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A SENSITIVITY ANALYSIS ON MULTICOMMODITY NETWORK FLOWS

A THESIS

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CHAPTER I

INTRODUCTION

Within the numerous applications of network flow theory, some of the most important are those in which a number of distinct commodities flow through the network. Examples of such problems are abundant in traffic problems, communication network problems, and various other related areas.

The different types of multicommodity problems can, mathematically, be formulated as linear programs. One of them consists in maximizing the values of flows subject to the arc capacity constraints. Procedures have been suggested to solve this linear program, using the special structure of its constraint matrix. The most typical procedure is that proposed by Ford and Fulkerson (4).

Practical problems that are formulated as mathematical models are seldom completely solved as soon as a given method identifies the optimal solution for the model. The parameters of the model are seldom known with complete certainty. Therefore, it is usually advisable to perform a sensitivity analysis to determine the effect on the optimal solution if particular parameters take on other possible values. A second situation where additional computations are required is where changes must be made in the original model, either because errors and omissions were discovered or because new information indicates that the estimates of the parameter values should be revised.

The objective of this research is to perform a sensitivity analysis on multicommodity network flows, specifically, the problem of adding arcs to the network; given an optimal solution and a new set of arcs for a multicommodity network determine which arcs of the new set can be added to the network to obtain the "best" improvement of the solution. This research would be helpful, for instance, in a transportation system in deciding where to build new roads.

Let us now review some preliminary concepts and basic definitions to a better formulation of the problem and its solution.

Definition

A network, $G = (N, E)$, consists of a finite set, N, of u elements, N_i, $i = 1...u$, and a subset, E, of the pairs, (N_i, N_j), of the elements in N.

In a graph, N is a set of nodes or vertices and E is a set of arcs or edges connecting the nodes.

The arcs can be ordered pairs or unordered pairs, and the arc is correspondingly called directed or undirected.

Definition

An undirected network is a network with all of the arcs undirected.

Definition

When only one commodity flows between two appropriate nodes, N_s and N_t, called respectively the source and the sink, the network is a single-commodity network.

Definition

A multicommodity network is a network with more than one commodity flowing between appropriate pairs of sources and sinks.

For a multicommodity network, there are two distinguished sets, $S \subset N$ and $T \subset N$, each containing exactly q elements specified by $S = \{s_1, s_2, \ldots, s_q\}$ and $T = \{t_1, t_2, \ldots, t_q\}$. Each distinguished pair of nodes (s_j, t_j) is associated with a different commodity.

Definition

Associated with every arc is a non-negative real number, $b(N_i, N_j)$, defined as the arc capacity. The value $b(N_i, N_j)$ can be thought of as the maximum amount of flow which could be transported by the arc.

Definition

A proper disconnecting set for q pairs of nodes is a subset of E the removal of which will make s_j disconnected from t_j, $j = 1 \ldots, q$, and no proper subset of which will have this property.

Definition

A minimum proper disconnecting set separating s_j from t_j, $j = 1 \ldots, q$, is a proper disconnecting set such that the sum of the capacities of the arcs in the proper disconnecting set is minimal over all disconnecting sets. Let this set be denoted by (S,T).

Definition

Let $b(S,T)$ be the sum of the capacities of the arcs in the minimum proper disconnecting set which disconnects s_j from t_j, $j = 1 \ldots, q$. More simply, we will refer to this sum as the capacity of the minimum cut.

Definition

A chain from s_j to t_j is an uninterrupted sequence of nodes and arcs beginning at s_j and terminating at t_j.

Definition

Let us consider the problem of maximizing the sum of the flows of different commodities. For each commodity there may exist many chains joining the source to the sink. The problem is to select chains for each commodity such that the arc capacities are not violated and the sum of the flows in all the chains selected is maximum. Let $a_1 \ldots a_m$ be a list of the arcs of the network G(N,E), with arc capacities, $b_1 \ldots b_m$. A chain in the network can be represented by an m-vector with 1 in a component if the arc is used and 0 if the arc is not used in the chain. Let $p_1 \ldots p_n$ be a list of all vectors that represent all chains which join the source to the sink, for the various commodities. From now on we will refer to these vectors as the chains of the network. Let us define an arc-chain incidence matrix $A = \|a_{ij}\|$ as follows:

$$a_{ij} = \begin{cases} 1, \text{ if the arc } a_i \text{ is in the chain } p_j. \\ 0, \text{ otherwise.} \end{cases}$$

If x_j is the amount of flow in chain p_j, then the problem may be formalized as follows:

$$\text{Maximize:} \quad z_p = \sum_{j=1}^{n} x_j \tag{1.1}$$

$$\text{Subject to:} \quad \sum_{j=1}^{n} a_{ij} x_j \leqq b_i \qquad (i = 1 \ldots m)$$

$$x_j \geqq 0$$

Note that it is immaterial whether the problem involves directed or undirected arcs.

The dual formulation of (1.1) is:

$$\text{Minimize:} \quad \sum_{i=1}^{m} \pi_i b_i \tag{1.2}$$

$$\text{Subject to:} \quad \sum_{i=1}^{m} \pi_i a_{ij} \geqq 1 \qquad (j = 1 \ldots n)$$

$$\pi_i \geqq 0$$

Consider the network shown in Figure 1. In this example the maximum flow z_p^* is equal to 9/2, i.e. 3/2 along each unique path from each source to its respective sink. Also, $(S,T) = \{a_1, a_4\}$ with $b(S,T) = 5$. The optimal dual variables π_i^* associated with their respective arcs are: $\pi^* = 0$, $\pi_2^* = \pi_3^* = \pi_4^* = 1/2$. It is well known that the optimal dual variable, π_i^*, indicates the expected rate of change, in the objective function z_p^*, as b_i varies. For instance, if the capacity of the arc a_3 is changed from 3 to 4, the expected change in z_p^* is 1/2, and this is in effect the change, a flow of 2 along the path from s_1 to t_1, a flow of 2 along the path from s_3 to t_3, and a flow of 1 along the path from s_2 to t_2, z_p^* is now equal to 5. However, if the capacity of the arc a_3 is changed from 3 to 5, the actual change of the objective function is still 1/2.

Assume the directed arc $a_5 = (s_1, x)$ is added to the network shown in Figure 1. If the capacity of a_5 is 2, it is easy to see that the optimal solution z_p^* increases by one; a flow of 2 from s_1 to t_1 and through a_3 and a_5, a flow of 1/2 from s_1 to t_1 using the remaining unique path from s_1 to t_1, a flow of 1/2 from s_3 to t_3, and a flow of 5/2 from s_2 to t_2, z_p^* is then equal to 11/2.

If the capacity of a_5 is now 3, it is easy to see that the optimal solution z_p^* increases by 3/2.

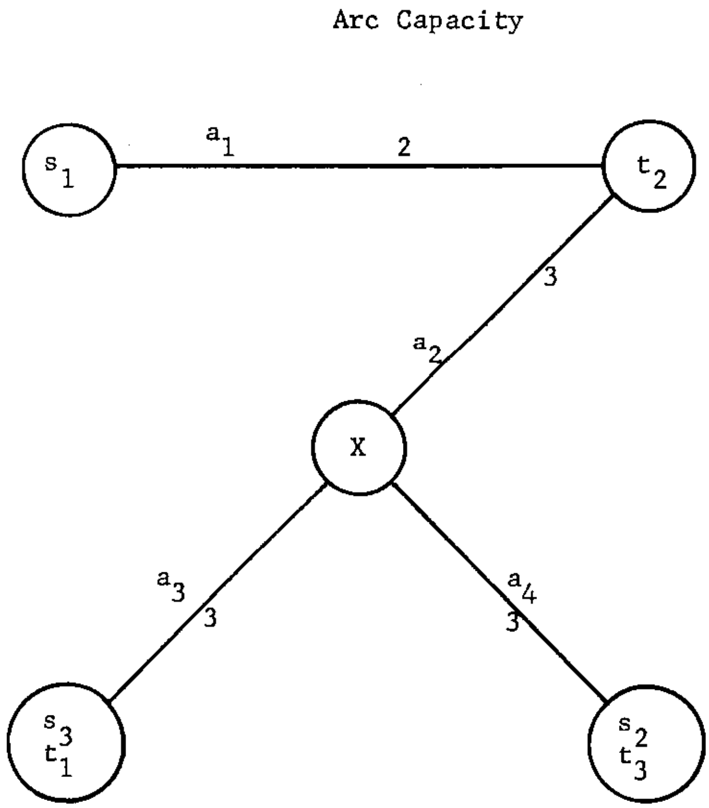

Figure 1. An Example of Three-Commodity Flow

In the two preceding cases, the increment of the objective function was 1/2 times the increment from 0 of the capacity of the arc a_5, it seems reasonable to expect that the dual variable corresponding to a_5 is 1/2, because it indicates the expected change in the objective function.

Suppose now that the directed arc $a_6 = (t_2, s_2)$ is added to the network shown in Figure 1. When this arc is added to the network no increment of the objective function can be obtained.

If the directed arc $a_7 = (s_3, t_3)$ is added to the network shown in Figure 1, the change in the objective function is equal to the change of the capacity of a_7 from 0 to b_7, and the dual variable corresponding to a_7 would be equal to one.

If the dual variables of the arcs a_5, a_6, and a_7 are, respectively, 1/2, 0, and 1, we will say that a_7 is the best of these three arcs, because when this arc is added to the network the expected change in the objective function is the largest one.

In this thesis, two efficient methods for finding the dual variables associated with a new set of arcs for a given network are presented. We also present a method for adding the best set of arcs to a given network.

CHAPTER II

ON MULTICOMMODITY NETWORK FLOWS

In this chapter several theorems and lemmas, relative to the underlying structure of the problem, are presented and evaluated.

To attack the problem of adding arcs to a network, three principal aspects are studied in this chapter; these are:

i. Bounds in the objective function of the multicommodity maximum flow problem.

ii. Properties of the dual variables.

iii. Relationship between the maximum flow and multicommodity disconnecting set.

If we add an arc, a_{m+1}, with capacity b_{m+1}, to a network, the new optimal solution z_1^* has an upper and lower bound. Let us show these bounds in the following theorem:

Theorem (2.1): If a new arc, a_{m+1}, with capacity b_{m+1}, is added to a network, G(N,E), with optimal flow, Z_0^*, then the optimal solution to $G(N,E + a_{m+1})$, Z_1^*, will be:

$$Z_0^* \leqq Z_1^* \leqq Z_0^* + b_{m+1}$$

Proof:

(i) $Z_0^* \leqq Z_1^*$ is obvious since any solution to G(N,E) is the solution to $G(N,E + a_{m+1})$.

(ii) $Z_1^* \leqq Z_0^* + b_{m+1}$.

Suppose $Z_1^* > Z_0^* + b_{m+1}$

C_1: the set of all chains passing through a_{m+1}.

C_2: the set of chains not passing through a_{m+1}.

$f(C_1)$: the sum of the flow chains of the optimal solution for $G(N,E + a_{m+1})$ for chains in C_1.

$f(C_2)$: the sum of the flow chains of the optimal solution for $G(N,E + a_{m+1})$ for chains in C_2.

Then:

$$f(C_1) + f(C_2) = Z_1^* > Z_0^* + b_{m+1}$$

now

$$f(C_1) \leqq b_{m+1}$$

and

$$f(C_1) + f(C_2) > Z_0^* + b_{m+1}$$

therefore

$$f(C_2) > Z_0^*$$

But this contradicts the fact that Z_0^* is an optimal solution of the original network. We therefore conclude that

$$Z_1^* \leqq Z_0^* + b_{m+1}$$

Q.E.D.

Consider the multicommodity flow problem (1.1) in matrix notation:

$$\text{Maximize: } \bar{1}\, x$$

$$\text{Subject to: } Ax \leqq b$$

$$x \geqq 0$$

where:

$\bar{1}$ is the sum vector

$\bar{1} = (1,\ldots 1)$

and $\quad A = \|a_{ij}\| \quad a_{ij}$ equals zero or one.

The dual problem associated with (1.1) is:

$$\text{Minimize: } \pi\, b$$

$$\text{Subject to: } \pi\, A \geqq \bar{1}$$

$$\pi \geqq 0$$

Theorem (2.2): The value of the dual variables, in the optimal solution, cannot be greater than one.

Proof:

Assume $\pi^* = (\pi_1^* \ldots \pi_m^*)$ is an optimal dual solution with some π_k^* greater than one. Let $P_k = \{p_j^k\}$ be the set of all chains passing through the arc a_k.

Then:

$$\pi^* p_j^k > 1 \qquad \forall\ p_j \in P_k$$

Let us choose another solution to the dual problem, $\pi' = (\pi_1' \ldots \pi_m')$ where

$$\pi_i' = \begin{cases} 1 & i = k \\ \pi_i^* & \text{otherwise} \end{cases}$$

then

$$\pi'\ p_j^k \geqq 1 \qquad \forall\ p_j \in P_k$$

and

$$\pi' b < \pi^* b$$

therefore π' is a feasible and better solution to the dual problem than

π^*. But this is a contradiction. We therefore conclude that

$$\pi_i^* \leq 1 \qquad \forall\, i,\ i = 1 \ldots m$$

Q.E.D.

Definition

We will say that a disconnecting set, C, cuts a chain, p_j, n times when n arcs of p_j belong to C.

Theorem (2.3): If the maximal flow, Z*, to (1.1) equals the capacity of minimum cut, C_0, any cut C_1 that cuts some chain carrying flow more than once cannot be minimal.

Proof:

Assume that:

(i) p_k is cut two times by C_1

(ii) f_k, the flow in chain p_k, is greater than zero

(iii) u_{ij} is the flow in arc (N_i, N_j)

Now:

$$u_{ij} \leq b(N_i,N_j) \qquad \forall (N_i,N_j) \in E$$

then:

$$u_{ij} \leq b(N_i,N_j) \qquad \forall (N_i,N_j) \in C_1$$

$$\sum_{(N_i,N_j) \in C_1} u_{ij} \leq \sum_{(N_i,N_j) \in C_1} b(N_i,N_j)$$

but

$$\sum_{(N_i,N_j) \in C_1} u_{ij} = Z^* + f_k$$

and

$$Z^* + f_k > \sum_{(N_i,N_j) \in C_0} b(N_i,N_j)$$

since

$$Z^* = \sum_{(N_i,N_j)\ \varepsilon C_0} b(N_i,N_j) \qquad \text{and } f_k > 0$$

therefore:

$$\sum_{(N_i,N_j)\ \varepsilon C_0} b(N_i,N_j) < Z^* + f_k \leqq \sum_{(N_i,N_j)\ \varepsilon C_1} b(N_i,N_j)$$

$$\sum_{(N_i,N_j)\ \varepsilon C_0} b(N_i,N_j) < \sum_{(N_i,N_j)\ \varepsilon C_1} b(N_i,N_j)$$

C_1 is no minimal.

Q.E.D.

Consider the network shown in Figure 2. The maximum flow $z_p^* = 14$, and $C_0 = \{a_1, a_3, a_4, a_7\}$ is a disconnecting set. Now $b(C_0) = 14$, then C_0 is a minimum disconnecting set. Therefore, in this network the maximum flow equals the capacity of the minimum cut.

We can observe that every chain carrying flow has only one of its arcs in C_0.

Consider the disconnecting set $C_1 = \{a_1, a_3, a_5, a_6, a_7\}$, $b(C_1) = 18$, C_1 is not minimal.

The carrying flow chain (s_2, s_3, t_2) is cut two times by C_1.

Corollary (2.1): If the maximum flow to (1.1) equals the capacity of the minimum cut, there is no path carrying flow cut more than once by any minimum cut.

The proof is obvious from theorem (2.3).

Theorem (2.4): If the maximal flow to (1.1) equals the capacity of the minimum cut, there exists an integer, 0-1, optimal solution to the dual problem, 1's at the arcs of the minimum cut and 0's otherwise.

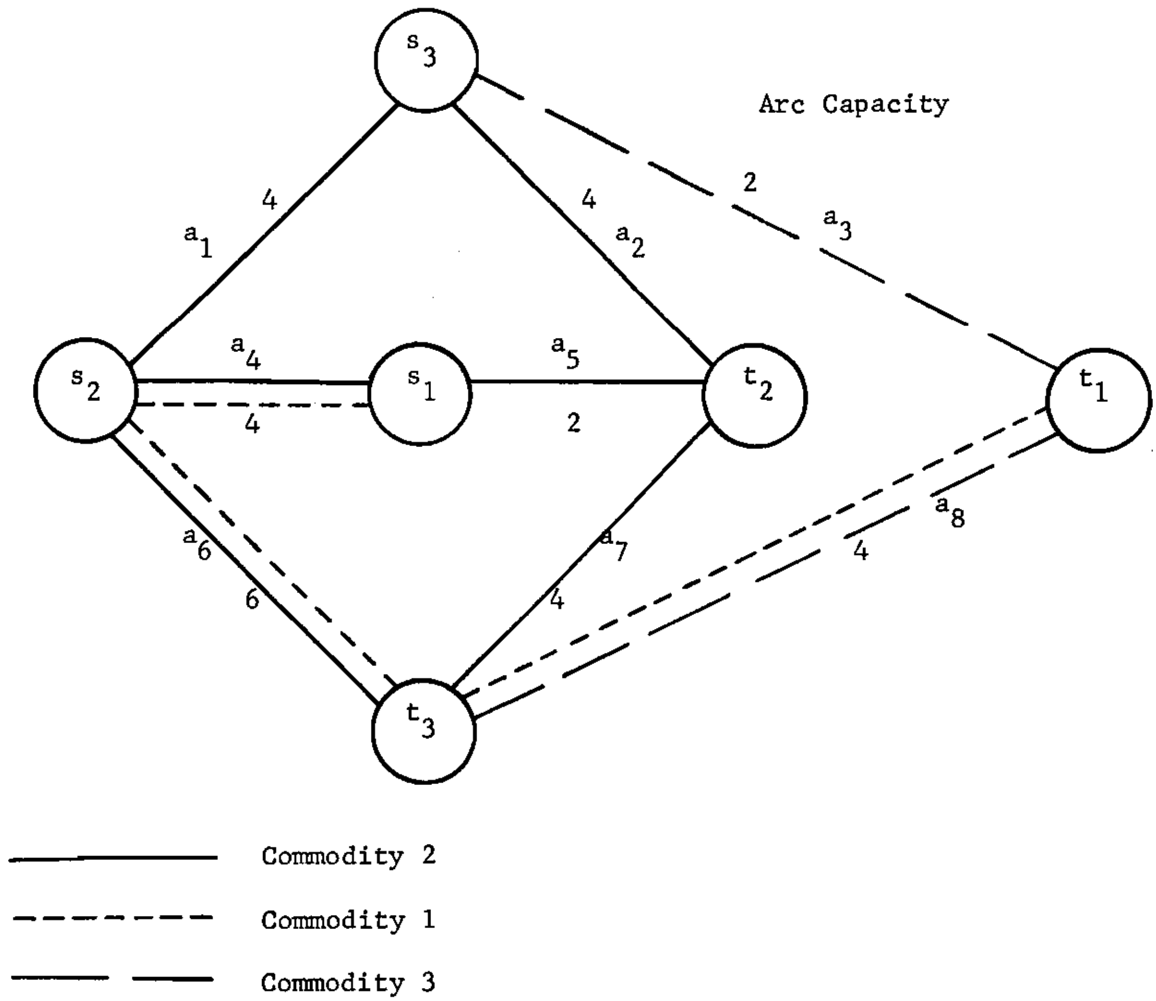

Figure 2. A Three-Commodity Network, where the Maximum Flow Equals the Capacity of the Minimum Disconnecting Set

Proof:

Assume $\pi = (\pi_c, \gamma)$ are the dual variables, π_c the dual variables associated with arcs of the minimum cut and γ otherwise.

$$\pi_c = \bar{1}, \qquad \gamma = 0$$

(i) $\pi = (\pi_c, \gamma)$ is a feasible solution, since $\pi p_j \geqq 1 \quad \forall\, p_j$ because any path p_j is cut by the minimal disconnecting set.

(ii) The solution is minimal, since the dual objective function Z_D is:

$$Z_D = \pi b \qquad b = (b_c, b_\gamma)$$

$$= (\pi_c, \gamma)(b_c, b_\gamma)$$

$$= \pi_c b_c$$

$$= \bar{1}\, b_c = \sum_{(N_i,N_j)\, \varepsilon C_0} b(N_i,N_j)$$

by hypothesis:

$$\sum_{(N_i,N_j)\, \varepsilon C_0} b(N_i,N_j) = Z_p^*$$

therefore, Z_D is optimal.

Q.E.D.

The arc-chain formulation for the minimum disconnecting set (see Bellmore, Greenberg, and Jarvis, p. 431) is:

$$\text{Minimize:} \quad \sum_{i-1}^{m} b_i d_i$$

$$\text{Subject to:} \quad \sum_{i=1}^{m} d_i a_{ij} \geqq 1 \qquad j = 1\ldots n \tag{2.1}$$

$$d_i = 0,1 \qquad i = 1\ldots m$$

if $\quad d_i = 1 \rightarrow a_i \in C_0$

if $\quad d_i = 0 \rightarrow a_i \notin C_0$

The set of constraints say that, for any given chain, at least one arc must belong to the disconnecting set.

The linear programming problem corresponding to (2.1) becomes:

$$\text{Minimize:} \quad \sum_{i=1}^{m} b_i \pi_i$$

$$\text{Subject to:} \quad \sum_{i=1}^{m} \pi_i a_{ij} \geqq 1 \qquad j = 1 \ldots n \tag{2.2}$$

$$\pi_i \geqq 0 \qquad i = 1 \ldots m$$

Equation (2.2) is exactly the dual problem of (1.1) and by theorem (2.2) the π's cannot be greater than one. Therefore, if (2.2) has an integer solution it has to be a 0-1 integer solution.

Theorem (2.5): If the solution to the dual problem (2.2) is an integer solution, then the maximum flow equals the minimum cut.

Proof:

If the best feasible solution of a linear programming problem is an integer solution, it must be the best feasible solution to the corresponding integer linear programming problem.

Therefore

$$\pi^* = d^*$$

$$\pi_i^* = d_i^* = \begin{cases} 1 & a_i \in C_0 \\ 0 & \text{otherwise} \end{cases}$$

$$Z_D^* = \pi^* b = \sum_{(N_i, N_j) \in C_0} b(N_i, N_j)$$

but the optimal solution to the dual, Z_D^*, has to be equal to the optimal solution to the primal. If Z_p^* is the maximum flow

$$Z_P^* = Z_D^*$$

Q.E.D.

By theorems (2.4) and (2.5), we can establish the following theorem:

Theorem (2.6): The maximum flow of problem (1.1) is equal to the capacity of the minimum cut, if and only if, the solution to the dual problem (2.2) is an integer solution.

Definition

An arc is called a saturated arc, when the flow passing through this arc is equal to its capacity.

Corollary (2.2): If the maximum flow to (1.1) is equal to the capacity of the minimum cut, the arcs of this minimal cut have to be saturated arcs.

Proof:

Let $C_0 = \{a_1 \ldots a_r\}$ be the minimal cut.

Suppose all arcs in C_0 are saturated arcs, except a_r, in the optimal solution

C_1: The set of chains using a_r in the optimal solution.

C_2: The set of chains not using a_r in the optimal solution.

f_i^*: The amount of flow in i^{th} arc in the optimal solution.

Then

$$b(a_i) = f_i^* \qquad i = 1 \ldots r-1$$

$$b(a_r) > f_r^*$$

therefore

$$\sum_{a_i \in C_0} b(a_i) > \sum_{a_i \in C_0} f_i^*$$

but

$$\sum_{a_i \varepsilon C_0} f_i^* \geqq Z_P^* \qquad \text{(maximal flow)}$$

then

$$\sum_{a_i \varepsilon C_0} b(a_i) > Z_P^*$$

But this is a contradiction. We therefore conclude that

$$b(a_i) = f_i^* \quad \forall\, i \qquad i = 1 \ldots r.$$

Q.E.D.

The converse to corollary (2.2) is not true. That is, if the arcs of the minimal cut are saturated arcs, the maximum flow is not always equal to the capacity of the minimal cut. Looking at the counter example of Figure 3, the maximum flow is equal to three, and the minimal cut is equal to four. The arcs of the minimum cut are saturated arcs.

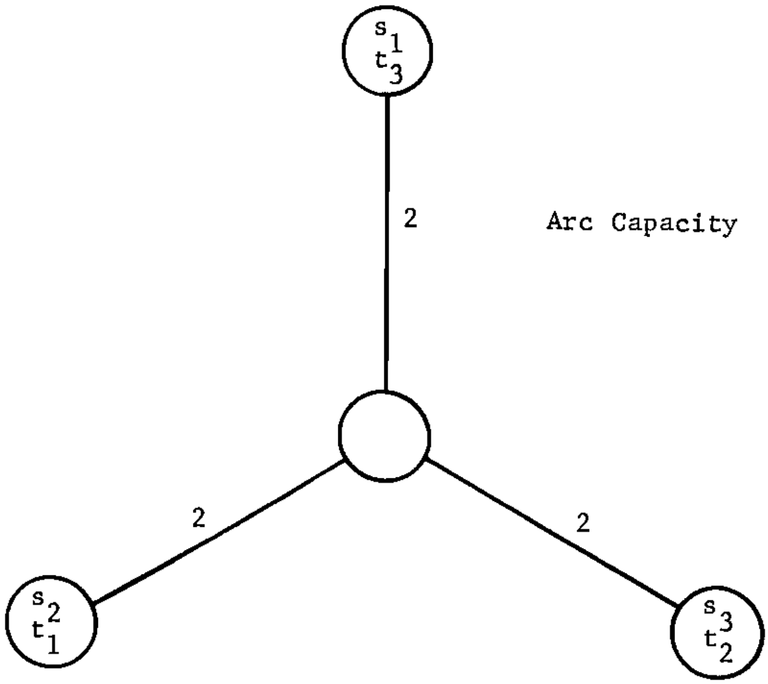

Figure 3. A Three-Commodity Counterexample

CHAPTER III

ADDING ARCS TO A MULTICOMMODITY NETWORK

In this chapter, a method is presented for determining which arc of a set of arcs must be put in a given network to obtain the best improvement of the solution. Two other methods are developed for finding the dual variables associated with a new set of arcs when added to a given network.

The multicommodity flow problem formulated in (1.1) is solved in (4). We will repeat here some of the most important aspects for ease in developing and justifying the methods.

The multicommodity aspect does not appear explicitly in (1.1), but it is contained in the structure of the matrix $A = \|a_{ij}\|$. Assuming that we have m columns which form a starting basis of (1.1)(we can start with the slack variables as the basic variables), we can solve (1.1) and get the price vector $\pi = (\pi_1 \ldots \pi_m)$, where each π_i corresponds to a specific row. The relative cost of every nonbasic column p_j is given by $\bar{c}_j = c_j - \pi p_j$. If $\bar{c}_j \leqq 0$, then the current basis is optimal. If $\bar{c}_j > 0$, then that column should be brought into the basis. Now the problem is to find $\bar{c}_j$. If we interpret π_i as the lengths of the arcs, then πp_j is the length of the chain which is represented by the column p_j. Note that c_j, the cost associated with the variable x_j, is equal to 1 for all x_j, then the problem reduces to find the length of the chains.

Using the revised simplex method, π will appear in the cost row of the slack variables. Therefore we do not need to list all the columns representing chains leading from different sources to different sinks. At each stage of the computation we use the revised simplex method and keep a martix of size $(m + 1) \times (m + 1)$. If some π_i is negative, then we choose the corresponding slack column as a pivot column. If all π_i are non-negative, we consider π_i as the lengths of the arcs and find the shortest chain leading from the source to the sink for each commodity. If the shortest chain of every commodity is of length one or more, it implies that $\bar{c}_j \leqq 0$ for all j, and the current basis is optimum. Each column should be updated before adding to the tableau; that is, the vector entering the basis has to be expressed in terms of the current basis, for doing the pivot operations. Therefore given an optimal solution to (1.1)

$$\pi p_j \geqq 1 \qquad j = 1 \ldots n \tag{3.1}$$

Note that using the revised simplex method we have:

$$\begin{bmatrix} 1 & \pi \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} -1 \\ p_j \end{bmatrix} = \begin{bmatrix} -(1-\pi p_j) \\ B^{-1} p_j \end{bmatrix} = \begin{bmatrix} -\bar{c}_j \\ \bar{p}_j \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & \pi \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} \pi b \\ B^{-1} b \end{bmatrix} = \begin{bmatrix} Z \\ \bar{b} \end{bmatrix}$$

Let us call D the optimal basis,

$$D = \begin{bmatrix} 1 & -c_B \\ 0 & B \end{bmatrix} \qquad \text{then} \qquad D^{-1} = \begin{bmatrix} 1 & \pi \\ 0 & B^{-1} \end{bmatrix}$$

Lemma (3.1): Given an optimal solution for (1.1) and a new arc a_{m+1}, a feasible basis and its inverse can be obtained from D and D^{-1}, respectively, by utilizing the slack column associated with the new arc.

Proof:

Construct D_1 as follows:

$$D_1 = \begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix} \quad \text{then} \quad D_1^{-1} = \begin{bmatrix} D^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

We will show that D_1 is a feasible basis since:

$$D_1^{-1}\ b_1 = \begin{bmatrix} D^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ b \\ b_{m+1} \end{bmatrix} = \begin{bmatrix} 1 & \pi & 0 \\ 0 & B^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ b \\ b_{m+1} \end{bmatrix}$$

$$D_1^{-1}\ b_1 = \begin{bmatrix} Z \\ \overline{b} \\ b_{m+1} \end{bmatrix}$$

and $\overline{b} \geqq 0$ and $b_{m+1} = 0$.

Q.E.D.

The matrix D_1^{-1} indicates that the dual variable, at this stage of the computation, associated with the new arc a_{m+1} is zero.

The generalization of lemma (3.1) is obvious; we will only say, when r arcs, $a_{m+1} \ldots a_{m+r}$, are added to the original network, which optimal solution is known,

D_r^{-1} has the form:

$$D_r^{-1} = \begin{bmatrix} D^{-1} & 0 \\ 0 & I_r \end{bmatrix}$$

The slack variables at these new arcs will be the new basic variables and the dual variable associated with each new arc is zero.

Definition

When an optimal solution for (1.1) is obtained, the dual variables indicate the potential rate of change of the objective function as the capacity of the arcs varies. We are interested in getting a similar measure for arcs not in the network.

Assume that we have a set of arcs, A, candidate to put in the network. As these arcs are not in the network, we will say that the capacity of the new arcs is equal to zero.

Suppose we put a specific arc, a_k, in the network, with capacity $b(a_k) = 0$, and get the optimal solution. The dual variable, π_k, associated with a_k indicates the expected rate of change of the solution per unit of change of the capacity $b(a_k)$. The dual variable π_k measures the desirability of the arc a_k.

We say that a_h is the best arc if

$$\pi_h = \max_{a_j \in A} (\pi_j)$$

If we put a_h in the network, this arc gives us the best potential improvement of the solution.

Theorem (3.1): Given an optimal solution to the multicommodity flow problem (1.1), and a new arc, a_{m+1}, with $b(a_{m+1}) = 0$, the optimal dual variable π^*_{m+1} is equal to:

$$\pi^*_{m+1} = \begin{cases} 1 - \ell_k & \ell_k < 1 \\ 0 & \text{otherwise} \end{cases}$$

where ℓ_k is the length of the shortest path in $G_1(N_1, E_1)$, $N_1 = N$,

$E_1 = E + a_{m+1}$ and $b(a_{m+1}) = 0$.

Proof:

By lemma (3.1):

$$D_1^{-1} = \begin{array}{c} \\ \\ s_{m+1} \end{array}\left[\begin{array}{cc|c} 1 & \pi & 0 \\ 0 & B^{-1} & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \qquad b_1 = \left[\begin{array}{c} 0 \\ b \\ \hline b_{m+1} \end{array}\right]$$

Let $H = \{h_j\}$ be the set of new chains created by the new arc, a_{m+1}

$$h_j = [h_{1j} \ldots h_{mj}, 1]$$

Note: One is in the m+1 position of all h_j because we place the new arc in the m+1 position.

$$\ell_j = (\pi, 0)\, h_j$$
$$\bar{c}_j = 1 - \ell_j$$

(i) If $\ell_j \geqq 1, \forall_j \rightarrow \bar{c}_j \leqq 0 \ \forall_j$, the solution is dual feasible and by lemma (3.1) it is primal feasible; therefore, the solution is optimal and $\pi^*_{m+1} = 0$.

(ii) $\ell_j < 1$ for some j

say $\ell_k = \min\limits_j (\ell_j)$

then $\bar{c}_k = 1 - \ell_k > 0$

$$\bar{c}_k = \max_j (\bar{c}_j)$$

therefore, h_k is a candidate for entering the basis;

$$\bar{h}_k = \begin{bmatrix} B^{-1} & 0 \\ 0 & 1 \end{bmatrix} h_k$$

$$\bar{h}_k = [\bar{h}_{1k} \ldots \bar{h}_{mk}, 1]$$

$$\min \frac{\bar{b}_i}{\bar{h}_{ki} > 0} = 0 \qquad i = m+1, \text{ since } b(a_{m+1}) = 0$$

therefore, s_{m+1} leaves the basis, and

$$D_r^{-1} = \begin{array}{c} \\ \\ h_k \end{array}\begin{bmatrix} 1 & \pi & \bar{c}_k \\ & & -h_{1k} \\ 0 & B^{-1} & \\ & & -\bar{h}_{mk} \\ 0 & 0 & 1 \end{bmatrix} \qquad \bar{b}_1 = \begin{bmatrix} Z' \\ \bar{b}_1 \\ \bar{b}_m \\ 0 \end{bmatrix}$$

the solution is optimal because it is primal feasible since the value of variables remains the same.

It is dual feasible since: the dual variables π_i remain the same for i = 1...m and $\pi_{m+1} = \bar{c}_k = 1 - \ell_k$ then $\pi^* = (\pi, \bar{c}_k)$.

Consider:

$P = \{p_j\}$ the set of chains not passing through a_{m+1}

$H = \{h_j\}$ the set of chains passing through a_{m+1}

$$\bar{c}'_j = 1 - \pi^* p_j \leqq 0 \;\forall\; p_j \in P$$

$$\bar{c}'_j = 1 - \pi^* h_j = 1 - (\pi, \bar{c}_k)[h_{1j} \ldots h_{mj}, 1]$$

$$= 1 - \ell_j - \bar{c}_k$$

$$= \bar{c}_j - \bar{c}_k$$

$$\bar{c}'_j \leqq 0$$

Q.E.D.

Selecting the Best Potential Arc

Now, if we want to choose the best arc from a set, A, of arcs, we find the optimal dual variable associated with each arc and pick the largest. In this problem we are finding a set of shortest paths connecting the different pairs of nodes indicated by the arcs in the set A, and leading from a source to its corresponding sink.

There exists a fast and easy method to find the shortest length between any pair of nodes in a network. This is the Floyd Algorithm. The shortest length between any pair of nodes is given in matrix form, Floyd's matrix. Floyd's algorithm is given in Appendix A.

Suppose a_k is a new directed arc added to a given network G(N,E)

$$a_k \in A \qquad a_k = (N_p, N_q) \qquad \text{then}$$

$$\ell_k = \min_j (\ell s_j p + \ell q t_j)$$

where:

ℓ_k: length of the shortest path, using a_k, from the source, s_j, to its corresponding sink, t_j

$\ell s_j p$: length of the shortest path from the source s_j to the node N_p

$\ell q t_j$: length of the shortest path from the node N_q to the sink t_j.

A method for adding the best arc to a given network can be described now:

Step 1. Calculate Floyd's matrix.

Step 2. a) Calculate the shortest length ℓ_i when a_i is in the network and no other new arc is in the network, for all $a_i \in A$.

b) Calculate the dual variable π_i associated with a_i:

$$\pi_i = \begin{cases} 1 - \ell_i & \ell_i < 1 \\ 0 & \text{otherwise} \end{cases}$$

Step 3. Select the best arc, a_k^*, such

$$\pi_k^* = \max_i (\pi_i)$$

EXAMPLE I

Consider the network shown in Figure 4. There are three new directed arcs: a_1 : $(s_2, 3)$, a_2 : $(1,4)$, a_3 $(4,t_1)$. Choose the best arc for adding to the network.

Step 1. Calculate the Floyd matrix (see Table 1).

Table 1. Floyd Matrix for Example I

	s_1	s_2	1	2	3	4	5	6	t_1	t_2
s_1	0	.5	.4	.3	.4	.6	.5	.6	1.0	1.1
s_2	.5	0	.3	.2	.3	.5	.4	.5	.9	1.0
1	.4	.3	0	.1	.2	.2	.1	.2	.6	.7
2	.3	.2	.1	0	.1	.3	.2	.3	.7	.8
3	.4	.3	.2	.1	0	.2	.1	.2	.6	.7
4	.6	.5	.2	.3	.2	0	.1	.2	.6	.7
5	.5	.4	.1	.2	.1	.1	0	.1	.5	.6
6	.6	.5	.2	.3	.2	.2	.1	0	.6	.5
t_1	1.0	.9	.6	.7	.6	.6	.5	.6	0	1.1
t_2	1.1	1.0	.7	.8	.7	.7	.6	.5	1.1	0

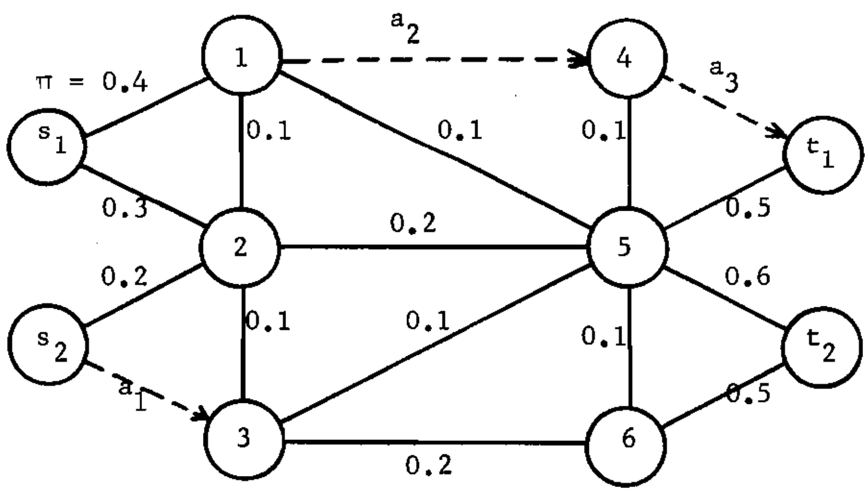

Figure 4. Example I Network

Step 2.

$$\ell_1 = \min\ (0.5 + 0.6,\ 0 + 0.7)$$

$$\ell_1 = 0.7 \qquad \rightarrow \pi_1 = 0.3$$

$$\ell_2 = \min\ (0.4 + 0.6,\ 0.3 + 0.7)$$

$$\ell_2 = 1.0 \qquad \rightarrow \pi_2 = 0$$

$$\ell_3 = \min\ (0.6 + 0,\ 0.5 + 1.1)$$

$$\ell_3 = 0.6 \qquad \rightarrow \pi_3 = 0.4$$

Step 3.

$$\pi^* = \max\ (0.3,\ 0,\ 0.4)$$

$$\pi^* = 0.4$$

Arc a_3 is the best potential arc.

Dual Variables for Non-present Arcs

Given an optimal solution to the multicommodity flow problem (1.1) and a set of arcs, $a_{m+1} \ldots a_{m+k}$, added to the network, G(N,E), the optimal dual variables associated with each arc, π^*_{m+j}, can be obtained, of course, using the revised simplex algorithm. By the generalization of lemma (3.1), a feasible basis, D, and its inverse, D_k^{-1}, exist:

$$D_k^{-1} = \begin{bmatrix} 1 & \pi & 0 \\ 0 & B^{-1} & 0 \\ 0 & 0 & I_k \end{bmatrix}$$

As before, we are assuming the arcs have zero capacity; therefore, at any state of the computation in the revised simplex algorithm we can do the pivot operation by any of the last k rows, provided that the entering variable has the corresponding component greater than zero. The π's associated with the old arcs do not change. Then, the revised simplex algorithm deals with the last k rows and the last k columns, finding a basic set of paths until the length of the shortest path is not less than one. Now the length of those basic paths has to be equal to one.

Given the characteristics of the problem, we can describe a method for finding the optimal dual variables associated with a set of new arcs, $a_1 \ldots a_k$; added to a given network.

METHOD I

Start with the slack variables, $s_1 \ldots s_k$, and $\pi_1 = \ldots \pi_k = 0$

Step 1. Find the shortest path, p, and its length, ℓ, in $G_1 = (N_1, E_1)$

$$N_1 = N$$

$$E_1 = E + a_1 + \ldots + a_k$$

if $\ell \geqq 1$ terminate

otherwise $\bar{\ell} = 1 - \ell$

choose j such $a_j \in p$

Step 2. $p_j = p$ (p_j = basic path)

$\pi_j = \pi_j + \bar{\ell}$

Step 3. Calculate the length of the basic paths, $\ell(p_i)$ if

$\ell(p_i) = 1 \ \forall\ i \quad i = 1 \ldots k$ go to step 1

otherwise go to step 4.

Step 4. $\pi_i = \pi_i + (1 - \ell(p_i)) \ \forall\ i \quad i = 1 \ldots k$

and go to step 3.

EXAMPLE II

Consider the network shown in Figure 5. There are three new arcs

$$a_1 : (s_2, 3),\ a_2 : (1,4),\ a_3 : (4, t_1)$$

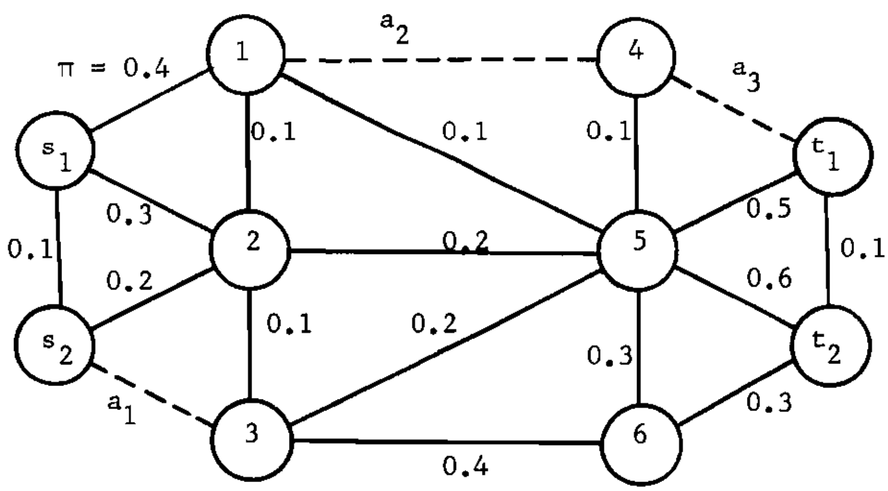

Figure 5. Example II and Example III Network

Find the optimal dual variables associated with these new arcs.

METHOD I

Step 1. $p = (s_1, s_2), (s_2, 3), (3,2), (2,1), (1,4), (4, t_1)$

$\ell = 0.3 \qquad \overline{\ell} = 1.0 - 0.3 = 0.7$

$a_1 \varepsilon p$

Step 2. $p_1 = p$

$\pi_1 = 0 + 0.7$

$\pi_1 = 0.7$

Step 3. $\ell(p_1) = 1.0$

Step 1. $p : (s_1, 1), (1,4), (4, t_1)$

$\ell = 0.4 \qquad \overline{\ell} = 1.0 - 0.4 = 0.6$

$a_3 \varepsilon p$

Step 2. $p_3 = p$

$\pi_3 = 0 + 0.6$

$\pi_3 = 0.6$

Step 3. $\ell(p_1) = 0.1 + 0.7 + 0.1 + 0.1 + 0 + 0.6$

$\ell(p_1) = 1.6$

$\ell(p_3) = 0.4 + 0 + 0.6$

$\ell(p_3) = 1.0$

Step 4. $\pi_1 = 0.7 + (1.0 - 1.6)$

$\pi_1 = 0.1$

Step 3. $\ell(p_1) = 0.1 + 0.1 + 0.1 + 0.1 + 0 + 0.6$

$\ell(p_1) = 1.0$

$\ell)p_3) = 1.0$

Step 1. $p : (s_2, 3), (3,6), (6, t_2)$

$\ell = 0.8 \qquad \bar{\ell} = 1.0 - 0.8 = 0.2$

$a_1 \in p$

Step 2. $p_1 = p$

$\pi_1 = 0.1 + 0.2$

$\pi_1 = 0.3$

Step 3. $\ell(p_1) = 0.3 + 0.4 + 0.3$

$\ell(p_1) = 1.0$

$\ell(p_3) = 0.4 + 0 + 0.6$

$\ell(p_3) = 1.0$

Step 1. $p : (s_1, 1), (1,5), (5, t_1)$

$\ell = 1.0$ terminate

$\pi_1^* = 0.3, \pi_2^* = 0, \pi_3^* = 0.6$

Lemma (3.2): Given an optimal solution to the multicommodity flow problem (1.1) and a set of arcs, $a_{m+1} \ldots a_{m+k}$, added to the network, $G(N,E)$, the optimal dual variables, π^*_{m+j}, associated with each arc are equal to:

$$\pi^*_{m+j} = \begin{cases} 1 - \ell_j & \ell_j < 1 \\ 0 & \text{otherwise} \end{cases}$$

where ℓ_j is the length of the shortest path in $G_j(N, E_j)$, $E_j = E + a_{m+1} \ldots + a_{m+j}$ and the lengths of the arcs, $a_{m+1}, \ldots, a_{m+j-1}, a_{m+j}$ are $\pi^*_{m+1} \ldots \pi^*_{m+j-1}$, 0, respectively.

Proof:

The proof follows from theorem (3.1). After a_{m+1} has been introduced, the solution obtained is an optimal solution, with the dual variable

π^*_{m+1} at the arc a_{m+1}.

Therefore we can again by theorem (3.1) introduce the arc a_{m+2}, find ℓ_2 and π^*_{m+2}, and so forth until all arcs have been introduced.

METHOD II

Another method for finding the optimal dual variables associated with a set of arcs, $a_1 \ldots a_k$, added to a given network can be described now:

Let $j = 0$

Step 1. $j = j + 1$

if $j = k + 1$ terminate.

put a_j in $G(N,E)$

$\pi_j = 0$

Calculate the length of the shortest path, ℓ.

Step 2. if $\ell \geqq 1$ go to step 1

otherwise $\bar{\ell} = 1 - \ell$

$\pi_j = \pi_j + \bar{\ell}$

go to step 1.

EXAMPLE III

Consider the same network shown in Figure 5. There are three new arcs $a_1 : (s_2, 3)$, $a_2 : (1,4)$, $a_3 : (4, t_1)$.

Find the optimal dual variables associated with these new arcs.

METHOD II

Step 1. $j = 1$

a_1 in G, $\pi_1 = 0$

shortest length, $\ell = 0.7$

Step 2. $\bar{\ell} = 1.0 - 0.7 = 0.3$

$\pi_1 = 0.3$

Step 1. $j = 2$

a_2 in G_1, $\pi_2 = 0$

shortest length, $\ell = 1.0$

Step 2. $\ell = 1.0$

Step 1. $j = 3$

a_3 in G_2, $\pi_3 = 0$

shortest length, $\ell = 0.4$

Step 2. $\ell = 1.0 - 0.4 = 0.6$

$\pi_3 = 0.6$

Step 1. $j = 4$

$j > 3$ terminate

$\pi_1^* = 0.3$, $\pi_2^* = 0$, $\pi_3^* = 0.6$

Method II is easier and faster than Method I. We know at each stage of the computation of Method II the definitive values of the dual variables associated with the arcs which have been introduced; when an arc has been introduced and its dual variables have been found, this arc becomes an old arc for the next iteration. We do not have to worry any more about this arc in the entire computation using Method II. When Method I is used, the task for finding the shortest path is more laborious than in Method II because all arcs are present in the network from the beginning of the computation. The values of the dual variables may be changed at any iteration and their definitive values are known only at the end of the computation.

CHAPTER IV

SELECTING THE BEST SET OF ARCS

In this chapter a method, using the branch and bound technique, for selecting the best set of k arcs from a set of r arcs is developed.

Let us consider the following lemma.

Lemma (4.1): Given an optimal solution for the multicommodity flow problem (1.1) and a new set of arcs added to the network, the minimum value of the optimal dual variable, associated with every new arc a_j, is the complement to one of the length of the shortest path in the network, when a_j is the only new arc in the network.

Assume ℓ'_j is the length of the shortest path, when a_j is in the network, and π^*_j is the optimal dual variable associated with a_j, when all arcs are in the network, then

$$\pi^*_j \geqq \max\,(0,\ 1-\ell'_j)$$

Proof:

If $\quad \ell'_j > 1$

$\quad \pi^*_j \geqq 0$

which is true in any case, if $\ell'_j < 1$;

suppose $\quad \pi^*_j < 1 - \ell'_j$

or $\quad \ell'_j + \pi^*_j < 1$

this implies that, in the optimal solution, there exists a chain with length less than one. But this is a contradiction. We therefore conclude that

$$\pi_j^* \geqq 1 - \ell_j' \qquad \text{when } \ell_j' < 1$$

Corollary (4.1): If the new set of arcs, A, for adding to an optimal multicommodity network is $A = \{a_1 \ldots a_k\}$

then
$$\sum_{i=1}^{k} \pi_i^* \geqq \sum_{i=1}^{k} \bar{\ell}_i'$$

where $\bar{\ell}_i' = 1 - \ell_i'$ and ℓ_i' is the length of the shortest path when a_j is the only new arc in the network.

Proof:

By lemma (4.1)

$$\pi_i^* \geqq \bar{\ell}_i'; \qquad \text{and} \qquad \pi_j^* \geqq 0$$

therefore
$$\sum_{i=1}^{k} \pi_i^* \geqq \sum_{i=1}^{k} \bar{\ell}_i'$$

Corollary (4.2):
$$\sum_{i=1}^{k} \pi_i^* \leqq k$$

Proof:

By theorem (2.2)

$$\pi_i \leqq 1 \ \forall\ i$$

therefore
$$\sum_{i=1}^{k} \pi_i^* \leqq k$$

<u>Definition</u>

Suppose a set of k arcs, with zero capacity, has been added to a given network and the optimal solution for this new network has been obtained. By extension of theorem (2.1), the new optimal solution will be

$$Z' \leqq Z_0^* + \sum_{i=m+1}^{m+k} b_i$$

Let the capacity of each new arc be subject to a small change e, such that the solution be still primal feasible. The π's do not depend on the capacity of the arcs; therefore, the solution is still optimal. The optimal dual variable, π_i, corresponding to the new arc, a_i, gives the rate of change of the objective function as the capacity of the arc a_i changes from zero to the value e. In this case, the new optimal solution will be

$$Z' = \sum_{i=1}^{m+k} \pi_i b_i$$

$$Z' = Z^* + \sum_{i=1}^{k} \pi_i b_i$$

$$Z' = Z^* + e\ (\pi_1 + \ldots + \pi_k)$$

$$\text{Max} \qquad Z' = Z^* + e \max\ (\pi_1 + \ldots + \pi_k)$$

Therefore, we obtain a potential maximum of the objective function when we maximize the sum of the dual variables corresponding to a set of new k arcs. We say that we obtain a potential maximum of the objective function because we are assuming that

(i) The solution remains primal feasible.

(ii) The changes in the capacities are small enough to consider them equal to e.

Given these assumptions, we define as the best set of arcs that set of arcs with the largest sum of their optimal dual variables. We say that the best improvement of the solution is obtained when the best set of arcs is added to the network.

Suppose we want to select from a set of r arcs the best set of k arcs. Based on corollaries (4.1) and (4.2), we can use the branch and bound technique to select the best set of arcs.

Assume that a lower bound on the optimal value of the objective function is available. This usually is the value of the objective function for the best feasible solution identified so far. The first step is to partition the set of all feasible solutions into several subsets, and for each one, an upper bound is found for the value of the objective function of the solutions within that subset. Those subsets whose upper bounds are less than the current lower bound are excluded from further consideration. One of the remaining subsets is then partitioned further into several subsets. Their upper bounds are obtained in turn and used as before to exclude some of these subsets from further consideration. This process is repeated again and again until a feasible solution is found such that the corresponding value of the objective function is not less than the upper bound for any subset.

EXAMPLE IV

Consider Figure 6 and select the best two arcs from the three new arcs, a_1, a_2, a_3.

$$\ell_1' = 0.3 \qquad \bar{\ell}_1' = 0.7$$

$$\ell_2' = 0.9 \qquad \bar{\ell}_2' = 0.1$$

$$\ell_3' = 0.5 \qquad \bar{\ell}_3' = 0.5$$

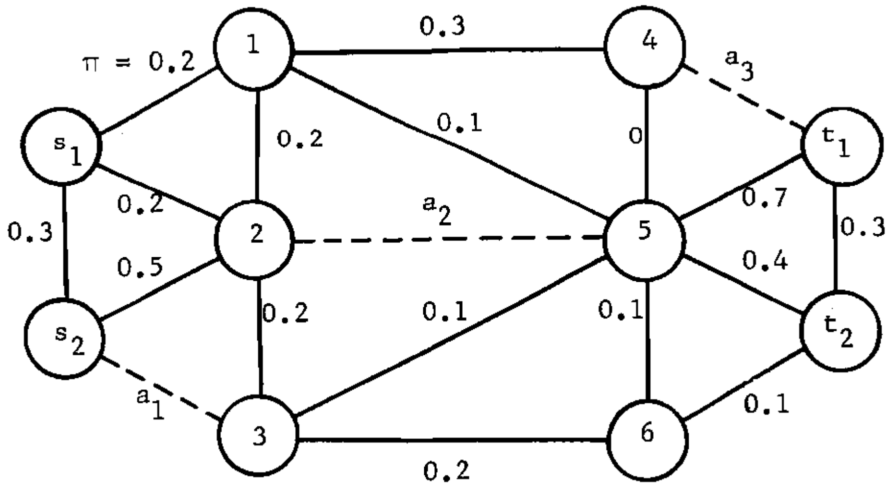

Figure 6. Example IV Network

A lower bound is:

$$\overline{\ell}'_1 + \overline{\ell}'_3 = 0.7 + 0.5 = 1.2$$

At the last step, the current lower and upper bounds are equal, so the feasible solution corresponding to this upper bound is the desired optimal solution. Therefore, the best set of two arcs is $\{a_1, a_3\}$, as indicated by the tree of Figure 7.

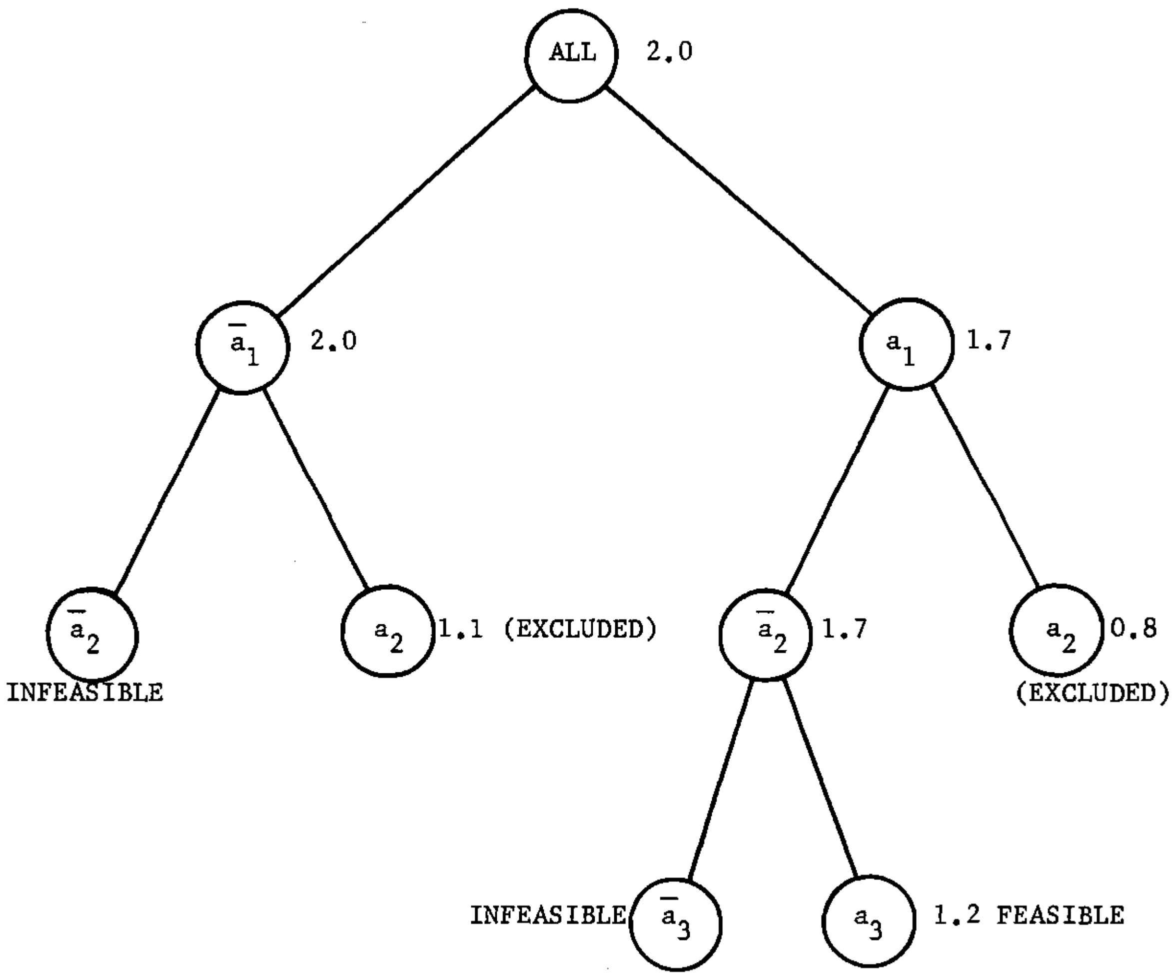

Figure 7. Decision Tree for Selecting the Best Set of Arcs

CHAPTER V

ON THE MIN-COST MULTICOMMODITY FLOW PROBLEM

Preliminary Concepts

In this chapter an extension for finding the optimal dual variables of a set of new arcs will be shown when the problem is the minimum-cost multicommodity flow problem.

Let us consider the minimum-cost multicommodity flow problem (9). The min-cost flow problem may also be formulated by an extension of Ford and Fulkerson's arc-chain formulation for the maximum flow problem (4).

Suppose we have an enumeration of the arcs $a_1 \ldots a_m$. Let $p_1^k \ldots p_{N_k}^k$ be the set of all chains joining source and sink for commodity k. Then we may describe the arc-chain incidence matrix A^k

$$A^k = \|a_{ij}^k\| \qquad \begin{array}{l} i = 1 \ldots m \\ j = 1 \ldots N_k \\ k = 1 \ldots q. \end{array}$$

where

$$a_{ij}^k = \begin{cases} 1 & \text{if } a_i \in p_j^k \\ 0 & \text{otherwise} \end{cases}$$

Let the capacity of arc a_i be b_i and its associated cost c_i, and let x_j^k be the flow of commodity k in chain p_j^k.

The flows must satisfy the capacity constraints and the flow requirement r_k. The cost, C_j^k, associated with any chain is given by

$$\sum_{i=1}^{n} c_i a_{ij}^k$$

The min-cost flow problem can be formulated as follows:

$$\text{Minimize:} \quad \sum_{k=1}^{q} \sum_{j=1}^{N_k} \sum_{i=1}^{m} c_i a_{ij}^k x_j^k$$

$$\text{Subject to:} \quad \text{a)} \quad \sum_{k=1}^{q} \sum_{j=1}^{N_k} a_{ij}^k x_j^k \leqq b_i \qquad (5.1)$$

$$\text{b)} \quad \sum_{j=1}^{N_k} x_j^k = r_k$$

$$x_j^k \geqq 0$$

$$i = 1 \ldots m$$

$$j = 1 \ldots N_k$$

$$k = 1 \ldots q$$

In matrix notation the problem is

$$\text{Minimize:} \quad Cx$$

$$\text{Subject to:} \quad \text{a)} \quad Ax \leqq b \qquad (5.2)$$

$$\text{b)} \quad Bx = r$$

$$x \geqq 0$$

where

$$A = (A' \ldots A^q)$$

$$B = \begin{bmatrix} \bar{1} & & 0 \\ & & \\ 0 & & \bar{1} \end{bmatrix}$$

$C = cA \qquad c = (c_1 \dots c_m)$

$X = (x' \dots x^q)$

The dual problem of (5.2) is

$$\begin{aligned} \text{Minimize:} \quad & (\pi, \alpha)\ [b, r] \\ \text{Subject to:} \quad & (\pi, \alpha) \begin{bmatrix} A \\ B \end{bmatrix} \geqq -C \qquad (5.3) \\ & \pi \geqq 0,\ \alpha \gtrless 0 \end{aligned}$$

An equivalent formulation of (5.3) is:

$$\begin{aligned} \text{Maximize:} \quad & (\bar{\pi}, \bar{\alpha})\ [b, r] \\ \text{Subject to:} \quad & (\bar{\pi}, \bar{\alpha}) \begin{bmatrix} A \\ B \end{bmatrix} \leqq C \qquad (5.4) \\ & \bar{\pi} \leqq 0,\ \bar{\alpha} \lesseqgtr 0 \end{aligned}$$

where $\bar{\pi} = -\pi \qquad \bar{\alpha} = -\alpha$

Suppose we have a basic feasible solution and the m+q simplex multiplier $\bar{\pi}_1 \dots \bar{\pi}_m$, $\bar{\alpha}_1 \dots \bar{\alpha}_q$. Then the variable x_j^k may be introduced into the basis if

$$\sum_{i=1}^{m} c_i a_{ij}^k - \sum_{i=1}^{m} \bar{\pi}_i a_{ij}^k - \bar{\alpha}_k < 0,$$

i.e., if
$$\sum_{i=1}^{m} (c_i - \bar{\pi}_i)\ a_{ij}^k - \bar{\alpha}_k < 0 \qquad (5.5)$$

A shortest chain algorithm, attaching lengths $(c_i - \bar{\pi}_i)$ to arcs a_i

(i = 1...m) may be used to search for a chain p_j^k satisfying (5.5) for commodity k. We can see that the lengths $(c_i - \bar{\pi}_i)$ are non-negative, because c_i are assumed non-negative and $-\bar{\pi}_i = \pi_i$ are also non-negative values.

Adding Arcs to the Multicommodity Network

Suppose we have an optimal solution for the min-cost multicommodity flow problem (5.1). Using the revised simplex method the optimal inverse matrix, D^{-1}, has the form

$$D^{-1} = \left[\begin{array}{cc|c} 1 & -\bar{\pi} & -\bar{\alpha} \\ \hline 0 & B^{-1} & P \\ \hline 0 & R & Q \end{array}\right]$$

Suppose a new arc a_{m+1} is added to the given network. Let c_{m+1} be its associated cost, let $b_{m+1} = 0$ be its capacity. As an extension of lemma (3.1), a feasible basis D, and its inverse D_1^{-1} can be obtained from D and D^{-1}, by utilizing the slack column associated with the new arc a_{m+1}. D_1^{-1} will have the form

$$D_1^{-1} = \begin{array}{c} \\ \\ \\ \\ \\ s_{m+1} \\ \\ \\ \\ \end{array} \left[\begin{array}{c|cc|c} 1 & -\bar{\pi} & 0 & -\bar{\alpha} \\ \hline & & 0 & \\ & B^{-1} & \vdots & P \\ & & 0 & \\ 0 & 0 \; . \; . \; . & 0 \; 1 & 0 \; \ldots \; 0 \\ \hline & & 0 & \\ & R & \vdots & Q \\ & & 0 & \end{array}\right]$$

Therefore, the dual variable, $\bar{\pi}_{m+1}$, associated with a_{m+1} is zero.

If

$$\sum_{i=1}^{m+1} (c_i - \bar{\pi}_i)\, a_{ij}^k \geqq \bar{\alpha}_k \ ,$$

for $\forall$ j, j = 1...N_k and $\forall$ k, k = 1...l, the solution is optimal and the optimal dual variable π^*_{m+1} is equal to zero. Let us assume ther exist some chains p_j^k for which

$$\sum_{i=1}^{m+1} (c_i - \bar{\pi}_i)\, a_{ij}^k - \bar{\alpha}_\alpha < 0$$

Note: if p_j^k exists, it has to use the arc a_{m+1}.

$$p_j^k = [C_j^k,\ p_{ij}^k \ldots p_{mj}^k,\ 1,\ 0,\ldots 1 \ldots 0]$$

$$\bar{p}_j^k = D_1^{-1}\ p_j^k$$

$$\bar{p}_j^k = \left[(C_j^k - \sum_{i=1}^{m+1} \bar{\pi}_i\, p_{ij}^k - \bar{\alpha}_k),\ \bar{p}_{ij}^k \ldots \bar{p}_{mj}^k,\ 1,\ \bar{p}_{m+1j}^k \ldots \bar{p}_{m+qj}^k\right]$$

let

$$\ell_j^k = C_j^k - \sum_{i=1}^{m+1} \bar{\pi}_i p_{ij}^k - \bar{\alpha}_k = \sum_{i=1}^{m+1} (c_i - \bar{\pi}_i)\, p_{ij}^k - \bar{\alpha}_k$$

Now let us take p_h^o a vector for entering the basis such that

$$\ell_h^o = \min_{j,k} \ell_j^k$$

then D_1^{-1} becomes

$$D_1^{-1} = \left[\begin{array}{c|ccc|c} 1 & & -\bar{\pi} & -\bar{\pi}_{m+1} & -\bar{\alpha} \\ \hline & & & -\bar{p}_{1,h} & \\ 0 & & B^{-1} & & P \\ & & & -p_{m,h} & \\ 0 & 0 & \ldots\ldots & .0 \; 1 & 0 \; . \; . \; . \; 0 \\ \hline & & & -\bar{p}_{m+1,h} & \\ & & & \vdots & \\ 0 & & R & -\bar{p}_{m+q,h} & Q \end{array}\right]$$

where $-\bar{\pi}_{m+1} = -\ell_h^o$, the original dual variables do not change and the new dual variable $\bar{\pi}_{m+1}$ is equal to ℓ_h^o. Again, as in theorem (3.1), we show that the solution which has been obtained is an optimal solution.

1. The solution is primal feasible, since the capacity of the new arc is zero.

2. The solution is dual feasible

i) For those chains not using a_{m+1} the value of $\sum_{i=1}^{m+1} (c_i - \bar{\pi}_i)$ a_{ij}^k is not less than $\bar{\alpha}_k$, since we start with an optimal solution and the dual variables have not changed for these arcs.

ii) Suppose a chain using a_{m+1} for which $\sum_{i=1}^{m+1} (c_i - \bar{\pi}_i)\ a_{ij}^k$ $- \bar{\alpha}_k$ less than zero exists. Let this chain be p_j^k

$$p_j^k = [C_j^k,\ p_{ij}^k, \ldots p_{mj}^k,\ 1,\ 0..1,..0]$$

the pricing operation gives us

$$\bar{c}_j^k = c_j^k - \sum_{i=1}^{m} \bar{\pi}_i\ p_{ij}^k - \bar{\pi}_{m+1} - \bar{\alpha}_k < 0$$

$$\bar{c}_j^k = \ell_j^k - \ell_h^o < 0 \qquad \text{or} \qquad \ell_j^k < \ell_h^o \ .$$

But this is a contradiction.

We therefore conclude that

$$\bar{c}_j^k \geqq 0$$

Q.E.D.

Summing up, if a new arc is added to an optimal min-cost network, the optimal dual variable associated with this arc is $- \ell_h^o$ where

$$\ell_h^o = \min_{j,k} \ell_j^k , \qquad \ell_j^k < 0$$

$$\ell_j^k = \sum_{i=1}^{m+1} (c_i - \bar{\pi}_i) \ a_{ij}^k - \bar{\alpha}_k$$

$$\bar{\pi}_{m+1} = 0$$

EXAMPLE V

Consider the network shown in Figure 8. The cost and the dual variables corresponding to an optimal solution are indicated in the figure. From the solution $\bar{\alpha}_1 = 7$, $\bar{\alpha}_2 = 6$, find the optimal dual variable corresponding to the arc $a_6 = (s_2, t_1)$, which has a capacity equal to zero and $c_6 = 1$.

$$c_1 - \bar{\pi}_1 = 4 - 0 \qquad = 4$$

$$c_2 - \bar{\pi}_2 = 2 - 0 \qquad = 2$$

$$c_3 - \bar{\pi}_3 = 1 - (-6) \quad = 7$$

$$c_4 - \bar{\pi}_4 = 2 - (-1) \quad = 3$$

$$c_5 - \bar{\pi}_5 = 1 - 0 \qquad = 1$$

$$c_6 - \bar{\pi}_6 = 1 - 0 \qquad = 1$$

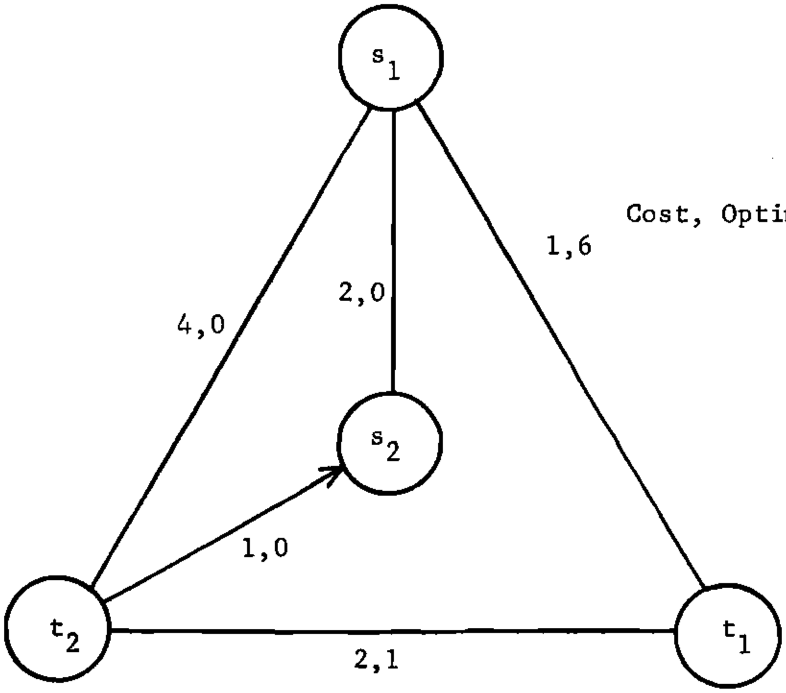

Figure 8. Example V Network

$$\ell_h^o = \min\left(\sum_{i=1}^{6} (c_i - \bar{\pi}_i)\, a_{ij}^k - \bar{\alpha}_k\right)$$

$$\ell_h^o = (3 - 7) = - 4$$

$$\pi^*_{m+1} = 4$$

The new entering chain is the chain of the arcs a_2 and a_6 for commodity one.

A method equivalent to Method II presented in Chapter III can be used to obtain the optimal dual variables associated with a set of new arcs added to a min-cost multicommodity network.

After a_{m+1} has been introduced, the obtained solution is an optimal solution, with the dual variable π^*_{m+1} in the arc a_{m+1}.

If now a_{m+2} is wanted in the network, we are in the same situation we were in for adding a_{m+1}; therefore, we can obtain π^*_{m+2} and so forth until all arcs of the set have been introduced.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

The principal results of this thesis are:

a. A method for adding the best arc from a set of new arcs to a given network has been presented.

b. Two methods for calculating the dual variables associated with non-present arcs have been developed and evaluated.

c. A method for adding the best set of k arcs from a set of r arcs to a given network has been developed.

d. An extension has been shown to use similar techniques to obtain the optimal dual variables associated with non-present arcs for the min-cost multicommodity flow problem.

e. Several theorems and lemmas in the general area of multicommodity network flows have been presented and proved.

Further research is recommended in two general areas:

a. An investigation to extend the methods described in this research to consider the maximum flow, minimum cost multicommodity problem. This is the problem of finding the flow chains over a given network at minimum cost, when the existing flow is the maximum flow. An investigation in this area must begin searching for an appropriate formulation of the maximum flow, minimum cost multicommodity problem.

b. An investigation using the theorems and lemmas of Chapter II to consider the relationship between the maximum flow and the multicommodity disconnecting set, determining for instance when the maximum flow is equal to the capacity of the minimum cut or equivalently by theorem (2.6) when a linear programming problem has an integer optimal solution.

APPENDIX A

THE SHORTEST PATHS BETWEEN ALL PAIRS OF NODES OF A NETWORK

The following is the Floyd algorithm to find the shortest path between all pairs of nodes of a network (see Dreyfus, p. 401).

Assume a network with N nodes and d_{ij} the directed distance from N_i to N_j.

Step 0. Construct the N X N matrix D.

$$D = \|d_{ij}\| = D^o = \|d^o_{ij}\|$$

(i) if (N_i, N_j) does not exist, $d_{ij} = \infty$

(ii) $d_{ii} = 0$.

Step k. Construct the N X N matrix D^k.

$$D^k = \|d^k_{ij}\|$$

$$d^k_{ij} = \min\,(d^{k-1}_{ij}\ ,\ d^{k-1}_{ik} + d^{k-1}_{kj})$$

Terminate after D^{N-1} has been constructed.

N matrices are constructed sequentially. The k^{th} such matrix can be interpreted as giving the lengths of the shortest allowable paths between all node pairs (N_i, N_j), where only paths with intermediate nodes belonging to the set of nodes 1 through k are allowed.

The D^{N-1} matrix indicates the length of the shortest paths between all pairs of nodes of the network, but it does not indicate the paths.

If this information is required, an additional calculation has to be made through the computation of the matrices.

Let us keep this information in matrix form as follows:

Step 1. $$H^o = \|h^o_{ij}\|$$

$$h^o_{ij} \begin{cases} = i & \text{for } d_{ij} < \infty \\ = - & \text{otherwise} \end{cases}$$

Step k. $$H^k = \|h^k_{ij}\|$$

$$h^k_{ij} \begin{cases} = h^{k-1}_{ij} & \text{if } d^k_{ij} = d^{k-1}_{ij} \\ = k & \text{otherwise} \end{cases}$$

Terminate when $k = N - 1$

$$H^{N-1} = \|h^{N-1}_{ij}\|$$

Assume $$h^{N-1}_{ij} = k$$

This means that the node k precedes node j in the shortest path from node i to node j. In the same form from H^{N-1}, we can obtain the preceding node to node k in the shortest path from node i to node k, and going back we can obtain all the intermediate nodes in the shortest path from node i to node j from H^{N-1}.

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