AN ANALYTIC INVESTIGATION OF THE POINT OF INFLECTION OF TRACE CURVES

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Edward J. Staros

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AN ANALYTIC INVESTIGATION OF THE POINT
OF INFLECTION OF TRACE CURVES

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GLOSSARY

A	Length of coupler of a four-bar mechanism.
B	Length of follower of a four-bar mechanism.
C	Length of fixed link of a four-bar mechanism.
K	Constant.
m	Imaginary link connecting pin joints 1A and BC.
q	$\sqrt{1 + B^2 + C^2 - A^2}$
RC1	Crank range corresponding to follower travel from the right to the left extreme positions.
RC2	Crank range corresponding to follower travel from the left to the right extreme positions.
RF	Range of follower.
X_N	Normalized displacement of crank.
Y_N	Normalized displacement of follower.
α	Clockwise angular displacement of crank measured from left end of X-axis.
α_1	Clockwise angular displacement of crank corresponding to the rightmost position of the follower.
α_2	Clockwise angular displacement of crank corresponding to the leftmost position of the follower.
β	Clockwise angular displacement of follower measured from left end of X-axis.
β_1	Rightmost clockwise angular displacement of follower.
β_2	Leftmost clockwise angular displacement of follower.
ω_2	Angular velocity of crank.
ω_4	Angular velocity of follower.

SUMMARY

One of the methods for synthesizing four-bar mechanisms is the "curve and trace" matching method, by which the designer matches his required function trace against the catalogued trace curves of known mechanisms. The purpose of this investigation was to determine analytically the significance of the slope of trace curves, and to formulate a method for the solution of the point of inflection of these curves.

The catalogue of trace curves investigated concerns four-bar crank-and-rocker mechanisms. The present investigation was dictated by the need of further understanding of the trace curves to facilitate the designer in synthesizing mechanisms of this type.

The investigation revealed that the slope of the trace curve is directly proportional to the angular velocity of the follower, and the concavity of the curve is an indication of acceleration or deceleration of the follower. Specifically, a curve which is concave upward indicates acceleration, and a curve which is concave downward indicates deceleration.

The point of inflection of the trace curves, which indicates the condition of zero acceleration and maximum velocity of the follower, was established by the iteration method using an IBM 650 computer. An exact analytical solution was not possible because the second derivative of the trace curve equation was a transcendental equation which did not lend itself to simplification. It was also shown that the point of

inflection did not coincide with the intersection of the trace curve and the diagonal reference line of the unit square, as it was intuitively believed.

The above computer program was extended to include a positive check that the point so determined is indeed the point of zero acceleration. This was accomplished by showing that the same point is the point of maximum velocity of the follower.

The determination of the phase location of the mechanism at the extreme velocities of the follower is an additional contribution to mechanism synthesis, because as far as it is known, an exact solution for the phase at which the extreme velocities occur has not been formulated. The computer program allows the determination of the phase angle and the value of the extreme velocities to within computer accuracy.

In addition, an equation is presented for the determination of the angular acceleration of the follower for any phase angle of the crank. This equation is applicable to any four-bar mechanism, provided that the motion of the crank is constant.

CHAPTER I

INTRODUCTION

One of the methods for synthesizing four-bar mechanisms is known as the "curve and trace" matching method. The designer plots the curve of the path desired on a unit square and then compares it against catalogued paths of known mechanisms.

The present investigation is concerned with the catalogue compiled by Johnson (1)* which consists of 146 mechanisms of the crank-and-rocker type. The designation "crank-and-rocker" means that the crank rotates

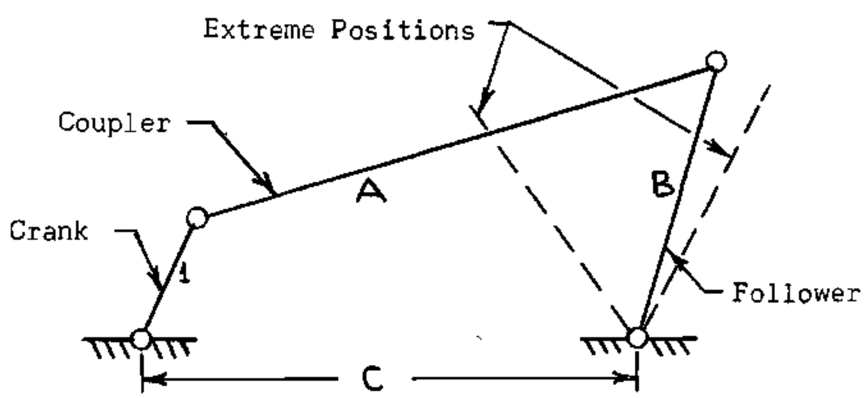

Figure 1. Crank-and-Rocker Mechanism.

through 360 degrees while the follower can only oscillate. Figure 1 is an illustration of a crank-and-rocker mechanism.

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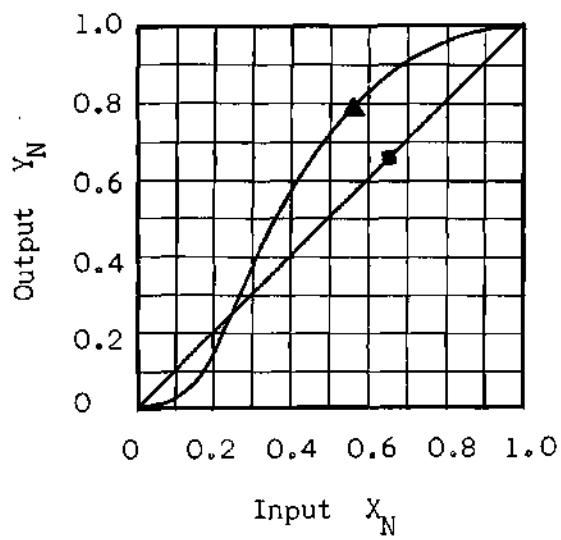
*Numbers in parentheses refer to items in Bibliography.

The motion of the mechanism is represented in a graphic form which is known as a trace curve. The trace curve shows the normalized motion of the follower corresponding to the crank input, plotted on a unit square. Figure 2 is an illustration of a trace curve. A complete explanation of this method of motion representation is given in Chapter II. A brief explanation follows:



■ Constant motion of crank

▲ Normalized motion of follower

Figure 2. Trace Curve.

As the crank rotates through 360 degrees, the follower will travel from one extreme position to the other, and then return to the first position. Corresponding to the full range of the follower from one extreme to the other, there is a range through which the crank travels. This range is not normally half of the 360 degree full range of the crank, although the sum of the two crank ranges, corresponding to one full oscillation of the follower, is equal to 360 degrees. Therefore, two trace curves are plotted for each mechanism corresponding to the two ranges of the crank.

Considering one of the crank ranges, normalized plotting on a unit square means that each point on the curve represents the per cent of the total travel of the crank through that range versus the corresponding per cent of total travel of the follower. The straight diagonal line on the unit square represents the constant motion of the crank. Throughout this paper, the plotted normalized motion of the follower will be referred to as a trace curve.

The problems to be investigated are the significance of the slope of the trace curve and the location of the point of inflection on this curve. Johnson has logically assumed that the concavity of the trace curve indicates acceleration when it is concave upward and deceleration when it is concave downward. It has also been suggested that the point of inflection of the curve coincides with the intersection of the trace curve and the diagonal reference line. Formal proof of either assumption has not been attempted, and it is very important to the designer that he has a better understanding of the significance of the trace curve.

The significance of the slope of trace curves has not been previously investigated as far as it is known. Therefore, there exists no background information. As it will be shown later, the slope of the trace curve is directly proportional to the angular velocity of the follower. Therefore, Johnson's assumption about the concavity of the curve is correct.

Since the rate of change of the slope is an indication of acceleration, then the point of inflection can be established by determining either the position at which the follower acceleration is zero or the position at which the follower velocity is maximum.

A review of the existing literature disclosed only one paper on the determination of extreme velocities written by Freudenstein (2). Dr. Freudenstein proved that at the position of extreme velocity of the follower, the collineation axis is perpendicular to the connecting link. Therefore, if an extreme velocity were desired, a mechanism could be graphically, not analytically, synthesized. He also developed a method by which the value of the extreme velocity ratio of a known mechanism can be obtained. This method depends on the solution of a sixth order equation, and only the value of the extreme velocity is obtained, but not the position at which it occurs.

The phase angle at which the maximum velocity occurs is of primary importance in solving for the point of inflection of the trace curve. Therefore, a method will be formulated in this paper for the solution of the phase angle at which the acceleration of the follower is zero and the velocity is maximum.

CHAPTER II

REVIEW OF TRACE CURVES

For the complete understanding of the present investigation, it is necessary to summarize part of Johnson's work. This chapter will present the main points of his investigation, together with the equations he derived. Proof of the equations will not be given, but they have been verified by the writer.

Figure 3 shows a crank-and-rocker mechanism at the two extreme positions of the follower.* Angle β_1 designates the extreme right

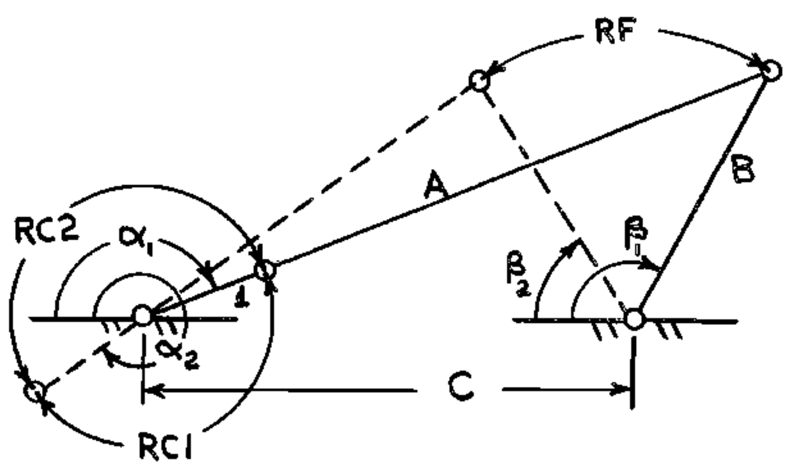


Figure 3. Crank-and-Rocker Mechanism at Extreme Positions.

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*The crank is always assumed to be equal to unity.

position of the follower, and angle α_1 the corresponding position of the crank. Angle β_2 designates the extreme left position of the follower, and angle α_2 the corresponding position of the crank. RC1 designates the crank range corresponding to follower travel from the right to the left extreme positions, and RC2 designates the crank range corresponding to follower travel from the left to the right extreme positions. RF designates the follower range. Therefore:

$$RC1 = \alpha_2 - \alpha_1$$

$$RC2 = 2\pi - RC1$$

$$RF = \beta_1 - \beta_2$$

The following equations were established for the solution of the extreme positions of the follower and for the two crank ranges:

By the cosine law:

$$\cos(\pi - \alpha_1) = \frac{(A+1)^2 + C^2 - B^2}{2(A+1)C} \quad (1)$$

$$\cos(2\pi - \alpha_2) = \frac{(A-1)^2 + C^2 - B^2}{2(A-1)C} \quad (2)$$

$$\cos\beta_1 = \frac{B^2 + C^2 - (A+1)^2}{2BC} \quad (3)$$

$$\cos\beta_2 = \frac{B^2 + C^2 - (A-1)^2}{2BC} \quad (4)$$

From the Pythagorean relations the above equations were converted into:

$$\pi - \alpha_1 = \tan^{-1}\frac{\sqrt{4(A+1)^2C^2 - [(A+1)^2 + C^2 - B^2]^2}}{(A+1)^2 + C^2 - B^2} \tag{5}$$

$$2\pi - \alpha_2 = \tan^{-1}\frac{\sqrt{4(A-1)^2C^2 - [(A-1)^2 + C^2 - B^2]^2}}{(A-1)^2 + C^2 - B^2} \tag{6}$$

$$\beta_1 = \tan^{-1}\frac{\sqrt{4B^2C^2 - [B^2 + C^2 - (A+1)^2]^2}}{B^2 + C^2 - (A+1)^2} \tag{7}$$

$$\beta_2 = \tan^{-1}\frac{\sqrt{4B^2C^2 - [B^2 + C^2 - (A-1)^2]^2}}{B^2 + C^2 - (A-1)^2} \tag{8}$$

The conversion to arctangent was necessary because the IBM 650 computer includes a direct subroutine for the solution of angles when in the above form.

Figure 4, showing a crank-and-rocker mechanism at an arbitrary position, was used for the development of the general equation of follower displacement in terms of crank displacement.

The equation for the imaginary link $\underline{m}$ was obtained by the cosine law:

$$m^2 = 1^2 + C^2 + 2C\cos\alpha \tag{9}$$

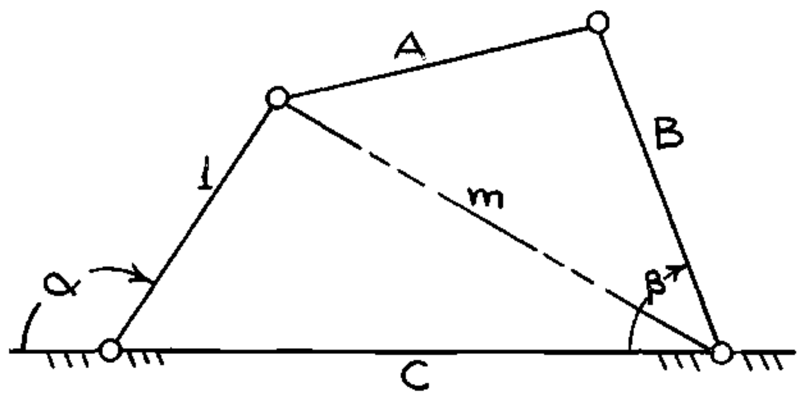


Figure 4. Crank-and-Rocker Mechanism at an Arbitrary Position.

The displacement equation of the follower was presented in the following two forms:

$$\beta = \tan^{-1}\frac{\sin\alpha}{C+\cos\alpha} + \cos^{-1}\frac{q^2+2C\cos\alpha}{2mB} \qquad (10)$$

$$\beta = \tan^{-1}\frac{\sin\alpha}{C+\cos\alpha} + \tan^{-1}\frac{\sqrt{4m^2B^2-(q^2+2C\cos\alpha)^2}}{q^2+2C\cos\alpha} \qquad (11)$$

where:

$$q^2 = 1^2 + B^2 + C^2 - A^2 \qquad (12)$$

Equation (11) was again necessary for the solution of β by the IBM 650 computer.

A computer program was written by which the solution of the above equations was obtained for 146 crank-and-rocker mechanisms. The data for the normalized representation of the follower motion were obtained as follows:

The value in radians of crank range RC1 was divided into ten h-equal increments. The follower phase angle was obtained by solving equation (11) for angles α equal to $\alpha = \alpha_1$, $\alpha = \alpha_1 + h$, $\alpha = \alpha_1 + 2h, \ldots, \alpha = \alpha_2$. The coordinates for the normalized representation for crank range RC1 were then obtained from:

Crank input: $$X_{N_1} = \frac{\alpha - \alpha_1}{RC1} \tag{13}$$

Follower output: $$Y_{N_1} = \frac{\beta_1 - \beta}{RF} \tag{14}$$

The data for the normalized representation for crank range RC2 were similarly obtained, with the exception that range RC2 was subdivided into ten equal increments, and the starting value for α was α_2. The coordinates for the normalized representation for crank range RC2 were obtained as follows:

Crank input: $$X_{N_2} = \frac{\alpha - \alpha_2}{RC2} \tag{15}$$

Follower output: $$Y_{N_2} = \frac{\beta - \beta_2}{RF} \tag{16}$$

Figures 5 and 6 represent a typical normalized motion representation of a mechanism as found in Johnson's study. A total of 146 such representations are included in his work.

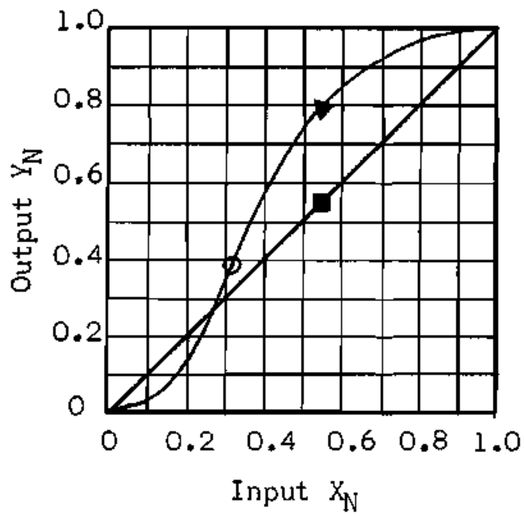


Figure 5. Trace, Crank Range RC1.

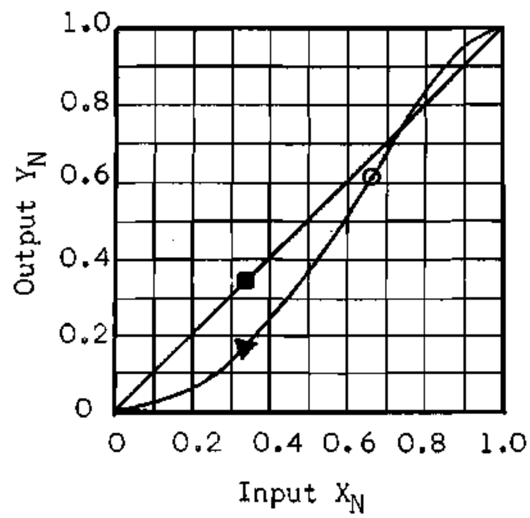


Figure 6. Trace, Crank Range RC2.

■ Crank's motion

▲ Follower's motion

⊙ Point of inflection

The straight diagonal line in the trace square represents the constant motion of the crank, and the curved line shows the normalized motion of the follower.

CHAPTER III

THE SIGNIFICANCE OF THE SLOPE

Method of Attack.--The significance of the slope of a trace curve will be investigated in the following manner:

The Cartesian equation of the trace curve for each crank range will be established in terms of X_N and Y_N. The slope of the trace curve, $(dY_N)/(dX_N)$, will be obtained by differentiating Y_N with respect to X_N. The angular displacement of the follower will then be differentiated with respect to time, establishing the angular velocity equation of the follower. This equation will be compared with the slope equations and conclusions drawn.

Crank Range RC1.--From equations (13) and (14) it is seen that the parametric equations of the trace curve are:

$$X_N = \frac{\alpha - \alpha_1}{RC1} \qquad\qquad Y_N = \frac{\beta_1 - \beta}{RF}$$

where β is a function of α as given by equation (10).

Rearranging:

$$X_N = \frac{\alpha}{RC1} - \frac{\alpha_1}{RC1}$$
$$Y_N = \frac{\beta_1}{RF} - \frac{f(\alpha)}{RF} \tag{17}$$

Since the angles α_1, RC1, and RF are constant for any given mechanism, the following substitutions will be made:

$$K_1 = \frac{1}{RC1} \qquad K_2 = \frac{\alpha_1}{RC1} \qquad K_3 = \frac{\beta_1}{RF} \qquad K_4 = \frac{1}{RF}$$

It should be noted that the above constants are positive. Substituting the constants and equation (10) into equations (17):

$$X_N = K_1\alpha - K_2 \tag{18}$$

$$Y_N = K_3 - K_4\left[\tan^{-1}\frac{\sin\alpha}{C+\cos\alpha} + \cos^{-1}\frac{q^2+2C\cos\alpha}{2B\sqrt{1+C^2+2C\cos\alpha}}\right] \tag{19}$$

Equations (18) and (19) are the parametric equations of the RC1 trace curve with the variable α being the parameter. From equation (18):

$$\alpha = \frac{X_N + K_2}{K_1} \tag{20}$$

The Cartesian equation of the RC1 trace curve is:

$$Y_N = K_3 - K_4\left[\tan^{-1}\frac{\sin\left(\frac{X_N+K_2}{K_1}\right)}{C+\cos\left(\frac{X_N+K_2}{K_1}\right)} + \right. \tag{21}$$

$$\left. + \cos^{-1}\frac{q^2+2C\cos\left(\frac{X_N+K_2}{K_1}\right)}{2B\sqrt{1+C^2+2C\cos\left(\frac{X_N+K_2}{K_1}\right)}}\right]$$

The slope of the trace curve is obtained by differentiating Y_N with respect to X_N. The differentiation and simplification steps are shown in the Appendix, beginning on page 27. The final form of the slope equation of the trace curve for crank range RC1 is:

$$\frac{dY_N}{dX_N} = -\frac{K_4}{K_1}\left[\frac{1 + C\cos\alpha}{m^2} + \right. \tag{22}$$

$$\left. + \frac{C\sin\alpha\,(m^2 + A^2 - B^2)}{m^2\sqrt{4A^2m^2 - (m^2 + A^2 - B^2)^2}}\right]$$

Crank Range RC2.--From equations (15) and (16), the parametric equations of the trace curve are:

$$X_N = \frac{\alpha - \alpha_2}{RC2} \qquad\qquad Y_N = \frac{\beta - \beta_2}{RF}$$

where again β is a function of α as defined by equation (10). Rearranging:

$$X_N = \frac{\alpha}{RC2} - \frac{\alpha_2}{RC2}$$

$$Y_N = \frac{f(\alpha)}{RF} - \frac{\beta_2}{RF} \tag{23}$$

For simplification:

$$K_5 = \frac{1}{RC2} \qquad K_6 = \frac{\alpha_2}{RC2} \qquad K_7 = \frac{1}{RF} \qquad K_8 = \frac{\beta_2}{RF}$$

It is again noted that the above constants are positive. Substituting the constants into equations (23):

$$X_N = K_5 \alpha - K_6$$
$$Y_N = -\left[K_8 - K_7 f(\alpha)\right] \tag{24}$$

The Cartesian equation of the RC2 trace curve is:

$$Y_N = -\left\{K_8 - K_7\left[\tan^{-1}\frac{\sin\left(\frac{X_N + K_6}{K_5}\right)}{C + \cos\left(\frac{X_N + K_6}{K_5}\right)} + \cos^{-1}\frac{q^2 + 2C\cos\left(\frac{X_N + K_6}{K_5}\right)}{2B\sqrt{1 + C^2 + 2C\cos\left(\frac{X_N + K_6}{K_5}\right)}}\right]\right\} \tag{25}$$

Equation (25) is similar to equation (21) with the exception of the sign. Then, based on equation (22), and by observing the correct substitution of constants and reversal of sign, the equation for the slope of the trace curve for crank range RC2 is:

$$\frac{dY_N}{dX_N} = \frac{K_7}{K_5}\left[\frac{1 + C\cos\alpha}{m^2} + \right. \tag{26}$$

$$\left. + \frac{C\sin\alpha(m^2 + A^2 - B^2)}{m^2\sqrt{4A^2m^2 - (m^2 + A^2 - B^2)^2}}\right]$$

Angular Velocity Equation.--Equation (10), which is the angular displacement equation of the follower in terms of angular displacement of the crank, is repeated below:

$$\beta = \tan^{-1}\frac{\sin\alpha}{C + \cos\alpha} + \cos^{-1}\frac{q^2 + 2C\cos\alpha}{2mB}$$

It should be noted that the angles α and β (Figure 4) are actually the clockwise angular displacements of the crank and follower respectively, measured from the horizontal reference line. Differentiation of the displacement equation with respect to time t will result into the angular velocity equation of the follower with respect to the angular velocity of the crank. The following notations will be used:

$$\omega_2 = \frac{d\alpha}{dt} = \text{Constant} = \text{Angular velocity of crank}$$

$$\omega_4 = \frac{d\beta}{dt} = \text{Angular velocity of follower .}$$

The differentiation and simplification is shown in the Appendix, beginning on page 32. The final form of the angular velocity equation is shown below:

$$\omega_4 = \omega_2 \left[\frac{1 + C\cos\alpha}{m^2} + \right. \tag{27}$$

$$\left. + \frac{C\sin\alpha\,(m^2 + A^2 - B^2)}{m^2\sqrt{4A^2m^2 - (m^2 + A^2 - B^2)^2}} \right]$$

Comparison of Results.--A comparison of equations (22), (26), and (27) reveals the following:

The bracketed portion of the three equations is identical, and the quantities K_4 / K_1, K_7 / K_5, and ω_2 are positive constants. Therefore, the slope of the trace curves is directly proportional to the angular velocity of the follower.

The above statement justifies Johnson's assumption that the concavity of the trace curves is an indication of acceleration or deceleration. Because, the derivative of equation (27) with respect to time is the angular acceleration of the follower, and by rules of mathematics, the sign of the second derivative of a curve is an indication of the type of concavity. Therefore, when the trace curve is concave upward, the sign of the second derivative must be positive, hence acceleration. Conversely, when the trace curve is concave downward, the sign of the second derivative is negative, hence deceleration.

The following explanation is not necessary for the interpretation of the significance of the slope, but it will justify the fact that the sign of equation (22) is negative. Johnson, in plotting the trace curve for range RC1, used as an ordinate the following:

$$Y_{N_1} = \frac{\beta_1 - \beta}{RF}$$

Conventional sign observation dictates that since the clockwise rotation was assumed positive, then counterclockwise rotation will be indicated by a negative sign. As the crank rotates clockwise in crank range RC1, the follower rotates counterclockwise (Figure 3). Therefore, correct sign convention requires that the ordinate of the RC1 trace curve be taken as:

$$Y_{N_1} = \frac{\beta - \beta_1}{RF}$$

which is a negative quantity, since $\beta_1 > \beta$. Therefore, with β varying from β_1 to β_2, the quantity $(\beta - \beta_1)$ will be an increasing quantity in the negative direction. But Johnson was interested primarily in the normalized motion of the follower as it increased from zero to one, regardless of whether it is a positive or a negative increase. Therefore, he wisely arranged the ordinate equation to indicate for simplicity a positive increase. Then, the negative sign of equation (22) is simply an indication of the above sign reversal, and equation (22) is a true mathematical representation of the slope of the RC1 trace curve.

CHAPTER IV

DETERMINATION OF THE POINT OF INFLECTION

Method of Attack.--The determination of the point of inflection will be accomplished as follows:

Equation (27), the follower angular velocity equation, will be differentiated with respect to time, resulting in the follower angular acceleration equation. Since it has been established that the concavity of the trace curves indicates acceleration or deceleration of the follower, then the point of the curve, to the left and right of which the senses of concavity are opposite, is the point of inflection. Therefore, the follower acceleration is zero at this point.

The acceleration equation will be set equal to zero and solved for the crank phase angle α. The solution will be accomplished by the use of the IBM 650 computer. The computer program will include routines to solve for the follower angle β and the coordinates X_N and Y_N of the point of inflection.

To check the validity of the above results, the computer program will be extended to include a routine by which it will be shown that the point of inflection as found above, is the point at which the angular velocity of the follower is maximum.

The Angular Acceleration Equation.--The angular acceleration equation of the follower is obtained by differentiating equation (27) with respect to time. The mathematical calculations are cumbersome, and they are shown in the Appendix, beginning on page 35. The following notation is used:

$$\frac{d^2\beta}{dt^2} = \frac{d\omega_4}{dt} = \text{Angular acceleration of follower.}$$

The final form of the acceleration equation is shown on the next page:

$$\frac{d^2\beta}{dt^2} = \frac{\omega_z^2 C}{m^4\left[4A^2m^2-\left(A^2+m^2-B^2\right)^2\right]^{3/2}} \Bigg\{\left[4A^2m^2-\left(A^2+m^2-B^2\right)^2\right]\Big[\sin\alpha\left(1-C^2\right)\sqrt{4A^2m^2-\left(A^2+m^2-B^2\right)^2} + m^4\cos\alpha + \left(A^2-B^2\right)\left(m^2\cos\alpha+2C\sin^2\alpha\right)\Big] + 2Cm^2\sin^2\alpha\left(m^2+A^2-B^2\right)\left(A^2-m^2+B^2\right)\Bigg\} \tag{28}$$

It should be noted that equation (28), although cumbersome, provides a means for the solution of the angular acceleration of the follower at any crank phase angle α.

Solution for the Point of Inflection.--Equation (28) is set equal to zero:

$$\left[4A^2m^2-(A^2+m^2-B^2)^2\right]\left[\sin\alpha\,(1-C^2)\sqrt{4A^2m^2-(A^2+m^2-B^2)^2}\;+\right.$$

$$\left.+\,m^4\cos\alpha+(A^2-B^2)(m^2\cos\alpha\;+2C\sin^2\alpha)\right]\;+$$

$$+\,2Cm^2\sin^2\alpha\;(m^2+A^2-B^2)(A^2-m^2+B^2)\;=\;0$$

Dividing by $\left[4A^2m^2-(A^2+m^2-B^2)^2\right]$:

$$\sin\alpha(1-C^2)\sqrt{4A^2m^2-(A^2+m^2-B^2)^2}\;\;+ \tag{29}$$

$$+\,m^4\cos\alpha\;+\;(A^2-B^2)(m^2\cos\alpha+2C\sin^2\alpha)\;+$$

$$+\;\frac{2Cm^2\sin^2\alpha\;(m^2+A^2-B^2)(A^2-m^2+B^2)}{4A^2m^2-(A^2+m^2-B^2)^2}\;=\;0$$

Attempts to simplify the above transcendental equation, to the extent that an exact solution may be possible, failed. Therefore, equation (29) was retained in its present form, and its solution was accomplished by an iteration programmed for the IBM 650 computer.

The flow chart and the computer program are shown in detail in the Appendix, beginning on page 40. A brief explanation follows:

The program is designed to obtain solutions for as many crank-and-rocker mechanisms as desired, but for one crank range at a time. The exchange of five IBM card orders converts the program from one crank range to the other.

The initial portion of the program is identical to Johnson's, and it solves equations (5) through (8) for the angles α_1, α_2, β_1, β_2, RC1, and RC2. The discussion will now pertain to crank range RC1. The program solves equation (29) by iteration. The starting value for angle α is the angle $(\alpha_1 + 0.01)$. The 0.01 radian was added to α_1 to ensure that the iteration begins beyond the point of zero follower velocity. Angle α is increased gradually until the sign of equation (29) changes. The value of angle α, which caused the change of sign, is recorded, and it represents the crank phase angle at which the acceleration of the follower is zero. The accuracy of the solution is within 0.000001 radian. With angle α known, the computer solves equation (10) for the follower angle β, and equations (13) and (14) for the normalized coordinates of the point of inflection.

In order to check the validity of the answers obtained, the computer program is extended to solve the velocity equation of the follower, equation (27). Three values of the angle α are used in this solution, the angle corresponding to the inflection point as found above, and two angles 0.001 radian on either side of the above angle.

<u>Results</u>.--Five mechanisms, selected at random from Johnson's work, were used in this investigation. The results are tabulated in the Appendix, beginning on page 51.

Results indicate that the normalized coordinates X_N and Y_N of the point of inflection are not equal to each other. Therefore, the point of inflection does not occur at the intersection of the trace curve and the diagonal reference line. Figures 5 and 6 represent one of the mechanisms investigated ($A = 2.0$, $B = 2.0$, $C = 2.0$), and the points of inflection are shown at their correct location.

Reference is again made to results, page 51. The phase angles corresponding to the points of inflection, together with the values of the maximum velocity for each crank range, are new contributions, as far as known, to the solution of extreme velocities and the phases at which they occur. The accuracy of these solutions is limited only by the capacity of the computer used. It should be noted that the negative sign of the extreme velocities in crank range RC1, signifies counterclockwise rotation of the follower.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions.--As a result of the investigation of the trace curves the following is concluded:

The slope of the trace curve is directly proportional to the angular velocity of the follower.

The concavity of the trace curve is an indication of angular acceleration or deceleration of the follower. Acceleration is indicated when the curve is concave upward, and deceleration is indicated when the curve is concave downward.

The point of inflection of the trace curve, which is the point to the left and right of which the senses of concavity are opposite, is not located at the point of intersection of the trace curve and the diagonal reference line. An exact determination of the point of inflection is not possible, but an IBM 650 computer program is provided by which the point of inflection can be located to within 0.000001 radian of the crank phase angle.

The IBM 650 program is designed to solve for the extreme velocities of the follower as well as for the phase angles at which they occur. Due to the excellent accuracy and the speed of the computer, this method is very useful in obtaining the above information.

Recommendations.--It is recommended that the IBM 650 computer program presented in this investigation be used to obtain the location of the point of inflection, as well as the extreme velocities and the phases at which they occur, for all 146 mechanisms catalogued in Johnson's work. This information might be useful to a designer, if he were to synthesize mechanisms by the trace and deviation method. The computer program is available at the Georgia Institute of Technology.

A P P E N D I X

DIFFERENTIATION OF TRACE CURVE EQUATION

Equation (21), which is the Cartesian equation of the RCl trace curve is shown below:

$$Y_N = K_3 - K_4 \left[\tan^{-1} \frac{\sin\left(\frac{X_N + K_2}{K_1}\right)}{C + \cos\left(\frac{X_N + K_2}{K_1}\right)} + \cos^{-1} \frac{q^2 + 2C\cos\left(\frac{X_N + K_2}{K_1}\right)}{2B\sqrt{1 + C^2 + 2C\cos\left(\frac{X_N + K_2}{K_1}\right)}} \right] \qquad (21)$$

The slope of the trace curve will be obtained by differentiating Y_N with respect to X_N based on the following differentiation formulae:

$$\frac{d}{dx}\left(\tan^{-1} u\right) = \frac{1}{1 + u^2} \frac{du}{dx}$$

$$\frac{d}{dx}\left(\cos^{-1} u\right) = -\frac{1}{\sqrt{1 - u^2}} \frac{du}{dx} \qquad (30)$$

Differentiating and simplifying the first term of equation (21):

$$\left.\frac{dY_N}{dX_N}\right)_1 = -K_4\left[\frac{1}{1+\dfrac{\sin^2\dfrac{X_N+K_2}{K_1}}{\left(C+\cos\dfrac{X_N+K_2}{K_1}\right)^2}}\right]\cdot$$

$$\cdot\left[\frac{\dfrac{1}{K_1}\cos\dfrac{X_N+K_2}{K_1}\left(C+\cos\dfrac{X_N+K_2}{K_1}\right)+\dfrac{1}{K_1}\left(\sin\dfrac{X_N+K_2}{K_1}\right)\left(\sin\dfrac{X_N+K_2}{K_1}\right)}{\left(C+\cos\dfrac{X_N+K_2}{K_1}\right)^2}\right]$$

Substituting: $\alpha = \dfrac{X_N+K_2}{K_1}$

$$\left.\frac{dY_N}{dX_N}\right)_1 = -\frac{K_4}{K_1}\left[\frac{1}{(C+\cos\alpha)^2+\sin^2\alpha}\right]\left[\cos\alpha\,(C+\cos\alpha)+\sin^2\alpha\right] =$$

$$= -\frac{K_4}{K_1}\left[\frac{C\cos\alpha+\cos^2\alpha+\sin^2\alpha}{C^2+2C\cos\alpha+\cos^2\alpha+\sin^2\alpha}\right]$$

From trigonometry: $\sin^2\alpha+\cos^2\alpha = 1$

Thus: $$\left.\frac{dY_N}{dX_N}\right)_1 = -\frac{K_4}{K_1}\left[\frac{1+C\cos\alpha}{1+C^2+2C\cos\alpha}\right]$$

From equation (9): $m^2 = 1+C^2+2C\cos\alpha$

Thus: $$\left.\frac{dY_N}{dX_N}\right)_1 = -\frac{K_4}{K_1}\left[\frac{1+C\cos\alpha}{m^2}\right] \qquad (31)$$

The differentiation and simplification of the second term of equation (21) begins on the next page:

$$\left.\frac{dY_N}{dX_N}\right)_2 = K_4\left[\frac{1}{\sqrt{1-\dfrac{\left(q^2+2C\cos\frac{X_N+K_2}{K_1}\right)^2}{4B^2\left(1+C^2+2C\cos\frac{X_N+K_2}{K_1}\right)}}}\right]\cdot$$

$$\cdot\left[\frac{\left(-\frac{2C}{K_1}\sin\frac{X_N+K_2}{K_1}\right)2B\sqrt{1+C^2+2C\cos\frac{X_N+K_2}{K_1}}-\left(q^2+2C\cos\frac{X_N+K_2}{K_1}\right)B\left(1+C^2+2C\cos\frac{X_N+K_2}{K_1}\right)^{-1/2}\left(-\frac{2C}{K_1}\sin\frac{X_N+K_2}{K_1}\right)}{4B^2\left(1+C^2+2C\cos\frac{X_N+K_2}{K_1}\right)}\right]$$

$$\left.\frac{dY_N}{dX_N}\right)_2 = \frac{K_4}{K_1}\left[\frac{2B\sqrt{1+C^2+2C\cos\alpha}}{\sqrt{4B^2(1+C^2+2C\cos\alpha)-(q^2+2C\cos\alpha)^2}}\right]\cdot$$

$$\cdot\left[\frac{(-2C\sin\alpha)2B\sqrt{1+C^2+2C\cos\alpha}-(q^2+2C\cos\alpha)\dfrac{B(-2C\sin\alpha)}{\sqrt{1+C^2+2C\cos\alpha}}}{\left(2B\sqrt{1+C^2+2C\cos\alpha}\right)^2}\right]$$

$$\left.\frac{dY_N}{dX_N}\right)_2 = \frac{K_4}{K_1}\left[\frac{1}{\sqrt{4B^2m^2-(q^2+2C\cos\alpha)^2}}\right]\left[\frac{-4BCm\sin\alpha+\dfrac{2CB\sin\alpha(q^2+2C\cos\alpha)}{m}}{2Bm}\right]$$

$$\left.\frac{dY_N}{dX_N}\right)_2 = -\frac{K_4}{K_1}\left[\frac{2Cm^2\sin\alpha - C\sin\alpha\,(q^2 + 2C\cos\alpha)}{m^2\sqrt{4B^2m^2 - (q^2+2C\cos\alpha)^2}}\right]$$

$$\left.\frac{dY_N}{dX_N}\right)_2 = -\frac{K_4}{K_1}\left[\frac{C\sin\alpha\,(2m^2 - q^2 - 2C\cos\alpha)}{m^2\sqrt{4B^2m^2 - (q^2+2C\cos\alpha)^2}}\right]$$

From equation (12): $q^2 = 1 + B^2 + C^2 - A^2$

$$\left.\frac{dY_N}{dX_N}\right)_2 = -\frac{K_4}{K_1}\left[\frac{C\sin\alpha\,(2m^2 - 1 - B^2 - C^2 + A^2 - 2C\cos\alpha)}{m^2\sqrt{4B^2m^2 - (1+B^2+C^2-A^2+2C\cos\alpha)^2}}\right]$$

$$\left.\frac{dY_N}{dX_N}\right)_2 = -\frac{K_4}{K_1}\left[\frac{C\sin\alpha\,(2m^2 - m^2 - B^2 + A^2)}{m^2\sqrt{4B^2m^2 - (B^2 - A^2 + m^2)^2}}\right]$$

$$\left.\frac{dY_N}{dX_N}\right)_2 = -\frac{K_4}{K_1}\left[\frac{C\sin\alpha\,(m^2 + A^2 - B^2)}{m^2\sqrt{4B^2m^2 - (B^2 - A^2 + m^2)^2}}\right] \qquad (32)$$

The radical term of equation (32) will be rewritten as follows:

$$4B^2m^2 - (B^2 - A^2 + m^2)^2 = 4B^2m^2 - B^4 - A^4 - m^4 - 2B^2m^2 + 2B^2A^2 + 2A^2m^2 =$$
$$= 4A^2m^2 - B^4 - A^4 - m^4 + 2B^2m^2 + 2B^2A^2 - 2A^2m^2 = 4A^2m^2 - (m^2 + A^2 - B^2)^2$$

$$\sqrt{4B^2m^2 - (B^2 - A^2 + m^2)^2} = \sqrt{4A^2m^2 - (m^2 + A^2 - B^2)^2} \qquad (33)$$

Substituting equation (33) into equation (32):

$$\left.\frac{dY_N}{dX_N}\right)_2 = -\frac{K_4}{K_1}\left[\frac{C\sin\alpha(m^2+A^2-B^2)}{m^2\sqrt{4A^2m^2-(m^2+A^2-B^2)^2}}\right] \quad (34)$$

The slope equation of the trace curve of crank range RC1 is obtained by adding equations (31) and (34):

$$\frac{dY_N}{dX_N} = -\frac{K_4}{K_1}\left[\frac{1+C\cos\alpha}{m^2} + \frac{C\sin\alpha(m^2+A^2-B^2)}{m^2\sqrt{4A^2m^2-(m^2+A^2-B^2)^2}}\right] \quad (22)$$

DERIVATION OF ANGULAR VELOCITY EQUATION

Equation (10), the angular displacement equation of the follower, is repeated below:

$$\beta = \tan^{-1}\frac{\sin\alpha}{C+\cos\alpha} + \cos^{-1}\frac{q^2 + 2C\cos\alpha}{2mB} \qquad (10)$$

Differentiating and simplifying the first term of equation (10):

$$\left.\frac{d\beta}{dt}\right)_1 = \left[\frac{1}{1+\frac{\sin^2\alpha}{(C+\cos\alpha)^2}}\right]\left[\frac{(C+\cos\alpha)\cos\alpha\frac{d\alpha}{dt} + (\sin\alpha)(\sin\alpha)\frac{d\alpha}{dt}}{(C+\cos\alpha)^2}\right]$$

Notation:

$$\omega_2 = \frac{d\alpha}{dt} = \text{Constant} = \text{Angular velocity of crank}$$

$$\omega_4 = \frac{d\beta}{dt} = \text{Angular velocity of follower}$$

$$\left.\omega_4\right)_1 = \frac{\omega_2\left(C\cos\alpha + \cos^2\alpha + \sin^2\alpha\right)}{(C+\cos\alpha)^2 + \sin^2\alpha}$$

$$\left.\omega_4\right)_1 = \frac{\omega_2\left(1 + C\cos\alpha\right)}{C^2 + 2C\cos\alpha + \cos^2\alpha + \sin^2\alpha} = \frac{\omega_2\left(1 + C\cos\alpha\right)}{1 + C^2 + 2C\cos\alpha}$$

$$\omega_4\Big)_1 = \frac{\omega_2 (1 + C\cos\alpha)}{m^2} \tag{35}$$

Differentiating and simplifying the second term of equation (10):

$$\frac{d\beta}{dt}\Big)_2 = -\left[\frac{1}{\sqrt{1 - \frac{(q^2 + 2C\cos\alpha)^2}{4m^2B^2}}}\right]\left[\frac{-2C\sin\alpha\frac{d\alpha}{dt}(2mB) - (q^2 + 2C\cos\alpha)2B\frac{dm}{dt}}{4m^2B^2}\right]$$

Note:

$$\frac{dm}{dt} = \frac{d}{dt}(1 + C^2 + 2C\cos\alpha)^{1/2} = \frac{1}{2}(1 + C^2 + 2C\cos\alpha)^{-1/2}(-2C\sin\alpha)\frac{d\alpha}{dt} =$$

$$= -\frac{C\sin\alpha\frac{d\alpha}{dt}}{\sqrt{1 + C^2 + 2C\cos\alpha}} = -\frac{C\omega_2\sin\alpha}{m}$$

Therefore:

$$\omega_4\Big)_2 = -\left[\frac{2mB}{\sqrt{4m^2B^2 - (q^2 + 2C\cos\alpha)^2}}\right]\left[\frac{-4mBC\omega_2\sin\alpha + \frac{2BC\omega_2\sin\alpha(q^2 + 2C\cos\alpha)}{m}}{4m^2B^2}\right]$$

$$\omega_4\Big)_2 = \frac{\omega_2\left[2m^2C\sin\alpha - C\sin\alpha(q^2 + 2C\cos\alpha)\right]}{m^2\sqrt{4m^2B^2 - (q^2 + 2C\cos\alpha)^2}}$$

$$\omega_4\Big)_2 = \frac{\omega_2 \; C\sin\alpha \; (2m^2 - q^2 - 2C\cos\alpha)}{m^2\sqrt{4m^2B^2 - (q^2 + 2C\cos\alpha)^2}}$$

From equation (12): $q^2 = 1 + B^2 + C^2 - A^2$

$$\omega_4\Big)_2 = \frac{\omega_2 C\sin\alpha \left(2m^2 - 1 - B^2 - C^2 + A^2 - 2C\cos\alpha\right)}{m^2\sqrt{4m^2B^2 - \left(1 + B^2 + C^2 - A^2 + 2C\cos\alpha\right)^2}}$$

$$\omega_4\Big)_2 = \frac{\omega_2 C\sin\alpha \left(2m^2 - m^2 - B^2 + A^2\right)}{m^2\sqrt{4m^2B^2 - \left(m^2 + B^2 - A^2\right)^2}}$$

Substituting for the radical from Equation (33):

$$\omega_4\Big)_2 = \frac{\omega_2 C\sin\alpha \left(m^2 + A^2 - B^2\right)}{m^2\sqrt{4A^2m^2 - \left(m^2 + A^2 - B^2\right)^2}} \tag{36}$$

The angular velocity of the follower is obtained by adding equations (35) and (36):

$$\omega_4 = \omega_2 \left[\frac{1 + C\cos\alpha}{m^2} + \frac{C\sin\alpha \left(m^2 + A^2 - B^2\right)}{m^2\sqrt{4A^2m^2 - \left(m^2 + A^2 - B^2\right)^2}} \right] \tag{27}$$

DERIVATION OF ACCELERATION EQUATION

Equation (27), the angular velocity equation of the follower, is repeated below:

$$\omega_4 = \omega_2 \left[\frac{1 + C\cos\alpha}{m^2} + \frac{C\sin\alpha(m^2 + A^2 - B^2)}{m^2\sqrt{4A^2m^2 - (A^2 + m^2 - B^2)^2}} \right] \tag{27}$$

Notation:

$$\frac{d^2\beta}{dt^2} = \frac{d\omega_4}{dt} = \text{ Angular acceleration of follower}$$

The derivative of $\underline{m}^2$ with respect to time $\underline{t}$ is:

$$\frac{dm^2}{dt} = \frac{d}{dt}(1 + C^2 + 2C\cos\alpha) = -2C\sin\alpha\frac{d\alpha}{dt} = -2C\omega_2\sin\alpha$$

Differentiating the first term of equation (27) with respect to time $\underline{t}$:

$$\left.\frac{d^2\beta}{dt^2}\right)_1 = \omega_2 \left[\frac{m^2(-C\sin\alpha)\frac{d\alpha}{dt} - (1 + C\cos\alpha)\frac{dm^2}{dt}}{m^4} \right]$$

$$\left.\frac{d^2\beta}{dt^2}\right)_1 = \omega_2 \left[\frac{-\omega_2 m^2 C\sin\alpha - (1+C\cos\alpha)(-2C\omega_2 \sin\alpha)}{m^4} \right]$$

$$\left.\frac{d^2\beta}{dt^2}\right)_1 = \omega_2^2 \left[\frac{C\sin\alpha\,(-m^2+2+2C\cos\alpha)}{m^4} \right]$$

$$\left.\frac{d^2\beta}{dt^2}\right)_1 = \omega_2^2 \left[\frac{C\sin\alpha\,(-1-C^2-2C\cos\alpha+2+2C\cos\alpha)}{m^4} \right]$$

$$\left.\frac{d^2\beta}{dt^2}\right)_1 = \omega_2^2 \left[\frac{C\sin\alpha(1-C^2)}{m^4} \right] \tag{37}$$

Differentiating the second term of equation (27):

$$\left.\frac{d^2\beta}{dt^2}\right)_2 = \omega_2 \left[\frac{m^2\sqrt{4A^2m^2-(A^2+m^2-B^2)^2}\,\frac{d}{dt}C\sin\alpha(m^2+A^2-B^2) - C\sin\alpha(m^2+A^2-B^2)\frac{d}{dt}\,m^2\sqrt{4A^2m^2-(A^2+m^2-B^2)^2}}{m^4\left[4A^2m^2-(A^2+m^2-B^2)^2\right]} \right] \tag{38}$$

One term of equation (38) is:

$$\frac{d}{dt}C\sin\alpha(m^2+A^2-B^2) = C\cos\alpha\,(m^2+A^2-B^2)\frac{d\alpha}{dt} + C\sin\alpha\frac{dm^2}{dt} =$$

$$= \omega_2 C\cos\alpha\,(m^2+A^2-B^2) - C\sin\alpha\,(2C\omega_2\sin\alpha) =$$

$$= \omega_2\left[C\cos\alpha\,(m^2+A^2-B^2) - 2C^2\sin^2\alpha\right] \tag{39}$$

Another term of equation (38) is:

$$\frac{d}{dt} m^2 \left[4A^2m^2-(A^2+m^2-B^2)^2\right]^{1/2} = \frac{dm^2}{dt}\sqrt{4A^2m^2-(A^2+m^2-B^2)^2} +$$

$$+m^2\frac{1}{2}\left[4A^2m^2-(A^2+m^2-B^2)^2\right]^{-1/2}\left[4A^2\frac{dm^2}{dt}-2(A^2+m^2-B^2)\frac{dm^2}{dt}\right] =$$

$$=(-2C\omega_2\sin\alpha)\sqrt{4A^2m^2-(A^2+m^2-B^2)^2} +$$

$$+\frac{m^2(-2C\omega_2\sin\alpha)\left[2A^2-(A^2+m^2-B^2)\right]}{\sqrt{4A^2m^2-(A^2+m^2-B^2)^2}} =$$

$$=\frac{-2C\omega_2\sin\alpha}{\sqrt{4A^2m^2-(A^2+m^2-B^2)^2}}\left[4A^2m^2-(A^2+m^2-B^2)^2 + \right. \tag{40}$$

$$\left. + m^2(A^2-m^2+B^2)\right]$$

Substituting equations (39) and (40) into equation (38):

$$\left.\frac{d^2\beta}{dt^2}\right)_2 = \frac{\omega_2^2}{m^4\left[4A^2m^2-(A^2+m^2-B^2)^2\right]}\left\{\left[m^2\sqrt{4A^2m^2-(A^2+m^2-B^2)^2}\right]\cdot\right.$$

$$\cdot\left[C\cos\alpha(m^2+A^2-B^2)-2C^2\sin^2\alpha\right]+\left[\frac{2C^2\sin^2\alpha(m^2+A^2-B^2)}{\sqrt{4A^2m^2-(A^2+m^2-B^2)^2}}\right]\cdot$$

$$\left.\cdot\left[4A^2m^2-(A^2+m^2-B^2)^2+m^2(A^2-m^2+B^2)\right]\right\}$$

$$\left.\frac{d^2\beta}{dt^2}\right)_2 = \frac{\omega_2^2}{m^4\left[4A^2m^2-(A^2+m^2-B^2)^2\right]^{3/2}}\left\{m^2\left[4A^2m^2-(A^2+m^2-B^2)^2\right]\left[C\cos\alpha\,(m^2+A^2-B^2)-2C^2\sin^2\alpha\right]+\right.$$

$$\left.+2C^2\sin^2\alpha\,(m^2+A^2-B^2)\left[4A^2m^2-(A^2+m^2-B^2)^2+m^2(A^2-m^2+B^2)\right]\right\}$$

$$\left.\frac{d^2\beta}{dt^2}\right)_2 = \frac{\omega_2^2}{m^4\left[4A^2m^2-(A^2+m^2-B^2)^2\right]^{3/2}}\left\{\left[4A^2m^2-(A^2+m^2-B^2)^2\right]\left[m^2C\cos\alpha\,(m^2+A^2-B^2)-\right.\right.$$

$$\left.\left.-2C^2m^2\sin^2\alpha+2C^2\sin^2\alpha\,(m^2+A^2-B^2)\right]+2C^2\sin^2\alpha\,(m^2+A^2-B^2)\,m^2\,(A^2-m^2+B^2)\right\}$$

$$\left.\frac{d^2\beta}{dt^2}\right)_2 = \frac{\omega_2^2}{m^4\left[4A^2m^2-(A^2+m^2-B^2)^2\right]^{3/2}}\left\{\left[4A^2m^2-(A^2+m^2-B^2)^2\right]\left[m^4C\cos\alpha+m^2C\cos\alpha\,(A^2-B^2)-\right.\right.$$

$$\left.\left.-2C^2m^2\sin^2\alpha+2C^2\sin^2\alpha\;m^2+2C^2\sin^2\alpha\,(A^2-B^2)\right]+2C^2m^2\sin^2\alpha\,(m^2+A^2-B^2)(A^2-m^2+B^2)\right\}$$

$$\left.\frac{d^2\beta}{dt^2}\right)_2 = \frac{\omega_2^2 C}{m^4\left[4A^2m^2-(A^2+m^2-B^2)^2\right]^{3/2}} \left\{\left[4A^2m^2 - \right.\right. \tag{41}$$

$$\left. - (A^2+m^2-B^2)^2\right]\left[m^4\cos\alpha + (A^2-B^2)(m^2\cos\alpha + 2C\sin^2\alpha)\right] +$$

$$\left. + 2Cm^2\sin^2\alpha\,(m^2+A^2-B^2)(A^2-m^2+B^2)\right\}$$

The angular acceleration of the follower is obtained by adding equations (37) and (41):

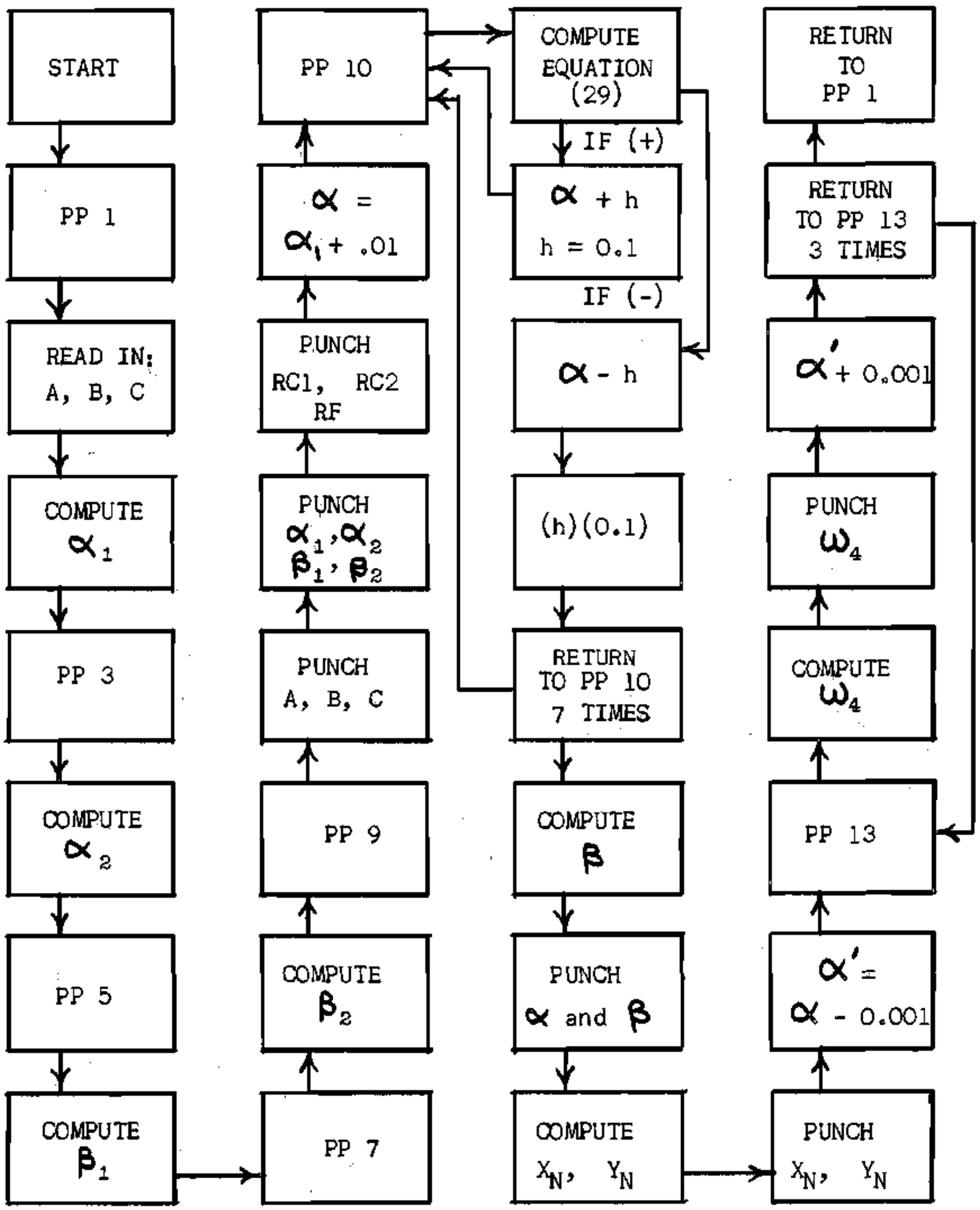
$$\frac{d^2\beta}{dt^2} = \frac{\omega_2^2 C}{m^4\left[4A^2m^2-(A^2+m^2-B^2)^2\right]^{3/2}} \left\{\left[4A^2m^2 - \right.\right. \tag{28}$$

$$- (A^2+m^2-B^2)^2\Big]\Big[\sin\alpha\,(1-C^2)\sqrt{4A^2m^2-(A^2+m^2-B^2)^2} +$$

$$+ m^4\cos\alpha + (A^2-B^2)(m^2\cos\alpha + 2C\sin^2\alpha)\Big] +$$

$$\left. + 2Cm^2\sin^2\alpha\,(m^2+A^2-B^2)(A^2-m^2+B^2)\right\}$$

FLOW CHART FOR IBM 650 PROGRAM



THE IBM 650 COMPUTER PROGRAM

Crank Range RC1.--The following computer orders pertain to crank range RC1:

+ 9	800	001	000	PP 1
+ 7	000	500	502	Read in A, B, C
- 1	500	901	000	000: $A - 1$
+ 2	000	000	570	570: $(A-1)^2$
+ 2	501	501	571	571: B^2
+ 2	502	502	572	572: C^2
+ 1	500	901	000	000: $A + 1$
+ 2	000	000	573	573: $(A + 1)^2$
+ 2	572	405	574	574: $4C^2$
+ 2	574	573	575	575: $4C^2(A + 1)^2$
+ 1	572	573	000	000: $(A+1)^2 + C^2$
- 1	000	571	576	576: $(A+1)^2 + C^2 - B^2$
+ 2	576	576	000	000: $[(A+1)^2 + C^2 - B^2]^2$
- 1	575	000	000	000: $4C^2(A+1)^2 - [(A+1)^2 + C^2 - B^2]^2$
+ 0	300	000	000	000: $\sqrt{4C^2(A+1)^2 - [(A+1)^2 + C^2 - B^2]^2}$
+ 3	000	576	577	577: $\tan(\pi - \alpha_1)$

+ 8	700	577	002	If not negative, go to PP 2
+ 0	305	577	000	000: $(\pi - \alpha_1)$
- 2	000	901	520	520: α_1
+ 8	000	000	003	Transfer to PP 3
+ 9	800	002	000	PP 2
+ 0	305	577	000	000: $(\pi - \alpha_1)$
- 1	403	000	520	520: α_1
+ 9	800	003	000	PP 3
+ 2	574	570	610	610: $4C^2(A - 1)^2$
+ 1	570	572	000	000: $(A - 1)^2 + C^2$
- 1	000	571	578	578: $(A - 1)^2 + C^2 - B^2$
+ 2	578	578	000	000: $[(A - 1)^2 + C^2 - B^2]^2$
+ 1	610	000	000	000: $4C^2(A - 1)^2 - [(A - 1)^2 + C^2 - B^2]^2$
+ 0	300	000	000	000: $\sqrt{4C^2(A - 1)^2 - [(A - 1)^2 + C^2 - B^2]^2}$
+ 3	000	578	579	579: $\tan(2\pi - \alpha_2)$
+ 8	700	579	004	If not negative, go to PP 4
+ 0	305	579	000	000: $(2\pi - \alpha_2)$
- 1	403	000	521	521: α_2
+ 8	000	000	005	Transfer to PP 5
+ 9	800	004	000	PP 4
+ 0	305	579	000	000: $(2\pi - \alpha_2)$

- 1	904	000	521	521: α_2
+ 9	800	005	000	PP 5
+ 2	574	571	580	580: $4B^2C^2$
+ 1	571	572	581	581: $B^2 + C^2$
- 1	581	573	582	582: $B^2 + C^2 - (A+1)^2$
+ 2	582	582	000	000: $[B^2 + C^2 - (A+1)^2]^2$
- 1	580	000	000	000: $4B^2C^2 - [B^2 + C^2 - (A+1)^2]^2$
+ 0	300	000	000	000: $\sqrt{4B^2C^2 - [B^2 + C^2 - (A+1)^2]^2}$
+ 3	000	582	583	583: $\tan \beta_1$
+ 8	700	583	006	If not negative, go to PP 6
+ 0	305	583	000	000: $-\beta_1$
+ 1	403	000	522	522: β_1
+ 8	000	000	007	Transfer to PP 7
+ 9	800	006	000	PP 6
+ 0	305	583	522	522: β_1
+ 9	800	007	000	PP 7
- 1	581	570	584	584: $B^2 + C^2 - (A-1)^2$
+ 2	584	584	000	000: $[B^2 + C^2 - (A-1)^2]^2$
- 1	580	000	000	000: $4B^2C^2 - [B^2 + C^2 - (A-1)^2]^2$
+ 0	300	000	000	000: $\sqrt{4B^2C^2 - [B^2 + C^2 - (A-1)^2]^2}$
+ 3	000	584	585	585: $\tan \beta_2$

	+ 8	700	585	008	If not negative, go to PP 8
	+ 0	305	585	000	000: $-\beta_2$
	+ 1	403	000	523	523: β_2
	+ 8	000	000	009	Transfer to PP 9
	+ 9	800	008	000	PP 8
	+ 0	305	585	523	523: β_2
	+ 9	800	009	000	PP 9
	- 1	521	520	524	524: $\alpha_2 - \alpha_1$ = RC1
	- 1	904	524	525	525: 2π - RC1 = RC2
	- 1	522	523	526	526: $\beta_1 - \beta_2$ = RF
	+ 7	300	500	502	Punch A, B, C
	+ 7	300	520	523	Punch α_1, α_2, β_1, β_2
	+ 7	300	524	526	Punch RC1, RC2, RF
	+ 2	500	500	586	586: A^2
	+ 2	586	405	587	587: $4A^2$
	- 1	586	571	588	588: $A^2 - B^2$
	+ 1	572	901	589	589: $1 + C^2$
	- 1	901	572	590	590: $1 - C^2$
	+ 2	902	502	591	591: 2C
	+ 1	571	586	592	592: $A^2 + B^2$
(1a)	+ 1	520	406	527	527: $\alpha_1 + 0.01 = \alpha$

+ 9	800	010	000	PP 10
+ 1	527	900	527	527: α
+ 0	303	527	593	593: $\sin\alpha$
+ 0	304	527	594	594: $\cos\alpha$
+ 2	591	594	000	000: $2\ C\cos\alpha$
+ 1	000	589	595	595: $1 + C^2 + 2C\cos\alpha = m^2$
+ 2	587	595	596	596: $4\ A^2m^2$
+ 1	588	595	597	597: $A^2 + m^2 - B^2$
+ 2	597	597	000	000: $(A^2 + m^2 - B^2)^2$
- 1	596	000	598	598: $4A^2m^2 - (A^2 + m^2 - B^2)^2$
+ 0	300	598	599	599: $\sqrt{4A^2m^2 - (A^2 + m^2 - B^2)^2}$
+ 2	590	593	000	000: $(1 - C^2)(\sin\alpha)$
+ 2	000	599	600	600: $\sin\alpha\,(1 - C^2)\sqrt{4A^2m^2-(A^2+m^2-B^2)^2}$
+ 2	594	595	601	601: $m^2\cos\alpha$
+ 2	595	601	602	602: $m^4\cos\alpha$
+ 2	593	593	000	000: $\sin^2\alpha$
+ 2	000	591	603	603: $2C\sin^2\alpha$
+ 1	603	601	000	000: $2C\sin^2\alpha + m^2\cos\alpha$
+ 2	000	588	604	604: $(A^2 - B^2)(m^2\cos\alpha + 2C\sin^2\alpha)$
- 1	592	595	000	000: $A^2 + B^2 - m^2$
+ 2	000	597	000	000: $(A^2 - m^2 + B^2)(A^2 + m^2 - B^2)$

	+ 2	000	603	000	000: $2C \sin^2\alpha(A^2 - m^2 + B^2)(A^2 + m^2 - B^2)$
	+ 2	000	595	000	000: $2Cm^2\sin^2\alpha(A^2 - m^2 + B^2)(A^2 + m^2 - B^2)$
	+ 3	000	598	605	605: $\frac{2Cm^2\sin^2\alpha(A^2-m^2+B^2)(A^2+m^2-B^2)}{4A^2m^2 - (A^2+m^2-B^2)^2}$
	+ 1	600	602	000	000: (600) + (602)
	+ 1	000	604	000	000: (600) + (602) + (604)
(2a)	+ 1	000	605	000	000: Sum of equation (29), (neg)
(3a)	- 2	000	901	606	606: Sum of equation (29), (pos)
	+ 1	527	407	527	527: $\alpha + 0.1$
	+ 8	700	606	010	If not negative, go to PP 10
	+ 2	407	902	000	000: (0.1)(2.0)
	- 1	527	000	527	527: $\alpha - 0.1$
	+ 2	407	408	407	407: 0.01
	+ 8	100	007	010	Return to PP 10 seven times
	+ 1	527	409	527	527: α (inflection point)
	+ 2	408	901	407	407: 0.1
	+ 1	502	594	000	000: $C + \cos\alpha$
	+ 3	593	000	607	607: $\frac{\sin\alpha}{C + \cos\alpha}$
	+ 0	305	607	608	608: $\tan^{-1}\frac{\sin\alpha}{C + \cos\alpha} = \beta'$
	+ 1	571	595	000	000: $B^2 + m^2$
	- 1	000	586	609	609: $B^2 + m^2 - A^2$
	+ 3	599	609	611	611: $\frac{\sqrt{4A^2m^2 - (A^2+m^2-B^2)^2}}{B^2+m^2-A^2}$

	+ 8	700	611	011	If not negative, go to PP 11
	+ 0	305	611	000	000: $- \tan^{-1} \frac{\sqrt{4A^2m^2 - (A^2 + m^2 - B^2)^2}}{B^2 + m^2 - A^2} = -\beta''$
	+ 1	403	000	612	612: β''
	+ 8	000	000	012	Transfer to PP 12
	+ 9	800	011	000	PP 11
	+ 0	305	611	612	612: β''
	+ 9	800	012	000	PP 12
	+ 1	608	612	528	528: β (inflection point)
	+ 7	300	527	528	Punch α and β
(4a)	- 1	527	520	000	000: $\alpha - \alpha_1$
(5a)	+ 3	000	524	529	529: $\frac{\alpha - \alpha_1}{RC1} = X_N$
(6a)	- 1	522	528	000	000: $\beta_1 - \beta$
	+ 3	000	526	530	530: $\frac{\beta_1 - \beta}{RF} = Y_N$
	- 1	529	530	531	531: $X_N - Y_N$
	+ 7	300	529	531	Punch X_N, Y_N, $X_N - Y_N$
	- 1	527	410	527	527: $\alpha - 0.001 = \alpha'$
	+ 9	800	013	000	PP 13
	+ 1	527	900	527	527: α'
	+ 0	303	527	613	613: $\sin \alpha'$
	+ 0	304	527	614	614: $\cos \alpha'$
	+ 2	502	614	615	615: $C \cos \alpha'$

+ 1	615	901	616	616: $1 + C \cos\alpha'$
+ 2	902	615	000	000: $2C \cos\alpha'$
+ 1	000	589	617	617: $1 + C^2 + 2C \cos\alpha' = m^2$
+ 3	616	617	618	618: $\frac{1 + C \cos\alpha}{m^2}$
+ 1	617	588	619	619: $m^2 + A^2 - B^2$
+ 2	619	613	000	000: $\sin\alpha (m^2 + A^2 - B^2)$
+ 2	000	502	620	620: $C \sin\alpha (m^2 + A^2 - B^2)$
+ 2	587	617	621	621: $4A^2m^2$
+ 2	619	619	000	000: $(A^2 + m^2 - B^2)^2$
- 1	621	000	000	000: $4A^2m^2 - (A^2 + m^2 - B^2)^2$
+ 0	300	000	000	000: $\sqrt{4A^2m^2 - (A^2 + m^2 - B^2)^2}$
+ 2	000	617	000	000: $m^2 \sqrt{4A^2m^2 - (A^2 + m^2 - B^2)^2}$
+ 3	620	000	621	621: $\frac{C \sin\alpha (m^2 + A^2 - B^2)}{m^2 \sqrt{4A^2m^2 - (A^2 + m^2 - B^2)^2}}$
+ 1	618	621	532	532: $\omega_4 = \omega_2$ [equation (27)]
+ 7	300	532	532	Punch ω_4
+ 1	527	410	527	527: $\alpha' + 0.001$
+ 8	200	003	013	Return to PP 13 three times
+ 8	000	000	001	Return to PP 1

CONSTANT STORAGE

403	1	+ 3	141	592	750	403:	π
405	2	+ 4	000	000	050	405:	4.00
		+ 1	000	000	048	406:	0.01
407	1	+ 1	000	000	049	407:	0.1
408	1	+ 1	000	000	049	408:	0.1
409	1	+ 1	000	000	044	409:	0.000001
410	1	+ 1	000	000	047	410:	0.001
0000						START	
500	3	+ 2	000	000	050	500:	A = 2.0
		+ 2	000	000	050	501:	B = 2.0
		+ 2	000	000	050	502:	C = 2.0
500	3	+ 2	500	000	050	500:	A = 2.5
		+ 3	000	000	050	501:	B = 3.0
		+ 2	000	000	050	502:	C = 2.0
500	3	+ 3	000	000	050	500:	A = 3.0
		+ 3	000	000	050	501:	B = 3.0
		+ 2	000	000	050	502:	C = 2.0
500	3	+ 3	500	000	050	500:	A = 3.5
		+ 2	500	000	050	501:	B = 2.5
		+ 4	500	000	050	502:	C = 4.5
500	3	+ 3	500	000	050	500:	A = 3.5
		+ 4	000	000	050	501:	B = 4.0
		+ 2	000	000	050	502:	C = 2.0

End of RC1 Program

Crank Range RC2.--To convert the above program to the RC2 crank range, replace computer orders (1a), (2a), (3a), (4a), (5a), and (6a) with the following orders:

(1b)	+ 1	521	406	527	527: $\alpha = \alpha_2 + 0.01$
(2b)	+ 1	000	605	606	606: Sum of equation (29), (pos)
(3b)					No replacement order for (3a)
(4b)	- 1	527	521	000	000: $\alpha - \alpha_2$
(5b)	+ 3	000	525	529	529: $\dfrac{\alpha - \alpha_2}{RC2} = X_{N_2}$
(6b)	- 1	528	523	000	000: $\beta - \beta_2$

TABULATED RESULTS

Table 1. Mechanism 47*

A	2.0000000
B	2.0000000
C	2.0000000
α_1	2.4188584
α_2	4.9650692
β_1	1.6961242
β_2	0.5053605
RC1	2.5462108
RC2	3.7369745
RF	1.1907637

Point of inflection data:

Item	Crank range RC1	Crank range RC2
α	3.2282148	7.4267745
β	1.2302094	1.2302094
X_N	0.3178670	0.6587429
Y_N	0.3912739	0.6087261
ω_4 at $\alpha - 0.001$	$-(1.0221998)\omega_2$	$(0.5054896)\omega_2$
ω_4 at α (maximum)	$-(1.0222028)\omega_2$	$(0.5054897)\omega_2$
ω_4 at $\alpha + 0.001$	$-(1.0222000)\omega_2$	$(0.5054896)\omega_2$

*The number indicates the page in Johnson's thesis on which this mechanism appears.

Table 2. Mechanism 93

A	2.5000000
B	3.0000000
C	2.0000000
α_1	2.1151405
α_2	4.2362700
β_1	1.5082556
β_2	0.4604934
RC1	2.1211295
RC2	4.1620558
RF	1.0477622

Point of inflection data:

Item	Crank range RC1	Crank range RC2
α	3.0313589	7.0498358
β	1.0089961	1.1374647
X_N	0.4319484	0.6760039
Y_N	0.4765008	0.6461116
ω_4 at $\alpha - 0.001$	$-(1.0423688)\omega_2$	$(0.3788068)\omega_2$
ω_4 at α (maximum)	$-(1.0423723)\omega_2$	$(0.3788069)\omega_2$
ω_4 at $\alpha + 0.001$	$-(1.0423686)\omega_2$	$(0.3788068)\omega_2$

Table 3. Mechanism 143

A	3.0000000
B	3.0000000
C	2.0000000
α_1	2.3288371
α_2	4.5870612
β_1	1.8234766
β_2	0.7227342
RC1	2.2582241
RC2	4.0249612
RF	1.1007423

Point of inflection data:

Item	Crank range RC1	Crank range RC2
α	3.1980717	7.2711886
β	1.3465113	1.4234790
X_N	0.3849196	0.6668704
Y_N	0.4333124	0.6366110
ω_4 at $\alpha - 0.001$	$-(1.0095151)\omega_2$	$(0.4076617)\omega_2$
ω_4 at α (maximum)	$-(1.0095181)\omega_2$	$(0.4076618)\omega_2$
ω_4 at $\alpha + 0.001$	$-(1.0095151)\omega_2$	$(0.4076617)\omega_2$

Table 4. Mechanism 199

A	3.5000000
B	2.5000000
C	4.5000000
α_1	2.5786325
α_2	5.8321585
β_1	1.2893163
β_2	0.4510268
RC1	3.2535260
RC2	3.0296593
RF	0.8382895

Point of inflection data:

Item	Crank range RC1	Crank range RC2
α	3.7752046	7.3581208
β	0.9701496	0.8620423
X_N	0.3677770	0.5036746
Y_N	0.3807356	0.4903026
ω_4 at $\alpha - 0.001$	$-(0.4127342)\omega_2$	$(0.4280565)\omega_2$
ω_4 at α (maximum)	$-(0.4127345)\omega_2$	$(0.4280567)\omega_2$
ω_4 at $\alpha + 0.001$	$-(0.4127342)\omega_2$	$(0.4280564)\omega_2$

Table 5. Mechanism 233

A	3.5000000
B	4.0000000
C	2.0000000
α_1	2.0469154
α_2	4.0997849
β_1	1.5864220
β_2	0.5367502
RC1	2.0528695
RC2	4.2303158
RF	1.0496718

Point of inflection data:

Item	Crank range RC1	Crank range RC2
α	3.0173042	6.8383733
β	1.0638820	1.2021392
X_N	0.4726987	0.6473721
Y_N	0.4978127	0.6339020
ω_4 at $\alpha - 0.001$	$-(1.0546955)\omega_2$	$(0.3509620)\omega_2$
ω_4 at α (maximum)	$-(1.0546990)\omega_2$	$(0.3509621)\omega_2$
ω_4 at $\alpha + 0.001$	$-(1.0546955)\omega_2$	$(0.3509620)\omega_2$

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2. Freudenstein, Ferdinand, "On the Maximum and Minimum Velocities and the Accelerations in Four-Link Mechanisms," The American Society of Mechanical Engineers, Vol. 78, 1956, pp. 779-787.