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Exact and Approximate Expressions for Bubble/Particle Collision

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Abstract

The flotation microprocess of collision is investigated and an exact expression for the probability of collision ($P_c$) is developed based on the intermediate flow of Yoon and Luttrell (1). This expression for $P_c$ only assumes that the bubble and particle are spherical and that the particle radius is less than the bubble radius (i.e., $R_p < R_B$). In addition to removing the requirement that $R_p << R_B$, the influence of a particle settling velocity is also included in the model development. The expression for $P_c$ is shown to be a function of three dimensionless groups: (i) the magnitude of the dimensionless particle settling velocity, $|G|$; (ii) the bubble Reynolds number, $Re_B$; and (iii) the ratio of particle to bubble radius, $R_p/R_B$.

The probability of collision model is compared to available experimental data and good agreement is shown. A parametric study is also completed for $0 \leq |G| \leq 1$, $0 \leq Re_B \leq 500$, and $0.001 \leq R_p/R_B < 1$. In general, $P_c$ is independent of $Re_B$ when $R_p/R_B \lesssim 0.03$, the particle settling velocity is important for small values of $R_p/R_B$, and $R_p/R_B$ dominates as $R_p/R_B \to 1$.

Key Words: capture; collision; flotation; microprocess probability; particle settling velocity
Introduction

Flotation separation is used in many industries such as mineral processing, petrochemical refining, water treatment, and pulp and paper manufacturing. In the paper industry, flotation is used in paper recycling to separate inks and other contaminants from useable cellulose fiber. This separation process is called flotation deinking, and is accomplished by injecting air bubbles into an agitated liquid tank containing suspended cellulose fibers and contaminant particles (including inks and toners). The air bubbles preferentially attach to hydrophobized contaminant particles and transport them to the froth layer where they may be easily removed.

The basic viewpoint that has been taken in modeling the overall flotation separation process is that it is a multi-stage probability process consisting of a sequence of microprocesses with associated probability measures. This sequence includes the approach of a particle to an air bubble, the subsequent interception of that particle by the bubble, the sliding of the particle along the surface of the thin liquid film that separates the particle from the bubble, film rupture, the subsequent formation of a three-phase contact between the bubble, particle, and liquid, and the stabilization of the bubble/particle aggregate (with its subsequent transport to the froth layer for removal from the flotation cell).

Probability measures, which are associated with some of the elementary microprocesses have appeared in many places in the literature. In this paper we develop new, exact, expressions for $P_c$, the microprocess probability of collision (or capture) of a particle by a bubble. In the analysis to follow, all particles and all bubbles in any given volume of the flotation cell are assumed to be perfectly spherical.

As indicated in Fig. 1, only those particles which approach a rising bubble within a streaming tube of limiting capture radius $R_c$ can collide with or be intercepted by a bubble.
Once an expression has been determined for $R_c$, the probability $P_c$ is then computed to be the ratio of the number of particles with $R_p < R_B$ which encounter a bubble per unit time to the number of particles which approach a bubble in a stream tube with cross section equal to $\pi(R_p + R_B)^2$; this ratio is easily determined to be given by

$$P_c = \left(\frac{R_c}{R_p + R_B}\right)^2$$

where $R_p$ and $R_B$ are the particle and bubble radius, respectively.

Many authors simply take $P_c = \left(\frac{R_c}{R_B}\right)^2$, e.g. Yoon and Luttrell (1); however, as these authors note, “the denominator should actually be $R_B + R_p$ but (the) equation holds when $R_B >> R_p$”. Because one of our goals is the derivation of exact expressions for $P_c$, we choose not to make any approximations which are based upon assumptions concerning the relative magnitudes of $R_p$ and $R_B$ until the final stages of the analysis.

The determination of an expression for $R_c$ in [1] is a nontrivial exercise which has occupied the attention of many researchers in colloidal hydrodynamics during the past six decades since the original work of Sutherland (2) (which dealt with potential flow around the bubble in the absence of both inertial forces and gravitational effects); principal contributions in this area include the work of Yoon and Luttrell (1, 3), Ahmed and Jameson (4), Schulze (5, 6), Flint and Howarth (7), Nguyen-Van and Kmet (8), Nguyen-Van (9), Weber (10), Weber and Paddock (11), Reay and Ratcliff (12), Dobby and Finch (13), Anfruns and Kitchener (14, 15), Spielman (16), and Michael and Norey (17). During the course of this analysis, we will have occasion to refer to specific results in several of the papers referenced above and, in particular, will indicate the manner in which many of those results are either special cases of or approximations to the more exact relations that are derived below.

The specific derivation of expressions for the capture radius $R_c$ is dependent upon the
basic assumptions one makes about the relationship between $R_p$ and $R_B$, the nature of the flow field in which the particle moves, and the role (or lack thereof) of inertial and gravitational forces in the process. At this stage of the overall flotation process, i.e., the approach of a particle to a bubble, only the long-range hydrodynamic interaction is taken into account as opposed to those short-range hydrodynamic interactions which must be considered once the particle has intercepted the bubble and begins the sliding process over the thin film which separates the particle from the bubble. A rather comprehensive discussion of the overall flotation deinking process may be found in (18-20).

Among the key parameters which arise in any discussion of the flow field in the neighborhood of a rising bubble are the bubble Reynolds number

$$Re_B = \frac{v_B d_B \rho_t}{\mu t}$$  \[2\]

and the Stokes number

$$St = \frac{\rho_p d_p^2 v_B}{g \mu t d_B} = \frac{Re_B \rho_p d_p}{9 \rho_l d_B^2}$$  \[3\]

which is the ratio of the inertial force of the particle to the viscous drag force of the bubble. In the above equations, $v_B$ is the bubble rise velocity, $d_B$ and $d_p$ are the bubble and particle diameter, $\rho_p$ and $\rho_t$ are the particle and liquid density, and $\mu_t$ is the liquid dynamic viscosity.

Much of the earlier literature on flotation processes was concerned with mineral flotation for which $0.1 < St < 1$ is a reasonable assumption; however, some of the later work in that area, as well as almost all the work on flotation deinking, has been concerned with the situation in which $St << 0.1$ so that inertial forces, in essence, no longer influence particle motion. Under these circumstances, it is still possible for particle paths to deviate slightly from the streamlines of the flow if one accounts for particle settling velocity.

In the present work three types of flow will be discussed: potential flow, Stokes flow, and
the intermediate flow of Yoon and Luttrell (1, 3); our main interest is in the latter class of flows as previously discussed in (18); the class of intermediate flows introduced in (1) has also been incorporated into the work of Schulze (5) and Nguyen-Van (9) and discussed in the recent survey paper of Matis and Zouboulis (21). For all three of the flows listed above we shall assume that the flow streamlines are symmetrical, fore and aft, with respect to the bubble surface; such an assumption was explicitly employed by Yoon and Luttrell (1) and implies that a grazing trajectory may be defined as the one which, at the bubble equator, passes within a distance of particle radius $R_p$ from the bubble surface (Fig. 1). Clearly, such a trajectory, when traced back an infinite distance from the bubble surface, passes precisely within a distance $R_c$ of the stagnation line of the flow which passes through the bubble center. Fore and aft asymmetry has been discussed by Clift et al. (22), and if one does not assume that the grazing trajectory occurs at $\theta = \pi / 2$ in Fig. 1, then a collision angle $\theta_c$ must be introduced, $\theta_c$ being the angle on the bubble surface, measured from the front stagnation point, over which particle interception by the surface is possible. The recent work of Nguyen-Van (9) indicates that $\theta_c = \pi / 2$ for the intermediate flow of Yoon and Luttrell (1), as well as for potential flow and creeping Stokes flow, is a reasonable assumption. Cases for which $\theta_c \neq \pi / 2$ have been discussed in (8-13).

In order to determine the trajectory of a particle approaching a rising bubble, one begins by considering, in Cartesian coordinates, the forces which act on a typical particle. In this paper we let $v_p$ represent the particle velocity, and $v_{px}$ and $v_{py}$, the $x$ and $y$ components, respectively, of the particle velocity field.

Accounting for the drag, buoyancy, and gravitational forces, a system of equations rep-
resenting the particle motion may be written as

\[
\begin{align*}
\frac{4}{3} \pi \rho_p R_p^3 \frac{dv_{px}}{dt} &= -f(v_{px} - u_x) \\
\frac{4}{3} \pi \rho_p R_p^3 \frac{dv_{py}}{dt} &= \frac{4}{3} \pi R_p \Delta g - f(v_{py} - u_y)
\end{align*}
\]  

[4]

where \( f \) is the friction factor and \( \Delta \rho = \rho_p - \rho_l \). For Stokesian particles it is well known that \( f = 6 \pi \mu \ell R_p \) in which case the drag force is given by \( \mathbf{F}_d = 6 \pi \mu \ell R_p \mathbf{v}_p \). For non-Stokesian particles we have, in general, \( \mathbf{F}_d = f \mathbf{v}_p \) while the coefficient of drag, \( C_D \), is defined to be

\[
C_D = \frac{f}{\frac{1}{2} \rho_l |\mathbf{v}_p| \pi R_p^2} 
\]  

[5]

In the Stokesian case, with \( f = 6 \pi \mu \ell R_p \) and \( C_D = C_{D}^{st} \), [5] yields

\[
C_{D}^{st} = 12 \nu \ell / R_p |\mathbf{v}_p| 
\]  

[6]

where \( \nu \ell \) is the liquid kinematic viscosity \( (\nu \ell = \mu \ell / \rho_l) \). If we define, in the usual manner, the Reynolds number for the particle to be

\[
Re_p = \frac{2 R_p |\mathbf{v}_p|}{\nu \ell} 
\]  

[7]

then [6] and [7] yield the widely known result (e.g., Cheremisinoff (23)) that \( C_{D}^{st} = 24 / Re_p \).

In the general case, however, it is easily seen that [5] and [7] combine so as to yield

\[
C_D = \frac{4 f}{(\pi \mu \ell R_p) Re_p} 
\]  

[8]

It is generally accepted (23) that \( C_D = C_{D}^{st} = 24 / Re_p \) holds for \( Re_p < 2 \). For the situation that is considered below, in which inertial forces acting on the particle are ignored (so that, in effect, the Stokes number \( St = 0 \)), the particle velocity corresponds to the particle settling velocity \( (\mathbf{v}_p = \mathbf{v}_{ps}) \). In this case it can be demonstrated (23) that

\[
C_D Re_p^2 = \frac{4}{3} Ar 
\]  

[9]
where the Archimedes number $Ar$ is the dimensionless parameter defined by

$$Ar = \frac{\Delta \rho \, a_p^3 \, g}{\rho \, \nu^2}$$  \hspace{1cm} [10]$$

For the Stokes' law range ($Re_p < 2$), the use of $C_D = C_D^{st} = \frac{24}{Re_p}$ in [9] leads to $Re_p = \frac{Ar}{18}$. In the intermediate or transitional range in which $2 < Re_p < 500$ empirical results must be used; from the results reported in (23) we infer that

$$C_D = \frac{18.5}{Re_p^{0.6}}, \quad 2 < Re_p < 500$$  \hspace{1cm} [11]$$

the use of which in [9] yields

$$Re_p = 0.152 Ar^{0.715}, \quad 2 < Re_p < 500$$  \hspace{1cm} [12]$$

By combining [8] with [9] we find that, in general

$$f = \frac{\pi \mu \ell R_p \, Ar}{3 \, Re_p^{18}}$$  \hspace{1cm} [13]$$

From [13], with $Re_p = \frac{Ar}{18}$ for Stokesian particles, we recover the usual friction factor $f = 6\pi \mu \ell R_p$ associated with the Stokes flow regime, while for $2 < Re_p < 500$ the required result for $f$ is obtained by combining [13] with the empirical relation [12].

For the analysis which follows, it is convenient to introduce the dimensionless factor

$$\lambda \equiv 6\pi \mu \ell R_p / f = 18Re_p / Ar$$  \hspace{1cm} [14]$$

by virtue of [13]. Clearly $\lambda = 1$ for Stokesian particles ($Re_p < 2$) while in the transitional domain ($2 < Re_p < 500$) $\lambda$ may be computed by combining [14] with the empirical relation [12].
Instead of working with the system [4] we may nondimensionalize the equations by introducing the variables

\[ v_p^* = \frac{v_p}{v_B}, \quad u^* = \frac{u}{v_B}, \quad t^* = \frac{tv_B}{R_B} \]  

[15]

A straightforward calculation, and using the dimensionless factor \( \lambda \), yields the system

\[
\begin{align*}
\lambda St \frac{dv_p^*}{dt^*} &= -(v_p^* - u_x^*) \\
\lambda St \frac{dv_p^y}{dt^*} &= \frac{2R_p^2 \Delta \rho g}{9\mu t v_B} \lambda - (v_p^y - u_y^*)
\end{align*}
\]

[16]

We now set

\[ \tilde{v}_p = -\frac{2R_p^2 \Delta \rho g}{9\mu t} \]  

[17]

and

\[ G = \frac{\tilde{v}_p}{v_B} \]  

[18]

According to our sign convention \( v_B > 0 \) so that \( v_p < 0 \); thus, we also have \( G < 0 \). In [17] and [18], \( \tilde{v}_p \) represents the (terminal) particle settling velocity for Stokesian particles, \( v_p = \lambda \tilde{v}_p \) is the true particle settling velocity, and \( G \) is the dimensionless particle settling velocity. For Stokesian particles, therefore, \( G = \tilde{v}_p/v_B \).

Using the definitions [17] and [18], in [16], assuming that \( St \simeq 0 \), so that inertial effects are discounted, and noting that \( -v_p = |v_p|, -G = |G| \), we easily find that

\[
\begin{align*}
v_p^* &= u_x^* \\
v_p^y &= |G| + u_y^*
\end{align*}
\]

[19]

The system [19] has appeared in Flint and Howarth (7) and Schulze (6); however, in these references it has been assumed that \( G \equiv \tilde{v}_p/v_B \), which is only valid for Stokes flow.
Exact and Approximate Expressions for $P_c$

We begin the analysis by recalling that in [1] $R_c$ represents the largest distance from the stagnation line through the center of the rising bubble, within which a particle path trajectory can pass so that the particle surface will graze the bubble surface at $\theta = \pi/2$, i.e., the maximal distance so that $r = R_B + R_p$ along the particle path trajectory when $\theta = \pi/2$. By virtue of [19], particle path trajectories are coincident with fluid streamlines when $G = 0$. Also, from Fig. 1, it is clear that there exists a smallest $r$, say, $r = r_c$ with the property that, along a particle path trajectory, an approaching particle will be at the distance $R_c$ from the stagnation line through the center of a rising bubble for all $r \geq r_c$. We now define $\theta_0$ by

$$\sin \theta_0 = \frac{R_c}{r_c}$$

[20]

and note that

$$\sin \theta = \frac{R_c}{r}, \quad \text{for all } r \geq r_c$$

[21]

Our first task is the derivation of an exact expression for $P_c$ for the case of the intermediate flow delineated in Yoon and Luttrell (1). The stream function for ‘intermediate flow’ (as given in (1)), has the form

$$\psi^{int} = v_B R_B^2 \sin^2 \theta \left[ \frac{1}{2} r^{*2} - \frac{3}{4} r^{*} + \frac{1}{4} r^{*3} \right] + R e^*_B \left( \frac{1}{r^{*2}} - \frac{1}{r^{*}} + r^{*} - 1 \right)$$

[22]

where

$$R e^*_B = \frac{1}{15} R e_B^{0.72}, \quad r^{*} = r/R_B$$

[23]

It is noted that the widely-used stream function empirically determined by Yoon and Luttrell (1) predicts a zero radial liquid velocity at $\theta = \pi/2$; however, experiments by Seeley et al. (24) show that this velocity is nonzero.
We now rewrite the system of equations [19] in ‘polar coordinates’ (actually, spherical coordinates projected onto the $x, y$ plane) as

\[
\begin{align*}
v_{\theta}^* &= |G| \sin \theta + u_{\theta}^* \\
v_{r}^* &= -|G| \cos \theta + u_{r}^*
\end{align*}
\]  

[24]

where

\[u_{\theta}^* = u_{\theta}/v_B, \quad u_{r}^* = u_{r}/v_B\]

[25]

and the subscripts $\theta$ and $r$ represent the angular and radial velocity components of the respective velocities.

The system [24] is identical to the similar (dimensionless) system in Flint and Howarth (7) except for the interpretation of $G$ that has already been noted. The dimensional form of [24] is

\[
\begin{align*}
v_{\theta} &= u_{\theta} + v_B|G| \sin \theta \\
v_{r} &= u_{r} - v_B|G| \cos \theta
\end{align*}
\]  

[26]

so that the radial and tangential components of the particle velocity field $v_p$ are computable once the radial and tangential components of the fluid velocity field have been specified; in [26], $v_BG = v_{ps} \equiv \lambda \tilde{v}_{ps}$, the (dimensional) particle settling velocity.

If $\Psi^*$ is the dimensionless particle trajectory stream function (see, e.g., Batchelor (25), §2.2) then

\[
\begin{align*}
v_{\theta}^* &= \frac{1}{r^* \sin \theta} \frac{\partial \Psi^*}{\partial r^*} \\
v_{r}^* &= -\frac{1}{r^* \sin \theta} \frac{\partial \Psi^*}{\partial \theta}
\end{align*}
\]  

[27]

and the dimensional form of the particle trajectory stream function is obtained from

\[\Psi = v_B R_B^2 \Psi^*\]

[28]
If \( \mathbf{u}^{\text{int}} \) is the fluid velocity field which corresponds to the intermediate flow of Yoon and Luttrell (1) then

\[
\begin{align*}
\begin{cases}
\frac{1}{v_B R_B^2} \frac{\partial \Psi^{\text{int}}}{\partial r} = u^\theta_{\text{int}} + v_B |G| \sin \theta \\
-\frac{1}{v_B R_B^2} \frac{1}{r^2 \sin \theta} \frac{\partial \Psi^{\text{int}}}{\partial \theta} = u^r_{\text{int}} - v_B |G| \cos \theta
\end{cases}
\end{align*}
\]  

[29]

However,

\[
\begin{align*}
\begin{cases}
 u^\theta_{\text{int}} = v_B R_B^2 \frac{1}{r \sin \theta} \frac{\partial \psi^{\text{int}}}{\partial r} \\
 u^r_{\text{int}} = -v_B R_B^2 \frac{1}{r^2 \sin \theta} \frac{\partial \psi^{\text{int}}}{\partial \theta}
\end{cases}
\end{align*}
\]  

[30]

where \( \psi^{\text{int}} \) is the dimensionless form of [22].

By combining [29] and [30] we easily obtain the system

\[
\begin{align*}
\begin{cases}
\frac{\partial \Psi^{\text{int}}}{\partial r} = \frac{\partial \psi^{\text{int}}}{\partial r} + \frac{|G|}{R^2_B} r \sin \theta \\
\frac{\partial \Psi^{\text{int}}}{\partial \theta} = \frac{\partial \psi^{\text{int}}}{\partial \theta} + \frac{1}{2} \frac{|G|}{R^2_B} r^2 \sin 2 \theta
\end{cases}
\end{align*}
\]  

[31]

Partially integrating these two equations and solving for the unknown constants yields

\[
\Psi^{\text{int}} = \psi^{\text{int}} + \frac{1}{2} \frac{|G|}{R^2_B} r^2 \sin^2 \theta
\]  

[32]

Using the appropriate expression for \( \Psi^{\text{int}} \) and rearranging, we obtain for the dimensional particle trajectory stream function associated with the intermediate flow of Yoon and Luttrell (1)

\[
\Psi^{\text{int}} = v_B R_B^2 \sin^2 \theta \left\{ \left[ \frac{1}{2}(1 + |G|) \right] \frac{r^2}{R^2_B} - \frac{3}{4} \frac{r}{R_B} + \frac{1}{4} \frac{R_B}{r} \right\} \\
+ Re_B^* \left[ \frac{R_B^2}{r^2} - \frac{R_B}{r} + \frac{r}{R_B} - 1 \right]
\]  

[33]

We observe that [33] reduces to the result cited in Flint and Howarth (7), for the case of Stokes' flow, with \( St = 0, G \neq 0 \), when \( Re_B^* = 0 \); however, \( Re_B^* = 0 \Leftrightarrow Re_B = 0 \) is precisely the condition under which the intermediate flow of Yoon and Luttrell (1) reduces to Stokes flow around the bubble.
We now employ $\Psi^{\text{int}}$, as given by [33], to compute $P_c$ for an intermediate flow around the bubble when $St = 0$ and $G \neq 0$. The grazing trajectory generated by the particle path stream function in [33] satisfies

$$\Psi^{\text{int}}(r, \theta) = \text{const.} \equiv \Psi^{\text{int}}\left( R_p + R_B, \frac{\pi}{2} \right)$$

Employing [34] in [33], assuming that $r \geq r_c$, so that $\sin \theta = R_c/r$, and then letting $r \to \infty$, we find that

$$R_c^2 = \frac{1}{1 + |G|} \left\{ \left[ (R_p + R_B)^2 (1 + |G|) - \frac{3}{2} R_B(R_p + R_B) + \frac{1}{2} \frac{R_B^3}{R_p + R_B} \right] \right.$$  

$$+ 2Re^*_B \left[ \frac{R_B^3}{(R_p + R_B)^2} - \frac{R_B^3}{R_p + R_B} + R_B(R_p + R_B) - R_B^2 \right] \right\}$$

Therefore, after some simplification, as an exact expression for $P_c$ in this case we obtain (recall that $P_c = R_c^2/(R_p + R_B)^2$):

$$P_c^{\text{int}} = \frac{1}{1 + |G|} \left[ \frac{1}{2(R_p + R_B)^3} \left\{ 2R_p^3 + 3R_p^2 R_B \right\} \right.$$  

$$+ \frac{2Re^*_B}{(R_p + R_B)^4} \left\{ R_B R_p^3 + 2R_B^2 R_p^2 \right\} \right] + \frac{|G|}{1 + |G|}$$

For intermediate flow, in the sense of Yoon and Luttrell (1), with the particle settling velocity assumed to be negligible, we may set $G = 0$ in [36] so as to obtain

$$\hat{P}_c^{\text{int}} = \frac{1}{2(R_p + R_B)^3} \left\{ 2R_p^3 + 3R_p^2 R_B \right\}$$  

$$+ \frac{2Re^*_B}{(R_p + R_B)^4} \left\{ R_B R_p^3 + 2R_B^2 R_p^2 \right\}$$

On the other hand, setting $Re^*_B = 0 \Leftrightarrow Re_B = 0$ in [36] yields the exact collision probability for Stokes' flow around the bubble with $St = 0, G \neq 0$, i.e.

$$P_c^{\text{st}} = \frac{1}{1 + |G|} \left[ \frac{1}{2(R_p + R_B)^3} \left\{ 2R_p^3 + 3R_p^2 R_B \right\} \right.$$  

$$+ \frac{|G|}{1 + |G|}$$
Finally, for Stokes' flow around the bubble with $St = 0$ and $G = 0$, [38] yields

$$
\dot{P}_{c}^{st} = \frac{1}{2(R_p + R_B)^3} \{2R_p^3 + 3R_p^2R_B\}
$$

[39]

The expressions [36] and [37] for intermediate flow and [38] and [39] for Stokes flow around the bubble are *exact relations* which depend only on the hypotheses that $St \approx 0$ and that the fluid flow streamlines are symmetric fore and aft of the bubble so that the collision angle $\theta_c = \pi/2$ (in both [37] and [39] we also assume that $G \approx 0$). To the best of the authors’ knowledge, the *exact* expressions in [36]-[39] have not appeared previously in the literature. What has appeared in the literature are approximate relations for $P_{c}^{int}$, $\dot{P}_{c}^{int}$, $P_{c}^{st}$, and $\dot{P}_{c}^{st}$ which depend on certain additional assumptions concerning the magnitudes $R_p$ and $R_B$ that have never been clearly delineated in the literature; these are summarized below.

From [36] and [38] we obtain the so-called limiting ‘efficiencies’

$$
E_{c}^{int} = \lim_{R_p \rightarrow 0} P_{c}^{int} = \frac{|G|}{1 + |G|}
$$

[40]

and

$$
E_{c}^{st} = \lim_{R_p \rightarrow 0} P_{c}^{st} = \frac{|G|}{1 + |G|}
$$

[41]

The result in [41] has appeared in Flint and Howarth (7). The result in [41] also follows from an approximate relation for $P_{c}^{int}$ which is listed in Table 1 of Nguyen-Van (9).

The most familiar approximate relation in the literature for the probability of collision is the one for Stokes flow, $\dot{P}_{c}^{st}$, which we indicate below. Actually the oldest form of approximate relation is that for potential flow $\dot{P}_{c}^{pot}$, $\dot{P}_{c}^{pot} = 3 \left( \frac{R_p}{R_B} \right)$, which was first given by Sutherland (2), where ‘pot’ denotes potential flow around the bubble and * indicates that $G \approx 0$, as well as $St \approx 0$.

To initiate the delineation of the various approximate results we assume in [39] that

$$
R_p + R_B \simeq R_B
$$

[42]
and

\[
\left( \frac{R_p}{R_B} \right)^3 \ll \left( \frac{R_p}{R_B} \right)^2
\]

[43]

In this case, we obtain from [39]

\[
\hat{P}_{c}^{st} \simeq \left( \frac{3}{2} \right) \frac{R_p^2}{R_B^2}
\]

[44]

a well-known result that has been often cited, e.g. Schulze (6), but never clearly identified as an approximate relationship; the same degree of approximation as that indicted in both [42] and [43] allows one to conclude, as a consequence of [38], that

\[
P_{c}^{st} \simeq \frac{3}{2(1 + |G|)} \left( \frac{R_p^2}{R_B^2} \right) + \frac{|G|}{1 + |G|}
\]

[45]

The result in [45] appears as the first entry of Table 1 in Nguyen-Van (9) but a derivation of this (albeit) approximate result does not appear in the reference cited there, i.e., in Gaudin (26). Turning to the exact expression for \(P_{c}^{int}\), i.e., [36] we now assume the validity of both [42] and [43] and, thus, deduce that

\[
P_{c}^{int} \simeq \frac{1}{1 + |G|} \left[ \left( \frac{3}{2} + 4Re^* \right) \frac{R_p^2}{R_B^2} \right] + \frac{|G|}{1 + |G|}
\]

[46]

The (approximate) result in [46] appears as the fourth entry in Table 1 of Nguyen-Van (9) but does not appear in the references cited there, e.g., in Yoon and Luttrell (1); what does appear in (1), albeit without a derivation, is the approximate result for \(\hat{P}_{c}^{int}\) which follows either from [46] by setting \(G = 0\), i.e.,

\[
\hat{P}_{c}^{int} = \left( \frac{3}{2} + 4Re^* \right) \frac{R_p^2}{R_B^2}
\]

[47]

or from the exact result [37] by employing the assumptions [42] and [43].
Model Validation

Direct experimental observations of the collision process are very complicated because it is difficult to isolate this microprocess from the other microprocesses in actual flotation separation. However, attempts to experimentally record $P_c$ have been made by a few researchers addressing mineral flotation (1, 3, 8, 9, 14, 15), and these data have been compared to the model presented above.

For these comparisons, considerable effort has been made to match the experimental conditions as closely as possible. Specific parameters of importance are the bubble rise velocity and the particle and fluid thermophysical properties. It was assumed that all experiments were performed in a fluid with properties corresponding to those of water. In all cases, the particular particles used in the experiments were identified by name, but when the density was not provided, a value was chosen based on available tabulated data. The most difficult parameter to match was the bubble rise velocity because this parameter was not always provided for each experimental condition.

Predictions were first compared to experimental data presented by Anfruns and Kitchener (14, 15). They experimentally studied the probability of collision as a single bubble rose through a dilute suspension of quartz particles with a measured size distribution. Five size fractions of quartz were used with mean diameters of 12.0, 18.0, 24.6, 31.4, and 40.5 $\mu m$. These particle diameters and a quartz density of 2.65 $g/cm^3$, obtained from Nguyen-Van and Kmet (8), were used in our predictions. The bubble rise velocity was obtained from data presented in (15) in which experimental results for $v_B$ were presented in terms of bubble diameter.

Figure 2 displays $P_c$ predictions made with [36] incorporating the above experimental
information, and compares the predictions to the experimental data presented by Anfruns and Kitchener (14, 15). In all cases, the $P_c$ calculations overpredict the experimental data. This is probably due to the experiments not sufficiently isolating $P_c$. In fact, Anfruns and Kitchener actually plot data as "Efficiency of Collection ($E_c$)," which implies that the experimental data may also include adhesion by sliding and stability effects. Since the overall collection efficiency is assumed to be the product of each flotation microprocess efficiency (6), the fact that the $P_c$ predictions overestimate the experimental data is not surprising. This discrepancy was also highlighted by Nguyen-Van and Kmet (8) in which they state: "In our opinion, the experimental results done by these authors [Anfruns and Kitchener] refer rather to [a] collection efficiency than to [a] collision one."

Yoon and Luttrell (1, 3) also present mineral particle flotation data to which $P_c$ model predictions are compared. The experimental set up in these experiments was similar to Anfruns and Kitchener (14, 15); however, they utilized very hydrophobic Buller seam coal particles with 0.13% ash content and mean diameters of 11.4, 31.0, and 40.1 $\mu$m in their experiments. According to Yoon and Luttrell (1), the probability of collection they recorded should closely match $P_c$ since the probability of adhesion by sliding for very hydrophobic particles should approach unity. In these comparisons, the particle density was specified to be 1.3 $g/cm^3$ and the bubble rise velocity was determined from a curve-fit to original bubble rise velocity data of Yoon and Luttrell\(^1\). Therefore, the bubble rise velocity was calculated from

$$v_B = 10.64(d_{B}^{13})$$  \[48\]

where $d_B$ and $v_B$ have units of $mm$ and $cm/s$, respectively.

\(^1\)Values provided by a reviewer.
Figure 3 compares our $P_c$ predictions to the results of Yoon and Luttrell (1, 3). Their predictions (i.e., [47]) are also included in Fig. 3. Our predictions do very well at predicting $P_c$, particularly for the smaller particle diameters of 11.4 and 31.0 $\mu$m. At $d_p = 40.1 \mu m$, [36] underpredicts the data slightly. However, the general trends are followed closely for all particle diameters considered. Our predictions do not differ significantly from the predictions of Yoon and Luttrell because the experimental conditions used to generate Fig. 3 include $R_p << R_B$ and $|G|$ on the order of 0.01 (or less), which satisfy the restrictions on [47].

Nguyen-Van (9) also presented $P_c$ experimental data for two different particle types; quartz ($\rho_p = 2.65$ g/cm$^3$, $R_p = 7.75 \mu m$) and galena ($\rho_p = 7.5$ g/cm$^3$, $R_p = 6.25 \mu m$). Property data were obtained from Nguyen-Van and Kmet (8). These experiments involved a fixed bubble held in place on a capillary tube with fluid flowing past the bubble. A dilute particle suspension was injected above the bubble from a second (movable) capillary tube and was entrained in the moving fluid. Particle collisions with the fixed bubble were visually observed. This method allowed for $R_c$ (see Fig. 1) to be experimentally determined. Since the bubble was fixed in these experiments, the bubble rise velocity was equivalent to the fluid velocity flowing past the bubble. The bubble rise velocity was obtained indirectly through $Re_B$ from the following relationship

$$R_B = \left[ \frac{9 \mu_l^2 Re_B (1 + 0.15 Re_B^{0.687})}{4 \rho_l g} \right]^{1/3}$$

[49]

which was presented in (8) and claimed to agree with experimental data.

Figure 4 presents the quartz data from Nguyen-Van (9). Nguyen-Van also developed a prediction for $P_c$, and this is also shown in the figure. This prediction includes the possibility that the maximum collision angle may be less than 90° from the stagnation point on the
bubble. This prediction has the form

\[ P_c = \frac{1}{1 + |G|} \left( \frac{R_p}{R_B} \right)^2 \left[ \frac{\sqrt{(X + C)^2 + 3Y^2} - (X + C)}{13.5Y^2} \right] \left[ \sqrt{(X + C)^2 + 3Y^2 + 2(X + C)} \right] \]

where

\[ C = \frac{2R_B^2 \Delta \rho g}{9\mu \varphi v_B} \]

\[ X = 1.5 \left[ 1 + \frac{3Re_B/16}{1 + 0.309Re_B^{0.694}} \right] \]

\[ Y = \frac{3Re_B/8}{1 + 0.217Re_B^{0.518}} \]

As one can easily see that this \( P_c \) prediction is rather complicated, but the Nguyen-Van \( P_c \) prediction follows the experimental data very closely. Our \( P_c \) prediction [36] has a much simpler form and also does a good job of following the data. The largest discrepancy is at the largest \( R_B \) values, but this is still within \( \sim 25\% \) of the experimental data. The deviation between our prediction and the experimental data may be due to the collision angle having an effect at these conditions. The inclusion of assumptions [42] and [43] yields [46], which is also shown in Fig. 4. This result does not significantly differ from that of [36] because the experimental conditions satisfy the assumptions incorporated into this approximation.

The predictions of Yoon and Luttrell (1) [47] are also shown in Fig. 4 and underpredict the experimental results, indicating that \( |G| \) has a significant effect for these experimental conditions.

Figure 5 reveals the same type of comparisons, but for the galena data of Nguyen-Van (9). Galena has a much larger density than that of quartz, so particle settling velocity is much more significant. This is evident by the fact that the Yoon and Luttrell (1) predictions significantly underpredict the experimental data. Our current \( P_c \) predictions [36] and [46] (with the associated assumptions) do a very good job of predicting the experimentally
determined $P_c$ values. The more complicated $P_c$ prediction of Nguyen-Van (9) also does a very good job.

In summary, our model for the probability of collision does a very good job of predicting available experimental results for $P_c$. The model is less complicated than that proposed by Nguyen-Van (9), but just as accurate, and is much improved over the model of Yoon and Luttrell (1).

**Parametric Variations**

The exact intermediate flow solution for $P_c$ [36] can be rearranged to show that three dimensionless groups of $R_p/R_B$, $Re_B$, and $|G|$ are the only parameters that influence the probability of collision. If $|G| \approx 0$, the particle settling velocity does not affect $P_c$. This result, and the approximate result of Yoon and Luttrell (1) [47], are shown in Fig. 6 for $Re_B \leq 500$. When $Re_B = 0$, Stokes flow conditions prevail (i.e., [39]), and a minimum $P_c$ results for all $R_p/R_B < 1$, with $P_c$ increasing as $R_p/R_B$ increases. Increasing $Re_B$ for a fixed $R_p/R_B$ increases $P_c$ and these values run parallel to those predicted by Stokes flow. The applicability of these results at $Re_B = 500$ is questionable because the stream function for intermediate flow was developed for $0 \leq Re_B \leq 100$ (1); however, Yoon and Luttrell (1) state that it “may be applicable for $Re_B > 100$, although no experimental (streamline) date [were] available in the present work.” In Fig. 6, unrealistic predictions from [36] ($P_c > 1$) result when $R_p/R_B \gtrsim 0.3$ and $Re_B = 500$. This result will be further discussed below. The exact and approximate solutions follow closely to one another for small values of $R_p/R_B$, and at $R_p/R_B = 0.1$, the approximate solution presented by Yoon and Luttrell (1) over predicts $P_c$ by approximately 25% when $Re_B = 0$ and by more than 35% when $Re_B = 500$. As expected, increasing $R_p/R_B$ further toward 1 increases this difference because the approximations of
Yoon and Luttrell (i.e., [42] and [43] with \(|G| = 0\)) are no longer valid. Applying these approximations when \(R_p/R_B \to 1\) results in [47] predicting \(P_c > 1\) for all \(Re_B\).

Figure 7 reveals the exact predictions for \(P_c\) [36] for \(0 \leq Re_B \leq 500\) and \(|G| = 0.1\). The exact prediction for Stokes flow [38] corresponds to \(Re_B = 0\). The bubble Reynolds number has a negligible effect on \(P_c\) when \(R_p/R_B \lesssim 0.03\), and a constant \(P_c\) results, which is a function of \(|G|\) (as shown in Fig. 8). When \(R_p/R_B \gtrsim 0.03\), \(P_c\) increases exponentially with increasing \(R_p/R_B\). Additionally, the increase in \(P_c\) is more abrupt as \(Re_B\) increases. When \(R_p/R_B = 1\), \(P_c \leq 1\) for \(Re_B \leq 100\). As previously stated, these predictions are questionable when \(Re_B > 100\) because the stream function used to generate [36] includes data only up to \(Re_B = 100\) [1]. In our predictions, when \(Re_B = 500\) and \(R_p/R_B \gtrsim 0.3\), \(P_c > 1\), but \(P_c\) is independent of \(Re_B\) when \(R_p/R_B \lesssim 0.03\) and [36] can be used outside its given \(Re_B\) range under these specific conditions.

Similar calculations to those presented above have been completed using [36] for fixed \(Re_B\) over the given \(R_p/R_B\) range and for selected values of \(|G|\). Figure 8 reveals one such plot for \(Re_B = 10\). The approximate solution [46], which incorporates assumptions [42] and [43], is also shown in Fig. 8. When \(|G| = 0\), the approximate solution corresponds to the predictions of Yoon and Luttrell [1]. When \(R_p/R_B\) is small, \(P_c\) increases with increasing \(|G|\) by several orders of magnitude when compared to the \(|G| = 0\) predictions, implying the particle settling velocity significantly enhances the collision probability when collision occurs between a particle that is much smaller than the colliding bubble. This would be particularly true for particles with a density much larger than that of water. The increase in \(P_c\) with increasing \(|G|\) is much smaller when a particle and bubble size are the same order of magnitude (with \(R_p < R_B\)), and as \(R_p/R_B \to 1\), \(P_c\) predictions approach the same value independent of \(|G|\). In Fig. 8, all \(P_c\) predictions are less than 1 for \(R_p/R_B \leq 1\),
except the approximate solution [46] when $R_p/R_B \to 1$ because the assumptions [42] and [43] are no longer valid. The approximate $P_c$ predictions [46] and the exact $P_c$ predictions [36] are equivalent for small $R_p/R_B$. When $R_p/R_B \approx 0.03$, 0.05, and 0.2, the approximate solution begins to deviate from the exact solution when $|G| = 0$, 0.01, and 1, respectively, and $Re_B = 10$. Also, as $R_p/R_B$ increases, the approximate solution asymptotes to the Yoon and Luttrell (1) solution [47], which does not include the effects of particle settling velocity. This figure reveals that particle settling velocity is important at small values of $R_p/R_B$ and assumptions [42] and [43] are not. Conversely, as $R_p/R_B \to 1$, assumptions [42] and [43] dominate and the inclusion of $|G|$ has only a secondary effect.

Additional calculations have been performed for fixed values of $R_p/R_B$ while both $|G|$ and $Re_B$ are varied. These predictions result in contour plots of $P_c$ for each fixed value of $R_p/R_B$. When $R_p/R_B = 0.1$ (Fig. 9), the contour lines are plotted with logarithmic increments, showing that $P_c$ varies by almost two orders of magnitude for the given conditions. For $R_p/R_B = 0.1$, $P_c$ is a strong function of $|G|$ for all values of $Re_B$. In contrast, $P_c$ is a function of $Re_B$ when $|G| < 0.2$ and is independent of $Re_B$ when $|G| \geq 0.2$.

At $R_p/R_B = 0.9$ (Fig. 10), $P_c$ contours are now plotted on a linear scale with major divisions (solid lines) corresponding to $P_c$ values in increments of 0.1 and minor divisions (dashed lines) representing $P_c$ values in increments of 0.05. Under these conditions, $P_c$ is independent of $Re_B$ only for small $Re_B$ and large $|G|$. Conversely, $P_c$ is independent of $|G|$ when $|G|$ is small and $Re_B$ is large. Therefore, as discussed earlier, when $R_p/R_B$ is large, particle settling velocity only plays a minor role and only when $Re_B$ is small and $|G|$ is large.

Summary

An exact solution for the probability of collision, $P_c$, has been developed based on the
intermediate flow field of Yoon and Luttrell (1). This solution is a function of three dimensionless parameters including the magnitude of the dimensionless particle settling velocity, $|G|$, the bubble Reynolds number, $Re_B$, and the ratio of particle to bubble radius, $R_p/R_B$. The resulting expression [36] only assumes that the bubble and particle are spherical and that $R_p < R_B$ (the restriction that $R_p << R_B$ is not required).

The new prediction for $P_c$ presented here does a good job of predicting available experimental data. The inclusion of the particle settling velocity is very important, particularly when the particles have a density much higher than that of water. Additionally, the form of $P_c$ derived here is much simpler than that proposed by Nguyen-Van (9), and just as accurate at predicting experimental results.

Selected $P_c$ predictions have also been presented using [36] for $0 \leq Re_B \leq 500$, $0 \leq |G| \leq 1$, and $0.001 \leq R_p/R_B < 1$. In general, $P_c$ is independent of $Re_B$ when $R_p/R_B \lesssim 0.03$, the particle settling velocity is important for small values of $R_p/R_B$, and $R_p/R_B$ dominates as $R_p/R_B \to 1$.

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REFERENCES


Figure Captions

Figure 1: Particle colliding with a bubble at $\theta_c = \pi/2$.

Figure 2: Experimental data for $P_c$ obtained by Anfruns and Kitchener (14, 15) and the associated numerical predictions from the $P_c$ model [36].

Figure 3: Comparison between the experimental $P_c$ data obtained from Yoon and Luttrell (1) and numerical predictions for $P_c$.

Figure 4: Comparison between experimental data obtained from Nguyen-Van (9) for quartz particles and numerical predictions for $P_c$.

Figure 5: Comparison between experimental data obtained from Nguyen-Van (9) for galena particles and numerical predictions for $P_c$.

Figure 6: Exact and approximate $P_c$ predictions with $0 \leq Re_B \leq 500$ and $|G| = 0$.

Figure 7: Exact $P_c$ predictions for $0 \leq Re_B \leq 500$ and $|G| = 0.1$.

Figure 8: Exact and approximate $P_c$ predictions for $|G| = 0, 0.01, \text{and } 1$ and $Re_B = 10$.

Figure 9: Contours of $P_c$ when $R_p/R_B = 0.1$. Note the $P_c$ scale is logarithmic.

Figure 10: Contours of $P_c$ when $R_p/R_B = 0.9$. Note the $P_c$ scale is linear.
Figure 1
Data from Anfruns and Kitchener (14, 15)

Quartz particles

\( \rho_p = 2.65 \text{ g/cm}^3 \)

--- Eq. [36]

- \( d_p = 40.5 \mu m \)
- \( d_p = 31.4 \mu m \)
- \( d_p = 24.6 \mu m \)
- \( d_p = 18.0 \mu m \)
- \( d_p = 12.6 \mu m \)

Figure 2
Data from Yoon and Luttrell (1)

Mean Particle Diameter

- □ 40.1 μm
- ○ 31.0 μm
- △ 11.4 μm

Figure 3
Quartz particles

\( R_p = 7.75 \ \mu m \)
\( \rho_p = 2.65 \ \text{g/cm}^3 \)

\[ P_c \]

Figure 4
Figure 5
Figure 6
Figure 7

$|G| = 0.1$

- $Re_B = 500$
- $Re_B = 100$
- $Re_B = 50$
- $Re_B = 10$
- $Re_B = 5$
- $Re_B = 1$
- $Re_B = 0$

$P_c$

$R_p/R_B$
Figure 8
Figure 10