DIFFUSION OF SOUND IN REVERBERANT ROOMS

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DIFFUSION OF SOUND IN REVERBERANT ROOMS

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SUMMARY

This thesis is concerned with the propagation of sound in enclosed rooms with partially absorptive walls. The primary physical quantity considered is a directional spectral energy density $S$ which represents the acoustic energy per unit volume per unit frequency bandwidth per unit solid angle of propagation direction, such that the sound in the room is considered as composed of many plane waves propagating in a wide variety of directions. The usual assumption inherent in most of the exciting literature is that $S$ be independent of both position and propagation direction, i.e. that the field be diffuse. The present work examines the validity of this assumption using fundamental energy conservation principles.

Variational expressions for the determination of $S$ are derived for rooms with either specularly reflecting walls or diffusely reflecting walls. It is shown that the familiar relation between energy density in a room and wall absorption follows if the trial $S$ in the variational indicator is initially assumed to be constant. Quantitative estimates of the nondiffuseness of sound fields in rooms are obtained using techniques from the calculus of variations. Numerical results and graphs are given. Particular results include the prediction that $S$ varies with propagation direction such
that it is largest in the direction towards the wall with the largest absorption and that $S$ tends to be nearly spatially uniform in the limit of small absorption coefficients but that the departure from spatial uniformity depends markedly on source location within the room.
CHAPTER I

INTRODUCTION

Room acoustics can be considered as the study and investigation of the behavior of sound waves in an enclosure. Although one may distinguish transient and steady state conditions, the latter are generally of most interest. Typical rooms usually have dimensions that are many times the wavelength of any audible sound waves that are emitted by the source. It is therefore reasonable to assume that even for the case where the source is emitting sound within a narrow band of frequencies, the source excites many normal modes of vibration of the room. When this is justified, it is customary to refer to the room as a "large room." When a steady sound source is radiating in a room, after a large number of successive reflections, there is a tendency towards an energy balance within the room. In ideal conditions, the sound in the room may be, in principle, assumed to be diffuse, i.e., the average energy density is uniform throughout the entire volume and there is an equal probability of energy flow in all directions. Since the concept of a diffuse field is an idealization, it is clear that it does not completely conform to physical reality, because in such a field there would be no net flow of power.
in any direction, but, in fact, at any point close to highly absorptive surfaces there must be a net flow of energy into the surface.\(^5\)

One of the concepts attributable to the reflection of sound at the surfaces of a room is that of reverberation. When a source in a room is suddenly turned off the energy density has a tendency to diminish in an exponential manner. This nearly exponential decay is attributable to the absorption of energy by the surfaces. That is, each incident ray loses a fraction of its energy when it reflects from the surface. This process continues until the sound becomes inaudible. The fact that the sound tends to persist after the source is stopped is called reverberation.

The nature of such a reverberation is a very important consideration in the design of an acoustically "good" room.\(^6\) An important step towards the development of engineering guidelines for design of rooms from the standpoint of acoustics was made some time ago by W. C. Sabine\(^7\) who introduced the concept of a reverberation time. By a series of experiments he arrived at the conclusion that the duration of audibility of residual sound is nearly the same everywhere in a room and independent of position of the source. Sabine defined the reverberation time of an enclosure as the time required for the sound pressure level to fall by 60 dB. This definition is most meaningful if the sound in the room is totally diffuse and if the energy is absorbed at a rate which
is proportional to the instantaneous energy in the room. Sabine also empirically obtained a relation for the reverberation time which is directly proportional to the product of wall area and its corresponding absorption coefficient. (The theoretical justification for this was subsequently given by Jaeger.\textsuperscript{8}) Sabine's relation is most nearly applicable in the limit of very small absorption coefficients. This is clear because, in the limit, where there is complete absorption, there are only direct waves from the source since there is no reflection energy and there could accordingly be nothing resembling an exponential decay of sound when the source is suddenly turned off. Also the concept is most ideally applicable for perfectly diffuse sound fields, which is an idealization with obvious limitations. Nevertheless, much subsequent work by many investigators has confirmed that Sabine's relation is very useful and reasonably suitable for large rooms with walls that are highly reflecting. In addition, considerable research has been done to modify Sabine's reverberation time formula to one more applicable for less lively rooms. Eyring's\textsuperscript{9} and Knudsen's\textsuperscript{10} theories represent some of these attempts.

Recently, some evidence\textsuperscript{10,11} has appeared which suggests that the lack of perfect diffuseness in a room may have appreciable effects on measurements of material properties or on those measurements of sound power output of sources. These derived measurements which employ reverberation
rooms assume the existence of a diffuse sound field. Thus more study about the manner of distribution of energy density in a room is desirable, both experimental and theoretical. Such studies may answer the questions of, for example, how diffuse is a given sound field and how the perturbation from this ideal diffuseness depends on the design of the room.

In the present thesis, an extensive analysis is given towards answering such questions. This analysis is based on a new formulation of room acoustics due to Pierce\textsuperscript{12}, the first detailed written exposition of which is given here. In particular, we develop a quantitative measure of the diffusivity of an imperfectly reverberant sound field (Chapter II). Then a theory of diffusion in reverberant rooms of arbitrary shape is presented based on the concept of a directional spectral energy density $S$ which represents the acoustical energy per unit volume per unit solid angle of propagation direction per unit frequency bandwidth\textsuperscript{13}. From the definition developed here, it is shown that there is a relation between the average energy density and the directional spectral energy density $S$. Then the law of conservation of energy within the room is expressed in terms of the directional spectral energy density.

Two idealized types of rooms are considered: a room with specularly reflecting walls (i.e., angle of incidence equals angle of reflection) and a room with diffusely
reflecting walls (energy reflected from any portion of the wall uniformly distributed in propagation direction).

In the formulation for conservation of energy within a room with diffuse reflecting walls the reflected directional spectral energy density $S_{\text{ref}}$ at any point on the walls is independent of direction, while in the corresponding formulation for the rooms with specularly reflecting walls $S$ depends on direction. Each conservation of energy law relating values of $S$ at different points of the room leads (Chapters III and IV) to a functional of the corresponding $S$ field which is stationary with respect to small variations in $S$. (A more general form of such variational expressions may be obtained where the directional spectral energy density is also a function of time. Such variational expressions would make it possible to study the nature of the transient response of the sound field in a room. However, in the present thesis, the variational principle is formulated only for the steady state case.) For a room with specularly reflecting walls, the corresponding variational indicator (V.I.) involves the directional distribution of energy in the room (because $S$ may possibly depend on direction). However, it is shown that the V.I. for $S$ in the case of a room with specularly reflecting walls, as a functional of values of $S$ corresponding to energy reflected from the walls, when restricted to trial functions $S$ which are independent of direction, reduces to that for the case of a room with
diffusely reflecting walls.

The utility of variational principles as a framework for choosing "good" approximate solutions to a problem should be clear to anyone familiar with, for example, the Rayleigh-Ritz method in mechanical vibration theory. Briefly, if a general category of trial $S$ is picked, a variational indicator enables one to choose the "best" $S$ out of such category. We accordingly examine a few general classes of trial functions to gain some insight into the actual sound field. For example, it is shown (Chapter V) that for each of the variational indicators derived here, under the assumption that $S$ be independent of position and direction, the stationarity of the variational indicator then requires a value for $S$ (called $S_0$) which is the same as is derived from equating sound power output of source to the energy absorbed per unit time by the walls in the steady state case. This value $S_0$ may be interpreted as the zeroth order solution to a sequence of approximate equations which may be derived from a variational indicator. To find the higher order corrections to the directional spectral energy density $S$ which embody the nondiffuseness of a sound field, one proceeds from the variational indicators with appropriate trial functions.

For the specular reflecting case, these techniques indicate (Chapter VI) that there may often be some directional preference in the sound propagation. For example, nonequal absorption coefficients on different walls of the room leads
to a perturbation from the zeroth order $S_0$. The result derived here is that the acoustical energy propagating in a room has a slight preference for directions pointing toward walls with greater absorption coefficients.

We subsequently consider (Chapter VII) the effects of room geometry, source position, and absorption coefficient on the spatial uniformity (as opposed to directional uniformity) of sound in a reverberant room. The general category of trial functions considered in this regard are those where $S$ is constant on each wall of the room (a rectangular room is considered) but not necessarily the same on any two walls. Then, the variational indicator (which in this case is the same, regardless of whether the walls reflect sound diffusely or specularly) is used to find a set of simultaneous equations, where the coefficients depend on room geometry, source position, and average absorption coefficient of all walls.
CHAPTER II

CONCEPT OF DIRECTIONAL SPECTRAL ENERGY DENSITY

We begin this chapter with definitions of the directional spectral energy density and vector acoustic intensity and with some discussion about these definitions. In Section 2.2 we find a relation between the acoustic intensity and the radiative sound energy transfer intensity. Conservation of energy laws in a reverberant room are discussed for different cases in the latter section of this chapter.

2.1 Definitions

A. Directional Spectral Energy Density

The directional spectral energy density $S$ represents the acoustical energy per unit volume per unit solid angle of propagation direction per unit frequency bandwidth. The quantity $S$ is a function of position $\mathbf{x}$, propagation direction $\mathbf{e}_k$, time $t$, and frequency $f$, i.e. $S = S(\mathbf{x}, \mathbf{e}_k, t, f)$. By this definition, the energy in volume $\Delta V$, propagating within the solid angle $\Delta \Omega_k$ and within the frequency bandwidth $\Delta f$ is equal to $S(\mathbf{x}, \mathbf{e}_k, t, f) \Delta V \Delta \Omega_k \Delta f$, as sketched in Figure 1.

Any sound field may be characterized by an energy density function $E(\mathbf{x}, t)$, representing the energy per unit volume near some point $\mathbf{x}$ at time $t$ such that
Figure 1. Sketch Illustrating the Definition of the Directional Spectral Energy Density $S$
(The energy in volume $\Delta V$ propagating within the solid angle $\Delta \Omega_k$ in direction $\hat{e}_k$ within the frequency band $\Delta f$ is equal to $S\Delta V\Delta \Omega_k \Delta f$)
where \( \rho_0 \) is the mass per unit volume, \( v = |\vec{v}| \) is the magnitude of the acoustic fluid velocity, \( p \) is the acoustic pressure and \( \rho_0 c \) is the characteristic impedance of the medium, while \( c \) is the speed of sound. The above identification follows from the well-known corollary of the linear acoustic equations that \( E \) satisfies a conservation of energy law, given by

\[
\frac{\partial^2 E}{\partial t^2} + \nabla \cdot (p \vec{v}) = D(\vec{x}, t) \tag{2.2}
\]

where \( p \vec{v} \) is the vector acoustic intensity and \( D \) represents the acoustic energy dissipated per unit time per unit volume.

Since \( E \) may be expected to vary both with time and with spatial coordinates, we may define a local average energy density as

\[
\overline{E}(\vec{x}, t) = \frac{1}{\Delta V \Delta t} \int_{\Delta V} \int_{\Delta t} E(\vec{x}, t + \tau) d^3 \vec{x} d\tau, \tag{2.3}
\]

where the integration extends over some volume \( \Delta V \) centered at \( \vec{x} \) and over some time interval \( \Delta t \) centered at \( t \). Here the dimensions of \( \Delta V \) are large compared to a representative wavelength \( \lambda \) but small compared to those of the room itself. The quantity \( \Delta t \) is presumed large compared to a wave period.
but small compared to the time duration of the source. The assumption is made that the value of $\mathbf{E}$ computed by Equation (2.3) is relatively insensitive to the choice of integration volume $\Delta V$ or of the time interval $\Delta t$ and that, moreover, it should be slowly varying with $\mathbf{x}$ over distances comparable to a representative wavelength. It should be slowly varying with time $t$ over time intervals comparable to a representative wave period.

We may next regard $\mathbf{E}$ as being a superposition of energies associated with various frequency bands and with various propagation directions, such that it may be written

$$
\mathbf{E}(\mathbf{x},t) = \int \int \mathbf{S}(\mathbf{x},\mathbf{e}_k, f, t) df \, d\Omega_k \tag{2.4}
$$

where the integration is over all positive frequencies and all solid angles. Here, in terms of the spherical coordinates $\theta_k$ and $\phi_k$ of propagation direction $\mathbf{e}_k$, the differential of solid angle is

$$
d\Omega_k = \sin \theta_k \, d\theta_k \, d\phi_k \tag{2.5}
$$

where $0 < \theta_k < \pi$, $0 < \phi_k < 2\pi$.

Consistent with the above definition and for small values of $\Delta V$, $\Delta f$ and $\Delta \Omega_k$, we can write

$$
\Delta E = \mathbf{S}(\mathbf{x},\mathbf{e}_k, f, t) \Delta V \Delta f \, \Delta \Omega_k \tag{2.6}
$$
where $\Delta E$ represents the energy in volume $\Delta V$ centered at $\hat{x}$ which is propagating in directions within a cone of solid angle $\Delta \Omega_k$ centered about direction $\hat{e}_k$ and within frequency band $\Delta f$ centered about $f$ averaged over a time interval large compared to $1/f$. Here $S$ may be interpreted as the quasi-limit of the ratio $\Delta E/(\Delta V \Delta f \Delta \Omega_k)$ as $\Delta V$, $\Delta f$ and $\Delta \Omega_k$ become small, but not infinitesimal, i.e. $\Delta f$ is still large compared to the separation between typical modal frequencies, $\Delta \Omega_k$ is still sufficiently large that $\Delta \Omega_k$ is big compared to $c^3/(f^3 V)$, where $V$ is the total volume of the room, $\Delta V$ has dimensions large compared to a wavelength.

A more precise definition of $S$ may be given in terms of an idealized measurement technique. For a rectangular volume $\Delta V$ centered at $\hat{x}$ of dimensions $l_x$, $l_y$, and $l_z$, the acoustic pressure $p(\hat{x}+\hat{\xi}, t+\tau)$, for all points $\hat{x}+\hat{\xi}$ in $\Delta V$ (centered at $\hat{x}$) and for all times $t+\tau$ where $-\Delta T/2 < \tau < \Delta T/2$, may be represented in a quadruple Fourier series as

$$i(\hat{k}_S \cdot \hat{\omega}_S \cdot t)$$

$$p(\hat{x}+\hat{\xi}, t+\tau) = \sum_{S} A_S e$$

(2.7)

where the symbol $S$ is an abbreviation for the set of four integral indices $n_x$, $n_y$, $n_z$, and $n_t$, the sum over each of which ranges from $-\infty$ to $\infty$. Here we have abbreviated

$$\hat{k}_S = \left(\frac{2\pi n_x}{l_x}\hat{e}_x + \frac{2\pi n_y}{l_y}\hat{n}_y + \frac{2\pi n_z}{l_z}\hat{n}_z\right)$$

(2.8)
\[ \omega_s = \frac{2\pi n_t}{\Delta t} \quad (2.9) \]

One may note that in Equation (2.7) the coefficient \( A_s \) can, in principle, be computed from the inverse relation

\[ A_s = \frac{1}{\Delta t \Delta V} \int p(x+\xi, t+\tau) e^{i(k_s \xi - \omega_s \tau)} \, d^3 \xi d\tau \quad (2.10) \]

Also, from the familiar linear acoustic relation between velocity \( \dot{v} \) and pressure \( p \) (which is \( \rho_o \frac{\partial \dot{v}}{\partial t} = -\nabla p \)), we can find the corresponding Fourier representation for \( \dot{v} \), i.e.

\[ \dot{v} = \sum_{s, \omega_s} \frac{k_s}{\rho_o \omega_s} A_s e^{i(k_s \xi - \omega_s \tau)} \quad (2.11) \]

Substituting from Equations (2.7) and (2.11) into (2.1), averaging it over time, and noting that \( A_s \) should be very small unless \( k_s^2/\omega_s^2 \) is very close to \( 1/c^2 \), the average energy density is given to a good approximation by

\[ E = \Sigma (1/\rho_o c^2) |A_s|^2 \quad (2.12) \]

Comparing this with Equation (2.6), we may identify

\[ S \Delta\Omega \Delta f = \Sigma \frac{1}{\rho_o c^2} |A_s|^2 \quad (2.13) \]
where the prime on the sum indicates that we include only those terms such that $|\omega_s|/2\pi$ lies between $f-\Delta f/2$ and $f+\Delta f/2$ and such that the direction of $\hat{k}_s/\omega_s$ lies within a cone of solid angle $\Delta\Omega_k$ centered about $\hat{e}_k$.

**B. Vector Acoustic Intensity**

The vector acoustic intensity $p^\dagger$ when averaged over time and $\Delta V$ may be expressed in terms of the Fourier coefficients $A_s$. The result, found from Equations (2.11) and (2.7), is

$$<p^\dagger> = \sum \frac{k_s}{p_0 \omega_s} |A_s|^2$$

(2.14)

or, with Equation (2.13),

$$<p^\dagger> = \int c \hat{e}_k S(x, \hat{e}_k, t, f) d\Omega_k df$$

(2.15)

### 2.2 Relation to Concept of Intensity Used in Radiative Heat Transfer

In radiative heat transfer an intensity $I$ (not to be confused with that defined in the previous section) is defined such that $In \cdot \hat{e}_k d\Omega_k df dA$ is the energy in frequency band $df$ which crosses area $dA$ per unit time with propagation direction centered about $\hat{e}_k$ within a cone of solid angle $d\Omega_k$. Here $\hat{n}$ is the unit normal to area $dA$ such that $\hat{n} \cdot \hat{e}_k$ is positive. If we want to define an analogous quantity (radiative sound energy intensity) for acoustics, then it is
easily demonstrated that one should set

\[ I = cS \]  

(2.16)

where \( c \) is the speed of sound and \( S \) is the directional spectral energy density. This follows from the consideration of a small volume of a cylinder with axis in the \( \hat{e}_k \) direction directly behind \( dA \) which is of length \( cdt \) and of oblique cross section \( dA \) and accordingly of volume \( \hat{n} \cdot \hat{e}_k cdt \ dA \). The average energy within the volume propagating within a solid angle \( d\Omega_k \) within a spectral band \( df \) is equal to \( \hat{S}\hat{n} \cdot \hat{e}_k cdt d\Omega_k df \). All of this energy (which is propagating in direction \( \hat{e}_k \)) crosses the surface in time \( dt \) since sound travels with speed \( c \). Thus the average time rate of energy crossing within these specifications is the above divided by \( dt \).

2.3 Conservation of Energy Laws

A. Without Wall Reflection

As we discussed previously, at any point in the room, the average acoustic energy density \( E \) in volume \( \Delta V \) propagating within a solid angle \( \Delta \Omega_k \) per unit frequency band is \( S \Delta V \Delta \Omega_k \). If one considers the location and direction of propagation of the same energy at successive times \( t \) and \( t+\tau \), one has

\[
S(\hat{x}, \hat{e}_k, t, f) \Delta V \Delta \Omega_k = \iint_{\Delta V \Delta \Omega_k} S(\hat{x}, \hat{e}_k, t, f) d^3x d\Omega_k \]  

(2.17)

\[
= \iint_{\Delta V \Delta \Omega_k} S(\hat{x}', \hat{e}_k', t+\tau, f) d^3x' d\Omega_k'
\]
where $\Delta V'$ and $\Delta \Omega'_k$ are the volume and solid angle range occupied by the same energy at time $t+\tau$. Here $\hat{x}$ is a position vector inside the volume $\Delta V$ and $S(\hat{x}; \hat{e}'_k, t+\tau, f)$ is the value of $S$ measured at time $t$ later than $t$. Here, $U=1$ if $\hat{x} = \hat{x} - c\tau \hat{e}'_k$ is within the volume $\Delta V$ and $\hat{e}'_k$ is within the solid angle $\Delta \Omega'_k$, otherwise $U=0$. This follows since the energy propagating in a fixed direction is expected to continue propagating with the same direction and the energy moves with speed $c$. In other words, the last integral of Equation (2.17) has nonzero integrand when $\hat{e}'_k$ is equal to $\hat{e}'_k$ and when $x' - c\tau \hat{e}'_k$ is in the volume $\Delta V$. Substituting for $\hat{x}'$ in Equation (2.17) and noting that $d\Omega'_k$ can be replaced by $d\Omega'_k$ we find

$$\int_{\Delta V} \int_{\Delta \Omega'_k} S(\hat{x}, \hat{e}_k, t, f) d^3x d\Omega'_k =$$

$$\int_{\Delta V} \int_{\Delta \Omega'_k} S(\hat{x} + c\tau \hat{e}_k, \hat{e}_k, t, f) d^3x d\Omega'_k$$

From this equation, by noting that integrations are over the same range of volume and solid angle, we obtain

$$S(\hat{x}, \hat{e}_k, t, f) = S(\hat{x} + c\tau \hat{e}_k, \hat{e}_k, t + \tau, f). \quad (2.18)$$

Since this equation is for arbitrary $\tau$, by differentiating with respect to $\tau$, and subsequently setting $\tau=0$, we find
\[
\mathbf{c} \mathbf{e}_k \cdot \nabla S + \partial S / \partial t = 0
\]

or

\[
\mathbf{e}_k \cdot \nabla S + \frac{1}{c} \frac{\partial S}{\partial t} = 0.
\]

This gives the conservation of energy as

\[
\nabla \cdot (\mathbf{e}_k S) + \frac{1}{c} \frac{\partial S}{\partial t} = 0 \tag{2.19}
\]

where \( \nabla \cdot (\mathbf{e}_k S) \) is the divergence of \( \mathbf{e}_k S \) and \( \nabla S \) is the gradient of \( S \). (Note that \( \mathbf{e}_k \) is in terms of coordinates independent of spatial coordinates.)

From Equation (2.18), it follows that, for any two points \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) in the room (see Figure 2), we can write

\[
S(\mathbf{x}_1, \mathbf{e}_k, t, f) = S(\mathbf{x}_2, \mathbf{e}_k, t+\Delta t, f) \tag{2.20}
\]

where

\[
\mathbf{x}_{12} = |\mathbf{x}_2 - \mathbf{x}_1| = c\Delta t \tag{2.21}
\]

and where \( \mathbf{e}_k \) and \( \Delta t \) are defined such that

\[
\mathbf{x}_2 = \mathbf{x}_1 + c\Delta t \mathbf{e}_k \tag{2.22}
\]
Figure 2. Conservation of Energy Without Wall Reflection.
(For the case that there is no reflection by the walls, the directional spectral energy density $S$ at point $x_1$ is equal to that at point $x_2$, with appropriate time shift, where $S_1 = S(x_1, e_k, t, f)$ and $S_2 = S(x_2, e_k, t+\Delta t, f).$)
B. Conservation of Energy Theorem for Specular Wall Reflection

When sound waves fall on a surface or object, their energy is partially reflected and partially absorbed. The sound absorbing property of the surface involved is described in terms of an absorption coefficient designated by a symbol $\alpha$. The "plane wave absorption coefficient" $\alpha$ is defined as the ratio of sound energy absorbed by the surface to the energy incident upon the surface. Another quantity of interest is the "reflection coefficient" $R$, often considered as a complex number to account for phase shift on reflection, whose magnitude is the ratio of the reflected wave amplitude to that of the incident wave. Generally, the use of the term reflection coefficient is restricted to the case where both incident and reflected waves are planar.

Since the sound energy is proportional to the square of the amplitude, the relation between $\alpha$ and $|R|$ is

$$\alpha = 1 - |R|^2$$

(2.23)

For the specular reflection case (angle of incidence equals angle of reflection) it can be shown from the definition of $\alpha$ and from the physical interpretation of the directional spectral energy density $S$, that, for any point $\mathbf{x}_g$ on the wall (see Figure 3), that the relation between the quantities of incident and reflected $S$ is as follows (for specular reflection case)
Figure 3. Sketch Illustrating a Room with Specular Reflecting Walls. (If a ray goes from point \( x_1 \) in direction \( e_{\text{in}} \) to point \( x_s \) (reflected point), then the reflected ray goes in direction \( e_{\text{ref}} \) to \( x_2 \))
\[ S(\hat{x}_s, \hat{e}_{\text{ref}}, t, f) = (1-a)S(\hat{x}_s, \hat{e}_{\text{in}}, t, f). \] (2.24)

Here the left side corresponds to the reflected energy and the \( S \) factor on the right side corresponds to incident energy. The quantities \( \hat{e}_{\text{in}} \) and \( \hat{e}_{\text{ref}} \) are the unit vectors in the incident and reflected directions, respectively. (They are related by the law of mirrors for specular reflection.)

From Equation (2.24) it follows that, if a specularly reflected ray goes from a point \( \hat{x}_1 \) in direction \( \hat{e}_{\text{in}} \) via wall reflection to \( \hat{x}_2 \), we have

\[ (1-a)S(\hat{x}_1, \hat{e}_{\text{in}}, t, f) = S(\hat{x}_2, \hat{e}_{\text{ref}}, t+\Delta t, f) \] (2.25)

where

\[ \Delta t = (\text{total length of reflected path})/c. \]

C. Conservation of Energy Theorem for Diffusely Reflected Sound

By definition, in diffuse reflection, there is a uniform distribution in propagation direction of a sound wave reflected from a surface\(^6\). Therefore, at any point on the wall, say \( \hat{x}_2 \) (see Figure 4), the output energy goes uniformly in all directions. In other words, the directional spectral energy density \( S_{\text{exit}} \) is independent of direction \( \hat{e}_k \) and we have \( S_{\text{exit}} = S(\hat{x}_2) \). At any point \( \hat{x}_2 \) on the wall we can generalize our definition of \( a \) to one applicable for
Figure 4. Sketch Illustrating a Room with Diffusely Reflecting Walls. (Reflected energy goes in all directions from any point on the wall (such as $x_2$) for the totally diffuse case.)
diffuse reflection such that

\[
\text{total reflected power} = (1-a)(\text{total incident power}).
\]

Substituting in the above equation for energy per unit time per unit frequency bandwidth from the intensity equation (2.15), and noting that only the exit energy density \( S_{\text{exit}}(\vec{x}_2) \) is uniform and separating out that incident energy which has been reflected from other surfaces, we find (for a closed surface) in the steady state that

\[
\int \int c \, S_{\text{exit}}(\vec{x}_2)(-\hat{n}_2 \cdot \hat{e}_{\text{exit}}) \, dA_2 \, d\Omega_{\text{exit}}
\]

\[
= (1-a) \int \int c \, S_{\text{exit}}(\vec{x}_1)(-\hat{n}_1 \cdot \hat{e}_{12}) \, dA_1 \, d\Omega_{\text{exit}}
\]

\[
+ (1-a)(\text{energy incident per unit time directly from source}) \quad (2.26)
\]

This identification follows since the left hand side is the total energy leaving all surfaces per unit time while \( c \, S_{\text{exit}}(\vec{x}_1)(-\hat{n}_1 \cdot \hat{e}_{12}) \, d\Omega_{\text{exit}} \) is the energy leaving area \( dA_1 \) per unit time in the direction within solid angle \( d\Omega_{\text{exit}} \) centered at the direction \( \hat{e}_{12} \) pointing from \( \vec{x}_1 \) to \( \vec{x}_2 \). The energy incident from \( dA_1 \) to \( dA_2 \) is just the latter with \( d\Omega_{\text{exit}} \) replaced by the solid angle subtended by \( dA_2 \) viewed from \( \vec{x}_1 \). Note that the quantities \(-\hat{n}_1 \cdot \hat{e}_{12}\) and \(-\hat{n}_2 \cdot \hat{e}_{\text{exit}}\) are
the cosine of angles between zero and $\pi/2$, so they are positive values. The quantity $d\Omega_{\text{exit}}$ is the solid angle that $dA_2$ subtends as seen from point $x_1$ and is equal to 

$$(\hat{e}_{12} \cdot \hat{n}_2 \ dA_2) \frac{r_{12}^2}{2}$$

while $r_{12}$ is equal to $|\vec{x}_2 - \vec{x}_1|$.

Since $S(\vec{x}_0)$ is independent of direction, the left hand side of the above equation may be partially integrated to give

$$cS(\vec{x}_2) dA_2 \int (-\hat{n}_2 \cdot \hat{e}_{\text{exit}}) d\Omega_{\text{exit}}$$

$$= cS(\vec{x}_2) dA_2 \int \int \cos \theta \sin \theta d\phi d\theta$$

$$+ \pi cS(\vec{x}_2) dA_2$$  \hspace{1cm} (2.27)

Therefore Equation (2.26) becomes

$$S(\vec{x}_2) = \frac{1 - \alpha}{\pi} \int S(\vec{x}_1) \frac{(\hat{n}_1 \cdot \hat{e}_{21}) (\hat{n}_2 \cdot \hat{e}_{12}) dA_1}{r_{12}^2}$$

$$+ \frac{1 - \alpha}{\pi c} \text{[energy incident per unit time per unit area directly from source]}$$  \hspace{1cm} (2.28)

which expresses the conservation of energy in the steady state condition.
D. Generalization Allowing for the Presence of an Omnidirectional Point Source

D.1 Source of Sound. For simplicity, we assume there is a single omnidirectional source of sound within the room. Energy emitted by the source per unit time per unit frequency bandwidth is denoted as \( \bar{W}(f,t) \). In the absence of wall reflections, the energy per unit volume per unit frequency bandwidth at a point of a net distance \( R \) from the source is readily seen to be

\[
E = \frac{\bar{W}}{4\pi R^2 c}
\]  

(2.29)

(This follows because of spherical spreading and because of the fact that energy moves with speed \( c \) which in turn requires the energy in any spherical shell of radius \( R \) and thickness \( c\Delta t \) (volume \( 4\pi R^2 c\Delta t \)) be equal to the energy \( \bar{W}\Delta t \) emitted by the source during an earlier time interval of net duration \( \Delta t \).)

The quantity \( E \) in Equation (2.29) is the integral over solid angle of propagation direction of the direct wave contribution to the directional spectral energy density \( S \). Since this direct wave is propagating away from the source, the corresponding value of \( S \) must be sharply peaked whenever \( \hat{e}_r \) coincides with the unit vector pointing away from the source. Thus, if we write
\[
S = \frac{W}{4\pi R^2 c} \quad \text{(directional factor)}, \quad (2.30)
\]

the directional factor should satisfy

\[
f(\text{directional factor}) \, d\Omega_k = 1
\]

and also should be of negligible value unless \( \theta = \theta_k', \phi = \phi_k' \),
where \( \theta \) and \( \phi \) are the corresponding spherical coordinates
of the unit vector pointing from the source towards the
point where \( S \) is to be measured. This later condition as
well as the integration condition (2.30) is satisfied if one
sets.

\[
\text{directional factor} = \frac{\delta(\theta - \theta_k) \delta(\phi - \phi_k)}{\sin\theta_k} \quad (2.31)
\]

where \( \delta(\theta - \theta_k) \) and \( \delta(\phi - \phi_k) \) are Dirac delta functions. One
may recall that the delta function \( \delta(x) \) is defined by

\[
\delta(x) = 0 \text{ for } x \neq 0 \text{ and } \quad \int_{-\infty}^{\infty} \delta(x) \, dx = 1.
\]

Substituting from Equation (2.31) into (2.30) we find

\[
S = \frac{W}{4\pi R^2 c} \frac{\delta(\theta - \theta_k) \delta(\phi - \phi_k)}{\sin\theta_k}
\]
Let us next recall that the previously derived energy conservation relations (2.25) and (2.28) either do not include waves coming directly from the source or else do not explicitly describe the dependence of source terms on $W$ and on source position. To find appropriate modifications in corporating the source presence, one approach is to first consider a very small sphere of radius $R$ centered at the source. The major contribution to $S$ at the surface of this sphere is that due to the direct wave, such that, from Equation (3.32), one has

\[ \hat{e}_k S = \hat{e}_R \frac{W}{4\pi R^2} \frac{\delta(\theta - \theta_k) \delta(\phi - \phi_k)}{\sin \theta_k} \]  

(2.33)

where $\hat{e}_R$ is the unit vector in the radial direction away from the source. The replacement of $\hat{e}_k$ by $\hat{e}_R$ follows since $S$ in Equation (2.33) is zero unless $\hat{e}_k$ is in the radial direction where $\theta = \theta_k$ and $\phi = \phi_k$. Multiplying both sides of Equation (2.33) by $\hat{n}dA$, where $\hat{n}$ is the outward pointing unit normal to the surface and $dA$ is the differential element of surface, one finds

\[ S \hat{e}_k \cdot \hat{n} \ dA = \hat{e}_R \cdot \hat{n} \frac{W}{4\pi R^2 c} \frac{\delta(\theta - \theta_k) \delta(\phi - \phi_k)}{\sin \theta_k} \ dA. \]

Integrating over the surface of the sphere and noting that the quantity $\hat{e}_R \cdot \hat{n}$ is unity and that $dA = R^2 \sin \theta d\phi d\theta$, one finds
\[ \int_S \hat{e}_k \cdot \hat{n} \, dA = \frac{W}{4\pi c} \int \frac{\delta(\theta-\theta_k) \delta(\phi-\phi_k)}{\sin \theta_k} \sin \theta d\phi d\theta. \]

The integral of the right side (by the definition of the delta function) is equal to unity, therefore

\[ \int_S \hat{e}_k \cdot \hat{n} \, dA = \frac{W}{4\pi c}. \]

Applying the divergence theorem to the above equation, we find

\[ \int_V \nabla \cdot (\hat{e}_k S) \, dV = \frac{W}{4\pi c} \]

which would be formally satisfied were the integrand replaced by \([W/(4\pi c)] \delta(\vec{r}-\vec{r}_o)\). This suggests that we replace Equation (2.19) by

\[ \nabla \cdot (\hat{e}_k S) + \frac{1}{c} \frac{\partial S}{\partial t} = \frac{W}{4\pi c} \delta(\vec{r}-\vec{r}_o) \quad (2.34) \]

where \(\vec{r}\) is the position vector and \(\vec{r}_o\) is the position of the source. The rationale for this replacement is that the left hand side should be zero everywhere except very near the source, while its integral over any small volume enclosing the source should be \(W/4\pi c\) (as discussed above), with the recognition that \(\int [\partial S/\partial t] \, dV\) approaches zero when the sphere radius shrinks to zero.

We next show that Equation (2.34) leads to a readily
recognizable energy conservation law for the room as a whole. Integrating Equation (2.34) over the volume of the room, we find

$$\int \nabla \cdot (\hat{e}_k \hat{n} S) \, dv + \frac{1}{c} \int \frac{\partial S}{\partial t} \, dv = W/4\pi c$$

which, by the divergence theorem, in turn becomes

$$\int \hat{e}_k \cdot \hat{n} S dA + \frac{1}{c} \frac{\partial}{\partial t} \int S dV = W/4\pi c.$$ 

Multiplying both sides of the above by the differential solid angle $d\Omega$ and subsequently integrating over all solid angles and noting that $\int d\Omega = 4\pi$, we find

$$\int \hat{e}_k \cdot \hat{n} S dA = \frac{1}{c} \frac{\partial}{\partial t} \int S dV d\Omega = \frac{W}{c}.$$ 

In the first term the integral may be interpreted as the difference of two terms (with $\hat{e}_k \cdot \hat{n} > 0$ and $\hat{e}_k \cdot \hat{n} < 0$, respectively) which, apart from a factor of $c$, represents energy incident on the walls per unit time and reflected from the walls per unit time. Thus the difference is just the $(1/c)$ times the energy absorbed per unit time. With the definition of $\alpha$, this term may accordingly be rewritten such that the conservation of energy law is

$$\int \int' (\hat{e}_k \cdot \hat{n}) \alpha S \, dA d\Omega + \frac{1}{c} \frac{\partial}{\partial t} E_T(f) = W/c \quad (2.35)$$

$$\hat{e}_k \cdot \hat{n} > 0$$
where

\[ E_T(f) = \int \int \int \frac{\partial}{\partial t} E_T(f) = W. \] (2.37)

\[ \int \int \int (\varepsilon_k \cdot \hat{n}) \, d\Omega + \frac{\partial}{\partial t} E_T(f) = W. \] (2.37)

The above equation shows the conservation of energy formula in the room. The first term of the left hand side represents the rate of energy lost through absorption at the walls. The second term is the rate of increase of energy in the room. The right side of the equation represents the power output by the source. (One should note that all of these energies are per unit frequency bandwidth.) From Equation (2.37) it is clear that, when the source is radiating, the total energy of the room may initially increase, until, after a certain time, a steady state is attained where the energy present in the room stays constant, i.e. \( (\partial / \partial t) E_T(f) = 0 \). In this steady state condition the walls absorb the same amount of energy per unit time as is
radiated by source. (In this analysis, we neglect absorption of energy within the air itself.)

D.2 Radiative Sound Energy Transfer for the Specularly Reflected Case. We next develop a relation (incorporating the source presence) between values of $S$ at two points $\hat{x}_1$ and $\hat{x}_2$ on the walls of the room. Let us suppose we want to find the spectral energy density $S$ at any point on the walls, say $\hat{x}_2$, with specified direction $\hat{e}_{k2}$, where $\hat{x}_1$ and $\hat{x}_2$ are successive reflection points of a given multireflected ray and where $\hat{e}_{k1}$ and $\hat{e}_{k2}$ are unit vectors pointing along the successive ray segments. We also define a unit vector $\hat{e}_{o2}$ along a ray from the source, which is located at $\hat{x}_o$, to the point $\hat{x}_2$ (see Figure 5). If we recall the rationales that led to Equations (2.20), (2.24), and (2.25), we may readily incorporate the source dependence by recognizing that (in the steady state)

$$S \text{ incident in direction } \hat{e}_{k2} \text{ at point } \hat{x}_2 = \text{ any } S \text{ due to direct wave from the source } + S \text{ propagating from } \hat{x}_1 \text{ to } \hat{x}_2 \quad (2.38)$$

In this relationship, the $S$ due to direct radiation from the source must correspond to $\hat{e}_k$ equal to $\hat{e}_{k2}$. With this substitution into Equation (2.32), we have (in an abbreviated notation)
Figure 5. The Presence of an Omnidirectional Point Source in a Room with Specular Reflecting Walls
(Energy density at $x_2$ in direction $\hat{e}_{k2}$ is the sum of energy from the source and energy from point $x_1$ in direction $\hat{e}_{k2}$.)
directional spectral energy density due to direct waves from the source

\[ = -\frac{W}{4\pi cr^2} \delta(\hat{e}_{o2} - \hat{e}_{k2}) \]

with the law of specular reflection, we may write the second term in Equation (2.38) as \((1-\alpha)S(x_1, \hat{e}_{kl}, t-r_{12}/c,f)\) (see Equation (2.25)) where \(r_{12}\) is the distance between \(x_1\) and \(x_2\). Therefore with appropriate allowances for propagation times, Equation (2.38) becomes

\[
S(x_2, \hat{e}_{k2}, t, f) = \frac{W(f, t-r_{o2}/c)}{4\pi cr_{o2}} \delta(\hat{e}_{o2} - \hat{e}_{k2})
\]

\[ + (1-\alpha)S(x_1, \hat{e}_{kl}, t-r_{12}/c,f) \]

where \(r_{o2}\) is the distance between the source and point \(x_2\). One should note that this relates two values of \(S_{in}\) (in for incident) rather than \(S_{exit}\) since both \(\hat{e}_{k2}\) and \(\hat{e}_{kl}\) point obliquely into (rather than out of) the walls at the corresponding points \(x_2\) and \(x_1\).
CHAPTER III

FORMULATION OF VARIATIONAL PRINCIPLE FOR THE SPECULAR REFLECTION CASE

In this chapter the problem of finding the directional spectral energy density $S$ is formulated as a problem in variational calculus. The reader may recall that the fundamental problem of the calculus of variations is to find a function $S$ of coordinates $\hat{x}$ and $t$ such that a functional, or a function of this function, has a stationary value for small variations in $S$. In order to state the problem in such terms, it is appropriate to formulate a variational indicator (V.I.) which is a functional of all values of $S(\hat{x}, \hat{e}_k)$ for points $\hat{x}$ on the walls, $\hat{e}_k$ either pointing obliquely into or obliquely out of the walls. In other words, this V.I. should be such that, for the correct $S(\hat{x}, \hat{e}_k)$, the V.I. is a maximum or minimum or, more generally, is stationary. The appropriate solution for $S(\hat{x}, \hat{e}_k)$ would be such that, if one replaces $S$ by $S + \xi$, the V.I. is unchanged to order $\xi$, providing $\xi$ is sufficiently small. In other words, V.I. is stationary in the sense that

$$V.I. = f[\text{correct } S + \xi(\text{arbitrary small mistake})]$$

$$= \text{second order in } \xi$$
where $\xi$ is sufficiently small. Hence, to find the "correct $S$" one can take a general trial function $S$ depending on a number of undetermined parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ and in turn determine these from the simultaneous equations

$$\frac{\partial}{\partial \lambda_i} (V.I.) = 0 \ (i=1,2,\ldots,n) \quad (3.1)$$

### 3.1 Variational Indicator as a Functional of Incident Energy

In order to formulate appropriate variation indicators we begin with Equation (2.40). This equation for the steady state case becomes

$$S(x_2, \hat{e}_{k2}) = (1-\alpha)S(x_1, \hat{e}_{k1}) + \left(\frac{W}{4\pi c r_2^2}\right) \delta(e_{o2} - \hat{e}_{k2}) \quad (3.2)$$

Let us first define the following unit vectors which are related to points $x_1$ and $x_2$ (see Figure 6). The unit vector from $x_1$ to $x_2$ is denoted by $\hat{e}_{12}$ (or $\hat{e}_{k2}$) while the unit vector from $x_2$ to $x_1$ is denoted by $\hat{e}_{21}$ or $\hat{e}_{k1}^\dagger$. With these definitions, the following relations should be obvious

$$\hat{e}_{21} = -\hat{e}_{12}, \quad \text{and} \quad \hat{e}_{k1} \cdot \hat{n}_1 = \hat{e}_{k1}^\dagger \cdot \hat{n}_1 \quad (3.3)$$

Multiplying both sides of Equation (3.2) by the variation $\delta S(x_1, \hat{e}_{k1})\hat{n}_1 \cdot \hat{e}_{k1} dA_1 d\Omega_1$ and integrating over solid angle and
Figure 6. Rays Movement in a Specular Reflecting Walls Room. (Source is located at \( \vec{r} \), while \( x_1 \), and \( x_2 \) are two points on the wall, \( \vec{n}_1 \) and \( \vec{n}_2 \) are unit normal vectors at \( x_1 \) and \( x_2 \).)
Here the prime on the integrals indicates the solid angle of integration be limited to directions $\hat{e}_{k1}$ pointing obliquely into the wall ($\hat{e}_{k1} \cdot \hat{n}_1 > 0$). Noting that the variation of the product $\delta(S_1 S_2)$ is just $S_1 \delta S_2 + S_2 \delta S_1$, we can write

$$\delta \mathcal{J}(x_1, e_{k1}) S(x_2, e_{k2}) \hat{n}_1 \cdot \hat{e}_{k1} dA_1 d\Omega_1$$

$$= \mathcal{J}'(1-\alpha) S(x_1, e_{k1}) \delta S(x_1, e_{k1}) \hat{e}_{k1} \cdot \hat{n}_1 dA_1 d\Omega_1$$

$$+ \frac{\mathcal{W}}{4\pi c} \int \int \delta S(x_1, e_{k1}) \delta(e_{k2} - e_{k02}) (\hat{e}_{k1} \cdot \hat{n}_1 / r^2) dA_1 d\Omega_1 \quad (3.4)$$

Next we show that the two integrals on the right hand side of the above equation are equal. By interchanging $\dot{x}_1$ and $\dot{x}_2$ in the second integral, from Figure 6 we find that the arguments of the integral changes as follows

$$S(\dot{x}_1, e_{k1}) \Rightarrow S(\dot{x}_2, e_{k2})$$
\[ S(x_2, e_{k2}) \Rightarrow S(x_1, e_{kl}) \]

\[ \hat{n}_1 \cdot e_{kl} = \hat{n}_1 \cdot e_{k1} \Rightarrow \hat{n}_2 \cdot e_{k2} \]

\[ dA_1 \Rightarrow dA_2 \]

\[ d\Omega_1 = \frac{\hat{n}_2 \cdot e_{k2} dA_2}{r_{12}} \Rightarrow d\Omega_2 = \frac{\hat{n}_1 \cdot e_{k1} dA_1}{r_{12}} \]

therefore the second integral of the right hand side of Equation (3.5), by interchanging \( \hat{x}_1 \) and \( \hat{x}_2 \), becomes

\[ \int \int S(\hat{x}_2, e_{k2}) [\delta S(\hat{x}_1, e_{kl})] \hat{n}_2 \cdot e_{k2} \frac{(\hat{n}_1 \cdot e_{k1}) dA_1}{r_{12}} dA_2 \]

\[ = \int \int S(\hat{x}_2, e_{k2}) [\delta S(\hat{x}_1, e_{kl})] \hat{n}_1 \cdot e_{k1} dA_1 d\Omega_1 \]

which is the same as the first integral on the right side of Equation (3.5). By this property, Equation (3.5) may be written,

\[ \delta \int \int S(\hat{x}_1, e_{kl}) S(\hat{x}_2, e_{k2}) \hat{n}_1 \cdot e_{k1} dA_1 d\Omega_1 \]

\[ = 2 \int \int S(\hat{x}_1, e_{kl}) [\delta S(\hat{x}_2, e_{k2})] \hat{n}_1 \cdot e_{k1} dA_1 d\Omega_1 \]

\[ = 2 \int \int S(\hat{x}_2, e_{k2}) [\delta S(\hat{x}_1, e_{kl})] \hat{n}_1 \cdot e_{k1} dA_1 d\Omega_1 \]

(3.6)
It accordingly follows that Equation (3.4) may be rewritten in the form

\[
\frac{1}{2} \delta \int S(\hat{x}_1, \hat{e}_{k1}) S(\hat{x}_2, \hat{e}_{k2}) \hat{n}_1 \cdot \hat{e}_{k1} \, dA_1 \, d\Omega_1
\]

\[
= \frac{1}{2} \delta \int (1 - \alpha) S(\hat{x}_1, \hat{e}_{k1}) S(\hat{x}_1, \hat{e}_{k1}) \hat{e}_{k1} \cdot \hat{n}_1 \, dA_1 \, d\Omega_1
\]

\[
+ \frac{W}{4\pi c} \int \delta S(\hat{x}_1, \hat{e}_{k1}) \delta (\hat{e}_{k2} - \hat{e}_{o2}) \frac{\hat{e}_{k1} \cdot \hat{n}_1}{r_{o2}^2} \, dA_1 \, d\Omega_1
\]

from which a variational indicator may be readily recognized as

\[
\text{V.I.} = \frac{1}{2} \int \int' S(\hat{x}_1, \hat{e}_{k1}) S(\hat{x}_2, \hat{e}_{k2}) \hat{n}_1 \cdot \hat{e}_{k1} \, dA_1 \, d\Omega_1
\]

\[
- \frac{1}{2} \int \int' (1 - \alpha) S(\hat{x}_1, \hat{e}_{k1}) S(\hat{x}_1, \hat{e}_{k1}) \hat{e}_{k1} \cdot \hat{n}_1 \, dA_1 \, d\Omega_1
\]

\[
- \frac{W}{2\pi c} \int \int' S(\hat{x}_1, \hat{e}_{k1}) \delta (\hat{e}_{k2} - \hat{e}_{o2}) \frac{\hat{e}_{k1} \cdot \hat{n}_1}{r_{o2}^2} \, dA_1 \, d\Omega_1 \quad (3.7)
\]

Substituting for \( d\Omega_1 = \hat{e}_{k2} \cdot \hat{n}_2 \, dA_2 / r_{12}^2 \) in the last term of the above V.I., and noting that \( d\Omega_2 = \hat{e}_{k1} \cdot \hat{n}_1 \, dA_1 / r_{12}^2 \), the integral of this term becomes

\[
\int \int' S(\hat{x}_1, \hat{e}_{k1}) \delta (\hat{e}_{k2} - \hat{e}_{o2}) \frac{\hat{e}_{k2} \cdot \hat{n}_2}{r_{o2}^2} \, dA_2 \, d\Omega_2
\]

This integrand is nonzero only if \( \hat{e}_{k2} \) is equal to \( \hat{e}_{o2} \) which
by previous definitions occurs where $\hat{e}_{o2}$ is equal to $\hat{e}_{l2}$ (see Figure 7). In other words, the integrand is nonzero only when the points $\hat{x}_1$, $\hat{x}_o$, and $\hat{x}_2$ are located along a straight line. When these conditions hold, we have

$$\hat{e}_{o2} = \hat{e}_{k2}, \text{ and } \hat{e}_{o1} = \hat{e}_{k1} \quad (3.8)$$

Using the well known properties of the delta function and substituting from (3.8), we accordingly find that the above integral becomes

$$\int S(\hat{x}_1, \hat{e}_{o1}) \frac{\hat{e}_{o2} \cdot \hat{n}_2 dA_2}{r_{o2}^2}.$$ 

However, from Figure 7, we find that

$$\frac{\hat{e}_{o2} \cdot \hat{n}_2}{r_{o2}^2} = \frac{\hat{e}_{o1} \cdot \hat{n}_1 dA_1}{r_{ol}^2}$$

$$= d\Gamma_o$$

so the integral becomes

$$\int S(\hat{x}_1, \hat{e}_{o1}) \frac{\hat{n}_1 \cdot \hat{e}_{o1}}{r_{ol}^2} dA_1.$$ 

Therefore, the variational indicator (3.7) can be taken as
Figure 7. Sketch Illustrating Alternate Interpretations of a Solid Angle Element $d\Omega$ (of rays pointing away from a source in a room).
3.2 Formulation of Variational Indicator as a Function of Reflected Energy (for Specular Reflected Case)

If one notes (again with reference to Figure 6) that 

\[ (1-a) S(x_1, e_k) = S(x_2, e_{k2}) \]

it is clear that Equation (3.2) can also be written in the form

\[ S(x_2, e_{k2}) = S(x_1, e_{k1}) + \frac{W}{4\pi c r_2} \delta (e_{o2} - e_{k2}). \]

If one next notes \( S(x_2, e_{k2}) = (1-a)S(x_2, e_{k2}) \), he can rewrite the above as

\[ \frac{1}{1-a} S(x_2, -e_{k2}^\dagger) = S(x_1, -e_{k1}) + \frac{W}{4\pi c r_2} \delta (e_{o2} - e_{k2}). \]

To derive a corresponding variational indicator, one can multiply both sides of the above by \( n_2 \cdot e_{k2} \delta S(x_2, -e_{k2}) dA_2 d\Omega_2 \) and integrate over area and solid angles, such that

\[ \int' \frac{1}{1-a} S(x_2, -e_{k2}^\dagger) \delta S(x_2, -e_{k2}) n_2 \cdot e_{k2} dA_2 d\Omega_2 \]
Each of these terms may be readily shown to be the variation of an integral (rather than an integral over an integrand with a variational quantity as a factor). This follows since the integral on the left hand side is unchanged when the direction polar angles are redefined such that \( \vec{e}_{k2} \) and \( \vec{e}^\dagger_{k2} \) are interchanged, while the first integral on the right is unchanged if the subscripts 1 and 2 are interchanged for essentially the same reason which allowed us to derive Equation (3.6). Thus we find

\[
\frac{1}{2} \delta ff' S(\vec{x}_1, -\vec{e}_{k1}) S(\vec{x}_2, -\vec{e}_{k2}) \hat{n}_2 \cdot \vec{e}_{k2} dA_2 d\Omega_2
\]

or

\[
\frac{1}{2} \delta ff' S(\vec{x}_1, -\vec{e}_{k1}) S(\vec{x}_2, -\vec{e}_{k2}) \hat{n}_2 \cdot \vec{e}_{k2} dA_2 d\Omega_2
\]

In regard to the last term in the right, we can also write

\[
\frac{W}{4\pi c} \delta ff' S(\vec{x}_2, -\vec{e}_{k2}) \delta (\vec{e}_{o2} - \vec{e}_{k2}) \frac{\hat{n}_2 \cdot \vec{e}_{k2}}{r_{o2}} dA_2 d\Omega_2
\]

or

\[
\frac{W}{4\pi c} \delta (\vec{x}_2, -\vec{e}_{o2}) \frac{\hat{n}_2 \cdot \vec{e}_{k2}}{r_{o2}} dA_2 d\Omega_2
\]
Therefore, Equation (3.10) becomes

\[
\frac{1}{2} \delta \mathcal{J}' \frac{1}{1 - \alpha} S(\hat{x}_2, -\hat{e}_{k2}) S(\hat{x}_2, -\hat{e}_{k2}) \hat{e}_{k2} \cdot \hat{n}_2 \, dA_2 \, d\Omega_2
\]

\[
= \frac{1}{2} \delta \mathcal{J}' \cdot S(\hat{x}_1, \hat{e}_{12}) S(\hat{x}_2, \hat{e}_{21}) \hat{n}_2 \cdot \hat{e}_{12} \, dA_2 \, d\Omega_2
\]

\[
+ \frac{W}{4\pi c} \delta \mathcal{J} \cdot S(\hat{x}_2, -\hat{e}_{02}) \frac{\hat{n}_2 \cdot \hat{e}_{02}}{r_{02}} \, dA_2
\]

Here, in the first term of the right, we have used the facts that \(-\hat{e}_{k2} = \hat{e}_{21}\) while \(-\hat{e}_{k2}^\dagger = \hat{e}_{12}\). From the above, the V.I. may be recognized as

\[
V.I. = \int \mathcal{J}' S(\hat{x}_1, \hat{e}_{12}) S(\hat{x}_2, \hat{e}_{21}) \hat{n}_2 \cdot \hat{e}_{12} \, dA_2 \, d\Omega_2
\]

\[
- \int \mathcal{J}' \frac{1}{1 - \alpha} S(\hat{x}_2, -\hat{e}_{k2}) S(\hat{x}_2, -\hat{e}_{k2}) \hat{e}_{k2} \cdot \hat{n}_2 \, dA_2 \, d\Omega_2
\]

\[
+ \frac{W}{2\pi c} \int S(\hat{x}_2, -\hat{e}_{02}) \frac{\hat{n}_2 \cdot \hat{e}_{02}}{r_{02}} \, dA_2
\]

\[
(3.11)
\]

This formulation gives the variational indicator as a functional of reflected energy while the variational indicator (3.9) is in terms of incident energy. In other words, the values of \(S\) which enter into the above are all for propagation directions which point obliquely out of rather than into the walls. With the subsequent application of the propagation laws and reflection formulas given in Chapter II, either
formulation may be regarded as a complete formulation for determining $S$ for any point within the room and for any propagation direction. (It is tacitly assumed that every point on the walls can be seen from a point in the interior of the room. This would appear to rule out rooms with very irregular shape.)
CHAPTER IV

FORMULATION OF VARIATIONAL PRINCIPLE FOR THE TOTALLY DIFFUSE CASE

In this chapter a variational principle is formulated for a diffuse room. As is described in Section 3-C of Chapter II, in a totally diffuse room the reflected directional spectral energy density $S_{\text{exit}}$ at any point on the walls is independent of direction (i.e. $S = S(x)$). It therefore appears most convenient in this case to seek a variational indicator for $S_{\text{exit}}$ corresponding to reflected waves rather than one for incident waves.

In Section 4.2, it is shown that the variational indicator for the specular reflection case (Equation (3.11)) when restricted to trial functions $S$ which are independent of direction reduces to that derived in this chapter for a room with diffusely reflecting walls.

4.1 Derivation of the Variational Indicator

Our starting point may be taken as Equation (2.28) of Chapter 2, which may be rewritten

$$S(x_2) = \frac{|R^2|}{\pi} \int S_{\text{exit}}(x_1) \frac{(\hat{n}_1 \cdot \hat{e}_{21}) (\hat{n}_2 \cdot \hat{e}_{12})}{r_{12}^2} \, dA_1$$

$$+ \frac{|R^2|}{\pi c} \frac{W \hat{n}_2 \cdot \hat{e}_{o2}}{4\pi r_{o2}^2}$$
Here we have abbreviated $|R^2|_2$ for $1-\alpha_2$ where $\alpha_2$ is the absorption coefficient at point $\hat{x}_2$ on the wall. (Apparently, in the case of a room with diffusely reflecting walls, one has no choice but to consider $\alpha$ independent of the direction of the incident wave.) The last term follows since the energy incident per unit time on area $dA_2$ should be $Wd\Omega_2/4\pi$ where $d\Omega_2 = \hat{n}_1 \cdot \hat{e}_{o2} dA_2 / r_{o2}^2$ should be the solid angle subtended by $dA_2$ viewed from the source. Multiplying both sides of the above by $\pi \delta_c S(\hat{x}_2)$ and dividing by $|R^2|_2$, and subsequently integrating over area $(dA_2)$, we find

\[
\int \frac{\pi}{|R^2|} S_{\text{exit}}(\hat{x}_2) \delta S(\hat{x}_2) dA_2 = \int \int S(x_1) \delta S(x_2) \frac{(\hat{n}_1 \cdot \hat{e}_{12})(\hat{n}_2 \cdot \hat{e}_{12})}{r_{12}^2} dA_1 dA_2
\]

\[
+ \frac{W}{4\pi c} \int \delta S_{\text{exit}}(\hat{x}_2) \frac{\hat{n}_2 \cdot \hat{e}_{o2}}{r_{o2}^2} dA_2
\]

Noting that $\delta \delta S = \frac{1}{2} \delta (S^2)$ (it may be derived by noting that $\delta (S^2) = (S+\delta S)^2-S^2$ which gives $\delta (S^2) = 2S \delta S$), we may rewrite the left hand side of the above equation as

\[
\int \frac{\pi}{2|R^2|} \delta S_{\text{exit}}^2(x_2) dA_2.
\]

Noting also that the integrand of the double integral above does not change if one interchanges $\hat{x}_1$ and $\hat{x}_2$, we may accordingly write
\[
\frac{\delta \Pi}{\delta \mathcal{S}} \int \frac{1}{|R^2|} S^2 (\mathbf{x}_2) dA_2^\text{exit} \\
= \frac{1}{2} \delta \int \frac{S(\mathbf{x}_1)}{\text{exit}} S(\mathbf{x}_2) \frac{(n_1 \cdot e_{21}) (n_2 \cdot e_{12})}{r_{12}^2} dA_1 dA_2 \\
+ \frac{W}{4\pi c} \delta \int \frac{S(\mathbf{x}_2)}{\text{exit}} \frac{\hat{n}_2 \cdot \hat{e}_{02}}{r_{02}^2} dA_2
\]

From this equation, the V.I. may be readily recognized as

\[
\text{V.I.} = \int \frac{S(\mathbf{x}_1)}{\text{exit}} S(\mathbf{x}_2) \frac{(n_1 \cdot e_{21}) (n_2 \cdot e_{12})}{r_{12}^2} dA_1 dA_2 \\
+ \frac{Wf}{2\pi c} \int \frac{S(\mathbf{x}_2)}{\text{exit}} \frac{\hat{n}_2 \cdot \hat{e}_{02}}{r_{02}^2} dA_2 \\
- \pi \int \frac{1}{|R^2|} S^2 (\mathbf{x}_2) dA_2^\text{exit}
\]

which is the variational indicator for finding \( S_{\text{exit}} \) for a room with diffusely reflecting walls.

### 4.2 Comparison Between Variational Indicators for Totally Diffuse and Specular Reflection Cases

If one starts with the V.I. for the reflected waves at the wall for the specular reflection case, i.e. Equation (3.11), and makes the apriori assumption that \( S \) is independent of direction, he finds
In the last term on the right the directional angle integration can be immediately performed using the fact that
\[ \int \hat{e}_{k2} \cdot \hat{n}_2 \, d\Omega_2 = \pi. \] Also the solid angle integration in the first term may be changed to an area integration with the prescription
\[ d\Omega_2 = \left[ \hat{\mathbf{n}}_1 \cdot \hat{e}_{21} / r_{12} \right]^2 \, dA. \] Thus the variational indicator reduces to

\[
\text{V.I.} = \iint \left( \mathbf{s}(\mathbf{x}_1) \mathbf{S}(\mathbf{x}_2) \hat{n}_2 \cdot \hat{e}_{12} \, dA_1 \, dA_2 \right) + \frac{W}{2\pi c} \int \mathbf{S}(\mathbf{x}_2) \frac{\hat{n}_2 \cdot \hat{e}_{o2}}{r_{o2}^2} \, dA_2 - \pi \int \frac{1}{1-\alpha} \mathbf{S}^2(\mathbf{x}_2) \, dA_2
\]

which is the same as the variational indicator in Equation (4.1), as can be seen by noting that \(|R^2| = 1-\alpha|.
CHAPTER V

ALTERNATE DERIVATIONS OF THE SABINE EQUATION

We begin this chapter with a brief review of a well known method for obtaining the Sabine's equation\(^7\) of room acoustics. What we call "Sabine's equation" is the differential equation for the conservation of energy in a room with the apriori assumption that the directional spectral energy density \(S\) be independent of propagation direction and position. Alternate derivations of Sabine's equation for the steady state case based on the variational indicators are subsequently given.

5.1 Derivation of Sabine's Equation From the Conservation of Energy Theorem

Let us first note (see, for example, Equation (2.30)) that, should \(S\) be independent of position and direction, one can simply write

\[
E(f) = 4\pi VS
\]

(5.1)

where \(4\pi\) represents total number of solid angles in a sphere and \(V\) is the volume of the room. Again \(E(f)\) represents the energy per unit bandwidth while \(S\) represents the energy per unit frequency bandwidth per unit solid angle per unit
volume. Substituting this into the conservation of energy relation (2.37), we find the Sabine equation as

$$\frac{2}{\rho} \frac{\partial}{\partial t} E_T + \frac{1}{T_R} E_T = W$$  \hspace{1cm} (5.2)$$

where the characteristic time $T_R$ is such that

$$\frac{1}{T_R} = \frac{c}{4\pi V} \int \int \hat{e}_k \cdot \hat{n} (1-|R|^2) dA d\Omega. \hspace{1cm} (5.3)$$

(The usual definition of reverberation time would be $6 \log_{10}$ times $T_R$, as explained below.)

In the steady state, we have

$$\frac{2}{\rho} \frac{\partial}{\partial t} E_T = 0$$

So Equation (5.2) becomes

$$E_T = T_R W.$$  

The quantity $T_R$ is the time constant of the process. If one substitutes for $E$ from Equation (5.1) into the above, he finds

$$S = T_R W/(4\pi V). \hspace{1cm} (5.4)$$
Similarly, the solution of the differential equation (5.2) for times after \( W \) has suddenly been turned off is

\[
E_T = E_{T,o} e^{-t/T_R}
\]  

(5.5)

where \( E_{T,o} \) is the energy per unit frequency band just before the source is shut off, and \( T_R \) is the time constant given by Equation (5.3). From Equation (5.5), we can find the so-called "reverberation time." The reverberation time \( T \) at a specific frequency is the time in seconds for intensity level, or sound pressure level, to decrease 60 decibels after the source is turned off\(^1\),\(^2\),\(^3\),\(^4\). By this definition, \( T \) is the time when \( E_T/E_{T,o} \) is \( 10^{-6} \) or when

\[
-60 = 10 \log_{10} e^{-T/T_R}
\]

which gives

\[
T = 13.8 \ T_R
\]

(5.6)

when \( T_R \) is given by Equation (5.3).

In the special case where \( |R^2| \) is constant, Equation (5.3) becomes

\[
\frac{1}{T_R} = \frac{ca}{4\pi V} \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\infty} \cos \theta \sin \theta d\theta\, d\phi\, dA
\]
\[ = \frac{cA\alpha}{4V} \]

or

\[ T_R = \frac{4V}{cA\alpha} \cdot \] (5.7)

Then, by substituting from Equation (5.7) into (5.4), we find

\[ S_{\text{Sabine}} = \frac{W}{\pi cA\alpha} \] (5.8)

We call this equation the steady state Sabine equation or Sabine's prediction for the directional spectral energy density in a room in the steady state. The corresponding \( S \) is called \( S_{\text{Sabine}} \). The total energy \( E \) in the room for the steady state case may be obtained by substituting for \( S \) as \( S_{\text{Sabine}} \) into Equation (5.1)

\[ E = \frac{4\pi V}{cA\alpha} \] (5.9)

If one substitutes the expression for \( T_R \) in Equation (5.7) into the reverberation time formula and inserts an appropriate value for the speed of sound in either metric or English units, he finds the same reverberation time equation as was originally derived empirically by W. C. Sabine, the theoretical basis of which is now quite well understood.\(^1,2,3,4\).
5.2 Derivation of Steady State Sabine Equation from the V.I. for a Room with Specularly Reflecting Walls

The variational indicator (3.1) with the apriori assumption that \( S \) be independent of position and direction becomes

\[
V.I. = s^2 \int \int \hat{n}_1 \cdot \hat{e}_{kl} \, dA_1 \, d\Omega_1 - s^2 \int \int (1-\alpha) \hat{e}_{kl} \cdot \hat{n}_1 \, dA_1 \, d\Omega_1
\]

\[
- \frac{W}{2\pi c} \int \frac{\hat{n}_1 \cdot \hat{e}_{cl} \, dA_1}{r_{ol}^2}
\]

Noting that the prime on the integrals indicates \( \hat{e}_{kl} \cdot \hat{n}_1 > 0 \), the first and second terms of the above may be simplified as

first term = \( s^2 \int \int \cos \theta \sin \theta \, d\phi \, d\theta \, dA \)

\[
= S^2 \pi A
\]

Here \( A \) is the total area of the walls in the room.

second term = \( S^2 (1-\alpha_{ave}) \pi A \)

where \( \alpha_{ave} \) is the average of \( \alpha \) in the room, i.e.

\[
\alpha_{ave} = \frac{1}{A} \int \alpha \, dA
\]
The third term of variational indicator is simply equal to \( \frac{2W}{c}S \). Therefore the V.I. becomes

\[
\text{V.I.} = S^{2} \pi A - S^{2} \pi A \left( 1 - \alpha_{\text{ave}} \right) - \frac{2W}{c}
\]

The requirement that this should be stationary with respect to small variations in \( S \) is satisfied if \( d(\text{V.I.})/dS = 0 \). This accordingly gives

\[
S_{\text{Sabine}} = \frac{W}{\pi c A \alpha_{\text{ave}}}
\]  \( (5.10) \)

Comparing Equations (5.8) and (5.10), we see that the values for \( S_{\text{Sabine}} \) obtained from the two different methods are the same.

5.3 Derivation of the Steady State Sabine Equation from the V.I. for a Room with Diffusely Reflecting Walls

The variational indicator (4.1) with the apriori assumption that \( S \) be independent of position becomes

\[
\text{V.I.} = S^{2} \int \frac{n_{1} \cdot e_{21} n_{2} \cdot e_{12}}{r_{12}^{2}} \, dA_{1} \, dA_{2}
\]

\[
+ \frac{W}{2 \pi c} S \int \frac{n_{2} \cdot e_{02}}{r_{02}^{2}} \, dA_{2} - \pi S^{2} \int \frac{dA}{|R^{2}|}
\]

Using the same method as done in the previous section to calculate the integrals, we find
V.I. = $S^2 \pi A + \frac{2W_S}{cS} - s^2 \pi \int dA$ 

This, in turn, is stationary if

$$S = \frac{W_S}{\pi cA \langle \frac{1}{|R^2|} \rangle_{ave} - 1}$$

where $\langle \frac{1}{|R^2|} \rangle_{ave}$ is the area averaged value of $\langle \frac{1}{|R^2|} \rangle$. While this is not exactly the same as Equation (5.12), one readily sees that it reduces to that previously derived if all the absorption coefficients are very small, since

$$\langle \frac{1}{|R^2|} \rangle_{ave} - 1 = \langle \alpha \rangle_{ave} + \langle \alpha^2 \rangle_{ave} + \langle \alpha^3 \rangle_{ave} + \ldots.$$ 

The distinction, however, points out the fact that the Sabine prediction as well as that above is not exact, albeit either may be a good approximation. The fact that the $S$ just derived above is somewhat smaller than $S_{\text{Sabine}}$ should not be surprising since the starting point was a variational indicator for $S_{\text{exit}}$. Thus, at best, we should only have a prediction of energy density with the direct wave omitted, i.e., for the reverberant portion of the field.
CHAPTER VI

ANALYSIS OF EFFECTS OF ROOM GEOMETRY AND ABSORPTION NONUNIFORMITY ON DIFFUSENESS OF SOUND

One virtue of a variational indicator is that it enables one to select, out of a class of possible approximate solutions, that approximate solution which in some sense is the best in its class. The present chapter uses this device to explore the notion that in a room with specularly reflecting walls there may be some directional preference in the sound propagation. In the first section of this chapter we limit our consideration to trial functions \( S = S_0 + \hat{S}_1 \cdot \hat{e}_k \) where \( S_0 \) and \( \hat{S}_1 \) are independent of position and direction and then find the "best" possible choice for \( S_0 \) and \( \hat{S}_1 \). In particular, our analysis predicts that, if the absorption coefficient \( \alpha \) is the same for all walls and is constant on each wall, then \( \hat{S}_1 = 0 \). In the second section, we assume that the absorption coefficient \( \alpha \) is not the same for two opposite walls of the room. In this case the prediction is that, in addition to \( S_0 \) being nonzero in the room, there is a nonzero \( \hat{S}_1 \) because of the nonconstant wall absorption. This implies that the acoustic energy propagating in the room has a slight directional preference and can not be perfectly diffuse.
6.1 First Approximation for the Directional Spectral Energy Density

We take our directional spectral energy density $S$, in a room with the same absorption coefficients $\alpha$ for all walls, to be of the following form

$$S = S_0 + \hat{s}_1 \cdot \hat{e}_k$$  \hspace{1cm} (6.1)

where

$$\hat{s}_1 = S_{1x} \hat{i} + S_{1y} \hat{j} + S_{1z} \hat{k}$$  \hspace{1cm} (6.2)

Here $S_0$ and $\hat{s}_1$ are assumed to be independent of position and direction. The unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are in directions $x$, $y$, and $z$. There are four unknowns $S_0$, and the three components of $\hat{s}_1$. An appropriate "best choice" for any of these four quantities may be found by substituting $S_0 + \hat{s}_1 \cdot \hat{e}_k$ into a suitable variational indicator and then requiring that the derivatives of this V.I. with respect to $S_0$, $S_{1x}$, $S_{1y}$ and $S_{1z}$ all respectively vanish. If one starts with the variational indicator of Equation (3.9), for example, one has

$$V.I. = \iint (S_0 + \hat{s}_1 \cdot \hat{e}_{k1})(S_0 + \hat{s}_1 \cdot \hat{e}_{k2}) \hat{e}_{k1} \cdot \hat{n}_1 dA_1 d\Omega_1$$

$$- \iint (1 - \alpha)(S_0 + \hat{s}_1 \cdot \hat{e}_{k1})(S_0 + \hat{s}_1 \cdot \hat{e}_{k1}^\dagger) \hat{e}_{k1} \cdot \hat{n}_1 dA_1 d\Omega_1$$
Noting that \( \mathbf{e}_{k2} = (e_{k2})_x \mathbf{i} + (e_{k2})_y \mathbf{j} + (e_{k2})_z \mathbf{k} \), we may write

\[
\mathbf{S}_1 \cdot \mathbf{e}_{k2} = \sum_{i=x,y,z} S_{1i} (\mathbf{e}_{k2})_i
\]

Using this expression and noting that \( \mathbf{e}_{k2} = -\mathbf{e}_{k1}^\dagger \), we may write the successive terms of the above V.I. as follows:

**First term**

\[
\text{first term} = S_0 \int \mathbf{S}_1 \cdot \mathbf{e}_{k1} \cdot \mathbf{n}_1 dA_1 d\Omega_1
\]

\[
+ \sum_{i,j=x,y,z} S_{1i} S_{1j} \int \mathbf{e}_{k1}^\dagger_i (\mathbf{e}_{k2})_j \mathbf{e}_{k1} \cdot \mathbf{n}_1 dA_1 d\Omega_1
\]

**Second term**

\[
\text{second term} = S_0 \int (1-\alpha) \mathbf{e}_{k1} \cdot \mathbf{n}_1 dA_1 d\Omega_1
\]

\[
+ \sum_i S_0 S_{1i} \int (1-\alpha) (\mathbf{e}_{k1})_i \mathbf{e}_{k1} \cdot \mathbf{n}_1 dA_1 d\Omega_1
\]

\[
+ \sum_j S_0 S_{1j} \int (1-\alpha) (\mathbf{e}_{k1})_j \mathbf{e}_{k1} \cdot \mathbf{n}_1 dA_1 d\Omega_1
\]

\[
+ \sum_{i,j} S_{1i} S_{1j} \int (1-\alpha) (\mathbf{e}_{k1})_i (\mathbf{e}_{k1})_j \mathbf{e}_{k1} \cdot \mathbf{n}_1 dA_1 d\Omega_1
\]

**Third term**

\[
\text{third term} = \frac{W}{2\pi c} S_0 \int \mathbf{e}_{o1} \cdot \mathbf{n}_1 dA_1
\]

\[
+ \frac{W}{2\pi c} \sum_i S_{1i} \int (\mathbf{e}_{o1})_i \mathbf{e}_{o1} \cdot \mathbf{n}_1 dA_1
\]

Some of the integrals appearing in the above are calculated in the previous chapter. For brevity, let us define the
other integrals as follows:

\[ I = \iint (1-\alpha) e_{kl} \cdot \mathbf{n}_1 \, dA_1 d\Omega_1 \]

\[ K_i = \iint (1-\alpha) (e_{kl})_i e_{kl} \cdot \mathbf{n}_1 \, dA_1 d\Omega_1 \]

\[ K_i^+ = \iint (1-\alpha) (e_{kl})_i^+ e_{kl} \cdot \mathbf{n}_1 \, dA_1 d\Omega_1 \]  \hspace{1cm} \text{(6.3)}

\[ L_{ij} = \iint (e_{kl})_i (e_{kl})_j e_{kl} \cdot \mathbf{n}_1 \, dA_1 d\Omega_1 \]

\[ M_{ij} = \iint (1-\alpha) (e_{kl})_i (e_{kl})_j^+ e_{kl} \cdot \mathbf{n}_1 \, dA_1 d\Omega_1 \]

\[ N_i = \int (e_{ol})_i \frac{e_{ol} \cdot \mathbf{n}_1}{r_{ol}} \, dA_1 \]

In terms of these abbreviations the variational indicator may be written

\[ \text{V.I.} = \sum_{i,j} 2 \pi A + \sum_{i} \sum_{j} S_{ij} S_{lj} L_{ij} - \sum_{i} \sum_{j} S_{ij} \sum_{i} S_{ij} K_i \]

\[ - \sum_{i} \sum_{j} S_{ij} L_{ij} M_{ij} - \frac{2W}{c} S_o - \frac{W}{2\pi c} \sum_{i} S_{li} N_i \]

\[ i,j = x,y,z \]

Differentiating this with respect to \( S_o, S_{lx}, S_{ly} \), and \( S_{lz} \) and subsequently setting each of the four expressions to zero, we find the following four simultaneous algebraic equations
for $S_0$ and the three components of $\hat{S}_1$

\[ 2(\pi A-I)S_0 - \sum (K_i + K_i^\dagger) S_{1i} = 2W/c \]

\[ -(K_x + K_x^\dagger)S_0 + \sum [L_{x i} + L_{x i} + M_{x i} + M_{x i}] = -W_{x}/2\pi c \]

\[ -(K_y + K_y^\dagger)S_0 + \sum [L_{y i} + L_{y i} + M_{y i} + M_{y i}] = -W_{y}/2\pi c \]

\[ -(K_z + K_z^\dagger)S_0 + \sum [L_{z i} + L_{z i} + M_{z i} + M_{z i}] = -W_{z}/2\pi c. \]

The above formulation holds regardless of whether or not $\alpha$ is constant. If $\alpha$ is constant and the room is rectangular, symmetry consideration should easily lead one to the conclusion that $\hat{S}_1$ is identically zero. This may not be quite as obvious if the room is not rectangular so we give a proof below of this fact which holds regardless of room shape. To this purpose we first show that for constant $\alpha$ the integrals $K_i$ and $K_i^\dagger$ are identically zero, so that $S_0$ may be found from the first equation alone.

The vector integral $\vec{K}$ (with three components $K_i$, $i=x,y,z$) may be written as

\[ \vec{K} = \iint (1-\alpha) \hat{e}_{ki} (\hat{e}_{kl} \cdot \hat{n}_l) dA_l d\Omega_l. \]

Since $\alpha$ at any point on the wall is presumed independent of incident direction at any point $\hat{n}_l$ on the wall, the angular
integration may be performed immediately. If one considers a local system of coordinates at point $x_1$ such that one of the unit vectors of this system is perpendicular to the area element $dA_1$ and pointing into the room, with the two other unit vectors tangent to $dA_1$ the integration is easily performed, giving

$$\hat{K} = \frac{2\pi}{3} \int (1-a) \hat{n}_1 \cdot \hat{e}_1 \, dA$$

so,

$$K_i = \frac{2\pi}{3} \int (1-a) \hat{n}_1 \cdot \hat{e}_i \, dA \quad (6.5)$$

$i=x,y,z$

With the assumption that $a$ is constant along the wall, the latter becomes

$$K_i = \frac{2\pi}{3} (1-a) \int \hat{n} \cdot \hat{e}_i \, dA$$

Applying the divergence theorem and noting that $V \cdot (\hat{e}_i) = 0$ we simply find that

$$K_i = 0 \quad i=x,y,z.$$

By the same way it can be shown that
By this result the first equation of (6.4) becomes

\[ 2(\pi A - I)S_0 = 2W/c. \]

Noting that the integral I from Equation (6.3) for a constant is equal to \((1-a)\pi A\), the above equation reduces to

\[ S_0 = \frac{W}{(\pi cAa)} \] \hspace{1cm} (6.6)

which is the same result as we have in Chapter V for \(S_{Sabine}\) in the steady state case (Equation (5.8)). The above reduces the problem of determining \(S_1^+\) in Equations (6.4) to that of solving three simultaneous equations. Noting that the integrals \(L_{ij}\) and \(M_{ij}\) are symmetric with respect to \(i,j\), these equations may be written as

\[ \sum (L_{ij} - M_{ij})S_{1i} = \frac{W_{ij}}{4\pi c} \hspace{1cm} j=x,y,z \]

\[ i=x,y,z \]

To show that \(S_{1i}^+, i=x,y,z\) is identically zero, let us first find the value of \(N_j\). From the last equation of (6.3) the vector integral \(\vec{N}\) (with components \(N_j, j=x,y,z\)) may be written

\[ \vec{N} = \int_{0}^{\varphi} \int_{0}^{\varphi} d\Omega \]
where $d\Omega_0 = (\hat{e}_{0l} \cdot \hat{n}_l \, d\Omega_l)/r_{0l}^2$ is substituted into the above. This vector integral is zero since the average of all directions over a total sphere is zero. Thus one has

$$N_i = 0 \quad i=x,y,z$$

Since the three simultaneous equations are therefore homogeneous, an obvious solution is

$$S_{1i} = 0 \quad i=x,y,z$$

This solution will, moreover, be unique if the determinant of the coefficients is nonzero. This does turn out to be the case, although the proof is omitted for brevity. In any event, one should be very surprised if the problem as posed should not lead to a unique solution. Note that these results depend strongly on the assumption that $a$ is the same constant for all walls.

### 6.2 Absorption Coefficient $a$ Not the Same on All Walls

In this section we assume that $a$ for any two opposite walls should be the same except for the walls which are perpendicular to the $\hat{e}_x$ direction. (The discussion here is restricted to the case of a rectangular room.) In other words we assume
\[
\alpha_x^+ \neq \alpha_x^-, \quad \alpha_y^+ = \alpha_y^-, \quad \alpha_z^+ = \alpha_z^-
\]

where \(\alpha_x^+, \alpha_y^+,\) and \(\alpha_z^+\) are the absorption coefficients of three different walls with unit outward normals pointing in the \(\hat{e}_x, \hat{e}_y,\) \(\hat{e}_z\) directions, respectively, while \(\alpha_x^-, \alpha_y^-,\) and \(\alpha_z^-\) are those for the corresponding three opposite walls.

Let us take \(S\) to be of the form

\[
S = S_0 + S_1 \hat{e}_k \hat{e}_x
\]

(Note that this assumes that \(S_1\) has only an \(x\) component. We could also assume at the outset that it has also \(y\) and \(z\) components, but for the case that \(\alpha_y^+ = \alpha_y^-\) and \(\alpha_z^+ = \alpha_z^-\) symmetry consideration quickly gives \(S_2 = S_3 = 0\) where \(S_2\) and \(S_3\) are assumed to be the components in the \(y\) and \(z\) directions.

Substituting the above into the variational indicator of Equation (3.9), we find

\[
\text{V.I.} = \iint (S_0 + S_1 \hat{e}_k \hat{e}_x) (S_0 + S_1 \hat{e}_k \hat{e}_x) \hat{e}_k \cdot \hat{n}_1 \, dA_1 \, d\Omega_1
\]

\[
- \iint (1-a) (S_0 + S_1 \hat{e}_k \hat{e}_x) (S_0 + S_1 \hat{e}_k \hat{e}_x) \hat{e}_k \cdot \hat{n}_1 \, dA_1 \, d\Omega_1
\]

\[
- \frac{W}{2\pi c} \int (S_0 + S_1 \hat{e}_0 \hat{e}_x) \frac{\hat{e}_0 \cdot \hat{n}_1}{r_{01}^2} \, dA_1
\]

Using the same definitions for integrals as is used in the previous section (Equations (6.3)), the above V.I. becomes
\[ V.I. = S_1^2 \pi A + S_1^2 L_{xx} - S_0^2 I - S_0 S_1 (K_x + K_x^+) \] (6.7)

\[ - S_0^2 M_{xx} - \frac{2W}{c} S_0 - \frac{W}{2\pi c} S_1 N_x \]

In the above V.I., the integrals which are independent of \( \alpha \) have the same value as are calculated in Section 6.1. The other integrals (i.e. those depending on \( \alpha \)) are calculated as follows:

The integral \( I \) simply becomes

\[ I = (1 - \alpha_{ave}) \pi A \]

where \( \alpha_{ave} \) is the area average of all \( \alpha \)'s.

The integral \( K_x \) may be calculated from Equation (6.5). This equation, for the case that \( i=x \), is zero unless \( \mathbf{\hat{n}} \) is in the direction (or in the opposite direction) of \( \mathbf{e}_x \). Therefore it becomes

\[ K_x = \frac{2\pi}{3} \left\{ \int_{A_x^+} (1 - \alpha_x^+) dA - \int_{A_x^-} (1 - \alpha_x^-) dA \right\} \]

which gives

\[ K_x = \frac{2\pi}{3} A_x (\alpha_x^- - \alpha_x^+) \]

where \( A_x \) is the area of the wall perpendicular to \( \mathbf{\hat{e}}_x \) (one
of two equal areas). The calculation for \( K^\dagger \) gives the same above value, i.e.

\[
K^\dagger = \frac{2\pi}{3} A_x (\alpha_x^- - \alpha_x^+).
\]

With the same method, the integral \( M_{xx} \) may be found to be

\[
M_{xx} = \frac{3\pi}{4} A_x (2\alpha_x^- - \alpha_x^+) - \frac{\pi}{4} (1 - \alpha_{ave}^-) A
\]

Substituting these into the V.I. of Equation (6.7), then differentiating with respect to \( S_0 \) and \( S_\perp \) and subsequently setting each of the expressions to zero, we find

\[
S_0 \pi A \alpha_{ave} = \frac{W}{c} - \frac{2\pi}{3} A_x (\alpha_x^- - \alpha_x^+) S_\perp
\]

\[
S_\perp \left\{ -\frac{\pi}{4} (A+x_A) - \frac{3\pi}{4} A_x (2\alpha_x^- - \alpha_x^+) + \frac{\pi}{4} (1 - \alpha_{ave}^-) A \right\}
\]

\[
= \frac{2\pi}{3} A_x (\alpha_x^- - \alpha_x^+) S_\perp.
\]

Since the absorption coefficients are presumed small, from the first equation we find to a good approximation

\[
S_0 = \frac{W}{(\pi c A \alpha_{ave})}
\]  

(6.8)

and, from the second one, we find

\[ \ldots \]
We see that $S_1$ is proportional to $(a_+^x-a_-^x)$ and we see that if $a_+^x = a_-^x$ then $S_1 = 0$, which is the result that we have in the previous section.

Note that if one applies the variational indicator of Equation (3.11) instead of Equation (3.9), he finds the same results as above for $S_0$ and $S_1$, only differing by a factor $(1-\alpha)$ (which is virtually equal to 1) for $S_0$.

6.3 The Effect of Room Geometry on the Diffusion of Reverberant Sound

To study the effect of room geometry on spatial uniformity of $S$, we consider a rectangular room with length $l$ in the $x$ and $y$ directions and length $l_z \neq l$ in the $z$ direction. Also we assume that the absorption coefficient $\alpha$ is the same for all walls. In the previous section we found that, under the assumption that the absorption coefficient be the same for all walls, the first correction to $S$, denoted by $S_1$, is zero, while the zeroth order quantity denoted by $S_0$ is unchanged. The absence of a nonzero correction is only true up to the first order. To find the lowest order nonzero correction when $\alpha$ is uniform, we expand $S$ up to the second order. That is, we write

$$S = S_0 + \tilde{S}_1 \cdot \vec{e}_k + \vec{e}_k \cdot S_2 \cdot \vec{e}_k \quad (6.10)$$
Here the first term may be considered as a "monopole type" term, the second term as a "dipole type" term and the third term as a "quadrupole type" term. With the assumption that the absorption coefficient $a$ is constant and the same for all walls, the second term of the above equation is identically zero (a result of Section 6.1). Thus we have

$$S = S_0 + \hat{e}_k \cdot \hat{e}_k$$  \hspace{1cm} (6.11)

In order to insure that $S_0$ is the directional average of $S$, we require, without loss of generality, that the trace of the dyadic $\$^2$ be zero. The above may be expanded in polar coordinates as

$$S = S_0 + (\$^2)^{xx}_{xx} \sin^2 \theta_k \cos^2 \theta_k + (\$^2)^{yy}_{yy} \sin^2 \theta_k \sin^2 \phi_k$$  \hspace{1cm} (6.12)

$$+ (\$^2)^{zz}_{zz} \cos^2 \theta_k$$

which by virtue of the assumed symmetry of the room reduces to

$$S = S_0 + \frac{1}{2} (\$^2)^{zz}_{zz} (3 \cos^2 \theta_k - 1)$$  \hspace{1cm} (6.13)

which, in turn, may be written in the form
If one substitutes this into the variational indicator of Equation (3.9) and then evaluates the corresponding integrals, he finds the latter is of the form

\[ V.I. = S_0^2 \pi A_\alpha + S_0 (S_2)_{zz} \pi (\ell^2 - \ell \ell_z) \alpha \]  \hspace{1cm} (6.15)

\[ - \frac{1}{2} (S_2)_{zz}^2 \pi (\ell^2 + \frac{3}{4} \ell \ell_z) \alpha - \frac{2W}{c} S_0 \]

where \( A_\alpha \) is the total area of the walls. The requirement that \( d(V.I.)/dS_1 = 0 \), where \( S_1 = S_0 \) or \( (S_2)_{zz} \), gives the two equations

\[ 2S_0^2 \pi A_\alpha - (S_2)_{zz} \pi (\ell^2 - \ell \ell_z) \alpha = \frac{2W}{c} \]  \hspace{1cm} (6.16a)

\[ S_0^2 \pi (\ell^2 - \ell \ell_z) \alpha + (S_2)_{zz} \pi (\ell^2 + \frac{3}{4} \ell \ell_z) = 0 \]  \hspace{1cm} (6.16b)

The second term of the first equation is small compared with the first term. Thus to a good approximation, we have

\[ S_0 = W/(\pi c A_\alpha) \]  \hspace{1cm} (6.17a)

\[ (S_2)_{zz} = -S_0 (\ell - \ell_z)/(\ell + \frac{3}{4} \ell_z) \]  \hspace{1cm} (6.17b)
This shows that, unless the room is perfectly cubical, there is a second order correction which is nonzero and which is directional (i.e., not diffuse). If $l > l_z$ the quantity $(\gamma_2)_{zz}$ is negative and $S$ is less than $S_0$ when $(e_k \cdot e_z)^2 > 1/3$ while it is greater than $S_0$ when the converse holds. This indicates that $S$ is larger for propagation towards those walls which are furthest apart. It is also of interest that the ratio of $(\gamma_2)_{zz}$ to $S_0$ is independent of the absorption coefficient. Thus, if one has a noncubical room, he can not increase the diffuseness of the sound by reducing the absorption of the wall.
CHAPTER VII

STUDY OF THE EFFECT OF ROOM GEOMETRY ON SPATIAL UNIFORMITY OF SOUND IN REVERBERANT ROOMS

In the present chapter we investigate various factors which may affect the spatial uniformity of sound in rooms. Since we are concerned with the spatial variation of the sound rather than its directional variation, we adopt the simplified view that we may explore the general features of deviations from spatial uniformity by restricting our attention to classes of trial solutions for $S$ which are independent of propagation direction. Thus, we assume that the quantity $S$ is independent of propagation direction, and for a rectangular room, as an additional simplifying assumption, we assume that it is constant on each wall of the room. Therefore there are six values of $S$'s for six walls which may be dependent on room geometry and source location. For any particular position of a source in a room, the values of the $S$'s may be found from a set of six simultaneous equations which may be written in matrix form as $[L]{S} = {B}$. We shall show that $[L]$ is a symmetric six by six matrix which depends on room shape and absorption coefficient, and $[B]$ is a column matrix whose elements are proportional to the solid angle subtended by the six walls.
In Sections 2 and 3 two methods for solving the above matrix equations are presented. Conclusions are summarized for various rooms with various locations of source and various values of absorption coefficient.

7.1 Derivation of Simultaneous Equations

We assume that the directional spectral energy density $S$ is constant on each wall therefore that there are six possibly different values of $S$. We denote these by $S_1$ to $S_6$ where $S_i$ corresponds to wall $i$, $i=1, \ldots, 6$. Since $S$ is assumed to be independent of direction, we can use the variational indicator of Equation (4.1) to find an appropriate "best" choice for each value (reflected waves) of $S$ on each wall. The variational indicator of Equation (4.1) for this case becomes

$$
v.I. = \sum_{i,j} \int \int S_i(\tilde{x}) S_j(\tilde{x}) \frac{(\hat{n}_i \cdot \hat{e}_{ij}) (\hat{n}_j \cdot \hat{e}_{ij})}{r_{ij}^2} \, dA_1 \, dA_j
$$

$$
+ \sum_i \frac{W}{2\pi c} \int S_i(\tilde{x}) \frac{\hat{n}_i \cdot \hat{e}_{oi}}{r_{oi}^2} \, dA_1 - \sum_i \frac{1}{n} \int \frac{1}{r_{i}^2} S_i^2(x) \, dA_i
$$

where the subscripts $i$ and $j$ refer to walls $i$ and $j$, respectively. With the assumption that $S$ be constant on each wall, this V.I. becomes
V.I. = \sum_{i=1}^{6} \sum_{j=1}^{6} S_i S_j K_{ij} + \frac{W}{2\pi c} \sum_{i=1}^{6} S_i \Omega_i - \sum_{i=1}^{6} \frac{\pi}{\langle |R_i^2| \rangle} S_i^2 A_i \tag{7.1}

where

\[ K_{ij} = \iint' \frac{(\hat{n}_i \cdot \hat{e}_{ij})(\hat{n}_j \cdot \hat{e}_{ij})}{x_{ij}^2} \, dA_i dA_j \tag{7.2} \]

and

\[ \Omega_i = \int \frac{\hat{n}_i \cdot \hat{e}_{oi}}{r_{oi}} \, dA_i \tag{7.3} \]

and where \( \langle |R_i^2| \rangle \) is the average over direction (weighted by \( \sin \theta \cos \theta \)) and over area of the ith wall. Differentiating with respect to each \( S_i \) and noting that the stationary property of the variational indicator requires that \( d(V.I.)/dS_i \) be zero, we find

\[ \sum_{j=1}^{6} (S_j K_{ij} + S_j K_{ji}) - \frac{2\pi}{\langle |R_i^2| \rangle} S_i A_i = -\frac{W}{2\pi c} \Omega_i. \tag{7.4} \]

\( i=1,2,\ldots,6 \)

Noting next that \( K_{ij} = K_{ji} \) (i.e. the integral in Equation (7.2) does not change with interchange of \( i \) and \( j \)) we find that the above equation becomes
where \( \alpha_i = 1 - \langle |R_i|^2 \rangle \). Noting that \( K_{ii} = 0 \) (since the quantity \( \vec{n}_i \cdot \vec{e}_i \) is the cosine between two perpendicular unit vectors corresponding to wall number \( i \)) the above equation may be written in matrix form as

\[
\begin{bmatrix}
-nA_1/(1-\alpha_1) & K_{12} & \cdots & K_{16} \\
K_{21} & -\pi A_2(1-\alpha_2) & K_{26} \\
\vdots & \vdots & \ddots & \vdots \\
K_{61} & \cdots \cdots \cdots & -\pi A_6(1-\alpha_6) \\
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
\vdots \\
S_6 \\
\end{bmatrix} = 
\begin{bmatrix}
\Omega_1 \\
\Omega_2 \\
\vdots \\
\Omega_6 \\
\end{bmatrix}
\] (7.6)

As we might expect, the square matrix is symmetric and depends on room geometry and absorption coefficient while the right hand side is proportional to the solid angles from source to each wall. If one divides both sides of the above equation by \( \pi \), calls the ratio of \( K_{ij}/\pi \) by \( F_{ij} \), and notes that \( 1/(1-\alpha) = 1+\alpha/(1-\alpha) \) he can write the above equations in the following form

\[
[[L_0] - [L_1]]\{S\} = \frac{-W}{4\pi c} \{\Omega\}
\] (7.7)

where
Before trying to solve Equations (7.7) for the $S$'s, it appears appropriate to first show that the matrix $[L_0]$ has determinant zero. Thus some numerical difficulties may be anticipated in inverting the matrix $[[L_0] - [L_1]]$ for the
case that absorption coefficients \( a_i \), \( i=1, \ldots, 6 \), are very close to zero.

To show that the determinant of the matrix \( [L_0] \) is zero, we apply a property of determinants which state that the value of a determinant does not change when any column vector is replaced by the sum of all column vectors (i.e., if \( a_{ij} \) is replaced by \( \sum_i a_{ij} \) for any fixed \( j \) and all \( i \)). To apply this theorem to the determinant \( [L_0] \) we replace the elements of the first column by the sum of all six columns such that we find

\[
|L_0| = \begin{vmatrix}
-A_1 + \sum_{j=1}^{6} F_{1j} & F_{12} & \cdots & F_{16} \\
-A_2 + \sum_{j=1}^{6} F_{2j} & -A_2 & \cdots & F_{26} \\
-A_6 + \sum_{j=1}^{6} F_{6j} & F_{62} & \cdots & -A_6
\end{vmatrix}
\]

Now we show that the elements of the first column are identically zero (i.e., \( \sum_{j=1}^{6} F_{1j} = A_1 \)). From Equation (7.9) we can write

\[
\sum_{j=1}^{6} F_{ij} = \sum_{j=1}^{6} \frac{1}{\pi} \int \int (\hat{e}_{i1} \cdot \hat{n}_1) (\hat{e}_{lj} \cdot \hat{n}_j) \frac{dA_j dA_i}{r_{ij}^2}
\]

Substituting for \( (\hat{e}_{lj} \cdot \hat{n}_j dA_j)/r_{ij}^2 = d\Omega \) where \( d\Omega = \sin\theta_1 d\phi d\theta \) and for \( \hat{e}_{j1} \cdot \hat{n}_1 = \cos\theta \), we find
\[
\sum_{i=1}^{6} \sum_{j=1}^{6} F_{ij} = \frac{1}{\pi} \int_{0}^{2\pi} \int_{0}^{\pi/2} \cos \theta \sin \theta \, d\theta \, d\phi \, d\phi_{1} = A_{1}
\]

(The same result may be found from Chapter V, Section 5.2 by noting that \( \sum_{i=1}^{6} \sum_{j=1}^{6} F_{ij} \) = total area of the walls.) The above equation may readily be generalized to the form

\[
\sum_{j=1}^{6} F_{ij} = A_{i} \quad i=1,2,\ldots,6
\]

By this result we see that all elements of the first column of \( L_{0} \) is zero. Thus

\[
\text{det.} [L_{0}] = 0 \quad (7.12)
\]

7.2 Solution of Equations by Perturbation Technique

In most cases of interest, the absorption coefficients are small and it would appear appropriate to apply a perturbation technique in which the solution for \( \{S\} \) is expanded in ascending series in the \( \alpha_{i}/(1-\alpha)'s \). Procedures for doing this are outlined below. Let us proceed under the assumption that we have already found the eigenvalues and eigenvectors of matrix \([L_{0}]\).

Let the eigenvalues of the matrix \([L_{0}]\) be denoted by \( \lambda_{i} \) and the corresponding eigenvectors by \( \{q_{i}\} \) where \( i=1,2,\ldots,6 \). Thus we can write
\[ [L_o] \{q_i\} = \lambda_i \{q_i\}. \quad (7.13) \]

A well known result in linear algebra is the fact that, if a symmetric matrix has determinant zero, then there is at least one eigenvector \( \{q_m\} \) corresponding to eigenvalue \( \lambda_m = 0 \). This eigenvector is readily seen to be

\[
\{q_m\} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{(constant)} \quad (7.14)
\]

(This is the only eigenvector with zero eigenvalue.) Another theorem states that the eigenvectors of a symmetric matrix may be chosen to be orthogonal, that is, one can write

\[
\{q_i\}^T \{q_j\} = 0, \quad (7.15)
\]

where \( \{q_i\} \) and \( \{q_j\} \) are any two different eigenvectors of matrix \( [L_o] \). Thus the \( \{q_i\} \)'s form a complete basis set for any six dimensional vector. Let us next expand \{S\} in the form

\[
\{S\} = \sum_{i=1}^{6} r_i \{q_i\}. \quad (7.16)
\]

Substituting from Equation (7.16) into Equation (7.7), we find that

\[
[[L_o] - [L]] \sum_{i=1}^{6} r_i \{q_i\} = -\frac{w}{4\pi^2 c} \{\Omega\} \quad (7.17)
\]
or, from Equation (7.13), that

\[ \sum_{i=1}^{6} \lambda_i {r_i}^T (q_i) = \sum_{i=1}^{6} {r_i}^T [L_1] (q_i) = \frac{-W}{4\pi^2 c} \{\Omega\} \quad (7.18) \]

Multiplying both sides of this equation by \( (q_j)^T \), we then have

\[ (q_j)^T \sum_{i=1}^{6} \lambda_i {r_i}^T (q_i) = (q_j)^T \sum_{i=1}^{6} {r_i}^T [L_1] (q_i) = \frac{-W}{4\pi^2 c} \{\Omega\} \quad (7.19) \]

Because of the orthogonality between eigenvectors (Equation (7.15)), the above simplifies to

\[ \lambda_j r_j (q_j)^T (q_j) = \sum_{i=1}^{6} {q_j}^T [L_1] (q_j) {r_i} = \frac{-W}{4\pi^2 c} (q_j)^T \{\Omega\}. \quad (7.20) \]

If we also choose the normalization of the vectors to be such that

\[ (q_j)^T (q_j) = 1, \quad (7.21) \]

Then Equation (7.20) may be rewritten as

\[ \lambda_j r_j = \sum_{i=1}^{6} (L_1)_{ji} r_i = p_j, \quad (7.22) \]

\[ j=1,2,\ldots,6 \]
where

\[(L_1)_{ji} = (q_j)^T L_1 q_i \]  \hspace{1cm} (7.23)

and

\[p_j = \frac{-W}{4\pi^2 c} (q_j)^T \Omega \]  \hspace{1cm} (7.24)

Next we solve Equation (7.22) for \( r_i \), then (by use of Equation (7.16)) we may find the values of \( S \). To find \( T_i \) from Equation (7.22), we use a perturbation technique as follows. The quantity \( r_i \) may be written as

\[ r_i = r_i^{(0)} + r_i^{(1)} + r_i^{(2)} + \ldots \]  \hspace{1cm} (7.25)

where the zero-order \( r_i^{(0)} \) correspond to unperturbed part (matrix \( L_0 \) ) and the other orders correspond to the perturbed part.

Equation (7.22) may be written as

\[ \lambda_j r_j - \epsilon \sum_{i=1}^{6} (L_1)_{ji} r_i = \epsilon p_j \]  \hspace{1cm} (7.26)

\[ j=1,2,\ldots,6 \]

where \( \epsilon \) (a formal ordering device) is equal to unity.

Consistent with this ordering scheme, the \( r_i \) may be written
as

\[ r_i = r_i^{(0)} + \varepsilon r_i^{(1)} + \varepsilon^2 r_i^{(2)} \]  

(7.27)

Substituting from (7.33) into (7.32) we find

\[ \lambda_j (r_j^{(0)} + \varepsilon r_j^{(1)} + \varepsilon^2 r_j^{(2)} + \ldots) - \varepsilon \sum_i (L_l)_{ji} r_i^{(0)} + \varepsilon r_i^{(1)} + \varepsilon^2 r_i^{(2)} + \ldots \]

(7.28)

\[ = \varepsilon \rho_j. \]

We require that this be satisfied identically to all orders of \( \varepsilon \). This gives the zeroth order equations as

\[ \lambda_j r_j^{(0)} = 0 \]  

(7.29)

and the first order equations as

\[ \lambda_j r_j^{(1)} - \varepsilon \sum_i (L_l)_{ji} r_i^{(1)} = \rho_j, \]  

(7.30)

\[ j = 1, 2, \ldots, 6 \]

the second order equations as

\[ \lambda_j r_j^{(2)} - \varepsilon \sum_{i=1}^{6} (L_l)_{ji} r_i^{(1)} = 0, \]  

(7.31)
or, generally, for \( n \geq 2 \)

\[
\lambda_j r_j^{(n)} - \sum (L_1)_{ji} r_i^{(n-1)} = 0. \quad (7.32)
\]

As we discussed before, one of the eigenvalues is zero (denoted here by \( \lambda_m \)). Therefore, from Equation (7.29), one requires

\[
r_i^{(0)} = 0. \quad (7.33)
\]

To find \( r_m^{(0)} \), we must solve the first order Equation (7.30). For \( j=m \) we have

\[
\lambda_m r_m^{(1)} = \sum_{i=1}^6 (L_1)_{mi} r_i^{(0)} = p_m. \nonumber
\]

The first term of this equation is zero since \( \lambda_m \) is zero and the only contribution to the second term comes from \( i=m \). Thus we find

\[
-(L_1)_{mm} r_m^{(0)} = p_m.
\]

or
\[ r_m^{(0)} = -\frac{p_m}{(L_m)_mm} \quad (7.34) \]

To find \( r_i^{(1)} \) we use again the first order Equation (7.30) for \( j \neq m \). We find

\[ \lambda_j r_j^{(1)} - (L_j)_jm r_m^{(0)} = p_j \]

\( j \neq m \)

From this, we find

\[ r_j^{(1)} = \frac{1}{\lambda_j} \left[ (L_j)_jm r_m^{(0)} + p_j \right] \quad (7.35) \]

\( j \neq m \)

Substituting for \( r_m^{(0)} \) from Equation (7.34) into the above, we obtain

\[ r_j^{(1)} = \frac{1}{\lambda_j} \left[ \frac{(L_j)_jm}{(L_m)_mm} p_m + p_j \right] \quad (7.36) \]

\( j \neq m \)

To find \( r_m^{(1)} \), we use second order Equation (7.31). For \( j = m \), we find

\[ -\sum_{i=1}^{6} (L_m)_mi r_i^{(1)} = 0 \]
which gives

\[(L_1)_{mm} r_m^{(1)} = - \sum_{i \neq m} \frac{(L_1)_{mi} r_i^{(1)}}{\lambda_i} \]

So

\[r_m^{(1)} = \sum_{i \neq m} \frac{(L_1)^{mi}}{(L_1)_{mm}} \left\{ - \frac{(L_1)^{im} p_m}{(L_1)_{mm}} + p_i \right\} \]

We stop our calculation with first order since the first order terms represent (for small \(a\)) the dominant contribution to the spatially inhomogeneous part of \(\{S\}\).

Substituting from Equation (7.31) into (7.16), up to first order we find

\[\{S\} = I \{r_0 + r_1^{(1)}\} \{q_i\} \]

where \(\{q_i\}\)'s are the normalized eigenvectors of matrix \([L_0]\), and where the \(r_0\)'s and \(r_1^{(1)}\)'s may be found from Equations (7.33) through (7.37).

To exhibit the numerical quantities for \(S\) from Equation (7.38) in a rectangular room, it is convenient to
consider the value of $S$ relative to what we called $S_{\text{Sabine}}$ (see Chapter V). The values of $S_{\text{Sabine}}$ for the totally diffuse case may be obtained from the variational indicator of Equation (7.1) with the assumption that all $S$'s are the same. Therefore, we would have

$$V.I. = S^2 \sum_{i=1}^{6} \sum_{j=1}^{6} K_{ij} + \frac{W}{2\pi G} S \sum_{i=1}^{6} \Omega_i$$

$$- \pi S^2 \sum_{i=1}^{6} \frac{1}{1-\alpha_i} A_i$$

where, by Equations (7.17) and (7.9) we have

$$\sum_{i=1}^{6} \sum_{j=1}^{6} K_{ij} = \pi A.$$

Here $A$ is the total area of walls in the room. Noting that in the V.I. above, $\sum_{i=1}^{6} \Omega_i = 4\pi$, we find

$$V.I. = \pi A S^2 + 2 \frac{W}{G} S - \pi A S^2 - \pi S^2 \sum_{i=1}^{6} \frac{\alpha_i}{1-\alpha_i} A_i$$

Differentiating with respect to $S$ gives the $S_{\text{Sabine}}$ as

$$S_{\text{Sabine}} = \frac{W}{\{\pi c \sum_{i} \frac{\alpha_i}{1-\alpha_i} A_i\}}$$

(7.39)

Dividing both sides of Equation (7.38) by $S_{\text{Sabine}}$, we find
\[ \{ S/S_{\text{Sabine}} \} = \left[ \frac{\pi c}{W} \sum \frac{\alpha_i A_i}{1-q_i} \right] \frac{6}{\sqrt{6}} \left\{ r_i^{(0)} + r_i^{(1)} \right\} \{ q_i \} \]

Noting that (from Equation (7.33)) all \( r_i^{(0)} \)'s are zero except for \( r_m^{(0)} \) and that the normalized eigenvector \( \{ q_m \} \) has elements equal to \( 1/\sqrt{6} \), the above equation becomes

\[ \{ S/S_{\text{Sabine}} \} = \left( \frac{\pi c}{W} \sum \frac{\alpha_i A_i}{1-q_i} \right) \left[ \frac{r_m^{(0)}}{\sqrt{6}} + \sum_{i=1}^{6} r_i^{(1)} \{ q_i \} \right] \]

If one evaluates the quantity \( r_m^{(0)} \) from Equation (7.34) and substitutes into the above equation, he finds that the product of the quantity in parenthesis with the first term is equal to unity. Thus the equation may be written as

\[ \{ S/S_{\text{Sabine}} \} = 1 + \left( \frac{\sqrt{6}/r_m^{(0)}}{r_m^{(0)}} \right) \sum_{i=1}^{6} r_i^{(1)} \{ q_i \} \quad (7.40) \]

From this, we can find the values of \( S/S_{\text{Sabine}} \) as a function of room geometry, absorption coefficient and source location.

### 7.3 Solution by Direct Matrix Inversion

We now reconsider Equation (7.6), which can be written in the form
where \( F_{ij} \) and \( \Omega_i \) are given by Equations (7.9) and (7.4). The above may also be written in an abbreviated form as

\[
[L]{S} = -\frac{W}{4\pi^2c}{\Omega} \tag{7.42}
\]

where \([L]\) is a square matrix, \({\Omega}\) and \({S}\) are column vectors. This has the formal solution

\[
{S} = \frac{-W}{4\pi^2c}[L]^{-1}{\Omega} \tag{7.43}
\]

Although this method is apparently less intricate than that which is discussed in the previous section, there is a disadvantage in the use of matrix inversion. That is, when \( \alpha \) is very close to zero, the determinant of matrix \([L]\) approaches zero and the computation of \([L]^{-1}\) becomes subject to large numerical errors.
7.4 Discussion of the Matrix \([L_0]\) and of the Column Vector \([\Omega]\)

A. Calculation of the Elements of the Matrix \([L_0]\)

The diagonal elements of matrix \([L_0]\) in Equation (7.8) are the areas of the walls, while the other elements, denoted by \(F_{ij}\), are given in Equation (7.9). This equation may be written in the following form

\[
F_{ij} = \frac{1}{\pi} \frac{1}{r_{ij}} \cos \theta_i \cos \theta_j dA_i dA_j
\]

where \(\cos \theta_i = \hat{e}_{ji} \cdot \hat{n}_i\) and \(\cos \theta_j = \hat{e}_{ij} \cdot \hat{n}_j\) (see Figure 8).

For two perpendicular rectangles with dimensions \(a, b\) and \(c, b\) where \(b\) is the length of their common side, \(F_{ij}\) may be represented\(^{21}\) as a function of \(L = c/b\) and \(N = a/b\). The equation for \(F_{ij}\) and numerical values are presented in Appendix A.

For two parallel rectangles with equal dimensions \(a, b\) and with a net distance \(c\) separating them, the quantity \(F_{ij}\) may be represented\(^{21}\) as a function of \(x = b/c\) and \(y = a/c\). The appropriate equation and some numerical values are also given in Appendix A.

B. Calculation of the Elements of the Column Vector \([\Omega]\)

One may rewrite Equation (7.3), which gives the elements of column vector \([\Omega]\), in the form
Figure 8. Sketch Illustrating Two Perpendicular Walls of a Rectangular Room (with common side $b$ where the quantity $F_{i,j} = \frac{1}{\pi} \int \cos \theta_i \cos \theta_j \, dA_i \, dA_j / r_{ij}^2$ may be written as a function of $L = c/b$ and $N = a/b$.)
\[ \Omega_i = \int \frac{\cos \theta_i}{r_{oi}^2} \, dA_i \]

where \( \cos \theta_i = \hat{n}_i \cdot \hat{e}_{oi} \). Here the unit vector \( \hat{n}_i \) is normal to the surface element \( dA_i \) and \( \hat{e}_{oi} \) is the unit vector from the source to the element \( dA_i \). In Appendix B it is shown that the solid angle \( \Omega \) subtended by each wall may be replaced by the sum of four (or two) solid angles \( \Omega \) subtended by subdivisions of the wall such that the source be located at one corner of a rectangle that has one common side with all subdivisions (see Appendix B). Techniques and accompanying plots may be found in the NASA report by Hamilton and Morgan. Extensive use of the graphs in that report was made in the numerical calculation presented here.

### 7.5 Some Representative Numerical Results

In our calculations of relative directional spectral energy density \( S/S_{\text{Sabine}} \) we considered three actual rooms which had previously been mentioned in the literature.

Room 1: dimensions 32 x 27 x 10 (ft\(^3\)) and \( \alpha = 0.12 \) (Ref. 22)

Room 2: dimensions 7.66 x 6.32 x 4.77 (m\(^3\)) and \( \alpha = 0.016 \) (Ref. 23)

Room 3: dimensions 20 x 14 x 8 (ft\(^3\)) and \( \alpha = 0.0977 \) (Ref. 1)

Here the absorption coefficient \( \alpha \) for each room is the area averaged value of the absorption coefficients of all surfaces.

For each room we considered two possible sets of
positions of the source.

Set 1 from point 0 (center of the room to point 0' (center of wall 3 (see Appendix C) with five equal increments.

Set 2 from point A to point A' (see Appendix C), again with five equal increments.

To investigate the dependence of $S/S_{\text{Sabine}}$ on absorption coefficient, we considered a number of alternate values of $\alpha$ for each room. For these various values of $\alpha$, the values of $S/S_{\text{Sabine}}$ for each wall were plotted versus the distance of the source from wall number 3. From these curves (see Figures 8 to 22), the following results may be noted.

1. When a source is at equal distance from two parallel walls in a rectangular room, the values of $S/S_{\text{Sabine}}$ at these two walls are equal.

2. When a source is in the center of a rectangular room, the largest value of $S/S_{\text{Sabine}}$ at the various walls is that corresponding to the wall with largest area.

3. The perturbation part of $S/S_{\text{Sabine}}$ (i.e., its difference from unity) for any given wall is nearly proportional to the ratio $\alpha/(1-\alpha)$. Thus for $\alpha$ very close to zero, the perturbation is very small.

4. By moving the position of a source closer to a wall, the value of $S/S_{\text{Sabine}}$ increases for that wall, while it decreases for the opposite wall. The rate of increase at the closer wall is apparently more than the rate of
decrease on the opposite wall.

5. For the parallel movement of a source to a wall, the value of $S/S_{\text{Sabine}}$ at the wall is maximum for the case that the source is located on a line normal to the wall at the center of the wall.
Figure 9. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.
Figure 10. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.
Figure 11. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.
Figure 12. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.

Figure 13. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the source in the center of the room, while position 6 corresponds to a source near the center of wall number 3.
Figure 14. Plots of Relative Energy Density Near Various Walls in a Room Versus Source Positions. Position 7 corresponds to a source at point A, while position 12 corresponds to a source near point A' (see Figure B1).
Figure 15. Plots of Relative Energy Density Near Various Walls in a Room versus Source Positions. Position 7 corresponds to a source at point A, while position 12 corresponds to a source near point A' (see Figure 81).
Figure 16. Plots of Relative Energy Density Near Various Walls in a Room versus Source Positions. Position 7 corresponds to a source at point A, while position 12 corresponds to a source near point A' (see Figure 61).
Figure 17. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.

Figure 18. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.
Figure 19. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.

Figure 20. Plots of Relative Energy Density Near Various Walls in a Room Versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.
Figure 21. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.
Figure 22. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.

Figure 23. Plots of Relative Energy Density Near Various Walls in a Room versus Source Position Relative to the Center of the Room. Position 1 corresponds to the center of the room, while position 6 corresponds to a source near the center of wall number 3.
CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

On the basis of the analysis in Chapters II to V, and from the results summarized in Chapters VI and VII, it appears possible to draw the following conclusions concerning steady state sound fields in reverberant rooms.

1. The concept of a directional spectral energy density $S$ may be used in the formulation of a theory of sound propagation in an enclosure. The laws of conservation of energy may be expressed in terms of the directional spectral energy density. The requirement that conservation hold in a local sense leads to a variational indicator. The form of the variational indicator depends on the nature of the wall reflection and on whether one expresses it in terms of incident or reflected wave fields.

2. The restriction of the class of trial functions $S$ to those which are directionally independent forces the variational indicator derived for the specular reflecting wall case to reduce to that derived for the diffusely reflecting wall case.

3. The existence of a variational indicator as a functional of $S$ enables one to make relatively simple estimates of the spatial and directional dependence of
\( S(\dot{x},\dot{e}_k) \) for any given room with given position of the source and for given values on the walls of the absorption coefficients.

4. According to the results in Chapter VI, there are a number of circumstances in which a reverberant field cannot be perfectly diffuse. These include the following:

a. When the absorption coefficients of the walls are not the same for all walls, there is a slight preference for propagation of energy towards directions pointing to the higher absorptive surfaces.

b. Even for the case that the absorption coefficients are not the same for entire walls, in a rectangular room, there is a second order perturbation from the ideal diffuseness for a noncubical room and the diffuseness of the sound cannot be increased by reducing the absorption of the walls.

5. From the analysis of the effect of room geometry, source location, and absorption coefficients on the spatial uniformity of sound in a room, the following results are concluded.

a. The values of the absorption coefficients have a dominant effect on the spatial uniformity of such a field. That is, the relative perturbation from ideal diffuseness is proportional to the average absorption coefficient and vanishes when the latter approaches zero.

b. The effect of source location is also important. The relative reverberant energy density (even excluding...
the direct wave) increases near any wall when the source is moved close to that wall. The rate of increase on the closest wall is greater than the rate of decrease on the opposite wall for the case of a rectangular room.

\(c\). In a rectangular room, when the source is located in the center of the room, the relative energy density is maximum nearest the walls which have maximum area.

As regards future study in this general area, we recommend some experimental work to measure spatial and directional dependence of \(S\) in a room. One may then compare the results with those predicted by the methods presented in this thesis. Also, it is recommended that one investigate the transient behavior of reverberant sound fields in rooms based on an extension of the variational formulation to include the transient case. Such a variational formulation has already been developed\(^{13}\) but its implications have not yet been fully explored.
APPENDIX A

THE EQUATIONS AND NUMERICAL VALUES FOR THE QUANTITY $F_{ij}$

IN RECTANGULAR ROOMS

For two perpendicular walls with dimensions $a,b$ and $b,c$, $F_{ij}$ is calculated\textsuperscript{21} as a function of $L = c/b$ and $N = a/b$. The equation is of the following form

$$F_{ij} = A \frac{1}{\pi L} \left[ L \tan^{-1} \left( \frac{1}{L} \right) + N \tan^{-1} \left( \frac{1}{N} \right) - (N^2 + L^2) \frac{1}{2} \tan^{-1} \left( \frac{1}{\sqrt{N^2 + L^2}} \right) \right]$$

$$+ \frac{1}{4} \log e \left\{ \frac{(1+N^2)(1+L^2)}{1+L^2+N^2} \left[ \frac{L^2(1+N^2+L^2)}{(1+N^2)(1+L^2)} \right] \frac{N^2(1+N^2+L^2)}{(1+N^2)(1+L^2)} \right\}$$

In the above, $F_{ij}$ is proportional to the area $A_i$ and has the dimensions of area, while $N$ and $L$ are dimensionless. We know that $F_{ij} = F_{ji}$, because by interchanging $i$ and $j$ (interchanging $L$ and $N$) we find that the quantity inside the big parenthesis does not change while the quantity outside the parenthesis remains the same (i.e. $A_i \frac{1}{\pi L} = A_j \frac{1}{\pi N} = \frac{b^2}{\pi}$).

In reference 21, plots are given of $F_{ij}$ as a function of $N$ and $L$.

For two parallel walls, each with dimensions $a,b$, and with a net distance $c$ separating them, $F_{ij}$ may be expressed as a function of $x = b/c$ and $y = a/c$. The result is
\[
F_{ij} = \frac{2ab}{\pi xy} \left\{ \log_e \left( \frac{(1+x^2)(1+y^2)}{1+x^2+y^2} \right)^{\frac{1}{2}} + y (1+x^2)^{\frac{1}{2}} \tan^{-1}\left( \frac{y}{\sqrt{1+x^2}} \right) + x (1+y^2)^{\frac{1}{2}} \tan^{-1}\left( \frac{x}{\sqrt{1+y^2}} \right) - y \tan^{-1} y - x \tan^{-1} x \right\}
\]

In reference 21, plots of $F_{ij}$ as a function of $x$ and $y$ are also given.

To find the numerical value of $F_{ij}$ for any specific room, one should note that, for any rectangular room, there are only four independent values of $F_{ij}$. That is, the other values may be found from the properties of the matrix $[L_0]$ (i.e., that it be symmetric and that its determinant be zero) and from the property of room symmetry, i.e., each two opposite walls have the same area). These four independent $F_{ij}$'s may be chosen as $F_{12}$, $F_{13}$, $F_{23}$, and $F_{36}$. The numerical values of these quantities for the three rooms, considered in this thesis, obtained from the plots in reference 21, are given in Table 1.
Table 1. Numerical Values of $F_{ij}/A_i$ for the Three Rooms Described in Chapter VII

<table>
<thead>
<tr>
<th></th>
<th>$F_{12}/A_1$</th>
<th>$F_{13}/A_1$</th>
<th>$F_{23}/A_2$</th>
<th>$F_{36}/A_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Room 1</td>
<td>0.1285</td>
<td>0.1073</td>
<td>0.1056</td>
<td>0.0630</td>
</tr>
<tr>
<td>Room 2</td>
<td>0.1840</td>
<td>0.1570</td>
<td>0.1420</td>
<td>0.1250</td>
</tr>
<tr>
<td>Room 3</td>
<td>0.1750</td>
<td>0.1160</td>
<td>0.1210</td>
<td>0.0750</td>
</tr>
</tbody>
</table>
APPENDIX B

FORMULATION AND THE NUMERICAL VALUES OF THE SOLID ANGLES SUBTENDED BY THE WALLS FOR THE THREE ROOMS

1. Formulation

For an arbitrary position of a source in a rectangular room, the solid angle subtended by a wall with dimensions, say $a, b$, may be found as follows. Suppose that the source is located at point $A$ at a net distance $L$ from the wall $(AA' = L)$ (see Figure B1). We consider a system of coordinates $\xi, \eta$ in the surface of the wall centered at point $A'$. The axes of this system of coordinates divide the rectangle $ab$ into four rectangles with dimensions $a', b', a'', b'$, and $a'', b''$. The element of solid angle subtended by $dA$ is

$$d\Omega = \frac{\cos \theta dA}{r^2}$$  \hspace{1cm} (1)

where $\cos \theta = L/r$. From the Figure 1-a we see that

$$r^2 = L^2 + R^2$$

$$R^2 = \xi^2 + \eta^2$$

Substituting from these into Equation (1) we find
Figure Bl. (a) Sketch Illustrating that the Solid Angle Subtended by a Wall May be Replaced by the Sum of 4 (or 2) Solid Angles Subtended by the Subdivisions of the Wall. (b) Different Positions of the Source are Shown as 1 to 12 in a Room with Dimensions a, b, c where the Bottom and Top Walls are Denoted by 1 and 4; Front and Back Walls, by 2 and 5; Left and Right Walls, by 3 and 6.
Thus the solid angle subtended by the wall area is

$$\Omega = \frac{a' b'}{L^2} \frac{d\xi d\eta}{(L^2 + \xi^2 + \eta^2)^3/2}$$

This may be divided into four subintegrals as

$$\Omega = \int \int d\Omega + \int \int d\Omega + \int \int d\Omega + \int \int d\Omega$$

where $d\Omega$ in each integral may be taken from Equation (2).

Each integral in the above denotes the solid angle subtended by a subdivision of the rectangle $ab$ when the source is located at one corner of a rectangle that has one common side with all four subrectangles. These integrals are denoted by $\phi$ such that

$$\phi_{ab} = \frac{a'b'}{L^2} \frac{L d\xi d\eta}{(L^2 + \xi^2 + \eta^2)^3/2}$$

Changing the variables $\xi$ and $\eta$ such that $\alpha = \frac{\xi}{L}$ and $\beta = \frac{\eta}{L}$ we find

$$\phi_{a'b'} = \int \int \frac{a'/L b'/L}{(\alpha^2 + \beta^2 + 1)^{3/2}} d\alpha d\beta$$
From this, $\phi_{a'b'}$ may be obtained as a function of $x = a'/L$ and $y = b'/L$

$$\phi = \tan^{-1} \left\{ \frac{xy}{\sqrt{(x^2 + y^2 + 1)}} \right\}$$

The values of $\phi/4\pi$ are given as a function of $x$ and $y$ in reference 21.

2. Numerical Values

For the three rooms, considered in the text, the corresponding values for $\Omega$ for various source locations are given in Tables 2 to 7. For each room, the values of the solid angles are calculated for two sets of positions of the source: set 1, where the source is on a line going from point 0 (center of the room) to point 0' (center of wall 3) with five equal increments, and set 2, where the source is on a line going from point A to point A', also with 5 equal increments. Each column of the table corresponds to a fixed position of the source and each row corresponds to a specific wall.
Table 2. Values in Sterradian of the Solid Angles Subtended by the Walls for Room 1, Set 1

<table>
<thead>
<tr>
<th>Source Position</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4.305</td>
<td>4.288</td>
<td>4.251</td>
<td>3.979</td>
<td>3.420</td>
<td>2.349</td>
</tr>
<tr>
<td>2</td>
<td>1.139</td>
<td>1.104</td>
<td>1.022</td>
<td>0.963</td>
<td>0.844</td>
<td>0.665</td>
</tr>
<tr>
<td>3</td>
<td>0.837</td>
<td>1.142</td>
<td>1.536</td>
<td>2.269</td>
<td>3.671</td>
<td>6.283</td>
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<td>4.305</td>
<td>4.288</td>
<td>4.251</td>
<td>3.979</td>
<td>3.420</td>
<td>2.349</td>
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<td>0.963</td>
<td>0.844</td>
<td>0.665</td>
</tr>
<tr>
<td>6</td>
<td>0.837</td>
<td>0.639</td>
<td>0.481</td>
<td>0.409</td>
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<td>0.253</td>
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Table 3. Values in Sterradian of the Solid Angles Subtended by the Walls for Room 1, Set 2

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<td>0.465</td>
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<td>Wall 3</td>
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<td>0.362</td>
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Table 4. Values in Steradian of the Solid Angles Subtended by the Walls for Room 2, Set 1

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Table 5. Values in Steradian of the Solid Angles Subtended by the Walls for Room 2, Set 2

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<th>6</th>
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Table 6. Values in Sterradian of the Solid Angles Subtended by the Walls for Room 3, Set 1

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<td>Wall 2</td>
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Table 7. Values in Sterradian of the Solid Angles Subtended by the Walls for Room 3, Set 2

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<td></td>
<td></td>
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