frac{1}{\pi a} \right) \sin (n\phi_j) \sin \frac{n\phi_{2j}}{a} \left( \frac{2}{\ell} \right) \int_0^\ell Y_j(x) \cos \alpha_m x \, dx,
\]

\[
X'_{jmn} = \left( \frac{1}{\pi a} \right) \cos (n\phi_j) \sin \frac{n\phi_{3j}}{a} \left( \frac{2}{\ell} \right) \int_0^\ell Y_j(x) \sin \alpha_m x \, dx.
\]
\[ Z'_{jmo} = 0, \quad (4.21) \]

\[ Z'_{jmn} = - \left( \frac{1}{\pi n c_{1j}} \right) \sin(n \phi_j) \sin \frac{n e_{1j}}{a} \left\{ \frac{2}{L} \int_0^L Z_j(x) \cos \alpha_m x \, dx \right\}, \quad (4.22) \]

\[ T'_{jmo} = 0, \quad (4.23) \]

\[ T'_{jmn} = - \left( \frac{2}{\pi n c_{2j}} \right) \cos(n \phi_j) \left\{ 1 - \cos \frac{n e_{2j}}{a} \right\} \left\{ \frac{2}{L} \int_0^L Q_j(x) \cos \alpha_m x \, dx \right\}, \quad (4.24) \]

\[ Y'_{jmo} = - \left( \frac{1}{\pi a_j} \right) \left\{ \frac{2}{L} \int_0^L Y_{jx}(x) \cos \alpha_m x \, dx \right\}, \quad (4.25) \]

\[ Y'_{jmn} = - \left( \frac{1}{\pi n c_{3j}} \right) \cos(n \phi_j) \sin \frac{n e_{3j}}{a} \left\{ \frac{2}{L} \int_0^L Y_j(x) \cos \alpha_m x \, dx \right\}, \quad (4.26) \]

\[ X'_{jmo} = 0, \quad (4.27) \]

\[ X'_{jmn} = - \left( \frac{1}{\pi n c_{4j}} \right) \sin(n \phi_j) \sin \frac{n e_{4j}}{a} \left\{ \frac{2}{L} \int_0^L X_j(x) \sin \alpha_m x \, dx \right\}. \quad (4.28) \]

It is assumed that the resultant forces \( X_j(x), Y_j(x), Z_j(x) \) and the resultant moment \( T_j(x) = Q_j(x)c_{2j} \) acting on the \( j \)th stringer may be expanded into Fourier series as odd or even functions with a period of \( 2L \) as follows:

\[ X_j(x) = \sum_{m=1}^{\infty} X_{jm} \sin \alpha_m x, \quad (4.29) \]
\[ Y_j(x) = \sum_{m=0}^{\infty} Y_{jm} \cos \alpha_m x, \quad (4.30) \]

\[ Z_j(x) = \sum_{m=0}^{\infty} Z_{jm} \cos \alpha_m x, \quad (4.31) \]

\[ T_j(x) = \sum_{m=0}^{\infty} T_{jm} \cos \alpha_m x, \quad (4.32) \]

where

\[ X_{jo} = 0, \quad (4.33) \]

\[ X_{jm} = \frac{2}{\ell} \int_0^\ell X_j(x) \sin \alpha_m x \, dx, \quad (4.34) \]

\[ Y_{jo} = \frac{1}{\ell} \int_0^\ell Y_j(x) \, dx, \quad (4.35) \]

\[ Y_{jm} = \frac{2}{\ell} \int_0^\ell Y_j(x) \cos \alpha_m x \, dx, \quad (4.36) \]

\[ Z_{jo} = \frac{1}{\ell} \int_0^\ell Z_j(x) \, dx, \quad (4.37) \]

\[ Z_{jm} = \frac{2}{\ell} \int_0^\ell Z_j(x) \cos \alpha_m x \, dx, \quad (4.38) \]

\[ T_{jo} = \frac{1}{\ell} \int_0^\ell T_j(x) \, dx, \quad (4.39) \]

\[ T_{jm} = \frac{2}{\ell} \int_0^\ell T_j(x) \cos \alpha_m x \, dx. \quad (4.40) \]
Inserting Equations (4.33) through (4.40) into (4.15) through (4.28) gives:

\[ Z_{jm} = f_1(n) \cos(n\phi_j)Z_{jm}, \]
\[ T_{jm} = f_2(n) \sin(n\phi_j)T_{jm}, \]
\[ Y_{jm} = f_3(n) \sin(n\phi_j)Y_{jm}, \]
\[ X_{jm} = f_4(n) \cos(n\phi_j)X_{jm}, \]
\[ Z'_{jm} = f_5(n) \sin(n\phi_j)Z_{jm}, \]
\[ T'_{jm} = f_6(n) \cos(n\phi_j)T_{jm}, \]
\[ Y'_{jm} = f_7(n) \cos(n\phi_j)Y_{jm}, \]
\[ X'_{jm} = f_8(n) \sin(n\phi_j)X_{jm}, \]

where the following notation has been introduced:

\[ f_1(n) = -\frac{1}{\pi n \epsilon_1} \sin \frac{n}{a} \epsilon_{ij}, \]
\[ f_2(n) = \frac{2}{\pi n \epsilon_2} \left( 1 - \cos \frac{n}{a} \epsilon_{2j} \right), \]
\[ f_3(n) = -\frac{1}{\pi c_{3j}} \sin \frac{n}{a} \epsilon_{3j} \]  
\( (4.51) \)

\[ f_4(n) = -\frac{1}{\pi c_{4j}} \sin \frac{n}{a} \epsilon_{4j} \]  
\( (4.52) \)

Therefore, the total surface loading on the cylinder is given by:

\[ p_x(x,\phi) = \sum_{j=1}^{K} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (s_j \cos \phi + \pi_j \sin \phi) \sin \alpha_m x, \]  
\( (4.53) \)

\[ p_\phi(x,\phi) = \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (s_j \sin \phi + \pi_j \cos \phi) \cos \alpha_m x, \]  
\( (4.54) \)

\[ p_z(x,\phi) = \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} ([\pi_j \cos \phi + \pi_j \sin \phi] \cos \alpha_m x \]  
\( (4.55) \)

where \( K \) is the total number of stringers attached to the shell.

**Solution of Donnell's Equations**

In considering a circular cylindrical shell under distributed loadings, one has to solve the complete, inhomogeneous Donnell equations (as given by (4.4), (4.5), and (4.6)) including the in-plane inertia terms. To solve these equations, the method of Fourier series was employed. In doing this, solutions are obtained which can be represented as follows:
\[ U = U_c + U_p, \quad (4.56) \]
\[ V = V_c + V_p, \quad (4.57) \]
\[ W = W_c + W_p, \quad (4.58) \]

where the subscripts \( c \) and \( p \) refer to the complementary and particular solutions, respectively. The particular solution is used to account for the distributed load and the complementary solution is required in order that the combination of the two solutions will satisfy the boundary conditions of the problem. Furthermore, for convenience the symmetric and antisymmetric parts of the complementary and particular solutions are treated separately. Then since the partial differential equations are linear, the general solution is recovered by superposing the symmetric and antisymmetric parts. If it happens that the particular solution also satisfies the boundary conditions, then it is not necessary to determine the complementary solution. For free vibration, we seek solutions for the middle displacements \((U,V,W)\) of the shell in the form:

\[
\begin{pmatrix}
U \\
V \\
W
\end{pmatrix} = \begin{pmatrix}
u \\
v \\
w
\end{pmatrix} \sin \omega t \quad (4.59)
\]

where \( \omega \) denotes the natural frequency of the composite structure.
According to Flügge [8], the symmetric part of the complementary solution of Equations (4.4), (4.5), and (4.6) can be written in the following form:

\[ u^c(x,\phi,t) = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} A_{sn} e^{\lambda s a} \cos n\phi \sin \omega t \]  

(4.60)

\[ v^c(x,\phi,t) = \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} B_{sn} e^{\lambda s a} \sin n\phi \sin \omega t \]  

(4.61)

\[ w^c(x,\phi,t) = \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} C_{sn} e^{\lambda s a} \cos n\phi \sin \omega t \]  

(4.62)

From (4.60), (4.61), and (4.62), we see that each displacement is a product of two functions, each of which is a function of one of the variables \( x \) or \( \phi \). The independent variable \( \phi \) represents a periodic motion with \( n \) waves around the circumference of the cylinder. Inserting (4.60), (4.61), and (4.62) into (4.4), (4.5), and (4.6) and setting \( P_x = P_{\phi} = P_z = 0 \) (complementary solution) results in a set of linear homogeneous equations in the constants \( A_{sn} \), \( B_{sn} \), and \( C_{sn} \) as follows:

\[ A_{sn} \left[ \frac{2\nu^2}{(1-\nu)} - \frac{(3-\nu)}{(1-\nu)} \right] + n^2 \left\{ 1 - \frac{\lambda^2 s}{n^2} \right\} + n^2 \left\{ 1 - \frac{(\lambda s a)^2}{n^2} \right\} \]  

(4.63)

\[ C_{sn} \lambda s \left[ \frac{2\nu^2}{1-\nu} + n^2 \left\{ 1 + \nu \frac{\lambda^2 s}{n^2} \right\} \right], \quad s=1,\ldots,8 \]
\[ B \left( \frac{2\Omega^2}{(1-\nu)} - \frac{(3-\nu)}{(1-\nu)} \Omega n^2 \left( 1 - \frac{\lambda_s^2}{n^2} \right) + n^4 \left( 1 - \frac{\lambda_s^2}{n^2} \right) \right) = \]

\[ C_{sn}(n) \left[ \frac{-2\Omega}{1-\nu} + n^2 \left( 1 - (2+\nu) \frac{\lambda_s^2}{n^2} \right) \right], \quad (4.64) \]

\[ s=1, \ldots, 8 \]

and,

\[ (1-\nu)(1-\nu^2)\lambda_s^4 = 2\Omega^2 - \Omega^2 \left[ 2+(3-\nu)n^2(1-\lambda_s^2/n^2) + 2kn^4\left(1-\lambda_s^2/n^2\right)^2 \right] + \]

\[ n^4(3-\nu)n^2(1-\lambda_s^2/n^2) - 2n^2(1-\nu^2\lambda_s^2/n^2) + \]

\[ (1-\nu)n^4(1-\lambda_s^2/n^2) + (3-\nu)kn^6(1-\lambda_s^2/n^2)^3 \] -

\[ (1-\nu)kn^8(1-\lambda_s^2/n^2)^4 \quad s=1, \ldots, 8 \quad (4.65) \]

where

\[ \Omega = (1-\nu^2)\rho a^2 \omega^2/E \quad \text{and} \quad k = h^2/12a^2. \]

Equation \((4.65)\) can be written as follows:
The complementary solution for the antisymmetric part of Equations (4.4), (4.5) and (4.6) can be written in the following form:

\[ u_a^c(x, \phi, t) = \{ \sum_{n=1}^{\infty} \sum_{s=1}^{s_n} A^s e^{\lambda_s x} \sin \frac{x}{k} \} \sin \omega t, \quad (4.67) \]

\[ v_a^c(x, \phi, t) = \{ \sum_{n=1}^{\infty} \sum_{s=1}^{s_n} B^s e^{\lambda_s x} \cos \frac{x}{k} \} \sin \omega t, \quad (4.68) \]

and

\[ w_a^c(x, \phi, t) = \{ \sum_{n=1}^{\infty} \sum_{s=1}^{s_n} C^s e^{\lambda_s x} \sin \frac{x}{k} \} \sin \omega t \quad (4.69) \]
where the superscript \( a \) represents the antisymmetric part of the solution. Inserting (4.67), (4.68), and (4.69) into (4.4), (4.5), and (4.6) and letting \( p_x = p_\phi = p_z = 0 \) result in a set of linear homogeneous equations in the constants \( A_{sn}, B_{sn}, \) and \( C_{sn} \) identical with (4.63), (4.64), and (4.65). In general the roots of Equation (4.66) will have the form:

\[
\lambda_s = (\psi_1^a i\theta_1), (\psi_2^a i\theta_2), (\psi_3^a i\theta_3), (\psi_4^a i\theta_4)
\]  

(4.70)

where \( \psi_1, \psi_2, \psi_3, \psi_4, \theta_1, \theta_2, \theta_3, \) and \( \theta_4 \) are real quantities.

The complementary solution corresponding to the roots of (4.70) may be written as follows:

\[
u_c^a(x, \phi, t) = \sum_{n=0}^{\infty} \left[ e^{\psi n a} \frac{x}{\ln a} \left( (A_{1} + A_{2}) \cos \theta_{1} \ln a \frac{x}{a} + (A_{1} - A_{2}) i \sin \theta_{1} \ln a \frac{x}{a} \right) + \right.
\]

\[
\left. e^{2n a} \frac{x}{2n a} [(A_{3} + A_{4}) \cos \theta_{3} \frac{x}{2n a} + (A_{3} - A_{4}) i \sin \theta_{3} \frac{x}{2n a}] + \right.
\]

\[
\left. e^{3n a} \frac{x}{3n a} [(A_{5} + A_{6}) \cos \theta_{5} \frac{x}{3n a} + (A_{5} - A_{6}) i \sin \theta_{5} \frac{x}{3n a}] + \right.
\]

\[
\left. e^{4n a} \frac{x}{4n a} [(A_{7} + A_{8}) \cos \theta_{7} \frac{x}{4n a} + (A_{7} - A_{8}) i \sin \theta_{7} \frac{x}{4n a}] \cos \ n \phi \right) \cos \ n \phi \}
\]

\[
\sin \omega t, \quad (n, 71)
\]
\[ \psi_c(x, \phi, t) = \sum_{n=1}^{\infty} \left\{ e^{\frac{\ln x}{a}} \left[ (B_1 + B_2) \cos \frac{\theta_1 x}{a} + (B_1 - B_2) i \sin \frac{\theta_1 x}{a} \right] \\
+ e^{\frac{\ln a}{2n}} \left[ (B_3 + B_4) \cos \frac{\theta_2 n x}{a} + (B_3 - B_4) i \sin \frac{\theta_2 n x}{a} \right] \\
+ e^{\frac{\ln a}{3n}} \left[ (B_5 + B_6) \cos \frac{\theta_3 n x}{a} + (B_5 - B_6) i \sin \frac{\theta_3 n x}{a} \right] \\
+ e^{\frac{\ln a}{4n}} \left[ (B_7 + B_8) \cos \frac{\theta_4 n x}{a} + (B_7 - B_8) i \sin \frac{\theta_4 n x}{a} \right] \right\} \sin n\phi \sin \omega t. \quad (4.72) \]

\[ \omega_c(x, \phi, t) = \sum_{n=0}^{\infty} \left\{ e^{\frac{\ln x}{a}} \left[ (C_1 + C_2) \cos \frac{\theta_1 x}{a} + (C_1 - C_2) i \sin \frac{\theta_1 x}{a} \right] \\
+ e^{\frac{\ln a}{2n}} \left[ (C_3 + C_4) \cos \frac{\theta_2 n x}{a} + (C_3 - C_4) i \sin \frac{\theta_2 n x}{a} \right] \\
+ e^{\frac{\ln a}{3n}} \left[ (C_5 + C_6) \cos \frac{\theta_3 n x}{a} + (C_5 - C_6) i \sin \frac{\theta_3 n x}{a} \right] \\
+ e^{\frac{\ln a}{4n}} \left[ (C_7 + C_8) \cos \frac{\theta_4 n x}{a} + (C_7 - C_8) i \sin \frac{\theta_4 n x}{a} \right] \right\} \cos n\phi \sin \omega t \quad (4.73) \]

Examination of Equation (4.73) shows that it is convenient to define a new set of arbitrary constants as follows:
\[ \bar{c}_1 = c_1 + c_2, \]
\[ \bar{c}_2 = (c_1 - c_2)i, \]
\[ \bar{c}_3 = c_3 + c_4, \]
\[ \bar{c}_4 = (c_3 - c_4)i, \]
\[ \bar{c}_5 = c_5 + c_6, \]
\[ \bar{c}_6 = (c_5 - c_6)i, \]
\[ \bar{c}_7 = c_7 + c_8, \]
\[ \bar{c}_8 = (c_7 - c_8)i. \]  

(4.74)

The constants \( A_i \) and \( B_i \) can be determined in terms of the \( C_i \) for each integer \( n \). Examination of (4.63) reveals that it may be written as follows:

\[ A_i = \alpha_i C_i \]  

(4.75)

where

\[ \alpha_i = \bar{a}_1 + i \bar{a}_2, \]
\begin{align*}
\alpha_2 &= \bar{\alpha}_1 + i\bar{\alpha}_2, \\
\alpha_3 &= \bar{\alpha}_3 + i\bar{\alpha}_4, \\
\alpha_4 &= \bar{\alpha}_3 - i\bar{\alpha}_4, \\
\alpha_5 &= \bar{\alpha}_5 - i\bar{\alpha}_6, \\
\alpha_6 &= \bar{\alpha}_5 - i\bar{\alpha}_6, \\
\alpha_7 &= \bar{\alpha}_7 + i\bar{\alpha}_8, \\
\alpha_8 &= \bar{\alpha}_7 - i\bar{\alpha}_8. \\
\end{align*}

(4.76)

By employing Equations (4.74), (4.75), and (4.76) the arbitrary coefficients of (4.71) can be written in the following manner:

\begin{align*}
A_1 + A_2 &= \bar{\alpha}_1 \bar{c}_1 + \bar{\alpha}_2 \bar{c}_2, \\
A_3 + A_4 &= \bar{\alpha}_3 \bar{c}_3 + \bar{\alpha}_4 \bar{c}_4, \\
A_5 + A_6 &= \bar{\alpha}_5 \bar{c}_5 + \bar{\alpha}_6 \bar{c}_6, \\
A_7 + A_8 &= \bar{\alpha}_7 \bar{c}_7 + \bar{\alpha}_8 \bar{c}_8, \\
\end{align*}
\[(A_1 - A_2)i = -\bar{\alpha}_2 \bar{C}_1 + \bar{\alpha}_1 \bar{C}_2,\]
\[(A_3 - A_4)i = -\bar{\alpha}_4 \bar{C}_3 + \bar{\alpha}_3 \bar{C}_4,\]
\[(A_5 - A_6)i = -\bar{\alpha}_6 \bar{C}_5 + \bar{\alpha}_5 \bar{C}_6,\]

and

\[(A_7 - A_8)i = -\bar{\alpha}_8 \bar{C}_7 + \bar{\alpha}_7 \bar{C}_8.\]  \hspace{1cm} (4.77)

Also, (4.64) has the form:

\[B_i = \beta_i C_i\]  \hspace{1cm} (4.78)

where

\[\beta_1 = \bar{\beta}_1 + i\bar{\beta}_2,\]
\[\beta_2 = \bar{\beta}_1 - i\bar{\beta}_2,\]
\[\beta_3 = \bar{\beta}_3 + i\bar{\beta}_4,\]
\[\beta_4 = \bar{\beta}_3 - i\bar{\beta}_4,\]
\[\beta_5 = \bar{\beta}_5 + i\bar{\beta}_6,\]
\[ \beta_6 = \tilde{\beta}_6 - i\tilde{\beta}_6, \]

\[ \beta_7 = \tilde{\beta}_7 + i\tilde{\beta}_8, \]

and

\[ \beta_8 = \tilde{\beta}_7 - i\tilde{\beta}_8. \] (4.79)

With the help of (4.74), (4.78), and (4.79), the arbitrary coefficients of (4.72) can be shown to be:

\[ B_1 + B_2 = \tilde{\beta}_1 \tilde{c}_1 + \tilde{\beta}_2 \tilde{c}_2 \]

\[ B_3 + B_4 = \tilde{\beta}_3 \tilde{c}_3 + \tilde{\beta}_4 \tilde{c}_4 \]

\[ B_5 + B_6 = \tilde{\beta}_5 \tilde{c}_5 + \tilde{\beta}_6 \tilde{c}_6 \]

\[ B_7 + B_8 = \tilde{\beta}_7 \tilde{c}_7 + \tilde{\beta}_8 \tilde{c}_8 \]

\[ (B_1 - B_2)i = \tilde{\beta}_1 \tilde{c}_2 + \tilde{\beta}_2 \tilde{c}_1 \]

\[ (B_3 - B_4)i = \tilde{\beta}_3 \tilde{c}_4 - \tilde{\beta}_4 \tilde{c}_3 \]

\[ (B_5 - B_6)i = \tilde{\beta}_5 \tilde{c}_6 - \tilde{\beta}_6 \tilde{c}_5 \]
Inserting Equations (4.77) and (4.80) into Equations (4.71), (4.72), and (4.73) yields the final form of the solution in terms of the new set of eight arbitrary constants $\tilde{C}_i$ and the real terms $\tilde{a}_i$ and $\tilde{\beta}_i$ as follows:

$$u^S = \left\{ \sum_{n=0}^{\infty} \left( e^{\psi_{ln}} \ln a \right) \left[ (a_{1n} \tilde{C}_n + \tilde{a}_n \tilde{C}_n) \cos \theta \ln \frac{x}{a} + \right. \right.$$  

$$+ (\tilde{a}_{1n} \tilde{C}_n - \tilde{a}_n \tilde{C}_n) \sin \theta \ln \frac{x}{a} \bigg] $$  

$$+ e^{\psi_{2n}} \frac{x}{a} \left[ (\tilde{a}_{3n} \tilde{C}_n + \tilde{a}_{4n} \tilde{C}_n) \cos \theta \ln \frac{x}{a} + (\tilde{a}_{3n} \tilde{C}_n - \tilde{a}_{4n} \tilde{C}_n) \sin \theta \ln \frac{x}{a} \right] $$  

$$+ e^{\psi_{3n}} \frac{x}{a} \left[ (\tilde{a}_{5n} \tilde{C}_n + \tilde{a}_{6n} \tilde{C}_n) \cos \theta \ln \frac{x}{a} + (\tilde{a}_{5n} \tilde{C}_n - \tilde{a}_{6n} \tilde{C}_n) \sin \theta \ln \frac{x}{a} \right] $$  

$$+ e^{\psi_{4n}} \frac{x}{a} \left[ (\tilde{a}_{7n} \tilde{C}_n + \tilde{a}_{8n} \tilde{C}_n) \cos \theta \ln \frac{x}{a} + \right.$$

$$+ (\tilde{a}_{7n} \tilde{C}_n - \tilde{a}_{8n} \tilde{C}_n) \sin \theta \ln \frac{x}{a} \bigg) \cos n\theta \sin \omega t \right\} $$

$$v^S = \left\{ \sum_{n=1}^{\infty} \left( e^{\psi_{ln}} \ln a \right) \left[ (\tilde{b}_{1n} \tilde{C}_n + \tilde{b}_n \tilde{C}_n) \cos \theta \ln \frac{x}{a} + \right. \right.$$  

$$+ (\tilde{b}_{1n} \tilde{C}_n - \tilde{b}_n \tilde{C}_n) \sin \theta \ln \frac{x}{a} \bigg] $$  

$$+ e^{\psi_{2n}} \frac{x}{a} \left[ (\tilde{b}_{3n} \tilde{C}_n + \tilde{b}_{4n} \tilde{C}_n) \cos \theta \ln \frac{x}{a} + (\tilde{b}_{3n} \tilde{C}_n - \tilde{b}_{4n} \tilde{C}_n) \sin \theta \ln \frac{x}{a} \right] $$

$$+ e^{\psi_{3n}} \frac{x}{a} \left[ (\tilde{b}_{5n} \tilde{C}_n + \tilde{b}_{6n} \tilde{C}_n) \cos \theta \ln \frac{x}{a} + (\tilde{b}_{5n} \tilde{C}_n - \tilde{b}_{6n} \tilde{C}_n) \sin \theta \ln \frac{x}{a} \right] $$

$$+ e^{\psi_{4n}} \frac{x}{a} \left[ (\tilde{b}_{7n} \tilde{C}_n + \tilde{b}_{8n} \tilde{C}_n) \cos \theta \ln \frac{x}{a} + (\tilde{b}_{7n} \tilde{C}_n - \tilde{b}_{8n} \tilde{C}_n) \sin \theta \ln \frac{x}{a} \right] $$
where the complex constants $A_{sn}$, $B_{sn}$, and $C_{sn}$ have been replaced by the real terms $\tilde{A}_{sn}$ and $\tilde{B}_{sn}$ and the real arbitrary constants $\tilde{\phi}_{ln}$, $\psi_{ln}$, $\psi_{2n}$, $\psi_{3n}$, and $\psi_{4n}$ are the real parts and $\phi_{ln}$, $\theta_{2n}$, $\theta_{3n}$, and $\theta_{4n}$ are the imaginary parts of the roots of Equation (4.66). The quantities $\tilde{A}_{ln}$ and $\tilde{B}_{ln}$ are given in Appendix E.

The functions $e^{\frac{x}{\ln a}} \cos \frac{\theta}{\ln a}$, $e^{\frac{x}{\ln a}} \sin \frac{\theta}{\ln a}$, $e^{\frac{x}{\ln a}} \cos \frac{\theta}{2n a}$, $e^{\frac{x}{\ln a}} \sin \frac{\theta}{2n a}$, $e^{\frac{x}{\ln a}} \cos \frac{\theta}{3n a}$, $e^{\frac{x}{\ln a}} \sin \frac{\theta}{3n a}$, $e^{\frac{x}{\ln a}} \cos \frac{\theta}{4n a}$, and $e^{\frac{x}{\ln a}} \sin \frac{\theta}{4n a}$ which occur in the complementary solution satisfy Dirichlet's conditions for each integer $n$ and are continuous in the closed interval $[0, \pi]$. Therefore, they can be expanded
in uniformly convergent Fourier cosine series by extending them as even functions as follows:

\[
\psi_{\ln a} \frac{x}{a} \cos \theta \ln \frac{x}{a} = \sum_{m=0}^{\infty} t_{1mn} \cos \alpha n x, \quad (4.84)
\]

\[
\psi_{\ln a} \frac{x}{a} \sin \theta \ln \frac{x}{a} = \sum_{m=0}^{\infty} t_{2mn} \cos \alpha n x, \quad (4.85)
\]

\[
\psi_{2n a} \frac{x}{2n a} \cos \theta \frac{x}{2n a} = \sum_{m=0}^{\infty} t_{3mn} \cos \alpha n x, \quad (4.86)
\]

\[
\psi_{2n a} \frac{x}{2n a} \sin \theta \frac{x}{2n a} + \sum_{m=0}^{\infty} t_{4mn} \cos \alpha n x, \quad (4.87)
\]

\[
\psi_{3n a} \frac{x}{3n a} \cos \theta \frac{x}{3n a} = \sum_{m=0}^{\infty} t_{5mn} \cos \alpha n x, \quad (4.88)
\]

\[
\psi_{3n a} \frac{x}{3n a} \sin \theta \frac{x}{3n a} = \sum_{m=0}^{\infty} t_{6mn} \cos \alpha n x, \quad (4.89)
\]

\[
\psi_{4n a} \frac{x}{4n a} \cos \theta \frac{x}{4n a} = \sum_{m=0}^{\infty} t_{7mn} \cos \alpha n x, \quad (4.90)
\]

\[
\psi_{4n a} \frac{x}{4n a} \sin \theta \frac{x}{4n a} = \sum_{m=0}^{\infty} t_{8mn} \cos \alpha n x, \quad (4.91)
\]

where
\[
\psi_{l_{mn}} = \frac{(-1)^m e^{\frac{m}{a} \ln a} \left[ \frac{\psi_{l_{mn}}}{a} \cos \theta_{l_{mn}} - \frac{\theta_{l_{mn}}}{a} \sin \theta_{l_{mn}} \right]}{a^2 + \left( \frac{\theta_{l_{mn}}}{a} + \frac{m}{\xi} \right)^2} - \frac{\psi_{l_{mn}}}{a}, \quad (4.92)
\]

\[
\psi_{l_{2mn}} = \frac{(-1)^m e^{\frac{m}{a} \ln a} \left[ \frac{\psi_{l_{mn}}}{a} \sin \theta_{l_{mn}} - \frac{\theta_{l_{mn}}}{a} \cos \theta_{l_{mn}} \right]}{a^2 + \left( \frac{\theta_{l_{mn}}}{a} + \frac{m}{\xi} \right)^2} + \frac{\theta_{l_{mn}}}{a} + \frac{m}{\xi}
\]

\[
\psi_{l_{3mn}} = \frac{(-1)^m e^{\frac{m}{a} \ln a} \left[ \frac{\psi_{l_{mn}}}{a} \sin \theta_{l_{mn}} - \frac{\theta_{l_{mn}}}{a} \cos \theta_{l_{mn}} \right]}{a^2 + \left( \frac{\theta_{l_{mn}}}{a} - \frac{m}{\xi} \right)^2} + \frac{\theta_{l_{mn}}}{a} - \frac{m}{\xi}
\]

with similar expressions for the Fourier coefficients \( t_{3mn}, t_{4mn}, t_{5mn}, t_{6mn}, t_{7mn}, \) and \( t_{8mn} \) which can be obtained from \( t_{1mn} \) and \( t_{2mn} \) by
by replacing $\psi_{1n}$ and $\theta_{1n}$ with either $\psi_{2n}$, $\theta_{2n}$ or $\psi_{3n}$, $\theta_{3n}$ or $\psi_{4n}$, $\theta_{4n}$.

The functions occurring in the complementary solution may also be expanded in uniformly convergent Fourier sine series as follows:

\[
\psi_{1n} \frac{x}{\tan \theta_{1n}} = \sum_{m=1}^{\infty} r_{1mn} \sin \alpha m x, \quad (4.94)
\]

\[
\psi_{2n} \frac{x}{\tan \theta_{2n}} = \sum_{m=1}^{\infty} r_{2mn} \sin \alpha m x, \quad (4.95)
\]

\[
\psi_{3n} \frac{x}{\tan \theta_{3n}} = \sum_{m=1}^{\infty} r_{3mn} \sin \alpha m x, \quad (4.96)
\]

\[
\psi_{4n} \frac{x}{\tan \theta_{4n}} = \sum_{m=1}^{\infty} r_{4mn} \sin \alpha m x, \quad (4.97)
\]

\[
\psi_{5n} \frac{x}{\tan \theta_{5n}} = \sum_{m=1}^{\infty} r_{5mn} \sin \alpha m x, \quad (4.98)
\]

\[
\psi_{6n} \frac{x}{\tan \theta_{6n}} = \sum_{m=1}^{\infty} r_{6mn} \sin \alpha m x, \quad (4.99)
\]

\[
\psi_{7n} \frac{x}{\tan \theta_{7n}} = \sum_{m=1}^{\infty} r_{7mn} \sin \alpha m x, \quad (4.100)
\]

\[
\psi_{8n} \frac{x}{\tan \theta_{8n}} = \sum_{m=1}^{\infty} r_{8mn} \sin \alpha m x, \quad (4.101)
\]

where
with similar expressions for the Fourier coefficients $r_{3mn}$, $r_{4mn}$, $r_{5mn}$, $r_{6mn}$, $r_{7mn}$, and $r_{8mn}$.

By employing Equations (4.84) through (4.103), the symmetric part of the complementary solution may be written as:

$$ u_c^s(x, \phi) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c_{1n} \alpha_{1n, \ln 1mn} - \alpha_{2n, 2mn} \sin \alpha_n x \cos n\phi + \ldots $$
\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \overline{c}_{2n}(\alpha_{1n} r_{2mn} + \tilde{\alpha}_{2n} r_{1mn}) \sin \alpha_m \times \cos n\phi +
\]

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \overline{c}_{3n}(\alpha_{3n} r_{3mn} - \tilde{\alpha}_{4n} r_{4mn}) \sin \alpha_m \times \cos n\phi +
\]

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \overline{c}_{4n}(\alpha_{5n} r_{4mn} + \tilde{\alpha}_{4n} r_{3mn}) \sin \alpha_m \times \cos n\phi +
\]

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \overline{c}_{5n}(\alpha_{5n} r_{5mn} - \tilde{\alpha}_{6n} r_{6mn}) \sin \alpha_m \times \cos n\phi +
\]

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \overline{c}_{6n}(\alpha_{5n} r_{6mn} + \tilde{\alpha}_{6n} r_{5mn}) \sin \alpha_m \times \cos n\phi +
\]

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \overline{c}_{7n}(\alpha_{7n} r_{7mn} - \tilde{\alpha}_{8n} r_{8mn}) \sin \alpha_m \times \cos n\phi +
\]

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \overline{c}_{8n}(\alpha_{7n} r_{8mn} + \tilde{\alpha}_{8n} r_{7mn}) \sin \alpha_m \times \cos n\phi, \quad (4.104)
\]

\[
v^c_c(x, \phi) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \overline{c}_{1n}(\tilde{\beta}_{1n} t_{1mn} - \beta_{2n} t_{2mn}) \cos \alpha_m \times \sin n\phi +
\]

\[
\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \overline{c}_{2n}(\tilde{\beta}_{1n} t_{2mn} + \beta_{2n} t_{1mn}) \cos \alpha_m \times \sin n\phi +
\]

\[
\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \overline{c}_{3n}(\tilde{\beta}_{3n} t_{3mn} - \beta_{4n} t_{4mn}) \cos \alpha_m \times \sin n\phi +
\]
\begin{align*}
&+ \phi u \cos \theta \sin^m \phi \cos n \phi \cos u \cos \theta \cos \phi \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \\
&+ \phi u \cos \theta \sin^m \phi \cos n \phi \cos u \cos \theta \cos \phi \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \\
&+ \phi u \cos \theta \sin^m \phi \cos n \phi \cos u \cos \theta \cos \phi \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \\
&+ \phi u \cos \theta \sin^m \phi \cos n \phi \cos u \cos \theta \cos \phi \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \\
&+ \phi u \cos \theta \sin^m \phi \cos n \phi \cos u \cos \theta \cos \phi \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \cos \theta \cos \phi \cos u \\
&= (\psi \times \mathbf{A})_n
\end{align*}
\[ u_p = \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \{ g_{5} Z_{jmn} + g_{6} T_{jmn} + g_{7} Y_{jmn} + g_{8} X_{jmn} \} \sin \alpha x \cos \phi, \quad (4.107) \]

\[ v_p = \sum_{j=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \{ g_{3} Z_{jmn} + g_{3} T_{jmn} + g_{4} Y_{jmn} + g_{5} X_{jmn} \} \cos \alpha x \sin \phi, \quad (4.108) \]

\[ w_p = \sum_{j=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{ g_{1} Z_{jmn} + g_{1} T_{jmn} + g_{2} Y_{jmn} + g_{7} X_{jmn} \} \cos \alpha x \cos \phi, \quad (4.109) \]

where \( g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, \) and \( g_9 \) are functions of \( m, n, \rho, \)
\( \nu, E, h, a, \) and \( w^2 \) given in Appendix F.

Therefore, the symmetric part of the general solution of (4.4),
(4.5), and (4.6) is given by:

\[ u^S = u_c + u_p, \quad (4.110) \]
\( v^s = v^s_c + v^s_p \) \hspace{1cm} (4.111)

\( w^s = w^s_c + w^s_p \) \hspace{1cm} (4.112)

The antisymmetric part of the complementary solution of Donnell's equations may be readily obtained by the replacement of \( \sin n\phi \) for \( \cos n\phi \) and \( -\cos n\phi \) for \( \sin n\phi \) and replacing \( \bar{c}_{sn} \) by \( \bar{c}'_{sn} \) in (4.104), (4.105), and (4.106). This solution is denoted by \( u^a_c, v^a_c, \) and \( w^a_c \). Similarly, the particular solution of (4.4), (4.5), and (4.6) for the antisymmetric part of the loading can be obtained by the replacement of \( \sin n\phi \) for \( \cos n\phi \) and \( -\cos n\phi \) for \( \sin n\phi \) in (4.107), (4.108), and (4.109). This solution is denoted by \( u^a_p, v^a_p, \) and \( w^a_p \).

Therefore, the antisymmetric part of the general solution of (4.4), (4.5), and (4.6) is given by:

\( u^a = u^a_c + u^a_p \) \hspace{1cm} (4.113)

\( v^a = v^a_c + v^a_p \) \hspace{1cm} (4.114)

and

\( w^a = w^a_c + w^a_p \) \hspace{1cm} (4.115)

and the general solution obtained by superposing solutions is given by:
Equations of Motion for the Stringers

The assumptions made concerning the stringers are the same as those given in Appendix A. V. Z. Vlasov's equations for extension, bending, and torsional vibrations of a thin-walled beam neglecting rotatory inertia (see Figure 11) are:

\[ u = u^S + u^a, \quad (4.116) \]

\[ v = v^S + v^a, \quad (4.117) \]

\[ w = w^S + w^a. \quad (4.118) \]

\[ E_j A_j \frac{d^2 u}{dx^2} + \rho_j A_j \omega^2 u_{o_j} = x_j, \quad (4.119) \]

\[ E_j I_j z_j \frac{d^4 v}{dx^4} + E_j I_j y_j \frac{d^4 w}{dx^4} + \rho_j I_j \omega^2 \frac{d^2 v}{dx^2} - \rho_j A_j \omega^2 v_{o_j} + \rho_j A_j a_j \omega^2 \theta_{o_j} = y_j, \quad (4.120) \]

\[ E_j I_j y_j \frac{d^4 w}{dx^4} + E_j I_j z_j \frac{d^4 v}{dx^4} + \rho_j I_j \omega^2 \frac{d^2 w}{dx^2} - \rho_j A_j \omega^2 w_{o_j} - \rho_j A_j a_j \omega^2 \theta_{o_j} = z_j, \quad (4.121) \]

and
\[ \frac{d^4 \theta_{oj}}{d^2 \theta_{oj}} + \frac{d^2 \theta_{oj}}{dx} - (GJ)_{ij} \frac{d^2 \theta_{ij}}{dx^2} + \rho_j A_j \omega^2 z_j v_{oj} - \rho_j A_j a_j \omega^2 w_{oj} - \rho_j I_{oj} \omega^2 \theta_{oj} = T_j + z_j Y_j, \quad (4.122) \]

where the following notation has been employed:

- \( u_{oj} \) = deflection of shear center in the x-direction,
- \( v_{oj} \) = deflection of shear center in the y-direction,
- \( w_{oj} \) = deflection of shear center in the z-direction,
- \( \theta_{oj} \) = rotation of the cross section,
- \( I_{yj}, I_{zj} \) = principal centroidal moments of inertia,
- \( A_j \) = cross sectional area,
- \( J \) = St. Venant constant for uniform torsion,
- \( \rho_j \) = mass density,
- \( I_{oj} \) = polar moment of inertia about the centroid,
- \( j \) = subscript indicating the stringer at location,
- \( X_j,Y_j \) = resultant longitudinal and tangential forces transmitted from the shell to the stringer,
- \( Z_j \) = resultant radial force,
- \( T_j \) = resultant torque,
- \( z_j \) = perpendicular distance from attachment line to shear center.
The first of V. Z. Vlasov's equations determines the longitudinal vibrations of the thin-walled rib and is not coupled to the other equations. The remaining three equations constitute a coupled system of ordinary linear differential equations with constant coefficients for the bending-torsional response of the stringers. Since each equation is of fourth order, their complementary solution will contain twelve arbitrary constants. Since the characteristic equation for this system will be a polynomial of degree 12, it is not possible to find a closed form expression for the solution. Furthermore, the precise form of the complementary solutions to these equations depends upon the relative magnitudes of the cross sectional constants.

For stringers with certain geometric properties, it is possible to uncouple the stringer equations of motion and thereby obtain a closed form solution. The three cases which fall into this category are given by
(1) the shear center of the stringer coincides with the centroid and the stringer has two axes of symmetry \( (I_{y'z'} = 0) \),

(2) the shear center lies on a vertical line above or below the centroid \( (a_y = 0, a_z \neq 0) \) and has a single axis of symmetry \( (I_{y'z'} = 0) \), and

(3) the shear center coincides with the centroid but the stringer has a nonsymmetrical cross section \( (I_{y'z'} \neq 0) \).

For the sake of lightweight construction and ease of fabrication, the stringers used in the aircraft industry are generally thin-walled members of which the \( (\text{I}) \) rectangular-shape, \( (\text{II}) \) hat-shape, and \( (\text{T}) \)-shape are frequent examples. These examples illustrate cases 1, 2, and 3, respectively. Therefore, these cases are technically important as well as of academic interest and are now discussed in some detail.

For the stringer with two axes of symmetry, the equations of motion for the bending-torsional response will be uncoupled. Inserting Equations (4.29), (4.30), (4.31), and (4.32) into Equations (4.119), (4.120), (4.121), and (4.122), and letting \( (a_y = a_z = I_{yz} = 0) \) yields the following set of equations:

\[
E_j A_j \frac{d^2 u_j}{dx^2} + \rho_j A_j \omega^2 u_{oj} = \sum_{m=0}^{\infty} X_{jm} \sin \frac{\text{m} x}{L},
\]

\[
E_j I_{jz} \frac{d^2 u_j}{dx^2} + \rho_j I_{jz} \omega^2 v_{oj} = \sum_{m=0}^{\infty} Y_{jm} \cos \frac{\text{m} x}{L},
\]
\[
E_j I_j \frac{d^2 \omega_{ij}}{dx^2} + \rho_j I_j \frac{d^2 \phi_{ij}}{dx^2} \omega^2 - \rho_j A_j \omega^2 \phi_{ij} = \sum_{m=0}^{\infty} z_{mj} \cos \frac{m \pi x}{l} \tag{4.125}
\]

\[
E_j I_j \frac{d^2 \phi_{ij}}{dx^2} - (Gj)_j \frac{d^2 \phi_{ij}}{dx^2} - \rho_j I_j \phi_{ij} = \sum_{m=0}^{\infty} T_{jm} \cos \frac{m \pi x}{l} + z_j \sum_{m=0}^{\infty} Y_{jm} \cos \frac{m \pi x}{l} \tag{4.126}
\]

The general solution of these equations is given by

\[
u_{ij}(x) = U_{ij} \cos \gamma_{ij} x + U_{ij} \sin \gamma_{ij} x + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{lm}^m \cos \frac{m \pi x}{l} \sin \frac{m \pi x}{n} \tag{4.127}
\]

\[
u_{ij}(x) = D_{ij} e^{\mu_{ij} x} + D_{ij} e^{-\mu_{ij} x} + D_{ij} \cos \mu_{ij} x + D_{ij} \sin \mu_{ij} x + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mu_{lm} \cos \alpha_{mn} x \tag{4.128}
\]

\[
\theta_{ij}(x) = B_{ij} \cosh \gamma_{ij} x + B_{ij} \sinh \gamma_{ij} x + B_{ij} \cos \gamma_{ij} x + B_{ij} \sin \gamma_{ij} x + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \xi_{mn} \cos \frac{m \pi x}{l} \sin \frac{m \pi x}{n} \tag{4.129}
\]

where
\[
\gamma_{4j} = \frac{\rho_j \omega^2}{E_j} , \quad \xi_{4m} = \frac{1}{E_j A_j (\gamma_{4j}^2 - \alpha_{4m}^2)} , \quad (4.131)
\]

\(\gamma_{2j}, \gamma_{3j}\) are positive real quantities defined by

\[
\gamma_{2j} = \left\{ \frac{(GJ)_j + \sqrt{(GJ)_j^2 + 4E_j^\ast C_{wj} \rho_j i_o j \omega^2}}{2E_j^\ast C_{wj}} \right\}^{\frac{1}{2}} , \quad (4.132)
\]

\[
\gamma_{3j} = \left\{ \frac{-(GJ)_j + \sqrt{(GJ)_j^2 + 4E_j^\ast C_{wj} \rho_j i_o j \omega^2}}{2E_j^\ast C_{wj}} \right\}^{\frac{1}{2}} , \quad (4.133)
\]

and

\[
\xi_{1j} = \left\{ \frac{-\rho \omega^2}{2E_j} + \sqrt{\frac{\rho}{E_j}} \omega^2 + \frac{4\rho_j A_j}{i z_j} \omega^2 \right\}^{\frac{1}{2}} , \quad (4.134)
\]

\[
\xi_{2j} = \left\{ \frac{\rho \omega^2}{2E_j} + \sqrt{\frac{\rho}{E_j}} \omega^2 + \frac{4\rho_j A_j}{i y_j} \omega^2 \right\}^{\frac{1}{2}} , \quad (4.135)
\]
If one neglects rotatory inertia of the stringers, then the solutions for \( v_{o_j}(x) \) and \( w_{o_j}(x) \) become:

\[
v_{o_j}(x) = D_{1j} \cosh y_{o_j} x + D_{2j} \sinh y_{o_j} x + D_{3j} \cos y_{o_j} x +
\]

\[
w_{o_j}(x) =
\]
\[ D_{ij} \sin \gamma_{oj} x + \sum_{m=0}^{\infty} \zeta_{om} N_{jm} \cos \frac{m\pi x}{L}, \quad (4.141) \]

\[ w_{oj}(x) = A_{1j} \cosh \gamma_{1j} x + A_{2j} \sinh \gamma_{1j} x + A_{3j} \cos \gamma_{1j} x + \]

\[ + A_{4j} \sin \gamma_{1j} x + \sum_{m=0}^{\infty} \zeta_{1m} Z_{jm} \cos \frac{m\pi x}{L}, \quad (4.142) \]

where

\[ \gamma_{oj} = \frac{\rho_{ij} \omega^2}{E_{ij} z_j}, \quad \zeta_{om} = \frac{1}{E_{ij} z_j (\alpha_m - \gamma_{oj})}, \quad (4.143) \]

and

\[ \gamma_{1j} = \frac{\rho_{ij} \omega^2}{E_{ij} y_j}, \quad \zeta_{1m} = \frac{1}{E_{ij} y_j (\alpha_m - \gamma_{1j})}, \quad (4.144) \]

For the case of a stringer with one axis of symmetry (i.e., \( \perp \perp \)) such that \( I_{yz} = a_y = 0 \) and \( a_z \neq 0 \), then the assumption is made that \( a_z = 0 \) and this case reduces back to case (1). The \( a_z \) term appears only in the \( m_{23} \) sub-matrix for the kinetic energy and for the stringers investigated \( \perp \perp \) its effect was insignificant. If the stringers are equally-spaced, then the contribution of this term is zero anyway.

For the case of the nonsymmetrical section in which the shear center coincides with the centroid \( (a_y = a_z = 0, I_{yz} \neq 0) \) then the bending vibrations in the two perpendicular directions are uncoupled with the torsional vibrations and the equations reduce to the problem
of "double coupling." For this case, the characteristic equation will be a polynomial of eighth order and one must resort to an approximate solution. However, if one neglects rotatory inertia of the stringers, then for this case, Equations (4.120) and (4.121) become for the homogenous part:

\[(D^4 - b_1)v + (b_2 D^4)w = 0, \quad (4.145)\]

\[(b_4 D^4)v + (D^4 - b_3)w = 0, \quad (4.146)\]

where \(D = \frac{d}{dx}\) and the constants \(b_1, b_2, b_3,\) and \(b_4\) are given as follows:

\[
b_1 = \frac{\rho A \omega^2}{E_i z_j}, \quad b_2 = \frac{I y z_i}{I z_j}, \quad b_3 = \frac{\rho A \omega^2}{E_j y_j}, \quad b_4 = \frac{I y z_i}{I y_j}. \quad (4.147)\]

Reduction of Equations (4.145) and (4.146) to "triangular form" yields the following two equations:

\[(D^4 - b_1)v + b_2 D^4 w = 0, \quad (4.148)\]

\[
[(1 - b_2 b_4)D^8 - (b_1 + b_2)D^4 + b_1 b_2]w = 0. \quad (4.149)\]

The characteristic equation for \(w\) can be written as:

\[r^8 - b_5 r^4 + b_6 = 0\]
the constants $b_5$ and $b_6$ are given by:

$$b_5 = \frac{I \cdot (E^2 I_{yz} I_{zj} + \rho_i A_i \omega^2)}{(I_{yz} I_{zj} - I_{yz}) E_j^i}, \quad (4.150)$$

and

$$b_6 = \frac{I \cdot (I_{yz} - I^2) \rho_i A_i \omega^2}{(I_{yz} I_{zj} - I_{yz}) E_j^i}, \quad (4.151)$$

This equation can be solved by making the substitution $\lambda = r^4$

which yields the following quadratic equation

$$\lambda^2 - b_5 \lambda + b_6 = 0 \quad (4.152)$$

which has solutions

$$\lambda = \frac{b_5 \pm \sqrt{b_5^2 - 4b_6}}{2}. \quad (4.153)$$

Let us denote the discriminant by $D$. For a given set of stringer parameters and a given $\omega$, one has the following three possibilities:

Case (i) if $D < 0$, then the roots are complex conjugates

$$\lambda_{1,2} = a_i \pm ia_j, \quad (4.154)$$
Case (ii) if $D_1 > 0$, then the roots are real and unequal

$$\lambda_1 = \alpha_1, \quad \lambda_2 = \alpha_2, \quad (4.155)$$

Case (iii) if $D_1 = 0$, then the roots are real, equal,

$$\lambda_1 = \alpha_1, \quad \lambda_2 = \alpha_1, \quad (4.156)$$

The roots of Equation (4.152) for case (i) can be put in the form:

$$r_{1,2} = \pm(i\gamma_5 + i\gamma_6), \quad (4.157)$$

$$r_{3,4} = \pm(i\gamma_6 + i\gamma_5), \quad (4.158)$$

$$r_{5,6} = \pm(i\gamma_5 + i\gamma_6), \quad (4.159)$$

$$r_{7,8} = \pm(-i\gamma_6), \quad (4.160)$$

For case (ii), the roots can be written as:

$$r_{1,2} = \pm\gamma_5; \quad r_{3,4} = \pm i\gamma_5; \quad r_{5,6} = \pm i\gamma_6; \quad r_{7,8} = i\gamma_6 \quad (4.161)$$

and for case (iii), we have repeated roots of the form
For any of these cases, the general solution for $w_{ij}$ may be written down. This solution is denoted by:

$$ w_{ij}(x) = \sum_{i=1}^{g} \sum_{j} A_{ij} w_{i}(x) $$  \hspace{1cm} (4.163)

where $w_{i}(x)$ denote the linearly independent functions of the complementary solution. The solution for $v$ may be recovered by standard techniques from Equation (4.148). The solution is denoted by:

$$ v_{ij}(x) = \sum_{i=1}^{g} \sum_{j} d_{ij} v_{i}(x) $$  \hspace{1cm} (4.164)

where $v_{i}(x)$ denote the functions occurring in the complementary solution and the $d_{ij}$ are known real constants which depend on the solution for $w_{ij}(x)$.

The complete solution is given by:

$$ v_{ij}(x) = \sum_{i=1}^{g} \sum_{j} d_{ij} v_{i}(x) - \sum_{m=0}^{\infty} \zeta_{0m} V_{m} \cos \alpha_{m} x $$

$$ + \sum_{m=0}^{\infty} \zeta_{0m} Z_{m} \cos \alpha_{m} x $$  \hspace{1cm} (4.165)

$$ w_{ij}(x) = \sum_{i=1}^{g} \sum_{j} A_{ij} w_{i}(x) + \sum_{m=0}^{\infty} \zeta_{1m} V_{m} \cos \alpha_{m} x $$

$$ - \sum_{m=0}^{\infty} \zeta_{1m} Z_{m} \cos \alpha_{m} x $$  \hspace{1cm} (4.166)
where

\[ \zeta_{om} = \left( \sum_{j} \xi_{yzm} a_{m} / \text{det} \right), \]  

(4.167)

and

\[ \xi_{lm} = \left( \sum_{j} \xi_{yzm} a_{m} / \text{det} \right), \]  

(4.168)

\[ \zeta_{om} = \left( \sum_{j} \xi_{yzm} a_{m} / \text{det} \right), \]  

(4.169)

\[ \xi_{lm} = \left( \sum_{j} \xi_{yzm} a_{m} / \text{det} \right), \]  

(4.170)

and

\[ \text{det} = \left( \sum_{j} \xi_{yzm} a_{m} / \text{det} \right) - \sum_{j} \xi_{yzm} a_{m} / \text{det}, \]  

(4.171)

### Boundary Conditions for Stringers

For free vibration, the general solutions of the stringer equations will be required to satisfy homogenous boundary conditions such as any combination of clamped supported, simply supported or free supported ends. Applying a given set of boundary conditions enables one to express the arbitrary constants occurring in the general solutions in terms of the loading parameters as follows:

\[ U_{ij} = \sum_{m=0}^{\infty} N_{im} z_{4m} x_{jm}, \quad (i=1,2) \]  

(4.172)
\[ B_{ij} = \sum_{m=0}^{\infty} G_{im} \varepsilon_{2m} T_{jm} + \sum_{m=0}^{\infty} G_{im} \varepsilon_{3m} Y_{jm}, \quad (i=1,2,3,4) \quad (4.173) \]

where the coefficients \( N_{im} \) and \( G_{im} \) have different forms for different boundary conditions. The arbitrary constants \( A_{ij} \) which occur in Equations (4.165) and (4.166) can be related to the external loading coefficients \( Z_{jm} \) and \( Y_{jm} \) in terms of a matrix equation as follows:

\[
\begin{bmatrix}
A_{1j} \\
A_{2j} \\
\vdots \\
A_{8j}
\end{bmatrix}
= 
\begin{bmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_8
\end{bmatrix}, \quad (j=1,2,\ldots,k) \quad (4.174)
\]

where the coefficient matrix \([B]_j\) and the external loading matrices \([Q]_j\) have different forms depending on the type of boundary conditions applied to the beam and the exact forms of the \( V(x) \) and \( W(x) \). Since the matrix \([B]\) is nonsingular, one can express the unknown constants in terms of the applied loading by the following equation:

\[
\begin{bmatrix}
A_{1j} \\
A_{2j} \\
A_{3j} \\
\vdots \\
A_{8j}
\end{bmatrix}
= [B]^{-1}_j \begin{bmatrix}
Q_{1j} \\
Q_{2j} \\
Q_{3j} \\
\vdots \\
Q_{8j}
\end{bmatrix}, \quad j=1,2,\ldots,k \quad (4.175)
\]

or
\[
\begin{pmatrix}
A_{1j} \\
A_{2j} \\
\vdots \\
A_{kj}
\end{pmatrix} = [\bar{\mathbf{u}}] \begin{pmatrix}
Q_{1j} \\
Q_{2j} \\
\vdots \\
Q_{kj}
\end{pmatrix}, \quad j=1,2,\ldots,k
\]

(4.176)

where \( u_{jk}^j \) denotes the matrix element of \([B]^{-1}_j\) in the \( i \)th row and the \( k \)th column and \( Q_{1j}, Q_{2j}, \ldots, Q_{kj} \) denote the applied loads. Equation (4.176) may be written as:

\[
A_{ij} = \sum_{m=0}^{\infty} \sum_{k=1}^{8} u_{ik}^j \tilde{K}_{im}^j Y_{jm} + \sum_{m=0}^{\infty} \sum_{k=1}^{8} u_{ik}^j K_{im}^j Z_{jm}
\]

(4.177)

where \( K_{im} \) and \( \tilde{K}_{im} \) will have different forms depending on the exact form of the boundary conditions employed. For the case when \( I_{yz} = 0 \) the arbitrary constants occurring in Equations (4.141) and (4.142) can be expressed in terms of the loading parameters by the following expressions:

\[
A_{ij} = \sum_{m=0}^{\infty} K_{im}^j \tilde{Z}_{jm}, \quad (i=1,2,3,4)
\]

(4.178)

\[
D_{ij} = \sum_{m=0}^{\infty} H_{im}^j \tilde{Y}_{jm}, \quad (i=1,2,3,4)
\]

(4.179)

For this case it is possible to find closed form expressions for the quantities \( K_{im}^j, H_{im}^j, \) and \( G_{im}^j \) for any given homogenous boundary conditions.

For the particular case in which both ends of the stringer are fixed (\( u = v = w = d\theta/dx = 0 = d\theta/dx = 0 \) at \( x = 0, l \)), the \( H_{im}^j, K_{im}^j, \) and
and $G_{im}$ have the following form:

(a) Clamped-clamped case (for $H_{im}$)

\[ H_{im} = F_{11} + F_{12} \cos m \pi, \text{ where} \]

\[ F_{11} = \frac{1 - \cos \gamma_{ij} \cosh \gamma_{ij} - \sin \gamma_{ij} \sinh \gamma_{ij}}{2(\cosh \gamma_{ij} \cos \gamma_{ij} - 1)}, \quad (4.180) \]

\[ F_{12} = \frac{\cosh \gamma_{ij} - \cos \gamma_{ij}}{2(\cosh \gamma_{ij} \cos \gamma_{ij} - 1)}, \quad (4.181) \]

\[ F_{21} = \frac{\sinh \gamma_{ij} \cos \gamma_{ij} + \sin \gamma_{ij} \cosh \gamma_{ij}}{2(\cosh \gamma_{ij} \cos \gamma_{ij} - 1)}, \quad (4.182) \]

\[ F_{22} = -\frac{\sin \gamma_{ij} + \sinh \gamma_{ij}}{2(\cosh \gamma_{ij} \cos \gamma_{ij} - 1)}, \quad (4.183) \]

\[ F_{31} = \frac{1 + \sinh \gamma_{ij} \sin \gamma_{ij} - \cosh \gamma_{ij} \cos \gamma_{ij}}{2(\cosh \gamma_{ij} \cos \gamma_{ij} - 1)}, \quad (4.184) \]

\[ F_{32} = -\frac{\alpha_{m} (\cos \gamma_{ij} - \cosh \gamma_{ij})}{2(\cosh \gamma_{ij} \cos \gamma_{ij} - 1)}, \quad (4.185) \]
\[ F_{41} = \frac{\sin \gamma_{j_1 l} \cosh \gamma_{j_1 l} + \sinh \gamma_{j_1 l} \cos \gamma_{j_1 l}}{2(\cosh \gamma_{j_1 l} \cos \gamma_{j_1 l} - 1)}, \quad (4.186) \]

\[ F_{42} = \frac{\sinh \gamma_{j_1 l} + \sin \gamma_{j_1 l}}{2(\cosh \gamma_{j_1 l} \cos \gamma_{j_1 l} - 1)} \]

(b) Clamped-clamped case (for \( K_{im} \))

\[ K_{im} = E_{11} + E_{12} \cos \pi m, \text{ where} \]

\[ E_{11} = \frac{1 - \cos \gamma_{j_1 l} \cosh \gamma_{j_1 l} - \sin \gamma_{j_1 l} \sinh \gamma_{j_1 l}}{2(\cosh \gamma_{j_1 l} \cos \gamma_{j_1 l} - 1)}, \quad (4.188) \]

\[ E_{12} = \frac{\cosh \gamma_{j_1 l} - \cos \gamma_{j_1 l}}{2(\cosh \gamma_{j_1 l} \cos \gamma_{j_1 l} - 1)}, \quad (4.189) \]

\[ E_{21} = \frac{\sinh \gamma_{j_1 l} \cos \gamma_{j_1 l} + \sin \gamma_{j_1 l} \cosh \gamma_{j_1 l}}{2(\cosh \gamma_{j_1 l} \cos \gamma_{j_1 l} - 1)}, \quad (4.190) \]

\[ E_{22} = \frac{-(\sin \gamma_{j_1 l} + \sinh \gamma_{j_1 l})}{2(\cosh \gamma_{j_1 l} \cos \gamma_{j_1 l} - 1)}, \quad (4.191) \]

\[ E_{31} = \frac{1 + \sinh \gamma_{j_1 l} \sin \gamma_{j_1 l} - \cosh \gamma_{j_1 l} \cos \gamma_{j_1 l}}{2(\cosh \gamma_{j_1 l} \cos \gamma_{j_1 l} - 1)}, \quad (4.192) \]
\[ E_{32} = -\frac{(\cos \gamma_{ij}^l - \cosh \gamma_{ij}^l)a_{im}}{2(\cosh \gamma_{ij}^l \cos \gamma_{ij}^l - 1)} \] 
(4.193)

\[ E_{41} = \frac{\sin \gamma_{ij}^l \cosh \gamma_{ij}^l + \sinh \gamma_{ij}^l \cos \gamma_{ij}^l}{2(\cosh \gamma_{ij}^l \cos \gamma_{ij}^l - 1)} \] 
(4.194)

\[ E_{42} = \frac{\sinh \gamma_{ij}^l + \sin \gamma_{ij}^l}{2(\cosh \gamma_{ij}^l \cos \gamma_{ij}^l - 1)} \] 
(4.195)

(c) Clamped-clamped case (for \( G_{lm} \))

\[ G_{lm} = L_{11} + L_{12} \cos \pi r, \]

where

\[ L_{11} = \frac{\gamma_{2j}^l \gamma_{3j}^l - \gamma_{3j}^l \gamma_{2j}^l \cosh \gamma_{2j}^l \cos \gamma_{3j}^l - \gamma_{3j}^l \gamma_{2j}^2 \sin \gamma_{3j}^l \sinh \gamma_{2j}^l}{2\gamma_{2j}^l \gamma_{3j}^l \cos \gamma_{3j}^l \cosh \gamma_{2j}^l - 2\gamma_{2j}^l \gamma_{3j}^l + (\gamma_{3j}^2 - \gamma_{2j}^2) \sin \gamma_{3j}^l \sinh \gamma_{2j}^l} \] 
(4.196)

\[ L_{12} = \frac{\gamma_{2j}^l \gamma_{3j}^l (\cosh \gamma_{2j}^l - \cos \gamma_{3j}^l)}{2\gamma_{2j}^l \gamma_{3j}^l \cosh \gamma_{2j}^l \cos \gamma_{3j}^l - 2\gamma_{2j}^l \gamma_{3j}^l + (\gamma_{3j}^2 - \gamma_{2j}^2) \sin \gamma_{3j}^l \sinh \gamma_{2j}^l} \] 
(4.197)

\[ L_{21} = \frac{\gamma_{2j}^l \gamma_{3j}^l \sin \gamma_{2j}^l \cos \gamma_{3j}^l + \gamma_{3j}^l \sin \gamma_{3j}^l \cosh \gamma_{2j}^l}{2\gamma_{2j}^l \gamma_{3j}^l \cosh \gamma_{3j}^l \cos \gamma_{2j}^l - 2\gamma_{2j}^l \gamma_{3j}^l + (\gamma_{3j}^2 - \gamma_{2j}^2) \sin \gamma_{3j}^l \sinh \gamma_{2j}^l} \] 
(4.198)
\[
L_{22} = \frac{(\gamma^2_3 \sin \gamma_3^l + \gamma_2^j \gamma_3^j \sinh \gamma_2^j)}{2 \gamma_2^j \gamma_3^j \cos \gamma_3^l \cosh \gamma_2^j - 2 \gamma_2^j \gamma_3^j \gamma_2^j - (\gamma_3^j - \gamma_2^j) \sin \gamma_3^l \sinh \gamma_2^j}.
\]

\[L_{31} = \frac{\gamma_2^j \gamma_3^j + \gamma_2^j \sin \gamma_2^j \sin \gamma_3^j - \gamma_2^j \gamma_3^j \cos \gamma_3^l \cosh \gamma_2^j}{2 \gamma_2^j \gamma_3^j \cos \gamma_2^j \cos \gamma_3^l - 2 \gamma_2^j \gamma_3^j + (\gamma_3^j - \gamma_2^j) \sin \gamma_3^l \sinh \gamma_2^j}.
\]

\[L_{32} = \frac{\gamma_2^j \gamma_3^j (\cos \gamma_3^l - \cosh \gamma_2^j)}{2 \gamma_2^j \gamma_3^j \cos \gamma_2^j \cos \gamma_3^l - 2 \gamma_2^j \gamma_3^j + (\gamma_3^j - \gamma_2^j) \sin \gamma_3^l \sinh \gamma_2^j}.
\]

\[L_{41} = \frac{\gamma_2^j \cos \gamma_3^l \sinh \gamma_2^j + \gamma_2^j \gamma_3^j \sin \gamma_3^l \cosh \gamma_2^j}{2 \gamma_2^j \gamma_3^j \cos \gamma_2^j \cos \gamma_3^l - 2 \gamma_2^j \gamma_3^j + (\gamma_3^j - \gamma_2^j) \sin \gamma_3^l \sinh \gamma_2^j}.
\]

\[L_{42} = \frac{\gamma_2^j \sin \gamma_2^j + \gamma_2^j \gamma_3^j \sin \gamma_3^l}{2 \gamma_2^j \gamma_3^j \cos \gamma_2^j \cos \gamma_3^l - 2 \gamma_2^j \gamma_3^j + (\gamma_3^j - \gamma_2^j) \sin \gamma_3^l \sinh \gamma_2^j}.
\]

where the subscript \(j\) may be omitted if all of the stringers are identical. The arbitrary constants \(U_{ij}\) and \(U_{2j}\) in Equation (4.127) are zero for the fixed-end case and therefore are not expressed in terms of the loading parameters \(X_{jm}\).
The functions which occur in the general solutions of the stringer equations can be expanded as even functions over the interval \([0, t]\) in a uniformly convergent Fourier cosine series. The expansions for the functions \(\cosh \gamma_{ij} x\), \(\sinh \gamma_{ij} x\), \(\cos \gamma_{ij} x\), and \(\sin \gamma_{ij} x\) which occur in the general solution of case (1) are given by:

\[
\cosh \gamma_{ij} x = \sum_{m=0}^{\infty} a_{jm} \cos \alpha_m x, \quad (4.204)
\]

\[
\sinh \gamma_{ij} x = \sum_{m=0}^{\infty} b_{jm} \cos \alpha_m x, \quad (4.205)
\]

\[
\cos \gamma_{ij} x = \sum_{m=0}^{\infty} c_{jm} \cos \alpha_m x, \quad (4.206)
\]

\[
\sin \gamma_{ij} x = \sum_{m=0}^{\infty} d_{jm} \cos \alpha_m x, \quad (4.207)
\]

where

\[
a_{jo} = \frac{\sinh \gamma_{ij} t}{\gamma_{ij} t}, \quad (4.208)
\]

\[
b_{jo} = \frac{(\cosh \gamma_{ij} t - 1)}{\gamma_{ij} t} \quad (4.209)
\]

\[
c_{jo} = \frac{\sin \gamma_{ij} t}{\gamma_{ij} t}, \quad (4.210)
\]
\[ d_{i_0} = \frac{1 - \cos \gamma_{i_j}}{\gamma_{i_j}} \]  
\[ a_{i_1} = \frac{2 \gamma_{i_1} (-1)^m \sinh(\gamma_{i_1})}{\xi(\gamma_{i_1}^2 + \alpha_m^2)} \]  
\[ b_{i_2} = \frac{2 \gamma_{i_2} (\cosh(\gamma_{i_2})(-1)^m - 1)}{\xi(\gamma_{i_2}^2 + \alpha_m^2)} \]  
\[ c_{i_3} = \frac{2 \gamma_{i_3} \sin(\gamma_{i_3})(-1)^m}{\xi(\gamma_{i_3}^2 + \alpha_m^2)} \]  
\[ d_{i_4} = \frac{2 \gamma_{i_4} (1 - \cosh(\gamma_{i_4})(-1)^m)}{\xi(\gamma_{i_4}^2 + \alpha_m^2)} \]

Therefore, for the case of no coupling the general solutions of the stringer equations are given by:

\[ u_{ij}(x) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} N_{ij} \zeta_{4m} x^j \sin(\alpha_m x) + \]  
\[ \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} N_{ij} \zeta_{4m} x^j \sin(\alpha_m x) + \]  
\[ \sum_{m=0}^{\infty} \zeta_{4m} x^j \sin(\alpha_m x), \quad \text{(4.216)} \]

\[ v_{ij}(x) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} H_{ij} \zeta_{4m} C_{ij} \cos(\alpha_m x) + \]
\begin{align*}
\sum_{m=0}^{\infty} \sum_{l=0}^{\infty} & K_{1l}^{\xi} \xi_{l}^{l} Z_{j}^{j} a_{m n}^{i} \cos(\alpha x) + \\
\sum_{m=0}^{\infty} \xi_{m}^{j l} Z_{j m}^{j} \cos(\alpha x),
\end{align*}
\quad (4.217)

w_{0j}(x) = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} K_{1l}^{\xi} \xi_{l}^{l} Z_{j}^{j} a_{m n}^{i} \cos(\alpha x) + \\
\sum_{m=0}^{\infty} \sum_{l=0}^{\infty} K_{2l}^{\xi} \xi_{l}^{l} Z_{j}^{j} b_{m n}^{i} \cos(\alpha x) + \\
\sum_{m=0}^{\infty} \sum_{l=0}^{\infty} K_{3l}^{\xi} \xi_{l}^{l} Z_{j}^{j} c_{m n}^{i} \cos(\alpha x) + \\
\sum_{m=0}^{\infty} \sum_{l=0}^{\infty} K_{4l}^{\xi} \xi_{l}^{l} Z_{j}^{j} d_{m n}^{i} \cos(\alpha x) + \\
\sum_{m=0}^{\infty} \xi_{m l}^{j} Z_{j m}^{j} \cos(\alpha x),
\end{align*}
\quad (4.218)

\theta_{0j}(x) = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} G_{1l}^{\xi} \xi_{l}^{l} 2m_{l} 2m^{j} (T_{j l} + Y_{j l}) \cos(\alpha x) + \\
\sum_{m=0}^{\infty} \sum_{l=0}^{\infty} G_{2l}^{\xi} \xi_{l}^{l} 2m_{l} 2m^{j} (T_{j l} + Y_{j l}) \cos(\alpha x) + \\
\sum_{m=0}^{\infty} \sum_{l=0}^{\infty} G_{3l}^{\xi} \xi_{l}^{l} 3m_{l} 3m^{j} (T_{j l} + Y_{j l}) \cos(\alpha x) + \\
\sum_{m=0}^{\infty} \sum_{l=0}^{\infty} G_{4l}^{\xi} \xi_{l}^{l} 3m_{l} 3m^{j} (T_{j l} + Y_{j l}) \cos(\alpha x) + \

Boundary Conditions for Shell

For free vibration, the general solutions of the shell equations will be required to satisfy homogenous boundary conditions of the form:

\[ L_i \{ u, v, w \} = 0, \quad (i = 1, 2, \ldots, 8) \]  \hspace{1cm} (4.220)

where the \( L_i \) are differential operators. Inserting Equations (4.116), (4.117), and (4.118) into the boundary conditions and using orthogonality conditions for \( \sin n\phi \) and \( \cos n\phi \) results in two sets of eight nonhomogenous equations for each integer \( n \) which relate the \( \tilde{C}_{sn} \) (symmetric part) and the \( \tilde{C}_{sn}' \) (antisymmetric part) to the applied loads \( \{ \bar{F} \}_s \) (symmetric) and \( \{ \bar{F} \}_a \) (antisymmetric), respectively. These equations may be written as two matrix equations as follows:

\[
[\tilde{D}] \begin{bmatrix} \tilde{C}_{1n} \\ \tilde{C}_{2n} \\ \vdots \\ \tilde{C}_{8n} \end{bmatrix} = \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \vdots \\ \bar{F}_8 \end{bmatrix}, \quad (n=0, 1, 2, \ldots) \]  \hspace{1cm} (4.221)
where the coefficient matrix $[D]$ and the external loading matrices $\{\bar{P}\}^s, \{\bar{P}\}^a$ have different forms depending on the type of boundary conditions applied to the shell. For the particular case of clamped-clamped supported boundaries, the matrices $\{\bar{P}\}^s$ and $\{\bar{P}\}^a$ have the following forms:
\[
\begin{align*}
\frac{1}{j} \sum_{m=0}^{\infty} \delta_{m,1} \cos \left( \frac{j \pi}{2} \right) \sin \left( \frac{m \pi}{2} \right) \frac{1}{j} \sum_{n=0}^{\infty} \delta_{n,1} \cos \left( \frac{n \pi}{2} \right) \sin \left( \frac{j \pi}{2} \right) \\
\frac{1}{j} \sum_{m=0}^{\infty} \delta_{m,1} \cos \left( \frac{j \pi}{2} \right) \sin \left( \frac{m \pi}{2} \right) \end{align*}
\]
Since the matrix $[\bar{D}]$ is nonsingular, one may express the unknown con-
stants in terms of the applied loads as:

\[
\begin{bmatrix}
\bar{C}_1 \\
\bar{C}_2 \\
\vdots \\
\bar{C}_8 \\
\end{bmatrix} = [\bar{D}]^{-1} \begin{bmatrix}
\bar{p}_1 \\
\bar{p}_2 \\
\vdots \\
\bar{p}_8 \\
\end{bmatrix}, \quad (n=0,1,2,\ldots) \quad (4.225)
\]

and

\[
\begin{bmatrix}
\bar{C}_1' \\
\bar{C}_2' \\
\vdots \\
\bar{C}_8' \\
\end{bmatrix} = [\bar{D}]^{-1} \begin{bmatrix}
\bar{p}_1 \\
\bar{p}_2 \\
\vdots \\
\bar{p}_8 \\
\end{bmatrix}, \quad (n=0,1,2,\ldots) \quad (4.226)
\]

or

\[
\begin{bmatrix}
\bar{C}_1 \\
\bar{C}_2 \\
\vdots \\
\bar{C}_8 \\
\end{bmatrix} = [\bar{S}] \begin{bmatrix}
\bar{p}_1 \\
\bar{p}_2 \\
\vdots \\
\bar{p}_8 \\
\end{bmatrix}, \quad (n=0,1,2,\ldots) \quad (4.227)
\]

and
where $G_{ij}$ denotes the matrix element if $[\bar{D}]^{-1}$.

Compatibility Relations

Each element of the composite structure has been described individually. To connect the thin-walled beams to the shell, we shall require that the following compatibility relations hold:

\begin{align*}
    w_{oj}(x) &= w(x,\phi_j), \quad (j=1, \ldots, K) \quad (4.229) \\
    v_{oj}(x) &= v(x,\phi_j) + \frac{v}{a} \frac{\partial w}{\partial \phi}(x,\phi_j), \quad (j=1, \ldots, K) \quad (4.230) \\
    \theta_{oj}(x) &= \frac{1}{a} \frac{\partial w}{\partial \phi}(x,\phi_j), \quad (j=1, \ldots, K) \quad (4.231) \\
    u_{oj}(x) &= u(x,\phi_j) + y_j \frac{\partial v}{\partial x}(x,\phi_j) + z_j \frac{\partial w}{\partial x}(x,\phi_j), \quad (j=1, \ldots, K) \quad (4.232)
\end{align*}

where $K$ is the total number of stringers attached to the shell. These compatibility relations relate the middle surface displacements of the shell at the line of attachment to the displacements of the shear center of the $j$th thin-walled rib.
Substitution of Equations (4.116), (4.117), (4.118) and Equations (4.216), (4.217), (4.218), (4.219) in conjunction with Equations (4.41) through (4.52) and Equations (4.227) and (4.228) into Equations (4.229), (4.230), (4.231), and (4.232) yields a doubly infinite system of simultaneous homogenous algebraic equations. These compatibility equations may be written in a single matrix equation as follows:

\[
\begin{bmatrix}
A_{QP} & B_{QP} & C_{QP} & K_{QP} \\
E_{QP} & F_{QP} & G_{QP} & H_{QP} \\
I_{QP} & J_{QP} & L_{QP} & M_{QP} \\
N_{QP} & O_{QP} & P_{QP} & Q_{QP}
\end{bmatrix}
\begin{bmatrix}
Z_p \\
T_p \\
Y_p \\
X_p
\end{bmatrix}
= \begin{bmatrix}
\bar{0} \\
\bar{0} \\
\bar{0} \\
\bar{0}
\end{bmatrix},
\]

(4.233)

where \( Z_p, T_p, Y_p \) and \( X_p \) are column vectors whose components are:

\[
\begin{align*}
Z_p &= Z_{j, k}, \\
T_p &= T_{j, k} \\
Y_p &= Y_{j, k} \\
X_p &= X_{j, k}
\end{align*}
\]

(4.234-4.237)

and \( j, k \) are related to \( P \) by
\[ j = 1 + \left\lceil \frac{(P-1)/\ell^*}{T} \right\rceil, \quad (j=1, \ldots, K) \tag{4.238} \]

\[ \ell = (P-1) - \ell^* \left\lceil \frac{(P-1)/\ell^*}{T} \right\rceil, \quad (\ell=1, 2, \ldots, \ell^*) \tag{4.239} \]

where \( \ell^* \) is the maximum value of \( \ell \) and the subscript \( T \) represents the operation of truncation after the decimal point. Similarly, the submatrices \( A_{QP}, B_{QP}, C_{QP}, \ldots, S_{QP} \) are given as follows:

\[ A_{QP} = A_{imj\ell^*} \tag{4.240} \]

\[ B_{QP} = B_{imj\ell^*} \tag{4.241} \]

\[ C_{QP} = C_{imj\ell^*} \tag{4.242} \]

\[ K_{QP} = K_{imj\ell^*} \tag{4.243} \]

\[ \vdots \]

\[ S_{QP} = S_{imj\ell^*} \tag{4.244} \]

and

\[ i = 1 + \left\lceil \frac{(Q-1)/\ell^*}{T} \right\rceil, \tag{4.245} \]

\[ m = (Q-1) - \ell^* \left\lceil \frac{(Q-1)/\ell^*}{T} \right\rceil, \tag{4.246} \]
where $P$ and $Q$ are integers such that

\begin{equation}
1 \leq P \leq K^*_E,
\end{equation}

\begin{equation}
1 \leq Q \leq K^*_E,
\end{equation}

and $K$ is the total number of stringers attached to the shell. For non-trivial solutions of $Z_{j_1}$, $T_{j_2}$, $Y_{j_2}$, and $X_{j_2}$, for $J=1,\ldots,K$ and $\lambda=0,1,\ldots,\lambda^*$ one requires that the determinant of the coefficient matrix of Equation (4.181) vanish. This yields the frequency equation, i.e.,

\begin{equation}
\begin{vmatrix}
A_{QP} & B_{QP} & C_{QP} & D_{QP} \\
E_{QP} & F_{QP} & G_{QP} & H_{QP} \\
I_{QP} & J_{QP} & L_{QP} & M_{QP} \\
N_{QP} & O_{QP} & P_{QP} & Q_{QP}
\end{vmatrix} = 0.
\end{equation}

Equation (4.249) defines a determinant of order $[4K(\lambda+1)] \times [4K(\lambda+1)]$. For example, if there are four stringers attached to the shell and if $\lambda=0,1,2,3$ then the size of the determinant is $(64) \times (64)$. Once the natural frequencies are found, then calculations can be performed for the mode shapes.
Numerical Example

To illustrate the applicability of the general linear theory developed, a numerical example is presented. This example should also help to clarify some minor details which were omitted from the formal development of the solution. For the free vibration analysis, a clamped-clamped, longitudinally-stiffened, thin cylindrical shell is considered. The boundary conditions for the shell are given by:

(i) \( u(0,\phi) = 0 \), (v) \( u(\ell,\phi) = 0 \),

(ii) \( v(0,\phi) = 0 \), (vi) \( v(\ell,\phi) = 0 \),

(iii) \( w(0,\phi) = 0 \), (vii) \( w(\ell,\phi) = 0 \),

(iv) \( \frac{\partial w}{\partial x}(0,\phi) = 0 \), (viii) \( \frac{\partial w}{\partial x}(\ell,\phi) = 0 \).

The boundary conditions for the thin-walled ribs are given by:

(i) \( u_{o,j}(0) = u_{o,j}(\ell) = 0 \),

(ii) \( v_{o,j}(0) = v_{o,j}(\ell) = 0 \),

(iii) \( \frac{dv}{dx}_{o,j}(0) = \frac{dv}{dx}_{o,j}(\ell) = 0 \),

(iv) \( w_{o,j}(0) = w_{o,j}(\ell) = 0 \),
\[
(v) \quad \frac{dw_{ij}}{dx}(0) = \frac{dw_{ij}}{dx}(l) = 0,
\]

\[
(vi) \quad \theta_{ij}(0) = \theta_{ij}(l) = 0,
\]

\[
(vii) \quad \frac{d\theta_{ij}}{dx}(0) = \frac{d\theta_{ij}}{dx}(l) = 0, \quad (4.251)
\]

where \( j = 1, \ldots, K \).

In order to keep the size of the determinant in reasonable limits while investigating the free vibration characteristics of a stiffened shell, a cylinder with four equally spaced stringers of the type used in reference [4] (hat-shaped) was chosen. A computer program was written for the general case. The numerical results presented are for the first few circumferential and axial modes.

The coefficients \( A_{OP}, B_{OP}, C_{OP}, \ldots, S_{OP} \) for the clamped-clamped case are given as follows:

\[
A_{OP} = A_{\text{imj}} = \left\{ K_{11} \xi_{1m} \psi_{1j} + K_{22} \xi_{1j} \psi_{1m} + K_{33} \xi_{1j} \psi_{1m} + \right. \\
\left. + K_{44} \xi_{1j} \psi_{1m} + \xi_{1j} \delta_{im} \right\}^{N^*}_{s=1} \sum_{s=1}^{N^*} \left( \frac{P_{snj} \cos n\phi_1}{snj \sin n\phi_1} \right)^{1/5mi} - \sum_{n=0}^{N^*} h_4(\cos n\phi_j \cos n\phi_1 + \sin n\phi_j \sin n\phi_1) \delta_{lm}, \quad (4.252)
\]
\[
B_{QP} = B_{imj}^* = \sum_{n=0}^{N} \sum_{s=1}^{8} \{Q_{snj} \cos n\phi_i + Q'_{snj} \sin n\phi_i\} a_{5mn} - \\
\sum_{n=0}^{N} h_s (\sin n\phi_j \sin n\phi_i - \cos n\phi_j \cos n\phi_i) \delta_{lm}, \tag{4.253}
\]

\[
C_{QP} = C_{imj}^* = \sum_{n=0}^{N} \sum_{s=1}^{8} \{R_{snj} \cos n\phi_i + R'_{snj} \sin n\phi_i\} a_{5mn} - \\
\sum_{n=0}^{N} h_s (\sin n\phi_j \cos n\phi_i - \cos n\phi_j \sin n\phi_i) \delta_{lm}, \tag{4.254}
\]

\[
K_{QP} = K_{imj}^* = \sum_{n=0}^{N} \sum_{s=1}^{8} \{S_{snj} \cos n\phi_i + S'_{snj} \sin n\phi_i\} a_{5mn} - \\
\sum_{n=0}^{N} h_s (\cos n\phi_j \cos n\phi_i + \sin n\phi_j \sin n\phi_i) \delta_{lm}, \tag{4.255}
\]

\[
E_{QP} = E_{imj}^* = \sum_{n=0}^{N} \sum_{s=1}^{8} \{P_{snj} \sin n\phi_i + P'_{snj} \cos n\phi_i\} a_{5mn} - \\
\sum_{n=0}^{N} h_s (\cos n\phi_j \sin n\phi_i + \sin n\phi_j \cos n\phi_i) \delta_{lm},
\]

\[
+ \sum_{n=0}^{N} h_s Z_i (n/2) (\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i) \delta_{lm} \tag{4.256}
\]

\[
P_{QP} = P_{imj}^* = \sum_{n=0}^{N} \sum_{s=1}^{8} \{Q_{snj} \sin n\phi_i + Q'_{snj} \cos n\phi_i\} a_{5mn} - \\
\sum_{n=0}^{N} h_s (\sin n\phi_j \sin n\phi_i + \cos n\phi_j \cos n\phi_i) \delta_{lm} + 
\]
\[ + \sum_{n=0}^{N^*} h_5 Z_i(n/a)(\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i) \delta_{m'} \] (4.257)

\[ G_{QP} = \sum_{n=0}^{N^*} \sum_{s=1}^{\infty} \{ R_{snj} \sin n\phi_i + R_{snj} \cos n\phi_i \} (a_{smn}) - \]

\[ - \sum_{n=0}^{N^*} h_3 (\sin n\phi_j \sin n\phi_i - \cos n\phi_j \cos n\phi_i) \delta_{m'} \]

\[ + \sum_{n=0}^{N^*} h_2 Z_i(n/a)(\cos n\phi_j \cos n\phi_i - \sin n\phi_j \sin n\phi_i) \delta_{m'}, \] (4.258)

\[ H_{QP} = \sum_{n=0}^{N^*} \sum_{s=1}^{\infty} \{ S_{snj} \sin n\phi_i + S_{snj} \cos n\phi_i \} (a_{smn}) - \]

\[ - \sum_{n=0}^{N^*} h_0 (\cos n\phi_i \sin n\phi_j - \sin n\phi_i \cos n\phi_j) \delta_{m'} \]

\[ - \sum_{n=0}^{N^*} h_0 (\sin n\phi_i \cos n\phi_j + \cos n\phi_i \sin n\phi_j) \delta_{m'} (n/a)Z_i, \] (4.259)

\[ \nu_{QP} = \sum_{n=0}^{N^*} \sum_{s=1}^{\infty} \{ P_{snj} \sin n\phi_i + P_{snj} \cos n\phi_i \} (n/a) t_{5mm} - \]

\[ - \sum_{n=1}^{N^*} h_4 \left( \frac{z_i}{a} \right) (n)(\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i) \delta_{m'}, \] (4.260)

\[ J_{QP} = \sum_{n=0}^{N^*} \sum_{s=1}^{\infty} \{ Q_{snj} \sin n\phi_i + Q_{snj} \cos n\phi_i \} (n/a) t_{5mm} - \]

\[ \zeta_{2m} \delta_{m'} \delta_{ij} + \sum_{n=1}^{N^*} \sum_{s=1}^{\infty} \{ Q_{snj} \sin n\phi_i + Q_{snj} \cos n\phi_i \} (n/a) t_{5mm} - \]
\[
\sum_{n=1}^{N^*} h_5 \left( \frac{z_i}{a} \right) (n) \cos \phi_j \sin \phi_i - \sin \phi_j \cos \phi_i \delta_{lm}^n \quad (4.261)
\]

\[
L_{QP} = \sum_{n=1}^{N^*} s_{nji} \sin \phi_i + R_{nji} \cos \phi_i \left( \frac{z_i}{a} \right) (n) t_{5mn} - \\
\sum_{n=1}^{N^*} h_6 \left( \frac{z_i}{a} \right) (n) \sin \phi_j \sin \phi_i + \cos \phi_j \cos \phi_i \delta_{lm}^n, \quad (4.262)
\]

\[
M_{QP} = \sum_{n=1}^{N^*} s_{nji} \sin \phi_i + \cos \phi_j \sin \phi_i \left( \frac{z_i}{a} \right) (n) t_{5mn} - \\
\sum_{n=1}^{N^*} h_7 (\sin \phi_j \cos \phi_i - \cos \phi_j \sin \phi_i \left( \frac{z_i}{a} \right) (n)) - \\
\sum_{n=1}^{N^*} h_8 (\sin \phi_j \cos \phi_i - \cos \phi_j \sin \phi_i \left( \frac{z_i}{a} \right) (n)), \quad (4.263)
\]

\[
N_{QP} = \sum_{n=1}^{N^*} s_{nji} \sin \phi_i \left( b_{smn} - (z_i a_m + y_i a_m) t_{smn} \right) + \\
P_{nji} \sin \phi_i \left( b_{smn} - (z_i a_m - y_i a_m) t_{smn} \right)
\]

\[
\sum_{n=1}^{N^*} h_i z_i a_m (\sin \phi_j \cos \phi_i + \cos \phi_j \sin \phi_i) \delta_{lm}^n - \\
\sum_{n=0}^{N^*} h_7 z_i a_m (\cos \phi_j \cos \phi_i + \sin \phi_j \sin \phi_i) \delta_{lm}^n + \\
\sum_{n=1}^{N^*} h_8 y_i a_m (\sin \phi_j \cos \phi_i - \cos \phi_j \sin \phi_i) \delta_{lm}^n, \quad (4.264)
\]
\[ C_{QP} = C_{i,m,l} = \sum_{n=0}^{N} \sum_{s=1}^{N} \{ C_{snjl} \cos \theta \{ b_{s,m} - (z_i \alpha_{m} + y_i \alpha_{n})t_{s,m} \} + Q_{snjl} \sin \theta \{ b_{s,m} - (z_i \alpha_{m} - y_i \alpha_{n})t_{s,m} \} \}
\]

\[ + \sum_{n=0}^{N} h_{z_i \alpha_{m}} (\sin \theta \sin \theta_i - \cos \theta \cos \theta_i) \delta_{lm} \]

\[ - \sum_{n=0}^{N} h_{y_i \alpha_{m}} (\sin \theta \sin \theta_i + \cos \theta \cos \theta_i) \delta_{lm} \]

\[ - \sum_{n=0}^{N} h_{y_i \alpha_{m}} (\sin \theta \sin \theta_i - \cos \theta \cos \theta_i) \delta_{lm}, \quad (4.265) \]

\[ U_{QP} = U_{i,m,l} = \sum_{n=1}^{N} \sum_{s=1}^{N} \{ R_{snjl} \cos \theta \{ b_{s,m} - (z_i \alpha_{m} + y_i \alpha_{n})t_{s,m} \} + R_{snjl} \sin \theta \{ b_{s,m} - (z_i \alpha_{m} - y_i \alpha_{n})t_{s,m} \} \}
\]

\[ + \sum_{n=0}^{N} h_{z_i \alpha_{m}} (\sin \theta \sin \theta_i + \cos \theta \cos \theta_i) \delta_{lm} \]

\[ - \sum_{n=0}^{N} h_{y_i \alpha_{m}} (\sin \theta \sin \theta_i + \cos \theta \cos \theta_i) \delta_{lm} \]

\[ - \sum_{n=0}^{N} h_{y_i \alpha_{m}} (\sin \theta \sin \theta_i - \cos \theta \cos \theta_i) \delta_{lm}, \quad (4.266) \]

\[ V_{QP} = V_{i,m,l} = \{ N_{1 \alpha} \xi_{4 \alpha} r_{4 \alpha} + N_{2 \alpha} \xi_{4 \alpha} r_{7 \alpha} + \xi_{4 \alpha} m \} \delta_{ij} + \]

\[ + \sum_{n=0}^{N} \sum_{s=1}^{N} \{ S_{snjl} \cos \theta \{ b_{s,m} - (z_i \alpha_{m} + y_i \alpha_{n})t_{s,m} \} + S_{snjl} \sin \theta \{ b_{s,m} - (z_i \alpha_{m} - y_i \alpha_{n})t_{s,m} \} \}
\]
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \sin n_{ij} \{ b_{ijmn} - (z_{i} a_{j} - y_{i} a_{j}) \} t_{ijmn} \\
- \sum_{n=0}^{N} h_{11} \{ \cos n_{i} \cos n_{j} + \sin n_{i} \sin n_{j} \} \delta_{il} - \\
+ \sum_{n=1}^{N} h_{10} z_{i} a_{m} \{ \sin n_{i} \cos n_{j} + \cos n_{i} \sin n_{j} \} \delta_{im} - \\
- \sum_{n=1}^{N} h_{12} y_{i} a_{m} \{ \sin n_{i} \cos n_{j} - \cos n_{i} \sin n_{j} \} \delta_{im},
\]

where the following notation has been introduced:

\[g_{3f_{1}} = h_{1},\]  \hspace{1cm} (4.268)

\[g_{3f_{2}} = h_{2},\]  \hspace{1cm} (4.269)

\[g_{4f_{3}} = h_{3},\]  \hspace{1cm} (4.270)

\[g_{5f_{1}} = h_{4},\]  \hspace{1cm} (4.271)

\[g_{1f_{2}} = h_{5},\]  \hspace{1cm} (4.272)

\[g_{2f_{3}} = h_{6},\]  \hspace{1cm} (4.273)

\[g_{5f_{1}} = h_{7},\]  \hspace{1cm} (4.274)

\[g_{5f_{2}} = h_{8},\]  \hspace{1cm} (4.275)
\[ g_6 f_3 = h_9, \quad (4.276) \]
\[ g_7 f_4 = h_{10}, \quad (4.277) \]
\[ g_8 f_4 = h_{11}, \quad (4.278) \]
\[ g_9 f_4 = h_{12}, \quad (4.279) \]

\[ a_{1mn} = t_{1mn} \left( \bar{\beta}_{1n} - \frac{Z_n}{a} \right) - t_{2mn} \bar{\beta}_{2n}, \quad (4.280) \]
\[ a_{2mn} = t_{1mn} \bar{\beta}_{2n} + t_{2mn} \left( \bar{\beta}_{1n} - \frac{Z_n}{a} \right), \quad (4.281) \]
\[ a_{3mn} = t_{3mn} \bar{\beta}_{3n} + t_{2mn} \left( \bar{\beta}_{1n} - \frac{Z_n}{a} \right), \quad (4.282) \]
\[ a_{4mn} = t_{3mn} \bar{\beta}_{4n} + t_{4mn} \left( \bar{\beta}_{3n} - \frac{Z_n}{a} \right), \quad (4.283) \]
\[ a_{5mn} = t_{5mn} \bar{\beta}_{5n} + t_{6mn} \left( \bar{\beta}_{5n} - \frac{Z_n}{a} \right), \quad (4.284) \]
\[ a_{6mn} = t_{5mn} \bar{\beta}_{6n} + t_{6mn} \left( \bar{\beta}_{5n} - \frac{Z_n}{a} \right), \quad (4.285) \]
\[ a_{7mn} = t_{7mn} \bar{\beta}_{7n} + t_{8mn} \left( \bar{\beta}_{7n} - \frac{Z_n}{a} \right), \quad (4.286) \]
\[ a_{8mn} = t_{7mn} \bar{\beta}_{8n} + t_{8mn} \left( \bar{\beta}_{7n} - \frac{Z_n}{a} \right), \quad (4.287) \]

\[ b_{1mn} = \bar{a}_{1mn} + \bar{a}_{2mn} \quad (4.288) \]
\begin{align*}
b_{2mn} &= \tilde{a}_{2n} r_{1mn} + \tilde{a}_{1n} r_{2mn}, \\
b_{3mn} &= \tilde{a}_{3n} r_{3mn} - \tilde{a}_{4n} r_{4mn}, \\
b_{4mn} &= \tilde{a}_{4n} r_{3mn} + \tilde{a}_{3n} r_{4mn}, \\
b_{5mn} &= \tilde{a}_{5n} r_{5mn} - \tilde{a}_{6n} r_{6mn}, \\
b_{6mn} &= \tilde{a}_{6n} r_{5mn} + \tilde{a}_{5n} r_{6mn}, \\
b_{7mn} &= \tilde{a}_{7n} r_{7mn} + \tilde{a}_{8n} r_{8mn}, \\
b_{8mn} &= \tilde{a}_{8n} r_{7mn} + \tilde{a}_{7n} r_{8mn},
\end{align*}

\begin{align*}
P_{snj \ell} &= \xi_{32} \delta h_{1j} \cos n\phi_j + \xi_{56} h_{4j} \cos n\phi_j + \xi_{57} h_{4j} (-1)^{\ell} \cos n\phi_j \\
&\quad + \xi_{57} h_{4j} (-1)^{\ell} \cos n\phi_j, \\
P'_{snj \ell} &= \xi_{52} \delta h_{1j} \sin n\phi_j + \xi_{53} h_{4j} \sin n\phi_j + \xi_{57} h_{4j} (-1)^{\ell} \sin n\phi_j \\
&\quad + \xi_{57} h_{4j} (-1)^{\ell} \sin n\phi_j, \\
Q_{snj \ell} &= \xi_{52} \delta h_{2j} \sin n\phi_j + \gamma_{53} h_{5j} \cos n\phi_j + \xi_{55} h_{2j} \sin n\phi_j (-1)^{\ell} \\
&\quad + \xi_{57} h_{5j} \cos n\phi_j (-1)^{\ell},
\end{align*}
\[ Q_{njl} = -\xi_{52} h_2 \delta_n \cos n\phi_j + \xi_{53} h_5 \sin n\phi_j, \]
\[ - \xi_{56} h_2 \cos n\phi_j(-1)^\ell + \xi_{57} h_5 \sin n\phi_j(-1)^\ell, \quad (4.299) \]
\[ R_{njl} = \xi_{52} h_3 \delta_n \sin n\phi_j + \xi_{53} h_6 \sin n\phi_j, \]
\[ + \xi_{56} h_3 (-1)^\ell \sin n\phi_j + \xi_{57} h_6 \sin n\phi_j(-1)^\ell, \quad (4.300) \]
\[ R'_{njl} = -\xi_{52} h_3 \delta_n \cos n\phi_j - \xi_{53} h_6 \cos n\phi_j, \]
\[ - \xi_{56} h_3 (-1)^\ell \cos n\phi_j - \xi_{57} h_6 \cos n\phi_j(-1)^\ell, \quad (4.301) \]
\[ S_{njl} = \xi_{52} h_{12} \delta_n \cos n\phi_j + \xi_{53} h_{10} \cos n\phi_j \]
\[ + \xi_{56} h_{12} \cos n\phi_j(-1)^\ell + \xi_{57} h_{10} \cos n\phi_j(-1)^\ell, \quad (4.302) \]
\[ S'_{njl} = \xi_{52} h_{12} \delta_n \sin n\phi_j + \xi_{53} h_{10} \sin n\phi_j \]
\[ + \xi_{56} h_{12} \sin n\phi_j(-1)^\ell + \xi_{57} h_{10} \sin n\phi_j(-1)^\ell, \quad (4.303) \]

where
\[ \delta_n = 0 \text{ if } n = 0 \]
\[ = 1 \text{ if } n \neq 0. \quad (4.304) \]
Numerical Results and Discussion

Computer programs were written to solve the eigenvalue problem (i.e., (4.249)) in the ALGOL for use with the Burrough's B5500 digital computer. Two numerical examples are considered. The first is for a simply supported cylindrical shell with four and eight internal stringers and the second is for a clamped-clamped shell with four internal stringers.

Free Vibration

Table 22 and Table 23 present a comparison of the theoretical frequencies as predicted by the general linear theory with the results of the energy method for four and eight equally spaced stringers on a simply supported cylindrical shell for both symmetric and antisymmetric modes. The theoretical calculations employed a 30-term series for the circumferential modes. Comparison of 25- and 30-term expansions showed excellent agreement for frequencies calculated (n<10). The comparison of the frequencies for the simply supported stiffened shell shows good agreement between both methods of solution. It should be noted that for the general theory there are some slight differences in frequencies between odd symmetric and antisymmetric circumferential modes (i.e., \( m=1, n=7 \)). Theoretically, these frequencies should be the same. The trouble stems from the approximation made of the twisting moment acting on the stringer. It was found that slightly better results could be obtained by considering a couple acting on the shell.
Figure 13 shows a comparison of some selected modes shapes as predicted by the general theory with the results of the Ritz method for four internal hat-shaped stringers. In general, very good agreement is noted between the two different methods. Better convergence of the modes was noted for the general linear theory as compared with the Ritz analysis (i.e., \( m=1, n=1 \)). The results are in more disagreement for comparison between the antisymmetric circumferential modes. Again, this is attributed to the mathematical representation of the twisting moment. Slightly better results may be obtained by expressing the twisting moment in terms of a singularity function.

Figure 14 shows a comparison of some selected circumferential mode shapes as predicted by the general linear theory with the results of the Ritz method for eight internal hat-shaped stringers. Again the results of the general theory show much better convergence properties than the Ritz method for the same number of terms in the series. The eigenvalues as predicted from the general theory were obtained to six decimal place accuracy as with only two decimal place accuracy from the Ritz method for the modes \( n=1,3 \). The mode shapes from the general theory are relatively smooth functions indicating that not enough terms were taken in the Ritz method to achieve a good approximation of the mode.

Table 24 shows a comparison of theoretical frequencies of the general theory for a clamped-clamped shell with four equally spaced flexible stringers versus those of the unstiffened shell using Donnell's shell theory for the lower axial and circumferential modes. Frequencies
are presented for both symmetric and antisymmetric modes. For the example considered, the stringers were very weak in torsion. This is reflected in the numerical results. The frequencies for the even antisymmetric circumferential modes are very nearly equal to those of the unstiffened cylindrical shell. These frequencies are slightly less than those for the unstiffened cylinder since the stringers contribute inertia to the structure but the contribution to the potential energy is insignificant since the stringers are so weak in torsion.

The frequencies for the even symmetric modes for the case when \( n=4 \) are greater than those of the unstiffened shell since all the stringers are in bending and contribute more potential energy to the structure than kinetic energy. Theoretical mode shapes for the axial modes \((m=1,2)\) and some lower circumferential modes are presented in Figure 15. Theoretical calculations employed six terms for the axial mode and 20 terms for the circumferential mode. It is necessary to take more terms to represent the circumferential mode shape since the presence of the stringers on the shell is to couple the unstiffened shell modes together as is seen from the Ritz analysis for the simply supported shell. Fewer terms are needed to represent the lower axial modes since these are relatively smooth curves. It should be noted that the integer \( n \) in Table 23 stands for the number of full waves around the circumference.
Table 22. Comparison of Theoretical Frequencies of General Theory versus Ritz Analysis for Four Equally-Spaced Stringers

<table>
<thead>
<tr>
<th>m=1</th>
<th>f in cps</th>
<th>m=2</th>
<th>f in cps</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>1</td>
<td>745</td>
<td>721</td>
<td>745</td>
</tr>
<tr>
<td>2</td>
<td>317</td>
<td>315</td>
<td>302</td>
</tr>
<tr>
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<td>157</td>
<td>157</td>
<td>157</td>
</tr>
<tr>
<td>4</td>
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</tr>
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<td>137</td>
<td>137</td>
<td>137</td>
</tr>
<tr>
<td>8</td>
<td>173</td>
<td>174</td>
<td>181</td>
</tr>
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<td>222</td>
<td>222</td>
</tr>
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<td>265</td>
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m=3

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<td>8</td>
<td>302</td>
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<td>287</td>
</tr>
<tr>
<td>10</td>
<td>321</td>
<td>322</td>
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</tbody>
</table>

a = Ritz analysis (symmetric modes).
b = General theory (symmetric modes).
c = Ritz analysis (antisymmetric modes).
d = General theory (antisymmetric modes).
Table 23. Comparison of Theoretical Frequencies of General Theory versus Ritz Analysis for Eight Equally-Spaced Stringers

<table>
<thead>
<tr>
<th>m=1</th>
<th>( f ) in cps</th>
<th>m=2</th>
<th>( f ) in cps</th>
</tr>
</thead>
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<td>a</td>
<td>b</td>
<td>c</td>
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<td>737</td>
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<td>313</td>
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<th>m=4</th>
<th>( f ) in cps</th>
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</tr>
<tr>
<td>6</td>
<td>392</td>
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<td>392</td>
</tr>
<tr>
<td>7</td>
<td>337</td>
<td>346</td>
<td>337</td>
</tr>
</tbody>
</table>

a = Ritz analysis (symmetric modes).
b = General theory (symmetric modes).
c = Ritz analysis (antisymmetric modes).
d = General theory (antisymmetric modes).
Table 24: Theoretical Frequencies of General Theory for a Clamped-Clamped Shell with Four Equally-Spaced Stringers

<table>
<thead>
<tr>
<th>m=1</th>
<th>f(cps)</th>
<th>m=2</th>
<th>f(cps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
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<td>1427</td>
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<tr>
<td>3</td>
<td>2329</td>
<td>2330</td>
<td>2336</td>
</tr>
<tr>
<td>4</td>
<td>4145</td>
<td>4137</td>
<td>4142</td>
</tr>
</tbody>
</table>

a = General theory (stiffened) symmetric modes.
b = General theory (stiffened) antisymmetric modes.
c = Unstiffened shell (Donnell's theory) [12].
Figure 13. Theoretical Mode Shapes for Four Stringers (Simply Supported Case)
KEY
--- Results of Ritz Analysis
----- Results of General Theory

Figure 13. Theoretical Mode Shapes for Four Stringers (Symmetric Modes) (Continued)
Figure 13. Theoretical Mode Shapes for Four Stringers (Symmetric Modes) (Continued)
Figure 14. Theoretical Circumferential Mode Shapes for Eight Stringers (Simply Supported Case)
Figure 15. Theoretical Axial and Circumferential Modes for a Clamped-Clamped Cylinder with Four Stringers
Figure 15. Theoretical Axial and Circumferential Mode Shapes for Clamped-Clamped Cylinder with Four Stringers (Continued)
CHAPTER V
SUMMARY OF CONCLUSIONS AND RECOMMENDATIONS

Summary of Conclusions

A theoretical analysis has been performed to predict the free vibration characteristics of a thin cylindrical shell stiffened by means of longitudinal ribs for any set of admissible boundary conditions. A mathematical model was developed using Donnell's linear thin shell theory and V. Z. Vlasov's equations for the stringers. The stringers are treated as discrete structural elements and are not assumed to be closely or equally spaced. Thus, there are no approximations due to "smearing" the stringer effects over the surface of the shell.

It was shown that for a shell with only a few stringers, it is possible to strongly couple the circumferential modes together which is in disagreement with the harmonic modes as predicted by the orthotropic theory. Also, it is possible for a stiffened cylindrical shell to exhibit two different natural frequencies depending upon whether the shell is excited in a symmetric or antisymmetric mode of vibration.

The results of the mathematical model were compared with the experimental results found in the literature and with the results of a Ritz analysis for simply supported cylinders. These results were found to be in excellent agreement.

No experimental results on the free vibration of longitudinally stiffened cylindrical shells have been found for boundary conditions
other than simply supported. Hence no comparison between theory and experiment has been possible. However, a numerical example is presented for a clamped-clamped stiffened cylinder and the results are compared with the experimental results for the same unstiffened shell to illustrate the use of the theory.

**Recommendations**

Further investigations might be made along the following lines.

A. An extension of the present analysis to the forced vibration case. Perhaps a good starting point would be to consider harmonic forcing functions and simply supported boundary conditions.

B. It is recommended that experimental data be obtained for boundary conditions other than simply supported with which to compare theoretical results.
APPENDIX
APPENDIX A

THIN-WALLED BEAM THEORY

The assumptions made concerning the thin-walled beams are as follows:

(1) The section contour is undeformable—the shape of the cross section does not change in its own plane.

(2) The shear strain at the middle surface of the thin-walled beam is assumed to be zero.

(3) The rotations of the cross section of the beam in its own plane are small.

(4) The thin-walled beams are homogenous and isotropic.

(5) The thin-walled beams are made of Hookean material.

(6) The cross section of the thin-walled beams are constant.

It is these assumptions which enable the equations of motion of a thin-walled beam to be reduced to a set of linear partial differential equations as was demonstrated by V. Z. Vlasov [10].

The following notation is employed (see Figure 15):

0 shear center of thin-walled beam.
C centroid of the cross section.
ξ displacement of shear center in y-direction.
η displacement of shear center in z-direction.
ζ displacement of the coordinate origin in x-direction.
θ angle of twist of cross section.
v, w \quad \text{in-plane displacements of a point on section contour along coordinate axes.}

u \quad \text{longitudinal displacement of point on stringer profile.}

By employing V. Z. Vlasov's assumptions for a thin-walled beam, one can assume displacements of the form:

\begin{align*}
    u & = \xi - y(s) \frac{\partial \xi}{\partial x} - z(s) \frac{\partial z}{\partial x} - \omega(s) \frac{\partial \theta}{\partial x} \\
    v & = \eta - [z(s) - a_z] \theta(x,t) \\
    w & = \eta + [y(s) - a_y] \theta(x,t)
\end{align*} \quad (A.1)

V. Z. Vlasov [10] showed that the strain energy of a thin-walled beam can be shown to be:

\begin{align*}
    V_s &= \frac{1}{2} \int_0^L \left\{ E A (\xi')^2 + E (\xi'')^2 I_z + E (\eta'')^2 I_y + 2E I_y z'' x'' + \\
    & \quad E (\theta'')^2 C_w + GJ(\theta')^2 \right\} dx \quad (A.4)
\end{align*}

where the prime (') denotes differentiation with respect to x and with the origin of coordinates at the centroid.

If the origin of coordinates is assumed to be located at the centroid then the kinetic energy of a stringer is given by:
\[ T_s = \frac{1}{2} \int_0^L \left( \rho A (\dot{\xi}^2 + \dot{\zeta}^2 + \dot{\eta}^2) + \rho I_z (\dot{\xi'})^2 + \rho I_y (\dot{\eta'})^2 + 2\rho I_{yz} (\dot{\xi'}) (\dot{\eta'}) \right. \\
\left. + (\dot{\xi'})^2 C_w + I_p (\dot{\psi})^2 - 2\dot{\psi} \dot{\delta} A + 2 \ddot{\psi} \dot{\delta} A \right) dx \]  

(A.5)

which includes the effects of rotatory inertia. If \( a_y = a_z = 0 \) then the shear center coincides with the centroid. If one neglects the rotatory inertia terms then the resulting equations are those used by Gere [18]. Vlasov states that studies in bending vibrations show that for most cases rotatory inertia is indeed negligible.
Figure 16. Stringer Displacements and Notation
APPENDIX B
DONELLI’S EQUATIONS OF MOTION

If one neglects the rotatory inertia effect, then the dynamic equations of motion of a cylindrical shell can be written as:

(inward w is positive)

\[
\frac{3N_x}{\partial x} + \frac{1}{a} \frac{3N_{\phi x}}{\partial \phi} + p_x = ph \frac{3^2u}{\partial t^2}, \quad (B.1)
\]

\[
\frac{1}{a} \frac{3N_{\phi}}{\partial \phi} + \frac{3N_{\phi x}}{\partial x} - \frac{Q_x}{a} + p_{\phi} = ph \frac{3^2v}{\partial t^2}, \quad (B.2)
\]

\[
\frac{3Q_x}{\partial x} + \frac{1}{a} \frac{3Q_{\phi}}{\partial \phi} + \frac{N_{\phi}}{a} + p_z = ph \frac{3^2w}{\partial t^2}, \quad (B.3)
\]

\[
\frac{3M_x}{\partial x} + \frac{1}{a} \frac{3M_{\phi}}{\partial \phi} - Q_{\phi} = 0, \quad (B.4)
\]

\[
\frac{3M_x}{\partial x} + \frac{1}{a} \frac{3M_{\phi}}{\partial \phi} - Q_{\phi} = 0, \quad (B.5)
\]

\[
N_{x \phi} - N_{\phi x} = 0. \quad (B.6)
\]

Eliminating \(Q_{\phi}\) and \(Q_x\) results in the following four equations
Figure 1.7(a). Shell Element Showing Stress Resultants

Figure 17(b). Shell Element Showing Stress Couples

Figure 17(c). Shell Element Showing Coordinate System
The stress resultants and stress couples are related to the strains and changes of curvature by:

\[
\begin{align*}
\frac{\partial N}{\partial x} + \frac{1}{a} \frac{\partial N_{\phi}}{\partial \phi} + p_x &= \rho h \frac{\partial^2 u}{\partial t^2}, \\
\frac{1}{a} \frac{\partial N}{\partial \phi} + \frac{1}{a} \frac{\partial N_{\phi}}{\partial x} + \frac{1}{a} \frac{\partial M_{\phi}}{\partial \phi} - \frac{1}{2} \frac{\partial M}{\partial \phi} + p_{\phi} &= \rho h \frac{\partial^2 v}{\partial t^2}, \\
\frac{\partial^2 M_{\phi}}{\partial x^2} + \frac{\partial^2 M}{\partial x \partial \phi} + \frac{\partial^2 M_{\phi}}{\partial \phi x} + \frac{\partial^2 M}{\partial \phi^2} + \frac{1}{a} N_{\phi} + p_z &= \rho h \frac{\partial^2 \omega}{\partial t^2}, \\
N_{\phi x} &= N_{\phi x},
\end{align*}
\]  

(B.10)

where
\{K_1, D_1\} = \{h, \frac{h^3}{12}\}E/(1-\nu^2), \tag{B.17}

and

\[ G = \frac{E}{2(1+\nu)}. \tag{B.18} \]

The strain-displacement relations are given by:

\[ e_{xx} = \frac{\partial u}{\partial x}, \tag{B.19} \]

\[ e_{\phi\phi} = \frac{1}{a} \frac{\partial v}{\partial \phi} - \frac{\partial w}{\partial \phi}, \tag{B.20} \]

\[ \gamma_{x\phi} = \frac{\partial w}{\partial x} + \frac{1}{a} \frac{\partial u}{\partial \phi}. \tag{B.21} \]

The moment-change in curvature relations as advanced by Donnell [20] are:

\[ K_x = -\frac{\partial^2 w}{\partial x^2}, \tag{B.22} \]

\[ K_\phi = -\frac{1}{a^2} \frac{\partial^2 w}{\partial \phi^2}, \tag{B.23} \]

\[ \tau = -\frac{2}{a} \frac{\partial^2 w}{\partial x \partial \phi}. \tag{B.24} \]

By substituting the curvature expressions (Equations B.22, B.23, and B.24) and the strain expressions (Equations B.19, B.20, and B.21) into the expressions for the stress resultants and stress couples and then
substituting the latter expressions into Equations (B.7), (B.8) and (B.9) yields:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{1 + v}{2a} \frac{\partial^2 v}{\partial x \partial \phi} - \frac{v}{a} \frac{\partial w}{\partial x} + \frac{1 - v}{2a} \frac{\partial^2 u}{\partial \phi^2} + \frac{(1-v^2)p}{Eh} \frac{\partial^2 u}{\partial t^2},
\]  

(B.25)

\[
\frac{1 + v}{2a} \frac{\partial^2 u}{\partial x \partial \phi} + \frac{1 - v}{2} \frac{\partial^2 v}{\partial x^2} + \frac{1}{a^2} \frac{\partial^2 v}{\partial \phi^2} - \frac{1}{a} \frac{\partial w}{\partial \phi} + \frac{(1-v^2)p}{Eh} \frac{\partial^2 v}{\partial t^2},
\]  

(B.26)

\[
\frac{v}{a} \frac{\partial u}{\partial x} + \frac{1}{a^2} \frac{\partial^2 u}{\partial \phi^2} - \frac{w}{a} - \frac{nh}{12} v^2 u + \frac{(1-v^2)p}{Eh} \frac{\partial^2 w}{\partial t^2}.
\]  

(B.27)

Applying \( \frac{\partial^2}{\partial x^2} \), \( \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2} \), and \( \frac{\partial^2}{\partial t^2} \) to Equation (B.25), solving in each case for the term involving \( n \), and substituting these expressions into the equation obtained by applying \( \frac{1}{a} \frac{\partial^2}{\partial x \partial \phi} \) to Equation (B.26) gives:

\[
\frac{v^4 u - v}{a} \frac{\partial^3 w}{\partial x^3} + \frac{1}{a^2} \frac{\partial^3 w}{\partial \phi^2} = - \frac{2(1+v)p}{E} \frac{\partial^2 u}{\partial t^2} + \frac{(1-v^2)p}{Eh} \frac{\partial^2 p_x}{\partial x^2} - \frac{(1-v^2)p}{Eh} \frac{\partial^2 p_x}{\partial x^2}.
\]  

(B.28)

Similarly, applying \( \frac{\partial^2}{\partial x^2} \), \( \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2} \), and \( \frac{\partial^2}{\partial t^2} \) to Equation (B.26), solving in each case for the term involving \( u \), and substituting these in Equation (B.25), after applying \( \frac{1}{a} \frac{\partial^2}{\partial x \partial \phi} \) to it, gives:
\[
\begin{align*}
\nabla^4 v - \frac{2 + v}{a^2} \frac{\partial^3 w}{\partial x^2 \partial \phi} - \frac{1}{a^4} \frac{\partial^3 w}{\partial \phi^3} &= \frac{2(1+v)}{E} \rho \frac{\partial^2}{\partial t^2} \left[ \frac{(1-v)^2 \rho}{E} \right] \frac{\partial^2 v}{\partial t^2} - \\
- \frac{3 - v}{2} \frac{\partial^2 v}{\partial \phi^2} + \left( \frac{1}{a^2} \right) \frac{\partial w}{\partial \phi} - \frac{(1-v^2)}{E \rho} \frac{\partial p}{\partial \phi} &= \frac{(1-v^2)}{E \rho} \frac{1}{a^2} \frac{\partial^2 p}{\partial \phi^2} - \\
\frac{(1-v^2)}{E \rho} \frac{2}{(1-v)} \frac{\partial^2 p}{\partial x^2 \partial \phi} + \frac{(1-v^2)}{E \rho} \frac{1}{a} \frac{\partial^2 p}{\partial x \partial \phi} = (B.29)
\end{align*}
\]

The third equation is obtained by applying $\frac{\nabla}{a} \frac{\partial}{\partial x}$ to Equation (B.28) and $\frac{1}{a^2} \frac{\partial}{\partial \phi}$ to Equation (B.29) and adding the results gives:

\[
\begin{align*}
\nabla^4 \left( \frac{\nabla}{a} \frac{\partial u}{\partial x} + \frac{1}{a^2} \frac{\partial v}{\partial \phi} \right) - \frac{\nabla^2}{a^2} \frac{\partial^4 w}{\partial x^2 \partial \phi^2} - \frac{2}{a^4} \frac{\partial^4 w}{a^2 \partial x \partial \phi^2} - \frac{1}{a^4} \frac{\partial^4 w}{a^2 \partial \phi^3} &= \\
- \frac{2(1+v)}{E} \rho \frac{\partial^2}{\partial t^2} \left[ \frac{v \partial u}{a \partial x} + \frac{1}{a^2} \frac{\partial v}{\partial \phi} \right] &= \\
- \frac{3 - v}{2} \frac{\partial^2}{\partial x^2} \left( \frac{v \partial u}{a \partial x} + \frac{1}{a^2} \frac{\partial v}{\partial \phi} \right) + \frac{\nabla^2}{a^2} \frac{\partial^2 w}{\partial x^2 \partial \phi^2} + \frac{1}{a^4} \frac{\partial^2 w}{\partial \phi^2} &= \\
- \frac{2(1+v)}{E} \rho \frac{\partial^2}{\partial t^2} \left[ - \frac{\nabla(v(1-v^2))}{aE} \frac{\partial p}{\partial x} - \frac{(1-v^2)}{Ea^2} \frac{\partial p}{\partial \phi} \right] &= \\
- \frac{(1-v^2)}{Ea} \frac{\partial^3 p}{\partial x^3} - \frac{2v}{a^3(1-v)} \frac{(1-v^2)}{E} \frac{\partial^3 p}{\partial x \partial \phi^2} &= \\
- \frac{1}{a^4} \frac{(1-v^2)}{E} \frac{\partial^3 p}{\partial \phi^3} - \frac{2}{a^2(1-v)} \frac{(1-v^2)}{E} \frac{\partial^3 p}{\partial x^2 \partial \phi} &= \\
+ \frac{1}{a^3} \frac{(1+v)}{E} \left[ \frac{(1-v^2)}{E} \right] \frac{\partial^3 p}{\partial x^2 \partial \phi} + \frac{v}{a^2(1-v)} \left[ \frac{(1-v^2)}{E} \right] \frac{\partial^3 p}{\partial x \partial \phi} &= (B.30)
\end{align*}
\]
The \( u \) and \( v \) terms in Equation (B.30) appear together in the expression:

\[
\frac{v}{a} \frac{\partial u}{\partial x} + \frac{1}{a^2} \frac{\partial v}{\partial \phi}
\]

If this expression is solved from Equation (B.27) in terms of \( w \) and its derivatives and the result substituted into Equation (B.30), we have:

\[
\frac{h^2}{12} w + \frac{1 - v^2}{a^2} \frac{\partial^4 w}{\partial x^4} - \frac{1 - v^2}{Eh} v^4 p_z = - \frac{2(1+v)}{E} \rho \frac{\partial^2}{\partial t^2} \left( \frac{1-v^2}{E} \rho \frac{\partial^2 p_z}{\partial t^2} \right)
\]

\[
- \frac{3-v^2}{2} \left\{ \frac{2(1-v^2)}{E} \frac{\partial^2 w}{\partial t^2} + \frac{w}{a^2} + \frac{h^2}{12} w - \frac{(1-v^2)}{Eh} p_z \right\} + \frac{(1-v)}{2} \frac{\partial^4 w}{\partial x^4} + \frac{1}{a^2} \frac{\partial^2 w}{\partial x^2} + \frac{1}{a^4} \frac{\partial^2 p_z}{\partial \phi^2}
\]

\[
- \frac{\nu(1-v)}{aEh} \frac{\partial p_x}{\partial x} - \frac{1}{Eh^2} \frac{\partial^3 p_x}{\partial \phi^3} - \frac{\nu(1-v^2)}{aEh} \frac{\partial^3 p_x}{\partial x^3} - \frac{2(1+v)}{E} \rho \frac{\partial^2}{\partial t^2} \left( \frac{1-v^2}{E} \rho \frac{\partial^2 p_z}{\partial t^2} \right)
\]

\[
\frac{2v(1-v^2)}{a^3(1-v)Eh} \frac{\partial^3 p_x}{\partial x^3} + \frac{1}{a^4} \frac{(1-v^2)}{Eh^3} \frac{\partial^3 p_x}{\partial \phi^3} + \frac{2(1-v^2)}{a^2(1-v)Eh} \frac{\partial^3 p_x}{\partial x^2 \partial \phi} + \frac{1}{a^3} \frac{(1+v)}{Eh} \frac{(1-v^2)}{\partial x^3} \frac{\partial^3 p_x}{\partial x^3 \partial \phi} + \frac{\nu(1+v)}{a(1-v)} \frac{(1-v^2)}{Eh} \frac{\partial^3 p_x}{\partial x^3 \partial \phi}.
\]  

(B.31)

After some simplification, we have:
\[ \frac{\hbar^2}{12} v^\theta w + \frac{(1-v^2)}{a^2} \frac{\partial^4 v}{\partial x^4} - \frac{(1-v^2)}{Eh} v^\theta p_z = \]

\[ - \frac{2(1+v)p}{E} \frac{a^2}{\partial t^2} \left[ \left( \frac{(1-v^2)p}{E} \frac{a^2}{\partial x^2} - \frac{(3-v)}{2} v^2 \right) \right. \]

\[ + \frac{(1-v)}{2} v^4 + \frac{v^2}{a} \frac{\partial^2 v^2}{\partial x^2} + \frac{1}{4} \frac{\partial^2 v^2}{\partial \phi^2} \]

\[ - \frac{2(1+v)p}{E} \frac{a^2}{\partial t^2} \left[ \frac{v(1-v)}{aeh} \frac{\partial^2 v}{\partial x^2} - \frac{(1-v^2)}{Eha^2} \frac{\partial^2 v}{\partial \phi^2} \right] \]

\[ - \frac{v(1-v^2)}{aeh} \frac{\partial^3 p_x}{\partial x^3} + \frac{(1-v^2)(1-v)}{a^3(1-v)eh} \frac{\partial^3 p_x}{\partial x \partial \phi^2} \]

\[ - \frac{1}{a^4 \frac{(1-v^2)}{Eh} \frac{\partial^3 p}{\partial \phi^3} - \frac{(2+v)(1-v^2)(1-v^2)}{a^2(1-v)eh} \frac{\partial^3 p}{\partial x^2 \partial \phi} \] (B.32)
APPENDIX C

MATRIX ELEMENTS IN RITZ ANALYSIS

This appendix contains expressions for the matrix elements appearing in the submatrices of the mass matrix of Equation (3.15) and the stiffness matrix given by Equation (3.14). The axial mode number \(m\) is considered as being given. The elements of the submatrices of the stiffness matrix are given as follows:

\[
k_{11}(j,n) = \delta_{jn} \left[ a^2 \alpha_m^2 + \frac{1}{2}(1-v)^2(1+\beta) \right] 
+ a^2 \alpha_m \sum_{i=1}^{K} D_{ij} \cos n\phi_i \cos j\phi_i, \tag{C.1}
\]

\[
k_{12}(j,n) = -\delta_{jn} \left[ a_j \frac{1}{2}(1+v)\alpha_m \right] + a^2 \alpha_m \sum_{i=1}^{K} D_{1j}(y_i) \sin n\phi_i \cos j\phi_i, \tag{C.2}
\]

\[
k_{13}(j,n) = \delta_{jn} \left[ -a_m - 3a^3 \alpha_m + \frac{(1-v)}{2} \alpha_m \beta \right] 
+ a^2 \alpha_m \sum_{i=1}^{K} D_{1j} Z_i \cos n\phi_i \cos j\phi_i. \tag{C.3}
\]
\[ k_{14}(j,n) = a_2^2 \sum_{i=1}^{K} D_{1i} \sin n\phi_i \cos j\phi_i \]  \hfill (C.4)

\[ k_{15}(j,n) = a_2^3 \sum_{i=1}^{K} D_{1i}(y_i^2) \cos n\phi_i \cos j\phi_i \]  \hfill (C.5)

\[ k_{16}(j,n) = a_2^3 \sum_{i=1}^{K} D_{1i} z_i \sin n\phi_i \cos j\phi_i \]  \hfill (C.6)

\[ k_{22}(j,n) = \delta_{jn} \left[ j^2 + \frac{1}{2}(1-v)a_2^2 + \beta(\frac{3}{2}(1-v)a_2^2) \right] \]

\[ + a_2^4 \sum_{i=1}^{K} D_{1i} y_i^2 \sin n\phi_i \sin j\phi_i \]

\[ + a_2^4 \sum_{i=1}^{K} D_{1i} z_i \sin(n\phi_i)\sin(j\phi_i) \]  \hfill (C.7)

\[ k_{23}(j,n) = \delta_{jn} \left[ 1 + 8(\nu a_2^2 + \frac{3}{2}(1-v)a_2^2) \right] \]

\[ + a_2^4 \sum_{i=1}^{K} D_{1i} y_i z_i \cos(n\phi_i)\sin(j\phi_i) \]

\[ + a_2^4 \sum_{i=1}^{K} D_{1i} y_i / z_i \cos n\phi_i \sin j\phi_i \]  \hfill (C.8)
\[ k_{24}(j,n) = +a^2_\alpha a^m_{i} \sum_{i=1}^{K} D_{1i}(y_i) \sin(n\Phi_i) \sin(j\Phi_i), \quad (C.9) \]

\[ k_{25}(j,n) = -a^2_\alpha a^m_{i} \sum_{i=1}^{K} D_{1i}(y_i^2) \cos(n\Phi_i) \sin(j\Phi_i) \]

\[ - a^2_\alpha a^m_{i} \sum_{i=1}^{K} D_{1i} I_{zi}/A_i \cos(n\Phi_i) \sin(j\Phi_i), \quad (C.10) \]

\[ k_{26}(j,n) = a^2_\alpha a^m_{i} \sum_{i=1}^{K} D_{1i}(y_i)Z_i \sin(n\Phi_i) \cos(j\Phi_i) \]

\[ + a^2_\alpha a^m_{i} \sum_{i=1}^{K} D_{1i} I_{yi}/A_i \sin(n\Phi_i) \sin(j\Phi_i) \]

\[ + a^2_\alpha a^m_{i} \sum_{i=1}^{K} D_{1i} I_{zi}/A_i \frac{1}{a} \sin(j\Phi_i) \cos(n\Phi_i), \quad (C.11) \]

\[ k_{33}(j,n) = \delta_{jn} \left[ 1 + a^4_\alpha (1+n^2) + a^4_\alpha \cos(n\Phi_i) \cos(j\Phi_i) + \right. \]

\[ + a^2_\alpha a^m_{i} \sum_{i=1}^{K} D_{1i}(Z_i^2) \cos(n\Phi_i) \cos(j\Phi_i) \]

\[ + a^2_\alpha a^m_{i} \sum_{i=1}^{K} D_{1i} I_{yi}/A_i \cos(n\Phi_i) \cos(j\Phi_i) \]

\[ + a^2_\alpha a^m_{i} \sum_{i=1}^{K} D_{3i}(j)(n) \sin(n\Phi_i) \sin(j\Phi_i) \]

\[ + a^2_\alpha a^m_{i} \sum_{i=1}^{K} D_{2i}(j)(n) \sin(n\Phi_i) \sin(j\Phi_i). \]
\[ + a^2 \alpha_m \sum_{i=1}^{K} D_{li} I_{yzi/A_i} \left( \frac{Z_i}{a} \right) \cos(j \psi_i) \sin(n \phi_i) \]

\[ - a^2 \alpha_m \sum_{i=1}^{K} D_{li} I_{zi/A_i} \left( \frac{Z_i}{a} \right)^2 \cos(j \psi_i) \sin(n \phi_i), \quad (C.12) \]

\[ k_{45}(j,n) = +a^2 \alpha_m \sum_{i=1}^{K} D_{li} I_{yzi/A_i} \sin(n \phi_i) \cos(j \psi_i), \quad (C.13) \]

\[ k_{35}(j,n) = a^2 \alpha_m \sum_{i=1}^{K} D_{li} (y_{zi}) Z_i \cos(n \phi_i) \cos(j \psi_i) \]

\[ + a^2 \alpha_m \sum_{i=1}^{K} D_{li} I_{yi/A_i} \sin(n \phi_i) \sin(j \psi_i) \]

\[ + a^2 \alpha_m \sum_{i=1}^{K} D_{li} I_{zi/A_i} \left( \frac{Z_i}{a} \right)(n) \sin(n \phi_i) \cos(j \psi_i), \quad (C.14) \]

\[ k_{36}(j,n) = a^2 \alpha_m \sum_{i=1}^{K} D_{li} (Z_i)^2 \sin(n \phi_i) \cos(j \psi_i) \]

\[ + a^2 \alpha_m \sum_{i=1}^{K} D_{li} I_{yi/A_i} \sin(n \phi_i) \cos(j \psi_i) \]

\[ - a^2 \alpha_m \sum_{i=1}^{K} D_{li} (j)(n) \cos(n \phi_i) \sin(j \psi_i) \]

\[ - a^2 \alpha_m \sum_{i=1}^{K} D_{li} (j)(n) \sin(n \phi_i) \cos(j \psi_i). \quad (C.15) \]

The elements of the submatrices of the mass matrix are given by the following expressions:
\[ m_{11}(j,n) = \delta_{jn} + \sum_{i=1}^{K} D_{4i} \cos(n\phi_i) \cos(j\phi_i), \quad (C.16) \]

\[ m_{12}(j,n) = \alpha_m \sum_{i=1}^{K} D_{4i} (y_i) \sin(n\phi_i) \cos(j\phi_i), \quad (C.17) \]

\[ m_{13}(j,n) = + \alpha_m \sum_{i=1}^{K} D_{4i} (Z_i) \cos(n\phi_i) \cos(j\phi_i), \quad (C.18) \]

\[ m_{14}(j,n) = \sum_{i=1}^{K} D_{4i} \sin(n\phi_i) \cos(j\phi_i), \quad (C.19) \]

\[ m_{15}(j,n) = \sum_{i=1}^{K} D_{4i} (y_i) \cos(n\phi_i) \cos(j\phi_i), \quad (C.20) \]

\[ m_{16}(j,n) = + \alpha_m \sum_{i=1}^{K} D_{4i} (Z_i) \sin(n\phi_i) \cos(j\phi_i), \quad (C.21) \]

\[ m_{22}(j,n) = \delta_{jn} (1+\beta) + \sum_{i=1}^{K} D_{4i} \left[ 1 + (y_i^2) + \alpha_m \frac{I_{z1}}{A_i} \sin(n\phi_i) \sin(j\phi_i) \right], \quad (C.22) \]

\[ m_{23}(j,n) = \delta_{jn} (2\beta_i) - \sum_{i=1}^{K} D_{4i} \left[ \frac{a_i}{\alpha} \right] (n) \sin(n\phi_i) \sin(j\phi_i) + \]

\[ \alpha_m \sum_{i=1}^{K} D_{4i} (y_i) (Z_i) \cos(n\phi_i) \sin(j\phi_i), \quad (C.23) \]
\[ m_{24}(j,n) = \alpha_m \sum_{i=1}^{K} D_{4i}(y_i)(n) \sin(n\phi_i) \sin(j\phi_i), \quad (C.24) \]

\[ m_{25}(j,n) = -\sum_{i=1}^{K} D_{4i}(1 + (y_i^2) + \frac{I_{zi}}{mA_i}) \cos(n\phi_i) \sin(j\phi_i), \quad (C.25) \]

\[ m_{26}(j,n) = \sum_{i=1}^{K} D_{4i}(\alpha_i^2 I_y/A_i + \alpha_i^2 (y_i)(Z_i)) \sin(n\phi_i) \sin(j\phi_i) + \]

\[ \sum_{i=1}^{K} D_{4i}(\frac{a_{zi}}{a})(n) \cos(n\phi_i) \sin(j\phi_i), \quad (C.26) \]

\[ m_{33}(j,n) = \delta_{jn}[1 + \beta(2\alpha_m^2 - j^2)] + \sum_{i=1}^{K} D_{4i}(1 + Z_i^2 + \alpha_m^2 I_{yi}/A_i) \cos(n\phi_i) \cos(j\phi_i) + \sum_{i=1}^{K} D_{4i}(j)(n) \left[ \frac{Z_i}{a} \right]^2 + \frac{(y_i/a)^2}{a} + \]

\[ I_{pi}/a^2 \sin(n\phi_i) \sin(j\phi_i) + \sum_{i=1}^{K} D_{4i}(\frac{y_i}{a}) \cos(n\phi_i) \sin(j\phi_i) \]

\[ + \sum_{i=1}^{K} D_{4i}(\frac{a}{a})(n) \sin(n\phi_i) \cos(j\phi_i) \]

\[ + \sum_{i=1}^{K} D_{4i}(\frac{a}{a})(n) \cos(j\phi_i) \sin(n\phi_i), \quad (C.27) \]

\[ m_{34}(j,n) = \alpha_m \sum_{i=1}^{K} D_{4i}(Z_i) \sin(n\phi_i) \cos(j\phi_i), \quad (C.28) \]

\[ m_{35}(j,n) = \sum_{i=1}^{K} D_{4i}(\frac{Z_i}{a})(n) \cos(n\phi_i) \sin(j\phi_i) \]

\[ - \alpha_m \sum_{i=1}^{K} D_{4i}(y_i)Z_i \sin(n\phi_i) \sin(j\phi_i) \]
\[ + \sum_{i=1}^{K} D_{ui} \left( \frac{y_i}{a} \right) (n) \cos(n\phi_i) \sin(j\phi_i), \]

\[ - \alpha_m^2 \sum_{i=1}^{K} D_{ui} \left( \frac{z_i}{A_i} \right) \cos(n\phi_i) \cos(j\phi_i), \quad (C.29) \]

\[ m_{36}(j,n) = \alpha_m^2 \sum_{i=1}^{K} D_{ui} \left( \frac{y_i}{a} \right)^2 (n) \cos(n\phi_i) \sin(j\phi_i) \]

\[ + \sum_{i=1}^{K} D_{ui} \left( \frac{y_i}{a} \right) (j) \sin(n\phi_i) \sin(j\phi_i) - \sum_{i=1}^{K} D_{ui} \left( \frac{y_i}{a} \right)^2 (j) (n) \cos(n\phi_i) \sin(j\phi_i) \]

\[ - \sum_{i=1}^{K} D_{ui} \left( \frac{z_i}{a} \right)^2 (n) \cos(n\phi_i) \sin(j\phi_i) - \sum_{i=1}^{K} D_{ui} \left( \frac{a y_i}{a} \right) (n) \cos(n\phi_i) \cos(n\phi_i) \quad (C.30) \]

The elements of the submatrices \( (m_{11}^a), (m_{12}^a), (m_{13}^a), (m_{22}^a), (m_{23}^a), \) and \( (m_{33}^a) \) which represent the antisymmetric circumferential modes of vibration may be obtained from the elements of the submatrices \( (m_{11}), (m_{12}), (m_{13}), (m_{22}), (m_{23}) \) and \( (m_{33}) \) by interchanging \( \sin(n\phi_i) \) and \( \cos(n\phi_i) \) and \( y_i \) with \( -h_i \). The constants appearing in the expressions for the mass and stiffness submatrices are given by:
where \((GJ)_i\) is the torsional stiffness and \(C_{wi}\) is the warping constant for the \(i\)th stringer. When Donnell's shell theory is used the expressions for the shell theory in Equations (C.1) to (C.15) are changed to:

\[
\begin{align*}
D_{11} &= \frac{E_i A_i (1-\nu^2)}{Ew h}, \quad (C.31) \\
D_{2i} &= \frac{(GJ)_i (1-\nu^2)}{Ew h^3}, \quad (C.32) \\
D_{3i} &= \frac{C_{wi} (1-\nu^2)}{Ew h^3}, \quad (C.33) \\
D_{4i} &= \frac{\rho_i A_i}{\rho w h}, \quad (C.34)
\end{align*}
\]

\[
\begin{align*}
k_{11}(j,n) &= \delta_{jn} \left\{ a_m^2 \alpha_j^2 + \frac{1}{2} (1-\nu)^2 \right\}, \quad (C.35) \\
k_{12}(j,n) &= -\delta_{nj} \left\{ a_j \frac{1}{2} (1+\nu) \right\} \alpha_m, \quad (C.36) \\
k_{13}(j,n) &= \delta_{jn} \{- a_m \alpha_m\}, \quad (C.37) \\
k_{22}(j,n) &= \delta_{jn} \left\{ j^2 + \frac{1}{2} (1-\nu) a_m^2 \alpha_m^2 \right\}, \quad (C.38) \\
k_{23}(j,n) &= \delta_{jn}(j), \quad (C.39) \\
k_{33}(j,n) &= \delta_{jn} [1 + \beta (n^4 a_m^4 + 2 a_m^2 n^2 a_m^2)], \quad (C.40)
\end{align*}
\]
where $\beta = h^2/12a^2$ and the beta terms represent the bending energy of the shell.
APPENDIX D

MATRIX ELEMENTS FOR CLAMPED-CLAMPED SUPPORTS

The nonzero elements of the coefficient matrix $[D]$ for clamped-clamped boundary are given as follows:

\[
\begin{align*}
d_{11} & = \tilde{a}_{1n}, \\
d_{12} & = \tilde{a}_{2n}, \\
d_{13} & = \tilde{a}_{3n}, \\
d_{14} & = \tilde{a}_{4n}, \\
d_{15} & = \tilde{a}_{5n}, \\
d_{16} & = \tilde{a}_{6n}, \\
d_{17} & = \tilde{a}_{7n}, \\
d_{18} & = \tilde{a}_{8n}, \\
d_{21} & = \tilde{b}_{1n}.
\end{align*}
\]
\[ d_{22} = \beta_{2n}, \]

\[ d_{23} = \beta_{3n}, \]

\[ d_{24} = \beta_{4n}, \]

\[ d_{25} = \beta_{5n}, \]

\[ d_{26} = \beta_{6n}, \]

\[ d_{27} = \beta_{7n}, \]

\[ d_{28} = \beta_{8n}, \]

\[ d_{31} = 1, \]

\[ d_{32} = 0, \]

\[ d_{33} = 1, \]

\[ d_{34} = 0, \]

\[ d_{35} = 1, \]

\[ d_{36} = 0, \]
\[ d_{37} = 1, \]
\[ d_{38} = 0, \]
\[ d_{41} = \phi_1, \]
\[ d_{42} = \theta_1, \]
\[ d_{43} = \psi_2, \]
\[ d_{44} = \theta_2, \]
\[ d_{45} = \theta_3, \]
\[ d_{46} = \psi_3, \]
\[ d_{47} = \psi_4, \]
\[ d_{48} = \theta_4, \]
\[ d_{51} = e^{\ln a} (\bar{\alpha}_1 \ln \cos \theta_1 \ln \frac{\ell}{a} - \bar{\alpha}_2 \sin \theta_1 \ln \frac{\ell}{a}), \]
\[ d_{52} = e^{\ln a} (\bar{\alpha}_2 \cos \theta_1 \ln \frac{\ell}{a} + \bar{\alpha}_1 \sin \theta_1 \ln \frac{\ell}{a}), \]
\[ d_{53} = e^{2n a} (\alpha_{3n} \cos \theta_{2n} \ell/a - \bar{\alpha}_{4n} \sin \theta_{2n} \ell/a), \]
\[ d_{54} = e^{2n a} (\alpha_{4n} \cos \theta_{2n} \ell/a + \bar{\alpha}_{3n} \sin \theta_{2n} \ell/a), \]
\[ d_{55} = e^{3n a} (\alpha_{5n} \cos \theta_{2n} \ell/a - \bar{\alpha}_{6n} \sin \theta_{3n} \ell/a), \]
\[ d_{56} = e^{3n a} (\alpha_{6n} \cos \theta_{3n} \ell/a + \bar{\alpha}_{5n} \sin \theta_{3n} \ell/a), \]
\[ d_{57} = e^{4n a} (\alpha_{7n} \cos \theta_{4n} \ell/a - \bar{\alpha}_{8n} \sin \theta_{4n} \ell/a), \]
\[ d_{58} = e^{4n a} (\alpha_{8n} \cos \theta_{4n} \ell/a + \bar{\alpha}_{7n} \sin \theta_{4n} \ell/a), \]
\[ d_{61} = e^{ln a} (\beta_{1n} \cos \theta_{1n} \ell/a - \bar{\beta}_{2n} \sin \theta_{1n} \ell/a), \]
\[ d_{62} = e^{ln a} (\beta_{2n} \cos \theta_{2n} \ell/a + \bar{\beta}_{1n} \sin \theta_{2n} \ell/a), \]
\[ d_{63} = e^{2n a} (\beta_{3n} \cos \theta_{2n} \ell/a - \bar{\beta}_{4n} \sin \theta_{2n} \ell/a), \]
\[ d_{64} = e^{2n a} (\beta_{4n} \cos \theta_{2n} \ell/a + \bar{\beta}_{3n} \sin \theta_{2n} \ell/a), \]
\[ d_{65} = e^{3n a} (\beta_{5n} \cos \theta_{3n} \ell/a - \bar{\beta}_{6n} \sin \theta_{3n} \ell/a), \]
\[ d_{66} = e^{\frac{\lambda}{a}} (\tilde{\beta}_5 \cos \theta_3 \frac{z}{a} + \tilde{\beta}_7 \sin \theta_3 \frac{z}{a}), \]
\[ d_{57} = e^{\frac{\lambda}{a}} (\tilde{\beta}_8 \cos \theta_4 \frac{z}{a} - \tilde{\beta}_6 \sin \theta_4 \frac{z}{a}), \]
\[ d_{68} = e^{\frac{\lambda}{a}} (\tilde{\beta}_7 \cos \theta_4 \frac{z}{a} + \tilde{\beta}_5 \sin \theta_4 \frac{z}{a}), \]
\[ d_{71} = e^{\frac{\lambda}{a}} \cos \theta_1 \frac{z}{a}, \]
\[ d_{72} = e^{\frac{\lambda}{a}} \sin \theta_1 \frac{z}{a}, \]
\[ d_{73} = e^{2\frac{\lambda}{a}} \cos \theta_2 \frac{z}{a}, \]
\[ d_{74} = e^{2\frac{\lambda}{a}} \sin \theta_2 \frac{z}{a}, \]
\[ d_{75} = e^{3\frac{\lambda}{a}} \cos \theta_3 \frac{z}{a}, \]
\[ d_{76} = e^{3\frac{\lambda}{a}} \sin \theta_3 \frac{z}{a}, \]
\[ d_{77} = e^{4\frac{\lambda}{a}} \cos \theta_4 \frac{z}{a}, \]
\[ d_{78} = e^{4\frac{\lambda}{a}} \sin \theta_4 \frac{z}{a}, \]
\[ d_{81} = \psi_{1n} e^{\ln a} \cos \theta_{1n} \frac{\ell}{a} - \theta_{1n} e^{\ln a} \sin \theta_{1n} \frac{\ell}{a}, \]

\[ d_{82} = \psi_{1n} e^{\ln a} \sin \theta_{1n} \frac{\ell}{a} + \theta_{1n} e^{\ln a} \cos \theta_{1n} \frac{\ell}{a}, \]

\[ d_{83} = \psi_{2n} e^{\ln a} \cos \theta_{2n} \frac{\ell}{a} - \theta_{2n} e^{\ln a} \sin \theta_{2n} \frac{\ell}{a}, \]

\[ d_{84} = \psi_{2n} e^{\ln a} \sin \theta_{2n} \frac{\ell}{a} + \theta_{2n} e^{\ln a} \cos \theta_{2n} \frac{\ell}{a}, \]

\[ d_{85} = \psi_{3n} e^{\ln a} \cos \theta_{3n} \frac{\ell}{a} - \theta_{3n} e^{\ln a} \sin \theta_{3n} \frac{\ell}{a}, \]

\[ d_{86} = \psi_{3n} e^{\ln a} \sin \theta_{3n} \frac{\ell}{a} + \theta_{3n} e^{\ln a} \cos \theta_{3n} \frac{\ell}{a}, \]

\[ d_{87} = \psi_{4n} e^{\ln a} \cos \theta_{4n} \frac{\ell}{a} - \theta_{4n} e^{\ln a} \sin \theta_{4n} \frac{\ell}{a}, \]

\[ d_{87} = \psi_{4n} e^{\ln a} \sin \theta_{4n} \frac{\ell}{a} + \theta_{4n} e^{\ln a} \cos \theta_{4n} \frac{\ell}{a}. \]
APPENDIX E

ALGEBRAIC EXPRESSIONS IN COMPLEMENTARY SOLUTION

This appendix contains expressions for the \( \alpha \)'s and \( \beta \)'s which occur in the complementary solutions to Donnell's equations. These are given as follows:

\[
\bar{\beta}_{1n} = \frac{Y_{11}Y_{13} - Y_{12}Y_{14}}{Y_{13}^2 + Y_{14}^2}, \quad (E.1)
\]

\[
\bar{\beta}_{2n} = \frac{Y_{12}Y_{13} + Y_{11}Y_{14}}{Y_{13}^2 + Y_{14}^2}, \quad (E.2)
\]

\[
\bar{\beta}_{3n} = \frac{Y_{21}Y_{23} - Y_{22}Y_{24}}{Y_{23}^2 + Y_{24}^2}, \quad (E.3)
\]

\[
\bar{\beta}_{4n} = \frac{Y_{22}Y_{23} + Y_{21}Y_{24}}{Y_{23}^2 + Y_{24}^2}, \quad (E.4)
\]

\[
\bar{\beta}_{5n} = \frac{Y_{31}Y_{33} - Y_{32}Y_{34}}{Y_{33}^2 + Y_{34}^2}, \quad (E.5)
\]

\[
\bar{\beta}_{6n} = \frac{Y_{32}Y_{33} + Y_{31}Y_{34}}{Y_{33}^2 + Y_{34}^2}, \quad (E.6)
\]
\[ \bar{a}_{7n} = \frac{Y_{41}Y_{43} - Y_{42}Y_{44}}{y_{43}^2 + y_{44}^2}, \quad (E.7) \]
\[ \bar{a}_{8n} = \frac{Y_{42}Y_{43} + Y_{41}Y_{44}}{y_{43}^2 + y_{44}^2}, \quad (E.8) \]
\[ \bar{a}_{9n} = \frac{x_{11}y_{13} + x_{12}y_{14}}{y_{13}^2 + y_{14}^2}, \quad (E.9) \]
\[ \bar{a}_{10n} = \frac{x_{12}y_{13} - x_{11}y_{14}}{y_{13}^2 + y_{14}^2}, \quad (E.10) \]
\[ \bar{a}_{11n} = \frac{x_{21}y_{23} + x_{22}y_{24}}{y_{23}^2 + y_{24}^2}, \quad (E.11) \]
\[ \bar{a}_{12n} = \frac{x_{22}y_{23} - x_{21}y_{24}}{y_{23}^2 + y_{24}^2}, \quad (E.12) \]
\[ \bar{a}_{13n} = \frac{x_{31}y_{33} + x_{32}y_{34}}{y_{33}^2 + y_{34}^2}, \quad (E.13) \]
\[ \bar{a}_{14n} = \frac{x_{32}y_{33} - x_{31}y_{34}}{y_{33}^2 + y_{34}^2}, \quad (E.14) \]
\[ \bar{a}_{15n} = \frac{x_{41}y_{43} + x_{42}y_{44}}{y_{43}^2 + y_{44}^2}, \quad (E.15) \]
\[
\alpha_{8n} = \frac{X_{42}v_{43} - X_{41}v_{44}}{v_{43}^2 + v_{44}^2}, \quad (E.15)
\]

where

\[
Y_{11} = n(n^2 - 2\Omega/(1-\nu) - (2+\nu)(\psi_{1n}^2 - \theta_{1n}^2)), \quad (E.17)
\]

\[
Y_{12} = \{2n(2+\nu)(\psi_{1n}^2 - \theta_{1n}^2)\}, \quad (E.18)
\]

\[
Y_{13} = \frac{-2n^2}{(1-\nu)} - \frac{(3-\nu)\Omega n^2}{(1-\nu)} + n^4 + (\psi_{1n}^2 - \theta_{1n}^2) \left(\frac{3-\nu}\Omega}{(1-\nu)} - 2n^2 + 1\right) \\
- 4\psi_{1n}^2 \theta_{1n}^2 + (\psi_{1n}^2 - \theta_{1n}^2)^2, \quad (E.19)
\]

\[
Y_{14} = \{4\psi_{1n}^2 \theta_{1n}^2 (\psi_{1n}^2 - \theta_{1n}^2) + \frac{(3-\nu)\Omega}{(1-\nu)} - 2n^2 \} (2\psi_{1n} \theta_{1n}), \quad (E.20)
\]

\[
Y_{21} = n(n^2 - 2\Omega/(1-\nu) - (2+\nu)(\psi_{2n}^2 - \theta_{2n}^2)), \quad (E.21)
\]

\[
Y_{22} = \{2n(2+\nu)(\psi_{2n}^2 - \theta_{2n}^2)\}, \quad (E.22)
\]

\[
Y_{23} = \frac{-2n^2}{(1-\nu)} - \frac{(3-\nu)\Omega n^2}{(1-\nu)} + n^4 + (\psi_{2n}^2 - \theta_{2n}^2) \left(\frac{3-\nu}\Omega}{(1-\nu)} - 2n^2 + 1\right) \\
- 4\psi_{2n}^2 \theta_{2n}^2 + (\psi_{2n}^2 - \theta_{2n}^2)^2, \quad (E.23)
\]

\[
Y_{24} = \{4\psi_{2n}^2 \theta_{2n}^2 (\psi_{2n}^2 - \theta_{2n}^2) + \frac{(3-\nu)\Omega}{(1-\nu)} - 2n^2 \} (2\psi_{2n} \theta_{2n}), \quad (E.24)
\]
\[ Y_{31} = n\{n^2 - 2\Omega/(1-v) - (2+v)(\psi_{3n}^2 - \theta_{3n}^2)\}, \quad (E.25) \]

\[ Y_{32} = \{2n(2+v)(\psi_{3n}^2)\}, \quad (E.26) \]

\[ Y_{33} = \left\{ \frac{2\Omega^2}{(1-v)} - \frac{(3-v)\Omega^2n^2}{(1-v)} + n^4 + (\psi_{3n}^2 - \theta_{3n}^2) \right\}^{(3-v)\Omega}{(1-v)} - 2n^2 + 1 - \{\psi_{3n}^2 + \theta_{3n}^2\}^2, \quad (E.27) \]

\[ Y_{34} = \{\psi_{3n}^2 - \theta_{3n}^2\} + \{\frac{(3-v)\Omega}{(1-v)} - 2n^2\} \cdot \{2\psi_{3n}^2 - \theta_{3n}^2\}, \quad (E.28) \]

\[ Y_{41} = n\{n^2 - 2\Omega/(1-v) - (2+v)(\psi_{4n}^2 - \theta_{4n}^2)\}, \quad (E.29) \]

\[ Y_{42} = \{2n(2+v) (\psi_{4n}^2)\}, \quad (E.30) \]

\[ Y_{43} = \left\{ \frac{2\Omega^2}{(1-v)} - \frac{(3-v)\Omega^2n^2}{(1-v)} + n^4 + (\psi_{4n}^2 - \theta_{4n}^2) \right\}^{(3-v)\Omega}{(1-v)} - 2n^2 + 1 - \{\psi_{4n}^2 + \theta_{4n}^2\}^2, \quad (E.31) \]

\[ X_{11} = \left[ (\psi_{1n}) \left\{ -\frac{2\Omega}{(1-v^2)} + n^2 \right\} \right] + v((\psi_{1n}^2) - \theta_{1n}^2) - 2\psi_{1n}^2 \theta_{1n}^2 \left\{ (\psi_{1n}^2 - \theta_{1n}^2) - 2\psi_{1n}^2 \right\}, \quad (E.32) \]
\( X_{12} = \left[ (\psi_{2n}^2 + n^2) + \nu(\psi_{1n}^2 - \psi_{1n}) + 2\psi_{1n}^2 \right] \), \hspace{1cm} (E.33)

\( X_{21} = \left[ (\psi_{2n}^2 + n^2) + \nu(\psi_{2n}^2 - \psi_{2n}) + 2\psi_{2n}^2 \right] \), \hspace{1cm} (E.34)

\( X_{22} = \left[ (\psi_{2n}^2 + n^2) + \nu(\psi_{2n}^2 - \psi_{2n}) + 2\psi_{2n}^2 \right] \), \hspace{1cm} (E.35)

\( X_{31} = \left[ (\psi_{3n}^2 + n^2) + \nu(\psi_{3n}^2 - \psi_{3n}) + 2\psi_{3n}^2 \right] \), \hspace{1cm} (E.36)

\( X_{32} = \left[ (\psi_{3n}^2 + n^2) + \nu(\psi_{3n}^2 - \psi_{3n}) + 2\psi_{3n}^2 \right] \), \hspace{1cm} (E.37)

\( X_{41} = \left[ (\psi_{4n}^2 + n^2) + \nu(\psi_{4n}^2 - \psi_{4n}) + 2\psi_{4n}^2 \right] \), \hspace{1cm} (E.38)

\( X_{42} = \left[ (\psi_{4n}^2 + n^2) + \nu(\psi_{4n}^2 - \psi_{4n}) + 2\psi_{4n}^2 \right] \). \hspace{1cm} (E.39)
APPENDIX F

FUNCTIONS IN PARTICULAR SOLUTION OF DONNELL'S EQUATIONS

This appendix contains expressions for the functions $g_1$, $g_2$, $g_3$, $g_4$, $g_5$, $g_6$, $g_7$, $g_8$, and $g_9$ which occur in the particular solutions of the Donnell's equations. These functions are given as follows:

$$g_1 = \frac{N_1}{D_1}, \quad (F.1)$$

where

$$N_1 = -(1+\nu)(3-\nu)(1-\nu^2)\rho a^2 \alpha_{mn}^2 (12Ea^4)(\omega^2) +$$

$$+ (24)(1+\nu)(1-\nu^2)\rho^2 a^4 \omega^2 + (1-\nu^2)12E^2 a^4 \alpha_{mn}^4,$$

and

$$D_1 = \omega^6 (-24(1+\nu)(1-\nu^2)\rho^3 \alpha_{mn}^2) + \omega^4 (24(1+\nu)(1-\nu^2)\rho^2 Eha^2 +$$

$$+ (1+\nu)(1-\nu^2)\rho^2 h^2 \alpha_{mn}^2 (2)Eha^4 + (1+\nu)(3-\nu)(1-\nu^2)\rho^2 \alpha_{mn}^2 (12)Eha^4) +$$

$$\omega^2 (-1+\nu)(3-\nu)\rho a^2 (12E^2 ha^2 - (1+\nu)(3-\nu)\rho h^2 \alpha_{mn}^6 E^2 ha^4$$

$$- 12(1+\nu)(1-\nu^2)\rho E^2 ha^4 \alpha_{mn}^4 + 24(1+\nu)(\nu^2)\rho E^2 ha^2 \alpha_m^2 + 24(1+\nu)\rho E^2 hn^2}$$
\[ + \{E^3 \alpha^3 \alpha^8_{mn} + 12(1-v^2)E^3 \alpha^2_{mn} \}, \]

\[ g_2 = \frac{N_2}{(1-v)D_1}, \quad (F.2) \]

where

\[ N_2 = -24(1+v)(1-v)(1-v)Ea^2 \rho_{mn}(\omega^2) + \]

\[ + n^3 (12)E^2 (1-v^2) (1-v) + (2+v)(1-v)^2(\alpha^2_m)(n)12E^2 a^2, \]

\[ g_3 = \frac{N_3}{D_3}, \quad (F.3) \]

where

\[ N_3 = (2+v)(\alpha^2_m)(n)g_1 E^2 a^2 + g_1 n^3 E^2 - 2(1+v)g_{mn} n a^2 E, \]

and

\[ D_3 = E^2 a^2 \alpha^4_{mn} - 2(1+v)(1-v)\rho a^4 \omega^2 - E^4 (1+v)(3-v)\rho a^2 \omega^2, \]

\[ g_4 = \frac{N_4}{D_4}, \quad (F.4) \]

where

\[ N_4 = \omega^2 \{2(1+v)(1-v^2)(1-v)\rho a^4 - 2(1+v)(1-v)(n)g_{2} \rho E h \} + \]
\[(2+\nu)(\alpha_m^2)(n)g_2E^2h(1-\nu)a^2 + n^3g_2E^2h(1-\nu) - (1-\nu^2)(n^2)E(1-\nu)a^2 \]

\[- 2(1-\nu^2)\alpha_m^2Ea^4,\]

and

\[D_4 = E^2h(1-\nu)a^4\alpha_{mn} - 2(1+\nu)(1-\nu^2)(1-\nu)\rho^2ha^4\omega^4 \]

\[- 2(1+\nu)\rho\alpha_{mn}^2 Eh (1-\nu)a^4\omega^2,\]

\[g_5 = \frac{N_5}{D_5}, \quad \tag{F.5}\]

where

\[N_5 = Ea^2\alpha_m^3g_1 - 2Ea(1+\nu)\rho\alpha_m^2g_1\omega^2 - E\alpha_m(n^2)g_1,\]

and

\[D_5 = E^2a^2\alpha_{mn}^4 + 2(1+\nu)(1-\nu^2)a^2\rho^2\omega^4 - E\alpha_m^2(1-\nu)(\alpha_m^2)g_2 = E^2a(1+\nu)(\rho^2\alpha_m^2)\omega^2,\]

\[g_6 = \frac{N_6}{D_6}, \quad \tag{F.6}\]

where

\[N_6 = E^2av(\alpha_m^2)g_2 - E^2(\alpha_m)(n^2)g_2 - 2Ea(1+\nu)\rho\alpha_m^2g_2,\]

and
$$D_6 = E^2a^2\alpha_{mn}^4 + 2a^2(l+v)(1-v)\rho^2\omega^4 - E\alpha^2(l+v)(3-v)(\alpha_{mn}^2)\omega^2,$$

$$g_7 = \frac{N_7}{D_1}, \quad (F.7)$$

where

$$N_7 = 2\rho\nu(1-v^2)\alpha_m E^3(\omega^2) - \nu(1-v^2)\alpha_m^2 E^2\alpha^3 - (1-v^2)(\alpha_m n^2)E^2\alpha$$

$$g_8 = \frac{N_8}{D_8}, \quad (F.8)$$

where

$$N_8 = (2+\nu)(\alpha_m^2)(n)E^2a^2h + (n)3\alpha_m^2E^2h\gamma_7 + E\alpha^3(l+v)^2(\alpha_m)(n),$$

$$D_8 = E^2a^4h\alpha_{mn}^4 + 2(l+v)(1-v^2)\alpha_m^4h\rho^2\omega^4 - E\alpha^4h(l+v)(3-v)\rho\omega^2\alpha_{mn}^2,$$

$$g_9 = \frac{N_9}{D_9}, \quad (F.9)$$

where

$$N_9 = g_7\nu(\alpha_m^3)E^2ha - g_7\alpha_m(n^2)E^2a^2h - 2g_7(l+v)\nu\rho\omega^2\alpha_m E^3h$$

$$+ 2(l+v)(1-v^2)\rho\omega^2a^2 - (1-v^2)\alpha_m^2E^2a^2 - 2(l+v)(n^2)E,$$

and

$$D_9 = E^2ha^2\alpha_{mn}^4 + 2(l+v)(1-v^2)^2\rho\omega^4ha^2 - Eha^2(l+v)(3-v)\rho\alpha_{mn}^2a^2.$$
\[ \alpha^4_{\text{mM}} = \left( \frac{\text{M}}{\xi} \right)^2 + \left( \frac{\pi}{a} \right)^2, \]
APPENDIX G

ANALYSIS FOR SIMPLY SUPPORTED STIFFENED CYLINDRICAL SHELLS

For the special case when both edges of the shell are simply supported, it is not necessary to determine an additional complementary solution since it is possible to find particular solutions which will satisfy all of the boundary conditions as well as the differential equations. Adopting the notation of Chapter IV, the general solution of the shell equations is given by:

\[ u(x, \phi) = \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_{5j} Z_{jmn} + g_{5j} T_{jmn} + g_{6j} Y_{jmn} + g_{6j} X_{jmn}) \cos n\phi \cos \frac{\pi n}{L} + \]

\[ \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_{5j} Z_{jmn} + g_{5j} T_{jmn} + g_{6j} Y_{jmn} + g_{6j} X_{jmn}) \sin n\phi \cos \frac{\pi n}{L}, \]

\[(G.1)\]

\[ v(x, \phi) = \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_{3j} Z_{jmn} + g_{3j} T_{jmn} + g_{4j} Y_{jmn} + g_{4j} X_{jmn}) \sin n\phi \sin \frac{\pi n}{L} - \]

\[ \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_{3j} Z_{jmn} + g_{3j} T_{jmn} + g_{4j} Y_{jmn} + g_{4j} X_{jmn}) \cos n\phi \sin \frac{\pi n}{L}, \]

\[(G.2)\]

\[ w(x, \phi) = \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_{1j} Z_{jmn} + g_{1j} T_{jmn} + g_{2j} Y_{jmn} + g_{2j} X_{jmn}) \cos n\phi \sin \frac{\pi n}{L} + \]

\[ \sum_{j=1}^{K} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (g_{1j} Z_{jmn} + g_{1j} T_{jmn} + g_{2j} Y_{jmn} + g_{2j} X_{jmn}) \sin n\phi \sin \frac{\pi n}{L}. \]

\[(G.3)\]
The general solutions of the stringer equations are given by:

\[ u_{ij}(x) = \sum_{m=0}^{\infty} \tau_{jm} x^m \cos \left( \frac{m\pi x}{l} \right), \]  
\[ v_{ij}(x) = \sum_{m=1}^{\infty} \tau_{jm} x^m \sin \left( \frac{m\pi x}{l} \right), \]  
\[ w_{ij}(x) = \sum_{m=1}^{\infty} \tau_{jm} x^m \sin \left( \frac{m\pi x}{l} \right), \]  
\[ \theta_{ij}(x) = \sum_{m=1}^{\infty} \tau_{jm} x^m \sin \left( \frac{m\pi x}{l} \right) - \frac{z_{jm}}{m} \sum_{m=1}^{\infty} \tau_{jm} x^m \sin \left( \frac{m\pi x}{l} \right). \]

The kinematical boundary conditions at the line of attachment are given by:

\[ w_{oi}(x) = w(x, \phi_i), \quad (i=1,2,\ldots,K) \]  
\[ \theta_{oi}(x) = \frac{1}{a} \frac{\partial w}{\partial \phi} \bigg|_{\phi=\phi_i}, \quad (i=1,2,\ldots,K) \]  
\[ v_{oi}(x) = v(x, \phi_i) + \left( \frac{z_{oi}}{a} \right) \frac{\partial w}{\partial \phi} \bigg|_{\phi=\phi_i}, \quad (i=1,2,\ldots,K) \]  
\[ u_{oi}(x) = u(x, \phi_i), \quad (i=1,2,\ldots,K) \]

where \( K \) is the total number of stringers attached to the shell.

Inserting Equations (G.1), (G.2), (G.3), and (G.4), (G.5), (G.6), (G.7) into (G.8), (G.9), (G.10), and (G.11) gives the following
Inserting Equations (G.1), (G.2), (G.3), and (G.4), (G.5), (G.6), (G.7) into (G.8), (G.9), (G.10), and (G.11) gives the following matrix equation:

\[
\begin{bmatrix}
A_p & B_p & C_p & K_p \\
E_p & F_p & G_p & H_p \\
I_p & J_p & L_p & M_p \\
N_p & O_p & R_p & S_p
\end{bmatrix}
\begin{bmatrix}
Z_p \\
T_p \\
Y_p \\
X_p
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]  \hspace{1cm} (G.12)

where \(Z_p, T_p, Y_p, \) and \(X_p\) are column vectors whose components are given by:

\[
Z_p = z_{jm},
\]  \hspace{1cm} (G.13)

\[
T_p = t_{jm},
\]  \hspace{1cm} (G.14)

\[
Y_p = y_{jm},
\]  \hspace{1cm} (G.15)

\[
X_p = x_{jm},
\]  \hspace{1cm} (G.16)

and \(m\) is a fixed integer (axial wave number) and \(j\) is related to \(P\) by:

\[
j = P - K[(P-1)/K]_T
\]  \hspace{1cm} (G.17)
and the subscript $T$ represents the operation of truncation after the decimal point. The coefficients $A_{ip}$, $B_{ip}$, etc. are given by:

$$A_{ip} = A_{ij} = \sum_{n=0}^{N} \left[ g_{2}^{1} \left( \cos n\phi_{j} \cos n\phi_{i} + \sin n\phi_{j} \sin n\phi_{i} \right) \right] - \delta_{ij} z_{jm},$$  \hspace{1cm} (G.18)

$$B_{ip} = B_{ij} = \sum_{n=0}^{N} \left[ g_{2}^{1} \left( \sin n\phi_{j} \cos n\phi_{i} - \cos n\phi_{j} \sin n\phi_{i} \right) \right], \hspace{1cm} (G.19)

$$C_{ip} = C_{ij} = \sum_{n=1}^{N} \left[ g_{2}^{1} \left( \sin n\phi_{j} \cos n\phi_{i} + \cos n\phi_{j} \sin n\phi_{i} \right) \right], \hspace{1cm} (G.20)

$$K_{ip} = K_{ij} = \sum_{n=0}^{N} \left[ g_{2}^{1} \left( \cos n\phi_{j} \cos n\phi_{i} + \sin n\phi_{j} \sin n\phi_{i} \right) \right], \hspace{1cm} (G.21)

$$E_{ip} = E_{ij} = \sum_{n=0}^{N} \left[ g_{2}^{1} \left( \sin n\phi_{j} \cos n\phi_{i} - \cos n\phi_{j} \sin n\phi_{i} \right) (n/a) \right], \hspace{1cm} (G.22)

$$F_{ip} = F_{ij} = \sum_{n=0}^{N} \left[ -g_{2}^{1} \left( \cos n\phi_{j} \sin n\phi_{i} + \sin n\phi_{j} \cos n\phi_{i} \right) \right], \hspace{1cm} (n/a) - \delta_{ij} z_{jm}^{*}, \hspace{1cm} (G.23)

$$G_{ip} = G_{ij} = \sum_{n=0}^{N} \left[ g_{2}^{1} \left( \cos n\phi_{j} \cos n\phi_{i} - \sin n\phi_{j} \sin n\phi_{i} \right) \right], \hspace{1cm} (n/a) - \delta_{ij} z_{jm}^{*}, \hspace{1cm} (G.24)

$$H_{ip} = H_{ij} = - \sum_{n=0}^{N} \left[ g_{2}^{1} \left( \sin n\phi_{j} \cos n\phi_{i} + \cos n\phi_{j} \sin n\phi_{i} \right) \right](n/a). \hspace{1cm} (G.25)
\[ I_{ip} = I_{ij} = \sum_{n=0}^{N} \left\{ g_{ij} f_{1}(\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i) \right\} - \sum_{n=0}^{N} \left\{ g_{ij} f_{1}(\cos n\phi_j \sin n\phi_i - \sin n\phi_j \cos n\phi_i) \right\} \left( \frac{z_i}{a} \right)(n), \]  

\[ J_{ip} = J_{ij} = \sum_{n=0}^{N} \left\{ g_{ij} f_{2}(\sin n\phi_j \sin n\phi_i + \cos n\phi_j \cos n\phi_i) \right\} - \sum_{n=0}^{N} \left\{ g_{ij} f_{2}(\sin n\phi_j \sin n\phi_i + \cos n\phi_j \cos n\phi_i) \right\} \left( \frac{z_i}{a} \right)(n), \]  

\[ L_{ip} = L_{ij} = \sum_{n=0}^{N} \left\{ g_{ij} f_{3}(\sin n\phi_j \sin n\phi_i - \cos n\phi_j \cos n\phi_i) \right\} - \sum_{n=0}^{N} \left\{ g_{ij} f_{3}(\sin n\phi_j \sin n\phi_i - \cos n\phi_j \cos n\phi_i) \right\} \left( \frac{z_i}{a} \right)(n) - \delta_{ij} \delta_{om}, \]  

\[ M_{ip} = M_{ij} = \sum_{n=0}^{N} \left\{ g_{ij} f_{4}(\cos n\phi_j \cos n\phi_i - \sin n\phi_j \sin n\phi_i) \right\} - \sum_{n=0}^{N} \left\{ g_{ij} f_{4}(\cos n\phi_j \cos n\phi_i - \sin n\phi_j \sin n\phi_i) \right\} \left( \frac{z_i}{a} \right)(n), \]
\[ N_{ip} = N_{ij} = \sum_{n=0}^{N} [g_{j} f_{1} (\cos n\phi_{j} \cos n\phi_{i} + \sin n\phi_{j} \sin n\phi_{i})], \quad (G.30) \]

\[ O_{ip} = O_{ij} = \sum_{n=0}^{N} [g_{j} f_{2} (\sin n\phi_{j} \cos n\phi_{i} - \cos n\phi_{j} \sin n\phi_{i})], \quad (G.31) \]

\[ R_{ip} = R_{ij} = \sum_{n=0}^{N} [g_{j} f_{3} (\sin n\phi_{j} \cos n\phi_{i} + \cos n\phi_{j} \sin n\phi_{i})], \quad (G.32) \]

\[ S_{ip} = S_{ij} = \sum_{n=0}^{N} [g_{j} f_{4} (\cos n\phi_{j} \cos n\phi_{i} + \sin n\phi_{j} \sin n\phi_{i})] \]

\[ - \tau_{4m} \delta_{ij}, \quad (G.33) \]

where \( i \) denotes the \( i \)th row.


Stephen Anthony Rinehart was born on September 23, 1941, in Coronado, California. He attended elementary schools in California, Virginia, and Florida. After his graduation from Pensacola Catholic High School in June, 1960, he entered Georgia Institute of Technology where he received his Bachelor of Science in Engineering Mechanics in June, 1964 and his Master of Science in Engineering Mechanics in June, 1966. He then enrolled as a doctoral student in the School of Engineering Mechanics.

On December 30, 1964, he married Cynthia Colleen Kennedy of Pensacola, Florida. They have one daughter, Sherry, born on November 9, 1967.