AN EXTENSION OF KAM THEORY TO QUASI-PERIODIC BREATHER SOLUTIONS IN HAMILTONIAN LATTICE SYSTEMS

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To YY and JG, in appreciation.

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SUMMARY

This thesis is concerned with the existence and linear stability of quasi-periodic solutions of the breather type for a one-dimensional lattice system of weakly-coupled anharmonic oscillators with interaction potential of the long-range type; namely, Hamiltonian systems whose Hamiltonian function is of the following form,

$$H = \sum_{n\in\mathbb{Z}}(\frac{p^2}{2} + V_n(q_n)) + \varepsilon \sum_{n\neq m} \frac{1}{p} C_{m,n}(q_m - q_n)^p \,,$$

where $p \in \mathbb{Z}$, $p \geq 3$ and $C_{m,n} = C_{n,m} = O(e^{-|m-n|})$; $W = \sum_{n\neq m} \frac{1}{p} C_{m,n}(q_m - q_n)^p$ is the so-called interaction potential. The study of a variety of physical and biological systems can be formulated in terms of a Hamiltonian system of the same type as the one above or of couplings of these; for example, the dynamics of thin layers of adsorbed atoms (adatoms) on crystal surfaces, the behavior of crystal lattices near a dislocation core (important in the understanding of the elastic properties of solids), the existence of localized electric currents in superconducting systems made out of Josephson-junction arrays, the generation and manipulation of localized excitations (intrinsic localized modes) in micromechanical cantilever arrays and, more recently, the breaking of macromolecules such as in the denaturation of the DNA molecule; to mention just a few.

In essence, all of the examples mentioned above are related to the mechanism of focusing and transport of energy. Typically, breathers are time-periodic or time-quasiperiodic solutions in extended Hamiltonian systems, which have the additional property of spatial localization; i.e., the decay of the amplitude of oscillation is exponential, super-exponential or other similar type of fast decay. This special class of solutions has been observed experimentally, modeled numerically and its existence has been rigorously established for certain continuous systems such as those modeled by PDEs of the wave, Schrödinger or beam types, and in discrete lattice systems such as the ones that arise from the examples mentioned in

the previous paragraph and which we will mention in more detail in chapter 3. Nevertheless, the existence of breathers in discrete systems is very different from their existence in continuous systems; for example, in continuous systems breathers are rather elusive, non-generic objects whereas in discrete systems they are more robust, in some sense, and their occurrence is more common; moreover, it usually is the case that their existence and linear stability can be simultaneously proved.

In this work we tackle the existence of breathers in Hamiltonian systems such as the one whose Hamiltonian function we introduced at the begining, using the KAM methodology; more precisely, we will consider an N-parametric family of real-analytic Hamiltonians of the form

$$H = \mathcal{N} + \mathcal{P},$$

where $N \in \mathbb{Z}$, $N > 1$, $\mathcal{N} = \langle \omega(\xi), I \rangle + \sum_{n\in\mathbb{Z}} \beta w_n \bar{w}_n$, $\beta > 0$ is a constant independent of the parameter $\xi \in \mathcal{O} \subset \mathbb{R}^N$ and $\mathcal{P} = \mathcal{P}(\theta, I, w, \bar{w}, \xi)$ is the perturbation term whose properties are inspired in and also generalize those of the perturbative interaction term, W, described previously; furthermore, $(\theta, I, w, \bar{w})$ belong to a complex neighborhood of $\mathbb{T}^N \times \{0\} \times \{0\} \times \{0\} \subset \mathbb{T}^N \times \mathbb{R}^N \times \ell^1 \times \ell^1$. We will prove that, provided that $\mathcal{N}$ and $\mathcal{P}$ satisfy certain non-degeneracy and regularity conditions and the perturbation $\mathcal{P}$ is small enough, there exists a family of Cantor sets $\mathcal{O}_\gamma \subset \mathcal{O}$, such that for each $\xi \in \mathcal{O}_\gamma$ and $\theta \in \mathbb{T}^N$, the quasi-periodic invariant torus $T_\xi = \{\omega(\xi)t + \theta\} \times \{0\} \times \{0\} \times \{0\}$ of the unperturbed system $H = \mathcal{N}$, will persist as an invariant torus T_ξ^γ of the perturbed system $H = \mathcal{N} + \mathcal{P}$, where $T_\xi^\gamma \sim T_\xi$ and the motion on T_ξ^γ is quasi-periodic with frequencies $\tilde{\omega} \sim \omega$; moreover, the normal coordinates w_n decay exponentially in $|n|$, uniformly in time. The precise formulation is made in Theorem B, ch. 4; we will sometimes refer to this theorem as our abstract KAM theorem.

The proof of Theorem B is the core of the thesis and in it we describe how to treat the combined problem of long-range interaction and normal frequencies with infinite multiplicities, which earlier results such as that in [108] (infinite multiplicities, nearest-neighbor interaction) or analogous KAM methods for PDEs such as [66, 65, 80, 79, 101, 17] (finite multiplicities) were not able to incorporate in the form we did here.

CHAPTER I

INTRODUCTION

Coincidentally, the beginning of the second half of the twentieth century saw two works whose impact in the field of dynamical systems was profound and long-lived.

The first of the two works we refer to above consists in the numerical study of a system of a finite number of particles by Fermi, Pasta and Ulam (FPU) in Los Alamos in 1955 (cf. [41]). In their work those authors carried out a Fourier-component study of a one-dimensional array of 64 particles with fixed ends, coupled by an anharmonic interaction, over periods of time which are much longer compared to the characteristic periods of the associated linear system. More precisely, if q_n is the displacement of the nth particle from its equilibrium position, the FPU system is Hamiltonian with a finite number of degrees of freedom and Hamiltonian function of the form

$$H = \sum_{n=1}^{N} \frac{p_n^2}{2} + \mathcal{W}(\{q_n\}_{n=1}^{N}),$$

where $N = 64$ and $\mathcal{W}$ is an interaction potential of one of three categories,

(i-ii) cubic (p=3) or quartic (p=4): $\mathcal{W} = \sum_{n=1}^{N} \frac{1}{p}(w_{n+1} - w_n)^p$,

(iii) broken-linear: $\mathcal{W} = \sum_{n=1}^{N}(\delta(q_{n+1} - q_n)(q_{n+1} - q_n)^2 + c(q_n)q_n)$, where δ and c are functions which depend on whether the quantities in parentheses are less than or greater than certain predetermined fixed values.

The purpose of the FPU numerical study was that of investigating the long-term effect (in the sense mentioned above) that nonlinear terms have on simple initial periodic solutions associated to the linear system.

The second of the two works is by A. N. Kolmogorov (1954) (cf. [57], English transaltion [58]) and is concerned with nearly-integrable Hamiltonian systems. In simple terms, Kolmogorov's theorem states that if an integrable Hamiltonian function is non-degenerate and sufficiently differentiable, then under small and differentiable enough perturbations, many of its quasi-periodic orbits will persist as quasi-periodic orbits of the full perturbed system;

moreover, such solutions live on a torus which they fill densely (cf. the "classical" KAM theorem in the next chapter or [36] p. 90).

The results of FPU and of Kolmogorov, challenged the general belief that a system injected with energy is to achieve its thermal equilibrium obeying the laws of heat diffusion, as one would expect from the 2nd law of Thermodynamics or the ergodic hypothesis (cf. [97]). In particular, the FPU experiment showed that it is possible for energy to be trapped in a system for a long period of time and thus slowing down its thermalization, in an apparent violation to the principle of equipartition of energy. More precisely, the calculations of Fermi, Pasta and Ulam showed that, starting with a sinusoidal mode of the linear system, instead of the energy being distributed equally over all modes in time ("thermalization"), it is distributed primarily on a few modes which may exchange their energies in a regular fashion.

As computers became more powerful and accessible, numerical studies gathered evidence which pointed out that the process by which energy is focused within a system thus preventing its relaxation is, in fact, very common. Under certain circumstances systems are capable of focusing and transporting energy and the mechanisms by which this is possible possess a relatively simple mathematical formulation.

Perhaps the two oldest examples of mathematical objects which reflect the property of energy localization and transport are solitons and breathers, which first appeared as special solutions of nonlinear wave equations such as KdV; we will elaborate more on these type of solutions in chapter 3 where we provide explicit formulae for them in the cases of the modified KdV and sine-Gordon equations. Solitons are traveling-wave solutions, they propagate throughout the medium without altering their shape. In contrast, breathers can be periodic or quasi-periodic in time and they may be anchored to a particular region in space or they may translate as well. Both solitons and breathers have the property of spatial localization, that is, they are expressed as highly localized disturbances in the medium.

Our work is concerned with solutions of the breather type in lattice systems; we will sometimes refer to this scenario as the discrete case. For a lattice system we will understand a one-dimensional array or alignment of oscillators (point masses) whose interactions with

the medium or substrate and with one another are modeled via an on-site potential (which might be different for each oscillator) and a coupling, or simply interaction, potential; respectively. More clearly, let q_n denote the displacement of the nth oscillator from its equilibrium position, then the equation that governs the motion of this oscillator is of the following form,

$$\ddot{q}_n + V_n'(q_n) \,=\, \sum_{k\geq 1}(\mathcal{W}'(q_{n+k}-q_n) - \mathcal{W}'(q_n - q_{n-k}))\,,$$

where V_n is the on-site potential that represents the interaction between the nth oscillator and the substrate and $\mathcal{W}$ is the interaction potential function that models how oscillators interact with one another ($\mathcal{W}'(x) = \frac{d}{dx}\mathcal{W}(x)$). The system of all equations of motion is Hamiltonian with Hamiltonian function given by

$$H \,=\, \sum_{n\in\mathbb{Z}}(\frac{p_n^2}{2} + V_n(q_n)) + \sum_{m\neq n}\mathcal{W}(q_m - q_n)\,. \tag{1.0.1}$$

In this setting, breathers (also called discrete breathers to make the distinction with their homologous solutions in the PDE setting –sometimes called the continuous case) are understood as time periodic or time quasi-periodic solutions whose amplitudes $|q_n|$ decay at an exponential or inverse-power rate as $|n| \to \infty$. The existence of this type of solutions is very well documented in numerical models in the physics literature (cf. [23, 81] and references therein). However, rigorous analytic proofs of their existence were not produced prior to the work of MacKay and Aubry (1994) [70]. Those authors used continuation methods from anti-integrability (or rather, anti-continuity) theory to prove the persistence of a one-lattice excitation mode when the interaction term $\mathcal{W}$ in (1.0.1) is "switched on", i.e., when $\mathcal{W} = \varepsilon W$ and $\varepsilon > 0$ is constant[1]. We observe that when $\varepsilon = 0$, under appropriate convexity properties of the on-site potentials, the system decouples and becomes integrable. The anti-continuity method of MacKay and Aubry is not able to handle more than two-excited sites at a time; however, these same authors suggested in their seminal paper that, by combining ideas from anti-integrability and KAM theories, it should be possible to prove

[1] the interaction potential that they considered, however, is of the nearest-neighbor type (cf. chapter 3), not of the long-range type as in (1.0.1)

the more general case of persistence of an arbitrary number of excited sites, a result which was strongly supported by numerical studies.

At the time of the publication of [70], the first works extending KAM theory to infinite-dimensional systems had already been in circulation for ten years. The development of KAM theory in infinite dimensions was started simultaneously by Vittot and Bellissard [100] (1985) on lattice systems, and by Frölich, Spencer and Wayne [43] (1986). Moreover, it was during the 1990's that the development of KAM theory in infinite dimensions experienced substantial progress in the works of (in alphabetical order) Bourgain, Chierchia, Craig, Kuksin, Pöschel, You and Wayne (cf. [59, 60, 61, 66, 33, 62, 101, 79, 17, 65, 80, 63, 28], see also [64], ch. 8).

In order to understand the main contribution of this thesis let us here briefly mention some results which we will come back to again later on in here and in more detail in chapter 3. The first KAM proof on existence of discrete breathers in lattice systems is due to Yuan [108] (2002) who considered a Hamiltonian system like (1.0.1) with $V_n(x) = V(x)$ for all n, such that $V(0) = V'(0) = 0$, $V''(0) = \beta^2$, $\beta > 0$ and $\mathcal{W}(q_m - q_n) = \varepsilon\delta_{m,n}W(q_m - q_n)$, where $\delta_{m,n} = 1$ if $|m - n| = 1$ and zero otherwise (this type of interaction potential is the so-called nearest-neighbor type); furthermore, $W(x) = O(|x|^3)$. The Hamiltonian function thus considered is

$$H \;=\; \sum_{n\in\mathbb{Z}} (\frac{1}{2}q_n^2 + V(q_n) + \varepsilon W(q_{n+1} - q_n))\,. \tag{1.0.2}$$

By selecting a finite number of lattice sites $\mathcal{J} = \{n_1, \ldots, n_N\}$, $N > 1$, one can write the Hamiltonian above in normal form,

$$H \;=\; \langle \omega(\xi), I\rangle + \sum_{n\in\mathbb{Z}_1} \frac{1}{2}\beta w_n \bar{w}_n + \varepsilon\mathcal{P}\,, \tag{1.0.3}$$

where $\mathcal{P}$ is a higher-order perturbative term and $\mathbb{Z}_1 = \mathbb{Z} \setminus \mathcal{J}$ (cf. [108] p. 67, or chapter 4 of this work). ξ plays the role of an N-parameter and β is the so-called normal frequency, the same for every n. Writing H in the normal form above immediately implies that previous KAM results derived from a PDE setting are not applicable in this case because such results require normal frequencies of finite multiplicity or satisfying certain asymptotic spectral conditions (cf. [108], or [64] p. 134). In the above cited work Yuan derived a

KAM theorem for (1.0.3) to treat the infinite-multiplicity case with short-range interaction, by which he established the existence of quasi-periodic solutions of the breather type. We remark at this point that the proof of Yuan's KAM theorem relies crucially on the fact that the Hamiltonian under consideration is of the short-range type. The main result in this thesis, Theorem B in chapter 4, consists in a KAM theorem inspired in a Hamiltonian system similar to (1.0.2) but with a long-ranged interaction potential; more precisely, in Theorem A of chapter 4, we consider a Hamiltonian with an interaction potential of the form

$$\varepsilon \sum_{m \neq n} W(q_m - q_n)\,,$$

where $W(x) = O(|x|^p)$ and $p \geq 3$. We will show that, after writing the Hamiltonian of Theorem A in normal form, one can see clearly that neither the KAM results from PDEs, nor the KAM theorem in [108] are applicable in this case to establish the existence of quasi-periodic motions because, in one hand, the spectrum of the associated linear problem is degenerate (only one frequency and thus it suffers from infinite muiltiplicity) and in the other hand, the proof of the analogous KAM theorem in [108] cannot be adapted to prove our abstract KAM theorem, Theorem B, because Yuan's proof makes strong use of the short-range property of his interaction potential. We will demonstrate that one can nevertheless abstract the properties of our physical Hamiltonian in Theorem A and formulate a more general type of Hamiltonian of the long-range type which, in spite of having one single normal frequency and thus exhibiting infinite multiplicity, it is still fit for us to apply the KAM methodology to prove the existence of quasi-periodic solutions of the breather type; provided that the perturbation term in the Hamiltonian satisfies certain growth conditions which are again inspired in the properties of the physical Hamiltonian in Theorem A. We will also motivate the study of the Hamiltonian in Theorem A by showing that it can be interpreted as a higher-order term in the expansion of a physical interaction potential from quantum physics; this interpretation provides another way of looking at systems of the Frenkel-Kontorova (FK) type or their extensions. Regarding the interpretation of interaction potentials, it usually is the case (although not always) that exact existence results of breathers in lattice systems use a FK type of interaction and leave

out an explanation as of why they are interested in this type of potential, perhaps guarded behind the well-known success of this model in explaining a large number of phenomena (cf. chapter 3). There is, however, a number of interaction potentials of physical origin (cf. [85]) for which rigorous existence results of breather solutions in associated lattice systems has not been done.

Chapters 2 and 3 are introductory in nature. In chapter 2 we introduce some of the basic mathematical formalism that will appesr in chapter 4 and discuss the classical results of KAM theory in finite dimensions, particularly so those regarding the persistence of lower-dimensional tori because they constitute the natural finite-dimensional framework in which to understand the analogous persistence problem in infinite dimensions and which amounts to the existence of quasi-periodic solutions, of which breathers are but a particular class. We start chapter 3 by introducing the notion of breathers following a rather chronological order, namely, we discuss the existence of intrinsic localized modes (ILMs) in PDEs such as Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV) and sine-Gordon (sG), with emphasis on solitons and time-periodic breathers. The existence of solitons and breathers in PDE systems was established rigorously via the inverse scattering method in the early 1970's although such types of solutions had already been identified earlier (cf. [1, 2] and references therein). However, with the exception of the sG equation, breather solutions in PDEs appear as rather non-robust objects in the sense of the results in [91, 94, 37, 13, 14]; this is to contrast with the existence of this same type of solutions in lattice systems, including those systems which originate as spatial discretizations (in one of two ways mentioned in chapter 3) of KdV, mKdV and sG equations. Indeed, the so-called continuous (PDE) and discrete (lattice) systems are very different from the dynamics point of view, in our work we will not be concerned with the continuous case and thus we will not try to elaborate on it but on the discrete case, in which it is well-known that the combination of discreteness and non-linearity are the main ingredients responsible for the ability of a system to support breathers (cf. [70, 69, 6]).

Lattice systems arise naturally as mathematical models in physics and in biology; to illustrate this point we have included in chapter 3 a few examples used in the study of

the dynamics of crystal lattices, electric circuits, and in mechanical and biological systems. Following our examples, we present a discussion of exact existence results of quasi-periodic breathers in Hamiltonian lattice systems with comments on the methods of proof (cf. [70, 10, 9, 108, 95]). Our choice of results is three-fold. First, we aim at providing a brief account of (selected) methods used in proving the existence of breathers in lattice systems; namely, anti-continuity (implicit-function methods; [70]), general normal-form theorems for abstract Hamiltonian systems in Banach spaces ([10, 9]), KAM ([108]) and finite-dimensional center-manifold reduction methods ([95, 54]). These methods are not exhaustive, for example, variational (minmax with constraint) methods have been left out. Even though variational methods are capable of obtaining solutions of physical interest and are suitable in the absence of parameters (such as those with weak coupling like [70, 108]), generally, however, with this type of method one can only establish the existence of single discrete breathers at the cost of rather complex proofs (cf. [6] and references therein). Second, even though we possess a precise mathematical definition of breathers, we believe that the list of works above also help in providing us with a dynamical characterization of such objects. Consider for instance the center-manifold reduction approach in [54, 95] which renders travelling breathers as homoclinic orbits to lower-dimensional tori; a notion one may develop the intuition for, after reading [108] or chapter 4 of this work (which establish the existence of ("true") quasi-periodic breathers) and then comparing the type of solutions found against those other, sometimes called, "breather-like", which start as true breathers but then break up and disappear via some type of phonon-radiation mechanism and which are more commonly found in the physics literature (e.g., cf. ch. 4 in [21] and references therein). Third and last, we hope that our selection of existence results will help in convincing the reader that our Theorem A in chapter 4 is the next (after [108]) natural, and perhaps we may also say necessary, step to take in the developing theory of breathers; namely, that of demonstrating the existence of this type of solutions in infinite-dimensional lattice systems with a long-range interaction term of the third order. To prove Theorem A, an abstract KAM theorem was developed which may be applied in both, lattice and PDE settings as well, making it an interesting result in its own right.

CHAPTER II

BRIEF REVIEW OF KAM THEORY IN FINITE DIMENSIONS

Chapter summary

Even though the main result of this work (cf. Theorem B, sec. 4.2) is concerned with KAM theory in infinite dimensions, its main ideas and method of proof can be better understood by looking into the case of the persistence of certain lower-dimensional tori in nearly integrable finite-dimensional Hamiltonian systems, which is in fact the scenario from which infinite-dimensional KAM theory drew its insight. The purpose of this chapter is that of briefly reviewing KAM theory in finite dimensions, in particular, later work on the perturbation of lower-dimensional tori. To this end, we will first introduce the basics of the Hamiltonian formalism. The type of reader interested in dynamical systems will find the exposition very familiar while the well-versed type may continue to the next chapter.

2.1 Hamiltonian systems: formalism and examples

Hamiltonian mechanics is just one among three approaches or models used to describe mechanical systems, the other two being the Newtonian and the Lagrangian, they all differentiate from one another depending on the principles or laws of motion that they observe. The Hamiltonian approach to mechanics requires the system under study to be endowed with a mathematical structure that we will briefly define in the next few pages.

Let us start by quickly recalling the derivation of Hamilton's equations from Lagrange's equations (cf. [40, 12]). In Lagrangian mechanics one assumes that the configuration or state of a physical system at time t is determined by a set of n coordinates, $q_1(t), \ldots, q_n(t)$, called the *position variables* at time t; we will assume that these coordinates are independent and complete; i.e., no one coordinate depends on the others and they fully determine the state of the system. The set of all possible configurations of the system is an n-dimensional (real)

manifold called the configuration space, whose local coordinates are the position variables. The evolution in time of the system is given by a curve $\gamma : t \mapsto (q_1(t), \ldots, q_n(t))$, with prescribed initial conditions $q_i(t_0) = q_i^0$, $i = 1, \ldots, n$. In Lagrangian mechanics the curve γ that describes the evolution of a system is obtained via the *principle of least action*; i.e., γ is such that it minimizes the path (also action) integral,

$$\gamma \,=\, \operatorname{argmin} \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt\,,$$

where $q = (q_1, \ldots, q_n)$. L is the so-called Lagrange function and is given by the difference between the total kinetic and potential energies of the system; we will assume that L is convex in $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_n)$. The minimization above is carried over some suitable space of functions so that the variational problem is well-defined. The condition for an extremum is thus given by the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \,=\, 0\,, \quad i = 1, \ldots, n\,.$$

The above system is equivalent to a $2n$-dimensional system of 1st-order ODEs; a point in phase space would have coordinates $(q, \dot{q})$. In the Hamiltonian formulation we use the position q and generalized momentum $p = (p_1, \ldots, p_n)$ as our local coordinates, where the momentum is defined as

$$p_i \,=\, \frac{\partial L}{\partial \dot{q}_i}\,, \quad i = 1, \ldots, n\,;$$

on the other hand, the Lagrangian is replaced by the *Hamiltonian function*, given by

$$H(p, q, t) \,=\, p\dot{q} - L(q, \dot{q}, t)\,,$$

where p is as defined above, then upon writing down the total differential of H and using the definition of the generalized momentum one can easily see that Lagrange's equations yield to *Hamilton's equations*

$$\dot{q} = \frac{\partial H}{\partial p}\,, \quad \dot{p} = -\frac{\partial H}{\partial q}\,, \tag{2.1.1}$$

together with $\partial H/\partial t = \partial L/\partial t$. When H is independent of t, Hamilton's equations become a $2n$-dimensional closed system of first-order equations. It turns out that, in the new

variables, the Hamiltonian becomes the sum of the kinetic, T, and potential V energies, i.e.,

$$H \,=\, T + V\,.$$

The above is a rather simplistic and quick approach to deriving Hamilton's equations. We now turn to explaining some of the geometric notation that accompanies Hamiltonian dynamics, for which we will need some abstraction. Associated to Hamilton's equations there is a $2n$-dimensional smooth manifold, M, where n stands for the number of *degrees of freedom* of the system. On M we define a *symplectic structure* as a symplectic, non-degenerate, closed form, ω^2; i.e., for any given $z \in M$, let T_zM denote the tangent space at z, then

$$\omega^2 : T_zM \times T_zM \to \mathbb{R}$$

is bilinear, antisymmetric ($\omega^2(u,v) = -\omega^2(v,u)$ for all $u, v \in T_zM$), if $\omega^2(u,v) = 0$ for any $v \in T_zM$ then $u = 0$, and $d\omega^2 \equiv 0$. The pair (M, ω^2) is called a *symplectic manifold.*

A theorem by G. Darboux (cf. [4] p. 230) states that in a neighborhood of any point $z \in M$ one can choose appropriate local coordinates, say $z = (p_1, \ldots, p_n, q_1, \ldots, q_n)$, such that ω^2 has the standard or canonical form

$$\omega^2 \,=\, \sum_{i=1}^{n} dp_i \wedge dq_i\,;$$

these coordinates are called symplectic or canonical.

The property of ω^2 of being non-degenerate allows us to identify vector fields and one-forms. This idea is central to the Hamiltonian formalism and thus we will elaborate more on it. Let $\xi \in T_zM$, the map $\xi \mapsto \omega^2(\cdot, \xi)$ from T_zM to T_z^*M is clearly linear; moreover, if ω^2 is non-degenerate, this map invertible and in fact an isomorphism. The previous observation motivates the following definition: let $TM = \cup\{T_zM : z \in M\}$ denote the tangent bundle of M, a map $X : M \to TM$ is called a vector field on M; the *inner product* of a vector field X and a non-degenerate two-form ω^2, is a 1-form, $i_X\omega^2$, defined by

$$i_X\omega^2(\cdot) \,:=\, \omega^2(X, \cdot)\,.$$

Now let $H : M \to \mathbb{R}$ be a smooth function defined on M, smoothness in this case refers to the local coordinates on M, dH is a covector field on M, $dH \in T^*M$, which we denote by

$dH = \sum_{i=1}^{n}(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i)$. We say that a vector field $X_H : M \to TM$ is a *Hamiltonian vector field* with the *energy function* H, if and only if

$$i(X_H)\omega^2 \,=\, dH\,, \tag{2.1.2}$$

and (M, ω^2, X_H) is called a *Hamiltonian system.* In particular, let us recall Hamilton's equations (2.1.1), in this case $X_H = \sum_{i=1}^{n}(\frac{\partial H}{\partial q_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial p_i})$ and one can show (cf. [12] p. 6) that Hamilton's equations are equivalent to (2.1.2); more precisely, if $\mathcal{J} : T^*M \to TM$ denotes the inverse of inner product defined by (2.1.2), then

$$\dot{z} \,=\, \mathcal{J}dH\,.$$

Another important geometric structure in Hamiltonian mechanics is the following. Suppose that F and G are smooth functions on M, the *Poisson bracket* of F and G is defined as

$$\{F, G\} := \omega^2(\mathcal{J}dG, \mathcal{J}dF)\,.$$

It is not difficult to show that the Poisson bracket is bilinear, skew-symmetric and that it satisfies the following properties:

(i) Leibniz rule: $\{F_1F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$.
(ii) Jacobi identity: $\{\{H, F\}, G\} + \{\{F, G\}, H\} + \{\{G, H\}, F\} = 0$.
(iii) Non-degenerate: if z is not a critical point of F, then there exists a smooth function G such that $\{F, G\}(z) \neq 0$.

The value of the Poisson bracket may also be calculated by $\{F, G\} = dF(\mathcal{J}dG)$, that is, as the value of the covector dF on the vector $\mathcal{J}dG$; in particular, the derivative of a function F in the direction of the Hamiltonian vector field $\mathcal{J}dH$ is $\dot{F} = \mathcal{J}dH$;

In symplectic local coordinates the notions defined above adopt a simple form; for example, Hamilton's equations are expressed in vector from as $\dot{z} = J\nabla H$ where $J \in \mathbb{R}^{2n\times 2n}$ is the *unit symplectic matrix* and is given by blocks by $J_{11} = J_{22} = 0$, $J_{12} = I = -J_{21}$, where 0 and I are the zero and identity matrices in $\mathbb{R}^{n\times n}$, respectively. Furthermore,

$\{F, G\} = (\nabla F)^\top J \nabla G$. Also, Hamilton's equations may be conveniently written as,

$$\dot{q}_i = \{p_i, H\}\,, \quad p_i = \{q_i, H\}\,, \quad i = 1, \ldots, n\,.$$

Examples of Hamiltonian systems pervade the natural sciences field, below we just mention three of them.

(A) System of coupled pendula

Consider a system of three rigid pendula hanging from a wire and whose planes of oscillation are parallel to each other. Assume that their masses are the same and that they are coupled by identical torsional springs whose effect is that of adding a torque that depends linearly on the phase difference between neighbors (cf. figure 1)

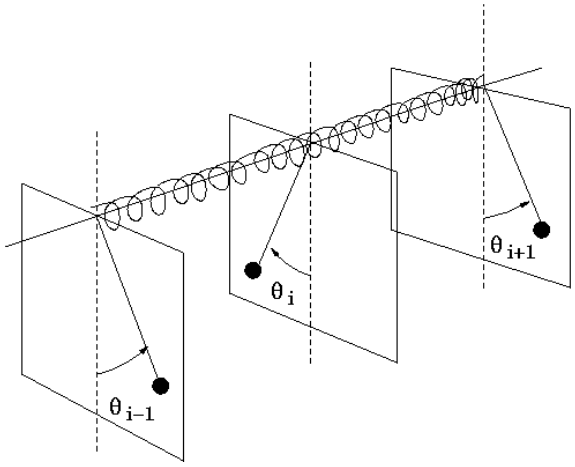

Figure 1: *A system of three identical rigid pendula linearly coupled by torsional springs.*

Let θ_i denote the phase of the ith pendulum, that is, its deviation from the vertical, then, in appropriate dimensionless variables, the system of equations for the individual torques is

$$\begin{aligned} \ddot{\theta}_1 + \sin\theta_1 &= \kappa(\theta_2 - \theta_1)\,, \\ \ddot{\theta}_2 + \sin\theta_2 &= \kappa(\theta_3 - \theta_2) - \kappa(\theta_2 - \theta_1)\,, \\ \ddot{\theta}_3 + \sin\theta_3 &= -\kappa(\theta_3 - \theta_2)\,. \end{aligned} \tag{2.1.3}$$

The system above is Hamiltonian, with conjugate variables $p_i = \dot{\theta}_i$, $q_i = \theta_i$, $i = 1, 2, 3$ and Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + (1 - \cos q_1) + (1 - \cos q_2) + (1 - \cos q_3) + \frac{\kappa}{2}((q_3 - q_2)^2 + (q_2 - q_1)^2)\,. \tag{2.1.4}$$

Moreover, the Poisson bracket between two functions F and G is defined by $\{F,G\} = (\nabla F)^\top J \nabla G = \sum_{i=1}^{3}(F_{q_i}G_{p_i} - F_{p_i}G_{q_i})$, where J is the canonical symplectic identity, thus $\dot{p} = \{H,p\}$ and $\dot{q} = \{H,q\}$, or if $z = (p,q)$ then $\dot{z} = J\nabla H(z)$.

Observe that in the limit $\kappa = 0$ the system (2.1.4) decouples to that of a system of three identical nonlinear oscillators,

$$\ddot{q}_i + \sin q_i \;=\; 0\,, \quad i = 1,2,3\,; \tag{2.1.5}$$

with Hamiltonians $H_i = \frac{1}{2}p_i^2 + (1-\cos q_i)$; the level sets $H_i = h_i^0$ give us the phase portraits for each oscillator. Let us quickly use this happenstance to illustrate the concept of integrability, which we will define more formally in the next section. Solving for p_i in the uncoupled system H_i we obtain $p_i = \frac{d}{dt}q_i = (2(h_i^0 - (1-\cos q_i)))^{\frac{1}{2}}$, which yields

$$t - t_0 \;=\; \int_{q_i^0}^{q_i} \frac{dq}{\sqrt{2(h_i^0 - (1-\cos q))}}\,,$$

where $q_i^0 = q_i(0)$. The integration procedure above is called *quadrature*, so one says that one has solved the system of one oscillator up to quadrature; however, the integral above cannot be solved explicitly and thus having an integral representation of the solution can prove not to be very useful; nevertheless, as we will see in the next section, knowing whether one can"integrate" the system is synonym to "simple dynamics" in an appropriate set of variables –think for example of the harmonic oscillator $\ddot{x} + a^2 x = 0$, of which the (2.1.5) is but a higher-order perturbation. In the trivial case of the harmonic oscillator it is possible to define polar coordinates such that the first-order Hamiltonian system translates to $\dot{r} = 0$ and $\dot{\theta} = -a$ (cf. [74] p. 6)

(B) The N-body problem of celestial mechanics

Perhaps the Hamiltonian system par excellence, the N-body problem in celestial mechanics consists in N point masses in a Newtonian reference frame, $\mathbb{R}^3$, interacting with one another via their mutual gravitational forces only. If m_i and $\mathsf{q_i} = (x_i, y_i, z_i)^\top$ stand for the mass and position vector, respectively, of the ith mass, then the system of equations that describe

the motion of the N particles is

$$m_i \ddot{\mathsf{q}}_\mathsf{i} \;=\; \sum_{\substack{1\le j\le N\\ j\neq i}} G m_i m_j \frac{\mathsf{q}_\mathsf{i}-\mathsf{q}_\mathsf{j}}{\|\mathsf{q}_\mathsf{i}-\mathsf{q}_\mathsf{j}\|^3} \;=\; \frac{\partial U}{\partial \mathsf{q}_\mathsf{i}}\,, \quad i=1,\ldots,N\,,$$

where G is the universal gravitational constant and U is the gravitational potential defined as

$$U \;=\; \sum_{1\le i<j\le N} \frac{G m_i m_j}{\|\mathsf{q}_\mathsf{i}-\mathsf{q}_\mathsf{j}\|}\,.$$

If we define the block-diagonal matrix $M_{ii}=m_i I_3$, where I_3 is the 3×3 identity matrix, and the $3N$-dimensional vector $\mathsf{q}^\top=(\mathsf{q_1}^\top,\ldots,\mathsf{q_N}^\top)$, then we can write the second-order system of equations compactly as $M\ddot{\mathsf{q}} = -\partial U/\partial \mathsf{q}$. This system is Hamiltonian with conjugate variables given by the positions q and momenta $\mathsf{p}^\top=(m_1\dot{\mathsf{x}}_\mathsf{i}^\top,\ldots,m_N\dot{\mathsf{x}}_\mathsf{N}^\top)$ and Hamiltonian

$$H \;=\; \sum_{i=1}^{N} \frac{\|\mathsf{p}_\mathsf{i}\|^2}{2m_i} - U\,,$$

thus

$$\dot{p}_i = \{p_i, H\}\,, \quad \dot{q}_i = \{q_i, H\}\,,$$

where J is the standard $6N\times 6N$ symplectic identity, the Poisson bracket is in its standard form $F(\mathsf{q},\mathsf{p})$ and $G(\mathsf{q},\mathsf{p})$ by $\{F,G\}=(\nabla F)^\top J\nabla G$; alternatively, if $z=(p,q)$ then

$$\dot{z} \;=\; J\nabla H\,.$$

2.2 Integrability and action-angle variables

Once the description of a system is cast in terms of a Hamiltonian system of equations, then there is the issue of "solving" these equations. But what do we mean by solving when, for instance, even the simple system of one nonlinear oscillator mentioned in example *A* of the previous section cannot be solved so as to obtain an explicit expression in terms of the time variable? Moreover, not being able to obtain explicit solutions does not mean that one cannot understand the system's dynamics; once again, recall the case of the single nonlinear oscillator where phase-space curves could be identified with the level curves of the Hamiltonian function –but then again there is the issue of high-dimensional phase spaces, etc. In Hamiltonian mechanics, the concept of "solvability" is replaced by that of "integrability"; this is the subject of this section.

The setting so far discussed is that of *autonomous systems*, i.e., the Hamiltonian does not depend explicitly of the time variable, this will be the case of this work and therefore, we will not discuss the more general case of time-dependent Hamiltonians. Associated to an autonomous Hamiltonian system of equations there are always conserved quantities, the Hamiltonian function being one of them (although it may be the only one, in mechanics it usually is the case that quantities such as the linear or angular momenta are preserved too, this is the case in the N-body problem of example B). We say that a function $F(p,q)$ is a *conserved quantity*, if its derivative in the direction of the Hamiltonian vector field is zero, i.e., if

$$\dot{F} \,=\, \{F, H\} = 0\,,$$

conserved quantities are also called *first integrals*. Loosely speaking, the more first integrals the easier the description of a system, as phase space (solution) curves will live in the intersection of the level sets of their first integrals. In Hamiltonian systems, for each first integral found one can effectively reduce the number of degrees of freedom by one, provided certain conditions are satisfied, we will discuss below.

We say that a Hamiltonian system (in general, any first-order system of ODEs) can be integrated by *quadratures* if its solutions can be found by a finite number of algebraic operations which include the inversion of functions, and by calculating integrals of known functions.

Essential to the idea of integrability by quadratures is the next definition. Assume that H is a Hamiltonian function that is defined on $\mathbb{R}^{2n}$ and that $z = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ are (global) symplectic coordinates. Let $F_1, \ldots F_k$ be smooth functions of z and define a level set of these functions as follows, let $f = (f_1, \ldots, f_n)$ then

$$M_f \,=\, \{z : F_i(z) = f_i\,, \quad i = 1, \ldots n\}\,. \tag{2.2.1}$$

We say that these functions are *independent on* M_f if their 1-forms dF_i are linearly independent at each point of M_f. We say that these functions are in *involution* if and only if all their Poisson-bracket pairs vanish identically; i.e., if and only if

$$\{F_i, F_j\} \,\equiv\, 0\,, \quad i, j = 1, \ldots, k\,.$$

Liouville's theorem states that if $F_1, \ldots, F_n$ are independent on M_f and in involution, then M_f is a smooth manifold invariant under the phase flow with Hamiltonian $H = F_1$. Moreover, if M_f is compact and connected then it is diffeomorphic to an n-dimensional torus $\mathbb{T}^n = \{(\phi_1, \ldots, \phi_n) \bmod 2\pi\}$ and the phase flow associated to the Hamiltonian H determines a quasi-periodic motion on M_f; i.e., $\dot{\phi}_i = \omega_i(f)$, $i = 1, \ldots, n$; furthermore, Hamilton's equations, $(\dot{p}, \dot{q}) = (-H_q, H_p)$, can be integrated by quadratures (cf. [4] p. 272 or [74], p. 38). If in addition, the frequencies $\omega_1(f), \ldots, \omega_n(f)$ rationally independent, then we say that the torus is *nonresonant*, on such a torus every phase trajectory is everywhere dense (cf. [5] p. 111). If M_f is not compact, then every one of its components is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for some positive integer k and it is possible to find coordinates $(\phi_1, \ldots, \phi_k) \bmod 2\pi$, $y_1, \ldots, y_{n-k}$ on which Hamilton's equations $\dot{z} = \{z, F_j\}$ take the form $\dot{\phi}_i = \omega_{ij}$ and $\dot{y}_l = c_{lj}$ (cf. [5] p. 110), this simple way of writing Hamilton's equations is discussed next.

Sometimes Hamilton's equations are not written in conjugate variables that facilitate their study; oftentimes it is desirable to find coordinates which will serve that pupose. But we are not interested in just any set of coordinates which will make the description of the dynamics simpler, we are interested in those which, in addition, preserve the symplectic structure. Suppose that on a symplectic manifold M it is possible to define two sets of local coordinates, (p, q) and (y, x); the transformation that maps one set of coordinates into the other, $g : (p, q) \mapsto (y, x)$, is called *canonical* if and only if there exists a smooth function F such that

$$dF(p, q) \;=\; ydx - pdq\,; \tag{2.2.2}$$

F is called *primitive function* of the transformation. Observe that $\omega^2_{p,q} = d(pdq) = d(ydx - dF) = d(ydx) = \omega^2_{y,x}$ thus, indeed, a canonical transformation preserves the symplectic structure. A necessary an sufficient condition for a transformation to be canonical is that if Γ is the Jacobian matrix of the transformation g, then

$$\Gamma^* J \Gamma \;=\; J$$

(J is the canonical symplectic matrix), a matrix that satisfies this condition is called symplectic, thus a transformation is canonical if and only if its Jacobian is symplectic. A

canonical transformation will preserve the canonical form of Hamilton's equations; in fact, if G is a smooth function of p and q, then

$$\dot{G}_{y,x} = \{G, H\}_{y,x} = \{G, H\}_{p,q} = \dot{G}_{p,q}\,,$$

where the subindex indicates with respect to what coordinates the quantities are being written. Lastly, we mention that condition (2.2.2) is sometimes written as

$$g^*\omega^2 \;=\; \omega^2$$

which says that the pull-back of ω^2 by g is again ω^2; that is again the preservation of the 2-form. The procedure by which one can "generate" canonical transformations is quite standard and we will not include it here (cf. [4] p. 258 or [5] p. 22).

Let us go back now to the case of a $2n$-symplectic manifold (M, ω^2) that has n independent first integrals in involution, $F_1, \ldots, F_n$. Now let M_f be as in (2.2.1) and assume that this invariant manifold is compact, then: *(1)* there is a small neighborhood of M_f in M which is diffeomorphic to $D \times \mathbb{T}^n$, where D is a small domain in $\mathbb{R}^n$ and *(2)* on $D \times \mathbb{T}^n$ there exist symplectic coordinates $I \in D$ and $\phi \in \mathbb{T}^n$, in which the first integrals become functions of I only; in particular, $H = H(I)$; moreover, $\omega^2 = \sum_{i=1}^n dI_i \wedge d\phi_i$ and Hamilton's equations turn into

$$\dot{I} = 0 \quad \dot{\phi} = \omega(I)\,,$$

the coordinates defined above are called *action-angle variables; I* is the so-called action variable which "labels" the tori. The construction of the action-angle variables is again standard (cf. [4] p. 282 and [5] p. 116); for example, if $\gamma_1, \ldots, \gamma_n$ are a basis for the one-dimensional cycles on the M_f (i.e., ϕ_i increases by 2π by going once around γ_i, while all the increase in all the other coordinates is zero) one can show that

$$I_i(f) \;=\; \frac{1}{2\pi} \oint_{\gamma_i} p\, dq\,.$$

2.3 The classical KAM theorem

Among the most influential papers in dynamical systems because of the amount of fundamental work that originated from it is [57, 58], due to A. N. Kolmogorov, published over half

a century ago. Since then, the work of Kolmogorov has been largely refined and extended by many authors. The "classical" KAM theorem we include below is due to Kolmogorov [57], Arnol'd [3] and Moser [75].

Let us consider a Hamiltonian function of the form

$$H(I,\phi,\varepsilon) \;=\; H_0(I) + \varepsilon H_1(I,\varphi,\varepsilon)\,, \tag{2.3.1}$$

and assume for the moment that H is analytic. When $\varepsilon = 0$, $H = H_0(I)$, the Hamiltonian system is completely integrable; the phase space is foliated by invariant tori $I = \textit{constant}$ and on each torus the motion is quasi-periodic with frequency

$$\omega_0(I) \;=\; \frac{\partial H_0}{\partial I}\,;$$

moreover, each nonresonant torus[1] is densely filled by every one of its phase trajectories while the remaining resonant tori are foliated by lower-dimensional tori.

The unperturbed Hamiltonian system, $H = H_0(I)$, is said to be *nondegenerate* if and only if its frequencies are *functionally independent;* i.e., if and only if there exists $\Omega \subset \mathbb{R}^n$ such that

$$\det\left(\frac{\partial \omega_0}{\partial I}\right) = \det\left(\frac{\partial^2 H_0}{\partial I^2}\right) \neq 0\,, \quad \forall\, I \in \Omega\,. \tag{2.3.2}$$

It turns out that, in nondegenerate integrable Hamiltonian systems, the set of all nonresonant tori is a full-measure dense set, thus the set of all resonant tori will have zero measure, however, it also is dense.

The classical KAM theorem states the following,

In the case of the Hamiltonian (2.3.1), assume (2.3.2), then there exists a family of Cantor sets $\Omega_\varepsilon \subset \Omega$ with $|\Omega/\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$, such that for every $I \in \Omega_\varepsilon$, the unperturbed torus $\mathbb{T}(I)$ persists, giving rise to a slightly deformed, analytic, quasi-periodic, invariant torus $\mathbb{T}_\varepsilon(I)$ of the full perturbed system. Moreover, for each $I \in \Omega_\varepsilon$, the perturbed torus preserves the frequency $\omega(I)$ of the corresponding unperturbed one.

[1] its frequencies $\omega_0(I) = (\omega_1^0(I), \ldots, \omega_n^0(I))$ are rationally independent

Pedagogical expositions on the different approaches to prove the classical KAM theorem above can be found in [36] and [102]. Improvements on the regularity requirements of the Hamiltonian are due to Rüsmann (1970) and (in various contexts) Pöschel (1980).

Among the studies on the problem of the persistence of a fixed Diophantine torus that preserves the toral frequency are Benettin, Galgani, Giorgilli and Strelcyn (1984) who used Lie transforms and follow the Kolmogorov's approach to prove the existence of a torus with a prescribed frequency (cf. [11]). Eliasson (1996; in fact, 1988) used a so-called "direct method" in the theory of small divisors to prove the convergence of Lindstedt series and thus the existence of quasi-periodic solutions for a nearly integrable real analytic Hamiltonian. Eliasson's work would inspire those of Chierchia and Falcolini (1994), who developed a method of weighted trees to exhibit the cancellation of resonant terms (cf. [27]), and of Gallavotti (1994) who used renormalization group theory ideas and tree expansions as in the previously cited work (cf. [45]), a more detailed work is also found in [46]. Other works on the direct method using renormalization group ideas are [44] and [24].

In the degenerate case, i.e., when (2.3.1) is not satisfied, the weakest assumption on the frequencies that still yields to persistence of tori is the so-called Rüssman condition (Rüssman 1990), which says that frequencies $\{\omega(I) : I \in \Omega\}$ should not lie on any hyperplane that contains the origin (cf. [86]). Other important generalizations of the classical KAM theorem above under degenerate conditions are due to Cheng and Sun (1994) who replaced the non-degeneracy condition by a condition on the range of the frequency map (cf. [26]); Xu, You and Qiu (1995) who showed that the Rüssman condition is equivalent to a condition on the rank of a certain matrix of the frequencies and their derivatives (cf. [105]); Sevryuk (1995) who proved that it suffices that the range of the frequency map is not confined to any hyperplane in the frequency space (cf. [92]); and Chow, Li and Yi (2002) who studied the persistence problem on a given sub-manifold in the action space and established conditions of the Rüsmann type (cf. [29]).

In addition to the weakening of the hypothesis of the KAM theorem above, advances in KAM theory have been driven by the study of numerous properties of the persisted tori as well as of the dynamics around them such as the destruction of resonant tori (cf. [93]) and

the regularity of the persisted tori (cf. [104]).

2.4 Persistence of lower-dimensional tori

We now turn to the discussion of persistence of lower-dimensional invariant tori of nearly integrable systems. One of the reasons this topic is so important in dynamics is because of its applications, particularly, those related to the construction of quasi-periodic solutions in infinite-dimensional Hamiltonian systems such as discrete lattice systems and Hamiltonian PDEs; our work deals with lattice systems and thus our need to include a few lines on the lower-dimensional tori case.

To fix ideas, consider the following real-analytic Hamiltonian,

$$H(x,y,z) \;=\; h(y,z) + P(x,y,z)\,, \tag{2.4.1}$$

where $(x,y,z) \in \mathbb{T}^n \times D \times \mathbb{R}^{2m}$, $D \subset \mathbb{R}^n$. We will assume the standard symplectic structure $\sum_{i=1}^n dx_i \wedge dy_i + \sum_{i=1}^m dz_i \wedge dz_{i+m}$, thus Hamilton's equations are

$$\dot{x} = h_y + P_y\,, \quad \dot{y} = -P_x\,, \quad \dot{z} = J(h_z + P_z)\,,$$

where J is the canonical symplectic identity. Assume now that

(i) $h_z(y,0) = 0$.

(ii) $h_{zz}(y,0) \neq 0$ (non-degeneracy).

Observe that the *(i)* and *(ii)* imply that the *unperturbed* Hamiltonian, $H_0 = h(y,z)$, has an invariant sub-manifold, $x \in \mathbb{T}^n$, $y \in D$, $z = 0$, which is foliated by tori $x \in \mathbb{T}^n$, $y = y_0$, $z = 0$, and on these tori the flow is given by $x(t) = x_0 + h_y(y_0,0)t$, $y = y_0$, $z = 0$. Thinking in KAM theory, we would then expect that a certain type of positive-measure Cantor sub-manifold would persist after the perturbation term is added back, provided, of course, that this term is small in some sense. In what follows we will refer to work by other authors who proved that indeed that intuition is right (cf. [107]). To this end, we first write the Hamiltonian in a form that is convenient when applying KAM methods; namely, let us expand the term $h(y,z)$ in (2.4.1) in a neighborhood of $z = 0$,

$$H(x,y,z) \;=\; h(y,0) + \frac{1}{2}\langle h_{zz}(y,0)z, z\rangle + O(z^3) + P(x,y,z)\,. \tag{2.4.2}$$

The expansion above says that, for the unperturbed case $P = 0$, and in particular near an unperturbed torus, $y = y_0$ and $z = 0$, the local normal behavior is dictated by $h_{zz}(y_0, 0)$, provided that this matrix is non-degenerate. If we now linearize the first two terms in (2.4.2) around $y = \xi$, we will obtain

$$H \;=\; N + P = \langle \omega(\xi), y \rangle + \frac{1}{2}\langle A(\xi)z, z\rangle + \mathcal{P}\,, \tag{2.4.3}$$

where $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2m}$, $\omega(\xi) = h_y(\xi, 0)$, $A(\xi) = h_{zz}(\xi, 0)$ and $\mathcal{P} = P + O(y^2) + O(z^3) + O(yz)$ (we have omitted a constant term that does not affect the dynamics). (2.4.3) is thus a perturbed system where the unperturbed Hamiltonian has been linearized in a neighborhood of the torus $\mathbb{T}^n \times \{y = \xi \in D\} \times \{z = 0\}$; note that ξ is regarded as a parameter. The above transforms the problem of persistence of invariant tori of a fixed Hamiltonian system, into the problem of persistence of an invariant torus of a family of perturbed linear Hamiltonian systems. Many results on KAM theory start with a Hamiltonian in the form (2.4.3), this is the case of the main abstract KAM presented in chapter 4.

When all the eigenvalues of $JA(\xi)$ are simple and purely imaginary, the unperturbed torus, $y = y_0$, $z = 0$, is said to be *elliptic*, in that case there is a linear symplectic coordinate transformation that writes (2.4.3) into the form

$$H \;=\; N + \mathcal{P} = \sum_{i=1}^{n} \omega_i(\xi) y_i + \frac{1}{2}\sum_{i=1}^{m} \Omega_i(\xi)(z_i^2 + z_{i+m}^2) + \mathcal{P}\,. \tag{2.4.4}$$

The following are the so-called *Melnikov's non-resonant conditions* that guarantee the persistence of lower-dimensional tori for (2.4.4),

1. $\langle k, \omega(\xi)\rangle + \Omega_i(\xi) \neq 0$.

2. $\langle k, \omega(\xi)\rangle + \Omega_i(\xi) + \Omega_j(\xi) \neq 0$ (no multiplicity).

3. $\langle k, \omega(\xi)\rangle + \Omega_i(\xi) - \Omega_j(\xi) \neq 0$, $|k| + |i - j| \neq 0$.

For the system (2.4.4), Melnikov (1965) and Eliasson (1988) established the existence of a real analytic invariant torus whose frequencies are close to the frequencies of the unperturbed torus, provided that the vector field of the perturbation is sufficiently small and under 1-3. This result, however, is not able to prescribe the frequencies of the perturbed torus (cf. [73, 39]). An infinite dimensional version of this theorem was later proved by Kuksin (1987) and Pöschel (1989) (cf. [59, 77]).

Using the nowadays called Craig-Wayne-Bourgain (CWB) method, Bourgain (1997) proved that one can dispense with the condition on pairs of normal frequencies; i.e., condition *2.* above (cf. [16]). Similar results were obtained by You and by Xu (1999) and Yu (2001) (cf. [106, 103], respectively).

When the eigenvalues of $JA(\xi)$ are away from the imaginary axis, the unperturbed torus is called *hyperbolic*. In such case, Moser (1967, [76]) and Graff (1974, [51]) proved that, provided that the frequencies $\omega(\xi)$ are non-degenerate and satisfy a Diophantine condition, the system (2.4.3) has an invariant torus with prescribed frequencies, see also Zehnder (1975, 1976, [109]). A generalization of these works to include non-Floquet, frequency varying normal forms and allowing degeneracy of the unperturbed frequencies was done by Li and Yi (2005, cf. [67]).

The problem of persistence of lower-dimensional tori under tangential non-degeneracy, i.e., the quadratic term in (2.4.3) is of the form $\frac{1}{2}\langle A(\xi)(y,z)^\top,(y,z)^\top\rangle$ and $A(\xi)$ is nonsingular in some domain, was studied by Li and Yi (2005) (cf. [67]). More recently, Han, Li and Yi (2006) considered a Hamiltonian system of the type (2.4.3) in which the matrix components of $A(\xi)$ are allowed to be singular, this is the so-called "normal degenerate case". Using the KAM methodology these authors were able to show that, under a weak Melnikov non-resonant condition and other conditions on the perturbative term, the majority of the unperturbed tori persist in the presence of a small perturbation, in addition, the tori that persist constitute a smooth family and carry quasi-periodic motions.

The extension of KAM theory to infinite-dimensional Hamiltonian systems was originally developed simultaneously by Bellissard and Vittot (1985, cf. [100] and by Fröhlich, Spencer and Wayne (1986, cf. [43, 42]). In the infinite-dimensional case, the problem of persistence of finite-dimensional tori, and thus of the existence of quasi-periodic motions, may be formulated as follows: let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, suppose that J is an anti-selfadjoint operator on $\mathcal{H}$ where we define the two-form $\omega^2 = -\langle J^{-1}du, du\rangle(\xi,\eta) := -\langle J^{-1}\xi,\eta\rangle$, for $\xi,\eta\in\mathcal{H}$. Let $u\in\mathcal{H}$ be such that

$$\dot{u}(t) \;=\; J(Au(t)+\varepsilon\nabla H(u(t))\,,$$

where A is a self-adjoint operator. The equation above is of the form

$$\dot{u}(t) = J\nabla\mathcal{K}_\varepsilon(u(t))\,,$$

where

$$\mathcal{K}_\varepsilon \;=\; \frac{1}{2}\langle Au, u\rangle + \varepsilon H(u)\,,$$

and thus it is Hamiltonian. Assume that J and A commute and that $\mathcal{H}$ admits an orthonormal basis $\{\varphi_j^\pm : j \geq 1\}$ such that

$$A\varphi_j^\pm = \lambda_j\varphi_j^\pm\,, \quad J\varphi_j^\pm = \mp\mu_j\varphi^\mp \quad j \geq 1\,;$$

therefore, the spectrum of JA is given by

$$\mathrm{Spec}(JA) \;=\; \{\pm\mathrm{i}\nu_j : j \geq 1, \nu_j = \lambda_j\mu_j\}\,.$$

Fix $N \in \mathbb{N}$ and let

$$\mathcal{H}_0 \;=\; \mathrm{span}\{\varphi_j^\pm : 1 \leq j \leq N\}\,,$$

observe that this space is invariant under the flow defined by $\mathcal{K}_0$; moreover, $\mathcal{K}_0$ is foliated by invariant N-tori

$$\mathbb{T}^N(I) \;:=\; \{\sum_{j=1}^N x_j^\pm\varphi_j^\pm : x_{j+}^2 + x_{j-}^2 = 2I_j \;\; \forall j\}\,,$$

where $I = (I_1, \ldots, I_N\} \in \mathbb{R}_+^N$; moreover, every torus is foliated by quasi-periodic solutions, $u(t) = U(\omega t)$, $\omega = (\lambda_1, \ldots, \lambda_N)$, of $\dot{u} = J\nabla\mathcal{K}_0$. The question being addressed is, which of those unperturbed tori ($\varepsilon = 0$) persist when $\varepsilon > 0$?

KAM theory in infinite dimensions evolved from answering the question above. The framework we presented is very general and is applicable to Hamiltonian PDEs and to lattice systems; in both cases, one must overcome a small divisor problem. In the discrete lattice case, most models of physical significance possess a bounded linear spectrum, whereas in the PDE one has an unbounded spectrum; in both cases infinite multiplicities is a common occurrence.

In chapter 3 we elaborate on the background and introduction to Hamiltonian lattice systems, whereas in chapter 4 we carry out explicitly the KAM methodology to establish

the existence of quasi-periodic motions within an infinite lattice system with a physical interpretation. In both chapters the reader will find plenty of references to the literature on quasi-periodic solutions in infinite-dimensional Hamiltonian systems arising from lattice systems and their treatment using, primarily, the KAM methodology.

The 1990's witnessed the development of infinite-dimensional KAM theory for Hamiltonian PDEs with the works of Kuksin, Wayne, Craig, Bourgain and Pöschel; in particular, the method to overcome the problems caused by infinite multiplicities is attributed to Bourgain, Craig and Wayne; cf. [59, 101, 66, 33, 19, 15, 79, 65, 80, 17]. Work on the nonlinear Schrödinger, wave, beam or KdV equations using either the Kolmogorov-Arnold-Moser or the Craig-Wayne-Bourgain methods can be found in [20, 18, 28, 49, 50, 48], to cite just a few references. See also [32, 64].

CHAPTER III

BREATHERS IN HAMILTONIAN NETWORKS

Chapter Summary

In this chapter we first introduce the concept of "breather solution" by considering two PDEs of prime importance, namely, Korteweg-de Vries and sine-Gordon equations. We provide explicit solutions to these equations of the soliton and breather types and give references to earlier work to sustain the general point of view that solutions of those types are in some sense uncommon within the PDE context. We then discretize the sine-Gordon equation to obtain a Hamiltonian system of a class that is in fact very popular in the physical and biological sciences, namely, the Frenkel-Kontorova model; we illustrate this with examples in section 3.2. As opposed to their continuous PDE counterparts, discrete systems that arise from discretization of wave equations as well as their generalizations, are well known to support solutions of the breather type, that is, time-(quasi)periodic and spatially localized. This is the topic of section 3.3 where we discuss earlier work on the existence of time-periodic and quasi-periodic breathers in Hamiltonian networks of a broad class that includes those networks discussed earlier in the chapter. Section 3.3 has the intention to emphasize the importance of the main result presented and proved in chapter 4. We remind the reader that this chapter does not intend to be an account of the state of the art on breathers.

3.1 Coherent structures in PDEs and in spatially extended Hamiltonian systems

The last twenty years have witnessed an increase in the interest in the study of coherent structures in spatially extended Hamiltonian systems. In its classical interpretation, a coherent structure is an organized repeating pattern that occurs within the structure of a physical system. In a broader sense, a coherent structure is also a spatially localized disturbance of the medium whose lifetime is long enough to be considered an important characteristic of the system. The latter interpretation is the one we shall adopt in this work. By spatially localized we mean that the extent of the perturbation is small compared

to the size of the medium. Examples of coherent structures abound, for instance, in the study of turbulent flows in incompressible Navier-Stokes equations or, more generally, in nonlinear PDEs. In what follows we will present two important examples of such structures.

One of the simplest and most studied examples of coherent structures are *solitons*, which arise as travelling wave solutions of integrable PDEs such as Korteweg-de Vries (KdV) or sine-Gordon (sG) equations, which we will discuss briefly below. Solitons are solutions of a particular type which reflects important symmetry properties of the structure of their mother PDEs; for example, the form in which they depend on the independent variables may be invariant under particular Lie group transformations; solutions with this property are called *similarity solutions* (cf. [38], p 6). For instance, the KdV equation appears in many problems of physical relevance, such as the formation of surface waves in water (cf [40], p 174) or in the long-distance transmission of signals over optical fibers. So if we let $u(x,t)$ represent the water surface profile at position x at time t, then u evolves according to the following model,

$$u_t + 6uu_x + u_{xxx} = 0\,, \tag{3.1.1}$$

for $(x,t) \in \mathbb{R} \times (0,\infty)$. The term u_{xxx} makes this equation be of the dispersive type, to understand this term, let us ignore the nonlinearity so the equation becomes Airy's equation, $u_t + u_{xxx} = 0$. The solution to Airy's equation can be written down explicitly using a Fourier transform, taking an initial condition $u(x,0) = f(x)$ and homogeneous conditions at infinity, $u,\ u_x,\ u_{xx} \to 0$ as $|x| \to \infty$ (cf. [38] p. 18). But we are more interested in the elementary observation that this linear PDE also admits complex-valued travelling-wave solution components of the form $\exp(\mathrm{i}(kx - \omega t))$ so long as the frequency ω and wave number $k \in \mathbb{Z}$ are related by $\omega(k) = k^3$. The nonlinear dependence of the frequency on the wave number implies that solutions which originate at time $t = 0$, $\exp(\mathrm{i}kx)$, with different wave numbers will propagate at different speeds, such phenomenon is known as *dispersion* and the form in which the frequency and the wave number depend on each other is known as the *dispersive relation*. The KdV equation is remarkable in the sense that it is completely integrable for practically any initial data (cf. [38] pp. 18, 31, 32, 71); moreover, like Airy's equation, it too admits (real-valued) solutions of the travelling-wave

type, $u(x,t) = v(kx - \omega t)$ for any k and $\omega \in \mathbb{R}$. Indeed, substituting the previous form of u in (3.1.1) one obtains the following ODE in v,

$$-\omega v' + 6kvv' + k^3 v''' \,=\, 0\,,$$

where $(\cdot)' = d/ds$, $s = kx - \omega t$. This equation can be integrated once to get

$$-\omega v + 3kv^2 + k^3 v'' \,=\, a\,.$$

Multiplying by v' and integrating once more yields

$$-\frac{\omega}{2}v^2 + kv^3 + \frac{k^3}{2}v'^{\,2} - av \,=\, b\,.$$

A traveling wave that satisfies that v, v', v'' vanish as $s \to \infty$ is called a *solitary wave* thus, imposing solitary wave conditions on u we get $a = b = 0$ and so

$$-\frac{\omega}{2}v^2 + kv^3 + \frac{k^3}{2}v'^{\,2} \,=\, 0\,.$$

Solving for v' and integrating once more we get

$$-\int \frac{k\,dv}{v\sqrt{\frac{\omega}{k} - 2v}} \,=\, s\,,$$

where the minus sign is adopted for convenience. Changing variables to $v = \frac{\omega}{2k}\operatorname{sech}^2\theta$ the integral above results in

$$s = \frac{2}{\sqrt{\omega}}k^{\frac{3}{2}}\theta + c\,,$$

where c is constant, thus

$$v \,=\, \frac{\omega}{2k}\operatorname{sech}^2\left(\frac{\sqrt{\omega}}{2k^{\frac{3}{2}}}(s-c)\right)$$

and changing back to original coordinates we obtain the following explicit expression for the solitary wave,

$$u(x,t) \,=\, 2^{-1}\frac{\omega}{k}\operatorname{sech}^2\left[2^{-1}\sqrt{\frac{\omega}{k}}\left(x - \frac{\omega}{k}t - c\right)\right].$$

Let us now consider the so-called modified KdV (mKdV) equation,

$$u_t + 6u^2u_x + u_{xxx} \,=\, 0 \tag{3.1.2}$$

in $\mathbb{R} \times (0,\infty)$, which has applications to nonlinear long-standing waves whenever there is polarity symmetry, this is the case in electrodynamics and in stratified films (cf. [31] and

references therein). Like the original KdV equation, the mKdV equation is completely integrable and admits traveling-wave soliton solutions. Indeed, the same type of integration carried out above yields in this case

$$u_{sol}(x,t) \;=\; \sqrt{\frac{\omega}{2}}\mathrm{sech}\left[\sqrt{\frac{\omega}{k}}(x-\frac{\omega}{k}t+c)\right]. \tag{3.1.3}$$

But the mKdV equation also supports a different type of solution (cf. [31]), namely,

$$u_{br}(x,t) \;=\; -4\alpha\,(\mathrm{sech}\,\theta)\left(\frac{\cos\phi+(\alpha/\beta)\sin\phi\,\tanh\theta}{1+(\alpha/\beta)^2\sin^2\phi\,\mathrm{sech}^2\theta}\right), \tag{3.1.4}$$

where $\theta=-2\beta x-8\beta(\beta^2-3\alpha^2)t+\theta_0$, $\phi=2\alpha x+8\alpha(3\beta^2-\alpha^2)t+\phi_0$ and α, β, θ_0 and ϕ_0 are real-valued parameters; this type of solution can be derived using, for example, the inverse scattering method (cf. [31] or [38]). A 3d projection of (3.1.3) and (3.1.4) is included in figure 2.

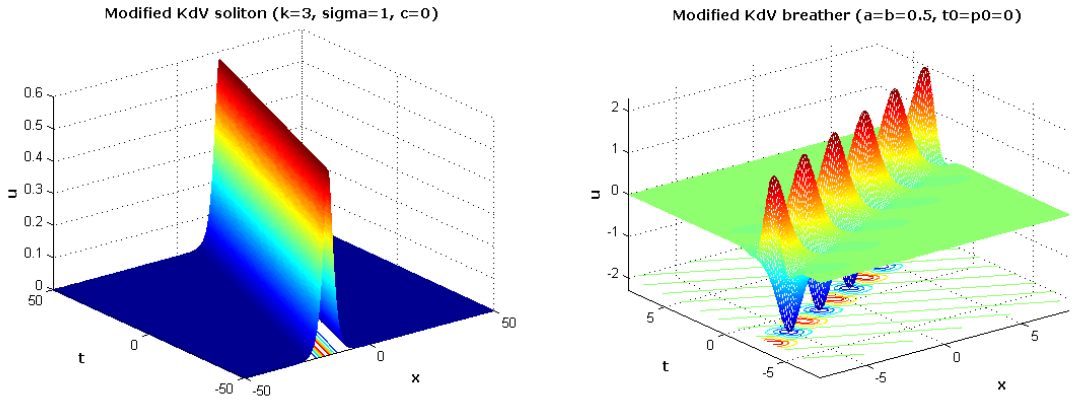

Figure 2: *Solutions to the modified KdV equation. 1a (left) soliton ($k=3$, $\sigma=1$, $c=0$), 1b (right) breather ($\alpha=\beta=0.5$, $t_0=p_0=0$).*

As (3.1.3) shows, solitons propagate without altering their shape; in contrast, the type of solutions represented by (3.1.4) do change their shape as they propagate throughout space, more precisely, in some appropriate moving reference frame these solutions will appear spatially localized like solitons, but unlike solitons they are time periodic or time quasi-periodic, a term that will be define more precisely later on. Solutions with the characteristics of spatial localization and time (quasi-)periodicity are named *breathers*.

The concept of breather was introduced in [1] (the authors also use the term "soliton

states") in connection to the sine-Gordon (sG) equation,

$$u_{tt} - u_{xx} + \sin(u) \;=\; 0\,, \tag{3.1.5}$$

which has applications in the study of transmission lines made out of Josephson junctions (see next section). Like the mKdV equation, the sG equation too admits a (kink-antikink) traveling-wave soliton solution

$$u_{kak} \;=\; 4\tan^{-1}\left(\frac{\sinh\frac{\nu t}{\sqrt{1-\nu^2}}}{\nu\cosh\frac{x}{\sqrt{1-\nu^2}}}\right)$$

and a (non-propagating) breather solution

$$u_{br} \;=\; 4\tan^{-1}\left(\frac{\varepsilon\sin\frac{t}{\sqrt{1+\varepsilon^2}}}{\cosh\frac{\varepsilon x}{\sqrt{1+\varepsilon^2}}}\right) \tag{3.1.6}$$

(cf. figure 3). Breather solutions had been known since before the works of [1, 2] showed that the mKdV, KdV and sG are among certain evolution equations which can be solved by the inverse scattering method (cf. [2] and references therein). In contrast to other methods used to solve these equations such as the Bäcklund-Darboux transform, inverse scattering is insightful in that it can be used to show that soliton and breather solutions can be associated to specific eigenvalue types; for example, [31] shows that in the case of the mKdV equation, breathers are obtained from complex-eigenvalue quartets whereas solitons can come from real-eigenvalue pairs; moreover, it is possible to obtain one type of solution from the other through bifurcation.

In contrast with the above, breather solutions in PDEs are exceptional. For instance, within the class of the Klein-Gordon (KG) equation ($u_{xx} - u_{tt} = g(u)$, $g(0) = 0$, $g'(0) > 0$), Segur and Kruskal ([91, 90]) (see also [56, 25]) considered the so-called ϕ^4 model,

$$u_{xx} - u_{tt} = 2u - 3u^2 + u^3\,, \tag{3.1.7}$$

and showed (arguably not rigorously as they could have missed a Cantor-like set of frequencies) that, in the limit of small amplitude and frequency just under that of spatially uniform infinitesimal oscillations ($\omega < \sqrt{2}$), even though it is possible to obtain a Fourier representation for any real-valued, smooth, time-periodic solution to (3.1.7), the asymptotic Fourier expansion does not correspond to a breather solution (spatially localized and time-periodic),

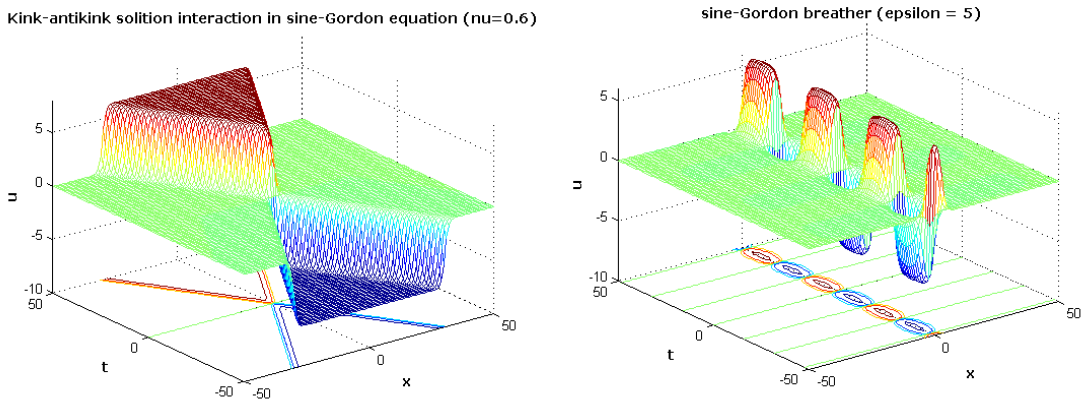

Figure 3: *Solutions to the sine-Gordon equations. 2a (left) Kink-antikink soliton solutions ($\nu = 0.6$). 2b (right) breather solutions ($\varepsilon = 5$).*

it does however represent approximate breathers to all orders. In the same vein, Sigal (cf. [94]) showed that periodic and quasi-periodc solutions of linear wave equations are unstable under generic nonlinear perturbations, he also showed that the majority of periodic and quasi-periodic solutions of nonlinear PDEs of the Schrödinger type are unstable as well. In relation to the sG equation, Denzler (cf. [37]) showed that, except for a one-dimensional linear space of perturbation functions, the family of breathers described by (3.1.6) does not persist under analytic perturbations up to first order of the perturbation parameter; Denzler's result is an extension of earlier work by Birnir, McKean and Weinstein (cf. [14]). In the same vein, a numerical study by Birnir (cf. [13]) showed that most perturbations of the sG equation will cause breathers to radiate, blow up and split into kink-antikink pairs. The sG equation is *practically* the only equation of its kind which will sustain non-radiating breathers (cf. [23] p. 29); nevertheless, one can still find in the literature oscillatory solutions that are long-lived and spatially localized and which some authors will refer to as "breather modes" or "breather-like" solutions.

KdV, mKdV and sG equations are examples of nonlinear Hamiltonian PDEs, that is, equations which (provided suitable boundary conditions are supplied) with the appropriate choice of symplectic Hilbert scale (cf. [64] p.15) can be written in the form $\partial_t \xi = J(\xi)\nabla h(\xi, t) =: V_h(\xi, t)$, where $J = (-\bar{J})^{-1}$ and $\bar{J}$ is an anti-self-adjoint operator depending smoothly on ξ; h is the C^1-smooth Hamiltonian function and V_h is a, say,

C^1-smooth, Lipschitz continuous vector field so the Hamiltonian ODE problem is well-posed (i.e., existence and uniqueness of solutions which depend smoothly on the initial condition is guaranteed). For instance, the Hamiltonian structure of the sG equation on the unit circle ($x \in S = \mathbb{R}/2\pi\mathbb{Z}$) becomes evident if one defines $-v = \partial_t u$, so that $\partial_t v = -u_{xx} + \sin(u)$, $\xi = (u, v)$, $J(u, v) = (-v, u)$ (thus $\bar{J} = J$) and $h = \frac{1}{2}\langle A\xi, \xi\rangle + H(\xi)$ where $A(u, v) = (-u_{xx}, v)$, $H(u, v) = -\int \cos u(x)\, dx$, thus $\nabla H(u, v) = (\sin u, 0)$, and the resulting Hamiltonian equations are $\partial_t \xi = J\nabla h = J(A\xi + \nabla H(\xi))$.

Whereas the works by Sigal and Denzler cited above indicate that in fact breathers are rare and non-robust solutions of nonlinear Hamiltonian PDEs, this is not quite the case in spatially discretized versions of those same PDEs. Consider for instance the following naïve semidiscretization of the sG equation where the spatial partial derivative is simply substituted by its three-point central difference approximation via the identifications

$$\begin{aligned} u(x_n, t) &\longrightarrow u_n(t)\,, \\ u_{xx}(x_n, t) &\longrightarrow \frac{u(x_{n+1}, t) - 2u(x_n, t) + u(x_{n-1}, t)}{h^2}\,. \end{aligned}$$

In that case one can readily see that (3.1.5) becomes the following nonlinear system of second-order coupled equations,

$$\ddot{u}_n - \frac{1}{h^2}(u_{n+1} - 2u_n + u_{n-1}) + \sin(u_n) \;=\; 0\,, \quad n \in \mathbb{Z}\,,$$

which we will prefer to rewrite as

$$\ddot{u}_n + V'(u_n) = \varepsilon\Delta_n(\{u\})\,, \quad n \in \mathbb{Z}\,, \tag{3.1.8}$$

where $V(x) = 1 - \cos(x)$, $\Delta_n(\{u\}) = -\partial W_{sG}(\{u\})/\partial u_n$, $W_{sG}(\{u\}) = \frac{1}{2}\sum_k (u_{k+1} - u_k)^2$ and $\varepsilon = \frac{1}{h^2}$. It is not difficult to see that, if we now define conjugate variables, $q_n = u_n$ and $p_n = \dot{q}_n$ (3.1.8) becomes a Hamiltonian system

$$\dot{q}_n \;=\; \frac{\partial H}{\partial p_n}\,, \quad \dot{p}_n \;=\; -\frac{\partial H}{\partial q_n}\,, \quad n \in \mathbb{Z}\,, \tag{3.1.9}$$

with formally defined Hamiltonian function

$$H = \sum_n (\frac{1}{2}p_n^2 + V(q_n)) + \varepsilon W_{sG}(\{q\})\,. \tag{3.1.10}$$

(3.1.10), or its associated system of equations (3.1.9) is an example of what we will call a one-dimensional (1d) *Hamiltonian lattice system* or *Hamiltonian network* (also *chain*). V is the so called *on-site substrate potential* or *self-interaction term,* its evaluation at q_n can be thought of as the contribution to the dynamics of the nth lattice point arising from its interaction with the medium. ε is the *coupling strength parameter,* setting $\varepsilon = 0$ uncouples the system, when $\varepsilon >> 1$ the chain is said to be *rigid* in which case the interaction among lattice points is stronger than their interaction with the substrate. W is the *coupling potential* and models the interaction among the lattice points, the particular form of this potential makes the coupling of the *nearest-neighbor* type, that is, the only lattice points that affect the dynamics of a particular point are those adjacent to it. Let us here add that the case when $\varepsilon = 0$ and $V(x) = \frac{1}{2}x^2$ for all n, is referred to as an uncoupled system of identical *harmonic* oscillators.

(3.1.10) is a Hamiltonian network of identical *anharmonic* oscillators with nearest-neighbor coupling, we will refer to it simply as the discrete sG system. Unlike its continuous analog, the sG equation, the discrete sG system is not exactly integrable and for this reason, most of the work done on this system is numerical. For example, a simple method to solve numerically the discrete sG system to find a solution of the soliton type can be found in [30]; this work uses a Runge-Kutta method to fourth order with initial conditions derived from a kink solution to the continuous case, using mid-point interpolation for the position and speed of the kink. The interplay between Hamiltonian PDEs and discrete Hamiltonian system is, by many reasons, an interesting subject in its own right but we will not discuss it here, we will simply limit ourselves to say that the passage from a discrete system to a PDE can be done in basically two ways. One way is the sometimes called *continuum limit approximation* and consists in simply reversing the naïve semidiscretization substitution mentioned above by replacing, modulo a spatial re-scaling, a nearest neighbor interaction term by a spatial second-order partial derivative thus, for example, (3.1.8) becomes the integrable sG equation (3.1.5). However, the semi-discretization method, although historically the first one to be used, does not take into account lattice-discreteness effects and sometimes leads to ill-posed nonlinear PDEs whose developing of singularities in finite time

prevents the formation of coherent structures. The second way to obtain a PDE from a lattice system was originally developed by P. Rosenau for the Fermi-Pasta-Ulam system (a 1d Hamiltonian network in the absence of a substrate potential) and the FK model with a generalized type of nearest neighbor interaction (cf. [82]). Rosenau's method is developed further in [84, 83] and it is nowadays known as the *quasi-continuum limit approximation*; this method takes into account lattice-discreteness effects by including a regularization step that consists in an expansion in the lattice spacing followed by the identification and inversion of an operator of the Schrödinger type, this inversion creates dispersive terms which compensate for the steepening of solutions caused by nonlinearity, thus making it possible for localized modes such as breathers or kinks to arise. The quasi-continuum limit, however, requires the assumption of a lattice-interaction term with a predominant linear part. The nonlinearity vs. dispersion mechanism responsible for the ability of a system to sustain localized modes such as solitons or breathers was identified once again years later when exact results on the existence of breathers in lattice systems emerged.

Despite of the fact that exact integrability is not a generic property of Hamiltonian networks, it is still possible to establish the existence of special solutions. For example, MacKay and Aubry (cf. [70]) proved the existence of time-periodic breather solutions for a large class of time-reversible Hamiltonian networks of which (3.1.10) is but a particular case, other works on existence of solutions for Hamiltonian networks of a more general type than the discrete sG system will be discussed in section 3.3. Other nonlinear PDEs whose discretization lead to Hamiltonian lattice systems are the nonlinear Klein-Gordon (NLKG) equation, $\frac{1}{c^2}\psi_{tt} - \nabla^2\psi + V'(\psi) = 0$, which arises as a generalization of the relativistic equation of a free charged particle in an electromagnetic field, and nonlinear Schrödinger (NLS) equations, $iA_t + A_{xx} \pm A|A|^2$, which govern the modulation of a weakly nonlinear wave packet in a moving medium, which are frequently used in nonlinear optics.

One may think system (3.1.8) is artificial in the sense that it came from the discretization of a PDE as opposed to being a mathematical model of a problem from the natural sciences, but, as we will see in the next section, the Hamiltonian network (3.1.10) pertains to a broad class of Hamiltonian networks that is commonly found in many problems in Physics and in

Biology.

3.2 Examples of Hamiltonian networks in Physics and in Biology

3.2.1 The Frenkel-Kontorova model

In 1938 Ya. Frenkel and T. Kontorova proposed a mathematical model to describe the dynamics of a crystal lattice in the vicinity of a dislocation core (cf. 3. in [21]), similar work by L. Prandtl and U. Dehlinger dates back to ten and nine years earlier, respectively (cf. 1. and 2. in [21]). By crystal lattice we shall understand a three-dimensional (3d) array of atoms and by dislocation we will understand a lower-dimensional array of atoms of a different nature, inserted within the 3d array. The success of the now celebrated Frenkel-Kontorova (FK) model can never be over-emphasized, being a one-dimensional model is simple and yet its proven universality extends beyond the realm of crystal lattice systems, it also set the cannon to model more complex systems.

In simple terms, the FK model considers an alignment, which we will from here on refer to as chain, of atoms having the same mass, called *effective atoms*, that are constrained to move in the direction of the chain, thus it is a one-dimensional model. In equilibrium, the effective atoms are assumed to lie equally spaced at a distance a_0 from their nearest neighbors, we will call this quantity the *lattice spacing*. The chain is immersed in a medium or *substrate* that interacts with the chain via a periodic on-site potential with period a_s, thus there are two different competing length scales in the FK model. We say that the system is *incommensurate* whenever a_0 and a_s are rationally independent, in such case, whereas the interatomic interaction favors an equidistant configuration, the interaction with the substrate favors regular spacing. In order to derive the equations of motion of the array, we first write down the total kinetic and potential energies as the sum of the individual energies of the atoms in the lattice thus, if we let x_n denote the position of the nth atom with respect to some fixed origin along the chain, the total kinetic energy of the system is

$$K = \sum_{n\in\mathbb{Z}} \frac{m_a}{2}\left(\frac{dx_n}{dt}\right)^2,$$

where m_a is the atomic mass (same for every n). The total potential energy is split into two different types, the potential energy that arises from the interaction between the substrate

and the atoms in the chain, V, and the potential energy that arises from the interaction among the atoms themselves, W. The classical FK model assumes that, for every n, the only atoms within the chain that affect the motion of the nth atom are its immediate neighbors, $n \pm 1$; moreover, the influence that immediate neighbors exert upon one another depends solely on the difference between their relative distance and their relative distance in equilibrium (a_0); furthermore, since equilibrium corresponds to a state of minimum energy and nearest-neighbor interactions are assumed to be the same for each pair, it is reasonable to assume that $W = W_{FK}$ where

$$W_{FK} \;=\; \sum_n W_{int}(x_{n+1} - x_n - a_0) \;=\; \sum_n \frac{g}{2}(x_{n+1} - x_n - a_0)^2\,,$$

g is the so-called *elastic constant* of the medium; note that $x_m > x_n$ whenever $m > n$. As for the interaction between the substrate and the chain, this is assumed to be a_s-periodic and the same for every atom, that is, $V = \sum_n V_{sub}(x_n)$, where $V_{sub}(x + a_s) = V_{sub}(x)$ for all x. In the classical FK model, atoms in the chain interact with the substrate via a sinusoidal potential, this corresponds to expanding V_{sub} into its Fourier series and keeping only the first harmonic, that is, $V_{sub}(x) \to (1 - \cos(\frac{2\pi}{a_s}x))$ (the one being added for physical significance), thus $V = V_{FK}$ where

$$V \;=\; V_{FK} \;=\; \sum_n \frac{\varepsilon_s}{2}(1 - \cos(\frac{2\pi}{a_s}x_n))\,,$$

ε_s is the so-called potential amplitude. Now, following the approach of classical mechanics, the equations of motion are $m_a\ddot{x} = -\nabla(V + W)$, that is

$$m_a\frac{d^2x_n}{dt^2} \;=\; g(x_{n+1} - 2x_n + x_{n-1}) - \frac{\varepsilon_s}{2}\frac{2\pi}{a_s}\sin(\frac{2\pi}{a_s}x_n)\,.$$

It is common to write the equations above in dimensionless variables via the substitutions $t \to (\frac{2\pi}{a_s})(\frac{\varepsilon_s}{2m_a})^{1/2}t$, $x_n \to \frac{2\pi}{a_s}x_n$, $a_0 \to \frac{2\pi}{a_s}a_0$, $g \to (\frac{a_s}{2\pi})^2 g(\frac{2}{\varepsilon_s})$, after which they become

$$\frac{d^2x_n}{dt^2} - g(x_{n+1} - 2x_n + x_{n-1}) + \sin(x_n) \;=\; 0\,, \quad n \in \mathbb{Z}\,, \tag{3.2.1}$$

this is the same type of system we encountered before in the semidiscretization of the sG equation, also note that this system is independent of the lattice spacing.

System (3.2.1) is the culmination of the classical FK model, many systems have been successfully modeled by it. In its original setting, the FK model was used to describe the dynamics around a dislocation core within a metal; dislocations are responsible for most mechanical properties of solids, the FK model was among the first in explaining a macroscopic property at an atomic level. A typical example of a dislocation is the one-edge dislocation, formed when a mono-atomic semi-infinite plane (the layer) is inserted into a perfect crystal lattice; in this case, the classical FK model has been used to describe the dynamics of the dislocation plane itself as well as the dynamics of the atoms in the direction perpendicular to the dislocation (cf. figures 4 and 5). Other settings in which the classical FK model has been used is that of atomic surfaces such as the hydrogen-bond chains on a deformable oxygen lattice (substrate) whose distortions tend to lower the activation barrier of protons and thus promote their motion along the chain; or adsorbed atomic layers. In the former case the effective atoms are "light" atoms forming a lower-dimensional subsystem of a larger lattice whose remaining part plays the role of the substrate and is made out of "heavier" atoms which are assumed fixed or their motion neglible; in the second case the effective atoms, called *adatoms*, form a thin layer that is deposited on a crystal surface made out of atoms which can only vibrate around their equilibrium positions and are thus considered fixed.

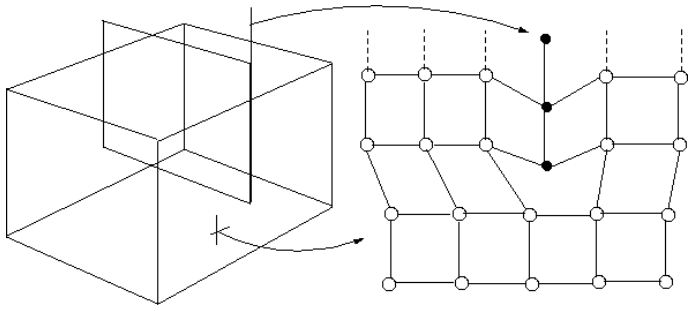

Figure 4: *Representation of a transversal view at a one-edge dislocation, in this case the FK model can be used to accurately describe the dynamics along the dislocation line as well as the dynamics in the direction perpendicular to it.*

In the remaining of this section we will discuss some features of the dynamics of special solutions of the FK system (3.2.1) or its approximations. Let us define q variables as the displacement of the atoms from their equilibrium positions, that is, $x_n = na_s + q_n$, $n \in \mathbb{Z}$.

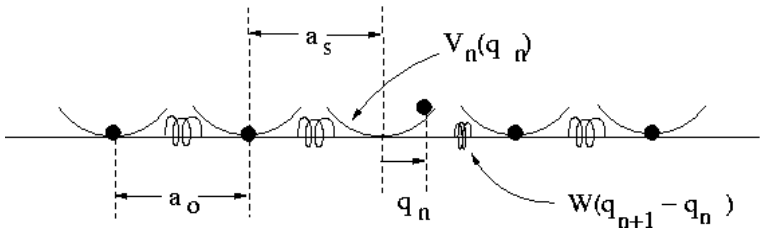

Figure 5: *The classical FK model in the commensurate case ($a_0 = a_s$), at equilibrium each adatom sits at a local minimum of the on-site potential, with only one atom per bottom well.*

Assume now that $a_s = 2\pi$, then $\sin(x_n) = \sin(q_n)$ and the linearization of (3.2.1) around its equilibrium ($q = 0$) is given by

$$\ddot{q}_n + q_n - g(q_{n+1} - 2q_n + q_{n-1}) = 0\,, \quad n \in \mathbb{Z}\,;$$

in the case of an infinite chain or a finite chain with periodic boundary conditions the system above admits solutions of the form

$$q_n(t) \propto e^{\mathrm{i}(\omega_{ph} t - \kappa n)}\,, \quad n \in \mathbb{Z} \tag{3.2.2}$$

provided

$$\omega_{ph}^2 = 1 + 2g(1 - \cos\kappa)\,. \tag{3.2.3}$$

(3.2.2) are the so-called linear excitation modes of the system and receive the name of *phonons*, (3.2.3) is the dispersion relation and κ is the dimensionless *wave number*, $|\kappa| < \pi$. The set

$$\mathcal{S} := \{\omega_{ph} : \omega_{ph}^2 = 1 + 2g(1 - \cos\kappa), \quad k \in \mathbb{Z}\} \tag{3.2.4}$$

is the *frequency spectrum* of the FK chain, one can see it consists of a finite band with a gap $(-1, 1)$ and *cut-off frequency* $\omega_{max} = \omega(\pi) = \sqrt{1 + 4g}$. The frequency spectrum plays an important role in the existence of breather solutions for systems of the FK type; this is intuitively clear if one thinks of a periodic or quasi-periodic solution in terms of its Fourier expansion. If the frequency of a Fourier mode lies within the frequency spectrum $\mathcal{S}$ resonance takes place and thus breather solutions are not sustained.

When the amplitudes of the q variables are not small the linear approximation is no longer valid and one must find other ways of studying the dynamics of the chain. A common

approach is that of considering the continuum or quasi-continuum limit approximations mentioned in the previous section, in which the system of equations (3.2.1) is replaced by a single PDE.

Just like in the case of the integrable sG equation, the FK model accepts kink solutions. A *FK kink* (*FK anti-kink*) corresponds to a localized compression (expansion) of the initially commensurate lattice's ground state, that is, kinks are born to an excess of atoms in the lattice and are associated a *topological charge* of $\sigma = 1$; similarly, anti-kinks are associated with a decrease in the number of atoms per unit length and have a topological charge of $\sigma = -1$. When studying their dynamics, both kinks and anti-kinks can be considered (approximately) as virtual particles. For instance, a kink solution of (3.2.1) is of the form $u_n = f(na_s - X)$, where f is a function describing the kink's shape, in the continuum limit approximation ($g \gg 1$) f coincides with a soliton solution to the integrable sG equation. X is the coordinate of the kink's center and in the case of a finite chain it is given by $\sigma \sum_n u_n + C$, where C is a constant that one can choose so that $X = 0$ corresponds to a minimum energy configuration. Finding the explicit form of the kink's shape function, f, can be done in two different ways: one way uses the quasi-continuum limit method of Rosenau and perturbation theory (cf [23], sec. 2.3) and the other one uses a Hamiltonian formalism (cf. [23], sec. 2.4.3, also Willis et al (1986) and Boesch et al (1988)). A characteristic of kinks in lattice systems that is exclusive to discreteness is the existence of the so-called Peierls-Nabarro (PN) potential which affects their dynamics, in particular, when the kink's energy is above that of the PN energy, the kink will "break out" and move along the chain with a varying (oscillating) speed. In general, steady-state solutions of a lattice system such as kinks are not found by themselves, more precisely, a moving kink will lose its kinetic energy by radiating linear phonon modes until its energy decreases below the PN energy at which point it becomes trapped again and continues to radiate phonon modes until it turns into a stationary state associated to a static configuration.

The numerical analysis of breather solutions, including their stability, is handled in the same way as in the case of kink solutions. To fix ideas, consider the following discrete NLS

equation,

$$\ddot{u}_n - g(u_{n+1} - 2u_n + u_{n-1}) + \omega_0^2 u_n + \alpha u_n^2 + \beta u_n^3 \,, \quad n \in \mathbb{Z} \,,$$

where α and β are parameters of the model. Letting as before a_s stand for the lattice spacing, linearizing the equation above and substituting a plane-wave solutions one obtains a dispersion relation,

$$\omega^2 \, = \, \omega_0^2 + 4g \sin^2 \frac{ka_s}{2} \,,$$

which indicates finite-gap spectrum with cut-off frequency $\omega_{max} = \sqrt{\omega_0^2 + 4g}$. Assuming $\omega_0^2 \gg g$ (weak coupling), one may look for slow-temporal approximate solutions of the following form,

$$u_n \, = \, \phi_n + \psi_n e^{-\mathrm{i}\omega_0 t} + \psi_n^* e^{\mathrm{i}\omega_0 t} + \xi_n e^{-2\mathrm{i}\omega_0 t} + \xi_n^* e^{2\mathrm{i}\omega_0 t} \, + \, \ldots$$

under the assumptions $\phi_n \sim \varepsilon^2$, $\xi_n \sim \varepsilon^2$, $\psi \sim \varepsilon$, $g \sim \varepsilon^2$, $\omega_0^2 \sim 1$, $\alpha \sim 1$, $\beta \sim 1$, $1/t \sim \varepsilon^2$. The ellipsis mean that higher harmonics will be ignored. Substituting the above solution into the system equations and keeping only lowest order terms in ε one obtains,

$$2\mathrm{i}\omega_0 \dot{\psi}_n + g(\psi_{n+1} - 2\psi_n + \psi_{n-1}) - 2\alpha(\phi_n \psi_n + \psi_n^* \xi_n) - 3\beta |\psi_n|^2 \psi_n \, = \, 0 \,, \quad n \in \mathbb{Z} \,,$$

together with the algebraic relations,

$$\phi_n \approx -\frac{2\alpha}{\omega_0^2} |\psi_n|^2 \quad \xi_n \approx \frac{\alpha}{3\omega_0^2} \psi_n^2 \,,$$

thus, to lowest order terms, the system of equations is,

$$\mathrm{i}\dot{\psi}_n + K(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \lambda |\psi_n|^2 \psi_n \, = \, 0 \,, \quad n \in \mathbb{Z} \,,$$

where, $K = g/2\omega_0$ and $\lambda = -(3\beta - 10\alpha^2/3\omega_0^2)/2\omega_0$. The nonlinear discrete system above is the one used in practice to study localized mode solutions of the original system; this same approach is followed in the study of other nonlinear discrete systems. The approximate equation last obtained accepts exact wave solutions $\psi_n(t) = \psi_0 e^{\mathrm{i}\theta_n}$, for $\theta_n = ka_s n - \omega t$ and $\omega = 4K \sin^2(ka_s/2) - \lambda \psi_0^2$. The linear stability analysis of these waves depends on k (cf. [23] pp. 74-75).

It is perhaps not difficult to realize the limitations of the classical FK model. For instance, the nearest-neighbor form of the coupling potential comes from the elastic properties

of the system, rather than from the electric interaction among the atoms. From a more subtle perspective, one may be interested in modeling the dynamics of multi-layers of adatoms deposited on a substrate of heavier atoms, in such case the multi-layer may be modeled as 1d chains of adatoms in parallel that interact with one another via weakly coupling, forming what it is known in the physics literature as quasi-1d chains. An example of a quasi-2d model of coupled adatomic chains can be found in [22], where the dynamics of kinks is considered on a system of coupled FK rigid chains. In [22] a kink is a soliton (also called topological soliton) which describes the transport of mass along a chain due to an excess of adatoms in a commensurate lattice; the rigidity of the lattice allows for a continuum approximation that transforms the system of chains into a sG system (cf. figure 6). We will come back to the form of the coupling potential when we discuss the system model studied in the next chapter.

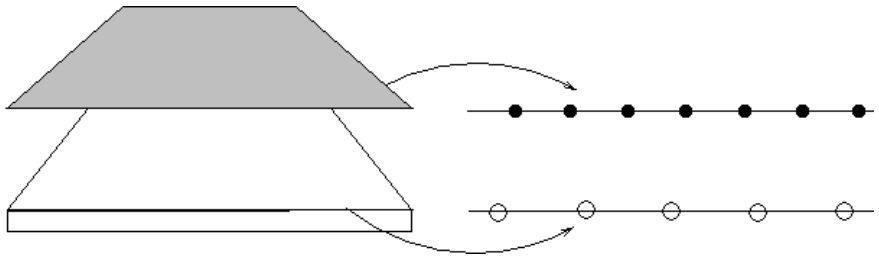

Figure 6: *A quasi-2d model is obtained when a double-layer lattice is adsorbed into a substrate of heavier atoms; the dynamics of the double layer can be modeled by coupling two FK chains where the coupling among adatoms is stronger than the effect that the substrate exerts on them.*

3.2.2 Micromechanical cantilever arrays

A typical michromechanical cantilever array consists of cantilevers of different lengths connected to one another by an overhang, forming a comb-like structure. Cantilevers are made of a thin film a few hundreds of nanometers thick and they may be of two different fixed lengths of the order of micrometers. For example, [87] describes a cantilever array made out of a Si_3N_4 film, $300nm$ thick; the cantilevers are of two different lenghts, $55\mu m$ and $50\mu m$, with a width and pitch of 15 and $40\mu m$, respectively. The experimental creation, detection and even manipulation of ILMs in this type of arrays has been reported in many

places before, this brief section is based on [87, 71, 88]. The type of ILMs produced in cantilever arrays behave like breathers for a relatively long period of time, they appear as a single sustained oscillation which is localized in the sense that it usually involves just a small number of elements in the array; such oscillations can be photographed and their characteristics measured. [87] discusses the case of a cantilever array bonded to a piezoelectric which acts as the driver of the system. At the beginning of the experiment the piezoelectric produces chirping which can be modeled as white noise, the chirping continues during a period of time in which several ILMs are created, then the piezoelectric enters a fixed frequency mode where ILMs which are in synchrony with the piezoelectric are fed their amplitudes which results in their locking, whereas asynchronous ILMs move across the array and eventually disappear. In order to observe ILMs, one focuses a laser beam on the array, the reflected beam is then captured by a camera which is not fast enough to measure the vibrations in the system, however, when several ILMs are locked, the reflected beam misses the camera which will then register a darker region at the site of the cantilever that supports the ILM. When the piezoelectric quits, those locked ILMs die out without hoping to other lattice sites. This relatively straightforward method to detect ILMs also serves to observe their interaction with one another; for example, once the piezoelectric starts vibrating at a constant frequency, those ILMs asynchronous with it may bounce on the locked ILMs, and when the piezoelectric quits, neighboring locked ILMs may repel each other an disperse.

In order to derive the dynamic equations of a cantilever array of two alternating length sizes, say a and b, let x_{ai} be the displacement of the free end of the ith cantilever from its position at equilibrium, then one may think of x_{ai} as the displacement of a point with mass m_a. The term to model the coupling among cantilevers varies and depends on the particular array one is considering. For the systems discussed in [87], their authors prosed that the coupling among cantilevers be of the nearest-neighbor type and harmonic, while the on-site interaction term is assumed to be anharmonic and produced by a fourth-order

term; therefore,

$$m_a\ddot{x}_{ai} + \frac{m_a}{\tau}\dot{x}_{ai} + k_{2a}x_{ai} + k_{4a}x_{ai}^3 + k(2x_{ai} - x_{bi} - x_{bi-1}) = m_a\alpha\,,$$
$$m_b\ddot{x}_{bi} + \frac{m_b}{\tau}\dot{x}_{bi} + k_{2b}x_i + k_{4b}x_{bi}^3 + k(2x_{bi} - x_{ai+1} - x_{ai}) = m_b\alpha\,,$$

for $i \in \mathbb{Z}$ and where α is a common acceleration term for each cantilever provided by the piezoelectric. τ is the energy lifetime. The fact that nearly identical experimental conditions produce the same type of excitations, suggests that the modes produced in this experiment are not due to impurities in the system.

The model above was revisited in [71] in the case of cantilevers of the same length,

$$\ddot{x}_i + \gamma\dot{x}_i + k_2x_i + k_4x_i^3 - k(x_{i+1} - 2x_i + x_{i-1}) = A(t)\,, \quad n \in \mathbb{Z}\,.$$

The authors used a method called nonlinear response manifold (NLRM) to develop a systematic way to find parameter regimes within which the system can develop ILMs. A simple example of the NLRM technique consists in driving the system with a periodic forcing such as $A(t) = A_0\cos(\omega_d t)$ and then using a Newton method to determine the dependence in A_0 of the amplitude of oscillation of a particular cantilever which oscillates periodically with the same frequency as the driver, say, $x_i = x_{i,0}f(\omega_d t)$, where f is a periodic function of period 2π. In this work it is concluded that breather solutions to the system of equations above can be obtained in a driven system with damping by choosing the frequency of the driver outside the phonon band of the system while its amplitude satisfies a certain multistability condition for the NLRM.

An experimental setting that allows the optical manipulation of ILMs and DBs is discussed in [88], which considers the case of cantiever arrays of two lengths. In this work, ILMs are created using the same process as described in [87]. Then an optically induced impurity mode is created in the array by means of focusing a laser on a few cantilevers, the laser heats up the cantilevers which reduces their spring constants (Young moduli). The laser is then moved toward the site that sustains the ILM and it is observed that, depending on whether the frequency of the impurity mode is below or above the band states, which in turn depends on the change in sign of the linear spring constant of the cantilevers being excited by the laser. It was observed that when the impurity mode is either above the

ILM/DB frequency or below the band states, it will repel the ILM, and if the impurity mode is above the band states but below the ILM/DB frequency, it will attract it. When the laser is turned off, the ILM/DB will stay at the place it was in the lattice before the laser was turned off. To characterize numerically their findings, the authors proposed the following model for the displacement of the cantilevers free ends which requires a six-neighbor interaction,

$$m_i\ddot{x}_i + \frac{m}{\tau}\dot{x}_i + k_{2Oi}(x_i - z) + k_{4Oi}(x_i - z)^3 + \sum_{j=1}^{6} k_{2Ij}(-x_{i+j} + 2x_i - x_{i-j}) + \sum_{j=1}^{6} k_{4Ij}((x_i - x_{i+j})^3 + (x_i - x_{i-j})^3) + \frac{\varepsilon_0 l_i w V^2}{2(d + x_i - z)^2} = 0\,, \quad i \in \mathbb{Z}\,.$$

$z(t)$ is the driving provided by a piezoelectric, for example $z(t) = z_0 \cos(2\pi\omega_d t)$, the O and I indices stand for "on-site" and "intersite". d is the separation between each cantilever and the substrate, ε_0 is the dielectric constant of the vacuum and l_i is the length of the cantilever, $l_i = a$ if i is odd and $l_i = b$ if i is even. The six-neighbor interaction was necessary in order to explain the experimental results obtained.

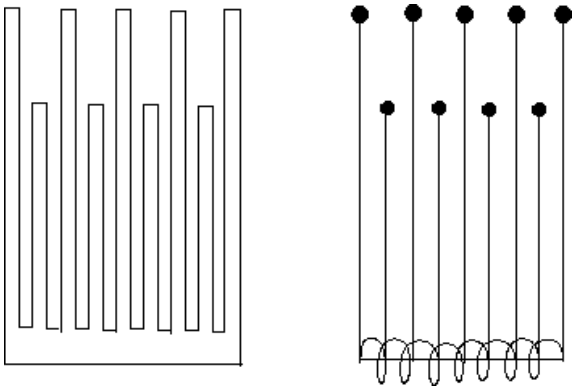

Figure 7: *A typical cantilever array. An initial white-noise like excitation of the system followed by a sustained single-frequency driving mode creates and feeds ILMs or DBs which will live for as long as the driver excites the array. The optical manipulation of these modes is achieved b means of a laser beam focused on some elements in the array.*

3.2.3 Josephson-junction arrays and Josephson ladders

Josephson arrays are one of the main physical settings in which the existence of ILMs, in particular breathers, has been predicted and observed experimentally. The literature on

Josephson arrays is vast, this short section is based on [72, 99, 21], further bibliography in the subject is cited therein. A Josephson tunnel junction (JJ) consists of two superconducting electrodes separated by a thin insulating barrier, this is the so-called SIS tunneling structure (superconductor-insulator-superconductor). The total current, I, through the junction is given by

$$I \,=\, C\frac{dV}{dt} + \frac{V}{R} + I_c \sin\varphi\,,$$

where φ is the gauge-invariant phase difference variable, $V = \frac{\Phi_0}{2\pi}\frac{d\varphi}{dt}$ is the voltage through the junction (Φ_0 is a constant called the flux quantum), I_c is the critical current or maximum supercurrent of the junction, and C and R are capacity and resistive constants, respectively. The equation above can be re-normalized to get

$$i \,=\, \ddot{\varphi} + \Gamma\dot{\varphi} + \sin\varphi \,=:\, \mathcal{N}(\varphi)\,, \tag{3.2.5}$$

where $\Gamma = \sqrt{\Phi_0/2\pi I_c C R^2}$. We recognize in (3.2.5) the equation of a forced and damped pendulum. It is possible to couple JJs using superconducting leads to form arrays of almost any desired size and geometry whose dynamics is like that of systems of nonlinear coupled oscillators. The equations governing the dynamics are derived from Kirchoff's laws for current and voltage, in addition, one has the so-called fluxoid quantization condition which states that for every loop ℓ in the array the following holds,

$$\sum_{j\in\ell} \varphi_j \,=\, 2\pi(n_\ell - f_\ell)\,,$$

where n_ℓ is an integer which comes from the multi-valuedness of the phase variables entering in the definition of φ, this number has no influence in the dynamics and can thus be ignored; f_ℓ is the total magnetic flux through the loop. One of the simplest JJ arrays consists in connecting several JJs in parallel (cf. figure 8a), the equations for the phases are then

$$\mathcal{N}(\varphi_j) \,=\, \lambda(\varphi_{j+1} - 2\varphi_j + \varphi_{j-1}) + i_{ext}\,, \quad j \in \mathbb{N}\,, \tag{3.2.6}$$

where λ is a self-inductance constant. In the case of an open array with N junctions without an external magnetic field, the equations above are implemented with boundary conditions $\varphi_0 = \varphi_{N+1} = 0$. In the case of circular arrays one imposes periodic boundary

conditions $\varphi_{j+N} = \varphi_j + 2\pi M$, where M is the number of kinks or fluxons trapped in the array. A breather solution in this case would correspond to one phase oscillating with a large amplitude while the amplitudes of the oscillations of the remaining phases remain small, a solution of this type is sometimes called *oscillobreather*.

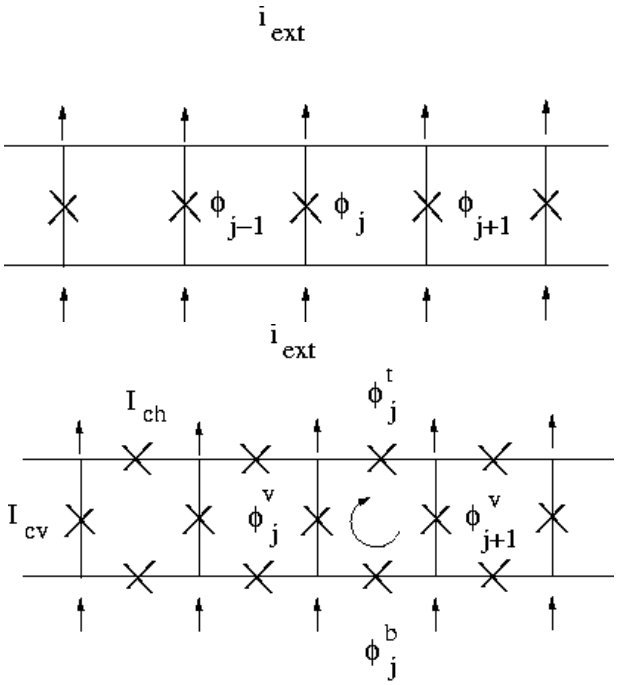

Figure 8: *(a) (top) A JJ parallel array biased by an external current i_{ext}. JJs are represented by crosses. The equations governing the dynamics of the phases form a damped FK system. (b) (bottom) JJ ladder, the experimental detection of breathers in this system is done by measuring voltages locally throughout the array.*

Even though (3.2.6) corresponds to a FK system with damping for which existence of breathers has been established theoretically; their experimental detection is virtually impossible (cf. [72]). Nevertheless, the experimental detection of breather solutions is possible in *JJ ladders*, (cf. figure 8b), whose mechanical analog is that of a set of coupled pendulums in parallel. Let φ_j^v, φ_j^b and φ_J^t stand for the phases of the vertical, bottom and top junctions, respectively, in the jth loop and define $\xi_j = \varphi_j^v + \varphi_j^t - \varphi_{j+1}^v - \varphi_j^b$, then, assuming zero external magnetic flux, the flux quantization condition in this case results in $\xi_j = -2\pi f_j^{ind}$, where f_j^{ind} is a magnetic flux term induced by currents circulating in the

array. The dynamic equations for the phases in this case are

$$\begin{aligned}\mathcal{N}(\varphi_j^t) &= -\frac{\lambda}{h}\xi_j\,,\\ \mathcal{N}(\varphi_j^v) &= \lambda(\xi_{j-1}-\xi_j)+i_{ext}\,,\\ \mathcal{N}(\varphi_j^b) &= \frac{\lambda}{h}\xi_j\,,\end{aligned}$$

for $j \in \mathbb{Z}$. h is parameter that measures the strength of the coupling between the vertical junctions because of the presence of the horizontal junctions; the strength is given by the quotient of the critical currents of the horizontal and vertical junctions, thus $h = I_{ch}/I_{cv}$. In the case of a finite array, the dynamic equations are usually added open boundary conditions, $\xi_0 = \xi_{N+1} = 0$. A breather solution in this case corresponds to a set of junctions being in a resistive state with nonzero mean voltage, $V_j \neq 0$, whereas the other junctions librate around an equilibrium solution ($V_j = 0$). This type of solutions, with localized voltages are called *rotobreathers* and their existence has been established numerically (cf. [72] and references therein).

3.2.4 DNA denaturation

Applications of breathers to the study of the dynamics of the double helix of DNA, has a long history and rests upon the basic idea that the process of denaturation or "melting" of the double helix by which base pairs are opened to allow their copying, can be initiated by a mechanism of energy localization and vibration along the DNA strands. For instance, the presence of "breather modes" along the DNA double helix is reported by Prohofsky et al in [81], the results of their experiments suggest a relationship between these modes and the mechanism of melting of the helix caused by the stretching of hydrogen bonds. The authors called breathing modes those normal vibrational modes of the entire DNA or RNA double helix which are characterized by large vibrations concentrated around the hydrogen bond sites and small vibrations elsewhere. These vibrations consist in two strands of the helix moving toward and away from each other. Such breather modes are believed to promote regions where melting was easier to induce by enzyme complexes via a resonance mechanism.

A mathematical model to study the dynamics of the DNA or RNA double helix is

studied by Zhang in [110] where the equations of motion for the bases consist in a system of coupled sG equations. The author studies a soliton-type mechanism by which energy travels and concentrates along the DNA chain. We explain briefly the mathematical model. Let the z-axis be the axis of the double helix and suppose that the plane of the nth base pair along this axis is perpendicular to it. Each base pair can be thought of as a pair of arrows, one pair per base pair, pointing at each other and each arrow anchored to a helix. Each arrow has associated to it an angle that measures the deviation of the arrow with respect to the line segment determined by the pair of points on both helices where the arrows are attached, one thus has one pair of angles per base pair; let φ_n and φ'_n be the angles at the nth site. The energy of each pair has two contributions: mechanical (vibrational) and an energy of interaction which comes from two sources, an interstrand interaction energy, $V(\varphi_n, \varphi'_n)$, and an energy of interaction with other base pairs. Therefore, the total energy of a DNA strand is

$$H = \sum_{n\in\mathbb{Z}} (\frac{1}{2} I(\dot{\varphi}_n^2 + \dot{\varphi}_n'^2) + V(\varphi_n, \varphi'_n) + \frac{1}{2} S(\varphi_n - \varphi_{n-1})^2 - \frac{1}{2} S(\varphi'_n - \varphi'_{n-1})^2) \,,$$

where I is an average moment of inertia from the rotation of the bases around axes parallel to the DNA axis and S is an energy constant intrinsic to the chain. The interstrand interaction energy is given by

$$\begin{aligned} V(\varphi_n, \varphi'_n) = {} & B(1 - \cos(\varphi_n - \varphi'_n)) + \lambda(1 - \cos\varphi_n) + \lambda(1 - \cos\varphi'_n) + \\ & \beta(3(1 - \cos\varphi_n \cos\varphi'_n) - (1 - \cos(\varphi_n - \varphi'_n))) \,, \end{aligned}$$

for B and λ constants of the model. To study the system of equations associated to H Zhang proposed a perturbative method (idem).

Another mathematical model to study vibrational modes in the DNA molecule was proposed and studied numerically by Dauxois and Peyrard in [34]. In their model, nucleotides on each DNA strand are seen as point masses whose significant vibration happens in the direction transversal to the main axis of the DNA (the $z-$axis in the model above), vibrations in the direction parallel to the DNA axis are disregarded. Using a Morse potential to model the interaction between bases in a pair, the Hamiltonian of the model is

$$H = \sum_{n\in\mathbb{Z}} (\frac{1}{2} m\dot{x}_n^2 + \frac{K}{2}(x_n - x_{n-1})^2 + \frac{1}{2} m\dot{y}_n^2 + \frac{K}{2}(y_n - y_{n-1})^2 + D(e^{-ay_n} - 1)^2) \,,$$

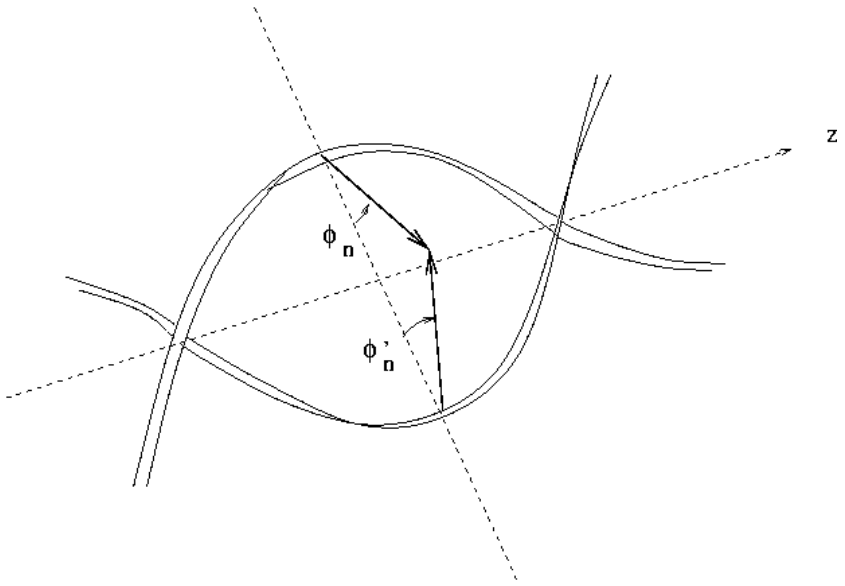

Figure 9: *Representation of the base-pair angles associated to the mathematical model of the vibration of the double helix. The vibrations of base pairs are assumed to take place on a plane perpendicular to the DNA's axis. Base pairs are represented by arrows in the figure whose angles with respect to their equilibrium position are labeled* φ_n *and* φ'_n

where $x_n = (w_n+v_n)/\sqrt{2}$, $y_n = (w_n-v_n)/\sqrt{2}$ and w_n and v_n are the nth nucleotide displacements from its equilibrium. In [34] Dauxois and Peyrard studied the thermal denaturation of the DNA molecule and thus follow a mechanical statistical approach. From the statistical point of view, the part of the Hamiltonian in the x_n variables, which corresponds to a chain of harmonic oscillators without a substrate potential, can be ignored.

Which Hamiltonian model is the one that describes the dynamics of the DNA molecule is a subject for debate and it also depends on which particular feature of the dynamics one would like to study. For example, in [35] a simple DNA model with harmonic nearest-neighbor interaction and anharmonic on-site potential of the 2-3 type is considered. It is shown numerically that the main contribution to the vibrations in the DNA chain comes from breather-like modes, that is, solutions of a Hamiltonian system which, for a long period of time, behave effectively like breathers. In appropriate dimensionless variables, the Hamiltonian studied in the above-cited work is

$$H \;=\; \frac{k}{\alpha^2}\sum_{n\in\mathbb{Z}}\left(\frac{1}{2}\dot{u}_n^2 + \frac{1}{2}(u_n - u_{n-1})^2 + \omega_d^2\left(\frac{1}{2}u_n^2 - \frac{1}{3}u_n^3\right)\right).$$

The examples discussed in the past three sections tell us how DBs or, in general, ILMs arise

in nature; at the same time, we hope to have provided a glimpse of their importance. But there are many other examples in which ILMs arise naturally and which lend themselves for theoretical, numerical and experimental studies. To mention just a few, [98] considers a 3d scalar lattice field and uses the rotation wave approximation technique (which discards high frequency components of the solution) together with lattice Green's function formalism, to numerically calculate stationary localized modes for two types of fourth-order interaction potentials. [89] reports an experimental procedure to observe countably many ILMs in an antiferromagnetic spin lattice. In the same vein, [96] uses spectroscopy methods to establish the existence of intrinsic localized vibrational-energy modes (multiphonon bound states) within a quasi-1d crystal.

3.3 Exact existence results

In this section we summarize earlier work by several authors which is mainly focused on establishing the existence of breathers in 1d lattice systems with Hamiltonian of a general form $H = K + \mathcal{V}$, where K is the kinetic energy term and $\mathcal{V}$ is the term representing the on-site and coupling potentials whose origin we explained in the previous section. We focus in particular on four earlier results, presented in chronological order, which pave the road to our main result in chapter four on the existence of quasi-periodic breathers and which we regard as the next natural and necessary step in the study of breathers in Hamiltonian networks.

In 1994, R. S. MacKay and S. Aubry published the first exact proof of the existence of time-periodic breathers in lattice systems (cf. [70]). MacKay and Aubry considered a time-reversible Hamiltonian network with one degree of freedom consisting of identical weakly coupled oscillators with symmetric nearest-neighbor coupling,

$$W_{MA} \,=\, \frac{\varepsilon}{2}\sum_{n\in\mathbb{Z}}(q_{n+1}-q_n)^2\,,$$

for a generally small constant $\varepsilon > 0$ and a twice-differentiable on-site potential, V, such that $V'(0) = 0$, $V''(0) = \omega_0^2 > 0$; that is, they studied a Hamiltonian system of equations

$$\ddot{q}_n + V'(q_n) \,=\, \varepsilon(q_{n+1} - 2q_n + q_{n-1})\,, \quad n \in \mathbb{Z}\,, \tag{3.3.1}$$

with associated formal Hamiltonian

$$H \,=\, \sum_{n\in\mathbb{Z}} (\frac{1}{2}p_n^2 + V(q_n) + \frac{1}{2}\varepsilon(q_{n+1}-q_n)^2)\,. \tag{3.3.2}$$

Observe that $(q_n = 0)_{n\in\mathbb{Z}}$ is clearly an equilibrium solution of (3.3.1); moreover, the linearization of this system about this trivial solution is

$$\ddot{u}_n + \omega_0^2 u_n \,=\, \varepsilon(u_{n+1} - 2u_n + u_{n-1})\,, \quad n \in \mathbb{Z}\,,$$

whose solutions are given by linear superpositions of dispersive waves $(u_n = e^{\mathrm{i}(kn-\omega t)})_{n\in\mathbb{Z}}$, $k \in \mathbb{Z}$, also known as *phonons*, provided that the following *dispersion relation* is satisfied,

$$\omega^2 = \omega_0^2 + \varepsilon 4\sin^2(k/2)\,.$$

When $\varepsilon = 0$ the system decouples; i.e., every oscillator acts independently from its neighbors; moreover, from the hypotheses on V it follows that, for any $n \in \mathbb{Z}$, $(q_n, p_n) = (0,0)$ is an isolated local minimum of the nth oscillator's energy, $H_n(p_n, q_n) = \frac{1}{2}p_n^2 + V(q_n)$; therefore, this point is a stable equilibrium point of the 2d system $\dot{q}_n = p_n$, $\dot{p}_n = -V'(q_n)$, in fact, a center, and every orbit $(q_n(t), p_n(t))$ that starts in a sufficiently close neighborhood of $(0,0)$ is periodic (cf. [52], pp. 176-177).

In order to state the main result in [70], we need to formalize the notions of nonresonance and anharmonicity, which up to this point we have been using rather informally. Consider the Hamiltonian system consisting of one oscillator and Hamiltonian function $H = \frac{1}{2}p^2 + V(q)$, where V is as described above thus, it is always possible to find a neighborhood of the origin $(0,0)$ where the level sets of H consist in closed curves that enclose the origin, so let $\mathcal{C}$ be one such level curve with energy energy $H = E$. We define action-angle variables on $\mathcal{C}$ in the usual way: the *action*, I, of $\mathcal{C}$ is the area enclosed by it, divided by 2π, i.e.,

$$I = \frac{1}{2\pi}\oint_{\mathcal{C}} p\,dq\,.$$

Now, $E = E(I)$ and it is a standard result that the angular frequency of the periodic orbit is given by

$$\omega(I) \,=\, \frac{dE}{dI}\,.$$

The *angle* variable, θ, is defined through the relation

$$\frac{d\theta}{dt} = \omega(I)\,.$$

Denote the transformation to action-angle variables as

$$q = Q(\theta, I) \quad \text{and} \quad p = P(\theta, I)\,.$$

The type of periodic solution considered by MacKay and Aubry is obtained as a continuation of the periodic solution which, at the anti-continuum limit $\varepsilon = 0$ (that is, the unperturbed case), consists of a single oscillator vibrating with frequency ω_0 while all the other lattice oscillators remain at rest in equilibrium. A solution of this kind, when only one oscillator vibrates is sometimes referred to as a *one-site breather*. A one-site breather solution of the unperturbed system with action I_0 is said to be *non-resonant* provided that

$$\omega(I_0) \neq \frac{\omega_0}{n}\,, \quad \text{for all } n \in \mathbb{N}\,.$$

The non-resonance condition has the purpose of avoiding resonances between the Fourier components of the solution and the linear phonons when the coupling is turned on, that is when $\varepsilon > 0$ and small. In addition, we say that a periodic orbit with action I_0 is *anharmonic* provided that

$$\frac{d\omega}{dI}(I_0) \neq 0\,;$$

this is a non-degeneracy condition of the dependence of the frequency on the amplitude of the oscillation.

In essence, MacKay and Aubry reformulated the problem of finding periodic solutions, as one of finding zeros of an operator $F : B_1 \times \mathbb{R} \to B_2$, $(z, \varepsilon) \mapsto w$, where $z = (z_n = (q_n, p_n))_{n\in\mathbb{Z}}$ and $w = (w_n = (u_n, v_n))_{n\in\mathbb{Z}}$ are given by

$$\begin{aligned} u_n(t) &= V'(q_n(t)) - \varepsilon(q_{n+1} - 2q_n + q_{n-1}) + \dot{p}_n(t)\,, \\ v_n(t) &= p_n(t) - \dot{q}_n(t)\,. \end{aligned}$$

B_1 and B_2 are appropriate Banach spaces of bounded infinite sequences of pairs of continuously differentiable (B_1), or just continuous (B_2), T-periodic functions which satisfy a time-reversibility (symmetry) property; namely, $q_n(-t) = q_n(t)$ and $p_n(-t) = -p_n(t)$;

similarly for w. Note that the set of T-periodic time-reversible solutions of (3.3.1) are in one-to-one correspondence with the zeros of F, i.e., with the set of all z such that $F(z,\varepsilon)=0$. The Theorem of MacKay and Aubry is the following,

Theorem 1 *(MacKay and Aubry, 1994)*

Let $V:\mathbb{R}\to\mathbb{R}$, $V\in C^r$, $r\geq 2$, such that $V'(0)=0$ and $V''(0)=\omega_0^2>0$. Let $\mathcal{C}$ be a periodic orbit associated to the one-degree-of-freedom Hamiltonian $H=\frac{1}{2}p^2+V'(q)$ and let I_0 be its action. Assume that $\mathcal{C}$ is non-resonant and anharmonic. Then the periodic orbit $z^o=(z_n^o=(q_n^o,p_n^o))_{n\in\mathbb{Z}}$ of the uncoupled system (3.3.2) with $\varepsilon=0$, given by $q_0^o(t)=Q(I_0,\omega(I_0)t)$ and $q_n^o\equiv 0$ for all $n\neq 0$, has a locally unique continuation as a periodic orbit $z=(z_n=(q_n,p_n))_{n\in\mathbb{Z}}$ of the coupled system (3.3.2) with $\varepsilon\neq 0$, with the same period $T=2\pi/\omega(I_0)$, provided that ε is small enough. Furthermore, z is C^r and for every $n\in\mathbb{Z}$, the C^1 norm of z_n decays exponentially as $n\to\infty$ for as long as DF remains invertible; more precisely, let ε_0 be such that $DF(\varepsilon)$ is invertible for $0<\varepsilon<\varepsilon_0$, then there exist constants C and $\lambda<1$, independent of ε, such that for any $\mu\in(\lambda,1)$ and for any $n\in\mathbb{Z}$,

$$|z_n|\ \leq\ |z_n(0)|e^{h\varepsilon}\mu^{|n|}\,,$$

where $h=C(1+\lambda)^2(\frac{2}{1+\lambda}+\frac{\mu}{1-\lambda\mu}+\frac{1}{\mu-\lambda})$.

Observe that z^o as defined in theorem 1 satisfies $F(z^o,0)=0$; therefore, provided that F is C^1 and $DF(z^o,0)$ is invertible, there exists a map $\varepsilon\mapsto z(\varepsilon)$ such that $F(z(\varepsilon),\varepsilon)=0$, for ε sufficiently small. It turns out that the exponential decay in space of the solutions follows from a general result that asserts that the elements of the inverse of $DF(z^o,0)$ decay exponentially with the distance between sites. The time-reversibility property is crucial when showing the invertibility of DF.

In their paper, MacKay and Aubry also point out methods to find an upper bound for the maximum magnitude of the coupling strength ε for which theorem 1 still holds. They also provide conditions for the existence of finitely many multi-site breathers, that is, the continuation of solutions which at the anti-continuum limit, correspond to finitely many

sites oscillating while all the others remain at rest; such conditions are that the frequencies of the excited sites at the anti-continuum limit must be in rational ratio (i.e., they must be commensurate) and their greatest common divisor must be non-resonant with the linear phonons of the system; in addition, the relative phases of the excited sites must be chosen so as to keep the time-reversibility property. Extensions to the case of unequal oscillators or longest-than-nearest-neighbor interactions are also suggested but, as in the case of multi-site breathers no proofs are provided. One must point out, however, that the non-resonance condition in the case of multi-site breathers may be hard to satisfy since the harmonics of the breather frequencies in the uncoupled case may densely fill the real axis and thus resonance with the linear spectrum may be inevitable. The methodology of MacKay and Aubry, using the implicit function theorem, has come to be known as the anti-integrability (cf. [7]) or, more suitably in this case, the *anti-continuity method* (cf. [8]).

Motivated by proving the exponential stability of time-periodic breathers conjectured in [70], D. Bambusi (cf. [10]) developed an alternative proof of the existence of time-periodic breathers in infinite lattice systems which is similar to the proof of MacKay and Aubry in what it uses Poincaré-continuation theorem ideas, but unlike the above-mentioned proof it also uses Nekhoroshev normal form ideas to develop a general theorem that gives a local normal form for abstract Hamiltonian systems in (possibly infinite-dimensional) Banach spaces that does not require the use of action-angle variables. The general normal form theorem developed in [10] is important in its own right as it bypasses issues such as the singularity of the action-angle coordinates or their explicit calculation required to check analyticity of the Hamiltonian; this theorem is also applicable in perturbed Hamiltonian systems near an unperturbed quasi-periodic orbit with a finite number of frequencies, an idea that the author exploits later on to prove the existence of quasi-periodic breathers in lattice systems (cf. [9]). In [10] Bambusi applies his normal form theorem to prove the existence of time-periodic breathers in the case of identical lattice oscillators with a *long-range* coupling potential

$$W_B = \frac{\varepsilon}{4} \sum_{m \neq n} \frac{1}{|m-n|^\alpha} (q_m - q_n)^2 \,,$$

where $\alpha > 1$ and ε is sufficiently small; observe that in the limit $\alpha \to \infty$ W_B becomes W_{MA}. W_B is second-order in the displacements q_n; we will call the coefficients $C_{mn} = 1/|m-n|^\alpha$, *power-law coupling coefficients*. The associated Hamiltonian and equations of motion are

$$\begin{aligned} H &= \sum_{n\in\mathbb{Z}} (\frac{1}{2}p_n^2 + V(q_n)) + \varepsilon \frac{1}{4} \sum_{m\neq n} \frac{1}{|m-n|^\alpha}(q_m - q_n)^2 \,, \\ \ddot{q}_n + V'(q_n) &= \varepsilon \sum_{k\geq 1} \frac{1}{k^\alpha}(q_{n+k} - 2q_n + q_{n-k}) \,, \end{aligned} \tag{3.3.3}$$

where V is assumed real analytic and, as it was the case before, $V'(0) = 0$ and $V''(0) = \omega_0^2 > 0$. In order to state Bambusi's theorem on the existence of periodic breathers for (3.3.3) we need to set some notation. First, the type of periodic orbits found in [10] correspond to solutions which at the anticontinuum limit are one-site breathers; that is, all sites are in equilibrium except for the zeroth lattice point which oscillates at frequency ω, such that ω is *Diophantine* with ω_0; i.e., we assume that there exists $\nu > 0$ such that

$$|\omega k_1 + \omega_0 k_2| \geq \frac{\nu}{|k|^2} \,, \quad \forall\, (k_1, k_2) \in \mathbb{Z}^2 \backslash \{0\} \,,$$

where $|k| = |k_1| + |k_2|$. At the anticontinuum limit the zeroth order oscillator's Hamiltonian is $H_0 = \frac{1}{2}p_0^2 + V(q_0)$. The hypotheses on V guarantee the existence of local action-angle variables $(I, \theta) \in \mathbb{R}_+ \times \mathbb{T}$; in these coordinates H_0 is a function of the action alone; i.e., $H_0 = h(I)$, and the equations of motion become $I(t) \equiv I_0$ and $\theta = \omega(I_0)t + \theta_0$, where $\omega(I_0) = \partial h/\partial I|_{I=I_0}$. Now define the associated phase space, $\mathcal{P} = \mathbb{R}_+ \times \mathbb{T} \times \ell^2 \times \ell^2$, where ℓ^2 is the space of square summable sequences. Denote by $\zeta = (I, \phi, \{p_k\}_{k\neq 0}, \{q_k\}_{k\neq 0})$ a point in the phase space $\mathcal{P}$ and by $\gamma_0(J) = \{(I, \theta, p, q) \in \mathcal{P} \,:\, I = J, \theta \in \mathbb{T}, p = q = 0\}$ the phase space trajectory of an unperturbed periodic orbit (one-site breather). $\mathcal{P}$ is then endowed with the norm

$$\|\zeta\|^2 \,=\, \|(I, \theta, \{p_k\}_{k\neq 0}, \{q_k\}_{k\neq 0})\|^2 \,=\, \max\{|I|^2, \varepsilon|\theta|^2, \frac{1}{2}\sum_{k\neq 0}(p_k^2 + \omega_0^2 q_k^2)\} \,;$$

we will denote by d be the distance induced by this norm.

Theorem 2 *(Bambusi, 1995)*

Consider the Hamiltonian system (3.3.3) where $V'(0) = 0$ and $V''(0) = \omega_0^2 > 0$. Let $\mathcal{P}$ be

the phase space defined above and d its distance function. Assume that there exists J such that

$$\omega = \omega(J) = \frac{\partial h}{\partial I}(J)$$

is diophantine with ω_0. Let $\tilde{E} = h(J)$, where h is the action-dependent energy of the zeroth oscillator in the unperturbed system. Then there exist positive constants ε_*, $C_1, \ldots C_6$ such that, whenever $0 < \varepsilon < \varepsilon_*$, on the energy surface $H = \tilde{E}$ there exists a periodic orbit with phase space trajectory γ such that

(i) $\forall\, \zeta \in \gamma$, $d(\zeta, \gamma_0) \leq C_1 \varepsilon^{1/2}$, γ_0 as defined above (one-site breather).

(ii) γ is exponentially stable, that is, for any initial data ζ_0 such that $d(\zeta_0, \gamma) \leq C_2\sqrt{\varepsilon}$, one has

$$d(\zeta(t), \gamma) \leq (1 + C_3\varepsilon^{1/6})(d(\zeta_0, \gamma) + C_4\sqrt{|t|}\varepsilon^{5/6} e^{-\frac{1}{2}\left(\frac{C_6}{\varepsilon}\right)^{1/6}}),$$

for all t such that $|t| \leq C_5 e^{\left(\frac{C_6}{\varepsilon}\right)^{1/6}}$.

The normal form theorem developed by Bambusi consists of a formal recursive algorithm to write the Hamiltonian in normal form up to any finite order in a way that the size of the remainder is exponentially small, we will not state his theorem here but refer the reader to [10]. Although this procedure is standard, the novelty introduced in [10] is the use of Fourier expansions.

The same normal form approach based on Poincaré-Lyapunov continuation techniques is used again in [9] to prove the existence of (time) *quasi-periodic* breathers for a large class of Hamiltonian lattice systems with *nearest-neighbor* interaction potential, whenever there are integrals of motion that are independent of the Hamiltonian and which arise from symmetries. This method is applied to revisit and refine earlier work by other authors on the DNLS and adiabatic Holstein quantum models in which the existence of quasi-periodic breathers with two non-resonant frequencies is proved; in these cases, the breather solutions found correspond to the electron's probability being concentrated at two lattice sites. In addition, a quasi-periodic breather with three non-resonant frequencies is obtained for the vector DNLS system. To finalize mentioning the work in [9] we remind that just as there are factors which favor the formation of breathers in infinite dimensional Hamiltonian systems

(see [70]), there are factors that hinder it. Two such unfavorable factors are pointed out in [9], the first factor is the coupling embedded in the nonlinearity and the second factor is resonance with the linear spectrum. While the latter factor can in principle be avoided if the spectrum is not continuous, the former factor can in some cases be bypassed when the system possesses symmetries; even though the presence of symmetries is not considered a generic property, there are many systems of physical significance which indeed possess a type of symmetry such as spatial translation and/or time invariance. The result we will discuss next, is a case in which the existence of breather solutions is facilitated due to the fact that the linear spectrum is not continuous.

It was pointed out by Aubry in [7] in regards to the limited applicability of the anti-continuity method to establish the existence of one-site breathers and, in a limited number of cases such as the DNLS equation (cf. [55]), two-frequency quasi-periodic breathers, that a general method to determine the existence of quasi-periodic breathers with any number of incommensurate frequencies should relate anti-continuity and KAM methodologies (cf. [7] p. 291). The first proof of the existence of quasi-periodic breathers with any finite number of incommensurate frequencies in infinite lattice systems via the KAM technique was developed by X. Yuan in [108]. Yuan considers weakly coupled lattice system with nearest-neighbor interaction potential of the following general form,

$$W_Y = \sum_{n\in\mathbb{Z}} W(q_{n+1} - q_n)\,,$$

and a Hamiltonian function and corresponding equations of motion given by

$$\begin{aligned} H &= \sum_{n\in\mathbb{Z}} (\frac{1}{2}p_n^2 + V(q_n) + \varepsilon W(q_{n+1} - q_n))\,, \\ \ddot{q}_n + V'(q_n) &= \varepsilon(W(q_{n+1} - q_n) - W(q_n - q_{n-1}))\,, \quad n \in \mathbb{Z}\,, \end{aligned} \tag{3.3.4}$$

where the potentials V and W are assumed to satisfy the following properties,

(i) V and W are analytic in a strip domain $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \delta_0\}$ for some constant $\delta_0 > 0$.

(ii) $V(0) = V'(0) = 0$, $V''(0) = \beta^2 > 0$ for $\beta > 0$, and $W(x) = O(|x|^3)$.

(iii) There exists a compact interval $I \subset \mathbb{R}_+$ such that for any $h \in I$ the equation $\frac{1}{2}y^2 + V(x) = h$ defines a simple closed curve, $\Gamma(h)$, that encloses the origin.

(iv) Let $\rho(h)$ be the area enclosed by $\Gamma(g)$, that is,

$$\rho(h) = \oint_{\Gamma(h)} y\,dx$$

and assume that $\rho'(h) \neq 0$, $\rho''(h) \neq 0$ for any $h \in \mathcal{I}$.

Observe that, since W is of third-order, the linearization of the equations of motion yields the same result as in the linearization of the unperturbed system, namely, $\ddot{q}_n + \beta^2 q_n = 0$ for all $n \in \mathbb{Z}$; therefore, the linear phonons of the system have only one type of frequency, $\omega_0 = \beta$, with infinite multiplicity thus, it seems plausible that the set of all the harmonics of the frequencies considered can avoid this small linear spectrum set and thus prevent occurrence of resonances. For a long time KAM theory had been well known for its use in determining the existence of families of quasi-periodic solutions on an invariant manifold associated to Hamiltonian systems with finitely many degrees of freedom. The extension of KAM theory in the 1980's to handle the infinite-dimensional case is attributed to the works of Vittot and Bellissard (V-B) (cf [V-B] in [108]) and Fröhlich, Spencer and Wayne (F-S-W) (cf. [43]). With this extension it is possible to establish the existence of *almost periodic solutions*, that is, solutions with infinitely many incommensurate frequencies, in systems of weakly-coupled oscillators with short-range coupling; more precisely, it is possible to use action-angle variables to reduce the Hamiltonian in (3.3.4) to a form in which the existence of almost periodic solutions can, in principle, be proved; however, in order to establish the existence of *quasi-periodic* breathers, one must be able to prove the existence of a *finite-dimensional* invariant torus for (3.3.4); therefore, the methods of V-B and F-S-W are not applicable in this case. The existence of quasi-periodic solutions to infinite-dimensional Hamiltonian systems, not necessarily of short range, was developed in the context of PDEs such as wave and NLS equations (cf. [66, 65, 80, 79, 101]). Once again, by use of action-angle variables, it is possible to write (3.3.4) in normal form as required in the previously mentioned works and thus the existence of quasi-periodic breathers seems plausible by the methods established therein; however, once again this is not the case since, in order for those methods to apply, all normal frequencies are required to have finite multiplicities and, as pointed out earlier, the linearization of the equations of motion in (3.3.4) reveal

that where $\omega_0 = \beta$ has infinite multiplicity. The contribution of [108] to the theory of breathers consisted in developing a normal-form abstract KAM theorem from which the existence of quasi-periodic breathers in the infinite-multiplicity case follows as a corollary. Since we will develop an abstract KAM theorem that will allow us to handle Hamiltonians of a more general type than that in (3.3.4) and the proof of our KAM theorem, although simpler, is similar in the sense that it too requires a Newton-type scheme and an infinite number of change of variables, we shall only state Yuan's existence theorem on quasi-periodic breathers; to this end we first need to add a few definitions.

Let $\mathcal{J} = \{j_1 \ldots, j_N\} \subset \mathbb{Z}$ be any finite sequence denoting oscillator positions, where $N \geq 1$, then equations of motion in (3.3.4) can then be regarded as a perturbation of the following system:

$$\begin{aligned} \ddot{x}_n + V'(x_n) &= 0\,, \quad n \in \mathcal{J}\,, \\ \ddot{x}_n + \beta^2 x_n &= 0\,, \quad n \in \mathbb{Z}' := \mathbb{Z}\backslash\mathcal{J}\,. \end{aligned} \tag{3.3.5}$$

Let ℓ be the space of sequences $u = (u_n = (x_n, y_n))_{n\in\mathbb{Z}'}$, $(x_n, y_n) \in \mathbb{C}^2$, and with finite norm $\|u\| := \sum_{n\in\mathbb{Z}'}(|x_n|^2 + |y_n|^2)e^{|n|/a} < \infty$, where a is a positive constant. Define the following inner product on ℓ, $\langle u, v\rangle_\ell := \sum_{n\in\mathbb{Z}'}(u_n \cdot v_n)e^{|n|/a}$, where " $\cdot$ " is the usual inner product in $\mathbb{C}^2$. Note that the origin is a stable center of the system (3.3.5); also, by preservation of the quantity $\frac{1}{2}y^2 + V(x) = h$, the first set of equations in (3.3.5) has an invariant N-torus $\Gamma(h_1) \times \cdots \times \Gamma(h_N)$ with frequency vector $\omega(\eta) = (H_0'(\rho(h_1)), \ldots, H_0'(\rho(h_N)))$, where H_0 stands for the inverse of $\rho(h)$; therefore, any solution starting on $T(\eta) = \Gamma(h_1) \times \cdots \times \Gamma(h_N) \times \{0\}$ is a breather solution to the unperturbed system (3.3.5). Yuan's theorem states that a large family of these invariant tori persist, slightly deformed, as quasi-periodic breather solutions of (3.3.4).

Theorem 3 *(Yuan, 2002)*

Let V and W satisfy properties (i) through (iv) above. Then, for any given integer $N \geq 1$, compact set $\Omega = I^N \subset \mathbb{R}^N_+$, and constant $0 < \gamma \ll 1$, there is a constant $\varepsilon^* = \varepsilon^*(\Omega, N, \gamma)$, small enough, such that whenever $0 < \varepsilon < \varepsilon^*$, there is a Cantor set $S \subset \Omega$, such that $m(S) = m(\Omega)(1 - O(\gamma))$, where m stands for Lebesgue measure, and a family of N-tori,

$T[S] = \cup_{\eta\in S} T(\eta) \subset \cup_{\eta\in\Omega} T(\eta)$, together with an analytic embedding $\Phi : T[S] \hookrightarrow \mathbb{R}^N \times \mathbb{T}^N \times \ell$ which is a higher-order perturbation of the inclusion map, $\Phi_0 : \cup_{\eta\in\Omega} T(\eta) \hookrightarrow \mathbb{R}^N \times \mathbb{T}^N \ell$; such that the restriction of Φ to each $T(\eta)$, $\eta \in S$, is an embedding of a rotational N-torus of (3.3.4). Moreover, any solution of (3.3.4) starting on $\Phi(T(\eta))$ corresponds to a quasi-periodic breather of frequencies ω^* such that $|\omega^* - \omega| = O(\varepsilon^{1/3})$ and super-exponential decay, i.e., $|x_{n+N}| \leq Ce^{-|n|/a}\varepsilon^{(1+\rho)^n/3}$, for fixed constants $C > 0$ and $0 < \rho < 1/9$.

The super-exponential decay mentioned in the theorem is one of the consequences of the interaction potential, W_Y, starting with a cubic term; this property of W_Y together with that of being short-range, is responsible for cancellations of terms during the KAM iterations and the weakening of the strength of resonance which, loosely speaking, make the Hamiltonian be effectively finite-dimensional, in which case the existence of quasi-periodic solutions is not prevented by resonances in the normal direction.

Almost-periodic breathers, i.e., spatially localized solutions with infinitely many incommensurate frequencies were studied much earlier in [43] for a Hamiltonian network of the form $H = \sum_{i\in\mathbb{Z}^d}(Q_i + \omega^2 P_i) + \varepsilon \sum_{\langle i,j\rangle} f_{\langle i,j\rangle}(Q_i, P_i, Q_j, P_j)$, where the frequencies ω_i's are assumed to be non-negative, independent and identically distributed random variables with smooth a distribution of fast decay at infinity, whereas the functions $f_{\langle i,j\rangle}$'s are assumed $O(P^2+Q^2)$ for Q and P sufficiently small. It is showed in [43] that there is a set $\Omega \subset \mathbb{R}^\infty_+$ of positive probability measure, such that for each realization of the frequencies $\omega \in \Omega$ there is an almost periodic breather. The reader will find work on spatial structures of a more general type in [78].

To finalize this section, we would like to briefly outline a finite-dimensional center manifold reduction technique used in recent years to establish the existence of small-amplitude time-periodic breathers in certain infinite one-dimensional lattice systems. This technique was developed by G. James, [53], who used it to provide a numerical calculation of time-periodic breathers in the FPU lattice; his method is in some sense an extension of the popular rotating wave approximation technique which is applicable in the case of an even coupling potential. The work of James is in turn based on work by Kirchgëssner, Mielke,

Ioos and Vanderbauwhede on center manifold theory for elliptic PDEs which established the existence of travelling wave solutions in infinite one-dimensional lattice systems (cf. Iooss (2000), Iooss and Kirchgässner (2000) in [53]). Let as before $(q_n)_{n\in\mathbb{Z}}$ represent the displacements of the lattice oscillators from their equilibrium positions, then the FPU system is

$$\ddot{q}_n \;=\; W'(q_{n+1}-q_n) - W'(q_n - q_{n-1})\,, \quad n\in\mathbb{Z}\,, \tag{3.3.6}$$

where the interaction potential W is assumed C^{k+1}, $k\geq 6$; moreover, $W(0)=W'(0)=0$ and $W''(0)=\alpha>0$. Observe that, for the lack of a parameter, the anticontinuity method cannot be used in this case. The property of the system (3.3.6) of being invariant under the translation $q_n \to q_n + c$, $c\in\mathbb{R}$, adds a degeneracy which translates into the center manifold being of a greater dimension; in this case, the author proposed a slight modification of his method which we just outline in here. Suppose that we are interested in finding periodic solutions with frequency ω. Since $W''(0)>0$, one can locally invert W' and introduce a 2π-periodic "force variable", $y_n(t)=\alpha^{-1}W'(q_n-q_{n-1})(t/\omega)$, which transforms (3.3.6) into another system with a FK-type right-hand side,

$$\alpha^{-1}\omega^2\frac{d^2}{dt^2}\mathcal{W}(y_n) \;=\; y_{n+1}-2y_n+y_{n-1}\,, \quad n\in\mathbb{Z} \tag{3.3.7}$$

where $\mathcal{W}(y)=(W')^{-1}(\alpha y)$ vanishes at zero. The y_n variables are 2π-periodic; moreover, it is possible to show that, for every n, the time average of y_n is independent of n; the author assumes it to be zero and, for simplicity, restricts the q_n variables to be even for all n. Knowing the force variables allows one to recover the q_n variables by integrating the equations of motion and using their parity. Next, by defining $Y_n=(y_n,y_{n-1})$ one can rewrite the equations of motion in the form

$$Y_{n+1} \;=\; F_\omega(Y_n)\,, \quad n\in\mathbb{Z}\,, \tag{3.3.8}$$

where F_ω is a C^{k-2} operator defined locally in a neighborhood of the origin and is given by

$$F_\omega(z,y) \;=\; \left(y\,,\,\alpha^{-1}\omega^2\frac{d^2}{dt^2}\mathcal{W}(y)+2y-z\right)\,,$$

with domain and range given by the Cartesian products of suitable Sobolev spaces of 2π-periodic functions. The spectrum of $DF_\omega(0)$ consists of an essential part at the origin and

infinitely many eigenvalues, σ_k, σ_k^{-1}, $k \geq 1$, which lie on the unit circle or on the real axis and have finite multiplicities; these eigenvalues satisfy the dispersion relation $\sigma^2 + (\alpha^{-1}\omega^2 k^2 - 2)\sigma + 1 = 0$. One then writes (3.3.8) in the form

$$Y_{n+1} = DF_{\omega_0} Y_n + N(Y_n \mu)\,, \quad n \in \mathbb{Z}\,, \tag{3.3.9}$$

where $\mu = \alpha^{-1}\omega^2 - 4$ and ω_0 is a critical value at which two eigenvalues coalesce on the real axis at $\sigma = -1$. The theory developed by James shows that for μ in some neighborhood of the origin, all solutions to (3.3.9) that remain in a neighborhood of the origin belong to a two-dimensional center manifold; this is what ultimately allows one to reduce the study of the dynamics of the original FPU system to that of studying a two-dimensional system. The theorem of James is also applicable to more general systems, including those with a non-zero on-site potential, provided certain conditions of symmetry (time-reversibility) and separability (of the spectrum of DF_ω) are satisfied. In abstract, the theorem of James, or rather, the part that involves the reduction of the dynamics to the center manifold, states the following,

Theorem 4a *(James, 2003)*

Let X be a Hilbert space and $L : X \to X$, be a closed linear operator with domain D, not necessarily dense in X. D is a Hilbert space continuously embedded in X, endowed with the scalar product $\langle u, v\rangle_D = \langle Lu, Lv\rangle_X + \langle u, v\rangle_X$. Let $\mathcal{U} \times \mathcal{V}$ be a neighborhood of the origin in $D \times \mathbb{R}^p$ and $N \in C^k(\mathcal{U} \times \mathcal{V}, X)$, $k \geq 2$ a nonlinear map such that $N(0,0) = 0$ and $D_u N(0,0) = 0$. We are to find sequences $(u_n)_{n\in\mathbb{Z}}$ in $\mathcal{U}$ satisfying the system of equations in X,

$$u_{n+1} \;=\; Lu_n + N(u_n, \mu) \quad n \in \mathbb{Z}\,, \tag{3.3.10}$$

for $\mu \in \mathcal{V}$ a parameter. Observe that when $\mu = 0$, $u_n = 0$ for all n is a fixed point of (3.3.10); moreover, by the implicit function theorem, this fixed point persists for $\mu \approx 0$ a smooth family of fixed points $u(\mu) \in D$ such that $u(0) = 0$. Assume that L has satisfies a spectral separation property, that is, its spectrum $\sigma(L)$ can be separated as follows, $\sigma(L) = \sigma_s \cup \sigma_c \cup \sigma_u$, such that $\sup_{z\in\sigma_s} |z| < 1$, $|z| = 1$ $\forall z \in \sigma_c$ and $\inf_{z\in\sigma_u} |z| > 1$. Then

there exists a neighborhood $\Omega\times\Lambda$ *of the origin in* $D\times\mathbb{R}^p$ *and a map* $\psi\in C_b^k(X_c\times\Lambda, D_h)$ *that satisfies* $\psi(0,0)=0$, $D_u\psi(0,0)=0$ *and is such that for all* $\mu\in\Lambda$ *there exists a manifold,*

$$\mathcal{M}_\mu \,=\, \{y\in D\,:\, y=x+\psi(x,\mu),\; x\in X_c\}\,,$$

with the following properties:

(i) $\mathcal{M}_\mu$ *is locally invariant under* $L+N(\cdot,\mu)$*; that is, for every* $y\in\mathcal{M}_\mu\cap\Omega$*,*

$$Ly+N(y,\mu)\in\mathcal{M}_\mu\,.$$

(ii) If $(u_n)_{n\in\mathbb{Z}}$ *is a solution to (3.3.10) such that* $u_n\in\Omega$ *for all* $n\in\mathbb{Z}$*, then* $u_n\in\mathcal{M}_\mu$ *for all* $n\in\mathbb{Z}$*; moreover, if* π_c *is the spectral projection operator associated to* σ_c *then* $u_n^c=\pi_c u_n$ *satisfies the recurrence relation on* $X_c=\pi_c X$

$$u_{n+1}^c \,=\, f(u_n^c,\mu)\,,\quad \forall\, n\in\mathbb{Z}\,, \tag{3.3.11}$$

where $f\in C^k((X_c\cap\Omega)\times\Lambda, X_c)$ *is given by* $f(\cdot,\mu)=\pi_c(L+N(\cdot,\mu))\circ(I+\psi(\cdot,\mu))$ *and* $f(\cdot,\mu)$ *is locally invertible.*

(iii) Conversely, if $\{u_n^c\}$ *is a solution to (3.3.11) such that* $u_n^c\in\Omega$ *for all* $n\in\mathbb{Z}$*, then* $u_n=u_n^c+\psi(u_n^c,\mu)$ *satisfies (3.3.10).*

James and Sire [95] applied the center manifold methodology to establish the existence of small-amplitude travelling breathers in Klein-Gordon (KG) chains with an exponentially small oscillatory tail. The KG chains they studied are of the form

$$\ddot{q}_n+V'(q_n) \,=\, \varepsilon(q_{n+1}-2q_n+q_{n-1})\,,\quad n\in\mathbb{Z}\,, \tag{3.3.12}$$

where $\varepsilon>0$ and $V\in C^k$, $k\geq 5$ is locally anharmonic: $V(x)=\frac{1}{2}x^2-\frac{a}{3}x^3-\frac{b}{4}x^4+h.o.t.$ A *travelling breather* is a solution which satisfies that for a fixed integer $p\geq 2$,

$$q_n(t) \,=\, q_{n-p}(t-T)\,,\quad n\in\mathbb{Z}\,, \tag{3.3.13}$$

and $\lim_{|t|\to\infty} q_n(t)=0$. When $p=1$ one has a travelling wave. In [95] the case $p=2$ is considered, thus, using (3.3.13), one can reduce (3.3.12) to

$$\ddot{q}_1 = -V'(q_1)+\varepsilon(q_2(t)-2q_1(t)+q_2(t+T))\,,$$

$$\ddot{q}_2 = -V'(q_2)+\varepsilon(q_1(t)=2q_2(t)+q_2(t+T))\,.$$

In order to apply center manifold reduction, set $\tau = t/T$ and define $(u_1(\tau), u_2(\tau)) = (x_1(\tau), x_2(\tau + \frac{T}{2}))$, then, renaming $\tau \to t$ and defining $U = (u_1, u_2, \dot{u}_1, \dot{u}_2, X_1(t,v), X_2(t,v))$, for $v \in [-\frac{1}{2}, \frac{1}{2}]$ and $X_i(t,v) = u_i(t+v)$, $i = 1, 2$, one obtains the following evolution problem

$$\dot{U} \;=\; LU + F(U)\,, \tag{3.3.14}$$

where $LU = (\dot{u}_1, \dot{u}_2, \alpha_1 u_1 + \alpha_2(\delta_{1/2} + \delta_{-1/2})X_2, \alpha_1 u_2 + \alpha_2(\delta_{1/2} + \delta_{-1/2})X_1, \partial_v X_1, \partial_v X_2)$, $\delta_a X_i(t,\cdot) = X_i(t,a)$, $\alpha_1 = -T^2(1+2\gamma)$, $\alpha_2 = T^2\gamma$; $F(U) = T^2(0,0,f(u_1),f(u_2),0,0)$ and $f(u) = au^2 + bu^3 + h.o.t.$ We note that if U is a solution to (3.3.14) then $U \in C^0(\mathbb{R}, D) \cap C^1(\mathbb{R}, H)$, where $H = \mathbb{R}^4 \times (C^0[-\frac{1}{2}, \frac{1}{2}])^2$ and $D = \{U \in \mathbb{R}^4 \times (C^1[-\frac{1}{2}, \frac{1}{2}])^2 \,:\, X_1(0) = u_1,\, X_2(0) = u_2\}$. L is a continuous map from D into H and $F : D \to D$ is C^{k-1}; moreover, $F(U) = O(\|U\|_D^2)$.

The evolution problem (3.3.14) is an ill-posed initial value problem in D; however, it still has bounded solutions for all $t \in \mathbb{R}$; moreover, center-manifold theory allows to reduce this system, locally, to a finite (8-)dimensional system of ordinary differential equations (see below).

Due in part to certain reversibility properties satisfied by (3.3.14), the spectrum of the linear operator L is symmetric with respect to the real and imginary axes; moreover, it satisfies the separability property required in [53], its eigenvalues are isolated and have finite multiplicities; furthermore, the central part of the spectrum, $\sigma = \mathrm{i}q$, satisfies the dispersion relation

$$(-q^2 + T^2(1+2\varepsilon))^2 \;=\; 4(\varepsilon T^2)^2 \cos^2(q/2)\,.$$

The set of equations $N(\mathrm{i}q, \varepsilon, T) = 0$, $dN(\mathrm{i}q, \varepsilon, T)/dq = 0$ define a so-called bifurcation curve which can be parametrized by q. On the bifurcation curve (except for countably many points) the part of the spectrum of L that lies on the imaginary axis consists of two distinct pairs of simple eigenvalues and one pair of double eigenvalues, $\pm\mathrm{i}q_0$, thus, the L-invariant subspace associated to the central part of the spectrum of L is 8-dimensional; call Δ be the region of Γ where this is the case. Now, let P represent the spectral projection onto this L-invariant 8-dimensional subspace and set $U_h = (I - P)U$, then a relation of the form $U_h(t) = \psi(U_c(t), \varepsilon, T)$ is valid locally, where U is a solution to (3.3.14). As for the central

projection $U_c = PU$, one has that

$$\frac{dU_c}{dt} \;=\; LU_c + PF(U_c + \psi(U_c, \varepsilon, T))\,. \tag{3.3.15}$$

Conversely, if U_c satisfies (3.3.15) then $U = U_c + \psi(U_c, \varepsilon, T)$ is a solution to (3.3.14). We summarize the main result in [95] as follows:

Theorem 4b *(Sire and James, 2004)*
For $(\varepsilon_0, T_0) \in \Delta_0 \subset \Delta$, consider $(\varepsilon, T) \approx (\varepsilon_0, T_0)$ such that L has four symmetric eigenvalues close to $\pm i q_0$ with nonzero real parts. Then, up to order four, the normal form expression of (3.3.15) admits homoclinic orbits to 2-dimensional tori. These solutions correspond to the principal part of travelling breather solutions to (3.3.12), superposed at infinity to an oscillatory tail.

As a final remark we mention that solutions to lattice systems that satisfy (3.3.13) and the spatial localization condition $\lim_{|n|\to\infty} u_n(t) = 0$ are called *exact travelling breathers*; the study of this type of breathers is somewhat delicate as they might not exist without an exponentially small oscillatory part whose presence violates the spatial localization. In their follow-up work [54], James and Sire used the center-manifold reduction technique to prove the existence of exact small-amplitude travelling breathers superposed on an exponentially small oscillatory tail for the case $p = 2$ for certain regions in the parameter space of the breather's period T and the coupling strength ε. The authors proved that, in the case of an even on-site potential, these solutions arise as homoclinic orbits to small periodic ones of a reduced (8-dimensional) reversible system of differential equations which, written in normal form, is integrable up to higher order terms.

CHAPTER IV

QUASI-PERIODIC BREATHERS IN HAMILTONIAN LATTICES

Chapter summary

In [7] S. Aubry suggested that in order to prove the existence of quasi-periodic breather solutions in lattice systems, a method was needed that related the concepts of KAM theory and of anti-integrability. In [108] X. Yuan adapted the infinite-dimensional KAM methodology to show that such solutions are possible in a network of weakly coupled anharmonic oscillators with nearest-neighbor interaction. In this chapter we further develop this approach by proving an abstract theorem of the infinite-dimensional KAM type (Theorem B), which, in particular, implies the existence of linearly stable quasi-periodic breathers for a 1d Hamiltonian network of weakly coupled anharmonic oscillators with *long-range* interaction potential, in the case of constant normal frequencies (Theorem A). Our abstract KAM theorem applies to a broader class of lattices which includes that considered by Yuan.

We will start by motivating the model considered in Theorem A, stating Theorem B and showing how the former follows as a corollary to the latter. The remaining sections in the chapter are dedicated to proving the abstract KAM theorem, in particular, the details of the first KAM step are presented in detail, followed by an iteration lemma, convergence and measure estimates. To avoid overcrowding, some technical steps were included in appendices A and B.

4.1 A prototype 1d lattice and its physical justification

In the rest of this work we will adopt the following notation, $(q_n)_{n\in\mathbb{Z}} = \{q_n\}$. Let, as before, $\{(p_n, q_n)\}$ denote the set of conjugate-variable pairs and consider a real analytic Hamiltonian of the following form,

$$H = \sum_{n\in\mathbb{Z}} (\frac{p_n^2}{2} + V_n(q_n)) + \varepsilon W(\{q_n\}), \tag{4.1.1}$$

where W is a coupling potential of the following long-range type,

$$W(\{q_n\}) \,=\, \sum_{m\neq n} \frac{1}{p} C_{m,n}(q_m - q_n)^p \,, \tag{4.1.2}$$

such that $p \geq 3$, $C_{m,n} = C_{n,m} = O(e^{-\beta|m-n|})$ are coupling coefficients of the exponential type, $m, n \in \mathbb{Z}$ and $\beta > 0$. We will assume that the on-site potentials V_n, $n \in \mathbb{Z}$, satisfy conditions (*V1*) and (*V2*) below:

(*V1*: local convexity) *for all* $n \in \mathbb{Z}$, $V_n(0) = V_n'(0) = 0$ *and* $V_n''(0) = \beta^2/2 > 0$.

Note that (*V1*) implies that the on-site potentials are locally convex and thus one can also assume, without loss of generality, that for any given $N \in \mathbb{Z}^+$ and set of indices $\mathcal{J} = \{n_1, \ldots, n_N\}$, there exists a compact interval $\mathcal{I} \subset \mathbb{R}^+$, possibly depending on $\mathcal{J}$, such that, for every $n_i \in \mathcal{J}$ and $h \in \mathcal{I}$, the equation

$$\frac{p^2}{2} + V_{n_i}(q) \,=\, h \,,$$

defines a simple and closed curve on the pq-plane, Γ_i, that encloses the origin.

(*V2*: anharmonicity) *Let*

$$2\pi\rho_i(h) \,\stackrel{def}{=}\, \oint_{\Gamma_i} p\, dq$$

be the area enclosed by Γ_i, *then*

$$\rho_i'(h) \neq 0 \quad \text{and} \quad \rho_i''(h) \neq 0 \quad \forall\, h \in \mathcal{I} \,.$$

We claim that our hypothesis (*V2*) is a reasonable one. Indeed, for example, if the Taylor expansion of $V_n(x)$ around $x = 0$ has at least one kth-order term with $k \geq 3$, then one can show (cf. [108], cor. 1.1) that in fact $\rho_n(h) = ah + bh^{k/2} + o(h^{k/2})$, where a and b are nonzero constants thus, if for example, V_n was even and had a fourth-order term, (*V2*) would be automatically satisfied. We say that V_{n_i} is *anharmonic* if $\rho_i''(h) \neq 0$ for all $0 < h \ll 1$, thus (*V2*) says that, the oscillators V_{n_i}, $n_i \in \mathcal{J}$, are anharmonic over $\mathcal{I}$. The anharmonicity condition will allow us to fix a finite number of frequencies close to zero that

we will use as parameters needed to apply the abstract KAM theorem stated in the next section. We note in passing that the sinusoidal on-site potential used in the classical FK model, $V_{FK} = 1 - \cos\frac{2\pi}{a_s}x$, is an example of an anharmonic potential.

Our choice of Hamiltonian (4.1.1) is motivated by the need to model physical systems such as those arising in the adsorption of atoms (adatoms) on crystal surfaces, these systems can exhibit different types of excitations depending on the charge of the adatoms, the nature of the substrate, etc. A popular lattice model in the analysis of adsorption problems consists of the adatoms which, in equilibrium, lie periodically along a straight line every a_s units of distance apart, and are subjected to an on-site potential described by a single smooth a_s-periodic function, each one of its energy wells contains exactly one adatom. More precisely, this model assumes that, for every $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, $V_n(x) := V(na_s + x) = V(x)$, $V(na_s) = V'(na_s) = 0$ and $V''(na_s) = \frac{\beta^2}{2} > 0$, so that *(V1)* is automatically satisfied. Once again, the classical FK model fits the description just given.

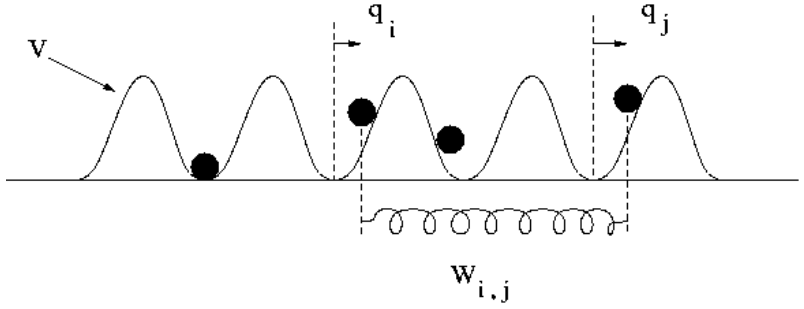

Figure 10: *Pictorial representation of Hamiltonian network (4.1.1). The on-site potentials are periodic-like, in the sense of (V1) and (V2), whereas the interaction potential is third-order or above and the interaction strength decreases exponentially in the sense of (4.1.2).*

As for our choice of interaction potential (4.1.2), there are basically four types of interaction potentials of physical relevance that arise in lattice adsorption models (cf. [23]) these are, *exponential:* $W_{\exp}(x) = W_0 e^{-\beta(x-a_s)}$, *power-law:* $W_{pl}(x) = W_0(a_s/x)^n$, *Morse potential:* $W_M(x) = W_0(e^{-2\beta(x-a_s)} - 2e^{-\beta(x-a_s)})$ and *double-well:* $W_{dw}(x) = W_0(\frac{1}{2}\beta^4(x - a_0)^4 - \beta^2(x - a_0)^2)$. In all these expressions, x represents the distance between any two adatoms (not necessarily adjacent) in the chain at a particular (fixed) time and W_0 is the energy of interaction between nearest-neighbor adatoms in equilibrium. These potentials

are commonly used in systems with long-range interactions, the exponential and power-law potentials are convex and repulsive, whereas Morse and double-well are nonconvex (cf. figure 11). Here we would like to focus on the exponential potential.

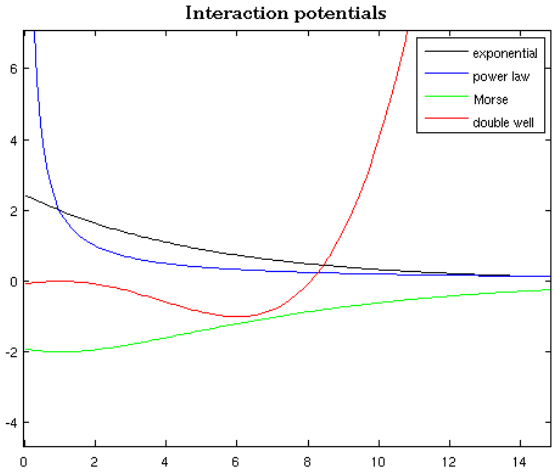

Figure 11: *Interaction potentials with parameters* $a_s = 1$, $\beta = 0.2$, $W_0 = 2$, $n = 1$.

The exponential potential,

$$W_{\exp}(x) \;=\; W_0 e^{-\beta(x-a_s)}\,, \tag{4.1.3}$$

is typically found in lattices with neutral atoms that interact with one another via their electronic clouds. Observe that if $x < a_s$, which can be the case when two adatoms are vibrating in adjacent energy wells, the system is at a higher energy configuration in regards to its equilibrium configuration, thus, in order to lower their energy, it is necessary for adatoms to move farther apart, at which point the interaction with the other adatoms in the chain becomes relevant. Note also that the farther apart, the smaller the interaction energy between adatom pairs.

To justify our choice of the interaction potential (4.1.2), we will write down the Taylor series of the exponential term in (4.1.3). To this end, let the equilibrium position of the 0th adatom be identified as the coordinate origin. Let x_m be the absolute position of the mth adatom with respect to the origin and let q_m be its relative position with respect to its equilibrium state; i.e., $x_m = ma_s + q_m$, then the distance between the mth and nth adatoms is given by $d_{m,n} = |x_m - x_n| = |m-n|a_s + \sigma(q_m - q_n)$ where $\sigma = \text{sign}(m-n)$. Substituting

$d_{m,n}$ in $W_{\exp}$ and adding over all different pairs to obtain the total interaction energy we get that

$$\begin{aligned} W &= \frac{1}{2}\sum_{m\neq n} W_{\exp}(|x_m - x_n|) \\ &= \frac{1}{2}\sum_{m\neq n} W_0 e^{-\beta(|m-n|a_s+\sigma(q_m-q_n)-a_s)} = \frac{1}{2}\sum_{m\neq n} W_0 e^{-\beta(|m-n|-1)a_s} e^{-\beta\sigma(q_m-q_n)} \\ &= \frac{1}{2}\sum_{m\neq n} W_0 e^{-\beta(|m-n|-1)a_s}(1-(\beta\sigma)(q_m-q_n)+\frac{1}{2!}(\beta\sigma)^2(q_m-q_n)^2 - \\ &\qquad\qquad \frac{1}{3!}(\beta\sigma)^3(q_m-q_n)^3+\cdots) \\ &= \sum_{m>n} W_0 e^{-\beta(m-n-1)a_s}(1-\beta(q_m-q_n)+\frac{\beta^2}{2!}(q_m-q_n)^2-\frac{\beta^3}{3!}(q_m-q_n)^3+\cdots). \end{aligned}$$

From the above we see that our interaction potential (4.1.2) corresponds to considering only the pth-order terms in the Taylor expansion of $W_{\exp}$ in relative coordinates. The idea of expanding in its Taylor series a fundamental interaction potential and considering only a particular higher-order term is frequently found in Physics and it has the purpose of identifying the effects of such term in the dynamics of the system. For instance, if in the expansion above one considers only the nearest-neighbor cubic terms together with all $(m \neq n)$ quadratic terms, a potential of the so-called Kac-Baker form is obtained (cf. [23] and references therein).

For the sole purpose of simplifying the exposition, from this point on till the end of this chapter, we will only consider the case $p = 3$ in (4.1.2); the proofs of theorems A and B, which we state in the next section, will still hold with minor obvious modifications. Thus, for those parameter values, the associated Hamilton's equations are

$$\frac{d^2 q_n}{dt^2} + V_n'(q_n) = \varepsilon \sum_{k=1}^{\infty} C_k[(q_{n+k}-q_n)^2 - (q_n - q_{n-k})^2], \quad n \in \mathbb{Z}, \tag{4.1.4}$$

where $C_k = O(e^{-\beta k})$, $\beta > 0$.

4.2 Statements of theorems A and B

Let A be a Lebesgue-measurable set and $|A|$ be its Lebesgue measure. For any given $k \geq 0$ we say that a function f is C^k in the sense of Whitney (also C^k-Whitney) in a compact set $P \subset \mathbb{R}^N$, if for every $x \in P$ one can find polynomials F_x of degree less than k such that

$f(x) = F_x(x)$, $x \in P$, and $|D^i F_x(y) - D^i F_x(x)| \leq |x-y|^{r-i}\sigma(|x-y|)$, $x, y \in P$, such that σ is a function that tends to zero.

Consider the Hamiltonian system (4.1.1) with interaction potential (4.1.2) and whose on-site potential satisfies $(V1)$ and $(V2)$. We will prove the following,

Theorem A. *For any integer $N > 1$, set of indices $\mathcal{J} = \{n_1, \ldots, n_N\}$ and $0 < r_0 \ll 1$, there exists $r_1 > r_0$ depending on $\mathcal{J}$, a family of Cantor sets $\mathcal{O}_\varepsilon \subset \mathcal{O} := \{\xi \in \mathbb{R}^N_+ : r_0 \leq |\xi| \leq r_1\}$ for $|\varepsilon| \ll 1$, with $|\mathcal{O} \setminus \mathcal{O}_\varepsilon| \to 0$ as $\varepsilon \to 0$ and Whitney smooth maps $\omega_* : \mathcal{O}_\varepsilon \to \mathbb{R}^N_+$, such that every $\xi \in \mathcal{O}_\varepsilon$ corresponds to a linearly stable, quasi-periodic breather $q(t) = \{q_n(t)\}$ of (4.1.4) with N-frequency $\omega_*(\xi)$. Moreover, $|q| = O(\sqrt{|\xi|})$ and $|q_n| \sim e^{-\beta|n|}$ for $|n| \gg 1$.*

The proof of theorem A will follow as a corollary of an abstract infinite-dimensional KAM type of theorem via appropriate normal form reductions. We advance that the abstract KAM theorem can in effect be applied to any Hamiltonian network of weakly coupled oscillators with constant frequencies and an interaction potential of the long-range type.

To state the abstract KAM theorem we need to first introduce some notation. Let $N \in \mathbb{Z}$, $N > 1$, and $r, s \in \mathbb{R}_+$ be given. Let $D(r,s)$ denote the complex neighborhood of $\mathbb{T}^N \times \{0\} \times \{0\} \times \{0\} \subset \mathbb{T}^N \times \mathbb{R}^N \times \ell^1 \times \ell^1$ given by

$$D(r,s) \,=\, \{(\theta, I, w, \bar{w}) : \, |\mathrm{Im}\theta| < r, \, |I| < s^2, \, \|w\| < s, \, \|\bar{w}\| < s\} \,,$$

where $|\cdot|$ is the sup-norm of complex vectors and $\|\cdot\|$ is the usual ℓ^1-norm. Let $\mathcal{O} \subset \mathbb{R}^N$ be such that $|\mathcal{O}| > 0$.

Let $F(\theta, I, w, \bar{w})$ be a real analytic function on $D(r,s)$ which depends C^1-Whitney smoothly on a parameter $\xi \in \mathcal{O}$. From now on, all dependencies on ξ will be assumed C^1-Whitney, thus all derivatives with respect to ξ shall be interpreted in this sense. The Taylor-Fourier series expansion of F in $\theta, I, w, \bar{w}$ is given by

$$F(\theta, I, w, \bar{w}) \,=\, \sum_{\alpha,\beta} F_{\alpha\beta} w^\alpha \bar{w}^\beta \,,$$

where $\alpha \equiv (\ldots, \alpha_n, \ldots)$, $\beta \equiv (\ldots, \beta_n, \ldots)$, $\alpha_n, \beta_n \in \mathbb{N}$, are multi-indices with only finitely

many nonzero components and

$$F_{\alpha\beta} = \sum_{k\in\mathbb{Z}^N, l\in\mathbb{N}^N} F_{kl\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k,\theta\rangle} ,$$

where $\langle\cdot,\cdot\rangle$ stands for the usual inner product in $\mathbb{C}^N$. Let us define the *weighted norm of* F as follows,

$$\|F\|_{D(r,s),\mathcal{O}} := \sup_{\substack{\|w\|<s\\ \|\bar{w}\|<s}} \sum_{\alpha,\beta} \|F_{\alpha\beta}\| \, |w^\alpha| \, |\bar{w}^\beta| , \tag{4.2.1}$$

where

$$\|F_{\alpha\beta}\| := \sum_{k,l} |F_{kl\alpha\beta}|_{\mathcal{O}} s^{2|l|} e^{|k|r} , \quad |F_{kl\alpha\beta}|_{\mathcal{O}} := \sup_{\xi\in\mathcal{O}} \{|F_{kl\alpha\beta}| + |\frac{\partial F_{kl\alpha\beta}}{\partial \xi}|\} .$$

In the case of a vector-valued function, say, $G : D(r,s)\times\mathcal{O} \to \mathbb{C}^N$, $N < \infty$, we will define its weighted norm as

$$\|G\|_{D(r,s),\mathcal{O}} := \sum_{i=1}^{N} \|G_i\|_{D(r,s),\mathcal{O}} .$$

The weighted norm of the Hamiltonian vector field

$$X_F = (F_I, -F_\theta, \{\mathrm{i}F_{w_n}\}, \{-\mathrm{i}F_{\bar{w}_n}\})$$

associated to a Hamiltonian function F on $D(r,s)\times\mathcal{O}$, will be defined by

$$\|X_F\|_{D(r,s),\mathcal{O}} := \|F_I\|_{D(r,s),\mathcal{O}} + \frac{1}{s^2}\|F_\theta\|_{D(r,s),\mathcal{O}} + \frac{1}{s}(\sum_n \|F_{w_n}\|_{D(r,s),\mathcal{O}} + \sum_n \|F_{\bar{w}_n}\|_{D(r,s),\mathcal{O}}) .$$

Associated with the symplectic structure $dI\wedge d\theta + \mathrm{i}\sum_{n\in\mathbb{Z}} dw_n \wedge d\bar{w}_n$, we will consider an N-parametric family of real-analytic Hamiltonians

$$\begin{aligned} H &= \mathcal{N} + \mathcal{P} , \\ \mathcal{N} &= \langle\omega(\xi), I\rangle + \sum_{n\in\mathbb{Z}} \Omega_n w_n \bar{w}_n , \\ \mathcal{P} &= \mathcal{P}(\theta, I, w, \bar{w}, \xi) , \end{aligned} \tag{4.2.2}$$

where $(I,\theta,w,\bar{w}) \in D(r,s)$, $\xi\in\mathcal{O}\subset\mathbb{R}^N$ is the multi-parameter, $\omega : \mathcal{O}\to\mathbb{R}^N$ is C^1-Whitney smooth, Ω_n, $n\in\mathbb{Z}$, are positive constants independent of ξ for all n, and $\mathcal{P}$ is real-analytic with respect to phase variables and C^1-Whitney smooth in ξ.

Observe that if we set $\mathcal{P}\equiv 0$, then $H = N$ and the associated Hamiltonian system is completely integrable and admits a family of quasi-periodic solutions $(\theta+\omega(\xi)t, 0, 0, 0)$

corresponding to invariant N-tori in phase space. To demonstrate the persistence of some of these N-tori, we need the following assumptions on $\omega(\xi)$, Ω_n and the perturbation P:

(*A1:* **Non-degeneracy of tangential frequencies)** $\omega : \mathcal{O} \to \mathbb{R}^N$ is C^1-Whitney smooth and non-degenerate, that is, there is a constant $\delta > 0$ such that

$$|\det(\frac{\partial \omega(\xi)}{\partial \xi})| \geq \delta\,, \quad \forall\, \xi \in \mathcal{O}\,.$$

(*A2:* **Constant normal frequencies)**

$$\Omega_n \equiv \beta > 0\,, \quad \forall\, n \in \mathbb{Z}\,.$$

(*A3:* **Reality and regularity of the perturbation)** $\mathcal{P}$ is real-analytic with respect to the space coordinates and C^1-Whitney smooth in ξ. In particular,

$$P \,=\, \bar{P}\,. \tag{4.2.3}$$

(*A4:* **Decay property of the perturbation)** we will assume that

$$\mathcal{P} \,=\, \breve{P} + \acute{P} + \grave{P}\,,$$

where

$$\breve{P} = \breve{P}(\theta, I, w, \bar{w}, \xi) \,=\, \breve{P}(\theta, I, 0, 0, \xi) \,+ \sum_{\substack{n \in \mathbb{Z} \\ \alpha_n + \beta_n \geq 1}} \breve{P}_n(\theta, I, \xi) w_n^{\alpha_n} \bar{w}_n^{\beta_n}\,,$$

is such that there exists $\beta > 0$ for which

$$\|\breve{P}_n(\theta, I, \xi)\| \,\leq\, e^{-\beta|n|} \quad \forall\, n \in \mathbb{Z}\,; \tag{4.2.4}$$

$$\acute{P} = \acute{P}(w, \bar{w}, \xi) \,=\, \sum_{\substack{n,m \in \mathbb{Z}, n \neq m \\ \alpha_n + \beta_n, \alpha_m + \beta_m \geq 1 \\ \alpha_n + \beta_n + \alpha_m + \beta_m \geq 3}} \acute{P}_{nm}(\xi) w_n^{\alpha_n} \bar{w}_n^{\beta_n} w_m^{\alpha_m} \bar{w}_m^{\beta_m}\,,$$

is such that

$$\|\acute{P}_{nm}(\xi)\| \,\leq\, e^{-\beta|n-m|} \quad \forall\, n, m \in \mathbb{Z}\ n \neq m\,; \tag{4.2.5}$$

and

$$\grave{P} = \grave{P}(w, \bar{w}, \xi) \,=\, \sum_{n \in \mathbb{Z}} O(|w_n|^3)\,. \tag{4.2.6}$$

Our abstract KAM theorem states the following,

Theorem B. *Consider the Hamiltonian* (4.2.2) *and assume* (*A*1)-(*A*4)*, then for fixed* $\gamma > 0$ *sufficiently small, there exists a positive constant* $\varepsilon = \varepsilon(\mathcal{O}, N, \gamma, r, s)$ *such that if* $\|X_{\mathcal{P}}\|_{D(r,s),\mathcal{O}} < \varepsilon$*, then the following holds: there exist Cantor sets* $\mathcal{O}_\gamma \subset \mathcal{O}$ *with* $|\mathcal{O} \setminus \mathcal{O}_\gamma| = O(\gamma)$ *and maps* $\Psi : \mathbb{T}^N \times \mathcal{O}_\gamma \to D(r,s)$, $\tilde{\omega} : \mathcal{O}_\gamma \to \mathbb{R}^N$*, which are real-analytic in* θ *and* C^1*-Whitney smooth in* ξ*, with* $\|\Psi - \Psi_0\|_{D(\frac{r}{2},0),\mathcal{O}_\gamma} \to 0$ *and* $|\tilde{\omega} - \omega|_{\mathcal{O}_\gamma} \to 0$ *as* $\gamma \to 0$*, where* Ψ_0 *is the trivial embedding:* $\mathbb{T}^N \times \mathcal{O} \to \mathbb{T}^N \times \{0,0,0\}$*, such that each* $\xi \in \mathcal{O}_\gamma$ *and* $\theta \in \mathbb{T}^N$ *correspond to a linearly stable,* N*-frequency quasi-periodic solution* $\Psi(\theta + \tilde{\omega}(\xi)t, \xi) = (\theta + \tilde{\omega}(\xi)t, I(t), \{w_n(t)\}, \{\bar{w}_n(t)\})$ *of the system of equations with Hamiltonian* (4.2.2)*; moreover,* $|w_n| \sim e^{-\beta|n|}$.

4.2.1 Proof of Theorem A

Recall that for simplicity of exposition we will take $p = 3$ in our interaction potential (4.1.2) and that we will assume that the on-site potentials satisfy conditions (*V1*) and (*V2*), then our Hamiltonian (4.1.1) becomes,

$$H = \sum_{n\in\mathbb{Z}} (\frac{p_n^2}{2} + V_n(q_n)) + \varepsilon \sum_{m\neq n} \frac{1}{3} C_{m,n}(q_m - q_n)^3 \,, \tag{4.2.7}$$

$C_{m,n} = C_{n,m} = O(e^{-\beta|m-n|})$. We will now derive action-angle-normal variables for (4.2.7) and will show that, in these variables, H is in the form described in Theorem B, thus implying the statement in theorem A. The procedure outlined below to obtain the action-angle-normal variables is standard (cf. Appendix B, se also [108] or [4] p. 279).

Let $\mathcal{J} = \{n_1, \ldots, n_N\}$ be as in Theorem A, then a second-order Taylor expansion of the terms of the on-site potential associated to the set $\mathbb{Z}_1 = \mathbb{Z} \setminus \mathcal{J}$ yields

$$H = \sum_{n\in\mathcal{J}} (\frac{p_n^2}{2} + V_n(q_n)) + \sum_{n\in\mathbb{Z}_1} (\frac{p_n^2}{2} + \frac{\beta^2 q_n^2}{2} + O(|q_n|^3)) + \varepsilon \sum_{m\neq n} \frac{1}{3} C_{m,n}(q_m - q_n)^3 \,. \tag{4.2.8}$$

Let $\mathcal{I} \subset \mathbb{R}^+$ and $\rho_j(h)$, $j = 1, \ldots, N$, be as in (*V2*); i.e., $\rho_j(h) = \oint_{\Gamma_j} p\, dq$ and Γ_j is the simple closed curve given by $\frac{p^2}{2} + V_{n_j}(q) = h$; the ρ_j's are bounded above and below by positive

quantities, say, $0 < r_0' < \rho_j < r_1$. Without loss of generality we will further assume that $\mathcal{I}$ is such that, for every $n_j \in \mathcal{J}$, ρ_j is invertible over $\mathcal{I}$ and that its inverse, $H_{0,j} : \mathbb{R}_+ \to \mathcal{I}$, is differentiable. Thus, letting $(q_n, p_n) = (\frac{1}{\sqrt{\beta}} v_n, \sqrt{\beta}\bar{v}_n)$ for $n \in \mathbb{Z}_1$ be our normal-coordinate change, a standard action-angle-normal coordinate-reduction procedure (cf. Appendix B) leaves (4.2.8) in the following form

$$H \,=\, \sum_{j=1}^{N} H_{0,j}(\rho_j) + \sum_{n\in\mathbb{Z}_1} \frac{\beta}{2}(v_n^2 + \bar{v}_n^2) + O(|v_n|^3) + \varepsilon \sum_{m\neq n} \frac{1}{3} C_{m,n}(q_m - q_n)^3 \,, \qquad (4.2.9)$$

note the obvious abuse of notation in the last term. Now let $v_n = (w_n + \bar{w}_n)/\sqrt{2}$ and $\bar{v}_n = (w_n - \bar{w}_n)/\mathrm{i}\sqrt{2}$ for every $n \in \mathbb{Z}_1$, so that $dv_n \wedge d\bar{v}_n = \mathrm{i} dw_n \wedge d\bar{w}_n)$, then

$$H \,=\, \sum_{j=1}^{N} H_{0,j}(\rho_j) + \sum_{n\in\mathbb{Z}_1} \beta w_n \bar{w}_n + O(|w_n + \bar{w}_n|^3) + \varepsilon \sum_{m\neq n} \frac{1}{3} C_{m,n}(q_m - q_n)^3 \,, \qquad (4.2.10)$$

In order to write (4.2.10) in the normal form contained in Theorem B, we first introduce parameters ξ_j by setting

$$\rho_j \,=\, \xi_j + I_j \,, \quad j = 1, \ldots, N \,,$$

where $\xi \in \mathcal{O} = \{\xi \in \mathbb{R}_+^N : 0 < r_0 \leq |\xi| \leq r_1\}$, and expand each $H_{0,j}(\xi_j + I_j)$ in Taylor series about $I_j = 0$, then, by disregarding constant terms that do not affect the dynamics, we obtain

$$H \,=\, \langle \omega(\xi), I\rangle + \sum_{j=1}^{N} O(|I_j|^2) + \sum_{n\in\mathbb{Z}_1} \beta w_n \bar{w}_n + O(|w_n + \bar{w}_n|^3) + \varepsilon \sum_{m\neq n} \frac{1}{3} C_{m,n}(q_m - q_n)^3 \,,$$

where

$$\omega(\xi) \,=\, (H_{0,1}'(\xi_1), \ldots, H_{0,N}'(\xi_N)) \,.$$

Observe that, because of the anharmonicity of V_{n_j}, given any $0 < r_0 \ll 1$, there is $r_1 > r_0$ such that $\omega(\xi)$ is a local diffeomorphism from $\mathcal{O} = \{\xi \in \mathbb{R}_+^N : r_0 \leq |\xi| \leq r_1\}$ to $\omega(\mathcal{O})$; therefore, *(A1)* is satisfied.

Now introduce the rescaling $I_j \to \varepsilon^{1/2} I_j$ and $(w_n, \bar{w}_n) \to (\varepsilon^{1/4} w_n, \varepsilon^{1/4} \bar{w}_n)$, thus

$$H = \sqrt{\varepsilon}\langle \omega(\xi), I\rangle + \varepsilon \sum_{j=1}^{N} O(|I_j|^2) + \sum_{n\in\mathbb{Z}_1} \{\beta\sqrt{\varepsilon} w_n \bar{w}_n + \varepsilon^{\frac{3}{4}} O(|w_n + \bar{w}_n|^3)\} + \varepsilon \sum_{m\neq n} \frac{1}{3} C_{m,n}(q_m - q_n)^3 \,.$$

Dividing by $\sqrt{\varepsilon}$ and a further rescaling, $\varepsilon^{-1/2}H \to H$, and $\varepsilon^{1/4} \to \varepsilon$ we arrive at

$$H = \langle \omega(\xi), I \rangle + \sum_{n \in \mathbb{Z}_1} \beta w_n \bar{w}_n + \varepsilon \left(\sum_{n \in \mathbb{Z}_1} O(|w_n + \bar{w}_n|^3) + \varepsilon \sum_{j=1}^{N} O(|I_j|^2) + \varepsilon \sum_{m \neq n} \frac{1}{3} C_{m,n} (q_m - q_n)^3 \right) . \tag{4.2.11}$$

The Hamiltonian above is of the form

$$H = \langle \omega(\xi), I \rangle + \sum_{n \in \mathbb{Z}_1} \beta w_n \bar{w}_n + \varepsilon P(\theta, I, w, \bar{w}, \xi) ,$$

thus it belongs in the parametric family described by (4.2.2), modulo renumbering, with $\Omega_n = \beta > 0$ for all $n \in \mathbb{Z}_1$, thus property (*A2*) is satisfied. Moreover, (4.2.4) and (4.2.5) follow from the exponential decay of the coupling coefficients and the presence of terms $O(|w_n + \bar{w}_n|^3)$ imply that (4.2.6) is also satisfied; furthermore, $\mathcal{P}$ is clearly real-valued. Therefore, the existence of quasi-periodic solutions for the Hamiltonian network (4.1.1) follows from theorem B.

□

4.3 KAM Step

In this section we present the KAM iteration scheme applied to (4.2.2). This is a succession of infinitely many steps (KAM steps or iterations) whose purpose is that of eliminating lower-order θ-dependent terms in P. At each KAM step the perturbation is made smaller at the cost of excluding a small-measure set of parameters. It will be shown that the KAM iterations converge and that, in the end, the total measure of the set of parameters that has been excluded is small. For simplicity in exposition, we have set $\beta = 1$ in (4.2.4) and in (4.2.5).

To begin the KAM iteration, let us set $r_0 = r$ and $\gamma_0 = \gamma$. We also recall that $\Omega_n = \beta$ for all n.

4.3.1 Normal form

In order to perform the KAM iteration scheme, we will first write the Hamiltonian (4.2.2) into a form that is more convenient for this purpose. Let $\varepsilon_* \sim \varepsilon^{\frac{5}{4}}$ and $K_0 = |\ln \varepsilon_*|$. The reader is asked to take note of the following constants that will appear later on, $\tilde{K}_0 = 5K_0$ and $\varepsilon_0 = \varepsilon_*^{\frac{5}{4}}$. We will also set s_0 such that $0 < s_0 < \min\{\varepsilon_0, s\}$.

Observe that, according to (4.2.5) and (4.2.6) in assumption (A4), it follows from lemma A.0.6 in Appendix A (generalized Cauchy inequality) that one can make s_0 smaller if necessary such that

$$\|X_{\acute{P}+\grave{P}}\|_{D(r_0,s_0),\mathcal{O}} \leq \varepsilon_* \, .$$

We now consider the term $\breve{P}$. According to (4.2.4) and the definition of norm (4.2.1) we have

$$\begin{aligned}
\breve{P} = & \ \breve{P}(\theta, I, 0, 0, \xi) + \sum_{\substack{n\in\mathbb{Z} \\ \alpha_n+\beta_n\geq 1}} \breve{P}_n(\theta, I, \xi) w_n^{\alpha_n} \bar{w}_n^{\beta_n} \\
= & \sum_{k,l} \breve{P}_{kl} I^l e^{\mathrm{i}\langle k,\theta\rangle} \ + \sum_{\substack{n,k,l \\ \alpha_n+\beta_n\geq 1}} \breve{P}_n^{kl\alpha_n\beta_n} I^l e^{\mathrm{i}\langle k,\theta\rangle} \ w_n^{\alpha_n} \bar{w}_n^{\beta_n} \ ,
\end{aligned}$$

where

$$\|\breve{P}_{kl}\| \leq e^{-|k|r_0} \ , \quad \|\breve{P}_n^{kl\alpha_n\beta_n}\| \leq e^{-|k|r_0} e^{-|n|} \ . \tag{4.3.1}$$

Let

$$R \ = \ \sum_{\substack{k \\ |l|\leq 1}} \breve{P}_{kl} I^l e^{\mathrm{i}\langle k,\theta\rangle} \ + \sum_{\substack{k,|n|\leq K_0 \\ 1\leq\alpha_n+\beta_n\leq 2}} \breve{P}_n^{k\alpha_n\beta_n} e^{\mathrm{i}\langle k,\theta\rangle} \ w_n^{\alpha_n} \bar{w}_n^{\beta_n} \, ,$$

so that

$$\breve{P} - R \ = \sum_{\substack{|n|>K_0 \\ \alpha_n+\beta_n\geq 1}} \breve{P}_n^{k\alpha_n\beta_n} e^{\mathrm{i}\langle k,\theta\rangle} \ w_n^{\alpha_n} \bar{w}_n^{\beta_n} + O(|I|^2) + O(|w|^3) \, ,$$

then it follows from (4.3.1) and lemma A.0.6 in Appendix A that one can make s_0 small enough so that

$$\|X_{\breve{P}-R}\|_{D(r_0,s_0),\mathcal{O}} \ \leq \ \varepsilon_* \, .$$

To handle the term R we will first construct a symplectic transformation $\Phi_* = \Phi_{F_*}^1$ defined

as the time-1 map of the Hamiltonian flow associated to a Hamiltonian F_* of the form

$$F_* = \bar{F}_* = \sum_{k\neq 0, |l|\leq 1} F_{kl} I^l e^{\mathrm{i}\langle k,\theta\rangle} + \sum_{k, |n|\leq K_0} (F_n^{k10} w_n + F_n^{k01} \bar{w}_n) e^{\mathrm{i}\langle k,\theta\rangle} + \sum_{k, |n|\leq K_0} (F_{nn}^{k20} w_n w_n + F_{nn}^{k02} \bar{w}_n \bar{w}_n) e^{\mathrm{i}\langle k,\theta\rangle} + \sum_{k\neq 0, |n|\leq K_0} F_{nn}^{k11} w_n \bar{w}_n e^{\mathrm{i}\langle k,\theta\rangle} ,$$

such that all resonant terms $\{\breve{P}_{kl} I^l e^{\mathrm{i}\langle k,\theta\rangle} : k \neq 0, |l| \leq 1\}$ and $\{\breve{P}_n^{k\alpha_n\beta_n} e^{\mathrm{i}\langle k,\theta\rangle} w_n^{\alpha_n} \bar{w}_n^{\beta_n} : k \neq 0, |n| \leq K_0, \alpha_n + \beta_n \leq 2\}$ will be eliminated, whereas terms $\{\breve{P}_{0l} I^l : |l| \leq 1\}$, $\{\breve{P}_{nn}^{011} w_n \bar{w}_n : |n| \leq K_0\}$ will be added to the normal form part of the new Hamiltonian. More precisely, let F_* satisfy the homological equation

$$\{N, F_*\} + R = \sum_{|l|\leq 1} \breve{P}_{0l} I^l + \sum_{|n|\leq K_0} \breve{P}_{nn}^{011} w_n \bar{w}_n . \tag{4.3.2}$$

One can show that this homological equation is solvable on the following parameter set

$$\mathcal{O}_* = \left\{ \xi \in \mathcal{O} : \begin{array}{l} |\langle k,\omega\rangle| \geq \frac{\gamma}{|k|^\tau}, \quad k \neq 0 \\ |\langle k,\omega\rangle + \beta| \geq \frac{\gamma}{|k|^\tau}, \\ |\langle k,\omega\rangle + 2\beta| \geq \frac{\gamma}{|k|^\tau}, \end{array} \right\} .$$

In this way we obtain a transformation Φ_* which transforms the Hamiltonian (4.2.2) into

$$H_* = H \circ \Phi_* = \mathcal{N}_* + \breve{P}_* + \acute{P} + \grave{P} = \bar{N}_* + \bar{\breve{P}}_* + \bar{\acute{P}} + \bar{\grave{P}}$$

where

$$\mathcal{N}_* = e_* + \langle \omega_*(\xi), I\rangle + \sum_{|n|\leq K_0} \Omega_n^* w_n \bar{w}_n + \sum_{|n|>K_0} \beta w_n \bar{w}_n ,$$

such that $\omega* = \omega + \breve{P}_{0l}(|l| = 1)$, $\Omega_n^* = \beta + \breve{P}_{nn}^{011}$, and

$$\breve{P}_* = \breve{P}^*(\theta, I, w_{n(|n|\leq K_0)}, \bar{w}_{n(|n|\leq K_0)}, \xi) + \sum_{\substack{|n|>K_0 \\ \alpha_n+\beta_n\geq 1}} \breve{P}_n^*(\theta, I, w_{m(|m|\leq K_0)}, \bar{w}_{m(|m|\leq K_0)}, \xi) w_n^{\alpha_n} \bar{w}_n^{\beta_n}$$

satisfies

$$\|\breve{P}_n^*(\theta, I, w_{m(|m|\leq K_0)}, \bar{w}_{m(|m|\leq K_0)}, \xi)\| \leq e^{-(|n|-K_0)} .$$

The first and second terms in the above expression for $\breve{P}_*$ come from $P\circ\Phi_*$ and $\breve{P}\circ\Phi_* + \acute{P}\circ\Phi_*$, respectively. Moreover, the decay property of $\breve{P}_n^*$ follows from the fact that Φ_* depends only on I, θ and w_m, $\bar{w}_m$ for $|m| \leq K_0$.

Now let us write the second term in $\breve{P}_*$ as follows,

$$\sum_{\substack{|n|>K_0\\ \alpha_n+\beta_n\geq 1}} \breve{P}_n^*(\theta, I, w_{m(|m|\leq K_0)}, \bar{w}_{m(|m|\leq K_0)}, \xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n} \;=$$

$$\sum_{\substack{|n|>5K_0\\ \alpha_n+\beta_n\geq 1}} \breve{P}_n^*(\theta, I, w_{m(|m|\leq K_0)}, \bar{w}_{m(|m|\leq K_0)}, \xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n} \;+$$

$$\sum_{\substack{K_0<|n|\leq 5K_0\\ \alpha_n+\beta_n\geq 1}} \breve{P}_n^*(\theta, I, w_{m(|m|\leq K_0)}, \bar{w}_{m(|m|\leq K_0)}, \xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n}\,.$$

It is not difficult to see that, on $D(r_0, s_0)\times\mathcal{O}_*$, the norm of the vector field associated with the first term above is bounded by ε_*^2; however, due to the multiple normal frequency condition *(A2)*, terms of the form $\breve{P}_{mn}^* w_m\bar{w}_n + \breve{P}_{nm}^* w_n\bar{w}_m$ in the second term will not be canceled by solving a homological equation, hence they need to be included in the normal form part of the Hamiltonian. More precisely, let $\tilde{K}_0 = 5K_0$ and

$$R_* \;=\; \sum_{k,|l|\leq 1} \breve{P}_{kl}^* I^l e^{\mathrm{i}\langle k,\theta\rangle} \;+\; \sum_{\substack{k,|m|\leq K_0, |n|\leq \tilde{K}_0\\ 1\leq\alpha_m+\beta_n\leq 2}} \breve{P}_{mn}^{*k\alpha_m\beta_n} e^{\mathrm{i}\langle k,\theta\rangle}\,(w_m^{\alpha_m}\bar{w}_n^{\beta_n} + \bar{w}_m^{\alpha_m} w_n^{\beta_n})$$

and

$$\begin{aligned} F_{**} = \bar{F}_{**} = &\sum_{k\neq 0,|l|\leq 1} f_{kl} e^{\mathrm{i}\langle k,\theta\rangle}\, I^l \;+\; \sum_{k,|n|\leq\tilde{K}_0} (f_n^{k10} w_n + f_n^{k01}\bar{w}_n) e^{\mathrm{i}\langle k,\theta\rangle} \;+\\ &\sum_{k,|m|\leq K_0,|n|\leq\tilde{K}_0} (f_{nm}^{k20} w_n w_m + f_{nm}^{k02}\bar{w}_n\bar{w}_m) e^{\mathrm{i}\langle k,\theta\rangle} \;+\\ &\sum_{\substack{k\neq 0\\ |m|\leq K_0,|n|\leq\tilde{K}_0}} (f_{nm}^{k11} w_n\bar{w}_n + f_{mn}^{k11} w_m\bar{w}_m) e^{\mathrm{i}\langle k,\theta\rangle} \;+\\ &\sum_{\substack{k\\ K_0<|n|\leq\tilde{K}_0}} (f_{nn}^{k20} w_n w_n + f_{nn}^{k02}\bar{w}_n\bar{w}_n) e^{\mathrm{i}\langle k,\theta\rangle} \;+\\ &\sum_{\substack{k\neq 0\\ K_0<|n|\leq\tilde{K}_0}} f_{nn}^{k11} w_n\bar{w}_n e^{\mathrm{i}\langle k,\theta\rangle} \end{aligned}$$

satisfy the homological equation

$$\{\mathcal{N}_*, F_{**}\} + R_* \;=\; \sum_{|l|\leq 1} \breve{P}_{0l}^* I^l \;+\; \sum_{\substack{|m|\leq K_0\\ |n|\leq\tilde{K}_0}} (\breve{P}_{nm}^{*011} w_n\bar{w}_m + \breve{P}_{nm}^{*011} w_m\bar{w}_n) \;+\; \sum_{K_0<|n|\leq\tilde{K}_0} \breve{P}_{nn}^{*011} w_n\bar{w}_n\,. \tag{4.3.3}$$

(4.3.3) is solvable on the domain

$$\mathcal{O}_0 = \left\{ \xi \in \mathcal{O}_* : \begin{array}{l} |\langle k, \omega_* \rangle| \geq \frac{\gamma}{|k|^\tau}\,, \quad k \neq 0 \\ |\langle k, \omega_* \rangle + \Omega_n^*| \geq \frac{\gamma}{|k|^\tau}\,, \quad |n| \leq K_0 \\ |\langle k, \omega_* \rangle + \beta| \geq \frac{\gamma}{|k|^\tau}\,, \\ |\langle k, \omega_* \rangle + \Omega_m^* + \Omega_n^*| \geq \frac{\gamma}{|k|^\tau}, \quad |m|, |n| \leq K_0 \\ |\langle k, \omega_* \rangle + \Omega_m^* + \beta| \geq \frac{\gamma}{|k|^\tau}\,, \quad |m| \leq K_0 \\ |\langle k, \omega_* \rangle + \Omega_m^* - \Omega_n^*| \geq \frac{\gamma}{|k|^\tau}\,, \quad k \neq 0,\, |m|, |n| \leq K_0 \\ |\langle k, \omega_* \rangle + \Omega_m^* - \beta| \geq \frac{\gamma}{|k|^\tau}, \quad k \neq 0,\, |m| \leq K_0 \\ |\langle k, \omega_* \rangle + 2\beta| \geq \frac{\gamma}{|k|^\tau}, \end{array} \right\}.$$

We now perform another symplectic transformation $\Phi_{**} = \Phi^1_{F_{**}}$, so that

$$H_0 \;=\; H_* \circ \Phi_{**} = \mathcal{N}_0 + \mathcal{P}_0 = \bar{\mathcal{N}}_0 + \bar{\mathcal{P}}_0\,,$$

where

$$\mathcal{N}_0 \;=\; e_0 + \langle \omega_0(\xi), I \rangle + \langle A^0 z^0, \bar{z}^0 \rangle + \sum_{|n| > \tilde{K}_0} \beta w_n \bar{w}_n\,,$$

and we have defined,

$$e_0 = \; e_* + \breve{P}^*_{00}\,,$$

$$\omega_0 = \; \omega_* + \breve{P}^*_{0l}(|l| = 1)\,,$$

and

$$\langle A^0 z^0, \bar{z}^0 \rangle \;=\; \sum_{|n| \leq \tilde{K}_0} \Omega_n^{**} w_n \bar{w}_n \;+ \sum_{\substack{m \neq n \\ |m| \leq K_0, |n| \leq \tilde{K}_0}} (\breve{P}^{*011}_{mn} w_m \bar{w}_n + \breve{P}^{*011}_{nm} w_n \bar{w}_m)\,,$$

where A^0 is a Hermitian matrix with $\dim(A^0) \leq \tilde{K}_0$ and we have set

$$z^0 \;=\; (\cdots, w_n, \cdots)_{|n| \leq \tilde{K}_0}\,, \quad \bar{z}^0 = (\cdots, \bar{w}_n, \cdots)_{|n| \leq \tilde{K}_0}\,,$$

and

$$\Omega_n^{**} \;=\; \Omega_n^* + \breve{P}^{*011}_{nn}\,.$$

Also,

$$\begin{aligned}
\mathcal{P}_0 &= \breve{P}_0 + \acute{P}_0 + \grave{P}_0\,,\\
\breve{P}_0 &= \breve{P}(\theta, I, z^0, \bar{z}^0, \xi) \;=\; \breve{P}^0(\theta, I, w_{n(|n|\le \tilde{K}_0)}, \bar{w}_{n(|n|\le \tilde{K}_0)}, \xi) \;+\\
&\qquad\qquad \sum_{\substack{|n|>\tilde{K}_0\\ \alpha_n+\beta_n\ge 1}} \breve{P}^0_n(\theta, I, w_{m(|m|\le \tilde{K}_0)}, \bar{w}_{m(|m|\le \tilde{K}_0)}, \xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n}\\
&\stackrel{\text{def}}{=}\; \breve{P}^0(\theta, I, z^0, \bar{z}^0, \xi) \;+\; \sum_{\substack{|n|>\tilde{K}_0\\ \alpha_n+\beta_n\ge 1}} \breve{P}^0_n(\theta, I, z^0, \bar{z}^0, \xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n}\,,
\end{aligned}$$

with

$$\|\breve{P}^0_n(\theta, I, z^0, \bar{z}^0, \xi)\| \le e^{-(|n|-\tilde{K}_0)}, \quad |n| > \tilde{K}_0\,;$$

$$\acute{P}_0 \;=\; \acute{P}(w, \bar{w}, \xi) = \sum_{\substack{n\ne m\\ \alpha_n+\beta_n, \alpha_m+\beta_m\ge 1\\ \alpha_n+\beta_n+\alpha_m+\beta_m\ge 3}} \acute{P}^0_{nm}(\xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n} w_m^{\alpha_m}\bar{w}_m^{\beta_m},$$

with

$$\|\acute{P}^0_{nm}(\xi)\| \le e^{-|n-m|}\,,$$

and

$$\grave{P}_0 \;=\; \grave{P}(w, \bar{w}, \xi) = \sum_{n\in\mathbb{Z}} O(|w_n|^3)\,.$$

Lastly, we also have that $\|X_{\mathcal{P}^0}\|_{D(r_0,s_0),\mathcal{O}_0} \le \varepsilon_*^{\frac{5}{4}} \stackrel{\text{def}}{=} \varepsilon_0$.

Now, suppose now that after the νth KAM step one arrives at the following Hamiltonian,

$$H \;\equiv\; H_\nu \;=\; \mathcal{N} + \mathcal{P} \;=\; \bar{\mathcal{N}} + \bar{\mathcal{P}} \;=\; \mathcal{N} + \breve{P} + \acute{P}_0 + \grave{P}_0\,,$$

where

$$\begin{aligned}
\mathcal{N} \;&=\; \mathcal{N}_\nu = \;\langle \omega(\xi), I\rangle + \langle Az, \bar{z}\rangle \;+\; \sum_{|n|>K} \beta w_n \bar{w}_n\,,\\
\breve{P} \;&=\; \breve{P}_\nu = \;\breve{P}(\theta, I, z^\nu, \bar{z}^\nu, \xi) \;+\; \sum_{\substack{|n|>K\\ \alpha_n+\beta_n\ge 1}} \breve{P}_n(\theta, I, z, \bar{z}, \xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n}\\
&\qquad =\; \breve{P}^\nu(\theta, I, z^\nu, \bar{z}^\nu, \xi) \;+\; \sum_{\substack{|n|>K_\nu\\ \alpha_n+\beta_n\ge 1}} \breve{P}^\nu_n(\theta, I, z^\nu, \bar{z}^\nu, \xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n}
\end{aligned}$$

are defined on a domain $D(r,s)\times\mathcal{O} = D(r_\nu, s_\nu)\times\mathcal{O}_\nu$, $K = K_\nu$ is a positive constant,

$$z = z^\nu = (\cdots, w_n, \cdots)_{|n|\le K}\,, \quad \bar{z} = \bar{z}^\nu = (\cdots, \bar{w}_n, \cdots)_{|n|\le K}\,,$$

$\mathcal{P} = \mathcal{P}^\nu$ is such that $\|X_{\mathcal{P}}\| < \varepsilon$ for some $\varepsilon = \varepsilon_\nu$ and

$$\|\breve{P}_n(\theta, I, z, \bar{z}, \xi)\|_{D(r,s),\mathcal{O}} \leq e^{-(|n|-K)}, \quad |n| > K .$$

In what follows, we will show how to construct a symplectic transformation, $\Phi = \Phi_\nu$, which, in smaller frequency and phase domains, carries the Hamiltonian $H = H_\nu$ into the next KAM cycle. We ask the reader to keep in mind that, in the remaining part of this section, all constants labeled $c_1 \ldots c_{12}$ are positive and independent of the iteration process. We will also denote the tensor (or direct) product between two matrices, $A = (a_{ij})$ and $B = (b_{nl})$, of sizes $m \times n$ and $k \times l$, respectively, by $A \otimes B$, and recall that this product results in a new $mk \times nl$ matrix given by

$$A \otimes B = (a_{ij}B) = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ldots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} .$$

We recall that it is custom to define the operator matrix norm by $\|M\| = \sup_{\|y\|=1} \|My\|$. In this work, however, if $A = (a_{ij})$, we will adopt the following definition for the operator norm of A,

$$\|A\| := \max\{\sup_i(\sum_j |a_{ij}|), \sup_j(\sum_i |a_{ij}|)\} .$$

Let $K_+ = 5K$, in the KAM step detailed below, terms w_n, $\bar{w}_n$ with $K < |n| \leq K_+$ will be added to the new normal components z^+, $\bar{z}^+$. To facilitate the calculations when solving a homological equation later on, we will also adopt the following notation when writing $\mathcal{N}$,

$$\begin{aligned} \mathcal{N} = \bar{\mathcal{N}} &= e + \langle \omega(\xi), I\rangle + \langle Az, \bar{z}\rangle + \sum_{K<|n|\leq K_+} \beta w_n \bar{w}_n + \sum_{|n|>K_+} \beta w_n \bar{w}_n \\ &\stackrel{\text{def}}{=} e + \langle \omega(\xi), I\rangle + \langle \tilde{A} z^+, \bar{z}^+\rangle + \sum_{|n|>K_+} \beta w_n \bar{w}_n , \end{aligned}$$

where $\tilde{A}$ is a Hermitian matrix with $\dim(\tilde{A}) \leq K_+$ and is given by

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & \beta I \end{pmatrix} .$$

We will also set $z^+ = (\cdots, w_n, \cdots)_{|n|\leq K_+}$, $\bar{z}^+ = (\cdots, \bar{w}_n, \cdots)_{|n|\leq K_+}$.

4.3.2 Truncation

Let us start by expanding $\breve{P}$, as obtained after the νth KAM iteration, into its Taylor-Fourier series,

$$\breve{P} = \sum_{k,l,\alpha,\beta} \breve{P}_{kl\alpha\beta} e^{\mathrm{i}\langle k,\theta\rangle}\ I^l z^\alpha \bar{z}^\beta + \sum_{\substack{k,l,n,\alpha,\beta \\ |n|>K, \alpha_n+\beta_n\geq 1}} \breve{P}_{kln\alpha\beta} e^{\mathrm{i}\langle k,\theta\rangle}\ I^l z^\alpha \bar{z}^\beta w_n^{\alpha_n} \bar{w}_n^{\beta_n}\ ,$$

where $k \in \mathbb{Z}^N, l \in \mathbb{N}^N$ and the multi-index α (β) runs over the set $\alpha \equiv (\cdots, \alpha_m, \cdots)_{|m|\leq K}$, $\alpha_m \in \mathbb{N}$ (resp. $\beta \equiv (\cdots, \beta_m, \cdots)_{|m|\leq K}$, $\beta_m \in \mathbb{N}$).

Now let R be the following truncation of $\breve{P}$:

$$\begin{aligned} R(\theta, I, z, \bar{z}, w, \bar{w}) = & \sum_{\substack{k \\ |l|\leq 1}} \breve{P}_{kl} e^{\mathrm{i}\langle k,\theta\rangle}\ I^l + \\ & \sum_k (\langle \breve{P}^{k10}, z\rangle + \langle \breve{P}^{k01}, \bar{z}\rangle) e^{\mathrm{i}\langle k,\theta\rangle} + \sum_{\substack{k \\ K<|n|\leq K_+}} (\breve{P}_n^{k10} w_n + \breve{P}_n^{k01} \bar{w}_n) e^{\mathrm{i}\langle k,\theta\rangle} + \\ & \sum_k (\langle \breve{P}^{k20} z, z\rangle + \langle \breve{P}^{k11} z, \bar{z}\rangle + \langle \breve{P}^{k02} \bar{z}, \bar{z}\rangle) e^{\mathrm{i}\langle k,\theta\rangle} + \\ & \sum_{\substack{k \\ K<|n|\leq K_+}} (\langle \breve{P}_n^{k20} z, w_n\rangle + \langle \breve{P}_n^{k11} z, \bar{w}_n\rangle + \langle \overline{\breve{P}_n^{(-k)11}} \bar{z}, w_n\rangle + \langle \breve{P}_n^{k02} \bar{z}, \bar{w}_n\rangle) e^{\mathrm{i}\langle k,\theta\rangle} + \\ & \sum_{\substack{k \\ K<|n|\leq K_+}} (\breve{P}_{nn}^{k20} w_n w_n + \breve{P}_{nn}^{k11} w_n \bar{w}_n + \breve{P}_{nn}^{k02} \bar{w}_n \bar{w}_n) e^{\mathrm{i}\langle k,\theta\rangle}\ . \end{aligned}$$

Remark 4.3.1 *We observe that, due to their decay property, terms in the Taylor-Fourier expansion of $\breve{P}$ corresponding to $|n| > K_+$ are small enough to be postponed to the next KAM step. Similarly, due to the decay property of $\breve{P}$ and the fact that $\acute{P}_0$ starts from third order terms, there are no coupling terms of the form $\sum_{\substack{n\neq m \\ K<|n|,|m|\leq K_+}} w_n \bar{w}_m$ in R. If $\acute{P}_0$ started from second-order terms, then the couplings between different oscillators would be so strong that the appearance of a continuous spectrum would become feasible.*

According to the normal form N, we may rewrite R as

$$\begin{aligned} R(\theta, I, z^+, \bar{z}^+) = & \ R_0 + R_1 + R_2 \\ = & \sum_{k,|l|\leq 1} P_{kl} e^{\mathrm{i}\langle k,\theta\rangle}\ I^l + \sum_k (\langle R^{k10}, z^+\rangle + \langle R^{k01}, \bar{z}^+\rangle) e^{\mathrm{i}\langle k,\theta\rangle} + \\ & \sum_k (\langle R^{k20} z^+, z^+\rangle + \langle R^{k11} z^+, \bar{z}^+\rangle + \langle R^{k02} \bar{z}^+, \bar{z}^+\rangle) e^{\mathrm{i}\langle k,\theta\rangle}\ , \end{aligned}$$

where

$$R^{k10} = \begin{pmatrix} \breve{P}^{k10} \\ \breve{P}^{k10}_n \end{pmatrix}_{K<|n|\leq K_+},$$

$$R^{k01} = \begin{pmatrix} \breve{P}^{k01} \\ \breve{P}^{k01}_n \end{pmatrix}_{K<|n|\leq K_+},$$

$$R^{k20} = \begin{pmatrix} \breve{P}^{k20} & \frac{1}{2}(\breve{P}^{k20}_n)^\top \\ \frac{1}{2}\breve{P}^{k20}_n & \breve{P}^{k20}_{nn} \end{pmatrix}_{K<|n|\leq K_+},$$

$$R^{k11} = \begin{pmatrix} \breve{P}^{k11} & (\overline{\breve{P}^{(-k)11}_n})^\top \\ \breve{P}^{k11}_n & \breve{P}^{k11}_{nn} \end{pmatrix}_{K<|n|\leq K_+},$$

$$R^{k02} = \begin{pmatrix} \breve{P}^{k02} & \frac{1}{2}(\breve{P}^{k02}_n)^\top \\ \frac{1}{2}\breve{P}^{k02}_n & \breve{P}^{k02}_{nn} \end{pmatrix}_{K<|n|\leq K_+}.$$

Remark 4.3.2 *It is clear that* $(R^{k20})^\top = R^{k20}$, $(R^{k02})^\top = R^{k02}$. *Moreover, since* $\bar{R} = R$,

$$\begin{aligned} \overline{P_{(-k)l00}} &= P_{kl}\,, \qquad \overline{R^{(-k)10}} = R^{k01}\,, \\ \overline{R^{(-k)01}} &= R^{k10}\,, \qquad \overline{R^{(-k)20}} = R^{k02}\,, \\ (\overline{R^{(-k)11}})^\top &= R^{k11}\,, \qquad \overline{R^{(-k)02}} = R^{k20}\,. \end{aligned} \tag{4.3.4}$$

Now, if we write $H = \mathcal{N} + R + (\mathcal{P} - R)$, then, from our definition of norms, it follows that

$$\|X_R\|_{D(r,s),\mathcal{O}} \leq \|X_P\|_{D(r,s),\mathcal{O}} \leq \varepsilon\,.$$

Note that

$$\mathcal{P} - R = \sum_{\substack{|n|>K_+ \\ \alpha_n+\beta_n\geq 1}} \breve{P}_n(\theta, I, z, \bar{z}, \xi) w_n^{\alpha_n} \bar{w}_n^{\beta_n} + O(|I|^2 + |I||w| + |w|^3)\,.$$

Now, let $r_+ = \frac{r}{2} + \frac{r_0}{4}$ and $\eta = \varepsilon^{\frac{1}{4}}$, then since

$$\|\breve{P}_n(\theta, I, z, \bar{z}, \xi)\| \leq e^{-(|n|-K)}\,,$$

one has (lemma A.0.6, Appendix A) that

$$\|X_{\mathcal{P}-R}\|_{D(r_+ + \frac{r-r_+}{2}, \eta s), \mathcal{O}} \leq \sum_{|n|>K_+} e^{-(|n|-K)} + O(s) \leq c_1 \varepsilon^{\frac{5}{4}} \,, \tag{4.3.5}$$

provided that

C0) $s \leq \varepsilon$.

4.3.3 The homological equation

Below we show that one can find a Hamiltonian function F, defined on a domain $D_+ = D(r_+, s_+)$ such that, the time-1 map $\Phi = \Phi_F^1$ associated to the Hamiltonian vector field X_F, is a (symplectic) map from D_+ to D which transforms H into H_+, the Hamiltonian of the next KAM cycle. Let us propose F to have the following form,

$$F(\theta, I, z^+, \bar{z}^+) \,=\, F_0 + F_1 + F_2 \,,$$

satisfy the homological equation,

$$\{N, F\} + R \,=\, \breve{P}_{00} + \langle \omega', I \rangle + \langle R^{011} z^+, \bar{z}^+ \rangle \,, \tag{4.3.6}$$

where

$$\begin{aligned}
F_0 &= \sum_{\substack{k \neq 0 \\ |l| \leq 1}} F_{kl} e^{\mathrm{i}\langle k, \theta \rangle} \, I^l, \\
F_1 &= \sum_{\substack{k \\ |n| \leq K_+}} (f_n^{k10} w_n + f_n^{k01} \bar{w}_n) e^{\mathrm{i}\langle k, \theta \rangle} \,= \sum_k (\langle F^{k10}, z^+ \rangle + \langle F^{k01}, \bar{z}^+ \rangle) e^{\mathrm{i}\langle k, \theta \rangle} \,, \\
F_2 &= \sum_{\substack{k \\ |m| \leq K, |n| \leq K_+}} (f_{nm}^{k20} w_n w_m + f_{nm}^{k02} \bar{w}_n \bar{w}_m) e^{\mathrm{i}\langle k, \theta \rangle} \\
&\qquad + \sum_{\substack{k \neq 0 \\ |m| \leq K, |n| \leq K_+}} (f_{nm}^{k11} w_n \bar{w}_m + f_{mn}^{k11} w_m \bar{w}_n) e^{\mathrm{i}\langle k, \theta \rangle} \\
&\qquad + \sum_{\substack{k \\ K < |n| \leq K_+}} (f_{nn}^{k20} w_n w_n + f_{nn}^{k02} \bar{w}_n \bar{w}_n) e^{\mathrm{i}\langle k, \theta \rangle} \;+ \sum_{\substack{k \neq 0 \\ K < |n| \leq K_+}} f_{nn}^{k11} w_n \bar{w}_n e^{\mathrm{i}\langle k, \theta \rangle} \\
&= \sum_k (\langle F^{k20} z^+, z^+ \rangle + \langle F^{k02} \bar{z}^+, \bar{z}^+ \rangle) e^{\mathrm{i}\langle k, \theta \rangle} \,+ \sum_{k \neq 0} \langle F^{k11} z^+, \bar{z}^+ \rangle e^{\mathrm{i}\langle k, \theta \rangle} \,, \\
\omega' &= \int \frac{\partial \breve{P}}{\partial I} d\theta |_{z^+ = \bar{z}^+ = w = \bar{w} = 0, I = 0} \cdot
\end{aligned}$$

Lemma 4.3.1 *Equation* (4.3.6) *is equivalent to the following system*

$$\begin{aligned}
&\langle k,\omega\rangle F_{kl} = \mathrm{i}\breve{P}_{kl}, \quad k\neq 0, |l|\leq 1\,,\\
&(\langle k,\omega\rangle I - \tilde{A})F^{k10} = \mathrm{i}R^{k10}\,,\\
&(\langle k,\omega\rangle I + \tilde{A})F^{k01} = \mathrm{i}R^{k01}\,,\\
&(\langle k,\omega\rangle I - \tilde{A})F^{k20} - F^{k20}\tilde{A} = \mathrm{i}R^{k20}\,,\\
&(\langle k,\omega\rangle I + \tilde{A})F^{k11} - F^{k11}\tilde{A} = \mathrm{i}R^{k11}\,, \quad k\neq 0\,,\\
&(\langle k,\omega\rangle I + \tilde{A})F^{k02} + F^{k02}\tilde{A} = \mathrm{i}R^{k02}\,.
\end{aligned} \tag{4.3.7}$$

Proof. (4.3.6) is equivalent to the following system of equations

$$\begin{aligned}
\{\mathcal{N},F_0\} + R_0 &= \breve{P}_{00} + \langle\omega', I\rangle\,,\\
\{\mathcal{N},F_1\} + R_1 &= 0\,,\\
\{\mathcal{N},F_2\} + R_2 &= \langle R^{011}z^+,\bar{z}^+\rangle\,.
\end{aligned} \tag{4.3.8}$$

By simple comparison of coefficients, we see that the first equation in (4.3.8) is equivalent to the first equation in (4.3.7). On the other hand,

$$\begin{aligned}
\{\mathcal{N},F_1\} = \mathrm{i}\sum_k (\langle\, \langle k,\omega\rangle F^{k10}, z^+\rangle - \langle\tilde{A}z^+, F^{k10}\rangle)e^{\mathrm{i}\langle k,\theta\rangle} \;+\\
\mathrm{i}\sum_k (\langle\, \langle k,\omega\rangle F^{k01}, \bar{z}^+\rangle + \langle\tilde{A}\bar{z}^+, F^{k01}\rangle)e^{\mathrm{i}\langle k,\theta\rangle}\\
= \mathrm{i}\sum_k \langle(\langle k,\omega\rangle I - \tilde{A})F^{k10}, z^+\rangle e^{\mathrm{i}\langle k,\theta\rangle} \;+\\
\mathrm{i}\sum_k \langle(\langle k,\omega\rangle I + \tilde{A})F^{k01}, \bar{z}^+\rangle e^{\mathrm{i}\langle k,\theta\rangle}\;,
\end{aligned}$$

and

$$\begin{aligned}
\{\mathcal{N}, F_2\} = & \ \mathrm{i}\sum_k (\langle\, \langle k,\omega\rangle F^{k20} z^+, z^+\rangle - \langle F^{k20} z^+, \tilde{A} z^+\rangle - \langle \tilde{A} z^+, (F^{k20})^\top z^+\rangle) e^{\mathrm{i}\langle k,\theta\rangle} \ + \\
& \qquad \mathrm{i}\sum_{k\neq 0} (\langle\, \langle k,\omega\rangle F^{k11} z^+, \bar{z}^+\rangle + \langle F^{k11} z^+, \tilde{A}\bar{z}^+\rangle - \langle \tilde{A} z^+, (F^{k11})^\top \bar{z}^+\rangle) e^{\mathrm{i}\langle k,\theta\rangle} \ + \\
& \qquad \mathrm{i}\sum_{k} (\langle\, \langle k,\omega\rangle F^{k02} \bar{z}^+, \bar{z}^+\rangle + \langle F^{k02} \bar{z}^+, \tilde{A}\bar{z}^+\rangle + \langle \tilde{A} \bar{z}^+, (F^{k02})^\top \bar{z}^+\rangle) e^{\mathrm{i}\langle k,\theta\rangle} \\
= & \ \mathrm{i}\sum_k (\langle\, \langle k,\omega\rangle F^{k20} z^+, z^+\rangle - \langle (\tilde{A} F^{k20} + F^{k20}\tilde{A}) z^+, z^+\rangle) e^{\mathrm{i}\langle k,\theta\rangle} \ + \\
& \qquad \mathrm{i}\sum_{k\neq 0} (\langle\, \langle k,\omega\rangle F^{k11} z^+, \bar{z}^+\rangle + \langle (\tilde{A} F^{k11} - F^{k11}\tilde{A}) z^+, \bar{z}^+\rangle) e^{\mathrm{i}\langle k,\theta\rangle} \ + \\
& \qquad \mathrm{i}\sum_{k} (\langle\, \langle k,\omega\rangle F^{k02} \bar{z}^+, \bar{z}^+\rangle + \langle (\tilde{A} F^{k02} + F^{k02}\tilde{A}) \bar{z}^+, \bar{z}^+\rangle) e^{\mathrm{i}\langle k,\theta\rangle} \\
= & \ \mathrm{i}\sum_k \langle (\langle k,\omega\rangle F^{k20} - \tilde{A} F^{k20} - F^{k20}\tilde{A}) z^+, z^+\rangle e^{\mathrm{i}\langle k,\theta\rangle} \ + \\
& \qquad \mathrm{i}\sum_{k\neq 0} \langle (\langle k,\omega\rangle F^{k11} + \tilde{A} F^{k11} - F^{k11}\tilde{A}) z^+, \bar{z}^+\rangle e^{\mathrm{i}\langle k,\theta\rangle} \ + \\
& \qquad \mathrm{i}\sum_{k} \langle (\langle k,\omega\rangle F^{k02} + \tilde{A} F^{k02} + F^{k02}\tilde{A}) \bar{z}^+, \bar{z}^+\rangle e^{\mathrm{i}\langle k,\theta\rangle} \ .
\end{aligned}$$

From the second and the third equations in (4.3.8), we see that F^{k10}, F^{k01}, F^{k20}, F^{k11}, F^{k02} satisfy the corresponding equations in (4.3.7).

□

Let

$$\mathcal{O}_+ = \left\{ \xi \in \mathcal{O} : \begin{array}{l} |\langle k,\omega\rangle^{-1}| \le \frac{|k|^\tau}{\gamma}\,, \quad k \neq 0 \\ \|(\langle k,\omega\rangle I + \tilde{A})^{-1}\| \le K_+^4 \frac{|k|^\tau}{\gamma}\,, \\ \|(\langle k,\omega\rangle I + \tilde{A}\otimes I + I\otimes \tilde{A})^{-1}\| \le K_+^8 \frac{|k|^\tau}{\gamma}\,, \\ \|(\langle k,\omega\rangle I + \tilde{A}\otimes I - I\otimes \tilde{A})^{-1}\| \le K_+^8 \frac{|k|^\tau}{\gamma}\,, \quad k \neq 0 \end{array} \right\},$$

then the first three equations in (4.3.7) can be solved in this region. Solvability of the remaining equations in (4.3.7) is a consequence of the following elementary result from matrix theory.

Lemma 4.3.2 *Let A, B, C be $n\times n$, $m\times m$ and $n\times m$ matrices, respectively, and let X be a $n\times m$ unknown matrix. Then the matrix equation*

$$AX - XB = C\,, \tag{4.3.9}$$

is solvable if and only if $I_m \otimes A - B \otimes I_n$ *is nonsingular. Moreover,*

$$\|X\| \leq \|(I_m \otimes A - B \otimes I_n)^{-1}\| \cdot \|C\| \,. \tag{4.3.10}$$

Proof. See [68, 106].

□

Note that $(F^{k20})^\top$ satisfies the same equation as F^{k20}, thus, by uniqueness of solutions, $F^{k20} = (F^{k20})^\top$; similarly, $F^{k02} = (F^{k02})^\top$.

Observe that, taking the conjugate transpose of the fourth equation in (4.3.7) and replacing k with $-k$ one obtains

$$(\overline{F^{(-k)20}})^\top(\langle -k, \omega\rangle I - (\overline{\tilde{A}})^\top) - (\overline{\tilde{A}})^\top(\overline{F^{(-k)20}})^\top = (-\mathrm{i})(\overline{R^{(-k)20}})^\top \,, \tag{4.3.11}$$

i.e.,

$$(\overline{F^{(-k)20}})^\top(\langle k, \omega\rangle I + (\overline{\tilde{A}})^\top) + (\overline{\tilde{A}})^\top(\overline{F^{(-k)20}})^\top \;=\; \mathrm{i}(\overline{R^{(-k)20}})^\top \,, \tag{4.3.12}$$

so

$$(\langle k, \omega\rangle I + (\overline{\tilde{A}})^\top)(\overline{F^{(-k)20}})^\top + (\overline{F^{(-k)20}})^\top(\overline{\tilde{A}})^\top \;=\; \mathrm{i}(\overline{R^{(-k)20}})^\top \,, \tag{4.3.13}$$

and using the fact that $(\overline{\tilde{A}})^\top = \tilde{A}$, $(\overline{F^{(-k)20}})^\top = \overline{F^{(-k)20}}$ and $(\overline{R^{(-k)20}})^\top = (R^{k02})^\top = R^{k02}$, it follows that

$$(\langle k, \omega\rangle I + \tilde{A})\overline{F^{(-k)20}} + \overline{F^{(-k)20}}\tilde{A} = \mathrm{i}R^{k02} \,; \tag{4.3.14}$$

i.e., $\overline{F^{(-k)20}}$ satisfies the same equation as F^{k02}, thus $\overline{F^{(-k)20}} = F^{k02}$.

If we now take the conjugate transpose of the fifth equation in (4.3.7) and replace k with $-k$,

$$(\overline{F^{(-k)11}})^\top(\langle -k, \omega\rangle I + (\overline{\tilde{A}})^\top) - (\overline{\tilde{A}})^\top(\overline{F^{(-k)11}})^\top = (-\mathrm{i})(\overline{R^{(-k)11}})^\top \,,$$

i.e.,

$$(\overline{F^{(-k)11}})^\top(\langle k, \omega\rangle I - (\overline{\tilde{A}})^\top) + (\overline{\tilde{A}})^\top(\overline{F^{(-k)11}})^\top = \mathrm{i}(\overline{R^{(-k)11}})^\top \,, \tag{4.3.15}$$

so

$$(\langle k, \omega\rangle I + (\overline{\tilde{A}})^\top)(\overline{F^{(-k)11}})^\top - (\overline{F^{(-k)11}})^\top(\overline{\tilde{A}})^\top = \mathrm{i}(\overline{R^{(-k)11}})^\top \,, \tag{4.3.16}$$

and since $(\overline{\tilde{A}})^\top = \tilde{A}$ and $(\overline{R^{(-k)11}})^\top = R^{k11}$,

$$(\langle k,\omega\rangle I + \tilde{A})(\overline{F^{(-k)11}})^\top - (\overline{F^{(-k)11}})^\top \tilde{A} = \mathrm{i}R^{k11}\,; \tag{4.3.17}$$

i.e., $(\overline{F^{(-k)11}})^\top$ satisfies the same equation as F^{k11}, thus $(\overline{F^{(-k)11}})^\top = F^{k11}$. In an entirely analogous way one can also show that also $\overline{F_{(-k)l}} = F_{kl}$, $\overline{F^{(-k)10}} = F^{k01}$, $\overline{F^{(-k)01}} = F^{k10}$ and $\overline{F^{(-k)02}} = F^{k20}$, thus

$$\bar{F} \;=\; F\,. \tag{4.3.18}$$

We are now going to estimate the norm of X_F and the order Φ_F^1, to this end, we will need the following

Lemma 4.3.3 *Let $A = (a_{ij}(\xi))_{K\times K}$ be an invertible matrix depending differentiably on a parameter $\xi \in \mathcal{O}$, and $\|A^{-1}\| \le L$, $\|\partial_\xi A\| \le M$, then*

$$\|\partial_\xi A^{-1}\| \le L^2 M\,. \tag{4.3.19}$$

Proof. Since $AA^{-1} = I$, then $(\partial_\xi A)A^{-1} + A(\partial_\xi A^{-1}) = 0$, hence $\partial_\xi A^{-1} = -A^{-1}(\partial_\xi A)A^{-1}$, thus

$$\|\partial_\xi A^{-1}\| \le \|A^{-1}\|^2\|\partial_\xi A\| \le L^2 M\,. \tag{4.3.20}$$

□

Lemma 4.3.4 *Let $D_i = D(r_+ + \frac{i}{4}(r-r_+), \frac{i}{4}s)$, $0 < i \le 4$. If*

C1) $K_+^{18} \le \varepsilon^{-\frac{1}{4}}$,

then there is a constant $c_2 > 0$ such that

$$\|X_F\|_{D_3,\mathcal{O}_+} \le c_2\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{3}{4}}\,.$$

Proof. From the definition of $\mathcal{O}_+$ we see that

$$\begin{aligned}
&\sup_{\xi\in\mathcal{O}_+} \|\partial_\xi\langle k,\omega\rangle\| \le |k|\,,\\
&\sup_{\xi\in\mathcal{O}_+} \|\partial_\xi(\langle k\omega\rangle I + \tilde{A})\| \le (|k| + K_+)\,,\\
&\sup_{\xi\in\mathcal{O}_+} \|\partial_\xi(\langle k,\omega\rangle I + \tilde{A}\otimes I + I\otimes\tilde{A})\| \le (|k| + K_+^2)\,,\\
&\sup_{\xi\in\mathcal{O}_+} \|\partial_\xi(\langle k,\omega\rangle I + \tilde{A}\otimes I - I\otimes\tilde{A})\| \le (|k| + K_+^2)\,.
\end{aligned}$$

Then, according to Lemma 4.3.3, Lemma 4.3.1 and Lemma 4.3.2, we have

$$
\begin{aligned}
|F_{kl}|_{\mathcal{O}_+} &\leq |\langle k,\omega\rangle|^{-2}|k||\breve{P}_{kl}|_{\mathcal{O}_+} \leq \gamma^{-2}|k|^{2\tau+1}|\breve{P}_{kl}|_{\mathcal{O}_+} \quad k\neq 0, \quad |l|\leq 1\,,\\
\|F^{kij}\|_{\mathcal{O}_+} &\leq \gamma^{-2}K_+^9|k|^{2\tau+1}\|R^{kij}\|_{\mathcal{O}_+}\,, \quad i\neq j, \quad 1\leq i+j\leq 2\,,\\
\|F^{k11}\|_{\mathcal{O}_+} &\leq \gamma^{-2}K_+^{18}|k|^{2\tau+1}\|R^{k11}\|_{\mathcal{O}_+} \quad (k\neq 0)\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{1}{s^2}\|F_\theta\|_{D_3,\mathcal{O}_+} &\leq \frac{1}{s^2}\Big(\sum_{k,|l|\leq 1}|F_{kl}|\cdot s^{2|l|}\cdot|k|\cdot e^{|k|(r-\frac{1}{4}(r-r_+))} \;+\\
&\qquad \sum_{1\leq i+j\leq 2}\sum_k(\|F^{kij}\|\cdot\|z^+\|)\cdot|k|\cdot e^{|k|(r-\frac{1}{4}(r-r_+))}\\
&\leq \frac{\gamma^{-2}K_+^{18}}{s^2}\Big(\sum_{k,|l|\leq 1}|\breve{P}_{kl}|\cdot s^{2|l|}\cdot|k|^{2\tau+2}\cdot e^{|k|(r-\frac{1}{4}(r-r_+))} \;+\\
&\qquad \sum_{1\leq i+j\leq 2}\sum_k(\|R^{kij}\|\cdot\|z^+\|)\cdot|k|^{2\tau+2}\cdot e^{|k|(r-\frac{1}{4}(r-r_+))}\Big)\\
&\leq c_3\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}K_+^{18}\|X_R\|\\
&\leq c_3\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{3}{4}}\,.
\end{aligned}
$$

A similar derivation yields,

$$
\|F_I\|_{D_3,\mathcal{O}_+} = \sum_{|l|=1}|F_{kl}|e^{|k|(r-\frac{1}{4}(r-r_+))} \leq c_4\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{3}{4}}\,.
$$

$$
\begin{aligned}
\|X_{F_1}\|_{D_3,\mathcal{O}_+} &\leq \frac{1}{s}\Big(\sum_n\|F_{1_{w_n}}\|+\sum_n\|F_{1_{\bar{w}_n}}\|\Big)\leq\frac{1}{s}\big(\|F_{1_{z^+}}\|+\|F_{1_{\bar{z}^+}}\|\big)\\
&\leq c_5\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}K_+^{10}\|X_{R_1}\|\leq c_5\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{3}{4}}\,.
\end{aligned}
$$

$$
\begin{aligned}
\|X_{F_2}\|_{D_3,\mathcal{O}_+} &\leq \frac{1}{s}\Big(\sum_n\|F_{2_{w_n}}\|+\sum_n\|F_{2_{\bar{w}_n}}\|\Big) \leq \frac{1}{s}\big(\|F_{2_{z^+}}\|+\|F_{2_{\bar{z}^+}}\|\big)\\
&\leq c_6\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}K_+^{10}\|X_{R_2}\| \leq c_6\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{3}{4}}\,.
\end{aligned}
$$

Putting together the estimates above, Lemma 4.3.4 follows.

□

Now let $D_{i\eta} = D(r_+ + \frac{i}{4}(r-r_+), \frac{i}{4}\eta s) =, 0<i\leq 4\,.$

Lemma 4.3.5 *If*

C2) $c_2\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{1}{2}} < 1\,,$

then

$$\Phi_F^t\,:\,D_{2\eta}\to D_{3\eta}\,,\quad -1\le t\le 1\,, \tag{4.3.21}$$

and moreover,

$$\|D\Phi_F^t - Id\|_{D_{1\eta}} < c_7\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{3}{4}}\,. \tag{4.3.22}$$

Proof. Let

$$\|D^mF\|_{D,\mathcal{O}_+} \;=\; \max\left\{\left\|\frac{\partial^{|i|+|l|+|\alpha|+|\beta|}F}{\partial\theta^i\partial I^l\partial(z^+)^\alpha\partial(\bar z^+)^\beta}\right\|_{D,\mathcal{O}_+}\;:\;|i|+|l|+|\alpha|+|\beta|=m\ge 2\right\}\,.$$

We note that F is a polynomial of order 1 in I and of order 2 in z^+, $\bar z^+$. It thus follows from Lemma 4.3.4 and Cauchy's inequality (lemma A.0.6 in Appendix A) that

$$\|D^mF\|_{D_2,\mathcal{O}_+} \;<\; c_8\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{3}{4}}\,,$$

for any $m\ge 2\,.$

Using the integral equation

$$\Phi_F^t \;=\; id\,+\int_0^t X_F\circ\Phi_F^s\,ds$$

and Lemma 4.3.4, one can see that $\Phi_F^t : D_{2\eta}\to D_{3\eta}\,,\ -1\le t\le 1\,.$ Indeed,

$$D\Phi_F^t \;=\; Id+\int_0^t (DX_F)D\Phi_F^s\,ds \;=\; Id+\int_0^t J(D^2F)D\Phi_F^s\,ds\,,$$

where J denotes the standard symplectic matrix. Let $c_7=2c_8$, it thus follows that

$$\|D\Phi_F^t - Id\|\le 2\|D^2F\| \;\le\; c_7\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{3}{4}}\,.$$

□

4.3.4 The new Hamiltonian

Let $\Phi=\Phi_F^1\,,\ s_+=\frac{1}{8}\eta s\,,\ D_+=D(r_+,s_+)$ and

$$\mathcal{N}_+ = e_+ + \langle\omega_+, I\rangle + \langle A^+z^+,\bar z^+\rangle + \sum_{|n|>K_+}\beta w_n\bar w_n\,,$$

$$\mathcal{P}^+ = \breve P_+ + \acute P_0 + \grave P_0\,,$$

where

$$
\begin{aligned}
e_+ &= e + \breve{P}_{00}\,,\\
\omega_+ &= \omega + \breve{P}_{0l}(|l|=1)\,,\\
A^+ &= \tilde{A} + R^{011}\,,\\
z^+ &= (\cdots, w_n, \cdots)_{|n|\le K_+}\,, \quad \bar{z}^+ = (\cdots, \bar{w}_n, \cdots)_{|n|\le K_+}\,,\\
\breve{P}_+ &= \int_0^1 \{(1-t)\{\mathcal{N},F\}+R,F\}\circ\Phi_F^t\,dt + (\breve{P}-R)\circ\Phi_F^1 + \\
&\qquad\qquad \int_0^1 \{\acute{P}_0+\grave{P}_0,F\}\circ\Phi_F^t\,dt\,.
\end{aligned}
$$

Thus, $\Phi : D_+ \times \mathcal{O}_+ \to D$ and, by Taylor's second-order formula,

$$
\begin{aligned}
H_+ \equiv\ & H\circ\Phi = (\mathcal{N}+R)\circ\Phi + (\mathcal{P}-R)\circ\Phi\\
&= \mathcal{N} + \{\mathcal{N},F\} + R + \int_0^1 (1-t)\{\{\mathcal{N},F\},F\}\circ\Phi_F^t\,dt + \\
&\qquad\qquad \int_0^1 \{R,F\}\circ\Phi_F^t\,dt + (\breve{P}-R)\circ\Phi_F^1 + (\acute{P}_0+\grave{P}_0)\circ\Phi_F^1\\
&= \mathcal{N} + \{\mathcal{N},F\} + R + \breve{P}_+ + \acute{P}_0 + \grave{P}_0\\
&= \mathcal{N}_+ + \mathcal{P}^+ + \{\mathcal{N},F\} + R - \breve{P}_{00} - \langle\omega', I\rangle - \langle R^{011}z^+, \bar{z}^+\rangle\\
&= \mathcal{N}_+ + \mathcal{P}^+\,.
\end{aligned}
$$

We will now show that H_+ has properties similar to those of H.

Observe that, since $(\overline{\tilde{A}})^\top = \tilde{A}$ and $(\overline{R^{011}})^\top = R^{011}$, then $(\overline{A^+})^\top = A^+$, i.e., A^+ is a Hermitian matrix. Then, from the assumptions on $\breve{P}$, we have that there is a constant $c_9 > 0$ such that

$$
|\omega_+ - \omega|_{\mathcal{O}_+} \le c_9\varepsilon\,, \quad \|A^+ - \tilde{A}\|_{\mathcal{O}_+} \le c_9\varepsilon\,.
$$

It thus follows that, if

C3) $c_9 K_+^{\tau+5}\varepsilon < \gamma - \gamma_+\,,$

then whenever $|k| \le K_+$,

$$|\langle k, \omega + P_{0l}\rangle^{-1}| \le \frac{|\langle k,\omega\rangle^{-1}|}{1 - |\langle k,\omega\rangle^{-1}| c_9 |k| \varepsilon} \le \frac{|k|^\tau}{\gamma_+}, \quad k \neq 0\,,$$

$$\|(\langle k, \omega + P_{0l}\rangle I + A^+)^{-1}\| \le \frac{\|(\langle k,\omega\rangle I + \tilde{A})^{-1}\|}{1 - \|(\langle k,\omega\rangle I + \tilde{A})^{-1}\| c_9 |k| \varepsilon} \le K_+^4 \frac{|k|^\tau}{\gamma_+}\,,$$

$$\begin{aligned}\|(\langle k, \omega + P_{0l}\rangle I + A^+ \otimes I + I \otimes A^+)^{-1}\| &\le \frac{\|(\langle k,\omega\rangle I + \tilde{A} \otimes I + I \otimes \tilde{A})^{-1}\|}{1 - \|(\langle k,\omega\rangle I + \tilde{A} \otimes I + I \otimes \tilde{A})^{-1}\| c_9 |k| \varepsilon} \\ &\le K_+^8 \frac{|k|^\tau}{\gamma_+}\,,\end{aligned}$$

$$\begin{aligned}\|(\langle k, \omega + P_{0l}\rangle I + A^+ \otimes I - I \otimes A^+)^{-1}\| &\le \frac{\|(\langle k,\omega\rangle I + \tilde{A} \otimes I - I \otimes \tilde{A})^{-1}\|}{1 - \|(\langle k,\omega\rangle I + \tilde{A} \otimes I - I \otimes \tilde{A})^{-1}\| c_9 |k| \varepsilon} \\ &\le K_+^8 \frac{|k|^\tau}{\gamma_+}, \quad k \neq 0\,.\end{aligned}$$

The above implies that, in the next KAM step, small denominator conditions are automatically satisfied when $|k| \le K_+$.

Let $R(t) = (1-t)(\mathcal{N}_+ - \mathcal{N}) + tR$, then $\mathcal{P}^+$ can be rewritten as

$$\begin{aligned}\mathcal{P}^+ &= \int_0^1 (1-t)\{\{\mathcal{N}, F\}, F\} \circ \Phi_F^t \, dt \, + \, \int_0^1 \{R, F\} \circ \Phi_F^t dt + (\mathcal{P} - R) \circ \Phi_F^1 \\ &= \int_0^1 \{R(t), F\} \circ \Phi_F^t \, dt + (\mathcal{P} - R) \circ \Phi_F^1\,,\end{aligned}$$

hence

$$X_{\mathcal{P}_+} \, = \, \int_0^1 (\Phi_F^t)^* X_{\{R(t),F\}} \, dt \, + \, (\Phi_F^1)^* X_{(\mathcal{P}-R)}\,.$$

Now, by Lemma 4.3.5, if

C4) $c_7 \gamma^{-2} (r - r_+)^{-(2\tau+N+1)} \varepsilon^{\frac{3}{4}} \le 1$,

then

$$\|D\Phi_F^t\|_{D_{1\eta}} \le 1 + \|D\Phi_F^t - I\|_{D_{1\eta}} \le 2\,, \quad -1 \le t \le 1\,.$$

Furthermore, by Lemma A.0.8 (cf. Appendix A) and (4.3.5), we also have

$$\begin{aligned}\|X_{\{R(t),F\}}\|_{D_{2\eta}} &\le \, c_{10} \gamma^{-2} (r - r_+)^{-(2\tau+N+1)} \eta^{-2} \varepsilon^{\frac{7}{4}}\,, \\ \|X_{(\mathcal{P}-R)}\|_{D_{2\eta}} &\le c_1 \, \varepsilon^{\frac{5}{4}}\,.\end{aligned}$$

Let $c_0 = \max\{c_1, \cdots, c_{10}, c_{11}, c_{12}\}$, where c_{11}, c_{12} will be defined later on, and let

$$\varepsilon_+ = 4c_0\gamma^{-2}(r - r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{5}{4}}\,,$$

then

$$\|X_{\mathcal{P}_+}\|_{D_+,\mathcal{O}_+} \leq 2c_1\varepsilon^{\frac{5}{4}} + 2c_{10}\gamma^{-2}(r - r_+)^{-(2\tau+N+1)}\varepsilon^{\frac{5}{4}} \leq \varepsilon_+\,.$$

The reality condition of $\mathcal{P}_+$ is verified easily because, if F satisfies $\bar{F} = F$ and G satisfies $\bar{G} = G$, then their Poisson bracket $\{F, G\}$ also satisfies $\overline{\{F, G\}} = \{\overline{F}, \overline{G}\} = \{F, G\}$.

We now examine the decay property of $\breve{P}_+$. More precisely, let us write

$$\breve{P}_+ = \breve{P}^+(\theta, I, z^+, \bar{z}^+, \xi) + \sum_{|n|>K_+, \alpha_n+\beta_n\geq 1} \breve{P}^+_n(\theta, I, z^+, \bar{z}^+, \xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n}\,,$$

we will show that

$$\|\breve{P}^+_n(\theta, I, z^+, \bar{z}^+, \xi)\|_{D_+,\mathcal{O}_+} \leq e^{-(|n|-K_+)}\,, \quad |n| > K_+\,.$$

Since F only involves the normal components $w_n, \bar{w}_n$ for $|n| \leq K_+$, so does $\{\mathcal{N}, F\}$; therefore, $\int_0^1 (1-t)\{\{\mathcal{N}, F\}, F\} \circ \Phi_F^t\, dt$ only involves the normal components $w_n, \bar{w}_n$ for $|n| \leq K_+$. Now, recall that

$$\grave{P}_0 \,=\, \sum_n O(|w_n|^3)\,,$$

thus $\{\grave{P}_0, F\}$ also only involves normal components $w_n, \bar{w}_n$ for $|n| \leq K_+$ only; consequently, the same is true about $\int_0^1 \{\grave{P}_0, F\} \circ \Phi_F^t\, dt$. Now, since R is a truncation of $\breve{P}$, in order to establish the decay property above, it suffices to consider the following two terms, $(\breve{P} - R)$ and $\int_0^1 \{\breve{P} + \acute{P}_0, F\} \circ \Phi_F^t\, dt$; let us take the first term. Recall that

$$\breve{P} \,=\, \breve{P}(\theta, I, z, \bar{z}, \xi) + \sum_{\substack{|n|>K \\ \alpha_n+\beta_n\geq 1}} \breve{P}_n(\theta, I, z, \bar{z}, \xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n}\,,$$

such that

$$\|\breve{P}_n(\theta, I, z, \bar{z}, \xi)\|_{D(r,s),\mathcal{O}_+} \,\leq\, e^{-(|n|-K)}\,,$$

and that

$$\acute{P}_0 \,=\, \sum_{\substack{n\neq m \\ \alpha_n+\beta_n, \alpha_m+\beta_m\geq 1 \\ \alpha_n+\beta_n+\alpha_m+\beta_m\geq 3}} \acute{P}^0_{nm}(\xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n} w_m^{\alpha_m}\bar{w}_m^{\beta_m}\,,$$

such that

$$\|\acute{P}^0_{nm}(\xi)\|_{D(r,s),\mathcal{O}_+} \leq e^{-|n-m|}\,.$$

Now remember that R only involves normal components $w_n, \bar{w}_n$ with $|n| \leq K_+$; therefore, terms corresponding to normal components $w_n, \bar{w}_n$ for $|n| > K_+$ in $\breve{P} - R$ are those corresponding for $|n| > K_+$ in $\breve{P}$, for which we already have the decay property

$$\|\breve{P}_n(\theta, I, z, \bar{z}, \xi)\|_{D(r_+,s_+),\mathcal{O}_+} \leq e^{-(|n|-K)} \leq e^{-(|n|-K_+)}\,.$$

It remains to obtain decay estimates for $\int_0^1 \{\breve{P} + \acute{P}_0, F\} \circ \Phi_F^t \, dt$. Once again, we only need to consider terms corresponding to normal components $w_n, \bar{w}_n$ with $|n| > K_+$; note however that F is independent of such components and thus so will be

$$\int_0^1 \{\breve{P}(\theta, I, z, \bar{z}, \xi) + \sum_{\substack{K<|n|<K_+ \\ \alpha_n+\beta_n \geq 1}} \breve{P}_n(\theta, I, z, \bar{z}, \xi) w_n^{\alpha_n} \bar{w}_n^{\beta_n}, F\} \circ \Phi_F^t dt\,.$$

By the same token,

$$\int_0^1 \{ \sum_{\substack{n\neq m, |n|, |m| \leq K_+ \\ \alpha_n+\beta_n, \alpha_m+\beta_m \geq 1 \\ \alpha_n+\beta_n+\alpha_m+\beta_m \geq 3}} \acute{P}^0_{nm}(\xi) w_n^{\alpha_n} \bar{w}_n^{\beta_n} w_m^{\alpha_m} \bar{w}_m^{\beta_m}, F\} \circ \Phi_F^t \, dt$$

is independent of normal components $w_n, \bar{w}_n$ for $|n| > K_+$. It only remains to consider the following terms

$$\begin{aligned}
&\int_0^1 \{ \sum_{\substack{|n|>K \\ \alpha_n+\beta_n \geq 1}} \breve{P}_n(\theta, I, z, \bar{z}, \xi) w_n^{\alpha_n} \bar{w}_n^{\beta_n}, F\} \circ \Phi_F^t \, dt \\
&\quad = \int_0^1 \sum_{\substack{|n|>K \\ \alpha_n+\beta_n \geq 1}} \{\breve{P}_n(\theta, I, z, \bar{z}, \xi), F\} \circ \Phi_F^t w_n^{\alpha_n} \bar{w}_n^{\beta_n} \, dt \\
&\quad = \sum_{\substack{|n|>K \\ \alpha_n+\beta_n \geq 1}} (\int_0^1 \{\breve{P}_n(\theta, I, z, \bar{z}, \xi), F\} \circ \Phi_F^t \, dt) w_n^{\alpha_n} \bar{w}_n^{\beta_n}\,, \qquad (4.3.23)
\end{aligned}$$

and

$$\int_0^1 \{ \sum_{\substack{n\neq m,|n|>K_+,|m|\leq K_+\\ \alpha_n+\beta_n,\alpha_m+\beta_m\geq 1\\ \alpha_n+\beta_n+\alpha_m+\beta_m\geq 3}} \acute{P}^0_{nm}(\xi) w_n^{\alpha_n}\bar{w}_n^{\beta_n} w_m^{\alpha_m}\bar{w}_m^{\beta_m}, F\} \circ \Phi_F^t \, dt$$

$$= \int_0^1 \sum_{|n|>K_+} \{ \sum_{\substack{|m|\leq K_+\\ \alpha_n+\beta_n,\alpha_m+\beta_m\geq 1\\ \alpha_n+\beta_n+\alpha_m+\beta_m\geq 3}} \acute{P}^0_{nm}(\xi) w_m^{\alpha_m}\bar{w}_m^{\beta_m}, F\} \circ \Phi_F^t w_n^{\alpha_n}\bar{w}_n^{\beta_n} \, dt$$

$$= \sum_{|n|>K_+} (\int_0^1 \{ \sum_{\substack{|m|\leq K_+\\ \alpha_n+\beta_n,\alpha_m+\beta_m\geq 1\\ \alpha_n+\beta_n+\alpha_m+\beta_m\geq 3}} \acute{P}^0_{nm}(\xi) w_m^{\alpha_m}\bar{w}_m^{\beta_m}, F\} \circ \Phi_F^t \, dt) w_n^{\alpha_n}\bar{w}_n^{\beta_n} \, . \quad (4.3.24)$$

Let

$$\tilde{P}_n = \breve{P}_n(\theta, I, z, \bar{z}, \xi) + \sum_{\substack{|m|\leq K_+\\ \alpha_n+\beta_n,\alpha_m+\beta_m\geq 1\\ \alpha_n+\beta_n+\alpha_m+\beta_m\geq 3}} \acute{P}^0_{nm}(\xi) w_m^{\alpha_m}\bar{w}_m^{\beta_m} \, .$$

We will combine (4.3.23) and (4.3.24) to obtain the decay property of

$$\sum_{|n|>K_+,\alpha_n+\beta_n\geq 1} (\int_0^1 \{\tilde{P}_n, F\} \circ \Phi_F^t \, dt) w_n^{\alpha_n}\bar{w}_n^{\beta_n} \, .$$

By relaxing decay properties of $e^{-(|n|-K)}$, $e^{-|n-m|}$ to $e^{-(|n|-K_+)}$ we have, by Lemma A.0.7 (see appendix A), that

$$\|\{\tilde{P}_n, F\}\|_{D(r-\sigma,\frac{1}{2}s)} \leq c_{11}\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\sigma^{-1}s^{-2}\varepsilon^{\frac{3}{4}}e^{-(|n|-K_+)} \, .$$

It follows by a Cauchy estimate (cf. lemma 4.5.3, Appendix A) that

$$\|X_{\{\tilde{P}_n,F\}}\|_{D(r-2\sigma,\frac{1}{4}s)} \leq c_{12}\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\sigma^{-2}s^{-4}\varepsilon^{\frac{3}{4}}e^{-(|n|-K_+)} \, ;$$

therefore, by Lemma 4.3.5, if

C5) $c_{11}\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\eta^{-2}\varepsilon^{\frac{3}{4}} \leq \frac{1}{2}$,

C6) $c_{12}c_2(\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\eta^{-2}\varepsilon^{\frac{3}{4}})^2 \leq \frac{1}{2}$,

then

$$\begin{aligned}
&\left\| \int_0^1 \{\tilde{P}_n, F\} \circ \Phi_F^t \, dt \right\|_{D(r_+,s_+)} \\
&\leq \|\{\tilde{P}_n, F\} \circ \Phi_F^t\|_{D(r_+,s_+)} \\
&\leq \|\{\tilde{P}_n, F\}\|_{D(r_+,s_+)} + \|\{\tilde{P}_n, F\} \circ \Phi_F^t - \{\tilde{P}_n, F\}\|_{D(r_+,s_+)} \\
&\leq \|\{\tilde{P}_n, F\}\|_{D(r_+,s_+)} + \|X_{\{\tilde{P}_n,F\}}\|_{D_{2\eta}} \|\Phi_F^t - id\|_{D_{1\eta}} \\
&\leq c_{11}\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\eta^{-2}\varepsilon^{\frac{3}{4}} e^{-(|n|-K_+)} + \\
&\qquad\qquad c_{12}c_2(\gamma^{-2}(r-r_+)^{-(2\tau+N+1)}\eta^{-2}\varepsilon^{\frac{3}{4}})^2 e^{-(|n|-K_+)} \\
&\leq e^{-(|n|-K_+)} .
\end{aligned}$$

Remark 4.3.3 *Note in this KAM step, the normal components $w_n, \bar{w}_n$ with $K < |n| \leq K_+$ are involved but, at this time, the perturbation is $O(\varepsilon)$, which means*

$$|w_n| \sim \varepsilon \sim e^{-|n|} .$$

Hence the breathers we obtained are exponentially localized in space. In [108] breathers super–exponentially localized in space are obtained, the reason behind obtaining this type of decay is that, in [108], a short-range coupling potential is considered, this translates into the growth of the normal components being at most linear; in our work normal components grow exponentially fast and thus breathers herein are exponentially localized in space. In fact, the method developed here can also handle super-exponential growth of the normal components; for example, if $\|\breve{P}_n\| \leq \frac{1}{|n|^\alpha}$, $\|\acute{P}_{nm}\| \leq \frac{1}{|n-m|^\alpha}$, $\alpha > 1$, then the breathers we would obtain are localized in space like $\frac{1}{|n|^\alpha}$ (cf. [47]).

This completes one step of KAM iterations.

□

4.4 Proof of Theorem B

Let $r_0, s_0, \varepsilon_0, \gamma_0, K_0, \mathcal{O}_0, H_0, \mathcal{N}_0, \mathcal{P}_0$ be as given in the beginning of Section 3. For each $\nu = 0, 1, \cdots$, we will label all index-free quantities by ν and all $+$-indexed quantities by

$\nu+1$, in section 4.3. This defines, for all $\nu = 1, 2, \cdots$, the following sequences:

$$
\begin{aligned}
r_\nu &= r_0(1-\sum_{i=2}^{\nu+1} 2^{-i})\,,\\
\varepsilon_\nu &= 4c_0\gamma_{\nu-1}^{-2}(r_{\nu-1}-r_\nu)^{-(2\tau+N+1)}\varepsilon_{\nu-1}^{\frac{5}{4}}\,,\\
\gamma_\nu &= \gamma_0(1-\sum_{i=2}^{\nu+1} 2^{-i})\,,\\
s_\nu &= \frac{1}{8}\eta_{\nu-1}s_{\nu-1} \;=\; 2^{-3\nu}(\prod_{i=0}^{\nu-1}\varepsilon_i)^{\frac{1}{4}}s_0\,, \quad \eta_\nu = \varepsilon_\nu^{\frac{1}{4}}\,,\\
K_\nu &= 5K_{\nu-1}\,,\\
D_\nu &= D(r_\nu, s_\nu)\,,\\
\tilde{D}_\nu &= D(r_{\nu+1}+\frac{1}{4}(r_\nu - r_{\nu+1}), \frac{1}{4}\eta_\nu s_\nu)\,,\\
H_\nu &= \bar{H}_\nu \;= \mathcal{N}_\nu + \mathcal{P}_\nu\,,\\
\mathcal{N}_\nu &= \bar{\mathcal{N}}_\nu \;= e_\nu + \langle\omega_\nu(\xi), I\rangle + \langle A^\nu z^\nu, \bar{z}^\nu\rangle + \sum_{|n|>K_\nu}\beta w_n\bar{w}_n\,,
\end{aligned}
$$

$$
\mathcal{O}_\nu = \left\{ \xi\in\mathcal{O}_{\nu-1} \,:\, \begin{array}{l} |\langle k,\omega_{\nu-1}\rangle^{-1}| \le \frac{|k|^\tau}{\gamma_{\nu-1}} \quad (k\neq 0)\\ \|(\langle k,\omega_\nu\rangle I + \tilde{A}^{\nu-1})^{-1}\| \le K_\nu^4\frac{|k|^\tau}{\gamma_{\nu-1}}\\ \|(\langle k,\omega_{\nu-1}\rangle I + \tilde{A}^{\nu-1}\otimes I + I\otimes\tilde{A}^{\nu-1})^{-1}\| \le K_\nu^8\frac{|k|^\tau}{\gamma_{\nu-1}}\\ \|(\langle k,\omega_{\nu-1}\rangle I + \tilde{A}^{\nu-1}\otimes I - I\otimes\tilde{A}^{\nu-1})^{-1}\| \le K_\nu^8\frac{|k|^\tau}{\gamma_{\nu-1}}\,, \quad k\neq 0 \end{array} \right\}\,,
$$

where

$$
\tilde{A}^{\nu-1} = \begin{pmatrix} A^{\nu-1} & 0\\ 0 & \beta I\end{pmatrix}\,.
$$

4.4.1 Iteration Lemma

The preceding analysis may be summarized as follows,

Lemma 4.4.1 *Given γ sufficiently small, there exists ε sufficiently small, then the following holds for all $\nu = 0, 1, \cdots$.*

a) *H_ν is real analytic on $D_\nu\times\mathcal{O}_\nu$,*

$$
\mathcal{N}_\nu = \bar{\mathcal{N}}_\nu = e_\nu + \langle\omega_\nu(\xi), I\rangle + \langle\tilde{A}^\nu z^{\nu+1}, \bar{z}^{\nu+1}\rangle + \sum_{|n|>K_{\nu+1}}\beta w_n\bar{w}_n\,,
$$

$$
\mathcal{P}^\nu = \bar{P}^\nu = \breve{P}_\nu + \acute{P}_0 + \grave{P}_0\,,
$$

and moreover,

$$
\begin{aligned}
&|\omega_{\nu+1}-\omega_\nu|_{\mathcal{O}_\nu}\le c_0\varepsilon_\nu\,,\\
&\|\tilde A^{\nu+1}-\tilde A^{\nu}\|_{\mathcal{O}_\nu}\le c_0\varepsilon_\nu\,,\\
&\|X_{P^\nu}\|_{D_\nu,\mathcal{O}_\nu}\le\varepsilon_\nu\,,\\
&\breve P_\nu=\breve P^\nu(\theta,I,z^\nu,\bar z^\nu,\xi)+\sum_{|n|>K_\nu,\alpha_n+\beta_n\ge1}\breve P^\nu_n(\theta,I,z^\nu,\bar z^\nu,\xi)w_n^{\alpha_n}\bar w_n^{\beta_n}\,,\\
&\acute P_0=\sum_{\substack{n\neq m\\ \alpha_n+\beta_n,\alpha_m+\beta_m\ge1\\ \alpha_n+\beta_n+\alpha_m+\beta_m\ge3}}\acute P^0_{nm}(\xi)w_n^{\alpha_n}\bar w_n^{\beta_n}w_m^{\alpha_m}\bar w_m^{\beta_m},\\
&\grave P_0=\sum_n O(|w_n|^3)\,,
\end{aligned}
$$

with

$$
\|\breve P^\nu_n(\theta,I,z^\nu,\bar z^\nu,\xi)\|_{D_\nu,\mathcal{O}_\nu}\le e^{-(|n|-K_\nu)}\quad\textit{and}\quad\|\acute P^0_{nm}(\xi)\|_{D_\nu,\mathcal{O}_\nu}\le e^{-|n-m|}\,.
$$

b) *There is a symplectic transformation*

$$
\Phi_\nu:\tilde D_\nu\times\mathcal{O}_{\nu+1}\to D_\nu
$$

such that

$$
\|D\Phi_\nu-I\|_{D_\nu\times\mathcal{O}_{\nu+1}}\le\varepsilon_\nu
$$

and

$$
H_{\nu+1}=\bar H_{\nu+1}=H_\nu\circ\Phi_\nu.
$$

Proof. It is suffices to verify conditions **C0**)–**C6**) for all $\nu=0,1,\cdots$.

By induction, suppose the lemma holds for some $\nu-1$, then it is not difficult to see that one can make $\gamma=\gamma_0$ small enough so that

$$
s_\nu=\frac18\eta_{\nu-1}s_{\nu-1}<\frac18\varepsilon_{\nu-1}^{\frac54}<\varepsilon_\nu\,,
$$

hence **C0)** holds for all ν.

Conditions **C1-6)** follow from the following two conditions,

D1) $K_{\nu+1}\le\varepsilon_\nu^{-\frac{1}{80}}$,

D2) $c_0\gamma_\nu^{-2}(r_\nu - r_{\nu+1})^{-(2\tau+N+1)}\varepsilon_\nu^{\frac{1}{4}} \le \frac{1}{2}$

for all $\nu = 0, 1, \cdots$.

Let us first take ε (hence ε_0) sufficiently small such that

$$\varepsilon_0 \,<\, \min\{\frac{\gamma_0^{10} r_0^{5(2\tau+N+1)}}{2^{5(2\tau+N+1)} c_0^5}(\Psi(r_0))^{-1}, \frac{\delta}{2}\}\,,$$

where

$$\Psi(r_0) \,=\, \prod_{i=1}^{\infty}[(r_{i-1} - r_i)^{-5(2\tau+N+1)}]^{(\frac{4}{5})^i}$$

which is easily seen to be well-defined. Then

$$c_0\gamma_0^{-2}(r_0 - r_1)^{-(2\tau+N+1)}\varepsilon_0^{\frac{1}{4}} \le \frac{1}{2}\,;$$

i.e., **D2**) holds for $\nu = 0$. Now recall that $K_0 = \ln \varepsilon_0^{-\frac{5}{4}}$, then we see that **D1**) also holds for $\nu = 0$.

Using an induction argument, one can show that, in fact, for any $\nu \ge 1$,

$$\begin{aligned} c_0\gamma_\nu^{-2}(r_\nu - r_{\nu+1})^{-(2\tau+N+1)}\varepsilon_\nu^{\frac{1}{4}} &= c_0\gamma_\nu^{-2}(r_\nu - r_{\nu+1})^{-(2\tau+N+1)} \\ &\qquad (4c_0\gamma_\nu^{-2}(r_{\nu-1} - r_\nu)^{-(2\tau+N+1)}\varepsilon_{\nu-1}^{\frac{5}{4}})^{\frac{1}{4}} \\ &\le\, (2^{4(2\tau+N+1)}c_0^5\gamma_\nu^{-10}(r_{\nu-1} - r_\nu)^{-5(2\tau+N+1)}\varepsilon_{\nu-1}^{\frac{5}{4}})^{\frac{1}{4}} \\ &\le\, (2^{4(2\tau+N+1)}c_0^5\gamma_0^{-10}\Psi(r_0)\varepsilon_0)^{\frac{1}{4}(\frac{5}{4})^\nu} \\ &\le\, (\frac{r_0^{5(2\tau+N+1)}}{2^{2\tau+N+1}})^{\frac{1}{4}(\frac{5}{4})^\nu} \le \frac{1}{2}\,, \end{aligned}$$

and

$$K_{\nu+1} \,=\, 5^{\nu+1}K_0 \le \varepsilon_\nu^{-\frac{1}{80}}\,,$$

i.e., **D1**) and **D2**) hold true.

□

4.4.2 Convergence

Let $\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{\nu-1}$, $\nu = 1, 2, \ldots$. An induction argument shows that $\Psi^\nu : \tilde{D}_\nu \times \mathcal{O}_{\nu+1} \to D_0$ and

$$H_0 \circ \Psi^\nu = H_\nu = N_\nu + P_\nu$$

for all $\nu = 1, 2, \ldots$.

Let $\tilde{\mathcal{O}} = \cap_{\nu=0}^{\infty} \mathcal{O}_\nu$, then (cf. Lemma 4.4.1 and [66, 79]) one can show that $H_\nu, e_\nu, \mathcal{N}_\nu, \mathcal{P}_\nu, \Psi^\nu$ and ω_ν converge uniformly on $D(\frac{1}{2}r_0, 0) \times \tilde{\mathcal{O}}$ to, say, H_∞, e_∞, $\mathcal{N}_\infty$, $\mathcal{P}_\infty$, Ψ^∞ and ω_∞, respectively, in which case it is clear that

$$\mathcal{N}_\infty = e_\infty + \langle \omega_\infty, I \rangle + \langle A^\infty z^\infty, \bar{z}^\infty \rangle \,.$$

Since

$$\varepsilon_\nu = 4c_0\gamma_{\nu-1}^{-2}(r_{\nu-1} - r_\nu)^{-(2\tau+N+1)}\varepsilon_{\nu-1}^{\frac{5}{4}} \leq (4c_0\gamma_0^{-2}\Psi(r_0)\varepsilon_0)^{(\frac{5}{4})^\nu} \,,$$

we have, by Lemma 4.4.1, that

$$X_{\mathcal{P}^\infty}|_{D(\frac{1}{2}r_0,0)\times\tilde{\mathcal{O}}} \equiv 0 \,.$$

Let Φ_H^t denote the flow of any Hamiltonian vector field X_H. Since $H_0 \circ \Psi^\nu = H_\nu$, then

$$\Phi_{H_0}^t \circ \Psi^\nu = \Psi^\nu \circ \Phi_{H_\nu}^t \,. \tag{4.4.1}$$

The uniform convergence of Ψ^ν and X_{H_ν} imply that one can pass the limit in the above and conclude that

$$\Phi_{H_0}^t \circ \Psi^\infty = \Psi^\infty \circ \Phi_{H_\infty}^t \,,$$

on $D(\frac{1}{2}r_0, 0) \times \tilde{\mathcal{O}}$. It thus follows that

$$\Phi_{H_0}^t(\Psi^\infty(\mathbb{T}^N \times \{\xi\})) \,=\, \Psi^\infty\Phi_{\mathcal{N}_\infty}^t(\mathbb{T}^N \times \{\xi\}) \,=\, \Psi^\infty(\mathbb{T}^N \times \{\xi\}) \,,$$

for all $\xi \in \tilde{\mathcal{O}}$. Hence $\Psi^\infty(\mathbb{T}^N \times \{\xi\})$ is an embedded invariant torus of the original perturbed Hamiltonian system at $\xi \in \tilde{\mathcal{O}}$. The frequencies $\omega_\infty(\xi)$ associated with $\Psi^\infty(\mathbb{T}^N \times \{\xi\})$ are slightly deformed from the unperturbed ones, $\omega(\xi)$; moreover, the behavior in the normal directions of the invariant tori $\Psi^\infty(\mathbb{T}^N \times \{\xi\})$ are governed by their normal frequency matrix $A^\infty = A^\infty(\xi)$, which is constant, from which their linear stability follows.

□

4.4.3 Measure estimates

For each $\nu = 0, 1, 2, \ldots$, recall that, for $|k| \leq K_{\nu+1}$, small denominator conditions are automatically satisfied. For $|k| > K_{\nu+1}$, let us consider the most complicated case, $\langle k, \omega_\nu \rangle I + \tilde{A}^\nu \otimes I - I \otimes \tilde{A}^\nu$. Because $\tilde{A}^\nu$ is Hermitian with $\dim(\tilde{A}^\nu) \leq K_{\nu+1}$, then $\langle k, \omega_\nu \rangle I + \tilde{A}^\nu \otimes I - I \otimes \tilde{A}^\nu$ is also Hermitian with $\dim(\langle k, \omega_\nu \rangle I + \tilde{A}^\nu \otimes I - I \otimes \tilde{A}^\nu) \leq K_{\nu+1}^2$. Now consider the following simple

Lemma 4.4.2 *Let $\mu_1, \cdots, \mu_K$ be the eigenvalues of a Hermitian matrix A and let P with $\bar{P}^\top P = I$ be such that $A = P^\top \Lambda P$, with $\Lambda = (\mu_j)_{1 \leq j \leq K}$. Let*

$$\min\{|\mu_1|, \cdots, |\mu_K|\} \geq l \,,$$

then

$$\|A^{-1}\| \leq \frac{K^2}{l} \,.$$

Proof. since $\bar{P}^\top P = I$, then $\|P\| \leq K$ and thus

$$\|A^{-1}\| \leq \|P\|^2 \|\Lambda^{-1}\| \leq \frac{K^2}{l} \,.$$

Let μ be an eigenvalue of $(\langle k, \omega_\nu \rangle I + \tilde{A}^\nu \otimes I - I \otimes \tilde{A}^\nu)$ then, according to Lemma 4.4.2, as long as $|\mu| \geq \frac{\gamma_\nu}{|k|^\tau}$ one will have that

$$\|(\langle k, \omega_\nu \rangle I + \tilde{A}^\nu \otimes I - I \otimes \tilde{A}^\nu)^{-1}\| \leq K_{\nu+1}^8 \frac{|k|^\tau}{\gamma_\nu} \,.$$

As a consequence of the above, we need to exclude the following parameter set

$$\mathcal{R}_k^{\nu 1}(\gamma_\nu) \,=\, \{\xi \in \mathcal{O}_{\nu-1} : |\mu| < \frac{\gamma_\nu}{|k|^\tau}\} \,.$$

For the same reason, we will also have to exclude the following parameter sets associated to eigenvalues ζ and λ of $(\langle k, \omega_\nu \rangle I + \tilde{A}^\nu)$ and $(\langle k, \omega_\nu \rangle I + \tilde{A}^\nu \otimes I + I \otimes \tilde{A}^\nu)$, respectively;

$$\begin{aligned}
\mathcal{R}_k^{\nu 2}(\gamma_\nu) &= \{\xi \in \mathcal{O}_{\nu-1} \,:\, |\langle k, \omega_\nu \rangle| < \frac{\gamma_\nu}{|k|^\tau}\} \,, \\
\mathcal{R}_k^{\nu 3}(\gamma_\nu) &= \{\xi \in \mathcal{O}_{\nu-1} \,:\, |\zeta| < \frac{\gamma_\nu}{|k|^\tau}\} \,, \\
\mathcal{R}_k^{\nu 4}(\gamma_\nu) &= \{\xi \in \mathcal{O}_{\nu-1} \,:\, |\lambda| < \frac{\gamma_\nu}{|k|^\tau}\} \,.
\end{aligned}$$

Then, the following is true for all $\nu = 0, 1, 2, \ldots,$

$$\mathcal{O}_\nu \subseteq \mathcal{O}_{\nu-1} \backslash \{ (\bigcup_{|k|>K_{\nu+1}} \mathcal{R}_k^{\nu 1}(\gamma_\nu) \} \bigcup \{ \bigcup_{|k|>K_{\nu+1}} \mathcal{R}_k^{\nu 2}(\gamma_\nu) \} \bigcup \{ \bigcup_{|k|>K_{\nu+1}} \mathcal{R}_k^{\nu 3}(\gamma_\nu) \} \bigcup \{ \bigcup_{|k|>K_{\nu+1}} \mathcal{R}_k^{\nu 4}(\gamma_\nu)) \} \,,$$

Now consider the resonant sets

$$\mathcal{R}^\nu = \{ \bigcup_{|k|>K_{\nu+1}} \mathcal{R}_k^{\nu 1}(\gamma_\nu) \} \bigcup \{ \bigcup_{|k|>K_{\nu+1}} \mathcal{R}_k^{\nu 2}(\gamma_\nu) \} \bigcup \{ \bigcup_{|k|>K_{\nu+1}} \mathcal{R}_k^{\nu 3}(\gamma_\nu) \} \bigcup \{ \bigcup_{|k|>K_{\nu+1}} \mathcal{R}_k^{\nu 4}(\gamma_\nu) \} \,,$$

It is clear that

$$\mathcal{O} \setminus \tilde{\mathcal{O}} \subseteq \bigcup_{\nu \geq 0} \mathcal{R}^\nu \,.$$

Lemma 4.4.3 *For fixed k and ν, there is a constant $C_1 > 0$ such that*

$$\left| (\mathcal{R}_k^{\nu 1}(\gamma_\nu) \bigcup \mathcal{R}_k^{\nu 2}(\gamma_\nu) \bigcup \mathcal{R}_k^{\nu 3}(\gamma_\nu) \bigcup \mathcal{R}_k^{\nu 4}(\gamma_\nu) \right| \leq C_1 \frac{\gamma}{|k|^{\tau-1}} \,.$$

Proof. Let μ be an eigenvalue of $(\langle k, \omega_\nu \rangle I + \tilde{A}^\nu \otimes I - I \otimes \tilde{A}^\nu)$, which is Hermitian. Then there is an eigenvector ψ with $\langle \psi, \psi \rangle = (\bar{\psi})^\top \psi = 1$, such that

$$\mu \;=\; \langle (\langle k, \omega_\nu \rangle I + \tilde{A}^\nu \otimes I - I \otimes \tilde{A}^\nu) \psi, \psi \rangle \,,$$

so that

$$\partial_\xi \mu \;=\; \langle \partial_\xi (\langle k, \omega_\nu \rangle I + \tilde{A}^\nu \otimes I - I \otimes \tilde{A}^\nu) \psi, \psi \rangle \,.$$

Thus

$$|\partial_\xi \mu| \;\geq\; |\langle \partial_\xi (\langle k, \omega_0 \rangle + \beta - \beta) \psi, \psi \rangle| - \varepsilon_0 |k| \geq \frac{\delta |k|}{2} \,.$$

The cases for $\mathcal{R}_k^{\nu 2}(\gamma_\nu)$, $\mathcal{R}_k^{\nu 3}(\gamma_\nu)$, $\mathcal{R}_k^{\nu 4}(\gamma_\nu)$ can be handled in an entirely analogous way. Thus lemma 4.4.3 is proven.

□

Lemma 4.4.4

$$\left| \mathcal{O} \setminus \tilde{\mathcal{O}} \right| \leq \left| \bigcup_{\nu \geq 0} \mathcal{R}^\nu \right| = O(\gamma) \,.$$

Proof. Let $\tau \geq N+2$, by Lemma 4.4.3, we have that

$$\left|\mathcal{O} \setminus \tilde{\mathcal{O}}\right| \leq \left|(\bigcup_{\nu\geq 0} \mathcal{R}^{\nu}\right| = O(\sum_{\nu\geq 0} \sum_{|k|>K_{\nu+1}} \frac{\gamma}{|k|^{\tau-1}}) = O(\sum_{\nu\geq 0} \frac{\gamma}{K_{\nu+1}}) = O(\gamma)\,.$$

□

And this completes the measure estimate and thus the proof of theorem B.

APPENDIX A

Lemma A.0.5

$$\|FG\|_{D(r,s),\mathcal{O}} \leq \|F\|_{D(r,s),\mathcal{O}}\|G\|_{D(r,s),\mathcal{O}}\,.$$

Proof. Since

$$(FG)_{kl\alpha\beta} = \sum_{k',l',\alpha',\beta'} F_{k-k',l-l',\alpha-\alpha',\beta-\beta'}G_{k'l'\alpha'\beta'}\,,$$

one has that

$$\begin{aligned}
\|FG\|_{D(r,s),\mathcal{O}} =& \sup_{\substack{\|w\|<s\\ \|\bar{w}\|<s}} \sum_{k,l,\alpha,\beta} |(FG)_{kl\alpha\beta}|s^{2l}|w^{\alpha}||\bar{w}^{\beta}|e^{|k|r} \\
\leq& \sup_{\substack{\|w\|<s\\ \|\bar{w}\|<s}} \sum_{k,l,\alpha,\beta}\sum_{k',l',\alpha',\beta'} |F_{k-k',l-l',\alpha-\alpha',\beta-\beta'}G_{k'l'\alpha'\beta'}|s^{2l}|w^{\alpha}||\bar{w}^{\beta}|e^{|k|r} \\
\leq& \|F\|_{D(r,s),\mathcal{O}}\|G\|_{D(r,s),\mathcal{O}}\,.
\end{aligned}$$

□

Lemma A.0.6 (*Generalized Cauchy inequalities*)

$$\begin{aligned}
\|F_{\theta}\|_{D(r-\sigma,s)} \leq& \frac{1}{\sigma}\|F\|_{D(r,s)}\,, \\
\|F_I\|_{D(r,\frac{1}{2}s)} \leq& \frac{4}{s^2}\|F\|_{D(r,s)}\,, \\
\|F_w\|_{D(r,\frac{1}{2}s)} \leq& \frac{2}{s}\|F\|_{D(r,s)}\,, \\
\|F_{\bar{w}}\|_{D(r,\frac{1}{2}s)} \leq& \frac{2}{s}\|F\|_{D(r,s)}\,.
\end{aligned}$$

Proof. See [79].

Let $\{F,G\}$ denote the Poisson bracket of smooth functions F and G; i.e.,

$$\{F,G\} = \langle\frac{\partial F}{\partial I},\frac{\partial G}{\partial\theta}\rangle - \langle\frac{\partial F}{\partial\theta},\frac{\partial G}{\partial I}\rangle + \mathrm{i}\sum_n(\frac{\partial F}{\partial w_n}\frac{\partial G}{\partial\bar{w}_n} - \frac{\partial F}{\partial\bar{w}_n}\frac{\partial G}{\partial w_n})\,.$$

Lemma A.0.7 *There exists a constant $c>0$ such that if*

$$\|F_n\|_{D(r,s)} < e^{-|n|}\,,\quad \|G\|_{D(r,s)} < \varepsilon\,,$$

then

$$\|\{F_n,G\}\|_{D(r-\sigma,\frac{1}{2}s)} < c\sigma^{-1}s^{-2}\|F_n\|_{D(r,s)}\|G\|_{D(r,s)} \leq c\sigma^{-1}s^{-2}\varepsilon e^{-|n|}\,.$$

Proof. By LemmaA.0.5 and LemmaA.0.6,

$$\begin{aligned}
\|\langle F_{n_I}, G_\theta\rangle\|_{D(r-\sigma,\frac{1}{2}s)} &< 4\sigma^{-1}s^{-2}\|F_n\|\cdot\|G\|\,,\\
\|\langle F_{n_\theta}, G_I\rangle\|_{D(r-\sigma,\frac{1}{2}s)} &< c\sigma^{-1}s^{-2}\|F_n\|\cdot\|G\|\,,\\
\|\sum_m F_{n_{wm}}G_{\bar{w}_m}\|_{D(r,\frac{1}{2}s)} &\le \sum_m \|F_{n_{wm}}\|_{D(r,\frac{1}{2}s)}\|G_{\bar{w}_m}\|_{D(r,\frac{1}{2}s)}\\
&\le \|F_{n_w}\|_{D(r,\frac{1}{2}s)}\|G_{\bar{w}}\|_{D(r,\frac{1}{2}s)}\\
&\le 4s^{-2}\|F_n\|\cdot\|G\|,\\
\|\sum_m F_{n_{\bar{w}m}}G_{w_m}\|_{D(r,\frac{1}{2}s)} &\le \sum_m \|F_{n_{\bar{w}m}}\|_{D(r,\frac{1}{2}s)}\|G_{w_m}\|_{D(r,\frac{1}{2}s)}\\
&\le \|F_{n_{\bar{w}}}\|_{D(r,\frac{1}{2}s)}\|G_w\|_{D(r,\frac{1}{2}s)}\\
&\le 4s^{-2}\|F_n\|\cdot\|G\|\,.
\end{aligned}$$

It follows that

$$\|\{F_n, G\}\|_{D(r-\sigma,\frac{1}{2}s)} < c\sigma^{-1}s^{-2}\|F_n\|_{D(r,s)}\|G\|_{D(r,s)} \le c\sigma^{-1}s^{-2}\varepsilon e^{-|n|}\,.$$

□

Lemma A.0.8 *There exists a constant $c>0$ such that if*

$$\|X_F\|_{D(r,s)} < \varepsilon'\,,\quad \|X_G\|_{D(r,s)} < \varepsilon''\,,$$

for some $\varepsilon', \varepsilon'' > 0$, then

$$\|X_{\{F,G\}}\|_{D(r-\sigma,\eta s)} < c\sigma^{-1}\eta^{-2}\varepsilon'\varepsilon''\,,$$

for any $0<\sigma<r$ and $0<\eta\ll 1$. In particular, if $\eta\sim\varepsilon^{\frac{1}{4}}$, $\varepsilon'\sim\varepsilon, \varepsilon''\sim\varepsilon^{\frac{3}{4}}$, then

$$\|X_{\{F,G\}}\|_{D(r-\sigma,\eta s)} \sim \varepsilon^{\frac{5}{4}}\,.$$

Proof. See [49].

□

APPENDIX B

This brief appendix details the first of a set of canonical transformations that wrote our Hamiltonian (4.1.1) in action-angle-normal coordinates. First recall that, for simplicity, we have set $p = 3$, so that

$$H \;=\; \sum_{n\in\mathcal{J}}(\frac{p_n^2}{2} + V_n(q_n)) + \sum_{n\in\mathbb{Z}_1}(\frac{p_n^2}{2} + \frac{\beta^2 q_n^2}{2} + O(|q_n|^3)) + \varepsilon \sum_{m\neq n} \frac{1}{3} C_{m,n}(q_m - q_n)^3 \,.$$

We fix $N \in \mathbb{Z}^+$, a set of indices $\mathcal{J} = \{n_1, \ldots, n_N\}$ and a compact interval $\mathcal{I} \subset \mathbb{R}_+$, possibly depending on $\mathcal{J}$ and contained in a sufficiently small neighborhood of the origin so that, for every $h \in \mathcal{I}$ and $n_j \in \mathcal{J}$, $\frac{p^2}{2} + V_{n_j}(q) = h$ defines a simple and closed curve, $\Gamma_j(h)$, around the origin in the pq-plane.

For every $j = 1, \ldots, N$, let

$$\rho_j(h) \;\stackrel{\text{def}}{=}\; \frac{1}{2\pi} \oint_{\Gamma_j(h)} p\, dq \,.$$

It follows from this definition and from hypotheses *(V1)* ($V_n(0) = V_n'(0) = 0$ and $V_n''(0) = \frac{\beta^2}{2} > 0$, $\forall\, n \in \mathbb{Z}$) and *(V2)* ($\rho_j'(h) \neq 0$ and $\rho_j''(h) \neq 0$ for all $h \in \mathcal{I}$) that we can assume, without any loss in generality, that $\rho_j(h) > 0$ for all $h \in \mathcal{I}$ and that ρ_j is a diffeomorphism over $\mathcal{I}$; so let $H_{0,j} : \mathbb{R}^+ \to \mathcal{I}$ be its inverse, thus

$$\frac{p^2}{2} + V_{n_j}(q) \;=\; H_{0,j}(\rho)\,, \quad j = 1, \ldots, N \,.$$

Consider for the moment $j \in \{1, \ldots, N\}$ fixed. Then, for any given ρ and q sufficiently small (say, $0 < \tilde{r}_0 \leq \rho \leq \tilde{r}_1$, where $0 < \tilde{r}_0 \ll 1$), there exists a unique Γ_j', clockwise-oriented arc of Γ_j that joins the positive p-semiaxis with the point (q, p) (cf. figure 12).

Let us now define a *generating function*, $S : (q, \rho) \mapsto S(q, \rho)$ by

$$S(q, \rho) \;\stackrel{\text{def}}{=}\; \oint_{\Gamma_j'} p\, dq \,.$$

and a map $\psi_0(\rho, \theta) \mapsto (p, q)$ given implicitly as follows,

$$\theta = \frac{\partial S(q,\rho)}{\partial \rho}\,, \qquad p = \frac{\partial S(q,\rho)}{\partial q}\,.$$

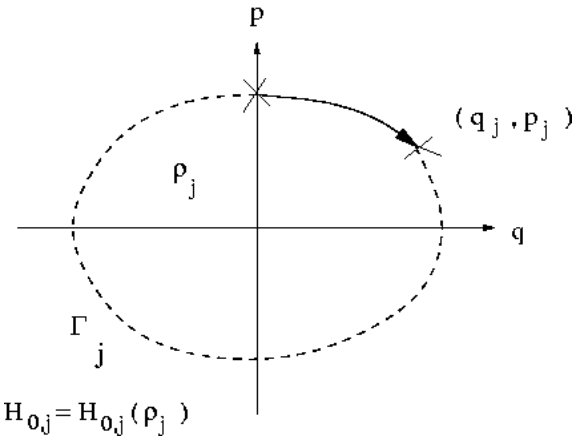

Figure 12: *Construction of action-angle coordinates.*

Note that $dS|_{\rho=constant} = p\,dq$ and thus S above is defined up to an integer multiple of the area enclosed by Γ_j; however, this integer multiple of the area does not enter in our definition $p = S_q(q, \rho, q)$ but it renders $\theta = S_\rho(q, \rho)$ multivalued; nevertheless, one can show that

$$\oint_{\Gamma_j} d\theta \,=\, 2\pi$$

(cf. [4] p. 281). The above outlines the construction of *action-angle variables* (ρ_j, θ_j), $j = 1, \ldots, N$. Observe that $dp \wedge dq = \frac{\partial^2 S}{\partial \rho \partial q} d\rho \wedge dq$ and that $d\rho \wedge d\theta = \frac{\partial^2 S}{\partial q \partial \rho} d\rho \wedge dq$; therefore, $dp \wedge dq = d\rho \wedge \theta$.

Finally, let Ψ be a map defined by

$$\Psi \,:\, \begin{cases} (p_n, q_n) = \psi_0(\rho_n, \theta_n) & n \in \mathcal{J} \\ (p_n, q_n) = (\sqrt{\beta}\bar{v}_n, \frac{1}{\sqrt{\beta}} v_n) & n \in \mathbb{Z}_1 \end{cases}$$

then it is clear that

$$\sum_{n\in\mathcal{J}} dp_n \wedge dq_n + \sum_{n\in\mathbb{Z}_1} dp_n \wedge dq_n = \sum_{j=1}^{N} d\rho_j \wedge d\theta_j + \sum_{n\in\mathbb{Z}_1} d\bar{v}_n \wedge dv_n \,;$$

that is, Ψ is symplectic. Therefore,

$$H \circ \Psi \,=\, \sum_{j=1}^{N} H_{0,j}(\rho_j) + \sum_{n\in\mathbb{Z}_1} \frac{\beta}{2}(v_n^2 + \bar{v}_n^2) + O(|v_n|^3) + \varepsilon \sum_{m\neq n} \frac{1}{3} C_{m,n}(q_m - q_n)^3 \,,$$

the last term must be understood as a function of $(\rho, \theta, v, \bar{v})$.

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