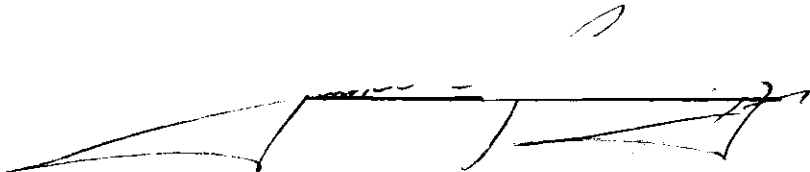


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A handwritten signature in black ink, consisting of a series of connected loops and lines, positioned below the text.

AN INVESTIGATION OF DETERMINISTIC LANCHESTER-TYPE
EQUATIONS OF WARFARE

A THESIS

Presented to

The Faculty of the Graduate Division

by

James C. Robinson, Jr.

In Partial Fulfillment

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AN INVESTIGATION OF DETERMINISTIC LANCHESTER-TYPE
EQUATIONS OF WARFARE

Approved:

1 1 2 0

Chairman

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CHAPTER I

INTRODUCTION

The Problem and Its Importance

In a battle between two opposing military forces, it is often desired to predict the "winner" of the battle, the losses sustained by each side, and the time length of the battle. With such predictions, a commander can either avoid a battle opportunity, or use the predictions in planning both tactics and logistics. With no predictions or estimates, the commander has no basis on which to plan tactical operations or logistical support of those operations.

Lanchester Equations

The most common mathematical models used for such predictions were first proposed by F. W. Lanchester around 1916 and are known as the Lanchester Equations of Warfare (1). These equations are simultaneous differential equations which describe the loss rates of the opposing sides over time.

Linear Law

This model applies to the situation where fire is placed on an area known to be occupied by the enemy, but without aiming at a specific enemy. The equations are:

$$\dot{x} = - Axy \quad (1)$$

and

$$\dot{y} = - Bxy \quad (2)$$

where A and B are constants, and x and y are the sizes of the opposing forces at any time t.

The linear law also obtains in the case of a duel between two individuals. The model then takes the form:

$$\dot{x} = -A \quad (3)$$

and

$$\dot{y} = -B \quad , \quad (4)$$

where A and B are constants. The solution of both these models, with time eliminated, is as follows:

$$A(y_0 - y) = B(x_0 - x) \quad (5)$$

This solution indicates that there is no particular advantage in concentrating forces. That is, the effect of force size is linear.

Square Law

The square law describes the case in which the combatants are exposed to each other and execute aimed fire at a given rate with a given single-shot hit probability. It is assumed that all combatants can fire upon any member of the opposition. The model is:

$$\dot{x} = -ay \quad (6)$$

and

$$\dot{y} = -bx \quad (7)$$

The steady state solution of this model is:

$$a(y_0^2 - y^2) = b(x_0^2 - x^2) \quad (8)$$

Under these conditions, the fighting strengths of the two forces are proportional to the square of the numerical strengths, which means that the effect of numerical advantage far outweighs that of individual fighting efficiency.

Research Objectives

The purpose of this research is to summarize the previous work in the field, and to extend the theory to a typical ground combat situation.

Two forms of the Lanchester type equations will be solved for several values of the parameters using electronic analog computer techniques. A third model, describing the assault of a defended position, will be formulated and solved by analog computer also. This model will incorporate aspects of both the other models considered.

The general objective is to solve each of the models considered using analog computer techniques.

The specific objective is to develop a model describing an attack upon a defended position which is more realistic than any previously proposed deterministic Lanchester type model.

The criteria for this study are that the solutions obtained and the model proposed have validity in the light of previous work in the field, and that the analog circuits designed produce reliable results.

Scope and Limitations

This study will be limited to the deterministic or "most likely" case. It is realized that the events which occur in warfare are probabilistic in nature, but combat is normally on such a scale that mean values are adequate for planning purposes. It will be assumed that the sizes of the forces are continuous rather than discrete in nature, and neither reinforcement nor noncombat operational losses will be considered.

CHAPTER II

LITERATURE SEARCH

Original Formulation

The work of F. W. Lanchester in the formulation and solution of the linear and square laws has already been cited. There is no doubt as to the originality of his work, but there is also evidence that others had worked out the same principles without the use of higher mathematics. A numerical demonstration of the square law by Rear Admiral Bradley A. Fiske, United States Navy, was published ten years prior to Lanchester's work (2). Fiske made the generalization that the decrease of a weaker fleet is geometrical rather than arithmetical.

Recent Developments

Developments since Lanchester's time have been in two main areas. These areas are: the strictly deterministic case such as described by the original equations; and analysis of the problem as a stochastic or probabilistic process.

Deterministic Case

The deterministic case can be further classified into those cases in which the forces of a side are completely homogeneous and those in which the forces are heterogeneous.

Homogeneous Forces. The most important assumption in this case is that each side is completely homogeneous within itself with regard to type of weapon, rate of fire, etc. This is, of course, a severely limiting assumption, but it facilitates analysis to a great degree.

One of the most comprehensive studies of homogeneous type models was made by Brackney (3). He proposed that a force is limited to one of three distinctive types of action. It must assault, defend with constant force density, or defend over a constant area. Each mode of action gives rise to a different loss rate. All possible combinations of these actions by the two opposing forces give nine different combat situations. Brackney developed steady state solutions of these nine situations for the survival fraction (x/x_0) , and the casualty multiple $((x_0 - x)/y_0)$. He recommends the following set of rules for the commander:

1. Assault under conditions which assure a superiority of force at the point or place chosen for action.
2. Assault, if possible, with sufficient speed and momentum to force the defender to retrench his position, else suffer a breakthrough.
3. Ideally, assault with such surprise, speed, and strength as to create at once a strong defensive position so that the enemy subsequently must either capitulate or himself attack at great disadvantage.

These rules were deduced from a comparison of the survival fractions and casualty multiples which result in each of the nine combat situations, assuming a value for the parameters.

The validity of Lanchester's equations was demonstrated by Engle (4) in an analysis of the capture of the island of Iwo Jima. He used the following model:

$$\dot{x} = R(t) - Ay \quad (\text{American}) \quad (9)$$

and

$$\dot{y} = -Bx \quad , \quad (\text{Japanese}) \quad (10)$$

where $R(t)$ is a reinforcement rate. Engle found a value of 0.0544 for A (the number of American casualties per day per effective enemy troop) and a value of 0.0106 for B (the number of enemy casualties per day per effective American troop). The value 5.1 for the ratio A/B seems high, but is not surprising, considering the fact that the Japanese held extremely strong defensive positions and were highly motivated. Engle also found that by utilizing the parameter values given above, the casualty rates predicted by equations 9 and 10 corresponded closely to known values.

An analysis of guerilla ambush tactics was made by Deitchman (5) using the model:

$$\dot{x} = - Axy \quad (\text{Ambusher}) \quad (11)$$

and

$$\dot{y} = - bx \quad (\text{Ambushed}) \quad (12)$$

In this situation, the "x" force ambushes the "y" force, starting to shoot while "y" is in full view. The "x" force is concealed and "y" must fire blindly into the area where "x" is suspected to be. The attrition rate of the ambusher is then proportional to both the density of the ambusher and the number of ambushed firing into the area occupied by the ambusher. Historical data is presented which shows that a high superiority of regular troops is required to assure victory against a guerrilla force. Of wars against guerrillas won by the regular troops, the regulars had a mean force superiority of 10 to 1. Of those lost to the guerrilla forces, the regulars had an advantage of 4 to 1.

Bach et al. (6) considered the problem of determining the force size the winner should commit in order to minimize total losses. The model used was as follows:

$$\dot{x} = -ay - cx \quad (13)$$

and

$$\dot{y} = -bx - dy \quad (14)$$

where the terms cx and dy are noncombat operational losses which are proportional to the force size involved. This model was solved by use of a lumped-parameter analog circuit.

Clark et al. (7) solved the square law for several values of the parameters using a standard electronic analog computer. This work is interesting in that the force sizes and the time scales are "normalized" to enable the solution graphs to apply to any force size. This is possible with the square law because the differential equations are linear in the variables x and y .

Heterogeneous Forces. It is rare in modern combat to find a completely homogeneous unit. Even in infantry squads there are semiautomatic rifles, automatic rifles, and grenade launchers, all having very different casualty-producing effects. In most cases of mutually supporting weapons systems, the total effect is nonlinear. That is, the total effect is not simply the sum of the effects of the weapons systems separately, but some nonlinear function thereof. A much used example is the tank-infantry team, the effect of which is greater than the sum of effects of tanks and infantry separately.

Most of the work done in the heterogeneous case assumes linearity of the effects, however. This is done by Weiss (8) in his consideration of a conflict which involves both "air" (aircraft and artillery) and "ground" (infantry and tanks). Another important condition in this analysis is that the air forces can attack both the air and ground forces of the enemy, but the ground forces are limited to attacking only the ground opposition. This problem was solved for optimum tactics, to include cost considerations.

Other special heterogeneous cases have been studied by Helmer (9), Snow (10), and Morse and Kimball (11).

Stochastic Models

Since chance plays such a large part in war, the outcome of a battle and the number of survivors may be viewed as random phenomena subject to the laws of probability.

The main work in this area has been done by Snow (10), Morse and Kimball (11), and Brown (12). Because of the mathematical complications in stochastic analysis, however, most workers in the field hold that the added realism is not worth the extra effort. Also it has been shown that the deterministic equations give solutions that agree quite well with the probabilistic solutions for expected values. For these reasons no further consideration will be given to stochastic models.

A Dissenting Opinion

Of the published work examined, only one analysis detracts substantially from the value of Lanchester's equations as predictive devices. Williard (13) made an analysis of land battles which occurred between 1618 and 1905 and concluded in part:

2. In general, force ratios had little to do with determining the outcome of the battles studied.
3. Lanchester's square law is the poorest among poor alternative choices of deterministic laws ...

These conclusions were based on an analysis of data of questionable validity. All the data was obtained from Militar-Historisches Kriegs-Lexicon by Bodart, written in 1908.

Discussion

Concerning the conclusions of Willard (13) and the validity of the data used, Von Clausewitz (14) wrote:

In order to show that for a long time the strength of the military forces was by no means regarded as a vital matter, we need only observe that in most and even in the most detailed histories of the wars of the eighteenth century, the strength of the armies is either not given at all, or only incidentally, and in no case is any special value set upon it....

Even Massenbach, in his manifold critical observations on the Prussian campaigns of 1793-1794 in the Vosges, talks a great deal about hills and valleys, roads and footpaths, but never says a syllable about the strength of either side....

Lastly, there are a number of instances to be found in which all the available forces were not really used in the battle, or in the war, because superiority in numbers was not considered to have that importance which in the nature of things belongs to it.

Thus it appears that many of the battles used by Willard in his analysis were fought under a set of false values.

The analog solution of the square law by Clark et al. (7) demonstrates good technique, but does not cover a very wide range of parameter values. This model will be solved over a wider range of values using a slightly different analog circuit.

Equations 11 and 12 describe not only the ambush situation, as previously stated, but also an attack on a concealed defender by a

force in the open. This model will be solved by electronic analog computer techniques.

Most researchers have attempted to apply one set of equations to a combat situation which changes. Brackney (3) considers only three modes of combat action, when in reality an assault or defense may entail at least two distinctive types of actions by each side. The casualty rates while the assault force is outside the defensive position area are certainly different from those produced if the assault breeches the defenses. As Brackney points out, the assault is the decisive action in warfare and is consequently worthy of further study.

A model of the assault which combines Lanchester's square law and equations 11 and 12 will be formulated. This model will be solved for several parameter values by analog computer.

CHAPTER III

PROCEDURE AND RESULTS

EquipmentAnalog Computer

The electronic analog computer facilities of the Georgia Tech Engineering Experiment Station were utilized. The major equipment consisted of an EASE Computer, manufactured by the Berkeley Division of Beckman Instruments, Inc., and an x-y Recorder, manufactured by Electro Instruments, Inc.

The EASE Computer consists of a control unit, setup panels, power supplies, operational amplifiers, function generators, potentiometers, ± 100 volt reference supply, variable dc voltage sources, and several types of nonlinear components. The availability of these nonlinear components allows the solution of relationships which would be virtually unsolvable by other methods.

The output device, or x-y Recorder, consists of a pen mounted on a vertical bar. The pen is movable along the bar in the vertical plane and the bar is movable in the horizontal plane. The relative positions of the pen and bar are determined by the voltages applied to their respective input terminals. These voltages may represent any two quantities taken from the computing circuits. The desired quantities are plotted with respect to each other on 10 x 15 inch graph paper as the computer solution progresses.

Digital Computer

The digital computer facilities of Georgia Tech's Rich Electronic Computer Center were used. The major equipment consisted of a Burroughs 220 Digital Computer with ALGOL compiler. All programming was in the Burroughs 220 ALGOL language. The input was on punched cards and the output was printed.

Computation MethodLanchester's Square Law (Model I)

In order to provide a general, or normalized, numerical solution, the following transformations were made:

Let

$$t = T/a$$

$$X = x/x_0$$

$$Y = y/y_0$$

$$\dot{X} = dX/dT$$

$$\dot{Y} = dY/dT$$

$$R = x_0/y_0$$

$$F = b/a$$

Where x_0 = size of x at time $t = 0$, y_0 = size of y at time $t = 0$, and a and b are constant coefficients.

Equations 6 and 7 then become:

$$\dot{X} = - (1/R)Y \quad (15)$$

$$\dot{Y} = - (FR)X \quad (16)$$

Analog Computer Solution. In order to insure that the plotter needle would track properly, the solution was slowed down by letting $T = \tau/10$. Then;

$$\dot{\bar{X}} = - (1/10 \cdot R) \bar{Y} \quad (17)$$

$$\dot{\bar{Y}} = - (RF/10) \bar{X} \quad , \quad (18)$$

where \bar{X} , \bar{Y} , $\dot{\bar{X}}$, and $\dot{\bar{Y}}$ are related to the computer time variable τ .

A schematic diagram of the computer network is shown in Figure 1. The values of R and F used, along with the computer settings are shown in Table 1. The solutions obtained are shown as Figures 2 through 6.

Digital Computer Solution. Equations 15 and 16 were solved for X and Y using Laplace transforms, giving:

$$X = \text{COSH} (T \sqrt{F}) - (1/R \sqrt{F}) \text{SINH} (T \sqrt{F}) \quad (19)$$

$$Y = \text{COSH} (T \sqrt{F}) - (R \sqrt{F}) \text{SINH} (T \sqrt{F}) \quad . \quad (20)$$

Numerical solutions for X and Y, over time T, were obtained for several values of R and F by digital computer and plotted on Figures 2 to 6 for comparison with the analog solutions. The ALGOL program is shown in Table 2.

Assault on a Defended Position (Model II)

This model, describing both an assault on a defended position and an ambush, is presented by equations 11 and 12 below:

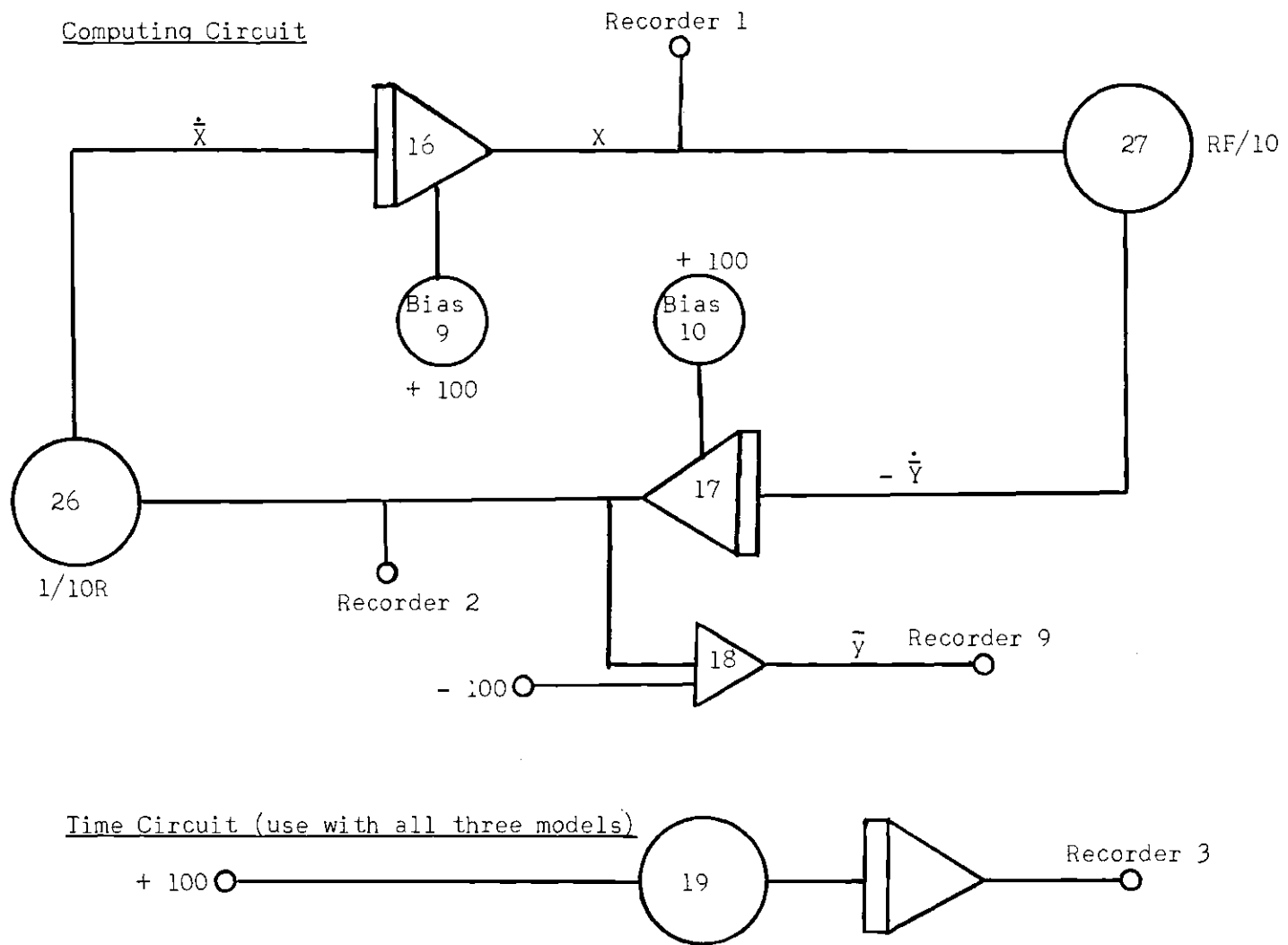


Figure 1. Analog Computer Schematic Diagram: Model I

Table 1. Analog Computer Settings: Model I

<u>Run</u>	<u>R</u>	<u>F</u>	<u>RF</u>	<u>Pot 26</u> <u>1/10R</u>	<u>Pot 27</u> <u>RF/10</u>	<u>Pot 19</u>	<u>Gain on</u> <u>Amplifier 16</u>
1	2.0	0.25	0.500	0.050	0.050	0.02	1
2	2.5	0.25	0.625	0.040	0.062	0.02	1
3	3.0	0.25	0.750	0.033	0.075	0.02	1
4	4.0	0.25	1.000	0.025	0.100	0.02	1
5	5.0	0.25	1.250	0.020	0.125	0.02	1
6	1.5	0.50	0.750	0.067	0.075	0.02	1
7	2.0	0.50	1.000	0.050	0.100	0.02	1
8	2.5	0.50	1.250	0.040	0.125	0.02	1
9	3.0	0.50	1.500	0.033	0.150	0.02	1
10	4.0	0.50	2.000	0.025	0.200	0.02	1
11	5.0	0.50	2.500	0.020	0.250	0.02	1
12	1.0	1.00	1.000	0.100	0.100	0.04	1
13	1.5	1.00	1.500	0.067	0.150	0.04	1
14	2.0	1.00	2.000	0.050	0.200	0.04	1
15	2.5	1.00	2.500	0.040	0.250	0.04	1
16	3.0	1.00	3.000	0.033	0.300	0.04	1
17	4.0	1.00	4.000	0.025	0.400	0.04	1
18	5.0	1.00	5.000	0.020	0.500	0.04	1
19	.75	2.00	1.500	0.133	0.150	0.04	1
20	1.0	2.00	2.000	0.100	0.200	0.04	1
21	1.5	2.00	3.000	0.067	0.300	0.04	1
22	2.0	2.00	4.000	0.050	0.400	0.04	1
23	2.5	2.00	5.000	0.040	0.500	0.04	1
24	3.0	2.00	6.000	0.033	0.600	0.04	1
25	4.0	2.00	8.000	0.025	0.800	0.04	1
26	5.0	2.00	10.000	0.020	0.999	0.04	1
					<u>RF/20</u>		
27	0.5	4.00	2.000	0.200	0.100	0.02	2
28	1.0	4.00	4.000	0.100	0.200	0.02	2
29	1.5	4.00	6.000	0.067	0.300	0.02	2
30	2.0	4.00	8.000	0.050	0.400	0.02	2
31	2.5	4.00	10.000	0.040	0.500	0.02	2
32	3.0	4.00	12.000	0.033	0.600	0.02	2
33	4.0	4.00	16.000	0.025	0.800	0.02	2
34	5.0	4.00	20.000	0.020	0.999	0.02	2

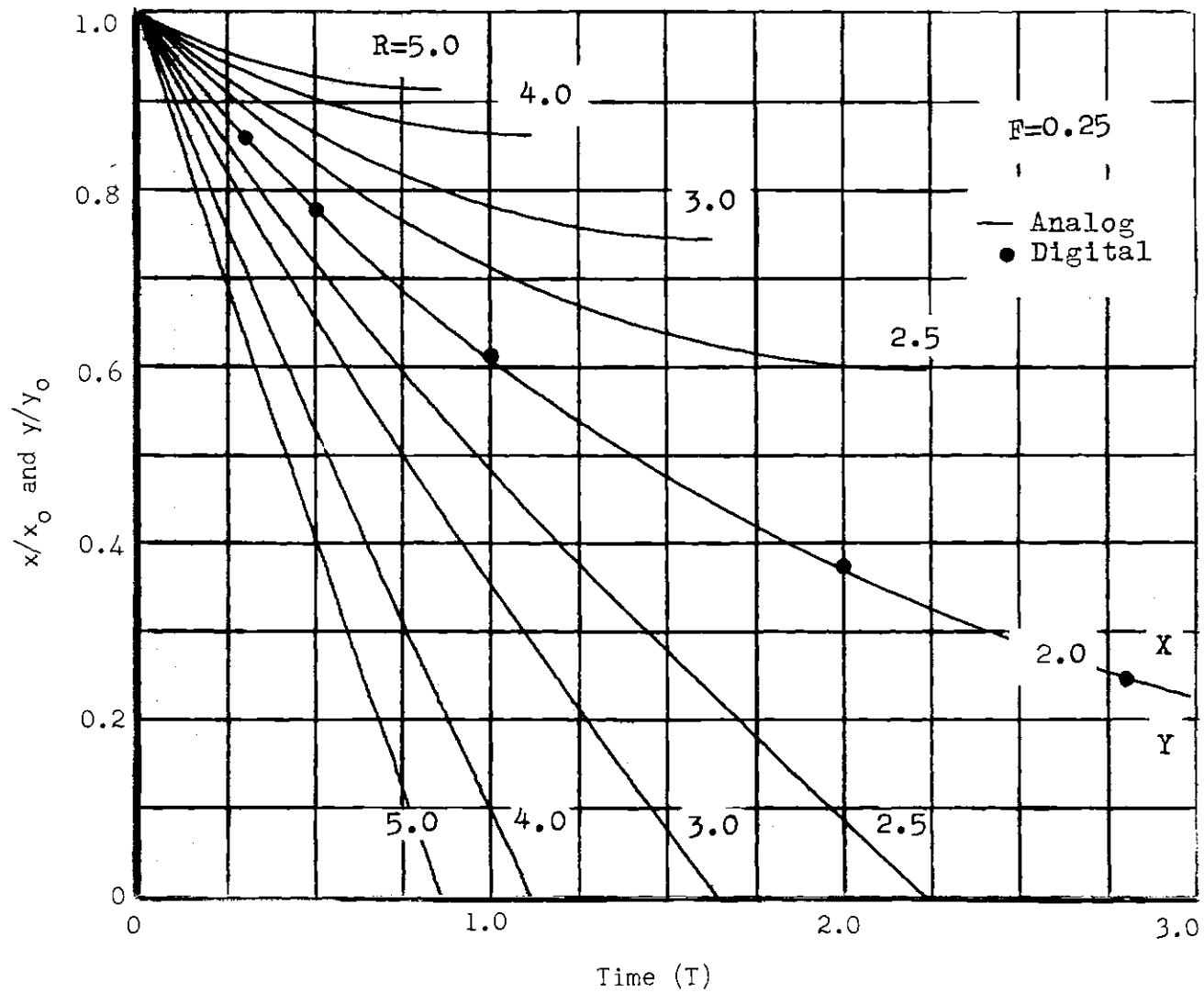


Figure 2. Model I Solution: $F = 0.25$

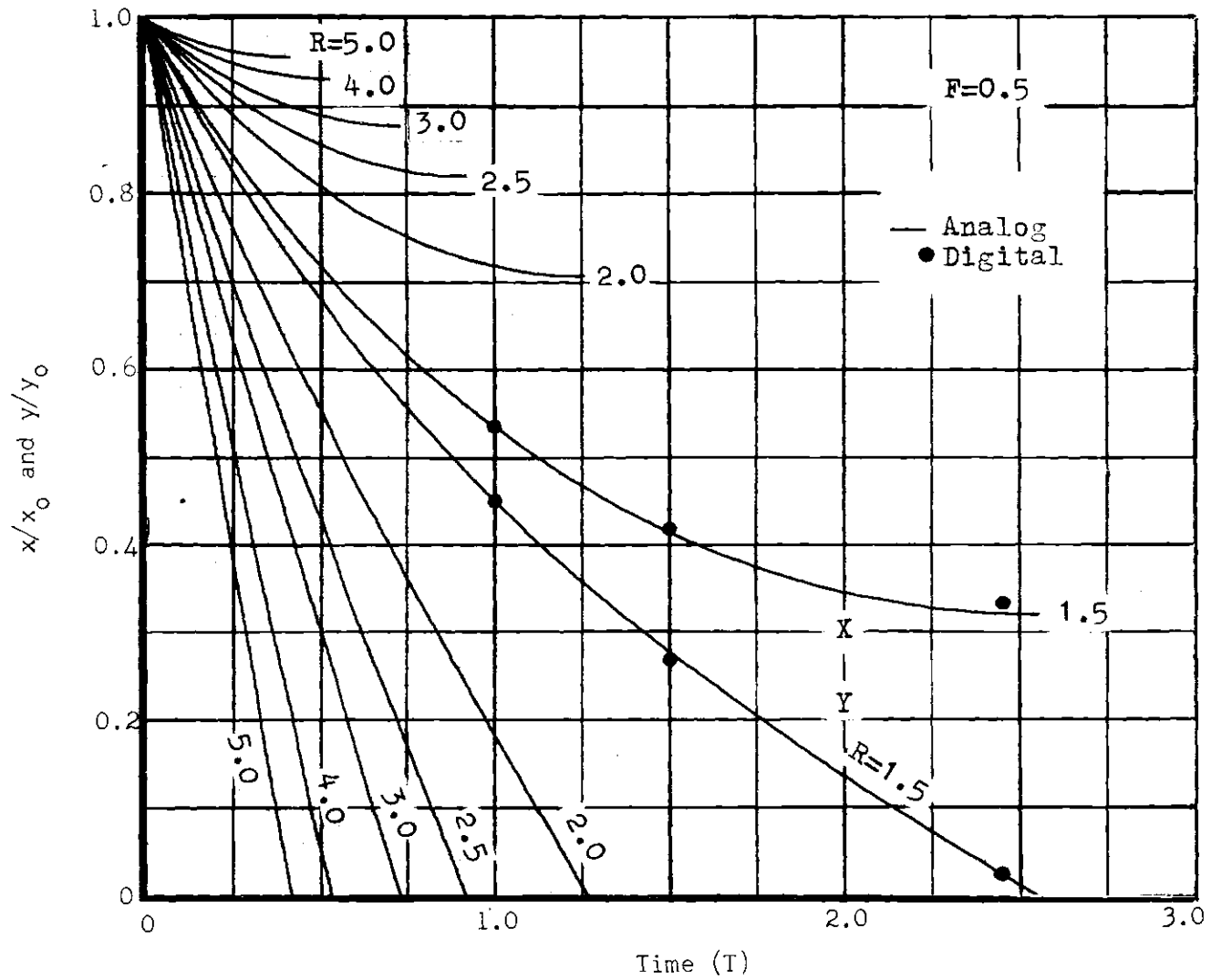


Figure 3. Model I Solution: $F = 0.50$

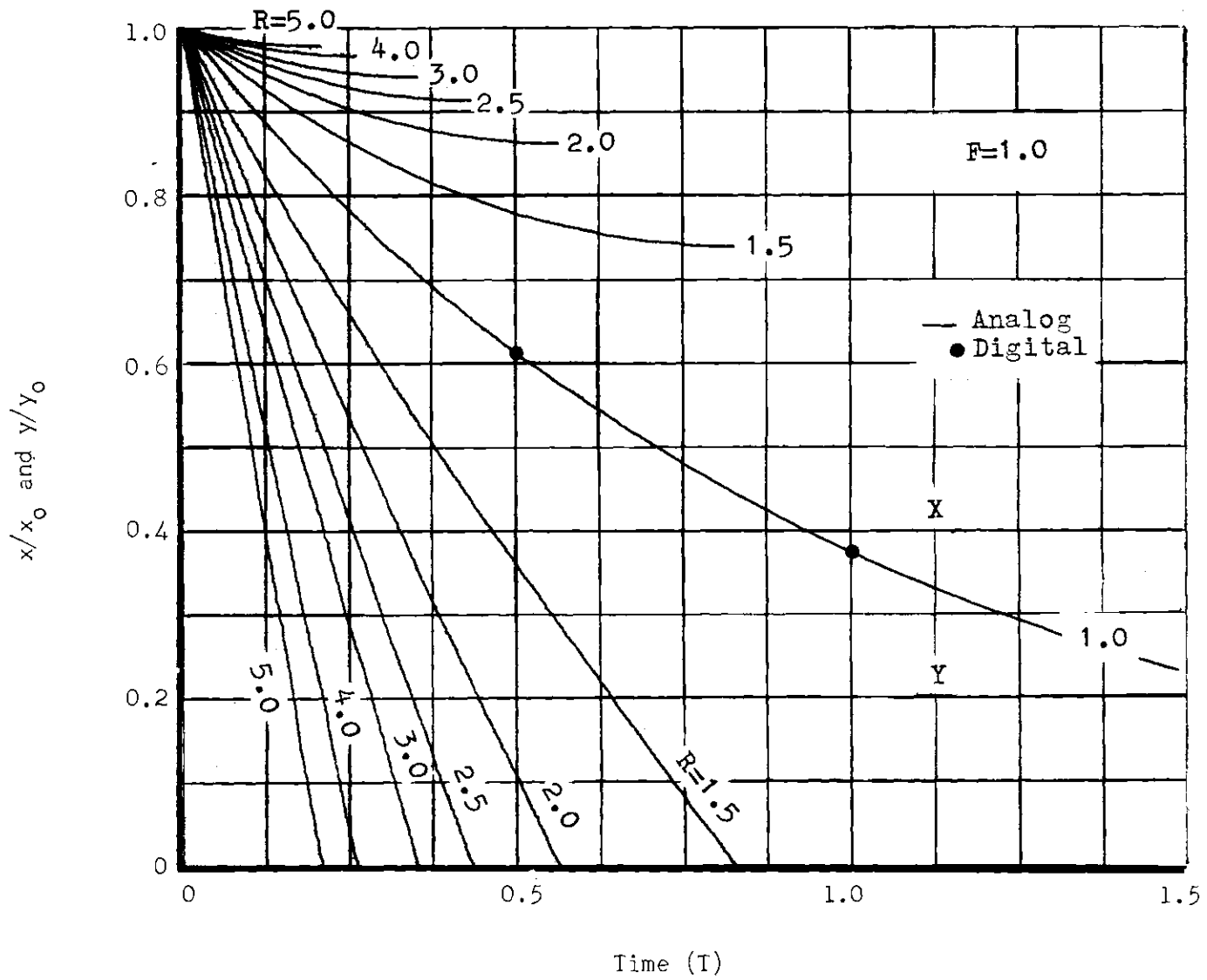


Figure 4. Model I Solution: $F = 1.00$

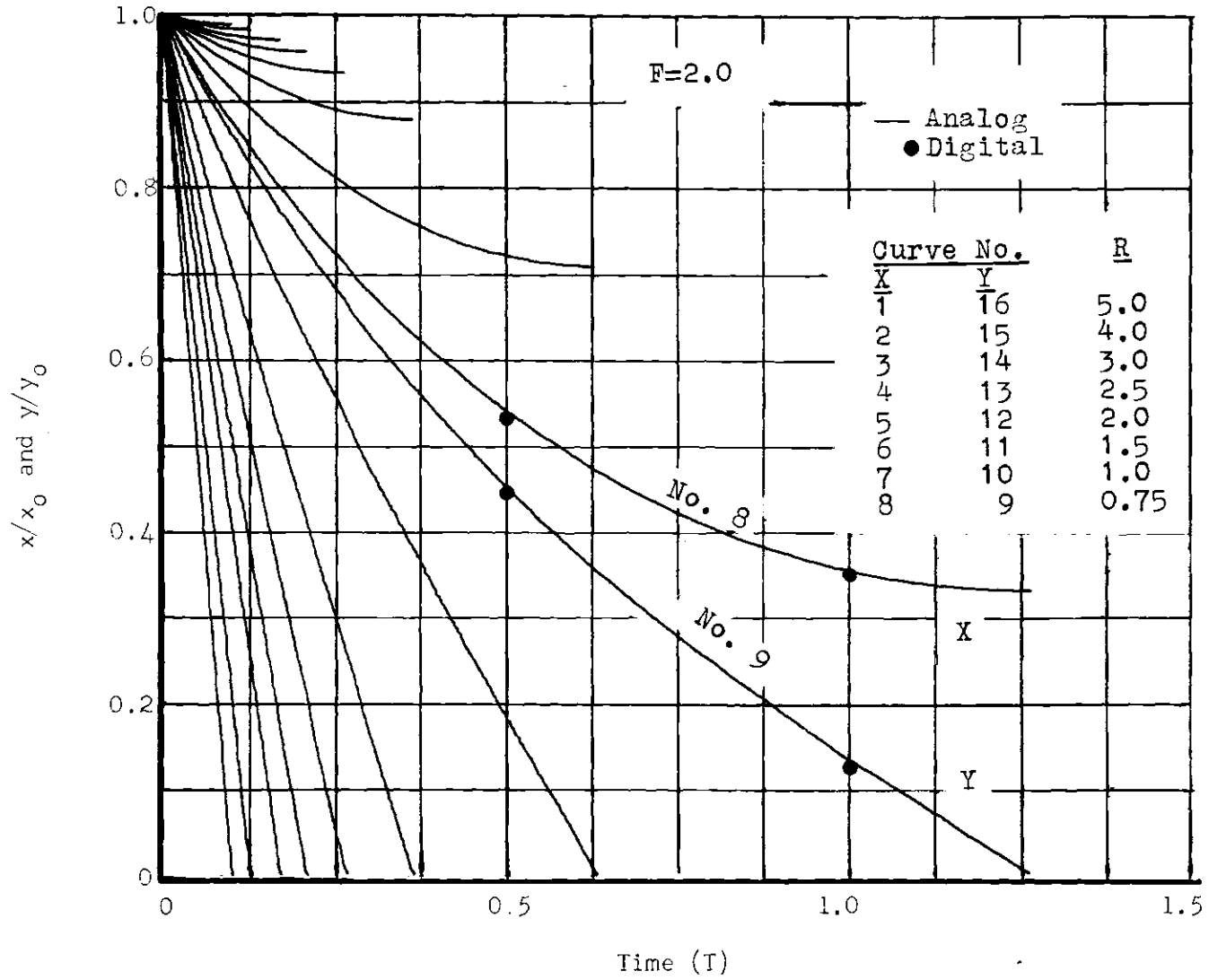


Figure 5. Model I Solution: $F = 2.00$

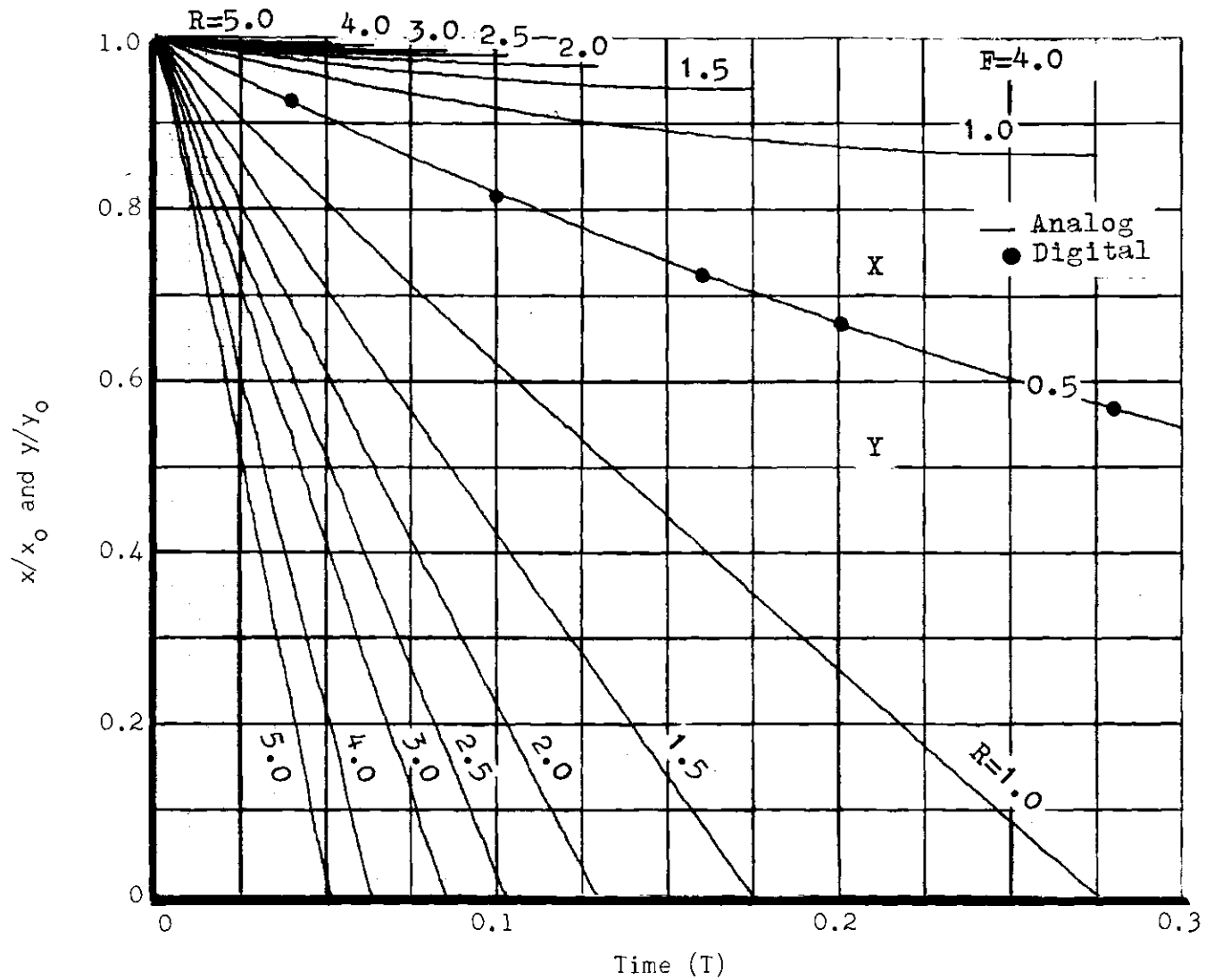


Figure 6. Model I Solution: $F = 4.00$

Table 2. Algol Program for Solution of Lanchester's
Square Law (Model I)

```

2 INTEGER COUNT, N                                     $
2 FLOATING T, X, Y, RATIO, R, F                       $
2 INPUT NUMBER(N)  $  READ($$NUMBER)                 $
2 COUNT=0                                             $
2     FIRST..          T=0                            $
2 INPUT DATA(R,F)  $          COUNT=COUNT + 1       $
2 IF COUNT EQL N+1  $          GO TO LAST             $
2 READ ($$DATA)                                       $
2 OUTPUT PARAM(R,F)                                  $
2 FORMAT TOP(B10,*R=*,X9.6,B4,*F=*,X9.6,WO)          $
2 WRITE ($$PARAM, TOP)                               $
2 FORMAT HEAD(B14,*T*,B17,*X*,B17,*Y*,B19,*X/Y*,WO) $
2 WRITE ($$HEAD)                                     $
2     STATE..
2 X=(COSH(T.SQRT(F)))-(1/(R.SQRT(F))).(SINH(T.SQRT(F))) $
2 Y=(COSH(T.SQRT(F)))-(R.SQRT(F)).(SINH(T.SQRT(F))) $
2 RATIO=X/Y                                          $
2 OUTPUT ANS(T,X,Y,RATIO)                            $
2 FORMAT BODY(B10,X9.5,B10,X8.5,B10,X8.5,B10,X10.5,WO) $
2 WRITE ($$ANS, BODY)                                $
2 T=T + 0.02                                         $
2 IF (X LEQ 0) OR (Y LEQ 0) OR (T GTR 3.0)$ GO TO FIRST $
2 GO TO START$
2 LAST..
                FINISH$

```

$$\dot{x} = - Axy \quad (\text{Defender}) \quad (11)$$

$$\dot{y} = - bx \quad (\text{Attacker}) \quad (12)$$

If the x side defends over a constant area, the coefficient A takes the form;

$$A = r_y \cdot A_{ex}/A_x \quad , \quad (21)$$

Where r_y = rate of fire for y

A_{ex} = exposed area per x unit

A_x = the total area occupied by x .

By letting $R = x_0/y_0$, $C = A_x/x_0$, and

$$D = r_y \cdot A_{ex}/y_0 \quad , \quad (22)$$

equations 11 and 12 may be transformed to ;

$$\dot{x} = - (D/RC) xy \quad (23)$$

$$\dot{y} = - bx \quad (24)$$

The analog computer schematic for solution of the model described by equations 23 and 24 is shown in Figure 7, and the parameter values and computer settings used are shown in Table 3. The solutions obtained are shown as Figures 8 through 11.

Proposed Assault Model (Model III)

The following model is proposed to describe the loss rates of the two sides in a battle between a force defending from a prepared position against an assault by a force in the open:

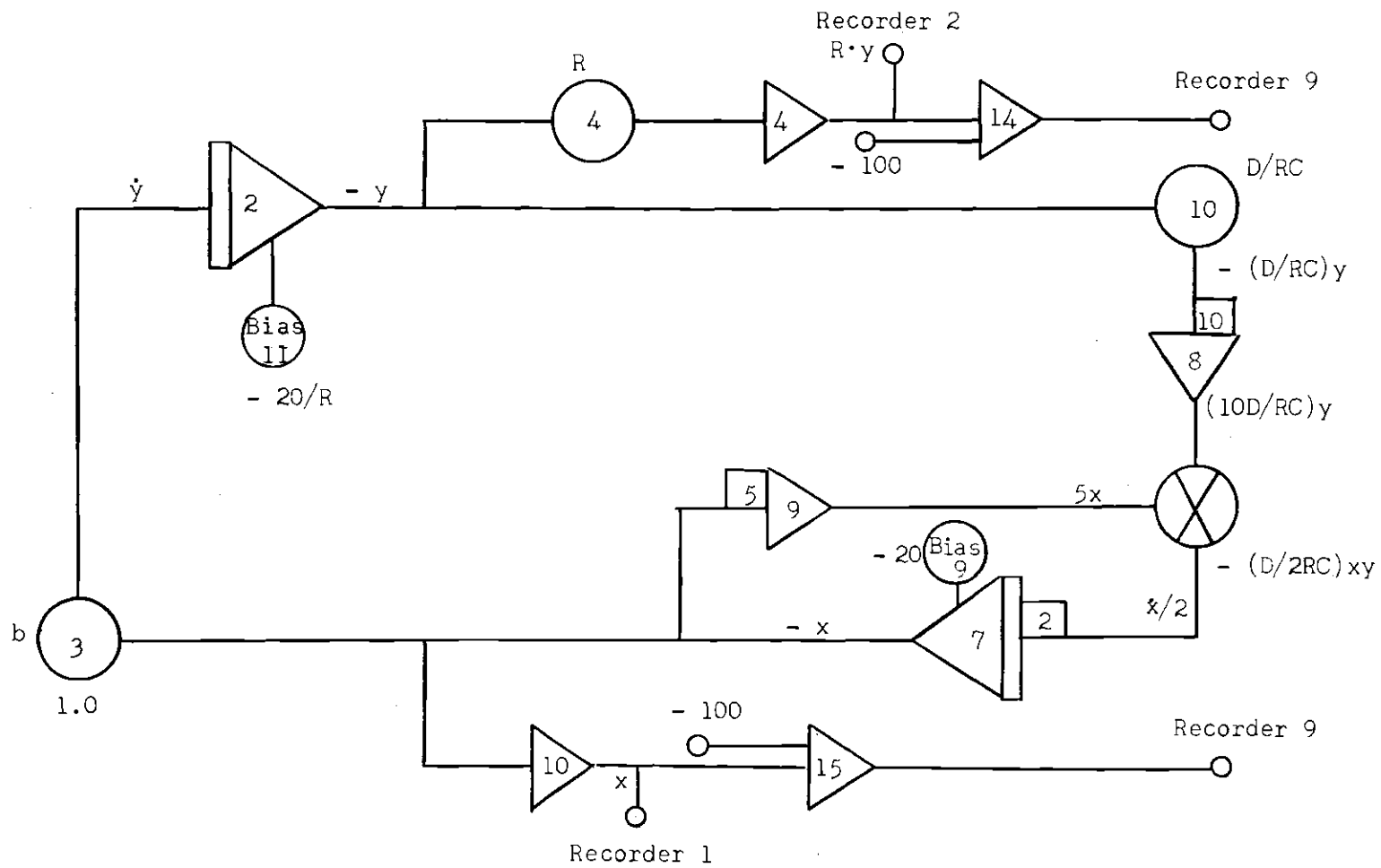


Figure 7. Analog Computer Schematic Diagram: Model II

Table 3. Analog Computer Settings: Model II

<u>Run</u>	<u>A_{ex}</u>	<u>1/R</u>	<u>X_o</u>	<u>Bias 11</u>	<u>Pot 10</u>	<u>Pot 4</u>
			$r_y = 100, c = 100$			
35	1.0	1.0	100	- 20	0.01	1.000
36	1.0	2.0	100	- 40	0.01	0.500
37	1.0	2.5	100	- 50	0.01	0.400
38	1.0	3.0	100	- 60	0.01	0.333
39	1.0	4.0	100	- 80	0.01	0.250
40	1.0	5.0	100	- 100	0.01	0.200
41	2.0	1.0	100	- 20	0.020	1.000
42	2.0	2.0	100	- 40	0.020	0.500
43	2.0	2.5	100	- 50	0.020	0.400
44	2.0	3.0	100	- 60	0.020	0.333
45	2.0	4.0	100	- 80	0.020	0.250
46	2.0	5.0	100	- 100	0.020	0.200
47	1.0	1.0	1000	- 20	0.001	1.000
48	1.0	2.0	1000	- 40	0.001	0.500
49	1.0	2.5	1000	- 50	0.001	0.400
50	1.0	3.0	1000	- 60	0.001	0.333
51	1.0	4.0	1000	- 80	0.001	0.250
52	1.0	5.0	1000	- 100	0.001	0.200
53	2.0	1.0	1000	- 20	0.002	1.000
54	2.0	2.0	1000	- 40	0.002	0.500
55	2.0	2.5	1000	- 50	0.002	0.400
56	2.0	3.0	1000	- 60	0.002	0.333
57	2.0	4.0	1000	- 80	0.002	0.250
58	2.0	5.0	1000	- 100	0.002	0.200

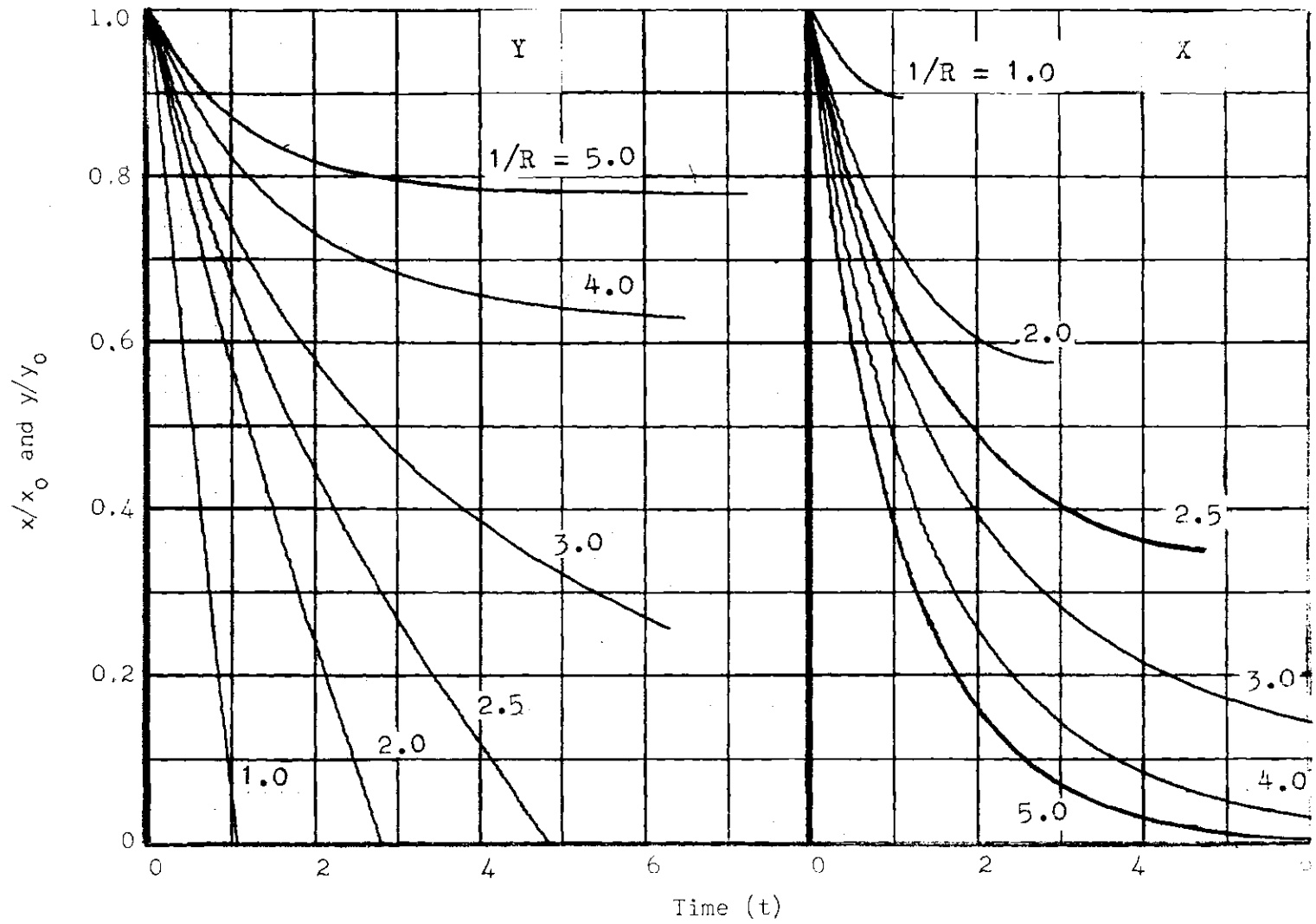


Figure 8. Model II Solution: $x_0 = 100, A_{ey} = 1.0$

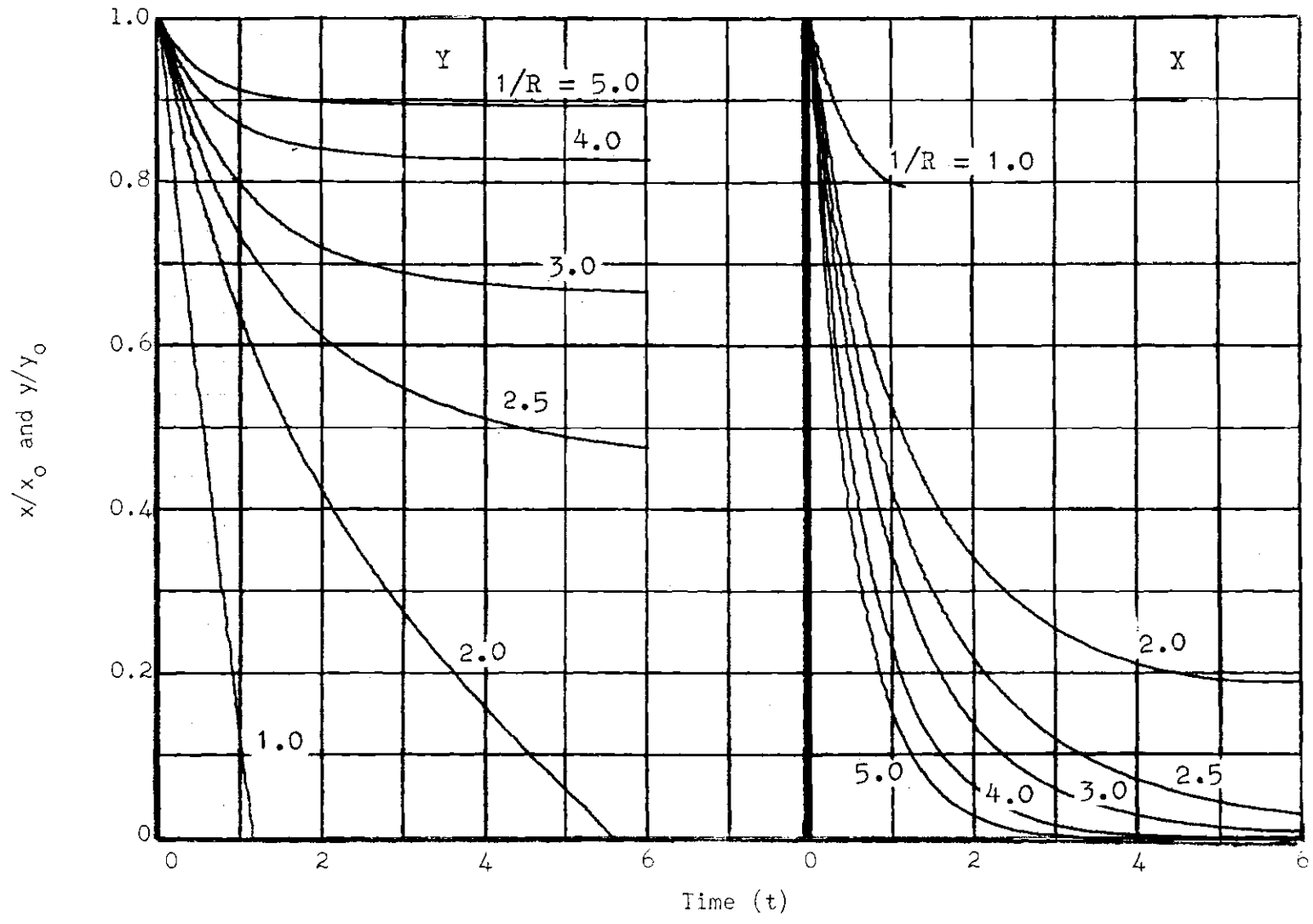


Figure 9. Model II Solution: $x_0 = 100, A_{ey} = 2.0$

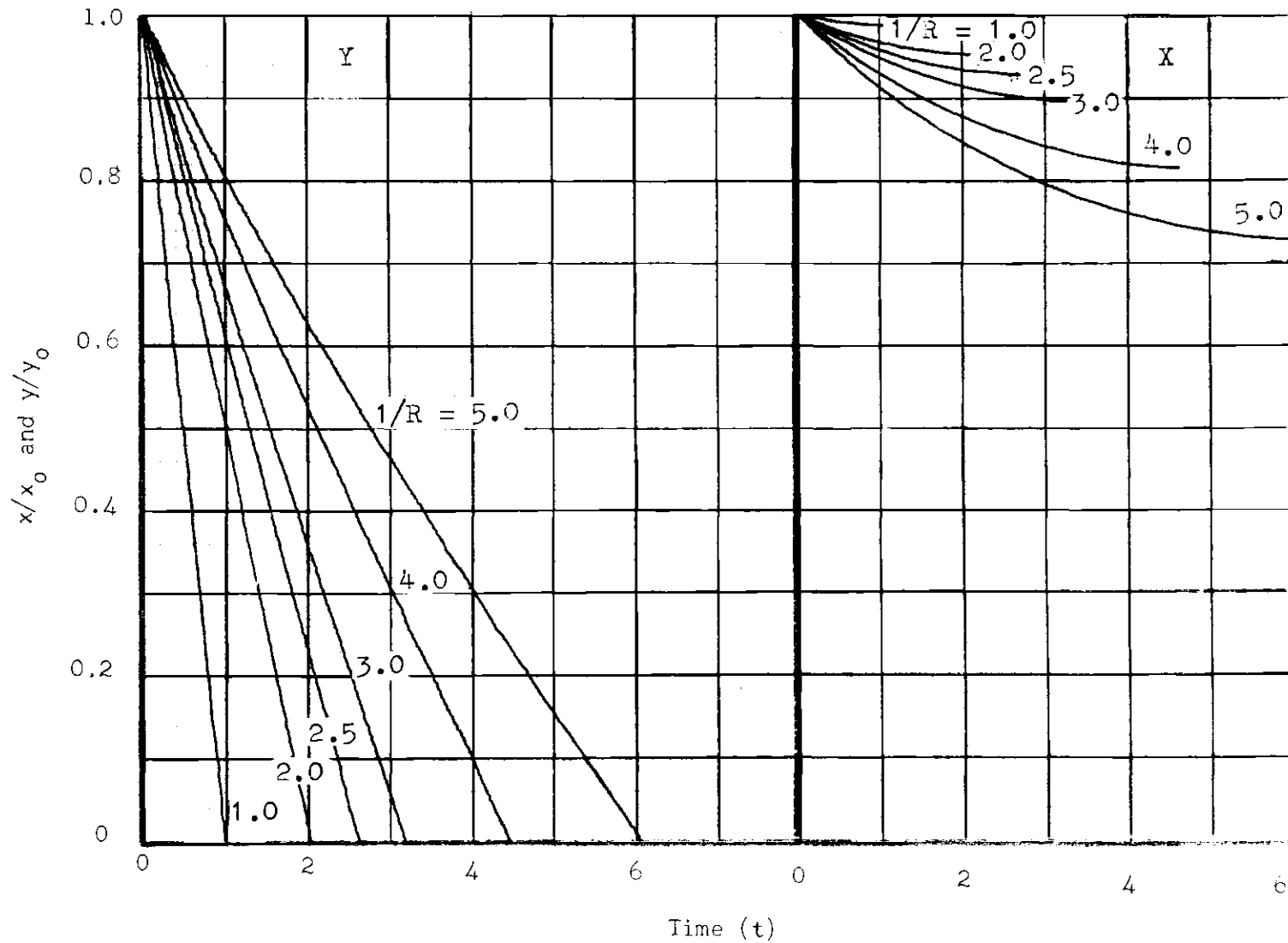


Figure 10. Model II Solution: $x_0 = 1000$, $A_{ey} = 1.0$

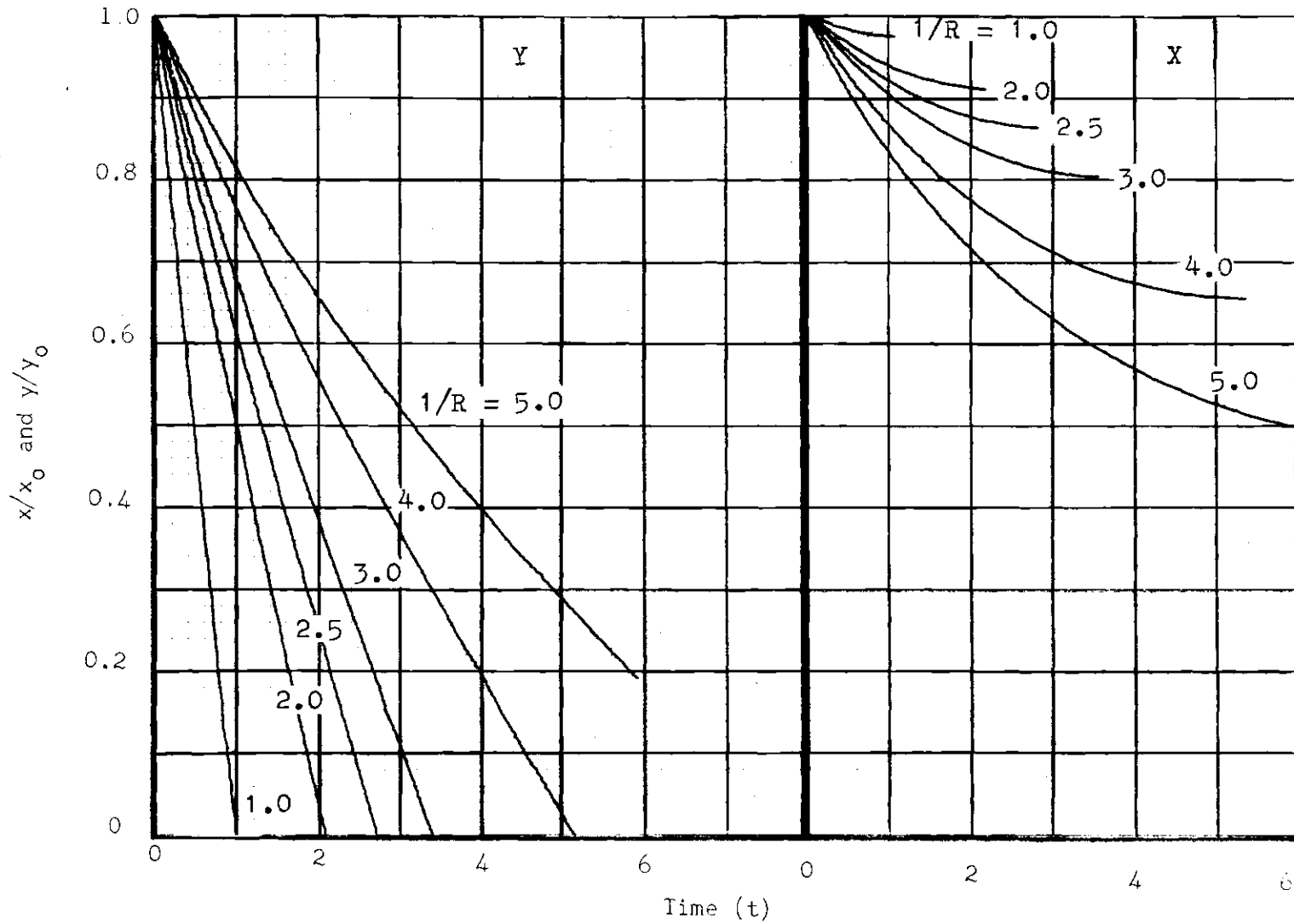


Figure 11. Model II Solution: $x_0 = 1000, A_{ey} = 2.0$

$$\dot{x} = -K_1(Axy) - K_2(ay) \quad (\text{Defender}) \quad (25)$$

$$\dot{y} = -bx \quad (\text{Attacker}) \quad (26)$$

where

$$0 \leq K_1 \leq 1 \quad , \quad (27)$$

$$0 \leq K_2 \leq 1 \quad , \quad (28)$$

and

$$K_1 + K_2 = 1 \quad (29)$$

This model takes into account the changing nature of an assault. At the beginning of an assault, the attacking force cannot see the defenders, but must fire blindly into the area being defended. The defender's loss rate due to this type of action is $-K_1(Axy)$, where K_1 indicates the part of the total loss rate attributable to unaimed fire. If the assault is successful, the defenders will gradually become visible to the attacking force and will be subjected to aimed fire. The loss rate due to aimed fire is $-K_2(ay)$, where K_2 represents the fraction of the total loss rate caused by aimed fire.

The coefficients K_1 and K_2 vary, of course, according to the physical actions of the attacking force. If the attacking force fails to attain aimed fire, $K_2 = 0$ and the model reduces to Equations 11 and 12 (i.e., Model II). If aimed fire is used exclusively, however, the model reduces to Lanchester's Square Law as represented in Equations 6 and 7 (i.e., Model I).

The course followed by most assaults, however, is one in which the attacking force gradually changes from unaimed to aimed fire as the defenders become visible, while the attacking force is under aimed fire throughout the battle. Description of this process by Equations 25 and 26 is attained by allowing K_1 and K_2 to vary between zero and one. It is assumed that K_1 is a function of the number of surviving defenders, x . Since the form of this relationship is not known, three representative functions will be used in solution of the model. For $x_0 = 1000$, these functions are:

$$K_1 = (1/3)\log(x + 1) \quad (30)$$

$$K_1 = x/1000 \quad (31)$$

$$K_1 = x^2/10^6 \quad (32)$$

representing logarithmic, linear, and quadratic relationships respectively. As x decreases with time, these functions give successively faster degeneration of K_1 from one to zero, corresponding to faster transition from unaimed to aimed fire.

The three K_1 functions, as generated by the computer, are compared in Figure 12. It should be noted that the logarithmic relationship is formed on the computer as a series of straight line segments rather than as a continuous curve. Also, at the initial condition of $x = x_0 = 1000$, K_1 has a value of $(1/3)\log(1001)$ instead of $(1/3)\log(1000)$. Neither approximation is of serious consequence, however; in the latter case particularly, the error introduced is only of the order of magnitude of the computer noise level.

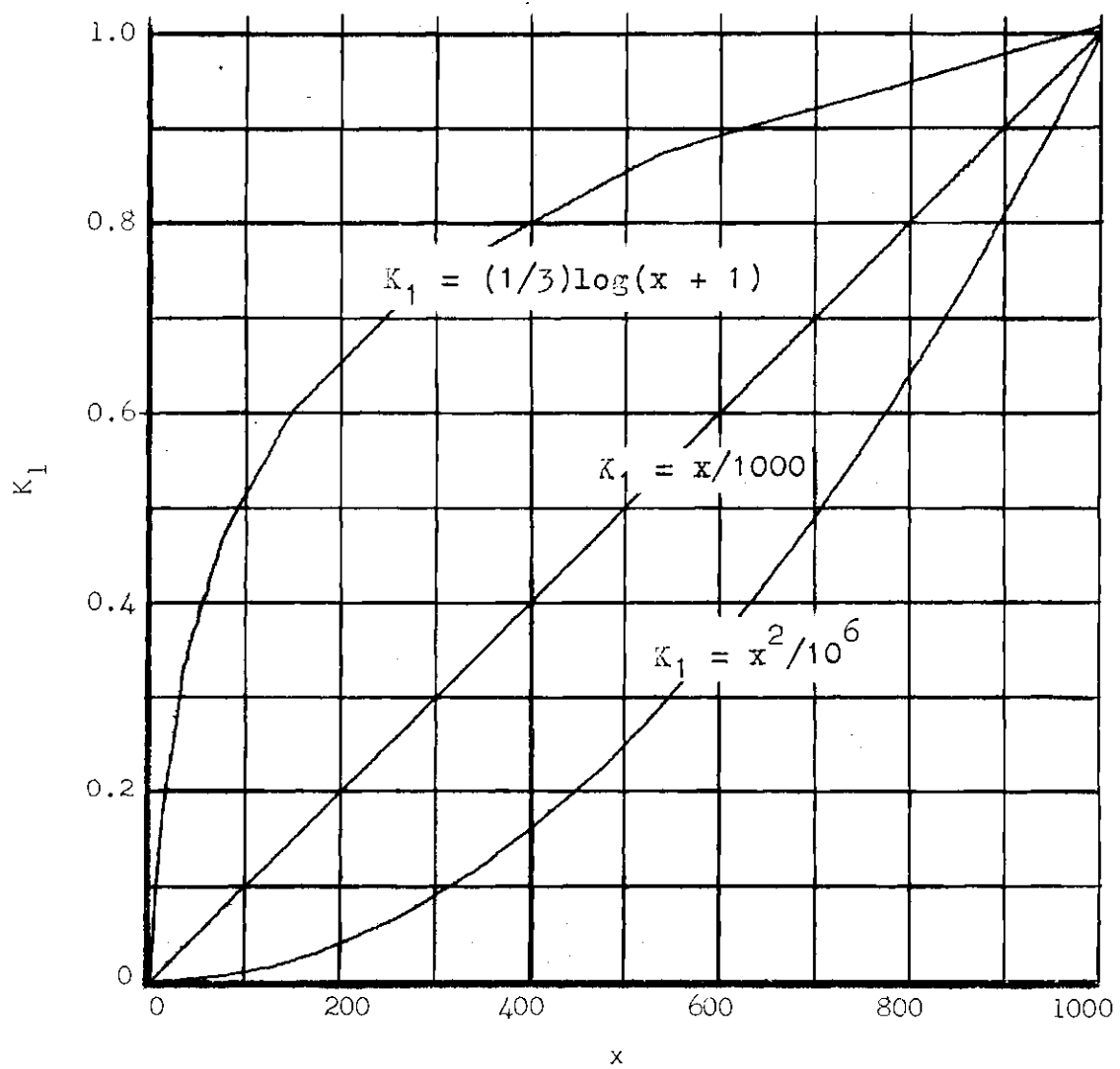


Figure 12. K_1 as a Function of x (Model III)

The model was solved by analog computer for several values of R , using each of the above functions for K_1 , and letting $a = 0.010$, $b = 0.030$, and $A = 0.0001$. A comparison of the results for $R = 0.20$ using each of the three K_1 functions is shown in Figure 13.

The analog computer schematic is shown in Figures 14 and 15. The solutions obtained are presented as Figures 16 and 17.

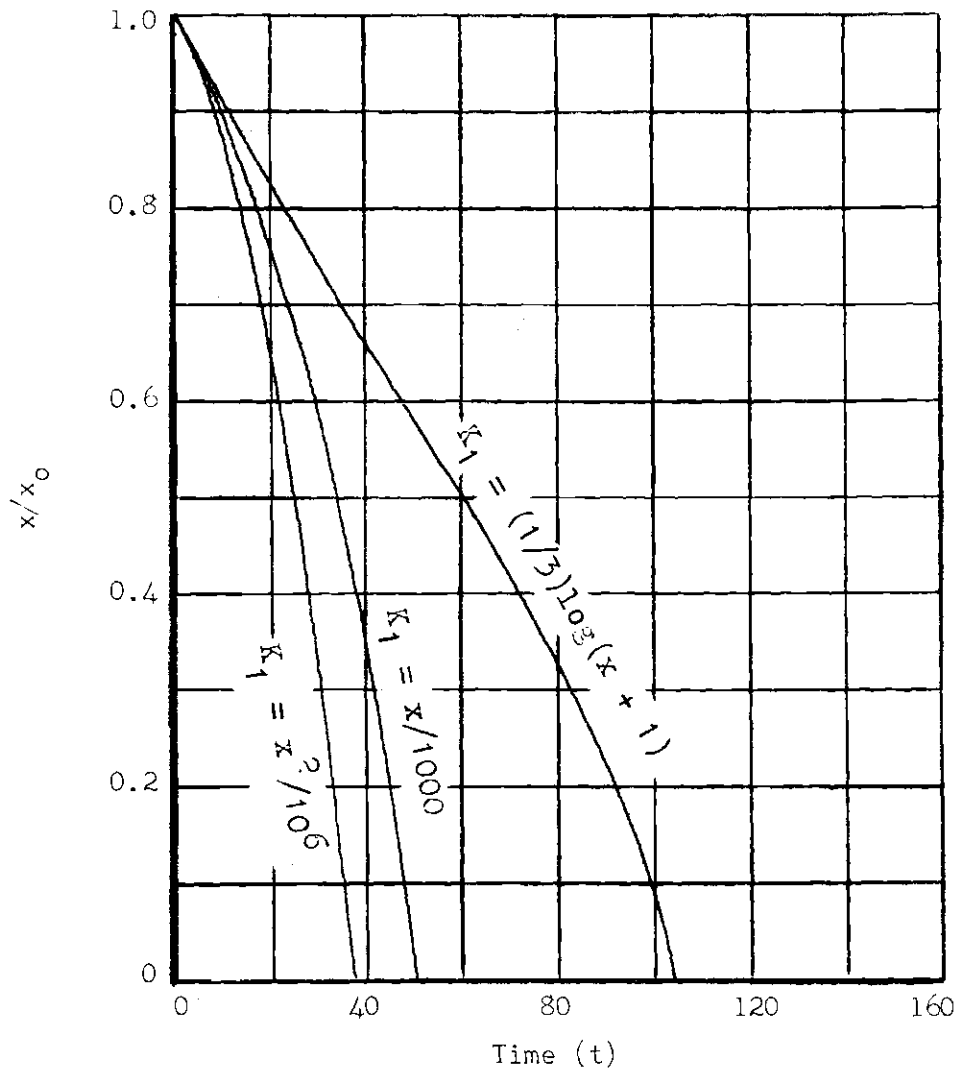


Figure 13. The Effect of the Functional Form of K_1 on Battle Duration
 ($a = 0.01$, $b = 0.03$, $A = 0.0001$, $l_R = 0.20$)

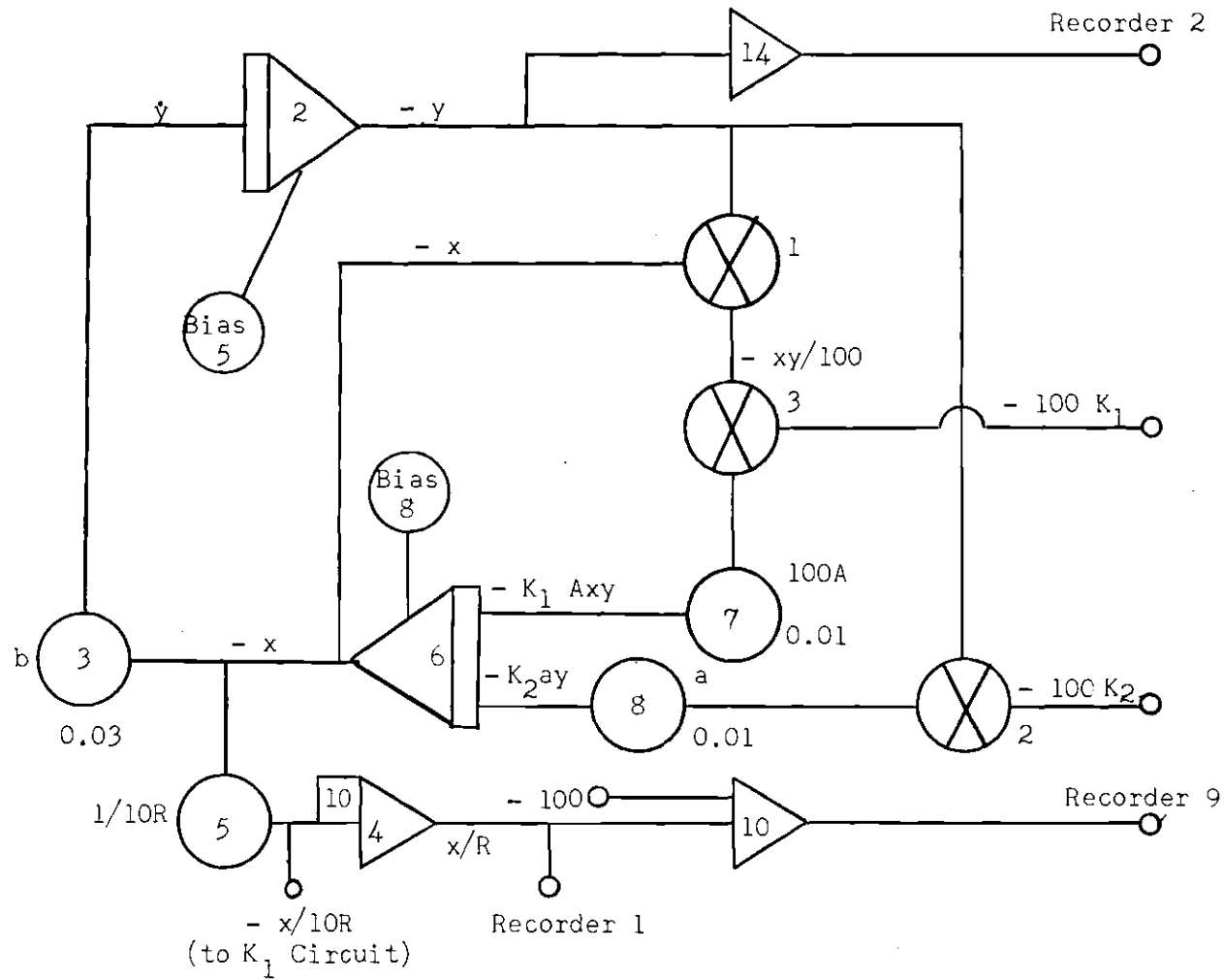
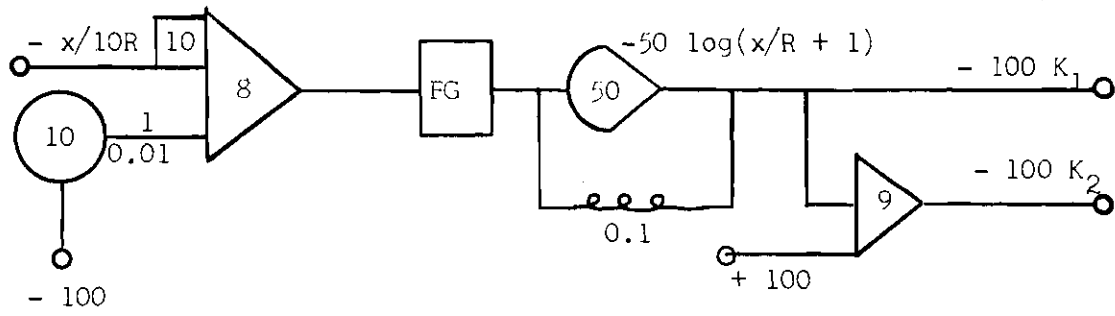
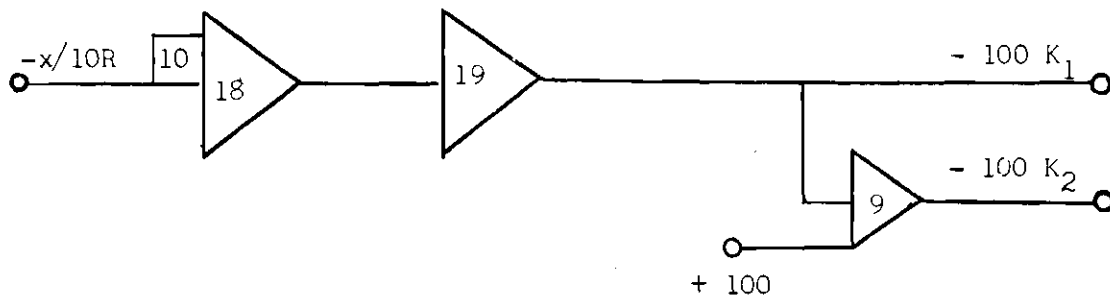


Figure 14. Analog Computer Schematic Diagram: Model III

Logarithmic



Linear



Quadratic

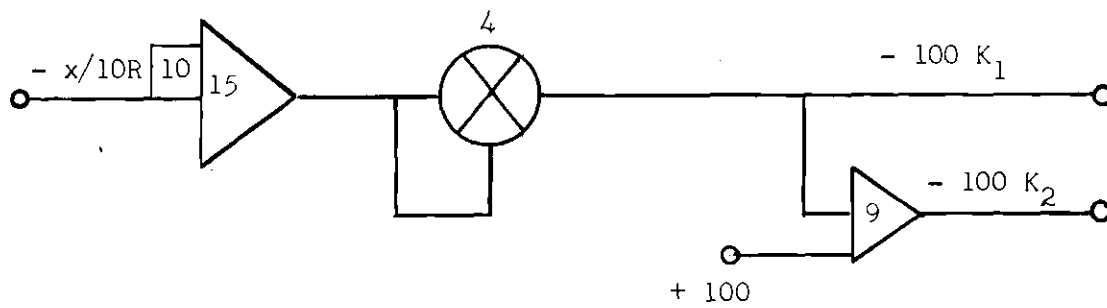


Figure 15. Analog Computer Schematic Diagram: $K_1 = f(x)$

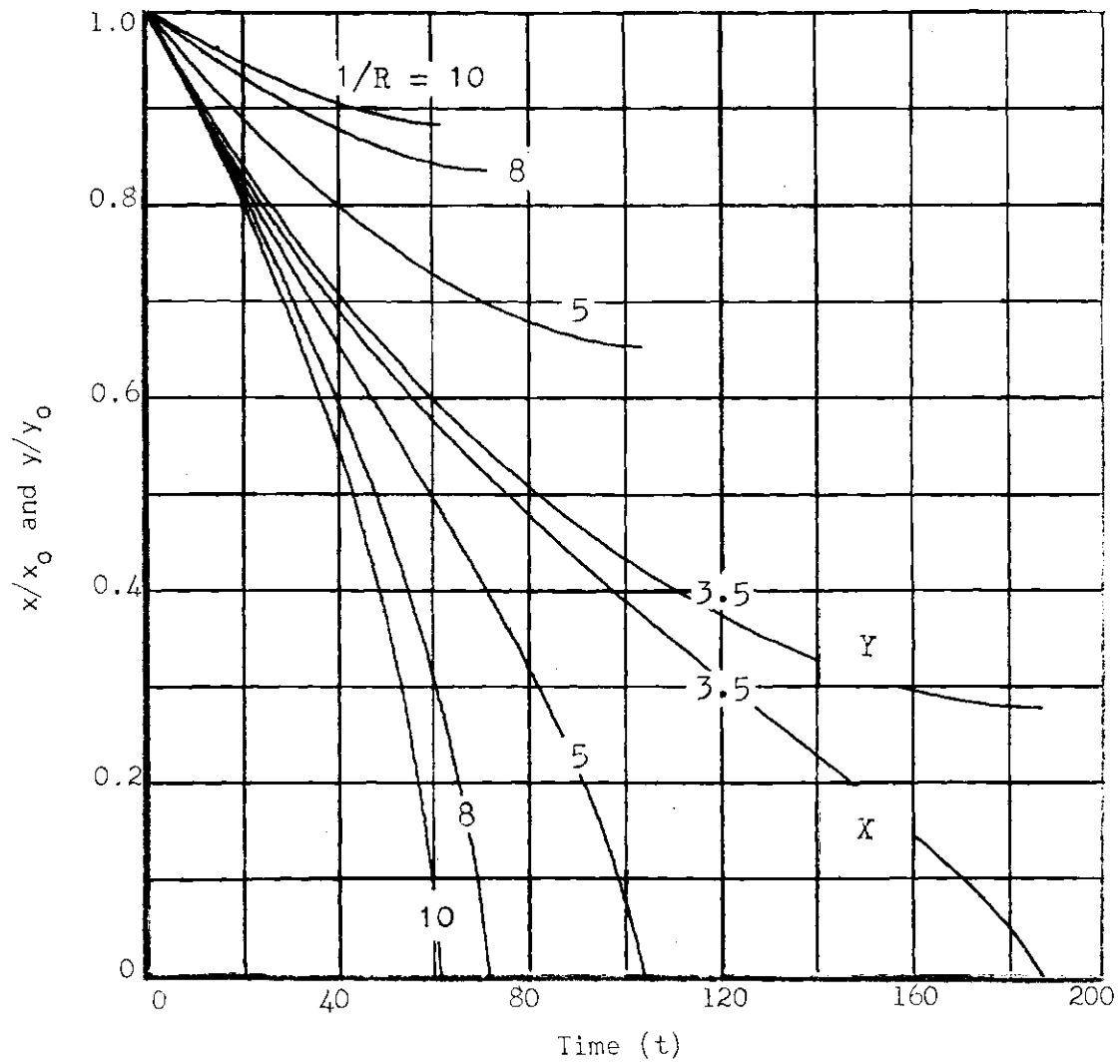


Figure 16. Model III Solution: $K_1 = (1/3)\log(x + 1)$ and $x_0 = 1000$

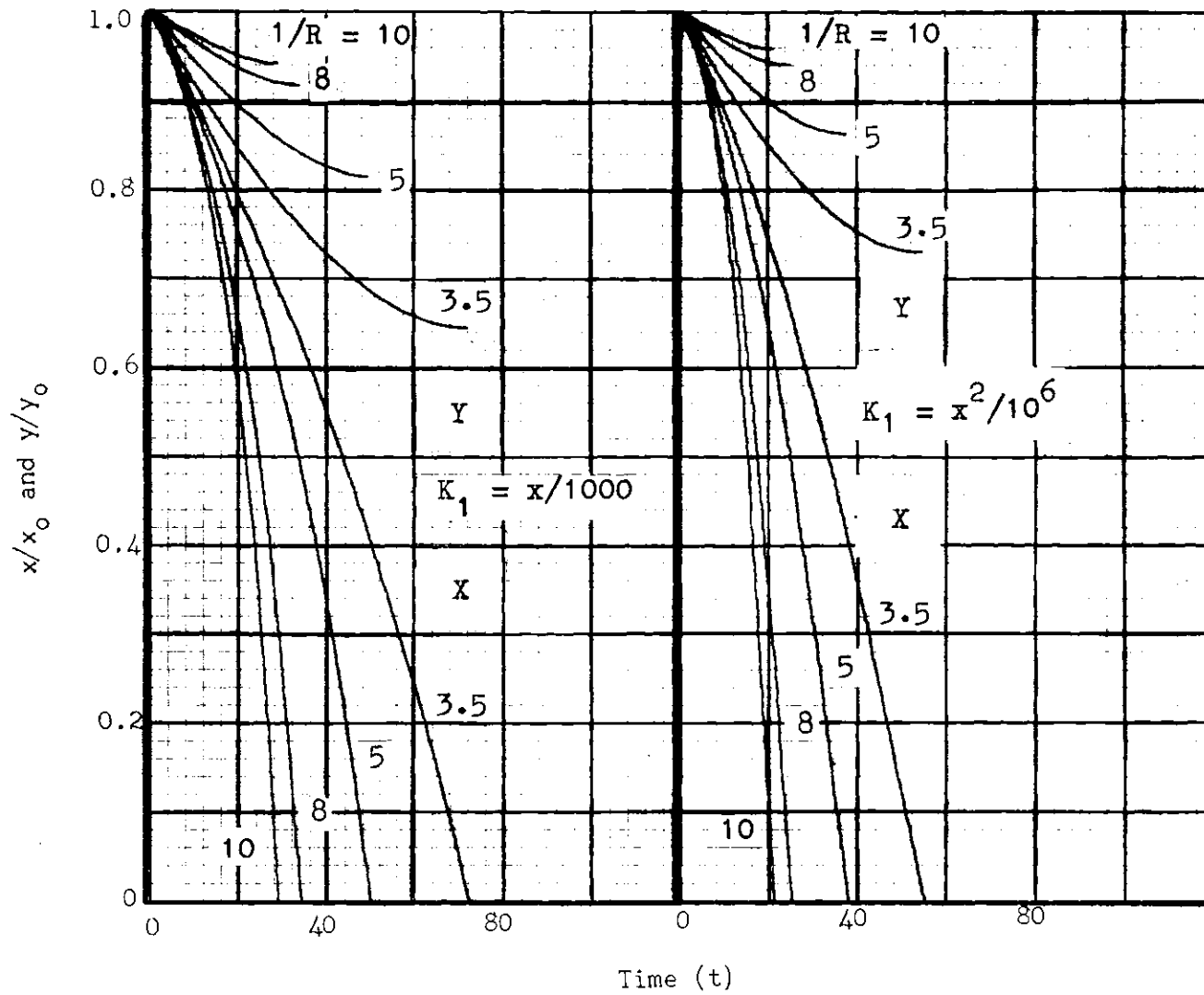


Figure 17. Model III Solution: $K_1 = x/1000$ and $K_1 = x^2/10^6$, $x_0 = 1000$

Table 4. Analog Computer Settings: Model III

<u>Run</u>	<u>f(x)</u>	<u>1/R</u>	<u>Pot 6</u>	<u>Bias 8</u>
59	$(1/3)\log(x+1)$	3.5	0.350	- 28.6
60	$(1/3)\log(x+1)$	5	0.500	- 20.0
61	$(1/3)\log(x+1)$	8	0.800	- 12.5
62	$(1/3)\log(x+1)$	10	1.000	- 10.0
63	$x/10^3$	3.5	0.350	- 28.6
64	$x/10^3$	5	0.500	- 20.0
65	$x/10^3$	8	0.800	- 12.5
66	$x/10^3$	10	1.000	- 10.0
67	$x^2/10^6$	3.5	0.350	- 28.6
68	$x^2/10^6$	5	0.500	- 20.0
69	$x^2/10^6$	8	0.800	- 12.5
70	$x^2/10^6$	10	1.000	- 10.0

CHAPTER IV

ANALYSIS

Solutions and Circuits

The criteria for the analog solutions are that they have validity in light of previous work in the field and that the analog circuits produce reliable results.

In the case of Lanchester's Square Law (Model I), these results and those obtained by Clark et al. (7) are in close agreement. For example, for $F = 1.0$ and $R = 1.0$, at time $T = 1.0$, Clark et al. (7) found the survival fraction for both sides to be approximately 0.4, as compared to 0.39 in this work. It is also noted that the digital computer solutions agree very closely with the analog solutions. However, there are no published numerical examples with which solutions of Model II and Model III can be compared.

Each circuit was designed so as to keep maximum voltages as near ± 100 volts as possible, thus introducing little error due to random voltage fluctuations. Solution stability was checked prior to and periodically during each computer session to insure reliable results.

Proposed Model (Model III)

The first two models studied considered the defender to be subjected to only aimed and unaimed fire respectively. However, neither of these models satisfactorily describes the actions which take place under realistic conditions.

The assault described by Model III starts with the attacker in the open firing blindly into the area occupied by the defender. As the assault progresses, the attackers locate the defenders and subject them to aimed fire. This is essentially what occurs in an actual assault and is much nearer reality than the assumption that only one type of loss rate is extant throughout the battle.

The results obtained upon solution of the model show that the duration of the battle is greatly affected by the functional form of K_1 . The logarithmic function produces the longest battle and the quadratic the shortest. This is as expected since the logarithmic function represents a slow transition from unaimed to aimed fire while the quadratic form of transition is much faster. The losses sustained by the attacker (when he wins) are also less when the transition to aimed fire is rapid.

Another important feature of this model is that for constant K_1 values of unity and zero, it reduces to the original assault model and Lanchester's Square Law respectively. The results of the model are then limited to values which lie between the results produced by the two earlier models.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

General Remarks

The most severe limitation of any mathematical model of warfare is that human behavior cannot be precisely predicted. In experiments involving chemical reactions, bacteria, and even some lower class mammals, reactions to certain stimuli are fairly consistent. This consistency is not present in individual soldiers or their commanders, however. Chemical reactions take place at rates and in proportions which are fixed by physical laws. No such laws govern the actions of men. Once the conditions of a chemical system are fixed, the reaction proceeds unalterably to a predictable conclusion. In a system embracing the battlefield, however, the reaction, or the battle, can be changed by the actions of the individuals within the system.

Because of this ability to influence events and the grave risks involved in any future military conflict, we should attempt to learn as much as possible about the mechanism of war. Part of this effort should be directed toward finding methods of predicting the course of a battle, given the initial conditions of the system. With these methods, and adequate knowledge of the enemy, military commanders could control the battlefield much more effectively.

Conclusions

The conclusions of this study could be summarized as follows:

The analog solutions obtained for each of the three models presented are valid in light of previous work, and the analog circuits used produce reliable results.

The model proposed for the assault of a defended position (Model III) is more realistic than any previously proposed deterministic Lanchester-type model. This realism may be enhanced by determination of the most appropriate functional form for K_1 .

Model III (Equations 25 and 26) also supports the doctrine of speed and shock action in the attack, while illustrating the benefits of cover and concealment to the defender. The most important point, however, is the value of numerical superiority in determining the victor, and the secondary importance of individual fighting ability.

Recommendations

It is recommended that verification of Lanchester-type equations and determination of parameter values be attempted using data from World War II, the Korean conflict, and the more recent operations in Viet Nam. A study to include beach landings, river crossings, ambushes, and the assault of defended hills and cities would be a significant addition to our knowledge of warfare.

It is further recommended that any such study be extended to the assault model proposed in this work and that the functional form of K_1 be determined, thus further explaining the actual casualty-producing process.

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