# THE NONCOMMUTATIVE GEOMETRY OF ULTRAMETRIC CANTOR SETS

A Thesis
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Mathematics

Georgia Institute of Technology
August 2008

# THE NONCOMMUTATIVE GEOMETRY OF ULTRAMETRIC CANTOR SETS

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# ACKNOWLEDGEMENTS

I would first like to thank my advisor Jean Bellissard for his guidance, his many suggestions, and his encouragement. I would also like to thank Matt Baker, Yuri Bakhtin, Stavros Garoufalidis, and Ian Putnam for serving on my thesis committee. In addition, I would like to thank Joseph Landsberg for his assistance in my first couple years at Georgia Tech.

I am extremely grateful to both my family and friends for their support. I know that I could not have finished without their help. I am grateful to my fellow members of the Math department, especially my fellow noncommutative geometers: Michael Burkhart, Ian Palmer, and Jean Savinien.

Thank you.

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# SUMMARY

In this thesis, an analogue of the Riemannian structure of a manifold is created for an ultrametric Cantor set $(C, d)$ using the tools of Noncommutative Geometry. Associated with $(C, d)$ is a weighted rooted tree, its Michon tree [53]. This tree allows to define a family of spectral triples $(\mathcal{C}_{\text{Lip}}(C), \mathcal{H}, D)$ using the $\ell^2$-space of its vertices, giving the Cantor set the structure of a noncommutative Riemannian manifold. Here $\mathcal{C}_{\text{Lip}}(C)$ denotes the space of Lipschitz continuous functions on $(C, d)$. The family of spectral triples is indexed by the space of *choice functions* which is shown to be the analogue of the sphere bundle of a Riemannian manifold. The Connes metric coming from the Dirac operator $D$ then allows to recover the metric on $C$. The corresponding $\zeta$-function is shown to have abscissa of convergence, $s_0$, equal to the *upper box dimension* of $(C, d)$.

Taking the residue at this singularity leads to the definition of a canonical probability measure on $C$. This measure in turn induces a measure on the space of choices. Given a choice, the commutator of $D$ with a Lipschitz continuous function can be interpreted as a directional derivative. By integrating over all choices, this leads to the definition of an analogue of the Laplace-Beltrami operator. This operator generates a Markov semigroup which plays the role of a Brownian motion on $C$.

This construction is applied to the simplest case, the triadic Cantor set where: (i) the spectrum and the eigenfunctions of the Laplace-Beltrami operator are computed, (ii) the Weyl asymptotic formula is shown to hold with the dimension $s_0$, (iii) the corresponding Markov process is shown to have an anomalous diffusion with $\mathbb{E}(d(X_t, X_{t+\delta t})^2) \simeq \delta t \ln(1/\delta t)$ as $\delta t \downarrow 0$.

Other classical examples of Cantor sets are shown to have a metric which is metrically equivalent to an ultrametric. These examples include: iterated function systems, cookie cutter systems, and the transversal of a repetitive, aperiodic Delone set of finite type. It is further shown that in the case of self-similar iterated functions systems that the measure constructed via the residue of the $\zeta$-function is in fact that Hausdorff measure. Finally, the case of the Fibonacci tiling is considered in detail and its $\zeta$-function is shown to have abscissa of convergence equal to its algorithmic complexity.

# CHAPTER I

# INTRODUCTION

## *1.1 The Classical Triadic Cantor Set*

The classical triadic Cantor set was introduced by George Cantor in 1883 in [10]. In his original construction, he defined the Cantor set as the set of numbers between 0 and 1 that can be written in the form $\sum_{k=1}^{\infty} 2\epsilon_k/3^k$ such that $\epsilon_k \in \{0,1\}$. A more geometric approach is given by the middle thirds construction which starts with the interval $[0,1]$ and removes its middle third $(1/3, 2/3)$. The process is then applied to the remaining two intervals. After proceeding indefinitely, the intersection of all such intervals gives the triadic Cantor set.

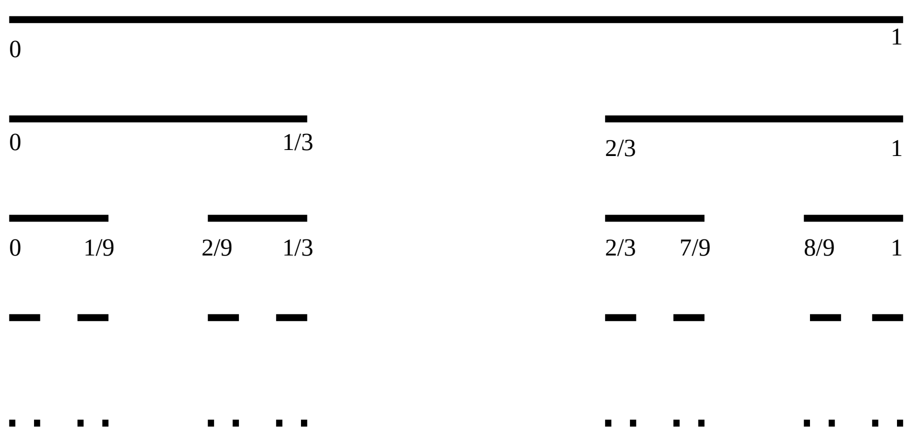


**Figure 1:** The Middle Thirds Construction of the Triadic Cantor Set

Cantor's original construction has been generalized into a more abstract definition of a Cantor set.

**Definition 1** *A* Cantor set *is a topological space that is non-empty, compact, perfect, totally disconnected and metrizable.*

That this definition effectively captures the essence of Cantor's original construction is given by the following theorem.

**Theorem 1 (Brouwer [9])** *Let $C$ be a Cantor set. Then $C$ is homeomorphic to the triadic Cantor set.*

**Proof:** See [3] Chapter 29 for a more modern proof. $\square$

According to this theorem, the Cantor set is essentially unique as a topological space. It is therefore the metric that provides the diversity seen in the various examples of Cantor sets. It is important that the metric is compatible with the topology.

**Definition 2** *Let $C$ be a Cantor set. A metric on $C$ will be called* regular *if it defines a topology on $C$ for which $C$ is a Cantor set.*

Because a specific Cantor set $C$ is really determined by the specific choice of a regular metric $d$ on $C$, we will often label a Cantor set by $(C, d)$. The first main step in this thesis will be to find a way to encode the metric. Unfortunately, we will not be able to do this for a general metric.

**Definition 3** *A metric $d$ on $C$ is an* ultrametric *if it satisfies the strong triangle inequality: for $x, y \in C$, $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $z \in C$.*

It turns out that restricting ourselves to working with ultrametric Cantor sets is not that severe of a restriction. In Sections 5.2.1,5.2.2, and 5.3.1, we will give three classes of Cantor sets whose natural metric is metrically equivalent to an ultrametric. Since many fractal geometric quantities (e.g. box dimension, Hausdorff dimension) are preserved under metric equivalence (see [26]), then we should be able to recover these quantities from an equivalent ultrametric.

In order to encode the Cantor set and its ultrametric, it is the viewpoint of this thesis that Cantor sets should be treated as the boundary of a tree. In the case of the triadic Cantor set, it is well know that $C_3$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$, that is as the boundary of the infinite dyadic tree. The intuition for this is given by the following picture.

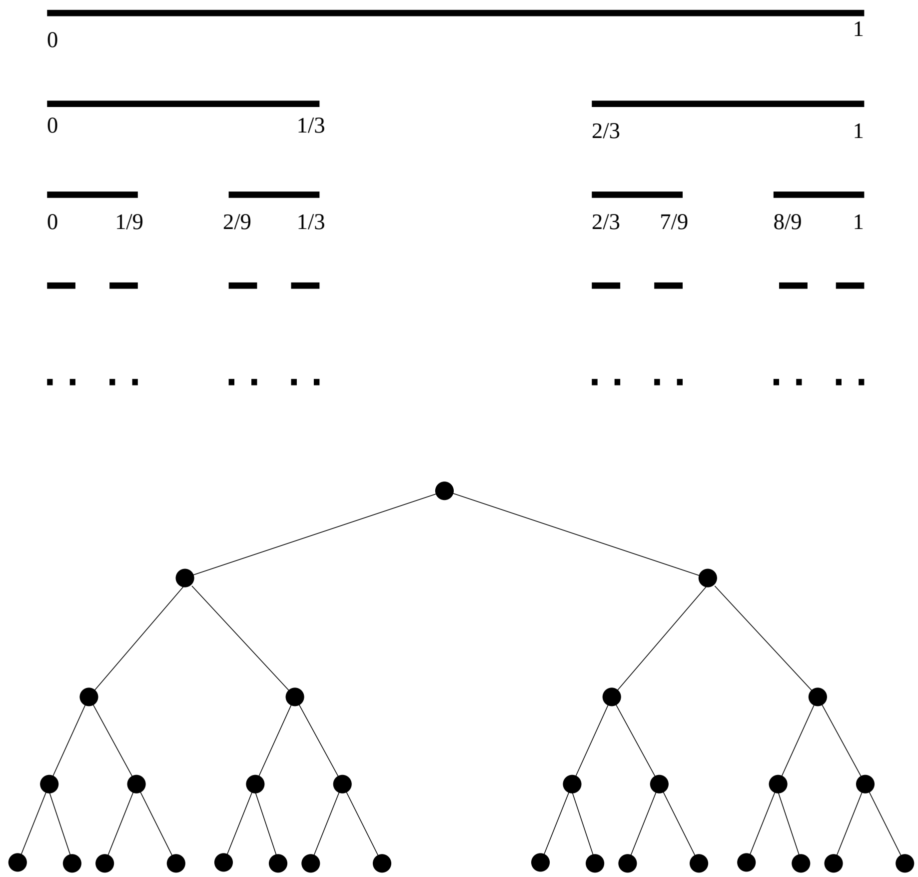


**Figure 2:** The Infinite Dyadic Tree and the Triadic Cantor Set

In Chapter 3, we show how to properly generalize this construction to any Cantor set with a regular ultrametric. Then using the work of Michon from [53], we show that a tree is a natural object to encode not only the topology but also the ultrametric.

## *1.2 Noncommutative Geometry*

Once we have shown that the tree can effectively capture the ultrametric and the topology, we will then apply the techniques of Noncommutative Geometry to the

Cantor set. One of the goals of Noncommutative Geometry is to find the proper analogues of classical geometric objects. At the foundation of Noncommutative Geometry lies the work of Alain Connes [18] who has shown that it is possible to generalize the notion of Riemannian manifold into the notion of a spectral triple. The spectral triple is a functional analytic object which allows to recover much of the information of the original Riemannian manifold. The actual definition of a spectral triple will be taken up in Section 4.1 but it is worth giving some insight into the validity of using a spectral triple in place of the actual Riemannian manifold. In [55], Rennie and Varilly have formalized Connes' idea by showing that a spectral triple along with several axiomatic conditions on the spectral triple is enough to recover the differentiable structure of a Riemannian manifold. Therefore, a spectral triple truly can serve as the proper generalization of Riemannian geometry.

In the case of a Cantor set $C$ with regular ultrametric $d$, the spectral triple will require Michon's construction of the tree associated to $C$. This will be done in Section 4.1.1 and is modeled after Connes work in [18] Chapter 4.3.$\epsilon$. Surprisingly, the spectral triple for a general ultrametric is not unique. The spectral triple will depend on a certain choice which will be properly defined in Section 4.1.1. We will then get a family of spectral triples indexed by the space of all possible choices. In the Noncommutative Riemannian structure, the space of choices will play the role of the sphere bundle. At this point, we must justify that our spectral triple is actually a 'good' spectral triple - that is, can it recover the geometric information of our fractal. It will be shown that the spectral triple can recover the fractal geometry. The appropriate way of recovering certain fractal geometric properties is summarized in Table 1 and will be shown in Sections 4.1.2, 4.2.3, 4.3.1.

Once it is determined that the spectral triple created in Section 4.1.1 is in fact a 'good' spectral triple, then we will proceed to carry the analogy with Riemannian geometry further. The first step is to appropriately generalize the Laplace-Beltrami

**Table 1:** A Summary of the Riemannian Analogues of Classical Fractal Geometric Properties

| **Fractal Geometric Property** | **Noncommutative Analogue** |
|---|---|
| Metric | $\rho(x, y)$ - the Connes metric |
| Upper Box Dimension | $s_0$ - the abscissa of convergence of the $\zeta$-function associated to $C$ |
| Hausdorff Measure | $\mu$ - the residue measure for the $\zeta$-function |

operator on a Riemannian manifold. This is taken up in Section 4.4. Here it is crucial to continue to think of the space of choices as the sphere bundle on $C$. The creation of the Laplacian will require the theory of Dirichlet forms as laid out in [32]. It is a subject of future research to discover the appropriate analogues of other objects from Riemannian geometry: curvature, connections, etc.

## *1.3 Further Results*

The theoretical tools developed in this thesis are then applied to the triadic Cantor set $C_3$ in Section 5.1. We explicitly compute how to recover the dimension and the Hausdorff measure. A formula for the Laplacian is also calculated and the appropriate analogue of the Weyl asymptotic formula is shown to hold for this Laplacian. It is then a small step to move in a probabilistic direction. Because the Laplacian and Brownian motion are intimately connected, we can use this close connection to construct a Markov process with values in $C_3$ that serves as the analogue of Brownian motion on the Cantor set. Diffusion on Cantor sets is not entirely new and has been studied in various contexts mostly as a non-Archimedean eld [1, 24, 47]. Del Muto and Figá-Talamanca have generalized this in [21, 29] for locally compact ultrametric spaces where the group of isometries is transitive and therefore allows to treat the Cantor set as an abelian group. In both cases, the construction of the diffusion relies heavily on the algebraic structure that is given to the space. It is important to note that the construction in this thesis requires no algebraic structure and comes solely from the metric. A nice aspect of the present approach is that it also allows to compute some of the asymptotics of the Brownian motion on $C_3$. These calculations are taken up in Section 5.1.2.

The work of this thesis was originally inspired by the desire to create a spectral triple for the transversal of an aperiodic, repetitive Delone set of finite type [4, 6, 5]. In this case, the transversal is a Cantor set. Moreover, there is a natural construction of the tree from its patches which in turn leads to a natural ultrametric on the transversal. Once the computations have been done for the simple example provided by $C_3$, the noncommutative construction is then applied to the transversal of the Fibonacci tiling. In this case, our construction is crucial as the transversal of the Fibonacci tiling has no obvious algebraic structure. In particular, as seen by its tree there is only one nontrivial isometry. It is then shown how to compute the $\zeta$-function for the Fibonacci tiling. Further results on the measure and the construction of the Laplacian are the subjects of future research.

# CHAPTER II

# MATHEMATICAL PRELIMINARIES

Let $\mathbb{C}$ denote the set of complex numbers. In this thesis, all Hilbert spaces will be assumed to be over $\mathbb{C}$ unless specified otherwise. Moreover, all Hilbert spaces will be assumed to be separable unless specified otherwise.

## *2.1* $C^*$*-algebras*

### 2.1.1 Basic Definitions and Examples

$C^*$-algebras are one of the starting points of Noncommutative Geometry. In this section, we introduce the basics of unital $C^*$-algebras. Most of this information is standard and can be found in [2, 19].

**Definition 4** *A* $C^*$-algebra *is a Banach algebra* $\mathcal{A}$ *with an involution* $*$ *such that* $||a^*a|| = ||a||^2$ *for every* $a \in \mathcal{A}$*. That is,* $\mathcal{A}$ *is an algebra over* $\mathbb{C}$ *that has a norm* $||\cdot||$ *relative to which* $\mathcal{A}$ *is a Banach space for which* $||ab|| \leq ||a||||b||$*. In addition, there is a map* $a \to a^*$ *from* $\mathcal{A}$ *to* $\mathcal{A}$ *such that for* $a, b \in \mathcal{A}$ *and* $\alpha \in \mathbb{C}$*: (i)* $(a^*)^* = a$*, (ii)* $(ab)^* = b^*a^*$*, and (iii)* $(\alpha a + b)^* = \bar{\alpha}a^* + b^*$*. An element* $1 \in \mathcal{A}$ *is said to be a* unit *if* $1 \cdot a = a \cdot 1 = a$ *for every* $a \in \mathcal{A}$*.*

The reader should note that $C^*$-algebras do not necessarily contain a unit. If $\mathcal{A}$ does contain a unit 1, then it is easy to check that $1^* = 1$. This implies that $||1||^2 = ||1^*1|| = ||1||$ and thus if $\mathcal{A}$ is not trivial then $||1|| = 1$. In this thesis, we will only work with non-trivial unital $C^*$-algebras.

**Example 1** *If* $\mathcal{H}$ *is a Hilbert space, then* $B(\mathcal{H})$*, the space of bounded linear operators on* $\mathcal{H}$*, is a* $C^*$*-algebra . The norm on* $B(\mathcal{H})$ *is the operator norm,* $||A||_{B(\mathcal{H})} :=$

$\sup\{||Ah||_{\mathcal{H}} : h \in \mathcal{H}, ||h||_{\mathcal{H}} \leq 1\}$. *Multiplication is given by multiplication of operators and* $A^*$ *is given by the adjoint of* $A$ *for* $A \in B(\mathcal{H})$. *The unit of* $B(\mathcal{H})$ *is given by the identity operator* $\mathbf{1}$.

It turns out that this example is quite canonical. In fact, we will see later that by the Gelfand-Naimark Theorem (Theorem 9) that every $C^*$-algebra is isometrically $*$-isomorphic to a $C^*$-algebra of operators on a Hilbert space.

**Example 2** *If* $X$ *is a compact Hausdorff space, then* $\mathcal{C}(X)$*, the space of continuous functions from* $X$ *to* $\mathbb{C}$*, is a* $C^*$*-algebra . The norm on* $\mathcal{C}(X)$ *is the sup norm,* $||f||_\infty := \sup\{|f(x)| : x \in X\}$. *For* $f, g \in \mathcal{C}(X)$ *and* $x \in X$*, multiplication is given by* $f \cdot g(x) = f(x)g(x)$ *and* $f^*(x) = \overline{f(x)}$. *The unit is given by the constant function* $1$.

It will be shown in Section 2.1.3 that every abelian $C^*$-algebra with unit is of this type.

### 2.1.2 The Spectrum

The spectrum is the single most useful tool in dealing with $C^*$-algebras and will be used throughout this thesis. To begin, let $\mathcal{A}$ be a $C^*$-algebra with unit 1. For $a \in \mathcal{A}$, the *spectrum* of $a$, $\sigma(a)$, is defined to be

$$\sigma(a) = \{\alpha \in \mathbb{C} : a - \alpha 1 \text{ is not invertible}\}.$$

The *resolvent set* of $a$, $\rho(a)$, is defined to be all $\alpha \in \mathbb{C}$ such that $\alpha \notin \sigma(a)$.

**Example 3** *If* $X$ *is a compact Hausdorff space, then for* $f \in \mathcal{C}(X)$, $\sigma(f) = Range(f)$. *For if* $\alpha = f(x_0)$ *then* $f - \alpha$ *has a zero and is therefore not invertible. On the other hand, if* $\alpha \notin Range(f)$*, then* $f - \alpha$ *is nonvanishing and therefore is invertible.*

The first fundamental theorem concerning the spectrum is the following.

**Theorem 2** *If $\mathcal{A}$ is a unital $C^*$-algebra, then for each $a \in \mathcal{A}$, $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$. Moreover, if $|\alpha| > ||a||$, then $\alpha \notin \sigma(a)$ and $z \to (z-\alpha)^{-1}$ is an $\mathcal{A}$-valued analytic function defined on $\rho(a)$.*

**Proof:** See [19] (Theorem VII.3.6). □

Because the spectrum of an element of a unital $C^*$-algebra is not empty, then the following useful quantity can be defined.

**Definition 5** *If $\mathcal{A}$ is a unital $C^*$-algebra and $a \in \mathcal{A}$ then the* spectral radius *of $a$, $r(a)$ is defined by $r(a) := \sup\{|\alpha| : \alpha \in \sigma(a)\}$.*

Because $\sigma(a)$ is compact, then $r(a)$ is finite and this supremum is attained. The following proposition gives a way to compute the spectral radius.

**Proposition 1 (Spectral Radius Formula)** *If $\mathcal{A}$ is a unital $C^*$-algebra and $a \in \mathcal{A}$ then $\lim_{n\to\infty} ||a^n||^{1/n}$ exists and*

$$r(a) = \lim_{n\to\infty} ||a^n||^{1/n}.$$

**Proof:** See [19] (Proposition VII.3.8). □

The formula in the previous proposition is called the *spectral radius formula.* It gives a crucial link between the algebraic properties and the properties of the norm for a $C^*$-algebra. The following definition specifies certain algebraic properties of elements of a $C^*$-algebra that will be used later.

**Definition 6** *If $\mathcal{A}$ is a unital $C^*$-algebra and $a \in \mathcal{A}$, then*

*(i) $a$ is* hermitian *if $a = a^*$.*

*(ii) $a$ is* normal *if $aa^* = a^*a$.*

*(iii) $a$ is* unitary *if $aa^* = a^*a = 1$.*

A simple application of the spectral radius formula then gives the following two corollaries.

**Corollary 1** *If $a \in \mathcal{A}$ is normal then $r(a) = ||a||$.*

**Proof:** Since $aa^* = a^*a$, then repeated use of the C$^*$-identity gives that

$$||a^2||^2 = ||a^2(a^*)^2|| = ||(aa^*)(aa^*)^*|| = ||aa^*||^2 = ||a||^4$$

and therefore $||a||^2 = ||a^2||$. By induction, $||a||^{2n} = ||a^{2n}||$ for $n \geq 1$. Thus, $||a|| = ||a^{2n}||^{1/2n}$ and $r(a) = \lim_{n\to\infty} ||a^n||^{1/n} = ||a||$. $\square$

One of the most important uses of the spectral radius formula is in the proof of the following lemma. This lemma is fundamental to $C^*$-algebras and says that the norm on a $C^*$-algebra is unique.

**Lemma 1** *Let $\mathcal{A}$ and $\mathcal{B}$ be unital $C^*$-algebras. If $h : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, then $||h(a)|| \leq ||a||$. In particular, if $h$ is a $*$-isomorphism, then $h$ is an isometry.*

**Proof:** If $\mathcal{A}$ has unit $1_{\mathcal{A}}$ and $\mathcal{B}$ has unit $1_{\mathcal{B}}$, then it is not necessarily true that $h(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. However, $h(1_{\mathcal{A}})$ is the identity for the $C^*$-algebra $\overline{h(\mathcal{A})}$. Since $h$ is a $*$-homomorphism to $\overline{h(\mathcal{A})}$ and the norm on this $C^*$-algebra is the same as the norm on $\mathcal{B}$, then we may assume that $\overline{h(\mathcal{A})} = \mathcal{B}$. If $a \in \mathcal{A}$ is such that $a - \lambda 1_{\mathcal{A}}$ is invertible with inverse $b$, then

$$1_{\mathcal{B}} = h(1_{\mathcal{A}}) = h(b(a - \lambda 1_{\mathcal{A}})) = h(b)(h(a) - \lambda 1_{\mathcal{B}})$$

and $h(a) - \lambda 1_{\mathcal{B}}$ is invertible. Therefore, $\rho_{\mathcal{A}}(a) \subset \rho_{\mathcal{B}}(h(a))$ and thus $\sigma_{\mathcal{B}}(h(a)) \subset \sigma_{\mathcal{A}}(a)$. Consequently, $r(h(a)) \leq r(a)$. Since $a^*a$ is hermitian, then by Corollary 1,

$$||h(a)||^2 = ||h(a^*a)|| = r(h(a^*a)) \leq r(a^*a) = ||a^*a|| = ||a||^2$$

and the lemma is proved. $\square$

### 2.1.3 Abelian $C^*$-algebras

Example 2 of Section 2.1.1 gave an example of an abelian $C^*$-algebra. Our next goal will be to show that every abelian $C^*$-algebra is of this type. Again, these results are standard and can be found in [2, 19]. Let $\mathcal{A}$ be an abelian $C^*$-algebra.

**Definition 7** *Let $\Sigma$ be the collection of all nonzero homomorphisms from $\mathcal{A} \to \mathbb{C}$. If $\Sigma$ is given the weak*-topology, then $\Sigma$ is called the* maximal ideal space *of $\mathcal{A}$.*

Recall that the weak*-topology is the smallest topology on the dual of $\mathcal{A}$, $\mathcal{A}^*$, such that for every $a \in \mathcal{A}$ the map $\Phi_a : \mathcal{A}^* \to \mathbb{C}$ defined by $\Phi_a(h) = h(a)$ is continuous. It is possible to put a norm on $\mathcal{A}^*$ by $||h|| = \sup\{|h(a)| : a \in \mathcal{A}, ||a|| \leq 1\}$ for $h \in \mathcal{A}^*$.

**Lemma 2** *If $\mathcal{A}$ is abelian and $h \in \Sigma$, then $||h|| = 1$.*

**Proof:** Let $a \in \mathcal{A}$ be such that $||a|| \leq 1$ and let $\lambda = h(a)$. If $|\lambda| > 1$, then $||a/\lambda|| < 1$. Then $1 - a/\lambda$ is invertible with inverse $\sum_{n=0}^{\infty}(a/\lambda)^n$. Let $b = \sum_{n=0}^{\infty}(a/\lambda)^n$. Then

$$h(1) = h(b(1 - a/\lambda)) = h(b) - h(ba/\lambda) = h(b) - h(b)\lambda/\lambda = 0$$

which contradicts the fact that $h$ is a non-zero homomorphism. Thus $|h(a)| \leq ||a|| \leq 1$ and $||h|| \leq 1$. Since $h(1) = 1 = ||1||$ then $||h|| = 1$. $\square$

This lemma can then be used to prove the following theorem.

**Theorem 3** *If $\mathcal{A}$ is an abelian $C^*$-algebra with unit, then its maximal ideal space $\Sigma$ is a compact Hausdorff space.*

**Proof:** The weak*-topology is a Hausdorff topology so $\Sigma$ is Hausdorff. Let $B$ be the unit ball in $\mathcal{A}^*$ and notice that by the previous lemma, $\Sigma \subset B$. By the Banach-Alaoglu Theorem, $B$ is weak*-compact and thus it is sufficient to show that $\Sigma$ is weak*-closed. Let $\{h_i\}$ be a net in $\Sigma$ (if $\mathcal{A}$ is separable, then a sequence can be used instead of a net). Then for $a, b \in \mathcal{A}$,

$$h(ab) = \lim_i h_i(ab) = \lim_i h_i(a)h_i(b) = h(a)h(b)$$

and $h$ is a homomorphism. Since $h(1) = \lim_i h_i(1) = 1$, then $h \in \Sigma$. Thus $\Sigma$ is a closed subset of a compact space and is thus compact. $\square$

Given a unital abelian $C^*$-algebra $\mathcal{A}$ and $a \in \mathcal{A}$, we can now define the *Gelfand transform* of $a$ as the map $\hat{a} : \Sigma \to \mathbb{C}$ defined by $\hat{a}(h) = h(a)$.

**Theorem 4** *If $\mathcal{A}$ is an abelian $C^*$-algebra with unit and $\Sigma$ is its maximal ideal space, then the Gelfand map $a \to \hat{a}$ is an isometric $*$-isomorphism from $\mathcal{A}$ to $\mathcal{C}(\Sigma)$.*

**Proof:** Let $\Gamma$ denote the Gelfand map so that $\Gamma(a) = \hat{a}$. We first show that $\hat{a}$ is continuous. If $h_i \to h \in \Sigma$ then $h_i \to h$ weak* in $\mathcal{A}^*$. Therefore, by the definition of the weak topology

$$\hat{a}(h_i) = h_i(a) \to h(a) = \hat{a}(h)$$

and $\hat{a}$ is continuous. Because each $h \in \Sigma$ is a homomorphism then

$$\hat{ab}(h) = h(ab) = h(a)h(b) = \hat{a}(h)\hat{b}(h)$$

for $a, b \in \mathcal{A}$. It is easy to see that $\Gamma$ is linear so therefore $\Gamma$ is a homomorphism.

The next step is then to show that $\Gamma(a^*) = \Gamma(a)^*$. To do this we will show that if $h \in \Sigma$ then $h(a^*) = \overline{h(a)}$. To begin, realize that every $a \in \mathcal{A}$ can be written as $a = x + iy$ where $x$ and $y$ are hermitian. Therefore, it is sufficient to show that $h(x)$ is real for $x$ hermitian. So let $x \in \mathcal{A}$ be such that $x = x^*$ and let $t \in \mathbb{R}$. If

$$u_t := e^{itx} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} x^n$$

then $||u_t|| \leq e^{||tx||}$ and is therefore well-defined. For $h \in \Sigma$,

$$h(u_t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} h(x^n) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} h(x)^n = e^{ith(x)}.$$

It is easy to see that $u_t^* = e^{-itx}$ and therefore that

$$u_t u_t^* = \sum_{n,m=0}^{\infty} \frac{(it)^{n+m}}{n!m!} x^n (-x)^m = \sum_{k=0}^{\infty} (it)^k x^k \sum_{n+m=k} \frac{(-1)^m}{n!m!} = 1$$

since $\sum_{n+m=k} (-1)^m/n!m! = (1-1)^k$ for $k > 0$. Similarly, $u_t^* u_t = 1$ and $u_t$ is unitary. Therefore, $||u_t|| = 1$ and consequently that $|h(u_t)| \leq \sup\{h(a) : a \in \mathcal{A}, ||a|| \leq 1\} = ||h|| = 1$ for every $t \in \mathbb{R}$. Since, $\Re(ith(x)) = -t\Im(h(x))$ then

$$e^{-t\Im(h(x))} = e^{\Re(ith(x))} = |e^{ith(x)}| = |h(u_t)| \leq 1.$$

Since $t$ is arbitrary, then this can only happen if $\Im(h(x)) = 0$. Thus $h(x)$ is real and therefore $h(a^*) = \overline{h(a)}$ for all $h \in \Sigma$ and $a \in \mathcal{A}$. Consequently, $\hat{a^*}(h) = h(a^*) = \overline{h(a)} = \overline{\hat{a}(h)}$ and $\Gamma(a^*) = \Gamma(a)^*$. So $\Gamma$ is a $*$-homomorphism.

Since $\mathcal{A}$ is abelian then each $a \in \mathcal{A}$ is normal. Therefore, by Corollary 1 $||a|| = r(a)$. On the other hand, by Example 3 we know that $\sigma(\hat{a}) = \text{Range}(\hat{a})$. Thus

$$||\hat{a}||_\infty = \sup\{|\hat{a}(h)| : h \in \Sigma\} = \sup\{|\lambda| : \lambda \in \text{Range}(\hat{a})\} = r(a).$$

Consequently, $||\hat{a}||_\infty = ||a||$ and $\Gamma$ is an isometry and injective.

Because $\Gamma$ is an isometry, then $\Gamma(\mathcal{A})$ is closed. Moreover, since $\overline{\Gamma(a)} = \Gamma(a^*)$, then $\overline{\Gamma(a)} \in \Gamma(\mathcal{A})$. For every $h \in \Sigma$, $h(1) = 1$ and therefore $\Gamma(1) = 1$. It is also clear that $\Gamma(\mathcal{A})$ separates the points of $\Sigma$. Consequently, the Stone-Weierstrass Theorem implies that $\Gamma(\mathcal{A}) = \mathcal{C}(\Sigma)$ and that $\Gamma$ is surjective. Thus, $\Gamma$ is a isometric $*$-isomorphism as desired. $\square$

### 2.1.4 The Continuous Functional Calculus in $C^*$-algebras

The Gelfand map from the previous section is a very powerful tool for working with $C^*$-algebras.

**Definition 8** *Let $\mathcal{B}$ be a unital $C^*$-algebra. For $a \in \mathcal{B}$, let $C^*(a)$ be defined to be the $C^*$-algebra generated by $1$ and $a$. That is, $C^*(a)$ is the closure in $\mathcal{B}$ of $\{p(a, a^*)\}$ where $p(z, \bar{z})$ is a polynomial in $z$ and $\bar{z}$.*

If $a$ is a normal element of $\mathcal{B}$ then $C^*(a)$ is an abelian $C^*$-algebra with unit. Thus Theorem 4 gives that $C^*(a)$ is isometrically $*$-isomorphic to $\mathcal{C}(\Sigma)$. The next proposition identifies $\Sigma$ in this case.

**Proposition 2** *Let $C^*(a)$ have maximal ideal space $\Sigma$. Then the map $\Phi : \Sigma \to \sigma(a)$ given by $\Phi(h) = h(a)$ is a homeomorphism.*

**Proof:** See [19] Proposition VIII.2.3. $\square$

**Definition 9** *Let $\mathcal{B}$ be a unital $C^*$-algebra and let $a \in \mathcal{B}$ be normal. Let $\Gamma^{-1} \circ \Phi^{\#} : \mathcal{C}(\sigma(a)) \to C^*(a) \subset \mathcal{B}$ be the isometric $*$-isomorphism given by Theorem 4 combined with Proposition 2. If $f \in \mathcal{C}(\sigma(a))$ define*

$$f(a) := \Gamma^{-1} \circ \Phi^{\#}(f).$$

*Then the map $f \to f(a)$ from $\mathcal{C}(\sigma(a)) \to \mathcal{B}$ is called the* functional calculus for *$a$.*

Now if $p(z, \bar{z})$ is a polynomial in $z$ and $\bar{z}$ then $\Gamma^{-1} \circ \Phi^{\#}(p(z, \bar{z})) = p(a, a^*)$. Consequently, for polynomials the functional calculus for $a$ is nothing new. The real power of the functional calculus is that it extends this to continuous functions. The following theorem shows that the spectrum transforms naturally under the functional calculus.

**Theorem 5 (Spectral Mapping Theorem)** *If $\mathcal{B}$ is a $C^*$-algebra and $a$ is a normal element of $\mathcal{B}$, then for every $f \in \mathcal{C}(\sigma(a))$,*

$$\sigma(f(a)) = f(\sigma(a)).$$

**Proof:** Because $\Gamma^{-1} \circ \Phi^{\#} : \mathcal{C}(\sigma(a)) \to C^*(a)$ is an isometric $*$-isomorphism, then $\sigma(f(a)) = \sigma(\Gamma^{-1} \circ \Phi^{\#}(f)) = \sigma(f)$. Since $\sigma(f) = \text{Range}(f) = f(\sigma(a))$, then the result follows. □

### 2.1.5 Positivity and the Polar Decomposition

In this section, we will use the functional calculus to study positivity. To begin, if $\mathcal{A}$ is a $C^*$-algebra, let Re $\mathcal{A}$ denote the set of hermitian elements of $\mathcal{A}$.

**Definition 10** *Let $\mathcal{A}$ be a $C^*$-algebra. For $a \in \mathcal{A}$, $a$ is called* positive *if $a \in$ Re $\mathcal{A}$ and $\sigma(a) \subset [0, \infty)$. If $a$ is positive then this is denoted by $a \geq 0$. We also write $a \geq b$ if $a - b \geq 0$. $\mathcal{A}_+$ denotes the set of all positive elements of $\mathcal{A}$.*

**Example 4** *Let $\mathcal{A} = \mathcal{C}(X)$ for some compact Hausdorff space $X$. $\mathcal{A}$ is a unital $C^*$-algebra and for $f \in \mathcal{A}$, $\sigma(f) =$ Range$(f)$ by Example 3. Therefore, $f \geq 0$ if and only if $f(x) \geq 0$ for all $x \in X$.*

For the other example we have been studying, the following theorem characterizes positivity.

**Theorem 6** *If $\mathcal{H}$ is a Hilbert space and $A \in B(\mathcal{H})$, then $A \geq 0$ if and only if $\langle Ah, h \rangle \geq 0$ for all $h \in \mathcal{H}$.*

**Proof:** See [19] Theorem VIII.3.8 or [58] Theorem 12.32. □

We will now use the functional calculus and positivity to get a polar decomposition of an operator on a Hilbert space. Recall that for $\lambda \in \mathbb{C}$, the polar decomposition is given by $\lambda = |\lambda|e^{i\theta}$. To begin, we will need to define the absolute value of an operator. In order to do so, we need the following theorem.

**Theorem 7** *If $\mathcal{A}$ is a $C^*$-algebra and $a \in \mathcal{A}$, then the following statements are equivalent.*
*(i) $a \geq 0$.*
*(ii) $a = b^2$ for some $b \in Re\ (\mathcal{A})$.*
*(iii) $a = x^*x$ for some $x \in \mathcal{A}$.*
*Moreover, if $a \geq 0$ then there is a unique $b \geq 0$ such that $a = b^2$.*

**Proof:** See [19] Section VIII.3. □

Let $|a|$ be defined to be the unique positive element such that $|a| = (a^*a)^{1/2}$. The previous theorem guarantees that $|a|$ is well-defined. To define the phase of the operator the following notion is needed.

**Definition 11** *A* partial isometry *is an operator $W \in B(\mathcal{H})$ such that for $h \in (\ker(W))^\perp$, $\|Wh\| = \|h\|$. $(\ker(W))^\perp$ is called the* initial space *of $W$ and $Range(W)$ is called the* final space *of $W$.*

**Theorem 8 (Polar Decomposition)** *If $A \in B(\mathcal{H})$, then there is a partial isometry $W$ with $(\ker(A))^\perp$ as its initial space and $\overline{Range(A)}$ as its final space such that $A = W|A|$. Moreover, if $A = UP$ where $P \geq 0$ and $U$ is a partial isometry with $\ker(U) = \ker(P)$, then $P = |A|$ and $U = W$.*

**Proof:** See [19] Theorem VIII.3.11. □

### 2.1.6 Representations and the Gelfand-Naimark Theorem

It was mentioned in Section 2.1.1 that Example 1 is a canonical example of a $C^*$-algebra. In this section, we explain this fact by presenting the Gelfand-Naimark-Segal Theorem.

**Definition 12** *A* representation *of a $C^*$-algebra $\mathcal{A}$ is a $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$. If $\pi$ is injective, then the representation is said to be* faithful*. If $\mathcal{A}$ has unit $\mathbf{1}$, then $\pi$ is called* non-degenerate *if $\pi(\mathbf{1}) = \mathbf{1}_{\mathcal{H}}$.*

**Theorem 9 (Gelfand-Naimark Theorem)** *Every unital $C^*$-algebra can be represented isometrically and $*$-isomorphically as a $C^*$-algebra of operators on a Hilbert space.*

**Proof:** See [2] Theorem 4.8.4. □

This theorem was important in the development of $C^*$-algebras because originally it was not understood whether abstract $C^*$-algebras were the same as $C^*$-algebras of operators on Hilbert space. The Gelfand-Naimark Theorem settled this by showing that they were in fact the same. It should be mentioned that the proof of this theorem is quite involved and that the proof also requires many concepts that will not be used in this thesis and consequently have been omitted.

## *2.2 Operator Algebras*

### 2.2.1 Compact Operators on Hilbert Spaces

Compact operators are the natural extension of matrices to infinite dimensional vector spaces. In this section, we only consider compact operators on Hilbert spaces. Most of the information in this section is standard and can be found in [19, 58].

**Definition 13** *Let $\mathcal{H}$ be a Hilbert space and let $U$ be the unit ball in $\mathcal{H}$. A linear transformation $T : \mathcal{H} \to \mathcal{H}$ is* compact *if the closure of $T(U)$ is compact. The set of compact operators on $\mathcal{H}$ is denoted by $\mathcal{K}(\mathcal{H})$.*

It is easy to see from the definition that $\mathcal{K}(\mathcal{H}) \subset B(\mathcal{H})$. In fact more can be said.

**Theorem 10** *$\mathcal{K}(\mathcal{H})$ is a closed two-sided $*$-ideal in $B(\mathcal{H})$. If $\mathcal{H}$ is separable, then $\mathcal{K}$ is the only nontrivial closed ideal of $B(\mathcal{H})$*

**Proof:** See [19] Proposition II.4.2, Theorem II.4.4 and Corollary IX.4.3. □

To show how compact operators are an extension of matrices, the following notion is needed.

**Definition 14** *Let $\mathcal{H}$ be a Hilbert space. A linear transformation $T : \mathcal{H} \to \mathcal{H}$ has* finite rank *if Range$(T)$ is finite dimensional. The set of bounded finite rank operators will be denoted by $\mathcal{K}_0(\mathcal{H})$.*

The notation of the previous definition is justified since $\mathcal{K}_0(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$. It is easy to see that $\mathcal{K}_0(\mathcal{H})$ is a two-sided $*$-ideal in $B(\mathcal{H})$. It is not closed.

**Theorem 11** *If $T \in B(\mathcal{H})$, then $T$ is compact if and only if there is a sequence $\{T_n\}$ of finite rank operators such that $\|T - T_n\|_{B(\mathcal{H})} \to 0$.*

**Proof:** See [19] Theorem II.4.4. □

The spectral properties of compact operators make them particularly nice to work with. Recall that for a Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ is a unital $C^*$-algebra. Therefore, for $T \in \mathcal{K}(\mathcal{H})$ it is possible to study $\sigma(T)$. The following theorem shows that $\sigma(T)$ is countable, compact, with at most one limit point, 0.

**Theorem 12** *If $\dim(\mathcal{H}) = \infty$ and $T \in \mathcal{K}(\mathcal{H})$, then one and only one of the following possibilities occurs:*

*(i)* $\sigma(T) = \{0\}$.

*(ii)* $\sigma(T) = \{0, \lambda_1, \ldots, \lambda_n\}$ *where* $\lambda_k \neq 0$ *and each* $\lambda_k$ *is an eigenvalue of* $T$ *with* $\dim\ker(T - \lambda_k) < \infty$.

*(iii)* $\sigma(T) = \{0, \lambda_1, \lambda_2, \ldots\}$ *where* $\lambda_k \neq 0$ *and each* $\lambda_k$ *is an eigenvalue of* $T$ *with* $\dim\ker(T - \lambda_k) < \infty$. *Moreover,* $\lim_{k\to\infty} \lambda_k = 0$.

**Proof:** See [19] Theorem VII.7.1. □

### 2.2.2 The Trace on Compact Operators

This section follows [20] (Section 18) and [33] (Chapter 7.C). Theorem 12 gives us a starting point to define a trace on compact operators. To begin, if $T$ is a positive compact operator then it is possible to list its nonzero eigenvalues (with multiple eigenvalue repeated) as $s_1 \geq s_2 \geq \cdots \geq s_k \geq \cdots \geq 0$. For a general compact operator $T$, $|T|$ is a positive operator by Theorem 7. Using the Spectral Mapping Theorem along with Theorem 12, it is easy to see that $|T|$ is also compact. Therefore, let $\{s_k(T)\}$ be defined to be the nonzero eigenvalues of $|T|$. Then $s_1(T) \geq s_2(T) \geq \cdots \geq s_k(T) \geq \cdots \geq 0$ and we can make the following definition.

**Definition 15** *For $1 \leq p < \infty$, define the* Schatten $p$-class, *$\mathcal{L}^p(\mathcal{H})$ to be the set of all $T \in \mathcal{K}$ such that*

$$||T||_p := \left( \sum_{k=1}^{\infty} s_k(T)^p \right)^{1/p} < \infty.$$

*$\mathcal{L}^1(\mathcal{H})$ will be called the* trace class *operators.*

Now, let $T \in \mathcal{L}^1(\mathcal{H})$ have eigenvalues $\{\lambda_k\}$ repeated according to multiplicity. Then

$$|\sum_{k=1}^{\infty} \lambda_k| \leq \sum_{k=1}^{\infty} |\lambda_k| = \sum_{k=1}^{\infty} s_k(T) < \infty$$

and therefore it is possible to define the *trace* of $T$ by, $\mathrm{Tr}\,(T) := \sum_{k=1}^{\infty} \lambda_k$. That the trace generalizes the classical trace on matrices is given by the following.

**Proposition 3** *If $A \in \mathcal{L}^1(\mathcal{H})$ and $\mathcal{E}$ is a basis for $H$, then*

$$\mathrm{Tr}\,(A) = \sum_{e \in \mathcal{E}} \langle Ae, e \rangle$$

**Proof:** See [20] Proposition 18.9. $\square$

The trace also has the following properties.

**Theorem 13** *Let $\mathcal{H}$ be a Hilbert space.*

*(i) $\mathcal{L}^1(\mathcal{H})$ is a two-sided $*$-ideal of $B(\mathcal{H})$ and $\|\cdot\|_1$ is a norm on $\mathcal{L}^1(\mathcal{H})$ such that $\|T\| = \|T^*\|$ for every $T \in \mathcal{L}^1(\mathcal{H})$.*

*(ii)* $\mathrm{Tr} : \mathcal{L}^1(\mathcal{H}) \to \mathbb{C}$ *is a positive definite linear functional. That is, if $T \in \mathcal{L}^1(\mathcal{H})$ and $T \geq 0$ with $T \neq 0$, then* $\mathrm{Tr}\,(T) > 0$.

*(iii) If $T \in \mathcal{L}^1(\mathcal{H})$, then* $\mathrm{Tr}\,(AT) = \mathrm{Tr}\,(TA)$ *and* $|\mathrm{Tr}\,(TA)| \leq \|T\|_1\|A\|$ *for every $A \in B(\mathcal{H})$.*

**Proof:** See [20] Theorem 18.11.

### 2.2.3 Unbounded Operators

In this thesis, not every operator will be bounded. In particular, the Laplacian will be an unbounded operator. This section presents some of the basics for unbounded operators as in [19, 58]. Let $\mathcal{H}, \mathcal{H}'$ be Hilbert spaces. A *(linear) operator* is a linear mapping $T : \mathcal{H} \to \mathcal{H}'$ such that $\mathrm{Dom}(T)$ is a linear subspace of $\mathcal{H}$ and $\mathrm{Range}(T)$ is contained in $\mathcal{H}'$. The *graph* of $T$ is the set of all $(h, Th) \in \mathcal{H} \times \mathcal{H}'$ where $h \in \mathrm{Dom}(T)$. A *closed* operator is one whose graph is closed in $\mathcal{H} \times \mathcal{H}'$. Another operator $S : \mathcal{H} \to \mathcal{H}'$ is an *extension* of $T$ if $\mathrm{Dom}(T) \subset \mathrm{Dom}(S)$ and $Sh = Th$ for all $h \in \mathrm{Dom}(T)$. An extension $S$ of $T$ is denoted by $T \subset S$. An operator is *closable* if it has a closed extension.

**Definition 16** *If $T : \mathcal{H} \to \mathcal{H}'$ is a densely defined operator, let*

$$Dom(T^*) := \{h' \in \mathcal{H}' : h \to \langle Th, h' \rangle \text{ is a bounded linear functional on } Dom(T)\}.$$

*For $g \in Dom(T^*)$, since $h \to \langle Th, g \rangle$ is a densely defined bounded linear functional on a subspace of $\mathcal{H}$ then by the Hahn-Banach Theorem this can be extended uniquely to a bounded linear functional on $\mathcal{H}$. By the Riesz Reperesentation Theorem there exists a unique $f \in \mathcal{H}$ such that $\langle Th, g \rangle = \langle h, f \rangle$. Denote this unique vector $f$ by*

*$T^*k = f$. That is,*

$$\langle Th, k\rangle = \langle h, T^*k\rangle.$$

*$T^*$ is called the* adjoint *of $T$.*

**Proposition 4** *If $T : \mathcal{H} \to \mathcal{H}'$ is a densely defined operator, then:*

*(i) $T^*$ is a closed operator,*

*(ii) $T^*$ is densely defined if and only if $T$ is closable,*

*(iii) if $T$ is closable, then its closure is $T^{**} := (T^*)^*$.*

**Proof:** See [19] Proposition X.1.6. □

For unbounded operators, self-adjointness is a more refined concept because the domain of the operator is not the whole Hilbert space.

**Definition 17** *Let $T : \mathcal{H} \to \mathcal{H}$ be a densely defined operator. $T$ is* symmetric *if $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in$ Dom$(T)$.*

It is not hard to see that for a symmetric operator $T$ that $\mathrm{Dom}(T) \subset \mathrm{Dom}(T^*)$ and that $T \subset T^*$. For a bounded operator $T$, $\mathrm{Dom}(T) = \mathcal{H}$ and thus $T^* = T$. However, for unbounded operators this is not necessarily true (see Example X.1.11 from [19] or Example 13.4 from [58]).

**Definition 18** *Let $T : \mathcal{H} \to \mathcal{H}$ be a densely defined operator. $T$ is* self-adjoint *if $T = T^*$.*

The following theorem is useful for showing that a symmetric operator is self-adjoint.

**Theorem 14** *Let $T : \mathcal{H} \to \mathcal{H}$ be a densely defined symmetric operator.*

*(i) If Dom$(T) = \mathcal{H}$, then $T$ is self-adjoint and $T \in B(\mathcal{H})$.*

*(ii) If $T$ is self-adjoint and one-to-one, then Range($T$) is dense in $\mathcal{H}$ and $T^{-1}$ is self-adjoint.*

*(iii) If Range($T$) is dense in $\mathcal{H}$, then $T$ is one-to-one.*

*(iv) If Range($T$) $= \mathcal{H}$, then $T$ is self-adjoint and $T^{-1} \in B(\mathcal{H})$.*

**Proof:** See [58] Theorem 13.11 or [19] Proposition X.2.4.

### 2.2.4 The Spectrum and Resolvent of Unbounded Operators

This section follows [63] Chapter VIII.

**Definition 19** *The* resolvent set, *$\rho(T)$ of a linear operator $T : \mathcal{H} \to \mathcal{H}$ is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda\mathbf{1}$ is a mapping of Dom($T$) with dense range whose inverse belongs to $B(\mathcal{H})$. This inverse will be denoted by $R(\lambda, T)$. The* spectrum *of $T$, $\sigma(T)$, is the complement of the resolvent set. The spectrum can be decomposed into three disjoint sets: $\sigma_p(T)$,$\sigma_c(T)$, and $\sigma_r(T)$. $\sigma_p(T)$ is the* point spectrum *of $T$ and consists of all $\lambda$ such that $T - \lambda\mathbf{1}$ is not invertible. $\sigma_c(T)$ is the* continuous spectrum *of $T$ and consists of all $\lambda$ such that $T - \lambda\mathbf{1}$ has an unbounded inverse with dense domain. $\sigma_r(T)$ is the* residual spectrum *of $T$ and consists of all $\lambda$ such that $T - \lambda\mathbf{1}$ has an inverse whose domain is not dense.*

The point spectrum of the previous definition is easily understood. From the definition, if $\lambda \in \sigma_p(T)$ then $\ker(T - \lambda\mathbf{1}) \neq 0$. Therefore, there exists $h \in \mathcal{H}$ with $h \neq 0$ such that $Th = \lambda h$. That is $T$ has an *eigenvector* $h$ with *eigenvalue* $\lambda$. Conversely, if there exists $h \in \mathcal{H}$ and $\lambda \in \mathbb{C}$ with $h \neq 0$ such that $Th = \lambda h$ then $T - \lambda\mathbf{1}$ is not invertible. Consequently, $\sigma_p(T)$ consists of eigenvalues.

The definition of the resolvent simplifies slightly if $T$ is assumed to be closed.

**Theorem 15** *Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a closed operator. Then for any $\lambda \in \rho(T)$, Dom($R(\lambda, T)$) $= \mathcal{H}$.*

**Proof:** See [63] Section VIII.1. □

Therefore, if $T$ is closed and $\lambda \in \rho(T)$ then $R(\lambda, T)$ is such that

$$R(\lambda, T)(T - \lambda\mathbf{1}) \subset (T - \lambda\mathbf{1})R(\lambda, T) = \mathbf{1}.$$

Some of the properties of the spectrum and resolvent of a bounded operator carry over into the unbounded case.

**Theorem 16** *Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a closed operator. Then $\rho(T)$ is an open subset of $\mathbb{C}$. Moreover, in each component of $\rho(T)$, $R(\lambda, T)$ is an analytic function of $\lambda$.*

**Proof:** The basic idea is that if $\lambda_0 \in \rho(T)$, then for $\lambda$ such that $|\lambda - \lambda_0| \|R(\lambda_0, T)\| < 1$,

$$R(\lambda_0, T) \sum_{k=0}^{\infty} (\lambda - \lambda_0)^n R(\lambda_0, T)^n$$

is an inverse for $T - \lambda\mathbf{1}$. For a full proof, see [63] Section VIII.2. □

It turns out that for $\lambda, \mu \in \rho(T)$, that $R(\lambda, T)$ and $R(\mu, T)$ are related by a very important equation called the resolvent equation.

**Lemma 3 (Resolvent Equation)** *If $\lambda, \mu \in \rho(T)$ are such that $R(\lambda, T)$ and $R(\mu, T)$ are everywhere defined bounded operators, then*

$$R(\lambda, T) - R(\mu, T) = (\lambda - \mu)R(\lambda, T)R(\mu, T).$$

**Proof:** Since $R(\mu, T)$ is everywhere defined then $(T - \mu\mathbf{1})R(\mu, T) = \mathbf{1}$ and

$$\begin{aligned} R(\lambda, T) &= R(\lambda, T)(T - \mu\mathbf{1})R(\mu, T) = R(\lambda, T)(T - \lambda\mathbf{1} + (\lambda - \mu)\mathbf{1})R(\mu, T) \\ &= R(\mu, T) + (\lambda - \mu)R(\lambda, T)R(\mu, T). \end{aligned}$$

□

Note that if $T$ is a closed operator then the hypothesis of the resolvent equation holds for all $\lambda \in \rho(T)$. The resolvent equation has several interesting consequences. First of all, if $T$ is closed then $R(\lambda, T)$ and $R(\mu, T)$ commute for $\lambda, \mu \in \rho(T)$. Secondly, if $T$ is closed and if $R(\lambda, T)$ is in some two-sided ideal of $B(\mathcal{H})$ then we automatically know that $R(\mu, T)$ is in this ideal for all $\mu \in \rho(T)$. In particular, if $R(\lambda, T)$ is compact for some $\lambda \in \rho(T)$ then it is compact for all $\lambda \in \rho(T)$. In this case, we say that $T$ had *compact resolvent*.

## *2.3 Probability*

### 2.3.1 Basic Notions

This section presents the basics of measure theoretic probability theory following [3, 56]. To begin, given a space $\Omega$ and a $\sigma$-algebra $\mathcal{F}$ on $\Omega$, a *probability measure* $\mathbb{P}$ is a positive finite measure on $\Omega$ such that $\mathbb{P}(\Omega) = 1$. $(\Omega, \mathcal{F}, \mathbb{P})$ is then called a *probability space* or *probability triple*. Elements $E$ of $\mathcal{F}$ are called *events* and $\mathbb{P}(E)$ is called the *probability of the event $E$*. If an event occurs with probability one, then the event is said to occur *almost surely* and is often written a.s.

Measurable functions from $\Omega$ to $\mathbb{R}$ are called *random variables* and are usually denoted by $X$ or $Y$. The integral of a random variable $X$ with respect to $\mathbb{P}$ is called the *expectation* of $X$ and is written $\mathbb{E}(X)$. That is,

$$\mathbb{E}(X) := \int_\Omega X d\mathbb{P} = \int_\Omega X(\omega) d\mathbb{P}(\omega).$$

For $A \in \mathcal{F}$, we write $\mathbb{E}(X; A)$ for $\int_A X d\mathbb{P}$. The *variance* of a random variable $X$ is defined by $\mathrm{Var}(X) := \mathbb{E}(X^2) - (\mathbb{E}(X))^2$. The *law* or *distribution* of a random variable $X$ is the probability measure $\mathbb{P}_X$ on $\mathbb{R}$ given by

$$\mathbb{P}_X(A) = \mathbb{P}(X \in A)$$

where the event $X \in A$ stands for $\{\omega : X(\omega) \in A\}$. For every random variable $X$, there is a $\sigma$-algebra associated to $X$, $\sigma(X)$, which is the smallest $\sigma$-algebra for which $X$ is measurable.

Two events are *independent* if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Two sub-$\sigma$-algebras $\mathcal{G}_1, \mathcal{G}_2$ of $\mathcal{F}$ are independent if every $A \in \mathcal{G}_1$ is independent of every $B \in \mathcal{G}_2$. Two random variables $X$ and $Y$ are independent if $\sigma(X)$ and $\sigma(Y)$ are independent. These definitions can also be generalized to multiple events. Let $\delta \in \{0, 1\}$ and let $A_i^0 = A_i$ and $A_i^1 = A_i^c$. Then $A_1, \ldots, A_n$ are independent if $\mathbb{P}(A_{i_1}^{\delta_1} \cap \cdots \cap A_{i_k}^{\delta_k}) = \mathbb{P}(A_{i_1}^{\delta_1}) \cdots \mathbb{P}(A_{i_k}^{\delta_k})$ for all $1 \leq i_1 \leq \cdots \leq i_k \leq n$ and $\delta_j \in \{0, 1\}$.

An extremely important concept in the realm of probability is the idea of conditional expectation of a random variable. Unfortunately, the existence of the conditional expectation must be presented as a theorem whose proof relies on the Radon-Nikodym Theorem.

**Theorem 17** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X$ be a random variable with $\mathbb{E}(|X|) < \infty$. Let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Then there exists a random variable $Y$ such that:*

*(i) $Y$ is $\mathcal{G}$-measurable;*

*(ii) $\mathbb{E}(|Y|) < \infty$;*

*(iii) for every $G \in \mathcal{G}$, $\mathbb{E}(Y; G) = \mathbb{E}(X; G)$.*

*Moreover, if $Z$ is another random variable with these properties then $Z = Y$ a.s. A random variable $Y$ with properties (i)-(iii) is called a* version of the conditional expectation $\mathbb{E}(X|\mathcal{G})$ of $X$ given $\mathcal{G}$ *and is written $Y = \mathbb{E}(X|\mathcal{G})$ a.s.*

**Proof:** See [56] Section II.39. □

For another random variable $Z$, $\mathbb{E}(X|\sigma(Z))$ is denoted by $\mathbb{E}(X|Z)$. The conditional expectation satisfies the following three important properties (see [56] Section II.41). First, if $X$ is $\mathcal{G}$-measurable then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$. Second, $\mathbb{E}(a_1X_1 + a_2X_2|\mathcal{G}) =$

$a_1\mathbb{E}(X_1|\mathcal{G}) + a_2\mathbb{E}(X_2|\mathcal{G})$ a.s. Third, if $X$ is independent of $\mathcal{G}$ then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

### 2.3.2 Daniell-Kolmogorov Consistency Theorem

The Daniell-Kolmogorov Consistency Theorem is an essential step in the construction of many stochastic processes. It will also be used to construct a measure on the space of choices in Section 4.3.2. We present a version of the theorem here as it will be used several times throughout this thesis. This version follows [56] Section II.30. The following notions will be necessary for the theorem. Given a metric space $E$, let $\mathfrak{B}(E)$ denote the $\sigma$-algebra of all Borel subsets of $E$. For a set $T$, let $E^T$ denote the set of all functions from $T$ to $E$. For $t \in T$, let $\pi_t : E^T \to E$ be defined to be the evaluation map $\pi_t(f) := f(t)$ for $f \in E^T$. Let $\mathcal{E}^T := \sigma(\{\pi_t : t \in T\})$. That is $\mathcal{E}^T$ is the smallest $\sigma$-algebra on $E^T$ such that $\pi_t$ is measurable as a map from $E^T$ to $E$. Let $\mathrm{Fin}(T)$ denote the the family of non-empty finite subsets of $T$.

**Theorem 18** *Let $E$ be a compact metrizable space, and let $\mathcal{E} = \mathfrak{B}(E)$. Let $T$ be a set. Suppose that for each $S$ in $Fin(T)$, there exists a probability measure $\mu_S$ on $(E^S, \mathcal{E}^S)$, and that the measure $\{\mu_S : S \in Fin(T)\}$ are* compatible *or* projective *in that*

$$\mu_U = \mu_V \circ (\pi_U^V)^{-1}$$

*holds whenever $U, V \in Fin(T)$ and $U \subset V$. Here $\pi_U^V$ is the restriction map from $E^V$ to $E^U$. Then there exists a unique measure $\mu$ on $(E^T, \mathcal{E}^T)$ such that*

$$\mu_S = \mu \circ \pi_S^{-1}$$

*on $(E^S, \mathcal{E}^S)$ where $\pi_S$ is the restriction map from $E^T$ to $E^S$.*

**Proof:** See [56] Section II.30. □

### 2.3.3 Stochastic Processes

Stochastic processes are the natural notion of a random occurrence with continuous time. In this section, we develop the basic notions of stochastic processes as in [3]. It is necessary in this thesis to consider random occurrences that do not necessarily have values in $\mathbb{R}$. Given a measurable space $(E, \mathcal{G})$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an *$E$-valued random variable* is a measurable map $X : \Omega \to E$.

**Definition 20** *Let $I$ be a set, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, and $(E, \mathcal{G})$ a measurable space. A quadruple $(\Omega, \mathcal{F}, \mathbb{P}, (X_t)_{t\in I})$, where $(X_t)_{t\in I}$ is a family of $E$-valued random variables on $\Omega$, is called a* stochastic process*. We call $I$ the* parameter *or* time *set and $E$ the* state space *of the stochastic process. For every $\omega \in \Omega$, the mapping $t \to X_t(\omega)$ is a called a* sample path *or* trajectory *of the process.*

In most cases, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given and $(\Omega, \mathcal{F}, \mathbb{P}, (X_t)_{t\in I})$ is often denoted by simply $(X_t)_{t\in I}$. There is also another way of looking at a stochastic process that ties in with the previous section. To do so let $J \in \mathrm{Fin}(I)$ and consider the measurable space $(E^J, \mathcal{G}^J)$. Then $X_J := \bigotimes_{t\in J} X_t$ is an $E^J$-valued random variable on $\Omega$. It is then possible to define a probability measure on $E^J$ by $\mathbb{P}_{X_J}$ where $\mathbb{P}_{X_J}(A) = \mathbb{P}(X_J \in A)$ for $A \in \mathcal{G}^J$. Now for $H, J \in \mathrm{Fin}(I)$ with $H \subset J$, there exists the projection map $\pi^J_H : E^J \to E^H$ by restriction. This map is measurable by the construction of $\mathcal{G}^J$ and $\mathcal{G}^H$. Since $X_H = \pi^J_H \circ X_J$, then $\mathbb{P}_{X_H} = \mathbb{P}_{X_J} \circ (\pi^J_H)^{-1}$. Therefore, $\{\mathbb{P}_{X_J} : J \in \mathrm{Fin}(I)\}$ is a family of projective measures. This family is called the *family of finite-dimensional distributions* of the process.

The Daniell-Kolmogorov Theorem then gives a converse to the above construction and allows to construct a stochastic process out of a family of finite dimensional distributions. The probability space is defined by letting $\Omega = E^I$, $\mathcal{F} = \mathcal{G}^I$, and $\mathbb{P} = \mathbb{P}_I$ - the unique measure given by the theorem. For $\omega \in \Omega$, $X_t(\omega) := \omega(t)$. This process is called the *canonical process* associated to a family of finite-dimensional distributions.

### 2.3.4 Markov Processes

Most of the interest in stochastic processes arises from studying a specific property shared by a class of stochastic processes. To this end, we now show a construction that gives a stochastic process that has the Markov property.

**Definition 21** *Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces. A* kernel from $(\Omega, \mathcal{F})$ to $(\Omega', \mathcal{F}')$ *is a function $K : \Omega \times \mathcal{F}' \to \mathbb{R}$ such that*

*(i) $\omega \to K(\omega, A')$ is $\mathcal{F}$-measurable for every $A' \in \mathcal{F}'$;*

*(ii) $A' \to K(\omega, A')$ is a measure on $\mathcal{F}'$ for every $\omega \in \Omega$.*

*A kernel $K$ is called* Markov (sub-Markov) *if $K(\omega, \Omega') = 1$ $(K(\omega, \Omega') \leq 1)$ for all $\omega \in \Omega$. If $\Omega = \Omega'$ and $\mathcal{F} = \mathcal{F}'$ then we say that this is a kernel on $\Omega$.*

We now develop an example of a kernel that will be used later. Given a $\sigma$-finite measure space $(\Omega, \mathcal{F}, \mu)$ and a $\mathcal{F} \otimes \mathcal{F}$-measurable function $k : \Omega \times \Omega \to \mathbb{R}$ such that $k \geq 0$, let

$$K(\omega, A) := \int_A k(\omega, \omega') d\mu(\omega').$$

Then by Fubini's Theorem and [57] Theorem 1.29, $K$ is a kernel on $\Omega$. We want to be studying stochastic processes so we will need a family of kernels.

**Definition 22** *Let $(P_t)_{t \in \mathbb{R}^+}$ be a family of kernels on a measurable space $(\Omega, \mathcal{F})$ indexed by $\mathbb{R}^+$. If for $\omega \in \Omega$ and $B \in \mathcal{F}$,*

$$P_{s+t}(\omega, B) = \int_\Omega P_s(\omega, d\omega') P_t(\omega', B)$$

*for all $s, t \in \mathbb{R}^+$ then we call $(P_t)_{t \in \mathbb{R}^+}$ a* semigroup of kernels *on $E$. If in addition all kernels $P_t$ are sub-Markov (Markov) then the semigroup is said to be* sub-Markov (Markov)*.*

The requirement that $P_{s+t}(\omega, B) = \int_\Omega P_s(\omega, d\omega')P_t(\omega', B)$ in the previous definition is known as the *Chapmann-Kolmogorov equation*. There is another way to look at the family of kernels $(P_t)_{t\in\mathbb{R}^+}$ that will be used later. For the characteristic function of $A$, $\chi_A$, we can define $(P_t\chi_A)(\omega) := P_t(\omega, A)$ for all $\omega \in \Omega$. This construction can easily be extended to all bounded measurable functions $f : \Omega \to \mathbb{R}$ by $(P_tf)(\omega) = \int_\Omega f(\omega')P_t(\omega, d\omega')$. Because $P_t(x, d\omega')$ is a measure and $f$ is measurable and bounded, then $P_tf$ is also a bounded measurable function. Therefore, $P_t$ can be seen as an operator on the space of bounded measurable functions on $\Omega$. As an operator, the Chapmann-Kolmogorov equations become $P_{s+t} = P_sP_t$ for all $s, t \in \mathbb{R}^+$. This semigroup property will be taken up in Section 4.4. The following theorem is the first step to construct a stochastic process out of a family of kernels.

**Theorem 19** *On a measurable space $(E, \mathcal{E})$, let $(P_t)_{t\in\mathbb{R}^+}$ be a Markov semigroup and let $\mu$ be a probability measure on $E$. For each $J \in Fin(\mathbb{R}^+)$ with $J = \{t_1, \ldots, t_n\}$, let*

$$\mathbb{P}_J(A) := \int_E \int_E \cdots \int_E \chi_A(x_1, \ldots, x_n)P_{t_n-t_{n-1}}(x_{n-1}, dx_n) \cdots P_{t_1}(x_0, dx_1)d\mu(x_0).$$

*Then $(P_J)_{J\in Fin(\mathbb{R}^+)}$ is a projective family of measures on $(E^{\mathbb{R}^+}, \mathcal{E}^{\mathbb{R}^+})$.*

**Proof:** See [3] Theorem 12.3.2. □

Let $E$ be a compact metrizable space. Then by the Daniell-Kolmogorov Theorem, there exists a probability measure $\mathbb{P}^\mu$ on $\Omega^{\mathbb{R}^+}$. The canonical process associated to this construction then gives a stochastic process $(X_t)_{t\in\mathbb{R}^+}$. A nice feature of this construction is that

$$
\begin{aligned}
\mathbb{E}(f(X_{t_1},\dots,X_{t_n})) &= \int_{E^{\mathbb{R}^+}} f(X_{t_1}(\omega),\dots,X_{t_n}(\omega))\mathbb{P}^{\mu}(\omega) \\
&= \int_{E^{\mathbb{R}^+}} f(\omega(t_1),\dots,\omega(t_n))\mathbb{P}^{\mu}(\omega) \\
&= \int_{E^{\mathbb{R}^+}} f(\omega(t_1),\dots,\omega(t_n))\mathbb{P}_{\{t_1,\dots,t_n\}}(\omega) \\
&= \int_E \cdots \int_E P_{t_n-t_{n-1}}(x_{n-1},dx_n)P_{t_{n-1}-t_{n-2}}(x_{n-2},dx_{n-1}) \\
&\quad \cdots P_{t_1}(x_0,dx_1)f(x_1,\dots,x_n)d\mu(x_0)
\end{aligned}
$$

This process then has an important property.

**Definition 23** *A stochastic process $(X_t)_{t\in I}$ on a totally ordered parameter set $I$ has the* elementary Markov property *if for every $B \in \mathcal{F}$ and every $s,t \in I$ with $s < t$,*

$$\mathbb{P}(X_t \in B|\mathcal{F}_s) = \mathbb{P}(X_t \in B|X_s), \quad \mathbb{P} - a.s$$

*where $\mathcal{F}_s = \sigma(\{X_r : r \leq s\})$.*

Conceptually, the Markov property means that a process does not have a memory of its past. It only depends on its current position.

**Theorem 20** *Let $(\Omega, \mathcal{F}, \mathbb{P}^{\mu}, (X_t)_{t\in\mathbb{R}^+})$ be a stochastic process with state space $(E,\mathcal{E})$ and $\mathbb{R}^+$ as parameter set, whose finite dimensional distributions are derived as in Theorem 19 from a Markov semigroup $(P_t)_{t\in\mathbb{R}^+}$ and a starting probability $\mu$ on $(E,\mathcal{E})$. Then the process has the elementary Markov property. Moreover, for $B \in \mathcal{E}$ and $s,t \in \mathbb{R}^+$ with $s < t$,*

$$\mathbb{P}^{\mu}(X_t \in B|\mathcal{F}_s) = P_{t-s}(X_s, B), \quad \mathbb{P}^{\mu} - a.s.$$

We now use the techniques of this section to construct Brownian motion.

**Example 5** *For $t \in (0,\infty)$, let $p_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by*

$$p_t(x,y) := (2\pi t)^{-1/2} \exp[-\frac{(x-y)^2}{2t}].$$

*Then, $p_t$ is often called the* Brownian transition density. *Since $p_t \geq 0$ and continuous then it defines a family $(P_t)_{t>0}$ by $P_t(x,A) := \int_A p_t(x,y)dy$ for $A \in \mathfrak{B}(\mathbb{R})$. Let $P_0 = 1$. Since $\int p_t(x,y)dy = 1$, then $P_t$ is Markov for all $t \geq 0$. Let $\delta_0$ be the original probability measure on $\mathbb{R}$. Then it turns out that the Daniell-Kolmogorov Theorem can be extended to Polish spaces and thus $(P_t)_{t\geq 0}$ defines a stochastic process $(X_t)_{t\geq 0}$ that has the Markov property. This process is such that for $t > s$,*

$$\begin{aligned}
\mathbb{E}^0(X_t - X_s) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} (x_2 - x_1) p_{t-s}(x_1,x_2) p_s(x_0,x_1) dx_1 dx_2 d\delta_0(x_0) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} (x_2 - x_1) p_{t-s}(x_1,x_2) p_s(0,x_1) dx_1 dx_2 \\
&= \int_{\mathbb{R}} p_s(0,x_1) \left( \int_{\mathbb{R}} (x_2 - x_1) p_{t-s}(x_1,x_2) dx_2 \right) dx_1.
\end{aligned}$$

*But $(x_2 - x_1)p_{t-s}(x_1,x_2)$ is an odd function in $x_2$ so $\mathbb{E}^0(X_t - X_s) = 0$. Also, for $t \geq 0$*

$$\begin{aligned}
\mathbb{E}^0((X_{t+s} - X_s)^2) &= \int_{\mathbb{R}}\int_{\mathbb{R}} (x_2 - x_1)^2 p_t(x_1,x_2) p_s(0,x_1) dx_1 dx_2 \\
&= \int_{\mathbb{R}} p_s(0,x_1) \left( \int_{\mathbb{R}} (x_2 - x_1)^2 p_t(x_1,x_2) dx_2 \right) dx_1 \\
&= t \int_{\mathbb{R}} p_s(0,x_1) = t.
\end{aligned}$$

*Thus $(X_t)_{t\geq 0}$ is such that $X_t - X_s$ is normally distributed with mean zero and variance $t - s$ for $t > s$. Moreover, $X_0 = 0$ a.s.*

**Definition 24** *A real-valued stochastic process $(W_t)_{t\geq 0}$ is a* Brownian motion *if it has the properties:*

*(i) $W_0(\omega) = 0$ for every $\omega \in \Omega$;*

*(ii) the map $t \to W_t(\omega)$ is a continuous function of $t$ for all $\omega \in \Omega$;*

*(iii) for every* $t, s \geq 0$*,* $W_{t+s} - W_s$ *is independent of* $\{W_r : 0 \leq r \leq s\}$ *and has a Gaussian distribution with mean* $0$ *and variance* $t$*.*

The process constructed in the example is therefore not a Brownian motion since it does not necessarily have continuous paths. However, it is a theorem of Kolmogorov-Centsov (see [44] Theorem 2.8) that the above process can be modified to have continuous paths and consequently a Brownian motion.

# CHAPTER III

# ROOTED TREES AND THE MICHON CORRESPONDENCE

## *3.1 Rooted Trees*

### 3.1.1 Basic Definitions

This section is a reminder about rooted trees following [8]. A graph is an ordered pair of disjoint sets $G = \{\mathcal{V}, \mathcal{E}\}$ where $\mathcal{E}$ is a subset of the set $\mathcal{V}^{(2)}$ of unordered pairs of $\mathcal{V}$. The set $\mathcal{V}$ is the set of *vertices* and the set $\mathcal{E}$ is the set of *edges*. $\mathcal{V}$ will be assumed to be a non-empty countable set. Moreover, $G$ will be assumed to be *simple*. That is, (i) there is no $e \in \mathcal{E}$ such that $e = \{v, v\}$ and (ii) if $e = \{v, w\}$ and $e' = \{v, w\}$ or $e' = \{w, v\}$ then $e = e'$. An edge $\{v, w\}$ is said to *join* or *link* $v$ and $w$ and will be denoted by $vw$. Thus $vw$ and $wv$ denote the same edge. For $u, v \in \mathcal{V}$ if $uv \in \mathcal{E}$ then $u$ and $v$ are said to be *adjacent* or *neighbors* and $u$ and $v$ are said to be *incident* with edge $uv$. The *degree* $|v|$ of a vertex $v \in \mathcal{V}$ is the number of edges $e \in \mathcal{E}$ such that $v$ is incident to $e$. In this thesis, it will be assumed that the degree is always finite.

A *walk* on a graph $G$ is a double sequence $\{(v_0, v_1, \cdots, v_{n-1}, v_n)\,; (e_1, e_2, \cdots, e_n)\}$ (where $n$ is finite or infinite) of incident vertices and edges linking them such that $e_i = v_{i-1}v_i$ for all $i > 0$. For a simple graph it is sufficient to specify the sequence of vertices. A *step* of the walk is a triple of the form $(v_{i-1}, v_i, e_i)$. The length of the walk is the number $n$ of steps making this walk. If the walk is finite, the first and the last vertices of the sequence are said to be *linked* by the walk. If $v$ is one of the vertices of the walk, then the latter is said to *pass* through $v$. A *path* is a walk with pairwise distinct vertices. The graph $G$ is *connected* if given any two vertices there is a finite path linking them.

A *cycle* is a finite walk with at least three steps, such that the first and the last vertices coincide and all other vertices are pairwise distinct. A *tree* is a connected graph with no cycle. A *rooted tree* is a pair $(\mathcal{T}, 0)$ where $\mathcal{T}$ is a tree and 0 is a vertex of $\mathcal{T}$ called the *root*. By abuse of notation, $\mathcal{T}$ will denote a rooted tree, and the root will be implicit. Since $\mathcal{T}$ is a tree and therefore contains no cycle, then given any pair of distinct vertices there is one and only one path linking them. In particular, there is a unique path linking the root to a given vertex. So that there is a one-to-one correspondence between the set of vertices and the set of finite paths starting at the root.

On a rooted tree $\mathcal{T}$ there is a partial order defined by $v \succeq w$ if the path from the root to $w$ necessarily passes through $v$. Then $w$ is called a *descendant* of $v$ and this will also be written as $w \preceq v$, while $v$ will be called an *ancestor* of $w$. If, in addition, $v, w$ are adjacent, then $v$ is called the *father* of $w$ and $w$ is called a *child* of $v$. If $v$ and $w$ are not related, then we write $v \nsim w$. The *height*, $\mathrm{ht}(v)$, of a vertex $v$ is the length of the unique path linking the root to $v$. Hence the root has height 0, its children have height 1 and so on.

### 3.1.2 The Boundary of a Rooted Tree

In this section $\mathcal{T}$ will denote a rooted tree with root 0. The set $\mathcal{V}$ of its vertices is endowed with the discrete topology. If $\mathcal{V}$ is finite, then $\mathcal{T}$ is compact with the discrete topology. However, if $\mathcal{V}$ is infinite it is certainly not compact. A compactification of the tree can be defined by considering the *boundary* $\partial\mathcal{T}$ of this tree defined as follows:

**Definition 25** *If $\mathcal{T}$ is a rooted tree, its boundary $\partial\mathcal{T}$ is the set of infinite paths starting at the root.*

If the set $\mathcal{V}$ of vertices of $\mathcal{T}$ is finite then $\partial\mathcal{T}$ is empty. This corresponds to the fact that $\mathcal{V}$ is already compact with the discrete topology. Therefore, in what follows we will only consider *infinite* trees, that is trees $\mathcal{T}$ such that $\mathcal{V}$ is infinite. In addition, a vertex is *dangling* if it has no child. Hence the boundary ignores dangling vertices. In what follows, only trees with no dangling vertices will be considered. This implies among other things that every finite path can be extended to an infinite path.

**Example 6** Let $T_2$ be the infinite binary rooted tree. That is, $T_2$ is the tree with a root and such that every vertex has exactly two children. Since every vertex has two children the edge linking it to one child will labeled by 0 and the other by 1. Hence any finite path starting at the root, and therefore any vertex, is labeled by a finite sequence of 0's and 1's. The root is given by the empty sequence. Thus $\mathcal{T}_2$ can be seen as the set of finite sequences of of 0's and 1's. Consequently, $\partial\mathcal{T}_2 = \{0,1\}^{\mathbb{N}}$. Let $\Theta : \partial\mathcal{T}_2 \to \mathbb{R}$ be defined by

$$\Theta((\epsilon_n)_{n\in\mathbb{N}}) := \sum_{i=0}^{\infty} \frac{2\epsilon_i}{3^{i+1}}.$$

Then $\Theta$ defines a one-to-one map from $\partial\mathcal{T}_2$ onto the classical triadic Cantor set.

The classical triadic Cantor set is constructed using gaps and intervals of the real line. The following definition gives the generalization of the interval in the case of a Cantor set.

**Definition 26** *Let $\mathcal{T}$ be a rooted tree. If $v$ is a vertex, $[v] \subset \partial\mathcal{T}$ denotes the set of infinite paths starting at the root and passing through $v$.*

**Proposition 5** *Let $\mathcal{T}$ be a rooted tree with no dangling vertex. Then, the set $\{[v]\,;\, v \in \mathcal{V}\}$ is a basis of open sets for a topology on $\partial\mathcal{T}$ for which $\partial\mathcal{T}$ is completely disconnected. For this topology $\partial\mathcal{T}$ is compact if and only if each vertex has at most a finite number of children. It has no isolated points if and only if each vertex has one descendant with at least two children.*

**Proof:** (i) Clearly the family covers $\partial\mathcal{T}$ since $[0] = \partial\mathcal{T}$. Moreover, if $v, w \in \mathcal{V}$ then either $v \nsim w$ or one of the two vertices is an ancestor of the other. In particular if, say $v \succeq w$, then $[v] \cap [w] = [w]$. If $v \nsim w$ then $[v] \cap [w] = \emptyset$. Thus, $\{[v]\,;\, v \in \mathcal{V}\}$ is indeed a basis for a topology on $\partial\mathcal{T}$.

(ii) Let $v \in \mathcal{V}$. Then let $\mathcal{V}(v)$ be the set of vertices with same height as $v$. Clearly if $w \neq v$ and $w \in \mathcal{V}(v)$ then $w$ is not comparable to $v$, hence $[v] \cap [w] = \emptyset$. Moreover, if $x \in \partial\mathcal{T}$ is an infinite path starting at the root, one of its vertices, say $w$, is such that $w \in \mathcal{V}(v)$ and $x \in [w]$. Consequently, the family $\{[w]\,;\, w \in \mathcal{V}(v)\}$ is a partition made of open sets. In particular, the complement of $[v]$ is the union of open sets and is open as well. Hence, for any vertex $v$, the set $[v]$ is a closed and open set (or a *clopen* set), so that $\partial\mathcal{T}$ is completely disconnected.

(iii) If there is a vertex $v$ having an infinite number of children, the family $\{[w]\,;\, w \text{ is a child of } v\}$ defines an open covering of $[v]$ from which no finite covering can be extracted since this is a partition. Thus $[v]$, which is closed, cannot be compact and thus $\partial\mathcal{T}$ cannot be compact either.

(iv) Conversely, let $\mathcal{T}$ be such that each of its vertices has only finitely many children and let $\mathcal{O}$ be an open cover of $\partial\mathcal{T}$. There exists an $N$ such that, for each $v \in \mathcal{V}$ of height $N$, there is an $O_v \in \mathcal{O}$ with $[v] \subset O_v$. Suppose not. Then there exists a sequence of vertices $v_0 v_1 \cdots$ such that each $[v_k]$ is not covered by any $O \in \mathcal{O}$. Moreover, this sequence actually gives an infinite path $\sigma = v_0' v_1' \cdots$ such that each $v_k'$ is not covered by any single $O \in \mathcal{O}$. This path is constructed as follows. One of the children of the root, called $v_1'$, must contain an infinite number of $v_k$. In the same way, one of the children of $v_1'$, called $v_2'$ must contain an infinite number of $v_k$. Proceeding recursively, an infinite sequence $v_0' v_1' \cdots$ is obtained such that (i) for each $n \geq 0$, $v_k'$ is a child of $v_{k-1}'$ and (ii) $[v_k']$ is not covered by any single $O \in \mathcal{O}$. Then $v_0' v_1' \cdots \in \partial\mathcal{T}$ and is not covered by $\mathcal{O}$ which contradicts the fact that $\mathcal{O}$ is an open cover. Consequently, since each vertex has only a finite number of children, then there are only a finite number of vertices of height $N$. Therefore, $\mathcal{O}$ has a finite subcover and $\partial\mathcal{T}$ is compact.

(v) Let $v$ be a vertex of $\mathcal{T}$ such that none of its descendants has more than one child. Then $[v]$ is reduced to one single path $x$ which is itself an open set. Hence $x$ is isolated. Conversely, if $x \in \partial\mathcal{T}$ is isolated, then $\{x\}$ is open, meaning that it contains at least one nonempty element of the basis. Hence there is $v \in \mathcal{V}$ such that $[v] \subset \{x\}$. But this can happen only if each descendant of $v$ has only one child, since otherwise, $[v]$ would contain at least two distinct infinite paths. □

**Definition 27** *A tree will be called Cantorian if it has a root, no dangling vertex and if each vertex has a finite number of children as well as a descendant with more than one child.*

**Remark 1** By Prop. 5 this definition is equivalent to $\partial\mathcal{T}$ is a Cantor set. Sometimes the condition that each vertex has a finite number of children is called locally finite.

Various surgical operations on a tree lead to similar boundaries. The first operation is *edge reduction*. Namely if there is a path $\gamma$ linking $v$ to one of its descendant's $w$ such that each vertex of this path distinct from $v, w$ has only one child, then the graph can be reduced by suppressing these vertices and replacing the path by one edge. Hence if $x \in \partial\mathcal{T}$ is any path passing through $v$ and $w$, it also automatically passes through all of the vertices of $\gamma$. Then it can also be reduced and the reduction operation gives a one-to-one mapping between the boundary of the initial tree and the boundary of the reduced one. In addition $[u] = [w]$ whenever $v \succeq u \succeq w$ and $u \neq w$, so that this mapping is actually a homeomorphism.

The opposite of edge reduction will be called *edge extension*. Namely any edge can be replaced by a finite path with the same end points so that each internal vertex of the path has only one child.

There is also the notion of *vertex extension*. Namely if $v$ is a vertex with at least three children then one child will be called $v_0$ and the others $v_1, \cdots, v_r$. Then a new vertex $u$ is created as a child of $v$ having $v_1, \cdots, v_r$ as children. Since each infinite path in the extended tree that passes through $v$ must eventually pass through exactly one of $v_0, \cdots, v_r$, then this vertex extension also defines a homeomorphism between the corresponding boundaries. In particular this implies the following proposition.

**Proposition 6** *Let $\mathcal{T}$ be a Cantorian tree. Then there is a map made up of the product of a (possibly infinite) family of edge reductions, edge extensions and vertex extensions, mapping $\mathcal{T}$ onto the binary tree $\mathcal{T}_2$ and defining a homeomorphism of their boundaries.*

Michon's correspondence will show how to associate a Cantorian tree to every Cantor set. Using this fact, this proposition is then one of the many ways of showing that every Cantor set is homeomorphic to $\{0, 1\}^{\mathbb{N}}$.

**Definition 28** *Let $\mathcal{T}$ be a Cantorian tree. If $A \subset \partial\mathcal{T}$ then a vertex $v$ is a common ancestor of $A$ if $A \subset [v]$. If $A$ has more than one point, its least common prefix (or l.c.p.) is the smallest of its ancestors. If $A = \{x, y\}$ the least common prefix will be denoted by $x \wedge y$.*

**Proposition 7** *Let $\mathcal{T}$ be a Cantorian tree. The l.c.p. of a subset $A \subset \partial\mathcal{T}$ with more than one point always exists and is unique.*

**Proof:** Since $[0] = \partial\mathcal{T}$ it follows that $A$ always admits the root as an ancestor. Now if $v$ and $w$ are both common ancestors of $A$, then since $A \subset [v] \cap [w]$ is non-empty it follows that one of the two vertices, say $v$ is an ancestor of the other, so that $A \subset [w] \subset [v]$. Hence the set of common ancestors of $A$ is totally ordered. Since it is at most countable, this set defines a path $x = (0 = v_0, v_1, \cdots, v_n)$. Since $A$ contains at least two distinct points, this path is automatically finite because otherwise the intersection $\bigcap_{i \geq 0}[v_i]$ would be reduced to $\{x\}$ and would contain $A$, a contradiction. Thus $v_n$ is the least common ancestor and is unique. $\square$

## 3.2 *Michon's Correspondence*

For the sake of the reader, this section recalls Michon's correspondence between regular ultrametrics on a Cantor set $C$, profinite structures on $C$, and weighted, rooted trees.

### 3.2.1 Ultrametrics and Profinite Structures

This section shows the correspondence between ultrametrics and profinite structures on $C$ as in [53] . Let $C$ be a Cantor set with regular metric $d$. Following [40], given $\epsilon > 0$ and $x, y \in C$ let an *$\epsilon$-chain* be a sequence $x_0 = x, x_1, \dots x_{n-1}, x_n = y$ of points in $C$ such that $d(x_i, x_{i+1}) < \epsilon$. This gives rise to an equivalence relation $\overset{\epsilon}{\sim}$ by defining $x \overset{\epsilon}{\sim} y$ if there is an $\epsilon$-chain between them. In such a case, $[x]_\epsilon$ will denote the equivalence class of $x \in C$. It is then possible to define the *separation* of $x$ and $y$ by $\delta(x, y) := \inf\{\epsilon : x \overset{\epsilon}{\sim} y\}$.

**Proposition 8** *Let $C$ be a Cantor set with regular metric $d$. Then the separation $\delta$ is the maximum ultrametric on $C$ dominated by $d$. Moreover, $\delta$ is regular.*

**Proof:** By [40] (Ch 29.3), $\delta$ is an ultrametric on the connected components. Since $C$ is totally disconnected then $\delta$ is an ultrametric on $C$. If $d(x, y) = \epsilon$ then $x \overset{\epsilon}{\sim} y$. Therefore, $\delta(x, y) \leq d(x, y)$. Now let $d'$ be another ultrametric on $C$ such that $d'(x, y) \leq d(x, y)$ for $x, y \in C$. Then for any $\epsilon$-chain $x_0 = x, \dots, x_n = y$,

$$d'(x, y) \leq \max\{d'(x_i, x_{i+1}) : 0 \leq i \leq n-1\} \leq \max\{d(x_i, x_{i+1}) : 0 \leq i \leq n-1\} < \epsilon.$$

Thus, $d' \leq \delta$. For a proof that $\delta$ is regular see [40]. □

It follows at once from the proposition that if $d$ is an ultrametric then $d = \delta$. From now on, we will be working with a Cantor set $C$ with regular ultrametric $d$.

**Definition 29** *A* profinite structure *on a Cantor set $C$ is given by an increasing family $\{R_\epsilon : \epsilon \in \mathbb{R}^+\}$ of equivalence relations on $C$ that satisfy the following properties:*

*(i) Each relation $R_\epsilon$ is open in $C \times C$ and for a certain $\epsilon$, $R_\epsilon = C \times C$;*

*(ii) The family is continuous on the left: $\bigcup_{\epsilon'<\epsilon} R_{\epsilon'} = R_\epsilon$;*

*(iii) $\bigcap_{\epsilon\in\mathbb{R}^+} R_\epsilon = \Delta$ (the diagonal of $C \times C$).*

**Proposition 9** *On a Cantor set $C$, there is a one-to-one correspondence between profinite structures and regular ultrametrics.*

**Proof:** Given a regular ultrametric $d$, the equivalence relation $\overset{\epsilon}{\sim}$ given by $\epsilon$-chains will be shown to be a profinite structure. (i) For $y \in [x]_\epsilon$, $B_\epsilon(y) := \{z \in C : d(z, y) < \epsilon\} \subset [x]_\epsilon$. Thus $[x]_\epsilon$ is open. Therefore $R_\epsilon = \bigcup_{x\in C}[x]_\epsilon \times [x]_\epsilon$ is open. A compact metric space is totally bounded, so there exists $\epsilon$ such that $R_\epsilon = C \times C$.

(ii)Let $x \overset{\epsilon}{\sim} y$. Then there exists $x_0 = x, x_1, \ldots, x_n = y$ with $d(x_i, x_{i+1}) < \epsilon$. If $\eta = (\max\{d(x_i, x_{i+1}) : 0 \le i < n\})/2$ then $x \overset{\eta}{\sim} y$ with $\eta < \epsilon$.

(iii)Suppose $[x]_0 := \bigcap_{\epsilon\in\mathbb{R}^+}[x]_\epsilon$ is the disjoint union of two closed sets $U$ and $V$. Since $C$ is compact, if both $U$ and $V$ are nonempty then there exists $u \in U$ and $v \in V$ such that $\mathrm{dist}(U, V) = d(u, v) > 0$. But then if $\eta = d(u, v)/2$ then $u \overset{\eta}{\nsim} v$. So $[x]_0$ must be connected. Thus since $C$ is totally disconnected, $[x]_0 = \{x\}$. Therefore, $\bigcap_{\epsilon\in\mathbb{R}^+} R_\epsilon = \Delta$.

Finally, given another regular ultrametric $d' \neq d$ then there exists $x, y \in C$ with $d(x, y) \neq d'(x, y)$. Suppose that $d(x, y) = \epsilon > d'(x, y) = \epsilon'$. If $\eta = (\epsilon + \epsilon')/2$, then $x \overset{\eta}{\sim}_{d'} y$ but $x \overset{\eta}{\nsim}_d y$ and therefore they give different profinite structures.

Conversely, given a profinite structure $\{R_\epsilon : \epsilon \in \mathbb{R}^+\}$ on $C$ let $d(x, y) := \inf\{\epsilon : x \overset{\epsilon}{\sim} y\}$. That $d(x, y) = 0$ if and only if $x = y$ follows from the fact that $\bigcap_{\epsilon\in\mathbb{R}^+} R_\epsilon = \Delta$. For $x, y, z \in C$, if $x \overset{\epsilon_1}{\sim} y$ and $y \overset{\epsilon_2}{\sim} z$ and if $\epsilon = \max\{\epsilon_1, \epsilon_2\}$, then $x \overset{\epsilon}{\sim} z$. Thus $d(x, z) \le \max\{d(x, y), d(y, z)\}$ and $d$ is an ultrametric. In order to show that $d$ is regular, let $id : C \to C$ be the identity map from $C$ with the original topology to

$C$ with the metric topology. First of all, if $x \overset{a}{\sim} y$ then by (ii) $x \overset{a-\delta}{\sim} y$ for some $\delta > 0$ and $d(x,y) < a$. Thus, $d(x,y) < \epsilon$ if and only if $x \overset{\epsilon}{\sim} y$. This gives that $B_a(x) = [x]_a$. In fact, $[x]_\epsilon$ is open in the original topology. This can be seen as follows. Let $(x,y) \in C \times C$. Since $R_\epsilon$ is open, then there exists an open set $V \subset C \times C$ such that $(x,y) \in V \subset R_\epsilon$. But $C \times C$ has the product topology and therefore there exists open sets $U_x, U_y \subset C$ such that $(x,y) \in U_x \times U_y \subset V$. For any $y \in U_y$, $(x,y) \in R_\epsilon$ and consequently $U_y \subset [x]_\epsilon$ and $[x]_\epsilon$ is open. Therefore, *id* is a continuous, bijective map from a compact space to a Hausdorff space and therefore a homeomorphism. Thus, $d$ is regular.

Given two different profinite structures $\{R_\epsilon\}$ and $\{R'_\epsilon\}$, then without loss of generality there exists $\epsilon > 0$ and $(x,y) \in R_\epsilon$ such that $(x,y) \notin R'_\epsilon$. Suppose $\{R_\epsilon\}$ gives ultrametric $d$ and $\{R'_\epsilon\}$ gives ultrametric $d'$. Then by (ii), $(x,y) \in R_{\epsilon-\delta}$ for some $\delta > 0$ and $d(x,y) < \epsilon \leq d'(x,y)$. Consequently, $d \neq d'$. □

### 3.2.2 Weighted, Rooted Trees

Using the results of the previous section, it is now possible to show the connection between Cantorian trees and ultrametrics on a Cantor set.

**Definition 30** *Let $\mathcal{T}$ be an infinite rooted tree with no dangling vertex. A* weight *on $\mathcal{T}$ is a function $\epsilon : \mathcal{V} \to \mathbb{R}^+$ that satisfies the following:*

*(i) If $v \succ v'$ then $\epsilon(v) > \epsilon(v')$.*

*(ii) For an infinite path $v_0 v_1 \cdots \in \partial\mathcal{T}$, $\lim_{n\to\infty} \epsilon(v_n) = 0$.*

*A rooted tree along with its weight function will be called a* weighted*, rooted tree.*

As mentioned previously, there are various surgical operations on trees that lead to the same boundary. Given a tree $\mathcal{T}$, any vertex with only one child can be reduced by the process of edge reduction. The weight function is then the restriction of the original weight function. A tree for which every vertex has at least two children will be called *reduced*.

**Proposition 10 (Michon's Correspondence)** *On a Cantor set $C$, there is a one-to-one correspondence between regular ultrametrics and reduced, weighted, rooted Cantorian trees. Moreover given a regular ultrametric $d$, the boundary $\partial\mathcal{T}$ of the corresponding weighted, rooted Cantorian tree is isometric to $(C, d)$. The weight function $\epsilon$ for $\mathcal{T}$ is such that $\epsilon(v) = \mathrm{diam}_d([v])$.*

**Proof:** Let $d$ be a regular ultrametric on $C$ and let $\{R_\epsilon\}$ be the profinite structure corresponding to $d$. We will first show that $R_\epsilon$ is closed. If $\{(x_n, y_n)\}_{n=1}^{\infty} \subset R_\epsilon$ converges to $(x, y)$ then there exists $N$ such that $x_N, y_N \in B_{\epsilon/2}(x) \times B_{\epsilon/2}(y)$. Thus $x_N \overset{\epsilon}{\sim} x$ and $y_N \overset{\epsilon}{\sim} y$. Since $x_N \overset{\epsilon}{\sim} y_N$, then by transitivity of the equivalence relation $x \overset{\epsilon}{\sim} y$ and $(x, y) \in R_\epsilon$. Thus $R_\epsilon$ is closed.

The tree $\mathcal{T}$ is built as follows. Let $\epsilon_0 = \inf\{\epsilon : R_\epsilon = C \times C\}$. Then $R_{\epsilon_0} \neq C \times C$ since $R_{\epsilon_0} = \bigcup_{\epsilon' < \epsilon_0} R_{\epsilon'}$. Similarly, let $\epsilon_{i+1} = \inf\{\epsilon : R_\epsilon = R_{\epsilon_i}\}$. Then $\{\epsilon_i\}_{i=0}^{\infty}$ is such that $R_{\epsilon_i} \neq R_{\epsilon_{i+1}}$. Let the root of $\mathcal{T}$ correspond to $C$ and let the vertices of height $n$ correspond to the equivalence classes of $R_{\epsilon_{n-1}}$. Let the edges be defined by $[x]_{\epsilon_j} \succeq [y]_{\epsilon_k}$ if and only if $[x]_{\epsilon_j} \supset [y]_{\epsilon_k}$. Then $\mathcal{T}$ is a rooted tree with no dangling vertex. As seen in the proof of the previous proposition, every equivalence class is clopen. Since $R_\epsilon$ is compact then there are only a finite number of equivalence classes. Thus each vertex has a finite number of children and has a descendant with more than one child. So, $\mathcal{T}$ is a Cantorian tree. In general, $\mathcal{T}$ is not reduced. However, since each vertex has a descendant with more than one child, edge reduction can be applied to each vertex with only one child without altering $\partial\mathcal{T}$. This will give a reduced tree $\mathcal{T}'$ with vertices $\mathcal{V}' \subset \mathcal{V}$ such that $\partial\mathcal{T}' = \partial\mathcal{T}$ as topological spaces.

Let $\Phi : \partial\mathcal{T}' \to C$ be defined by $\Phi(v_0 v_1 \cdots) = \bigcap_{i=1}^{\infty} [x_i]_{\epsilon_i}$ where $v_i = [x_i]_{\epsilon_i}$. This map is bijective and $\Phi^{-1}([x]_{\epsilon_i}) = [v]$ where $v = [x]_{\epsilon_i}$. Thus $\Phi$ is continuous and since $\partial\mathcal{T}'$ is compact, $\Phi$ is a homeomorphism. By abuse of notation, let $[v] = [x]_{\epsilon_i}$if $v = [x]_{\epsilon_i}$.

If $v = [x]_{\epsilon_k}$ then let $\epsilon(v) := \epsilon_{k+1}$. Since $\epsilon_k > 0$ for all $k$, then $\epsilon : \mathcal{V}' \to \mathbf{R}^+$. (i) follows automatically. (ii) Since $\epsilon([x]_{\epsilon_k}) \leq \epsilon_k$ and $\epsilon_k \to 0$ then $\lim_{k\uparrow\infty} \epsilon([x]_{\epsilon_k}) \leq \lim_{k\to\infty} \epsilon_k = 0$. So $\mathcal{T}'$ is a reduced, weighted, rooted Cantorian tree.

Let $\mathcal{T}$ be a reduced, rooted Cantorian tree with weight function $\epsilon$. For $x, y \in \partial\mathcal{T} =: C$, let $d(x, y) = \epsilon(x \wedge y)$ for $x \neq y$ and $d(x, x) = 0$. It is straightforward to show that $d$ is an ultrametric on $C$. Given $r > 0$ and $x \in C$, let $B_r(x) := \{y \in C : d(x, y) < r\}$. By (ii), $B_r(x)$ has more than one point, so let $v = l.c.p.(B_r(x))$. By the definition of $v$, for $y \in [v]$ there exists $z \in B_r(x)$ such that $x \wedge y \preceq x \wedge z$. Thus $d(x, y) \leq d(x, z) < r$ and therefore $[v] = B_r(x)$. Consequently, $B_r(x)$ is open in $\partial\mathcal{T}$ and $d$ is regular.

For $x, y \in [v]$ then $x \wedge y \preceq v$ and $d(x, y) = \epsilon(x \wedge y) \leq \epsilon(v)$. Thus, $\mathrm{diam}([v]) \leq \epsilon(v)$. Conversely, since $v$ has more than one child then there exists $x, y \in [v]$ such that $v = x \wedge y$. Therefore, $\epsilon(v) = d(x, y) \leq \mathrm{diam}(v)$ and $\epsilon(v) = \mathrm{diam}([v])$.

Starting with a regular ultrametric $d$ on $C$, let $d_\epsilon$ be the regular ultrametric obtained from the Cantorian tree $\mathcal{T}$ corresponding to $d$. Let $x, y \in C$. Then $d_\epsilon(x, y) = \epsilon(x \wedge y) = \epsilon_{k+1}$ if $x \wedge y = [x]_{\epsilon_k}$. So $x \overset{\epsilon_{k+1}}{\nsim} y$ but $x \overset{\epsilon_{k+1}+\delta}{\sim} y$ for $\delta > 0$. Since $d$ is an ultrametric then $d(x, y) = \epsilon_{k+1}$. Thus $d = d_\epsilon$ and $\partial\mathcal{T}$ is isometric to $C$.

Starting with a reduced, weighted, rooted Cantorian tree $\mathcal{T}$ let $\mathcal{T}_d$ be the tree obtained from the regular ultrametric $d$ corresponding to $\mathcal{T}$. Let $\Phi$ be the homeomorphism from $\partial\mathcal{T} \to \partial\mathcal{T}_d$. Let $\Psi : \mathcal{V} \to \mathcal{V}_d$ be defined by $\Psi(v) = l.c.p(\Phi([v]))$. Because each tree is reduced there is a one-to-one correspondence between clopen sets in the boundary and vertices, thus $\Psi$ is a bijection. Therefore the correspondence between reduced, weighted, rooted Cantorian trees and regular ultrametrics is indeed a bijection. □

**Remark 2** Since any Cantor set $C$ is metrizable, then let $d$ be a regular metric on $C$. The construction of a profinite structure from an ultrametric in the proof of Proposition 9 works even in the case that $d$ is simply a metric and not an ultrametric.

Moreover, the construction of the tree from the ultrametric in Proposition 10 would then construct a Cantorian tree $T$ such that $C$ is homeomorphic to $\partial\mathcal{T}$. Therefore, every Cantor set is homeomorphic to the boundary of a Cantorian tree. Of course, the weight function on $\mathcal{T}$ would give rise to an ultrametric and therefore does not necessarily allow one to recover $d$.

### 3.2.3 Saturated Ultrametrics

In Michon's correspondence, the correspondence was between a Cantor set $C$ with regular ultrametric $d$ and a reduced, weighted, rooted Cantorian tree. As will be seen later in the transversal of a Delone set in Section 5.3.1, sometimes the initial data is given by a Cantorian tree $\mathcal{T}$ and the metric is constructed using a given weight on this tree. This section classifies all regular ultrametrics that are given by weight functions on a fixed tree and consequently generalizes [22] Proposition 2.5.5.

**Definition 31** *An ultrametric $d$ will be called* saturated with respect to $\mathcal{T}$ *if* $d(x,y) = \mathrm{diam}([x \wedge y])$ *for all* $x, y \in \partial\mathcal{T}$.

**Corollary 2** *Let $\mathcal{T}$ be a reduced, rooted Cantorian tree and $d$ a regular ultrametric on $\partial\mathcal{T}$. Then $d$ is given by a weight function on $\mathcal{T}$ if and only if $d$ is saturated with respect to $\mathcal{T}$.*

**Proof:** Let $d$ be a regular ultrametric and let $\epsilon : \mathcal{V} \to \mathbb{R}^+$ be given by $\epsilon(v) = d(x,y)$ where $v = x \wedge y$. If $d$ is saturated with respect to $\mathcal{T}$ then $\epsilon$ is well-defined. Moreover, $\epsilon$ is a weight function. By Proposition 10, $d$ corresponds to $\mathcal{T}$ with weight $\epsilon$. Conversely, if $d$ is given by a weight function on $\mathcal{T}$ then clearly $d$ is saturated with respect to $\mathcal{T}$.
$\square$

### 3.2.4 Embedding of Ultrametric Cantor Sets

A simple application of Michon's correspondence is given by the following.

**Theorem 21** *Let $C$ be a Cantor set with regular ultrametric $d$. Let $\mathcal{T}$ with weight $\epsilon$ be the corresponding reduced, weighted, rooted Cantorian tree. If $\mathcal{V}_*$ denotes all the vertices of $\mathcal{T}$ except for the root, then there exists an isometric embedding of $C$ into the real Hilbert space $\ell^2_{\mathbb{R}}(\mathcal{V}_*)$.*

**Proof:** Let $x \in C$ and let $v_0 v_1 \cdots$ be the infinite path corresponding to $x$. Let

$$\Phi(x) := \sum_{n=0}^{\infty} \sqrt{\frac{\epsilon(v_n)^2 - \epsilon(v_{n+1})^2}{2}} |v_{n+1}\rangle$$

where $\{|v\rangle, v \in \mathcal{V}_*\}$ denotes the canonical basis of $\ell^2_{\mathbb{R}}(\mathcal{V}^*)$. If $v \neq v'$, then $\langle v, v' \rangle = 0$. Therefore,

$$||\Phi(x)||^2 = \sum_{n=0}^{\infty} \frac{\epsilon(v_n)^2 - \epsilon(v_{n+1})^2}{2} = \frac{\epsilon(v_0)^2}{2}$$

and $\Phi(x) \in \ell^2_{\mathbb{R}}(\mathcal{V}^*)$. Thus, $\Phi$ is well-defined. Let $y \in C$ with $y \neq x$. If $w_0 w_1 \cdots$ is the infinite path corresponding to $y$ then there exists an $n_0 > 0$ such that $w_n \neq v_n$ for $n > n_0$ and $w_n = v_n$ for $n \leq n_0$. Then $x \wedge y = v_{n_0}$ and $d(x, y) = \epsilon(v_{n_0})^2$. Moreover,

$$\Phi(x) - \Phi(y) = \sum_{n=n_0}^{\infty} \sqrt{\frac{\epsilon(v_n)^2 - \epsilon(v_{n+1})^2}{2}} |v_{n+1}\rangle - \sum_{n=n_0}^{\infty} \sqrt{\frac{\epsilon(w_n)^2 - \epsilon(w_{n+1})^2}{2}} |w_{n+1}\rangle$$

and consequently

$$\begin{aligned}
\|\Phi(x) - \Phi(y)\|^2 &= \sum_{n=n_0}^{\infty} \frac{\epsilon(v_n)^2 - \epsilon(v_{n+1})^2}{2} + \sum_{n=n_0}^{\infty} \frac{\epsilon(w_n)^2 - \epsilon(w_{n+1})^2}{2} \\
&= \frac{\epsilon(v_{n_0})^2}{2} + \frac{\epsilon(w_{n_0})^2}{2} \\
&= \epsilon(v_{n_0})^2 = d(x, y)^2.
\end{aligned}$$

Thus $\Phi$ is indeed an isometry. $\square$

# CHAPTER IV

# NONCOMMUTATIVE RIEMANNIAN GEOMETRY

## *4.1 A Spectral Triple*

Given Michon's correspondence, it is now possible to construct a spectral triple on a Cantor set $C$ with regular ultrametric $d$.

### 4.1.1 Construction of the Spectral Triple

**Definition 32** *An* odd spectral triple *for an involutive algebra* $\mathcal{A}$ *is a triple* $(\mathcal{A}, \mathcal{H}, D)$ *where* $\mathcal{H}$ *is a Hilbert space on which* $\mathcal{A}$ *has a representation* $\pi$ *by bounded operators.* $D$ *is a self-adjoint operator on* $\mathcal{H}$ *such that* $[D, \pi(a)]$ *is a bounded operator on* $\mathcal{H}$ *for all* $a \in \mathcal{A}$ *and such that* $D$ *has compact resolvent.* $D$ *is called the Dirac operator of the spectral triple.*

*An* even spectral triple *is an odd spectral triple along with a grading operator* $\Gamma : \mathcal{H} \to \mathcal{H}$. $\Gamma$ *is required to satisfy* $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $\Gamma D = -D\Gamma$, *and* $\Gamma\pi(a) = \pi(a)\Gamma$ *for all* $a \in \mathcal{A}$

We will first define our algebra. Recall that for $C$ with metric $d$, a *Lipschitz* function $f : C \to \mathbb{C}$ is a map for which there exists $c_0 > 0$ such $|f(x) - f(y)| \leq c_0\ d(x, y)$ for all $x, y \in C$. The *Lipschitz constant* of $f$ is the smallest $c$ such that $|f(x) - f(y)| \leq c\ d(x, y)$ for all $x, y \in C$. Let $\mathcal{C}_{\text{Lip}}(C)$ denote the set of all Lipschitz functions from $C$ to $\mathbb{C}$.

**Proposition 11** $\mathcal{C}_{\text{Lip}}(C)$ *is a dense* $*$*-subalgebra of* $\mathcal{C}(C)$.

**Proof:** Let $f, g \in \mathcal{C}_{\text{Lip}}(C)$. It is clear that if $a \in \mathbb{C}$ then $f + g, f^*, af \in \mathcal{C}_{\text{Lip}}(C)$. It is also clear that $\mathcal{C}_{\text{Lip}}(C) \subset \mathcal{C}(C)$. Let $f$ have Lipschitz constant $c$ and $g$ have Lipschitz constant $c'$. Since $f, g \in \mathcal{C}(C)$ and $C$ is compact, then there exists $M, M' > 0$ such that $|f(x)| \leq M$ and $|g(x)| \leq M'$ for all $x \in C$. Therefore, for $x, y \in C$

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| \\ &\leq |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)| \\ &\leq (M'c + Mc')d(x, y) \end{aligned}$$

and $fg \in \mathcal{C}_{\text{Lip}}(C)$. Now for $x \in C$, the function $f_x : C \to \mathbb{C}$ defined by $f_x(z) := d(x, z)$ is such that $f_x(x) \neq f_x(y)$ for $y \neq x$. Moreover, $f_x$ is Lipschitz with Lipschitz constant 1 by the triangle inequality. Consequently, $\mathcal{C}_{\text{Lip}}(C)$ separates points and thus is dense in $\mathcal{C}(C)$ by the Stone-Weierstrass Theorem. $\square$

Let $\mathcal{T}$ be the reduced, weighted, rooted Cantorian tree corresponding to the regular ultrametric $d$ via Michon's correspondence. Since $\mathcal{T}$ is Cantorian, the set of vertices, $\mathcal{V}$, is countable. Let $\mathcal{H} := \ell^2(\mathcal{V}) \otimes \mathbb{C}^2$. Thus an element $\psi \in \mathcal{H}$ is an ordered pair $(\psi_1, \psi_2)$ where $\psi_i \in \ell^2(\mathcal{V})$. Let $D$ be the operator on $\mathcal{H}$ given by

$$D\psi\ (v) := \frac{1}{\text{diam}(v)} \sigma_1 \psi(v) = \frac{1}{\text{diam}(v)} (\psi_2(v), \psi_1(v))$$

where $\sigma_1$ is the first Pauli matrix. The grading operator is the multiplication by $\Gamma := \mathbf{1} \otimes \sigma_3$ where $\sigma_3 = \text{diag}\{+1, -1\}$ is the third Pauli matrix. To define a representation on $\mathcal{C}_{\text{Lip}}(C)$ a notion of choice is required.

**Definition 33** *Let $C$ be a Cantor set with a regular ultrametric $d$. A* choice *function is a map $\tau : \mathcal{V} \mapsto C \times C$ such that, if $v \in \mathcal{V}$ and if $\tau(v) = (x, y)$, then both $x, y$ are in $[v]$ and $d(x, y) = \text{diam}[v]$. The set of choice functions on $C$ will be denoted by $\Upsilon(C)$.*

Let $\tau \in \Upsilon(C)$ be a choice function. In what follows $\tau(v) = (x, y)$ will be written $x = \tau_+(v), y = \tau_-(v)$. Then the $*$-representation $\pi_\tau : \mathcal{C}_{\text{Lip}}(C) \to B(\mathcal{H})$ is given by

$$\pi_\tau(f)\psi\ (v) = \begin{pmatrix} f(\tau_+(v)) & 0 \\ 0 & f(\tau_-(v)) \end{pmatrix} \psi(v).$$

**Proposition 12** *$\pi_\tau$ is a faithful $*$-representation of $\mathcal{C}(C)$ for all $\tau \in \Upsilon(C)$.*

**Proof:** That $\pi_\tau$ is a $*$-representation is obvious. It is bounded since $f$ is continuous and $C$ is compact. Let $f, g \in \mathcal{C}(C)$ be such that $\pi_\tau(f) = \pi_\tau(g)$. Then $f(\tau_+(v)) = g(\tau_+(v))$ for all $v \in \mathcal{V}$. For $x \in C$, there exists $v_0, v_1, \cdots \in \mathcal{V}$ such that $x \in [v_j]$ and $\text{diam}([v_j]) \to 0$. Then $f(x) = \lim_{j\to\infty} f(\tau_+(v_j)) = \lim_{j\to\infty} g(\tau_+(v_j)) = g(x)$. Thus $\pi_\tau$ is faithful. $\square$

Based on this proposition, $\pi_\tau$ is also a faithful representation on $\mathcal{C}_{\text{Lip}}(C)$.

**Proposition 13** *$(\mathcal{C}_{\text{Lip}}(C), \mathcal{H}, D)$ with grading operator $\Gamma$ is an even spectral triple for all $\tau \in \Upsilon(C)$.*

**Proof:** First of all, let $v \in \mathcal{V}$ and let $\psi_v$ be such that $\psi_v(w) = 0$ if $w \neq v$ and $\psi_v(v) = 1$. Then $\{(\psi_v, 0), (0, \psi_v)\}_{v\in\mathcal{V}}$ is a basis for $\mathcal{H}$. Now, $D(\psi_v, 0) = \text{diam}(v)^{-1}(0, \psi_v)$ and since $\text{diam}(v) > 0$ for all $v \in \mathcal{V}$ then $D$ is defined on this basis. Thus, $D$ is densely defined.

To show that $D$ is symmetric, let $\psi, \psi' \in \mathcal{H}$. Then,

$$\begin{aligned} \langle D\psi, \psi' \rangle_{\mathcal{H}} &= \sum_{v\in\mathcal{V}} (\text{diam}([v]))^{-1} \langle \sigma_1 \psi(v), \psi'(v) \rangle_{\mathbb{C}^2} \\ &= \sum_{v\in\mathcal{V}} (\text{diam}([v]))^{-1} \langle \psi(v), \sigma_1 \psi'(v) \rangle_{\mathbb{C}^2} \\ &= \langle \psi, D\psi' \rangle_{\mathcal{H}} \end{aligned}$$

because $\sigma_1$ is self-adjoint. Since $D$ is densely defined then $D$ is symmetric. By Theorem 14, if $\text{Range}(D) = \mathcal{H}$ then $D$ is self-adjoint. Let $\psi \in \mathcal{H}$ and let $\psi'(v) =$

$\operatorname{diam}(v)\sigma_1\psi(v)$. Then $D\psi'(v) = \psi(v)$ since $\sigma_1^2 = \mathbf{1}$. Now, since there exists $K$ such that $\operatorname{diam}(C) \leq K$, then

$$||\psi'||_{\mathcal{H}}^2 = \sum_{v\in\mathcal{V}} (\operatorname{diam}([v]))^2 ||\psi(v)||_{\mathbb{C}^2} \leq K^2 ||\psi||_{\mathcal{H}}^2$$

So, $\psi' \in \mathcal{H}$ and $\operatorname{Range}(D) = \mathcal{H}$. Thus $D$ is self-adjoint.

Let $v \in \mathcal{V}$. Then,

$$\begin{aligned}
([D, \pi_\tau(f)]\psi)(v) &= D(\pi_\tau(f)\psi(v)) - \pi_\tau(f)(D\psi(v)) \\
&= \frac{f(\tau_+(v)) - f(\tau_-(v))}{\operatorname{diam}([v])} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \psi(v) \\
&= \frac{f(\tau_+(v)) - f(\tau_-(v))}{d(\tau_+(v), \tau_-(v))} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \psi(v)
\end{aligned}$$

where the last line follows since $\tau$ is a choice function. Since $f$ is Lipschitz, then $||[D, \pi_\tau(f)]\psi||_{\mathcal{H}} \leq k||\psi||_{\mathcal{H}}$ where $k$ is the Lipschitz constant of $f$ and thus $[D, \pi_\tau(f)] \in \mathcal{B}(\mathcal{H})$.

To show that $D$ has compact resolvent, let $\psi \in \mathcal{H}$ and $v \in \mathcal{V}$. Note that $0 \in \rho(D)$. Then,

$$(D^{-1}\psi)(v) = \operatorname{diam}([v])\sigma_1\psi(v).$$

So for $\eta > 0$, let $(T^\eta\psi)(v) = (D^{-1}\psi)(v)$ if $\operatorname{diam}([v]) \geq \eta$ and $0$ otherwise. Now since there are only finitely many $v \in \mathcal{V}$ with $\operatorname{diam}([v]) \geq \eta$ then $T^\eta$ is finite rank. Consequently,

$$||T^\eta - D^{-1}||_{\mathcal{B}(\mathcal{H})} = \sup_{v\in\mathcal{V}} \{\operatorname{diam}([v]) : \operatorname{diam}([v]) < \eta\}.$$

Thus $||D^{-1} - T^\eta||_{\mathcal{B}(\mathcal{H})} < \eta$ and $\lim_{\eta\downarrow 0} T^\eta = D^{-1}$. Consequently $D^{-1}$ is compact and $D$ has compact resolvent.

Finally, it is clear that $\Gamma\pi_\tau(f) = \pi_\tau(f)\Gamma$ for all $f \in \mathcal{C}_{\text{Lip}}(C)$. That $\Gamma^* = \Gamma$ and $\Gamma^2 = \mathbf{1}$ follows from the fact that $\sigma_3^* = \sigma_3$ and $\sigma_3^2 = \mathbf{1}$. Lastly, since $\sigma_3\sigma_1 = -\sigma_1\sigma_3$ then $\Gamma D = -D\Gamma$. □

### 4.1.2 The Connes Distance

A good spectral triple should be able to recover some of the structure of the original space $C$. As shown in [18] Section 6.1, the classical Dirac operator $\mathcal{D}$ on a compact Riemannian spin manifold $M$ forms a spectral triple $(C^\infty(M), \mathcal{H}, \mathcal{D})$ where $\mathcal{H}$ is the space of $\mathcal{L}^2$-sections of some bundle. This spectral triple recovers the Riemannian metric $d$ on $M$ via the formula

$$d(x, y) = \sup\{|f(x) - f(y)| : \|[\mathcal{D}, f]\| \leq 1\}.$$

This formula is often called the *Connes distance*. In this section, we will show that the spectral triple given in the previous section can recover the metric when all possible choice functions are taken into account. To understand why all choice functions must be taken into account, it is necessary to understand the role of the space of choices.

In the Noncommutative Riemannian structure, the space of choices $\Upsilon(C)$ plays the role of the unit sphere subbundle of the tangent bundle. In particular the basic element of intuition is that a choice function is the analogue of a vector field of unit vectors on a manifold. With this intuition in mind, then $[D, \pi_\tau(f)]$ represents the directional derivative of $f$ in the direction of $\tau$. On $\mathbb{R}^d$, a function $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ is such that the gradient $\|\nabla f\|_\infty < 1$ if and only if the directional derivative $\|\partial_{\vec{v}} f\|_\infty < 1$ for every $\vec{v} \in \mathbb{R}$. Therefore, by this reasoning it is natural to expect that the metric can be recovered by using the Connes distance on functions $f \in \mathcal{C}(C)$ such that $\|[D, \pi_\tau(f)]\| < 1$ for every $\tau \in \Upsilon(C)$. This is shown in the following:

**Theorem 22** *Let $C$ be a Cantor set with a regular ultrametric $d$. Then $d$ coincides* *with the Connes distance $\rho$ defined by*

$$\rho(x,y) = \sup\{|f(x) - f(y)| \,:\, f \in \mathcal{C}_{\mathrm{Lip}}(C)\,,\ \sup_{\tau \in \Upsilon(C)} \|[D, \pi_\tau(f)]\| \,\leq\, 1\}$$

**Proof:** Let $x, y \in C$ and let $d_x : C \to C$ be given by $d_x(y) = d(x,y)$. Then $d_x$ is Lipschitz continuous. Let $\tau \in \Upsilon(C)$ and recall that this implies that $d(\tau_+(v), \tau_-(v)) = \mathrm{diam}(v)$. Then

$$\begin{aligned}
|||[D, \pi_\tau(d_x)]|||_{\mathcal{B}(\mathcal{H})} &= \sup_{v \in \mathcal{V}}\{\frac{|d_x(\tau_+(v)) - d_x(\tau_-(v))|}{\mathrm{diam}(v)}\} \\
&= \sup_{v \in \mathcal{V}}\{\frac{|d(x, \tau_+(v)) - d(x, \tau_-(v))|}{d(\tau_+(v), \tau_-(v))}\} \\
&\leq \sup_{v \in \mathcal{V}}\{\frac{d(\tau_+(v), \tau_-(v))}{d(\tau_+(v), \tau_-(v))}\} = 1
\end{aligned}$$

where the inequality follows from the triangle inequality. Consequently,

$$\sup_{\tau \in \Upsilon(C)} |||[D, \pi_\tau(d_x)]||| \leq 1$$

and $\rho(x,y) \geq |d_x(x) - d_x(y)| = d(x,y)$.

For $x, y \in C$, let $v \in \mathcal{V}$ be such that $v = x \wedge y$, so that $d(x,y) = \mathrm{diam}(v)$. Let $\tau$ be such that $\tau_+(v) = x$ and $\tau_-(v) = y$. Then for any $f \in \mathcal{C}_{\mathrm{Lip}}(C)$ such that $|||[D, \pi_\tau(f)]|||_{\mathcal{B}(\mathcal{H})}\} \leq 1$

$$\frac{|f(\tau_+(v)) - f(\tau_-(v))|}{\mathrm{diam}(v)} = \frac{|f(\tau_+(v)) - f(\tau_-(v))|}{d(\tau_+(v), d(\tau_-(v))} \leq 1.$$

This gives that $|f(x) - f(y)| \leq d(x,y)$ and therefore that $\rho(x,y) \leq d(x,y)$. $\square$

## *4.2 $\zeta$-Functions*

In this section, the Dirac operator $D$ is used to create a $\zeta$-function as formulated by Connes [18]. Since the Dirac operator is independent of the choice function, this

$\zeta$-function will also be independent of choice. When $C$ is a Cantor set with regular ultrametric $d$, then we will be able to recover the upper box dimension of $C$ as the abscissa of convergence of its $\zeta$-function.

### 4.2.1 The $\zeta$-function for $D$

Let $\mathcal{H}$ be the Hilbert space from the previously created spectral triple. Then for $\psi \in \mathcal{H}$, $(|D|^{-1}\psi)(v) = \mathrm{diam}(v)\psi(v)$. Let

$$\zeta(s) := \frac{1}{2}\mathrm{Tr}\left(|D|^{-s}\right) = \sum_{v\in\mathcal{V}} \mathrm{diam}(v)^s.$$

Then $\zeta$ is a Dirichlet series. By [39] (Ch. 2), as a function of the complex variable $s$, $\zeta$ either converges everywhere, nowhere, or in a half-plane given by $\Re(s) > s_0$. In this last case, $s_0$ is called the *abscissa of convergence*. Since $|D|^{-1}$ is compact then the eigenvalues of $|D|^{-1}$ are discrete. Therefore, let $\zeta(s) = \sum a_k \lambda_k^s$ where $\lambda_1 = \mathrm{diam}(C) > \lambda_2 > \cdots$ and $a_k$ is the multiplicity of $\lambda_k$, that is the number of $v \in \mathcal{V}$ with $\mathrm{diam}([v]) = \lambda_k$.

### 4.2.2 The Upper Box Dimension

In this section we recall the definition of the upper box dimension of a fractal. For a treatment of the many fractal dimensions, the reader can consult [26]. Let $X$ be a metric space with metric $d$. Let $N_\delta(X)$ be the least number of sets of diameter at most $\delta$ that cover $X$.

**Definition 34** *The* upper box dimension *is defined as*

$$\overline{\dim}_B(C) = \limsup_{\delta\downarrow 0} \frac{\log N_\delta(C)}{-\log\delta}$$

As shown in [26] Chapter 2.1, the upper box dimension satisfies the following dimension properties: monotonicity, zero on finite sets, and it gives dimension $n$ to open sets in $\mathbb{R}^n$ . Most importantly, it is invariant under bi-Lipschitz transformations. Therefore, if two different metrics on $X$ are metrically equivalent, then they have the same upper box dimension. The upper box dimension is also the largest of the typical fractal dimensions. In particular, it is greater than or equal to the Hausdorff dimension of $X$.

### 4.2.3 The Abscissa of Convergence

In this section, the abscissa of convergence of the $\zeta$-function of $D$ will be denoted by $s_0$. Also, $\zeta(s)$ will be written as $\sum a_k \lambda_k^s$. In order to prove that $s_0$ is equal to the upper box dimension in general, the following classical lemma on Dirichlet series is necessary.

**Lemma 4** *Let $\zeta(s) = \sum a_k \lambda_k^s$ be a Dirichlet series with abscissa of convergence $s_0$. Suppose further that all the $\lambda_k$ are positive, $\lambda_1 > \lambda_2 > \cdots$, and that $a_k > 0$ for all $k$. Then*

$$\limsup_{k\to\infty} \frac{\log \sum_{j=1}^{j=k} a_j}{-\log \lambda_k} = s_0$$

**Proof:** A proof of this can be found in [39] (Ch. 2.6). Note that the form of the Dirichlet series used there is slightly different than the one used here. $\square$

With this lemma in hand, it is now possible to prove the theorem. Let $\mathcal{T}$ be the tree corresponding to $(C, d)$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be the set of all distinct $\mathrm{diam}([v])$ for $v \in \mathcal{V}$ (these are also the distinct eigenvalues of $|D|^{-1}$). Let them be ordered such that $\lambda_1 > \lambda_2 > \cdots$. Let $M_n$ be such that every vertex with diameter at least $\lambda_n$ has at most $M_n$ children. Then the result is the following:

**Theorem 23** *If $(\log M_n)/(-\log \lambda_n) \to 0$ as $n \to \infty$, then $s_0 = \overline{\dim}_B(C)$.*

**Proof:** For any $\delta > 0$ such that $\lambda_n > \delta \geq \lambda_{n+1}$, $N_\delta(C) = N_{\lambda_{n+1}}(C)$ since there are no vertices with $\delta \geq \mathrm{diam}([v]) > \lambda_{n+1}$. Thus,

$$\frac{\log N_{\lambda_{n+1}}(C)}{-\log \lambda_{n+1}} \le \frac{\log N_\delta(C)}{-\log \delta} < \frac{\log N_{\lambda_{n+1}}(C)}{-\log \lambda_n}.$$

Let $M$ be such that every vertex has at most $M$ children. A minimal cover of $C$ with sets of diameter at most $\lambda_n$ must use every vertex of diameter $\lambda_n$. Thus, a cover of $C$ with sets of diameter at most $\lambda_{n+1}$ can be obtained by taking the children of each set of diameter $\lambda_n$. This cover, $\mathcal{O}$, is in fact minimal since no $O \in \mathcal{O}$ can cover two children of a vertex of diameter $\lambda_n$. Since every vertex of diameter at least $\lambda_n$ has at least 2 children and at most $M_n$ children, this gives

$$N_{\lambda_n} + a_n \le N_{\lambda_{n+1}} \le N_{\lambda_n} + (M_n - 1)a_n.$$

After iterating the procedure,

$$1 + \sum_{k=1}^{n} a_k \le N_{\lambda_{n+1}} \le 1 + (M_n - 1)\sum_{k=1}^{n} a_k.$$

where the 1 comes from the fact that $N_{\lambda_1} = 1$. For the binary tree, it is easy to check that these inequalities are in face equalities and therefore that this estimate is in some sense optimal. Since every cover of $C$ with sets of diameter at most $\lambda_{n+1}$ must use every vertex of diameter $\lambda_{n+1}$, then $N_{\lambda_{n+1}} \ge a_{n+1}$. Consequently, $N_{\lambda_{n+1}} \ge 1/2(a_{n+1} + 1 + \sum_{k=1}^{n} a_k)$. Thus,

$$\frac{\log 1/2(\sum_{k=1}^{j=n+1} a_k)}{-\log \lambda_{n+1}} \le \frac{\log N_\delta(C)}{-\log \delta} < \frac{\log(1 + (M_n - 1)\sum_{k=1}^{j=n} a_k)}{-\log \lambda_n}$$

Therefore, since $(\log(M_n - 1))/(-\log \lambda_n) \to 0$ as $n \to \infty$ then

$$\limsup_{n\to\infty} \frac{\log \sum_{k=1}^{j=n+1} a_k}{-\log \lambda_{n+1}} \le \limsup_{\delta\to 0} \frac{\log N_\delta(C)}{-\log \delta} \le \limsup_{n\to\infty} \frac{\log \sum_{k=1}^{j=n} a_k}{-\log \lambda_n}$$

and $\overline{\dim}_B(C) = s_0$. □

It is important to note that a special case of Theorem 23 is when there is a uniform bound on the number of children - this happens for the attractor of a self-similar iterated function system and the transversal of the Fibonacci tiling. In any case, the hypothesis says intuitively that the number of children can grow but it must be compensated for by a decrease in the size of the children.

## 4.3 Measure Theory on $C$

This section extends the study of the noncommutative geometry of a Cantor set $C$ by studying a measure $\mu$ that is naturally defined on $C$.

### 4.3.1 $\zeta$-regularity

In order to study more deeply the geometry of $C$ it is necessary to make some assumptions on $C$.

**Definition 35** *A Cantor set $C$ with regular ultrametric $d$ is* $\zeta$-regular *if the abscissa of convergence, $s_0$, of its $\zeta$-function is finite and if for any $f \in \mathcal{C}(C)$ and any $\tau \in \Upsilon(C)$*

$$\lim_{s \downarrow s_0} (s - s_0) \mathrm{Tr} \left( |D|^{-s} \pi_\tau(f) \right) \tag{1}$$

*exists.*

Given a $\zeta$-regular Cantor set and a choice function $\tau \in \Upsilon(C)$, it is then possible to define a measure $\mu_\tau$ on $C$ given by

$$\mu_\tau(f) \;=\; \int_C f d\mu_\tau = \lim_{s \downarrow s_0} \frac{\mathrm{Tr}\ (|D|^{-s} \pi_\tau(f))}{\mathrm{Tr}\ (|D|^{-s})}$$

**Theorem 24** *Let $C$ be a $\zeta$-regular Cantor set with a regular ultrametric $d$. Then $\mu_\tau$ is independent of the choice function $\tau$ and defines a regular Borel probability measure $\mu$ on $C$.*

**Proof:** Since the trace is linear and $|D|^{-s}$ is positive, then $\mu_\tau$ is a positive linear functional on $\mathcal{C}(C)$ and therefore by the Riesz Representation Theorem (see [57] Theorem 2.14) is a regular Borel measure on $C$. Since $\pi_\tau$ is faithful, then $\pi_\tau(1) = \mathbf{1}$ and $\mu_\tau$ is a probability measure on $C$. Let $\tau, \tau' \in \Upsilon(C)$ and $f \in \mathcal{C}_{\text{Lip}}(C)$ with Lipschitz constant $k$. For $\Re(s) > s_0$, since $|D|^{-s}$ is trace class and $\pi_\tau(f)$ is bounded, then $|D|^{-s}\pi_\tau(f)$ is trace class. Similarly $|D|^{-s}\pi_{\tau'}(f)$ is trace class. Therefore,

$$\begin{aligned}
|\mathrm{Tr}\left(|D|^{-s}\pi_\tau(f)\right) - \mathrm{Tr}\left(|D|^{-s}\pi_{\tau'}(f)\right)| &\leq \sum_{v\in\mathcal{V}} |f(\tau_+(v)) - f(\tau'_+(v))|\mathrm{diam}(v)^{\Re(s)} \\
&\quad + \sum_{v\in\mathcal{V}} |f(\tau_-(v)) - f(\tau'_-(v))|\mathrm{diam}(v)^{\Re(s)} \\
&\leq 2\sum_{v\in\mathcal{V}} k\mathrm{diam}(v)^{\Re(s)+1}.
\end{aligned}$$

Consequently,

$$|\mu_\tau(f) - \mu_{\tau'}(f)| = |\lim_{s\downarrow s_0} \frac{\mathrm{Tr}\ \left(|D|^{-s}\pi_\tau(f)\right) - \mathrm{Tr}\ \left(|D|^{-s}\pi_{\tau'}(f)\right)}{\mathrm{Tr}\ \left(|D|^{-s}\right)}| = 0$$

since $\mathrm{Tr}\left(|D|^{-s_0-1}\right) < \infty$. Since $\mathcal{C}_{\text{Lip}}(C)$ is dense in $\mathcal{C}(C)$ and $\pi_\tau$ is continuous for all $\tau \in \Upsilon(C)$, then $\mu_\tau$ and $\mu_{\tau'}$ are equal on $\mathcal{C}(C)$. $\square$

### 4.3.2 The Measure on the Space of Choices

In what follows, it will be necessary to have a measure on the spaces of choices, $\Upsilon(C)$. This measure will be created using the measure $\mu$ from the previous section. Recall that $\Upsilon(C)$ was the set of all functions $\tau : \mathcal{V} \to C \times C$ such that $\tau(v) \in [v] \times [v]$ and $d(\tau_+(v), \tau_-(v)) = \mathrm{diam}(v)$. Let $\mathcal{G} \subset \mathcal{V} \times \mathcal{V}$ be defined to be the set of all brothers. That is $(u, v) \in \mathcal{G}$ if $u$ and $v$ have the same parent and $u \neq v$. Let $\mathcal{G}_v$ be the set of all brothers whose parent is $v$. Now, $x, y \in [v]$ are such that $d(x, y) = \mathrm{diam}([v])$ if and only if there is a unique pair $(w, w') \in \mathcal{G}_v$ of distinct children (i.e. $w \neq w'$) of $v$ such that $x \in [w]$ and $y \in [w']$. Consequently

$$\Upsilon(C) = \prod_{v \in \mathcal{V}} \bigsqcup_{(w,w') \in \mathcal{G}_v} [w] \times [w'].$$

Therefore, define a measure $\nu_v$ on $\Upsilon_v(C) := \bigsqcup_{(w,w') \in \mathcal{G}_v} [w] \times [w']$ by

$$\nu_v = \frac{\mu \times \mu}{\sum_{(w,w') \in \mathcal{G}_v} \mu([w])\mu([w'])}.$$

This is then a probability measure on $\Upsilon_v(C)$. By Tychonoff's Theorem, $\Upsilon(C)$ is compact. By Urysohn's Metrization Theorem, $\Upsilon(C)$ is metrizable. Therefore, by the Daniell-Kolmogorov Consistency Theorem from Section 2.3.2, there is an extension of these measures to a probability measure $\nu$ on $\Upsilon(C)$. This measure $\nu$ is such that $\nu((\prod_{w \neq v} \Upsilon_w(C)) \times U_v) = \nu_v(U_v)$ for any $\nu_v$-measurable set $U_v$.

## *4.4 Dirichlet Forms and the Operator* $\Delta$

In this section, let $L^2_{\mathbb{C}}(C, d\mu)$ denote the Hilbert space completion of $\mathcal{C}(C, \mathbb{C})$ with respect to $\langle f, g \rangle = \int_C \bar{f} g d\mu$ and let $L^2(C, d\mu)$ denote the Hilbert space completion of $\mathcal{C}(C, \mathbb{R})$ with respect to the same inner product. Here $\mu$ is the residue measure that was created in Section 4.3.1. It is of interest to study Markovian semigroups of operators on $L^2(C, d\mu)$. As shown in [32], the study of Markovian semigroups is equivalent to studying the Dirichlet forms on $L^2(C, d\mu)$.

### 4.4.1 Dirichlet Forms

Given a real Hilbert space $\mathcal{H}$, a non-negative definite symmetric bilinear form densely defined on $\mathcal{H}$ is called a *symmetric form* on $\mathcal{H}$. Let $Q$ be a symmetric form on a Hilbert space $H$. If $\mathrm{Dom}(Q)$ is complete with respect to the metric given by $\langle f, g \rangle_1 = \langle f, g \rangle_H + Q(f, g)$ where $\langle \cdot, \cdot \rangle_H$ is the inner product on $H$ then $Q$ is called a *closed* form. Given a closed symmetric form $Q$ on $L^2(C, d\mu)$, then $Q$ is called

*Markovian* if $Q(\tilde{f},\tilde{f}) \leq Q(f,f)$ where $\tilde{f} = \min(\max(0,f),1)$. If $Q$ is not closed, the condition to be Markovian is more complicated; however, the previous condition is sufficient. A closed symmetric Markovian form is called a *(symmetric) Dirichlet form.* Given the formalism of the previous sections, it is possible to define a form $Q_s$ on $L^2_{\mathbb{C}}(C,d\mu)$ by

$$Q_s(f,g) := \frac{1}{2}\int_{\Upsilon(C)} \mathrm{Tr}\left(|D|^{-s}[D,\pi_\tau(f)]^*[D,\pi_\tau(g)]\right)d\nu(\tau).$$

It is now necessary to specify a domain for the form. Let $\mathcal{E} \subset L^2(C,d\mu)$ be the real linear space spanned by $\{\chi_v : v \in \mathcal{V}\}$ where $\chi_v$ is the characteristic function of $[v] \subset C$.

**Lemma 5** *$\mathcal{E}$ is dense in $L^2(C,d\mu)$.*

**Proof:** Let $f \in \mathcal{C}(C)$. Since $f$ is continuous and $C$ is compact, then $f$ is uniformly continuous. Consequently, for $\epsilon > 0$ there is a $\delta > 0$ such that if $d(x,y) < \delta$ then $|f(x) - f(y)| < \epsilon$. Let $v_1,\ldots,v_N$ be a partition of $C$ such that $\mathrm{diam}([v_i]) < \delta$. Let $\tau \in \Upsilon(C)$. Then if $g(x) := f(\tau_+(v_j))$ where $v_j$ is the unique vertex of the partition such that $x \in [v_j]$. Then $||f-g||_\infty < \epsilon$ and consequently $||f-g||_2 < \epsilon$. Thus $\mathcal{E}$ is dense in $\mathcal{C}(C)$. Since $\mathcal{C}(C)$ is dense in $L^2(C,d\mu)$ then $\mathcal{E}$ is dense in $L^2(C,d\mu)$. $\square$

Let $\mathrm{Dom}(Q_s) = \mathcal{E}$.

**Theorem 25** *Let $C$ be a $\zeta$-regular Cantor set with regular ultrametric $d$. Then the measure $\mu$ coming from the $\zeta$-function defines a measure $\nu$ on the space of choices $\Upsilon(C)$. Moreover, for all $s \in \mathbb{R}$ there is a closable Dirichlet form on the Hilbert space $L^2(C,\mu)$ defined by*

$$Q_s(f,g) := \frac{1}{2}\int_{\Upsilon(C)} \mathrm{Tr}\left(|D|^{-s}[D,\pi_\tau(f)]^*[D,\pi_\tau(g)]\right)d\nu(\tau)$$

*with $Dom(Q_s)$ a dense subspace of the real Hilbert space $L^2(C,\mu)$.*

**Proof:** Because the trace is linear and $\pi_\tau$ is linear, then $Q_s$ is bilinear. Now,

$$\begin{aligned}
|D|^{-s}[D,\pi_\tau(f)]^*[D,\pi_\tau(g)]\psi(v) &= \mathrm{diam}(v)^s\frac{f(\tau_+(v))-f(\tau_-(v))}{\mathrm{diam}(v)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^* \\
&\quad \frac{g(\tau_+(v))-g(\tau_-(v))}{\mathrm{diam}(v)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\psi(v) \\
&= \frac{(f(\tau_+(v))-f(\tau_-(v)))(g(\tau_+(v))-g(\tau_-(v)))}{\mathrm{diam}(v)^{2-s}}\psi(v)
\end{aligned}$$

Since this equation is symmetric in $f$ and $g$, then $Q_s$ is symmetric. Moreover,

$$\mathrm{Tr}\left(|D|^{-s}[D,\pi_\tau(f)]^*[D,\pi_\tau(g)]\right) = 2\sum_{v\in\mathcal{V}}\frac{(f(\tau_+(v))-f(\tau_-(v)))(g(\tau_+(v))-g(\tau_-(v)))}{\mathrm{diam}(v)^{2-s}}$$

and thus $Q_s$ is non-negative definite.

If $w \nsim v$, then $\chi_v(\tau_+(w)) = \chi_v(\tau_-(w)) = 0$ and thus $\chi_v(\tau_+(w)) - \chi_v(\tau_-(w)) = 0$. If $v \succeq w$, then $\chi_v(\tau_+(w)) = \chi_v(\tau_-(w)) = 1$ and thus $\chi_v(\tau_+(w)) - \chi_v(\tau_-(w)) = 0$. Therefore, since there are only finitely many $w \in \mathcal{V}$ with $w \succ v$ then $[D,\pi_\tau(\chi_v)]$ is finite rank for each characteristic function $\chi_v$ with $v \in \mathcal{V}$. Thus for $f \in \mathcal{E}$, $[D,\pi_\tau(f)]$ is finite rank and $Q_s(f,g) < \infty$ for all $g \in L^2(C,d\mu)$. Consequently, $Q_s$ is well-defined on $\mathcal{E}$.

Let now $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{E}$ such that $\lim_{n\to\infty}||f_n||_{L^2} = 0$ and $\lim_{n,m\to\infty} Q_s(f_n-f_m, f_n-f_m) = 0$. To show that $Q_s$ is closable, it is then necessary to show that $\lim_{n\to\infty} Q_s(f_n,f_n) = 0$. Since $\lim_{n\to\infty}||f_n||_{L^2} = 0$ there is a subsequence $f_{n_i}$ that converges pointwise $\mu$-a.e. to 0 (see [57] Thm. 3.12). In particular, thanks to the definition of the measure $\nu$ on the set of choices, $f_{n_i}(\tau_+(v)) \to 0$ for $\nu$-a.e. choice and for all $v \in \mathcal{V}$. Similarly for $\tau_-(v)$. So, given $\epsilon > 0$ let $N$ be such that

$Q_s(f_n - f_m, f_n - f_m) < \epsilon$ for $n, m > N$. Then for $m > N$,

$$Q_s(f_m, f_m) = \int_{\Upsilon(C)} \sum_{j=1}^{K} \mathrm{diam}(v_j)^{s-2}(f_m(\tau_+(v_j)) - f_m(\tau_-(v_j)))^2 d\nu(\tau).$$

Since $(f_m(\tau_+(v_j)) - f_m(\tau_-(v_j)))^2 =$

$$\liminf_{i\to\infty}(f_m(\tau_+(v_j)) - f_{n_i}(\tau_+(v_j)) - f_m(\tau_-(v_j)) + f_{n_i}(\tau_-(v_j)))^2$$

then using Fatou's lemma,

$$Q_s(f_m, f_m) \leq \liminf_{i\to\infty} Q_s(f_m - f_{n_i}, f_m - f_{n_i}) \leq \epsilon.$$

Thus $\lim_{m\to\infty} Q_s(f_m, f_m) = 0$ and $Q_s$ is closable.

The proof that $Q_s$ is Markovian is by inspection. Let $C_-, C_0, C_+$ denote the closed subsets of $C$ for which $f \leq 0, 0 \leq f \leq 1, 1 \leq f$. If $\tau_+(v) \in C_-$ and $\tau_-(v) \in C_-$ then $\tilde{f}(\tau_+(v)) = \tilde{f}(\tau_-(v)) = 0$ and thus $0 = |\tilde{f}(\tau_+(v)) - \tilde{f}(\tau_-(v))| \leq |f(\tau_+(v)) - f(\tau_-(v))|$. If $\tau_+(v) \in C_-$ and $\tau_-(v) \in C_0$ then

$$f(\tau_+(v)) \leq \tilde{f}(\tau_+(v)) = 0 \leq f(\tau_-(v)) = \tilde{f}(\tau_-(v))$$

and

$$|\tilde{f}(\tau_+(v)) - \tilde{f}(\tau_-(v))| = f(\tau_-(v)) \leq f(\tau_-(v)) - f(\tau_+(v)) = |f(\tau_+(v)) - f(\tau_-(v))|.$$

The remaining cases are proved similarly. Thus $Q_s(\tilde{f}, \tilde{f}) \leq Q_s(f, f)$. □

It is now possible to get a closed Dirichlet form using the following result.

**Theorem 26 ([32] Thm 2.1.1)** *Suppose $Q$ is a closable Markovian symmetric form on $L^2(X, m)$ where $X$ is a locally compact separable Hausdorff space and $m$ is a* *positive Radon measure on $X$ such that Supp$(m) = X$. Then its smallest closed extension is a Dirichlet form.*

### 4.4.2 Self-Adjoint Operators and Operator Semigroups

This section follows [32] (Ch 1.3). Let $H$ be a real Hilbert space.

**Definition 36** *A family $\{T_t, t > 0\}$ of linear operators is called a* strongly continuous, symmetric, contraction semigroup *if:*
*(i) each $T_t$ is a symmetric operator with Dom$(T_t) = H$.*
*(ii) semigroup property: $T_t T_s = T_{t+s}$ for $t, s > 0$.*
*(iii) contraction property: $\langle T_t f, T_t f\rangle \leq \langle f, f\rangle$ for all $f \in H$ and $t > 0$.*
*(iv) strong continuity: $\langle T_t f - f, T_t f - f\rangle \to 0$ as $t \downarrow 0$ for all $f \in H$.*

Let $\{T_t, t > 0\}$ be such a semigroup. Then the *generator* $A$ is an operator on $H$ defined by

$$Af := \lim_{t\downarrow 0} \frac{T_t f - f}{t}, \mathrm{Dom}(A) := \{f \in H : Af \text{ exists as a strong limit}\}.$$

In fact, there is a one-to-one correspondence between non-positive definite self-adjoint operators on $H$ and the family of strongly continuous, symmetric, contraction semigroups (see [32] Lemma 1.3.1 and Lemma 1.3.2). The correspondence from $A$ to $\{T_t\}$ is given by $T_t = \exp(tA)$.

Given a non-positive definite self-adjoint operator, let $Q(u, v) := \langle -Au, u\rangle$ with $\mathrm{Dom}(Q) := \mathrm{Dom}(\sqrt{-A})$. It turns out that $Q$ is a closed symmetric form on $H$. This correspondence is also one-to-one (see [32] Theorem 1.3.1). Starting with a closed, symmetric form $Q$ on $H$ the construction of $A$ is slightly more involved. Since $Q$ is closed, then $\mathrm{Dom}(Q)$ is a Hilbert space with norm $||g||_1 = ||g||_{L^2} + Q(g, g)$. Fix $f \in H$. Then $\langle \cdot, f\rangle$ is a bounded linear functional on $\mathrm{Dom}(Q)$. Therefore, let $Bf$ be the unique vector in $\mathrm{Dom}(Q)$ corresponding to this linear functional by the Riesz

Representation Theorem. Let $A := I - B^{-1}$. Then $A$ is the non-positive definite self-adjoint operator corresponding to $Q$.

Now let $H = L^2(X, m)$ where $X$ is a locally compact separable Hausdorff space and $m$ is a positive Radon measure on $X$ such that $\mathrm{Supp}(m) = X$. A bounded linear operator $S$ on $L^2(X, m)$ is called *Markovian* if $0 \le Sf \le 1, m$-a.e. whenever $f \in L^2(X, m)$ is such that $0 \le f \le 1$. A strongly continuous, symmetric, contraction semigroup $\{T_t\}$ such that $T_t$ is Markovian for each $t > 0$ is called a *Markovian semigroup.*

**Theorem 27 ([32] Thm 1.4.1)** *Let $X$ be a locally compact separable Hausdorff space and $m$ a positive Radon measure on $X$ such that Supp$(m) = X$. Then there is a one-to-one correspondence between Dirichlet forms on $L^2(X, m)$ and Markovian semigroups on $L^2(X, m)$.*

### 4.4.3 The Operators $\Delta_s$

Let $C$ be a $\zeta$-regular Cantor set with regular ultrametric $d$. Let $\mu$ be the measure constructed via the $\zeta$-function. Suppose $\mu$ is such that $\mathrm{Supp}(\mu) = C$. Then for $s \in \mathbb{R}$, the previous results give a non-positive definite self-adjoint operator $\Delta_s$ such that $T_t := \exp(t\Delta_s)$ is a Markovian semigroup. $\Delta_s$ is such that

$$\langle -\Delta_s f, g \rangle = \frac{1}{2} \int_{\Upsilon(C)} \mathrm{Tr}\left(|D|^{-s}[D, \pi_\tau(f)]^*[D, \pi_\tau(g)]\right) d\nu(\tau)$$

for $f, g \in \mathrm{Dom}(\Delta_s)$. It is important to note that $\mathcal{E} \subset \mathrm{Dom}(\Delta_s) \subset \mathrm{Dom}(\bar{Q}_s)$ where $\bar{Q}_s$ is the smallest closed extension of $Q_s$.

It is possible to calculate $\Delta_s \chi_v$ for $v \in \mathcal{V}$. Let $w \in \mathcal{V}$. Since $\chi_v(\tau_+(w)) - \chi_v(\tau_-(w)) = 0$ if $w \not\succ v$, then for $g \in \mathrm{Dom}(Q_s)$,

$$\langle -\Delta_s \chi_v, g \rangle = \sum_{w \succ v} \mathrm{diam}(w)^{s-2} \int_{\Upsilon(C)} (\chi_v(\tau_+(w)) - \chi_v(\tau_-(w)))(g(\tau_+(w)) - g(\tau_-(w))) d\nu(\tau).$$

However, since $\tau$ is only applied to $w$ then this becomes

$$= \sum_{w \succ v} \mathrm{diam}(w)^{s-2} \int_{\Upsilon_w(C)} (\chi_v(\tau_+(w)) - \chi_v(\tau_-(w)))(g(\tau_+(w)) - g(\tau_-(w)))d\nu_w(\tau).$$

By the very definition of $\nu_w$,

$$= \sum_{w \succ v} \frac{\mathrm{diam}(w)^{s-2}}{\sum_{(u,u') \in \mathcal{G}_w} \mu([u])\mu([u'])} \sum_{(u,u') \in \mathcal{G}_w} \int_{[u] \times [u']} (\chi_v(x) - \chi_v(y))(g(x) - g(y))d\mu(x)d\mu(y).$$

Given $w$ an ancestor of $v$, let $u_v$ be its child that is also an ancestor of $v$. Then for any other child $u$ of $w$, $\chi_u(x) = 0$ for $x \in [u]$. Thus this becomes

$$= \sum_{w \succ v} \frac{\mathrm{diam}(w)^{s-2}}{\sum_{(u,u') \in \mathcal{G}_w} \mu([u])\mu([u'])} 2 \sum_{(u_v,u') \in \mathcal{G}_w} \int_{[u_v] \times [u']} \chi_v(x)(g(x) - g(y))d\mu(x)d\mu(y).$$

Since $\bigcup_{(u_v,u') \in \mathcal{G}_w} [u'] = [w] \cap [u_v]^c$, then

$$= \sum_{w \succ v} \frac{\mathrm{diam}(w)^{s-2}}{\sum_{(u,u') \in \mathcal{G}_w} \mu([u])\mu([u'])} 2 \int_{[v]} d\mu(x) \int_{[w] \cap [u_v]^c} (g(x) - g(y))d\mu(y).$$

Consequently,

$$\langle -\Delta_s \chi_v, g \rangle = \sum_{w \succ v} \frac{\mathrm{diam}(w)^{s-2}}{\sum_{(u,u') \in \mathcal{G}_w} \mu([u])\mu([u'])} 2(\mu([w] \cap [u_v]^c)\langle \chi_v, g \rangle - \mu([v])\langle \chi_{[w] \cap [u_v]^c}, g \rangle)$$

and

$$\Delta_s \chi_v = - \sum_{w \succ v} \frac{\mathrm{diam}(w)^{s-2}}{\sum_{(u,u') \in \mathcal{G}_w} \mu([u])\mu([u'])} 2(\mu([w] \cap [u_v]^c)\chi_v - \mu([v])\chi_{[w] \cap [u_v]^c}). \tag{2}$$

An application of this formula is given by the following:

**Proposition 14** *The spectrum of $\Delta_s$ is pure point.*

**Proof:** Let $L_n \subset L^2(C, d\mu)$ be the space spanned by all $\chi_v$ such that $\mathrm{ht}(v) \leq n$.

Since $\mathcal{T}$ is Cantorian then $\dim(L_n) < \infty$. Moreover, $L_n \subset L_{n+1}$ and $\bigcup_n L_n$ is dense in $L^2(C, d\mu)$. Equation 2 then implies that $\Delta_s$ leaves each $L_n$ invariant. Because any finite rank operator is pure point, then $\Delta_s$ restricted to each finite dimensional $L_n$ is pure point. Consequently, $\Delta_s$ is pure point. $\square$

# CHAPTER V

# EXAMPLES AND APPLICATIONS

## 5.1 *The Triadic Cantor Set*

### 5.1.1 Eigenvalues and Eigenstates for $\Delta_s$ on $C_3$

This section will apply much of the previous machinery to the triadic Cantor set. Let $C_3$ denote the triadic Cantor set seen as a subset of the interval $[0,1]$. As seen in Example 6, $C_3$ is the boundary of the infinite binary tree $\partial\mathcal{T}_2$ and has a natural homeomorphism with $\{0,1\}^{\mathbb{N}}$ by

$$\phi(\omega) = \sum_{n=0}^{\infty} \frac{2\omega_n}{3^{n+1}}, \qquad \omega = \{\omega_n\}_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}.$$

Let $d$ be the regular ultrametric corresponding to the weight $\epsilon(v) = 3^{-\mathrm{ht}(v)}$. Then for $x, y \in C_3$,

$$\frac{d(x,y)}{3} \leq |x-y| \leq d(x,y)$$

and thus $d$ is metrically equivalent to the Euclidean metric. Then

$$\zeta(s) = \sum_{n=0}^{\infty} (\frac{2}{3^s})^n$$

and therefore has abscissa of convergence $s_0 = \ln 2/\ln 3$. This pole is clearly a simple pole. For any $v \in \mathcal{V}$,

$$\frac{1}{2}\mathrm{Tr}\,(|D|^{-s}\pi_\tau(\chi_v)) = \sum_{w\preceq v} \mathrm{diam}(w)^s = \mathrm{diam}(v)^s\zeta(s)$$

since the subtree starting at $v$ is identical to the tree starting at the root. Consequently, $\mu(\chi_v) = \mathrm{diam}(v)^{s_0}$. Thus $\mu(f)$ is defined on all characteristic functions and can be extended to all continuous functions. Therefore, $C_3$ is $\zeta$-regular and

$$\mu([v]) = \mathrm{diam}(v)^{s_0} = \frac{1}{3^{s_0 \mathrm{ht}(v)}} = \frac{1}{2^{\mathrm{ht}(v)}}.$$

Since $\mathrm{Supp}(\mu) = C_3$ then $\Delta_s$ can be defined on $L^2(C_3, d\mu)$. Equation 2 then gives that for $v = v_0 \cdots v_n \in \mathcal{V}$ with $n \geq 1$,

$$\Delta_s \chi_v = -\sum_{j=0}^{n-1} \frac{3^{j(2-s)}}{2^{-(2j+1)}} 2(2^{-(j+1)}\chi_v - 2^{-n}\chi_{[v]\cap[u_v]^c}).$$

Letting $\bar{a} = 1 - a$ for $a \in \{0, 1\}$ then this becomes

$$\Delta_s \chi_v = -2\sum_{j=0}^{n-1} \left(\frac{2}{3^{s-2}}\right)^j \chi_v + \frac{4}{2^n}\sum_{j=0}^{n-1} \left(\frac{4}{3^{s-2}}\right)^j \chi_{v_0 \cdots v_j \bar{v}_{j+1}}. \tag{3}$$

This formula can be used to find the eigenstates of $\Delta_s$.

**Definition 37** *Let $\mathcal{W}$ be the set of infinite sequences $\omega = \omega_1\omega_2 \cdots \in \{0, 1\}^{\mathbb{N}^+}$ such that all but a finite number of $\omega_k$'s are 0. Let $|\omega|$ be the maximum integer $k$ such that $\omega_k = 1$ with the convention that $|\omega| = 0$ if $\omega = 00\cdots$. The* Haar function *$\phi_\omega$ is defined by*

$$\phi_\omega = \sum_{v\in\{0,1\}^n} (-1)^{\omega\cdot v}\chi_v, \quad \omega \cdot v = \sum_{k=1}^{n} \omega_k v_k.$$

*for any $n \geq |\omega|$.*

Because $\chi_{v_1 \ldots v_N 0} + \chi_{v_1 \ldots v_N 1} = \chi_{v_1 \ldots v_N}$ and since if $N = |\omega|$ then $\omega_{N+m} = 0$ for $m > 0$,then

$$\sum_{v\in\{0,1\}^{N+1}} (-1)^{\omega\cdot v}\chi_v = \sum_{v\in\{0,1\}^N} (-1)^{\omega\cdot v}(\chi_{v_1\ldots v_N 0} + \chi_{v_1\ldots v_N 1}) = \sum_{v\in\{0,1\}^n} (-1)^{\omega\cdot v}\chi_v.$$

Therefore, $\phi_\omega$ does not depend on the choice of $n$ and $\phi_\omega$ is well-defined. The importance of the Haar functions comes from the following theorem.

**Theorem 28** *Let $C_3$ be the triadic Cantor set with the regular ultrametric $d$ given above. Let $\mu$ be its associated measure. Then*

*(i) The eigenstates of $\Delta_s$ are given by the Haar functions $\phi_\omega$ with $\omega \in \mathcal{W}$.*

*(ii) The eigenvalues of $\Delta_s$ are given by $\lambda_0 = 0$ and for $n \geq 1$*

$$-\lambda_n = -2\left(1 + 3^{s_0+2-s} + \cdots + \left(3^{s_0+2-s}\right)^{n-2} + 2\left(3^{s_0+2-s}\right)^{n-1}\right)$$

*(iii) The multiplicity of $\lambda_n$ is $2^{n-1}$ for $n \geq 1$ whereas $\lambda_0$ is simple.*

*(iv) For $s > s_0 + 2$, $\Delta_s$ is bounded and is a compact perturbation of a multiple of the identity.*

*(v) For $s \leq s_0 + 2$, $\Delta_s$ has compact resolvent.*

*(vi) For $s < s_0 + 2$, the density of states $\mathcal{N}(\lambda)$ given by the dimension of the spectral space corresponding to eigenvalues whose magnitude is less than or equal to $\lambda$ satisfies*

$$\mathcal{N}(\lambda) \overset{\lambda\uparrow\infty}{\sim} 2\left(\frac{\lambda}{2k}\right)^{s_0/(2+s_0-s)}(1 + o(1))$$

*where $k = 1/(1 - 3^{s-2-s_0}) + 1$.*

**Remark 3** On a compact Riemannian manifold $M$, the Laplacian is an unbounded operator with compact resolvent. Moreover, Weyl's theorem says that if $m$ is the dimension of $M$ then $\mathcal{N}(\lambda) \sim c_0\lambda^{m/2}$ as $\lambda \to \infty$ for an appropriate constant $c_0$. The constant $c_0$ is not arbitrary and actually gives the volume of the unit ball in the cotangent bundle over the manifold. In any case, the previous theorem shows that if $\Delta_s$ is interpreted as the Laplacian on a compact Riemannian manifold then $m = 2s_0/(2 + s_0 - s)$ gives the Riemannian dimension of this noncommutative manifold. By analogy, this suggests that $\Delta_{s_0}$ is the appropriate Laplacian on $C_3$ since it gives

Riemannian dimension $s_0$.

**Proof:** To begin let $\omega, \sigma \in \mathcal{W}$ and let $n = \max\{|\omega|, |\sigma|\}$. Then

$$\langle \phi_\omega, \phi_\sigma \rangle = \sum_{u,v \in \{0,1\}^n} (-1)^{\omega \cdot u + \sigma \cdot v} \langle \chi_u, \chi_v \rangle.$$

If $u \neq v$ then $[u] \cap [v] = \emptyset$. Since $\langle \chi_u, \chi_u \rangle = \mu([u]) = 2^{-n}$, then

$$\langle \phi_\omega, \phi_\sigma \rangle = \frac{1}{2^n} \sum_{u \in \{0,1\}^n} (-1)^{(\omega - \sigma) \cdot v}.$$

Because

$$\sum_{v \in \{0,1\}^n} (-1)^{(\omega-\sigma)\cdot v} = \prod_{k=1}^{n} (1 + (-1)^{\omega_k - \sigma_k}) = \prod_{k=1}^{n} 2\delta_{\omega_k, \sigma_k}$$

then $\langle \phi_\omega, \phi_\sigma \rangle = \delta_{\omega,\sigma}$ and the Haar functions are orthonormal. Let $v \in \mathcal{V}$ be such that $\mathrm{ht}(v) = n > 0$. Then from this same calculation,

$$\begin{aligned}
\frac{1}{2^n} \sum_{u \in \{0,1\}^n} (-1)^{v \cdot u} \phi_{u00\cdots} &= \frac{1}{2^n} \sum_{u,u' \in \{0,1\}^n} (-1)^{v \cdot u + u \cdot u'} \chi_{u'} \\
&= \sum_{u' \in \{0,1\}^n} \chi_{u'} \frac{1}{2^n} \sum_{u \in \{0,1\}^n} (-1)^{(v-u') \cdot u} \\
&= \sum_{u' \in \{0,1\}^n} \chi_{u'} \delta_{u',v} = \chi_v.
\end{aligned}$$

Thus, the Haar functions are in fact an orthonormal basis for $\mathcal{L}^2(C_3, d\mu)$.

Based on Equation 3, for $\omega \in \mathcal{W}$ with $|\omega| = n > 0$

$$
\begin{aligned}
-\Delta_s \phi_\omega &= \sum_{v\in\{0,1\}^n} (-1)^{\omega\cdot v} \Delta_s \chi_v \\
&= \sum_{v\in\{0,1\}^n} (-1)^{\omega\cdot v} \left( 2\sum_{j=0}^{n-1} \left(\frac{2}{3^{s-2}}\right)^j \chi_v - \frac{4}{2^n} \sum_{j=0}^{n-1} \left(\frac{4}{3^{s-2}}\right)^j \chi_{v_0\cdots v_j \bar{v}_{j+1}} \right) \\
&= 2\sum_{j=0}^{n-1} \left(\frac{2}{3^{s-2}}\right)^j \sum_{v\in\{0,1\}^n} (-1)^{\omega\cdot v} \chi_v \\
&\quad -\frac{4}{2^n} \sum_{j=0}^{n-1} \left(\frac{4}{3^{s-2}}\right)^j \sum_{v\in\{0,1\}^n} (-1)^{\omega\cdot v} \chi_{v_0\cdots v_j \bar{v}_{j+1}} \\
&= 2\sum_{j=0}^{n-1} \left(\frac{2}{3^{s-2}}\right)^j \phi_\omega - \frac{4}{2^n} \sum_{j=0}^{n-1} \left(\frac{4}{3^{s-2}}\right)^j \sum_{v\in\{0,1\}^n} (-1)^{\omega\cdot v} \chi_{v_0\cdots v_j \bar{v}_{j+1}}.
\end{aligned}
$$

However, if $j < n-1$ then

$$
\sum_{v\in\{0,1\}^n} (-1)^{\omega\cdot v} \chi_{v_0\cdots v_j \bar{v}_{j+1}} = \sum_{v_1,\ldots,v_{n-1}=0,1} (-1)^{\omega_1 v_1 + \cdots \omega_{n-1} v_{n-1}} \chi_{v_0\cdots v_j \bar{v}_{j+1}} \sum_{v_n=0,1} (-1)^{v_n} = 0
$$

since $\omega_n = 1$ and $\sum_{v_n=0,1} (-1)^{v_n} = 0$. Also, for $j = n-1$

$$
\sum_{v\in\{0,1\}^n} (-1)^{\omega\cdot v} \chi_{v_0\cdots v_j \bar{v}_{j+1}} = -\phi_\omega
$$

since $(-1)^{v_n} = -(-1)^{\bar{v}_n}$. Consequently,

$$
\begin{aligned}
\Delta_s \phi_\omega &= -2\sum_{j=0}^{n-1} \left(\frac{2}{3^{s-2}}\right)^j \phi_\omega - \frac{4}{2^n} \left(\frac{4}{3^{s-2}}\right)^{n-1} \phi_\omega \\
&= -\left( 2\sum_{j=0}^{n-1} \left(\frac{2}{3^{s-2}}\right)^j + 2\left(\frac{2}{3^{s-2}}\right)^{n-1} \right) \phi_\omega.
\end{aligned}
$$

Therefore, the Haar basis is an eigenbasis for $\Delta_s$ and the corresponding eigenvalues are precisely the $-\lambda_n$'s given in the statement of the theorem. Since there are exactly $2^{n-1}$ sequences $\omega \in \mathcal{W}$ with $|\omega| = n$ for $n > 0$ then the degeneracy of $-\lambda_n$ is $2^{n-1}$.

If $3^{s_0+2-s} < 1$, that is if $s > s_0 + 2$ then as $n \to \infty$,

$$-\lambda_n = -2\sum_{j=0}^{n-1} \left(3^{s_0+2-s}\right)^j + 2\left(3^{s_0+2-s}\right)^{n-1} \to -\frac{2}{1-3^{s_0+2-s}} =: -\lambda_\infty.$$

Hence, $\Delta_s$ is bounded and $\Delta_s + \lambda_\infty \mathbf{1}$ is compact.

If $s = s_0 + 2$ then $3^{s_0+2-s} = 1$ and $-\lambda_n = -2(n+1)$. Therefore, $(\Delta_s - 1)^{-1}$ has eigenvalues $-\lambda_n - 1$ and is compact. Consequently, since $-1 \in \rho(\Delta_s)$ then $\Delta_s$ has compact resolvent. If $s < s_0 + 2$ then $3^{s_0+2-s} > 1$ and

$$-\lambda_n = -2\left(3^{s_0+2-s}\right)^{n-1}\left(\frac{1-(3^{s-2-s_0})^n}{1-3^{s-2-s_0}} + 1\right).$$

Similar to the previous case, $(\Delta_s - 1)^{-1}$ is compact and $\Delta_s$ has compact resolvent. Moreover, if $N(\lambda)$ is such that

$$\lambda = 2\left(3^{s_0+2-s}\right)^{N(\lambda)-1}\left(\frac{1-(3^{s-2-s_0})^{N(\lambda)}}{1-3^{s-2-s_0}} + 1\right)$$

then if $k := 1/(1-3^{s-2-s_0}) + 1$,

$$N(\lambda) = 1 + \frac{\ln(\lambda + 2(3^{s_0+2-s} - 1)^{-1}) - \ln 2k}{\ln 2 - (s-2)\ln 3}.$$

Now,

$$\lim_{\lambda\to\infty}(N(\lambda) - \frac{\ln(\lambda/(2k))}{\ln 2 - (s-2)\ln 3}) = 0.$$

Thus, since

$$\mathcal{N}(\lambda) = 1 + \sum_{n\geq 1, \lambda_n \leq \lambda} 2^{n-1} = 2^{N(\lambda)}$$

then

$$\mathcal{N}(\lambda) \sim 2\left(\frac{\lambda}{2k}\right)^{s_0/(2+s_0-s)}(1+o(1))$$

as $\lambda \to \infty$ as desired. $\square$

### 5.1.2 Diffusion on $C_3$

Having computed the eigenstates and eigenvalues of $\Delta_s$, it is now possible to get an explicit description of its associated Markovian semigroup $\{\exp(t\Delta_s)\}_{t>0}$. In order to do so, let

$$\kappa_n(x,y) := \begin{cases} 1 & \text{if } d(x,y) = 3^{-n} \\ 0 & \text{otherwise} \end{cases} .$$

**Theorem 29** *Under the assumptions of Theorem 28 and for $s < s_0+2$, the following hold:*

*(i) Let the heat kernel $K_t(x,y)$ be defined by*

$$\langle f, e^{t\Delta_s} g\rangle = \int_{C_3\times C_3} f(x)K_t(x,y)g(y)d\mu(x)d\mu(y)$$

*for $f, g \in L^2(C_3, d\mu)$. Then, $K_t(x,y) = \sum_{n=0}^{\infty} \kappa_n(x,y)a_n(t,s)$ where*

$$a_n(t,s) = 1 - 2^n e^{-t\lambda_{n+1}} + \sum_{m=1}^{n} 2^{m-1}e^{-t\lambda_m}$$

*for $n \geq 1$ and $a_0 = 1 - e^{-t\lambda_1}$.*

*(ii) The Markovian semigroup $\{e^{t\Delta_s}\}$ defines a stationary Markov process $(X_t)_{t\geq 0}$ with values in $C_3$. This process satisfies the following for $s$ fixed:*

$$\mathbb{E}(d(X_{t_0}, X_{t_0+t})^\beta) \overset{t\downarrow 0}{\sim} \left(\frac{\lambda_1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2\cdot 3^\beta}\right)^n \left(2^n\lambda_{n+1} - \sum_{m=1}^{n} 2^{m-1}\lambda_m\right)\right) t(1+o(1))$$

*for $\beta > s_0 + 2 - s$ and*

$$\mathbb{E}(d(X_{t_0}, X_{t_0+t})^\beta) \overset{t\downarrow 0}{\sim} \frac{1}{2\beta\ln 3}\left(\frac{1}{1-3^{-\beta}} + 1\right)\left(1 - \frac{1}{3^{\beta+s_0}-1}\right) t\ln(1/t)\,(1+o(1))$$

*for* $\beta = s_0 + 2 - s$*. For* $\beta < s_0 + 2 - s$*,*

$$\mathbb{E}(d(X_{t_0}, X_{t_0+t})^\beta) = O(t^{\beta/(s_0+2-s)} \ln(1/t)).$$

**Remark 4** The previous section had suggested that $\Delta_{s_0}$ is the proper generalization of the Laplacian to the Cantor set. Classical Brownian motion on the real line is generated by the Laplacian and satisfies $\mathbb{E}(|X_{t_0} - X_{t+t_0}|^2) = |t|$. For $s = s_0$, $\mathbb{E}(d(X_{t_0}, X_{t_0+t})^2) \sim t \ln(1/t)$ and so there is a subdominant contribution by a term of order $\ln(1/t)$. For $\beta = 2$ this subdominant contribution only appears for $s \le s_0$ and therefore suggests that on the Cantor set something special is happening at $s = s_0$ as the subdominant term $t \ln(1/t)$ takes over from the term $t$ which dominates for $s > s_0$. A further understanding of this phenomenon needs to be investigated although presumably this logarithmic singularity comes from the fact that $X_t$ describes a jump process across the gaps of the Cantor set.

**Proof:** Because of the spectral decomposition of $\Delta_s$ given in Theorem 28,

$$e^{t\Delta_s} = \sum_{n=0}^{\infty} e^{-t\lambda_n} \Pi_n$$

where $\Pi_n$ is the spectral projection onto the eigenspace of $\Delta_s$ corresponding to the eigenvalue $-\lambda_n$. For $n = 0$, $\Pi_0 = |\phi_{00\cdots}\rangle\langle\phi_{00\cdots}| = |1\rangle\langle 1|$. For $n \ge 1$,

$$\Pi_n = \sum_{\sigma_1, \ldots, \sigma_{n-1} \in \{0,1\}} |\phi_{\sigma_1 \cdots \sigma_{n-1} 100\cdots}\rangle\langle\phi_{\sigma_1 \cdots \sigma_{n-1} 100\cdots}|$$

since the $\phi_{\sigma_1\cdots\sigma_{n-1}100\cdots}$ generate the eigenspace of Haar functions $\phi_\omega$ with $|\omega| = n$. By the definition of the Haar function,

$$\begin{aligned}
\Pi_n &= \sum_{\sigma_1,\ldots,\sigma_{n-1}\in\{0,1\}} \sum_{u_k,v_k\in\{0,1\},k=1,\ldots,n} \left(\prod_{k=1}^{n-1}(-1)^{(u_k-v_k)\sigma_k}\right)(-1)^{u_n-v_n}|\chi_u\rangle\langle\chi_v| \\
&= \sum_{u_k,v_k\in\{0,1\},k=1,\ldots,n} (-1)^{u_n-v_n}|\chi_u\rangle\langle\chi_v| \sum_{\sigma_1,\ldots,\sigma_{n-1}\in\{0,1\}} \prod_{k=1}^{n-1}(-1)^{(u_k-v_k)\sigma_k} \\
&= \sum_{u_k,v_k\in\{0,1\},k=1,\ldots,n} (-1)^{u_n-v_n}|\chi_u\rangle\langle\chi_v| \prod_{k=1}^{n-1} 2\delta_{u_k,v_k} \\
&= 2^{n-1} \sum_{u\in\{0,1\}^{n-1}} |\chi_{u0}\rangle\langle\chi_{u0}| - |\chi_{u0}\rangle\langle\chi_{u1}| - |\chi_{u1}\rangle\langle\chi_{u0}| + |\chi_{u1}\rangle\langle\chi_{u1}|.
\end{aligned}$$

Now $|\chi_u\rangle\langle\chi_v|$ is the operator with functional kernel $\chi_u(x)\chi_v(y)$. Because

$$\sum_{u\in\{0,1\}^{n-1}} \chi_{u0}(x)\chi_{u0}(y) + \chi_{u1}(x)\chi_{u1}(y) = \begin{cases} 1 & \text{if } d(x,y) \le 3^{-n} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{u\in\{0,1\}^{n-1}} \chi_{u0}(x)\chi_{u1}(y) + \chi_{u1}(x)\chi_{u0}(y) = \begin{cases} 1 & \text{if } d(x,y) = 3^{-n+1} \\ 0 & \text{otherwise} \end{cases}$$

then

$$\Pi_n(x,y) = \begin{cases} 2^{n-1} & \text{if } d(x,y) \le 3^{-n} \\ -2^{n-1} & \text{if } d(x,y) = 3^{-n+1} \\ 0 & \text{otherwise} \end{cases}$$

where $\Pi_n(x,y)$ is the functional kernel of the operator $\Pi_n$. Using the functions $\kappa_n$, this becomes

$$\Pi_n(x,y) = 2^{n-1}(-\kappa_{n-1}(x,y) + \sum_{m\ge n} \kappa_m(x,y)).$$

Therefore,

$$
\begin{aligned}
K_t(x,y) &= \Pi_0(x,y) + \sum_{n=1}^{\infty} e^{t\lambda_n}\Pi_n(x,y) \\
&= \sum_{n=0}^{\infty} \kappa_n(x,y) + \sum_{n=1}^{\infty} e^{t\lambda_n} 2^{n-1}\left(-\kappa_{n-1}(x,y) + \sum_{m\geq n} \kappa_m(x,y)\right) \\
&= \kappa_0(x,y)(1-e^{t\lambda_1}) + \sum_{n=1}^{\infty} \kappa_n(x,y)\left(1 - 2^n e^{t\lambda_{n+1}} + \sum_{m=1}^{n} 2^{m-1} e^{t\lambda_m}\right)
\end{aligned}
$$

as desired.

The definition of the stochastic process follows Section 2.3.4. It gives a way to evaluate $\mathbb{E}(d(X_{t_0}, X_{t_0+t})^{\beta})$ by

$$\mathbb{E}(d(X_{t_0}, X_{t_0+t})^{\beta}) = \int_{C_3}\int_{C_3}\int_{C_3} d(x,y)^{\beta} K_t(x,y) K_{t_0}(z,x) d\mu(x) d\mu(y) d\mu(z)$$

However, since the semigroup is Markov then $\int_{C_3} K_{t_0}(z,x) d\mu(z) = 1$ and

$$\mathbb{E}(d(X_{t_0}, X_{t_0+t})^{\beta}) = \int_{C_3\times C_3} K_t(x,y) d(x,y)^{\beta} d\mu(x) d\mu(y).$$

Because

$$\kappa_n(x,y) = \sum_{v\in\{0,1\}^n} \chi_{v0}(x)\chi_{v1}(y) + \chi_{v1}(x)\chi_{v0}(y)$$

then

$$\int_{C_3\times C_3} \kappa_n(x,y) d\mu(x) d\mu(y) = \sum_{v\in\{0,1\}^n} \frac{2}{2^{2n+2}} = \frac{1}{2^{n+1}}.$$

Thus since $\kappa_n(x,y) = 1$ if $d(x,y) = 3^{-n}$,

$$\mathbb{E}(d(X_{t_0}, X_{t_0+t})^{\beta}) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2\cdot 3^{\beta}}\right)^n a_n(t,s).$$

It is possible to rewrite $a_n(t,s)$ as

$$a_n(t,s) = 1 - 2^n e^{-t\lambda_{n+1}} + \sum_{m=1}^{n} 2^{m-1} e^{-t\lambda_m} = 2^n(1 - e^{-t\lambda_{n+1}}) - \sum_{m=1}^{n} 2^{m-1}(1 - e^{-t\lambda_m}).$$

Then for $t > 0$ and $\beta > s_0 + 2 - s$,

$$\begin{aligned}
\frac{1}{t}\mathbb{E}(d(X_{t_0}, X_{t_0+t})^{\beta}) &\leq \frac{1}{2t}\sum_{n=0}^{\infty} \frac{1}{3^{\beta n}}(1 - e^{-t\lambda_{n+1}}) \\
&\leq \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{3^{\beta n}}\lambda_{n+1} \\
&\leq \left(\frac{1}{1 - 3^{s-2-s_0}} + 1\right)\sum_{n=0}^{\infty} \frac{1}{3^{\beta n}}(3^{s_0+2-s})^n < \infty.
\end{aligned}$$

Therefore by dominated convergence,

$$\lim_{t\to 0} \frac{\mathbb{E}(d(X_{t_0}, X_{t_0+t})^{\beta})}{t} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2\cdot 3^{\beta}}\right)^n \lim_{t\to 0}\frac{a_n(t,s)}{t}.$$

But for $n \geq 1$,

$$\begin{aligned}
\lim_{t\to 0}\frac{a_n(t,s)}{t} &= \lim_{t\to 0}\left(2^n \frac{1 - e^{-t\lambda_{n+1}}}{t} - \sum_{m=1}^{n} 2^{m-1}\frac{1 - e^{-t\lambda_m}}{t}\right) \\
&= 2^n\lambda_{n+1} - \sum_{m=1}^{n} 2^{m-1}\lambda_m.
\end{aligned}$$

Consequently,

$$0 < \lim_{t\to 0}\frac{\mathbb{E}(d(X_{t_0}, X_{t_0+t})^{\beta})}{t} = \frac{\lambda_1}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{2\cdot 3^{\beta}}\right)^n \left(2^n\lambda_{n+1} - \sum_{m=1}^{n} 2^{m-1}\lambda_m\right) < \infty.$$

For $\beta = s_0 + 2 - s$, let $N_t = \ln(1/t)/(\beta\ln 3)$. First of all,

$$\frac{1}{2}\sum_{n=N_t+1}^{\infty}\left(\frac{1}{2\cdot 3^{\beta}}\right)^n a_n(t,s) < \frac{1}{2}\sum_{n=N_t+1}^{\infty}\left(\frac{1}{3^{\beta}}\right)^n = \frac{1}{2\cdot 3^{\beta N_t}}\sum_{n=1}^{\infty}\left(\frac{1}{3^{\beta}}\right)^n = \frac{t}{2}\frac{1}{3^{\beta}-1} = c_1 t$$

and since

$$a_n(t,s) > 2^n(1-e^{-t\lambda_{n+1}}) - (1-e^{-t\lambda_{n+1}})\sum_{m=1}^{n} 2^{m-1} = 1-e^{-t\lambda_{n+1}} > 1-e^{-t\lambda_1}$$

then

$$\begin{aligned}
\frac{1}{2}\sum_{n=N_t+1}^{\infty}\left(\frac{1}{2\cdot 3^\beta}\right)^n a_n(t,s) &> \frac{1}{2}(1-e^{-t\lambda_1})\sum_{n=N_t+1}^{\infty}\left(\frac{1}{2\cdot 3^\beta}\right)^n \\
&= \frac{1}{2}(1-e^{-t\lambda_1})\left(\frac{1}{2\cdot 3^\beta}\right)^{N_t}\frac{1}{2\cdot 3^\beta-1} \\
&= \frac{1}{2}(1-e^{-t\lambda_1})t^{1+s_0/\beta}\frac{1}{2\cdot 3^\beta-1} = c_2 t^{1+s_0/\beta}.
\end{aligned}$$

Both these terms are $o(t\ln(1/t))$ as $t \to 0$. By taking a Taylor expansion,

$$2^n t\lambda_{n+1} - 2^n\frac{t^2\lambda_{n+1}^2}{2} - t\sum_{m=1}^{n}2^{m-1}\lambda_m \le a_n(t,s) \le 2^n t\lambda_{n+1} - t\sum_{m=1}^{n}2^{m-1}\lambda_m + \frac{t^2}{2}\sum_{m=1}^{n}2^{m-1}\lambda_m^2.$$

Now

$$\begin{aligned}
\frac{1}{2}\sum_{n=1}^{N_t}\left(\frac{1}{2\cdot 3^\beta}\right)^n 2^n t^2\lambda_{n+1}^2 &= \frac{t^2}{2}\sum_{n=1}^{N_t}\left(\frac{1}{3^\beta}\right)^n\lambda_{n+1}^2 \\
&< 2t^2(\frac{1}{1-3^{-\beta}}+1)^2\sum_{n=1}^{N_t}3^{n\beta} \\
&< 2t^2(\frac{1}{1-3^{-\beta}}+1)^2\frac{1}{1-3^{-\beta}}3^{\beta N_t} = c_3 t
\end{aligned}$$

where $c_3 > 0$. Similarly, because

$$\begin{aligned}
\sum_{m=1}^{n}2^{m-1}\lambda_m^2 &< 4(\frac{1}{1-3^{-\beta}}+1)^2\sum_{m=1}^{n}2^{m-1}3^{2\beta(m-1)} \\
&< 4(\frac{1}{1-3^{-\beta}}+1)^2\frac{1}{1-3^{-\beta}2^{-1}}2^{n-1}3^{2\beta(n-1)},
\end{aligned}$$

then there exists $c_4 > 0$ such that

$$\sum_{n=1}^{N_t} \left( \frac{1}{2 \cdot 3^\beta} \right)^n \frac{t^2}{2} \sum_{m=1}^{n} 2^{m-1} \lambda_m^2 < c_4 t.$$

Therefore,

$$\mathbb{E}(d(X_{t_0}, X_{t_0+t})^\beta) > \frac{t}{2} \sum_{n=1}^{N_t} \left( \frac{1}{2 \cdot 3^\beta} \right)^n \left( 2^n \lambda_{n+1} - \sum_{m=1}^{n} 2^{m-1} \lambda_m \right) + c_2 t^{1+s_0/\beta} - c_3 t$$

and

$$\mathbb{E}(d(X_{t_0}, X_{t_0+t})^\beta) < \frac{t}{2} \sum_{n=1}^{N_t} \left( \frac{1}{2 \cdot 3^\beta} \right)^n \left( 2^n \lambda_{n+1} - \sum_{m=1}^{n} 2^{m-1} \lambda_m \right) + c_1 t - c_4 t.$$

Since,

$$\lambda_{n+1} = 2 \cdot 3^{\beta n} \left( \frac{1}{1 - 3^{-\beta}} + 1 \right) - \frac{2 \cdot 3^{-\beta}}{1 - 3^{-\beta}}$$

then

$$\begin{aligned}
\left( \frac{1}{2 \cdot 3^\beta} \right)^n \left( 2^n \lambda_{n+1} - \sum_{m=1}^{n} 2^{m-1} \lambda_m \right) &= 2 \left( \frac{1}{1 - 3^{-\beta}} + 1 \right) \left( 1 - \sum_{m=1}^{n} (2 \cdot 3^\beta)^{-m} \right) \\
&\quad - \frac{2 \cdot 3^{-\beta}}{1 - 3^{-\beta}} \left( \frac{1}{2 \cdot 3^\beta} \right)^n \\
&= 2 \left( \frac{1}{1 - 3^{-\beta}} + 1 \right) \left( 1 - \frac{1}{3^{\beta+s_0} - 1} \right) \\
&\quad + 2 \left( \frac{1}{1 - 3^{-\beta}} + 1 \right) \frac{1}{3^{\beta+s_0} - 1} \left( \frac{1}{3^{\beta+s_0}} \right)^n \\
&\quad - \frac{2 \cdot 3^{-\beta}}{1 - 3^{-\beta}} \left( \frac{1}{3^{\beta+s_0}} \right)^n \\
&= 2 \left( \frac{1}{1 - 3^{-\beta}} + 1 \right) \left( 1 - \frac{1}{3^{\beta+s_0} - 1} \right) \\
&\quad + 2 c_5 \left( \frac{1}{3^{\beta+s_0}} \right)^n
\end{aligned}$$

then

$$\frac{t}{2}\sum_{n=1}^{N_t}\left(\frac{1}{2\cdot 3^{\beta}}\right)^n\left(2^n\lambda_{n+1}-\sum_{m=1}^{n}2^{m-1}\lambda_m\right) = \left(\frac{1}{1-3^{-\beta}}+1\right)\left(1-\frac{1}{3^{\beta+s_0}-1}\right)tN_t$$
$$+c_5\frac{t}{3^{\beta+s_0}-1}(1-t^{1+s_0/\beta}).$$

Consequently,

$$\mathbb{E}(d(X_{t_0},X_{t_0+t})^{\beta}) \overset{t\downarrow 0}{\sim} \frac{1}{\beta\ln 3}\left(\frac{1}{1-3^{-\beta}}+1\right)\left(1-\frac{1}{3^{\beta+s_0}-1}\right)t\ln(1/t)\,(1+o(1))$$

for $\beta = s_0+2-s$.

For $\beta < s_0+2-s$, the proof that $\mathbb{E}(d(X_{t_0},X_{t_0+t})^{\beta}) = O(t^{\beta/(s_0+2-s)}\ln(1/t))$ uses the fact that $1-e^{-x} \leq x^{\alpha}$ for $0 \leq \alpha \leq 1$. Let $N_t = \ln(1/t)/(\beta\ln 3)$. If we let $\beta_0 = s_0+2-s$ then $\beta/\beta_0 < 1$ and

$$a_n(t,s) < 2^n(1-e^{-t\lambda_{n+1}}) \leq 2^n\left(t\lambda_{n+1}\right)^{\beta/\beta_0}.$$

Thus,

$$\frac{1}{2}\sum_{n=0}^{N_t}\left(\frac{1}{2\cdot 3^{\beta}}\right)^n a_n(t,s) < \left(\frac{1}{1-3^{-\beta_0}}+1\right)\sum_{n=0}^{N_t}t^{\beta/\beta_0} = c_6 t^{\beta/\beta_0}N_t$$

Again, we have that

$$\frac{1}{2}\sum_{n=N_t+1}^{\infty}\left(\frac{1}{2\cdot 3^{\beta}}\right)^n a_n(t,s) < \frac{1}{2}\sum_{n=N_t+1}^{\infty}\left(\frac{1}{3^{\beta}}\right)^n = \frac{1}{2\cdot 3^{\beta N_t}}\sum_{n=1}^{\infty}\left(\frac{1}{3^{\beta}}\right)^n = \frac{t}{2}\frac{1}{3^{\beta}-1} = c_1 t$$

since $a_n(t,s) < 2^n$. Consequently, $\mathbb{E}(d(X_{t_0},X_{t_0+t})^{\beta}) = O(t^{\beta/(s_0+2-s)}\ln(1/t))$ as desired. $\square$

### 5.1.3 The $p$-adic Integers and the Vladimirov Operator

In this section, we give a brief introduction to the $p$-adic integers and show the relationship between $\Delta_s$ and the Vladimirov operator [60]. To begin, we present some of the basics of the $p$-adic integers as in [59].

**Definition 38** *For any $n \in \{2, 3, \dots\}$, let $\mathbb{Z}_n$ be defined to be the set of all infinite sequences*

$$a_0 a_1 \dots a_m a_{m-1} \dots$$

*such that $a_i \in \{0, 1, \dots, n-1\}$. That is, $\mathbb{Z}_n = \{0, 1, \dots, n-1\}^{\mathbb{N}}$. Then $\mathbb{Z}_n$ is called the set of* $n$-adic integers.

It is possible to define an addition and multiplication on $\mathbb{Z}_n$ as follows. Let $x = a_0 a_1 a_2 \dots$ and $y = b_0 b_1 b_2 \dots$ be two elements of $\mathbb{Z}_n$. Then $x + y = c_0 c_1 c_2 \dots$ where $c_i$ is determined by:

(i) $c_i \in \{0, 1, \dots, n-1\}$

(ii) for each $m \in \mathbb{N}$,

$$\sum_{i=0}^{m} c_i n^i = \sum_{i=0}^{m} (a_i + b_i) n^i \pmod{n^{m+1}}$$

Similarly, $xy = d_0 d_1 d_2 \dots$ where $d_i$ is determined by:

(i) $d_i \in \{0, 1, \dots, n-1\}$

(ii) for each $m \in \mathbb{N}$,

$$\sum_{i=0}^{m} d_i n^i = \left( \sum_{i=0}^{m} a_i n^i \right) \left( \sum_{i=0}^{m} b_i n^i \right) \pmod{n^{m+1}}$$

**Proposition 15** *With the above addition and multiplication, $\mathbb{Z}_n$ is a commutative ring with $0 := 000\dots$ as a zero element and $1 := 100\dots$ as a unit. Moreover, if $p$ is a prime number then $\mathbb{Z}_p$ is an integral domain and $a_0 a_1 a_2 \dots$ has an inverse if and only if $a_0 \neq 0$*

**Proof:** see [59] Proposition 3.2 and Proposition 3.3.

Let $p$ be a prime number. Since $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}^{\mathbb{N}}$ then $\mathbb{Z}_p$ is homeomorphic to $\partial\mathcal{T}$ where $\mathcal{T}$ is the rooted tree such that every vertex has exactly $p$ children. It is then possible to assign an ultrametric, $|\cdot|_p$, to $\mathbb{Z}_p$ via the weight function $\epsilon(v) = p^{-\mathrm{ht}(v)}$ for $v \in \mathcal{V}$. Then, $|\cdot|_p$ is called the *p-adic valuation* on $\mathbb{Z}_p$. The $p$-adic numbers are the completion of $\mathbb{Q}$ with respect to this ultrametric $|\cdot|_p$ and $\mathbb{Z}_p$ is then the closed unit disc in $\mathbb{Q}_p$. The Vladimirov operator [60] is constructed using the field structure of $\mathbb{Q}_p$. It is defined by

$$(\mathcal{D}\psi)(x) = \frac{p^2}{p+1} \int_{\mathbb{Q}_p} \frac{\psi(x) - \psi(y)}{|x-y|_p^2} dy$$

where $\psi : \mathbb{Q}_p \to \mathbb{R}$ is a locally constant function with compact support and the measure $dy$ is the Haar measure on $\mathbb{Q}_p$. In particular, the measure $dy$ is such that $[v]$ has measure $p^{-\mathrm{ht}(v)}$ for $v \in \mathcal{V}$.

**Proposition 16** *Let $\mathcal{D}$ be the Vladimirov operator on $\mathbb{Q}_2$. For $z = v_0 v_1 \cdots \in \partial\mathcal{T}_2$ and $f \in \mathcal{E}$,*

$$(\mathcal{D}f)(z) = \frac{1}{3} \lim_{n\to\infty} \frac{1}{\mu([v_n])} \langle \chi_{v_n}, -\Delta_2 f \rangle.$$

**Proof:** From Section 4.4.3,

$$\langle \chi_{v_n}, -\Delta_2 f \rangle = \int_{[v_n]} d\mu(x) \sum_{j=0}^{n-1} \frac{4}{\mu([v_0 \cdots v_{j-1}])^2} \int_{[v_0 \cdots v_{j-1} \bar{v}_j]} f(x) - f(y) d\mu(y).$$

But for $x \in [v_0 \cdots v_j]$ and $y \in [v_0 \cdots v_{j-1} \bar{v}_j]$, $|x-y|_2 = \mu([v_0 \cdots v_{j-1}])$. Therefore,

$$\langle \chi_{v_n}, -\Delta_2 f \rangle = 4 \int_{[v_n]} d\mu(x) \int_{[v_n]^c} \frac{f(x) - f(y)}{|x-y|_2^2} d\mu(y)$$

and the result follows. $\square$

Because $|D|^{-1}D = F$ is the phase of the operator $D$, then this result shows that since the Vladimirov operator is constructed out of the phase then it does not take the metric on $C_3$ into account. This makes sense because the Vladimirov operator was created using the 2-adic metric which comes from the measure and not from the metric on $C_3$.

## *5.2 Iterated Function Systems*

This section is devoted to showing that regular, ultrametric Cantor sets are quite general. In fact, they encompass a wide variety of classical examples.

### 5.2.1 Iterated Function Systems

This example can be found in [27]. To begin, let $(X, d)$ be a complete metric space.

**Definition 39** *An* iterated function system (IFS) *consists of a family of contractions* $\{F_1, \ldots, F_m\}$ *on* $X$ *with* $m \geq 2$. *Recall that the requirement that* $F_i$ *is a contraction means* $d(F_i(x), F_i(y)) \leq a_i d(x, y)$ *for all* $x, y \in X$ *with* $0 < a_i < 1$.

The following theorem gives the most fundamental property of iterated function systems.

**Theorem 30** *Let* $\{F_1, \ldots, F_m\}$ *be an IFS on a complete metric space* $(X, d)$. *Then there exists a unique, non-empty compact set* $E \subset X$ *that satisfies*

$$E = \bigcup_{i=1}^{m} F_i(E)$$

**Proof:** See [26] (Theorem 9.1) □

The set $E$ given in the conclusion of the theorem is called the *attractor* of the IFS $\{F_1, \ldots, F_m\}$.

**Definition 40** *The attractor* $E$ *of an IFS* $\{F_1, \ldots, F_m\}$ *is called* self-similar *if* $F_i$ *is a similarity for* $i = 1, \ldots, m$. *Recall that* $F_i$ *is a similarity if* $d(F_i(x), F_i(y)) = r_i d(x, y)$ *for all* $x, y \in X$.

**Remark 5** There seems to be some disagreement about when to call a set self-similar. Notably, [46] uses self-similar for what has been termed an attractor here. Our definition follows [27].

Given an IFS $\{F_1, \ldots, F_m\}$ with attractor $E$, it is possible to encode $E$ in terms of the IFS. To begin, let $I_0$ consist of the empty sequence and $I_k = \{(i_1, \ldots, i_k) : 1 \leq i_l \leq m\}$ for $k > 0$. Then $F_{i_1} \circ \cdots \circ F_{i_k} \circ F_{i_{k+1}}(E) \subset F_{i_1} \circ \cdots \circ F_{i_k}(E)$ and $\operatorname{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(E)) \to 0$ as $k \to \infty$ since each $F_i$ is a contraction. Because each $F_{i_1} \circ \cdots \circ F_{i_k}(E)$ is nonempty, there exists a unique point

$$x_{i_1,i_2,\ldots} = \bigcap_{k=0}^{\infty} F_{i_1} \circ \cdots \circ F_{i_k}(E)$$

Moreover, since $E = \bigcup_{i=1}^{m} F_i(E)$ then for every point $x \in E$ there exists at least one sequence $i_1, i_2, \ldots$ such that $x \in F_{i_1} \circ \cdots \circ F_{i_k}(E)$ for $k \geq 0$. This sequence might not be unique.

**Definition 41** *Given an IFS $\{F_1, \ldots, F_m\}$ and its attractor $E$, then the IFS is said to satisfy the* strong separability condition *if $E = \bigsqcup_{i=1}^{m} F_i(E)$, i.e. the union is disjoint.*

Thus, if $\{F_1, \ldots, F_m\}$ satisfies the strong separability condition, then every $x \in E$ has a unique sequence associated to it. It is then possible to construct a tree $\mathcal{T}$ associated to $E$. The vertices of $\mathcal{T}$ will consist of finite sequences $(i_1, \ldots, i_k)$. The partial ordering is given by requiring that $(i_1, \ldots, i_k) \succeq (i_1, \ldots, i_k, \ldots, i_{k+l})$. $\mathcal{T}$ is then a reduced, rooted, Cantorian tree. Moreover, since $\{F_1, \ldots, F_m\}$ satisfies the strong separability condition then $E \simeq \partial\mathcal{T}$. If $\epsilon : \mathcal{V} \to \mathbb{R}^+$ is given by $\epsilon((i_1, \ldots, i_k)) = \operatorname{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(E))$ then $\epsilon$ is a weight function on $\mathcal{T}$. Let $d_\epsilon$ be the regular ultrametric corresponding to $\mathcal{T}$.

**Theorem 31** *Let $(C, d)$ be a self-similar Cantor set that is the attractor of the IFS $\{F_1, \ldots, F_m\}$. Suppose that $\{F_1, \ldots, F_m\}$ satisfies the strong separability condition.*

*Then $d$ is equivalent to the regular ultrametric $d_\epsilon$.*

**Proof:** Since $C$ is compact and the $F_i$'s are similarities with similarity ratio $r_i$, then

$$\mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(C)) = r_{i_1} \cdots r_{i_k} \mathrm{diam}(C).$$

Similarly,

$$\mathrm{dist}(F_{i_1} \circ \cdots \circ F_{i_k}(F_i(C)), F_{i_1} \circ \cdots \circ F_{i_k}(F_j(C))) = r_{i_1} \cdots r_{i_k} \mathrm{dist}(F_i(C), F_j(C)).$$

Therefore, let $x, y \in C$ with $x \wedge y = (i_1, \ldots, i_k)$. Then $x \in F_{i_1} \circ \cdots \circ F_{i_k}(F_i(C))$ and $y \in F_{i_1} \circ \cdots \circ F_{i_k}(F_j(C))$ with $i \neq j$. If $M = \min\{\mathrm{dist}(F_i(C), F_j(C)) : 1 \leq i, j \leq m, i \neq j\}$, then $M > 0$ since $\{F_1, \ldots, F_m\}$ satisfies the strong separability condition. Moreover,

$$\frac{M}{\mathrm{diam}(C)} d_\epsilon(x, y) \leq d(x, y) \leq d_\epsilon(x, y)$$

since $d_\epsilon(x, y) = \mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(C)) = r_{i_1} \cdots r_{i_k} \mathrm{diam}(C)$. Therefore, $d$ and $d_\epsilon$ are equivalent. $\square$

Given a self-similar Cantor set $(C, d)$ that is the attractor of an IFS $\{F_1, \ldots, F_m\}$ that satisfies the strong separability condition, it is now possible to study the zeta function associated to $d_\epsilon$. To begin,

$$\zeta(s) = \mathrm{diam}(C)^s \sum_{k=0}^{\infty} \sum_{(i_1, \ldots, i_k) \in I_k} (r_{i_1} \cdots r_{i_k})^s = \mathrm{diam}(C)^s \sum_{k=0}^{\infty} (r_1^s + \cdots + r_m^s)^k.$$

Thus $\zeta(s)$ converges for $\Re(s) > s_0$ where $s_0$ is such that $r_1^{s_0} + \cdots + r_m^{s_0} = 1$. This is the familiar *similarity dimension* of $C$ (see [26] Ch 9.2) and consequently the upper box dimension and similarity dimension coincide. In fact, it turns out that more is true. For a self-similar Cantor set that is the attractor of an IFS satisfying the strong separability condition, all the typical fractal dimensions coincide with $s_0$ (see [26]

Thm. 9.3).

### 5.2.2 Cookie Cutter Systems

Continuing with showing that the class of ultrametric Cantor sets is diverse, this section gives an example of an ultrametric Cantorian Julia set that comes from an iterated function system associated to a certain type of iterated rational function. This example can also be found in [27]. Let $X$ be a closed non-empty bounded interval in $\mathbb{R}$ and let $X_1, \ldots, X_m$ with $m \geq 2$ be non-empty closed subintervals. Let $f : \bigsqcup_{i=1}^m X_i \to X$ be such that $X_i$ is mapped bijectively onto $X$ for each $i = 1, \ldots, m$. Suppose further that $f$ has continuous second derivatives on $X$ and that $|f'(x)| > 1$ on $X_i$ for $i = 1, \ldots, m$. Thus $f$ is expanding at all points of $\bigsqcup_{i=1}^m X_i$. Since $f$ is bijective on $X_i$ for each $i = 1, \ldots, m$, it is possible to define $F_i : X \to X_i$ by $F_i = f^{-1}_{\restriction X_i}$. Because $|f'(x)| > 1$ then $|F_i'(x)| < 1$ for all $x \in \bigsqcup_{i=1}^m X_i$. By the Mean Value Theorem, $F_i$ is a contraction for all $i = 1, \ldots, m$. Thus $\{F_1, \ldots, F_m\}$ is an IFS. The attractor $C$ of this IFS is called the *repeller* of $f$. $C$ is called a *cookie-cutter set* and the IFS is called a *cookie-cutter system*. Since the union $\bigsqcup_{i=1}^m X_i = \bigsqcup_{i=1}^m F_i(X)$ is disjoint, then this IFS satisfies the strong separability condition and thus $C$ is a Cantor set. Let $\mathcal{T}$ and $d_\epsilon$ be defined as in the previous section. In this section, it will be shown that the Euclidean metric $d$ restricted to $C$ is metrically equivalent to $d_\epsilon$ on $C$. The proof requires the principle of bounded distortion as given in [27] (Ch. 4)

**Lemma 6 (Principle of Bounded Distortion)** *There exists $b_0 \in \mathbb{R}^+$ such that for all $k = 0, 1, \ldots$ and for all $(i_1, \ldots, i_k) \in I_k$,*

$$b_0^{-1} \leq \frac{\operatorname{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(X))}{\operatorname{diam}(X)} |(f^k)'(x)| \leq b_0 \tag{4}$$

*for all $x \in F_{i_1} \circ \cdots \circ F_{i_k}(X)$.*

With this lemma in hand, it is now possible to show that $d_\epsilon$ is equivalent to the Euclidean metric $d$ restricted to $C$.

**Theorem 32** *Let $C$ be a cookie cutter set. Then $d_\epsilon$ is equivalent to the Euclidean metric $d$.*

**Proof:** Let $M = \min\{\mathrm{dist}(F_i(C), F_j(C)) : i \neq j, 1 \leq i, j \leq m\}$. Let $y \in F_{i_1} \circ \cdots \circ F_{i_k} \circ F_i(C)$ and $z \in F_{i_1} \circ \cdots \circ F_{i_k} \circ F_j(C)$ be such that $i \neq j$. By definition, $d_\epsilon(y,z) = \mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(C))$. Now $|f^k(y) - f^k(z)| \geq M$. Since $y, z \in F_{i_1} \circ \cdots \circ F_{i_k}(C)$, then by the Mean Value Theorem there exists $x \in F_{i_1} \circ \cdots \circ F_{i_k}(X)$ such that $|f^k(y) - f^k(z)| = |y - z|(f^k)'(x)$. By the Principle of Bounded Distortion,

$$b_0^{-1} \leq |(f^k)'(x)| \frac{\mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(X))}{\mathrm{diam}(X)} \leq b_0.$$

This gives that,

$$b_0^{-1} \leq \frac{|f^k(y) - f^k(z)| \mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(X))}{|y - z| \mathrm{diam}(X)} \leq b_0.$$

Since $|f^k(y) - f^k(z)| \geq M$ and $\mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(C)) \leq \mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(X))$, then

$$\frac{M \mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(C))}{|y - z| \mathrm{diam}(X)} \leq \frac{|f^k(y) - f^k(z)| \mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(X))}{|y - z| \mathrm{diam}(X)} \leq b_0.$$

In particular,

$$\frac{M \mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(C))}{\mathrm{diam}(X)} b_0^{-1} \leq |y - z|$$

Since $y, z \in F_{i_1} \circ \cdots \circ F_{i_k}(C)$, then $|y - z| \leq \mathrm{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(C))$. Thus,

$$\frac{M}{\mathrm{diam}(X)} b_0^{-1} d_\epsilon(y,z) \leq |y - z| \leq d_\epsilon(y,z)$$

and the two metrics are equivalent. $\square$

**Example 7** Using the $\zeta$-function associated to $d_\epsilon$ it is possible to get well known bounds on the upper box dimension of a cookie cutter set. Let $f(x) = 5x(1-x)$ and let $X = [0,1]$ with $X_1 = [0, .28]$ and $X_2 = [.72, 1]$. This is a cookie cutter system with Cantor repeller $C$. Now $f$ has two inverses $F_1(x) = 1/2 + \sqrt{25-20x}/10$ and $F_2(x) = 1/2 - \sqrt{25-20x}/10$. Since these functions are monotone, then

$$\operatorname{diam}(F_{i_1} \circ \cdots \circ F_{i_k}(C)) = |F_{i_1} \circ \cdots \circ F_{i_k}(1) - F_{i_1} \circ \cdots \circ F_{i_k}(0)| = |(F_{i_1} \circ \cdots \circ F_{i_k})'(c)|$$

for some $c \in X$ by the Mean Value Theorem. But $F_1'(0) = 1/5 \leq |F_i'(x)| \leq 1/2.2 = F_1'(.28)$ for $x \in X_i$. Thus

$$\sum_{n=0}^{\infty} (\frac{2}{5^s})^n \leq \zeta(s) \leq \sum_{n=0}^{\infty} (\frac{2}{2.2^s})^n$$

Thus the abscissa of convergence $s_0$ for $\zeta(s)$ is such that

$$\frac{\log 2}{\log 5} \leq s_0 \leq \frac{\log 2}{\log 2.2}.$$

This also gives a bound on the upper box dimension for the Julia set of $f$ as well. In fact, for cookie cutter sets all the typical fractal dimensions coincide (see [27] Corollary 4.6) and consequently this bound holds for all the typical fractal dimensions.

### 5.2.3 Hausdorff Measure and Hausdorff Dimension

This section presents the basics of the Hausdorff measure (see [26]). Let $X$ be a metric space with metric $d$. If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most $\delta$ that covers $X$, then $\{U_i\}$ is called a $\delta$-*cover* of $X$. Let $s > 0$. For any $\delta > 0$, let

$$H^s_\delta(X, d) := \inf\{\sum \operatorname{diam}(U_i)^s\}$$

where the infimum is taken over all $\delta$ covers of $X$. Because the class of permissible covers of $X$ decreases as $\delta \downarrow 0$ then $H^s_\delta$ increases and therefore approaches a (possibly infinite) limit as $\delta \downarrow 0$. Then $H^s(X,d) := \lim_{\delta \downarrow 0} H^s_\delta(X)$ is called the *$s$-dimensional Hausdorff measure* of $X$. $H^s$ can be shown to be a measure. It has the following scaling property: $H_s(X, \lambda d) = \lambda^s H_s(X,d)$. That is changing the distance by a factor of $\lambda$ changes the measure by a factor of $\lambda^s$. From now on, when convenient $H^s(X,d)$ will be written as just $H^s(X)$.

By studying $H^s(X)$ as $s$ changes, it is possible to get another fractal dimension. First of all, $H^s(X)$ is non-increasing in $s$. In fact, it turns out that except for a single point $s_0$, $H^s(X)$ is either 0 or $\infty$. The *Hausdorff dimension* is defined to be $\dim_H(X) := \inf\{s : H^s(X) = 0\}$. At $s = \dim_H(X)$, $H^s(X)$ may either be $0, \infty$, or a positive real number. Hausdorff dimension satisfies the dimension properties mentioned earlier: monotonicity, zero on finite sets, and it gives dimension $n$ to open sets in $\mathbb{R}^n$. It is also invariant under bi-Lipschitz transformation and thus is the same for two equivalent metrics. As mentioned previously, the Hausdorff dimension is less than or equal to the upper box dimension.

### 5.2.4 The Measure on Self-Similar Cantor Sets

The Hausdorff dimension and the Hausdorff measure can be difficult to calculate in most cases. However, for self-similar Cantor sets the calculation is possible. Let $C$ be a self-similar Cantor set that is the attractor of an IFS $\{F_1, \ldots, F_n\}$ that satisfies the strong separability condition and has similarity ratios $r_1, \ldots, r_n$ . As mentioned previously, the Hausdorff dimension and the upper box dimension coincide with the unique $s_0 > 0$ that satisfies $r_1^{s_0} + \cdots + r_n^{s_0} = 1$. This is the *similarity dimension*. This

$s_0$ is also the abscissa of convergence of the $\zeta$-function associated to $C$. Now,

$$\begin{aligned}
\lim_{s \downarrow s_0}(s-s_0)\mathrm{Tr}\left(|D|^{-s}\right) &= \lim_{s \downarrow s_0}(s-s_0)\frac{\mathrm{diam}(C)^s}{1-(r_1^s+\cdots+r_n^s)} \\
&= \frac{\mathrm{diam}(C)^{s_0}}{-(r_1^{s_0}\ln r_1+\cdots+r_m^{s_0}\ln r_m)}.
\end{aligned}$$

Since the denominator is a sum of positive numbers, then $\lim_{s\downarrow s_0}(s-s_0)\mathrm{Tr}\ (|D|^{-s}) > 0$.

Because the subtree located at each vertex is just a scaled version of the tree $\mathcal{T}$, then for $v \in \mathcal{V}$ corresponding to the sequence $(i_1, \ldots, i_k)$

$$\begin{aligned}
\mu([v]) = \mu_\tau(\chi_{[v]}) &= \lim_{s\downarrow s_0}\frac{\mathrm{Tr}\ \left(|D|^{-s}\pi_\tau(\chi_{[v]})\right)}{\mathrm{Tr}\ (|D|^{-s})} \\
&= \lim_{s\downarrow s_0}(r_{i_1}\cdots r_{i_k})^s\frac{\mathrm{Tr}\ (|D|^{-s})}{\mathrm{Tr}\ (|D|^{-s})} = (r_{i_1}\cdots r_{i_k})^{s_0}.
\end{aligned}$$

Thus $\lim_{s\downarrow s_0}(s-s_0)\mathrm{Tr}\ (|D|^{-s}\pi_\tau(f))$ is defined for all simple functions $f$. By the continuity and the linearity of the trace and the representation, it is then be defined on all of $\mathcal{C}(C)$ and thus $C$ is $\zeta$-regular.

The following lemma often called the mass distribution principle will help show that this measure is in fact the Hausdorff measure.

**Lemma 7 (Mass Distribution Principle)** *Let $\mu$ be a measure on $C$ with support contained in $C$ such that $0 < \mu(C) < \infty$. Suppose that for some $s \geq 0$ there are numbers $c > 0$ and $\delta > 0$ such that*

$$\mu(U) \leq c\ \mathrm{diam}(U)^s$$

*for all sets $U$ such that* $\mathrm{diam}(U) \leq \delta$*. Then $H^s(C) \geq \mu(C)/c$ and $s \leq \dim_H(C)$.*

**Proof:** If $\{U_i\}$ is any $\nu$-cover of $C$ with $\nu < \delta$, then

$$0 < \mu(C) = \mu(\bigcup U_i) \leq \sum_i \mu(U_i) \leq c\sum_i \mathrm{diam}(U)^s.$$

Taking the infimum over all such $\{U_i\}$ shows that $H^s_\nu(C) \geq \mu(C)/c$ for $\nu < \delta$, so $H^s(C) \geq \mu(C)/c$. $\square$

**Theorem 33** *Let $C$ be the attractor of a self-similar iterated function system that satisfies the strong separability condition. Then the following are true:*
*(i) $C$ is a $\zeta$-regular Cantor set with its natural metric coming from the iterated function system;*
*(ii) up to a constant, $\mu$ is equal to the $s_0$-Hausdorff measure where $s_0$ is the similarity dimension of $C$.*

**Proof:** The proof of (i) was shown above. For part (ii), the Mass Distribution Principle guarantees that $H^{s_0}([v]) \geq \operatorname{diam}(C)^{s_0}\mu([v])$ for any $v \in \mathcal{V}$. However, for any height greater than $\operatorname{ht}(v)$ there is a cover $O$ of $[v]$ by taking all the descendants of $v$ of that height. If $v = (i_1, \ldots, i_k)$, then the cover $O$ is of the form $\{[v'] : v' = (i_1, \ldots, i_k, i_{k+1}, \ldots, i_{k+j})\}$. Thus,

$$\begin{aligned}
\sum_{v' \in O} \operatorname{diam}([v'])^{s_0} &= \sum_{(i_{k+1},\ldots,i_{k+m}) \in I_m} \operatorname{diam}([(i_1, \ldots, i_{k+j})])^{s_0} \\
&= \operatorname{diam}(C)^{s_0} \sum_{(i_{k+1},\ldots,i_{k+m}) \in I_m} (r_{i_1} \cdots r_{i_k} r_{i_{k+1}} \cdots r_{i_{k+j}})^{s_0} \\
&= \operatorname{diam}(C)^{s_0} (r_{i_1} \cdots r_{i_k})^{s_0} \sum_{(i_{k+1},\ldots,i_{k+j}) \in I_j} (r_{i_{k+1}} \cdots r_{i_{k+j}})^{s_0} \\
&= \operatorname{diam}(C)^{s_0} (r_{i_1} \cdots r_{i_k})^{s_0} (r_1^{s_0} + \cdots + r_n^{s_0})^j \\
&= \operatorname{diam}(C)^{s_0} (r_{i_1} \cdots r_{i_k})^{s_0}.
\end{aligned}$$

Consequently, $H^{s_0}([v]) \leq \operatorname{diam}(C)^{s_0}\mu([v])$ and thus $H^{s_0} = \operatorname{diam}(C)^{s_0}\mu$ on $C$. $\square$

## 5.3 Delone Sets, Tilings, and the Transversal

This section treats the case of Cantor sets arising as the transversal of a Delone set.

### 5.3.1 The Transversal of a Repetitive, Aperiodic Delone Set of Finite Type

The following definitions can be found in [6].

**Definition 42** *Let $\mathcal{L}$ be a discrete subset of $\mathbb{R}^d$.*

*1) $\mathcal{L}$ is* uniformly discrete *if there is an $r > 0$ such that every open ball of radius $r$ meets $\mathcal{L}$ in at most one point. $\mathcal{L}$ is then said to be $r$-discrete.*

*2) $\mathcal{L}$ is* relatively dense *if there is an $R > 0$ such that every closed ball of radius $R$ meets $\mathcal{L}$ in at least one point. $\mathcal{L}$ is then said to be $R$-dense.*

*3) $\mathcal{L}$ is a* Delone set *if it is both uniformly discrete and relatively dense. $\mathcal{L}$ is then said to be $(r, R)$-Delone if it is $r$-discrete and $R$-dense.*

*4) A Delone set $\mathcal{L}$ is of* finite type *whenever $\mathcal{L} - \mathcal{L} := \{x - y : x, y \in \mathcal{L}\}$ is locally finite.*

*5) $\mathcal{L}$ is* aperiodic *if there is no non-zero $a \in \mathbb{R}^d$ such that $\mathcal{L} + a = \mathcal{L}$.*

It is now possible to construct a tree $\mathcal{T}$ associated to an $(r_0, R)$-Delone set $\mathcal{L}$ of finite type. The key to forming the tree is the notion of a *patch*. Let $\mathcal{P}_r := \{(\mathcal{L} - x) \cap B(0, r) : x \in \mathcal{L}\}$ be the set of patches of radius $r$ for $r \geq 0$. Now $\mathcal{P}_r \subset (\mathcal{L} - \mathcal{L}) \cap \overline{B(0, r)}$ and since $\mathcal{L}$ is finite type then $(\mathcal{L} - \mathcal{L}) \cap \overline{B(0, r)}$ is finite. Thus $\mathcal{P}_r$ is finite for each $r \geq 0$.

**Lemma 8** *Let $\mathcal{L}$ be an $(r_0, R)$-Delone set of finite type and let $r > 0$. Then there exists $\delta > 0$ such that $\mathcal{P}_r = \mathcal{P}_{r-\delta}$.*

**Proof:** Let $p \in \mathcal{P}_r$. Then $p$ is a finite set since $\mathcal{L}$ is uniformly discrete. Let $m = \max\{||x|| : x \in p\}$. Now $m < r$. Thus if $\delta_p = (r - m)/2$, then $\delta_p > 0$. Also since $p \cap B(0, r - \delta_p) = p$, then $p \in \mathcal{P}_{r-\delta_p}$. If $\delta = \min\{\delta_p : \delta_p \in \mathcal{P}_r\}$, then $\delta > 0$ and $\mathcal{P}_r \subset \mathcal{P}_{r-\delta}$. However, any patch $p \in \mathcal{P}_r$ can restrict to a patch $p_{|s} = p \cap B(0, s)$ for $s < r$. Thus $|\mathcal{P}_r| \geq |\mathcal{P}_{r-\delta}|$ and $\mathcal{P}_r = \mathcal{P}_{r-\delta}$. $\square$

Because $\mathcal{L}$ is R-dense, then $\mathcal{P}_r \neq \mathcal{P}_{r+2R}$. Because of the previous lemma, then a maximal sequence $r_0 < r_1 < r_2 < \cdots$ exists such that $\mathcal{P}_{r_i} \neq \mathcal{P}_{r_{i+1}}$. In fact this maximal sequence could be given by $\{||a|| : a \in (\mathcal{L} - \mathcal{L}), a \neq 0\}$. From now on let $\mathcal{P}_n := \mathcal{P}_{r_n}$. As seen in the proof of the lemma, $\mathcal{P}_m$ projects onto $\mathcal{P}_n$ for $n < m$ and the projection $\pi_{m,n}$ is simply the restriction map. The tree $\mathcal{T}$ can be constructed by letting $\mathcal{V} = \bigcup_{k=0}^{\infty} \mathcal{P}_n$. Since $\mathcal{P}_0$ consists of just one patch corresponding to a single point, let this be the root. Then for $p, p' \in \mathcal{V}$, let $p \succeq p'$ if $p \in \mathcal{P}_m$ and $p' \in \mathcal{P}_n$ with $m > n$ and $\pi_{m,n}(p) = p'$. Thus the children of a vertex $p \in \mathcal{P}_n$ are the elements of $\mathcal{P}_{n+1}$ that restrict to $p$.

In order to study $\partial\mathcal{T}$, the idea of the Hull of a Delone set is necessary. In order to define the Hull, the Delone set must be put in the right framework. To every Delone set $\mathcal{L}$, it is possible to assign a measure

$$\nu^{\mathcal{L}} = \sum_{y \in \mathcal{L}} \delta_y$$

where $\delta_y$ assigns 1 to any set that contains $y$. Then $\mathcal{L}$ can be considered to be contained in $\mathfrak{M}(\mathbb{R}^d)$, the space of all measures on $\mathbb{R}^d$. Thinking of $\mathfrak{M}(\mathbb{R}^d)$ as the dual of $\mathcal{C}_c(\mathbb{R}^d)$, then $\mathfrak{M}(\mathbb{R}^d)$ can be topologized with the weak $*$-topology. This is the topology given by the family of seminorms, $\{\rho_f(\cdot)\}_{f \in \mathcal{C}_c(\mathbb{R}^d)}$ where $\rho_f(\nu) = |\nu(f)|$ for $\nu \in \mathfrak{M}(\mathbb{R}^d)$.

**Definition 43** *Given a Delone set $\mathcal{L}$, its* Hull $\Omega$ *is defined to be the closure in $\mathfrak{M}(\mathbb{R}^d)$ of $\{\nu^{\mathcal{L}+a} : a \in \mathbb{R}^d\}$ (i.e. the set of all its translates).*

If $\mathcal{L}$ is uniformly discrete then $\Omega$ is a compact subset of $\mathfrak{M}(\mathbb{R}^d)$ (see [5]). To each $\omega \in \Omega$, let $\mathcal{L}_\omega$ be the support of $\omega$. It is also true that if $\mathcal{L}$ is an $(r, R)$-Delone set and $\omega \in \Omega$, then $\mathcal{L}_\omega$ is also an $(r, R)$-Delone set (see [5]). It will be important in what follows to understand convergence in this topology. The following lemma shows that a sequence of Delone sets converges if it converges in every generic window.

**Lemma 9** *Let $\mathcal{L}$ be an $(r, R)$-Delone set with hull $\Omega$. Let $\mathcal{L}_1, \mathcal{L}_2, \dots \in \Omega$. Then $\nu^{\mathcal{L}_n}$ converges to $\omega$ if and only if for any open ball $B(0, R_0)$ with $\mathcal{L}_\omega \cap B(0, R_0) = \mathcal{L}_\omega \cap \overline{B(0, R_0)}$, $\mathcal{L}_n \cap B(0, R_0)$ converges to $\mathcal{L}_\omega \cap B(0, R_0)$ in the Hausdorff metric.*

**Proof:** Suppose $\nu^{\mathcal{L}_n}$ converges to $\omega$. Then $\omega \in \Omega$ and by the above discussion $\mathcal{L}_\omega$ is also an $(r, R)$-Delone set. Now $\nu^{\mathcal{L}_n} \to \omega$ in the weak $*$-topology if and only if $\nu^{\mathcal{L}_n}(f) \to \omega(f)$ for every $f \in \mathcal{C}_c(\mathbb{R}^d)$. To begin, let $p = B(0, R_0) \cap \mathcal{L}_\omega$. For $y \in p$ and $r/2 > \epsilon > 0$, let $g_y^\epsilon(x) = 1 - \max\{1 - ||x - y||/\epsilon, 0\}$. Then, $g_y^\epsilon(x) > 0$ if and only if $||x - y|| < \epsilon$. Since $g_y^\epsilon \in \mathcal{C}_c(\mathbb{R}^d)$, then $\nu^{\mathcal{L}_n}(g_y^\epsilon) \to \omega(g_y^\epsilon)$. But $\omega(g_y^\epsilon) = \sum_{x \in \mathcal{L}_\omega} g_y^\epsilon(x) = g_y^\epsilon(y) = 1$ since $\mathcal{L}_\omega$ is $r$-discrete and $\epsilon < r$. Thus $\nu^{\mathcal{L}_n}(g_y^\epsilon) \to 1$ and for $n \geq N_y$ there exists $x \in \mathcal{L}_n$ with $g_y^\epsilon(x) > 0$. This gives that $||x - y|| < \epsilon$. Since $\mathcal{L}_n$ is $r$-discrete, if $x, x \in \mathcal{L}_n$ with $||x - y|| < r/2$ then,

$$r < ||x - x'|| \leq ||x - y|| + ||x' - y|| < r/2 + ||x' - y||.$$

Thus, $||x' - y|| > r/2$ and this $x$ is unique. Moreover, $\mathrm{dist}(y, \mathcal{L}_n) < \epsilon$ for $n \geq N_y$. Because $\mathcal{L}_\omega$ is $r$-discrete, then $p = B(0, R_0 - \epsilon) \cap \mathcal{L}_\omega$ for $\epsilon$ small enough. So it can be assumed that in fact $x \in B(0, R_0) \cap \mathcal{L}_n$. Consequently if $N := \max\{N_y : y \in p\}$ then for $n \geq N$, $\mathrm{dist}(y, B(0, R_0) \cap \mathcal{L}_n) < \epsilon$ for all $y \in p$.

Because $p = \overline{B(0, R_0)} \cap \mathcal{L}_\omega$, there exists $r' > 0$ such that $B(0, R_0 + r') = p$. Let $h \in \mathcal{C}_c(\mathbb{R}^d)$ be such that $h$ is equal to 1 on $\overline{B(0, R_0)}$ and 0 outside of $B(0, R_0 + r')$. Then $\omega(h) = |p|$ where $|p|$ is the number of points in $p$. As seen above, to every $y \in p$ there is a unique $x \in B(0, R_0) \cap \mathcal{L}_n$ for $n \geq N$. Thus $|B(0, R_0) \cap \mathcal{L}_n| \geq |p|$. But since $\nu^{\mathcal{L}_n}(h) \to \omega(h)$, there exists $N'$ such that $|\nu^{\mathcal{L}_n}(h) - \omega(h)| < 1$ for $n \geq N'$. Thus $|B(0, R_0) \cap \mathcal{L}_n| = |p|$ for $n \geq N'$. Since $p$ and $B(0, R_0) \cap \mathcal{L}_n$ have the same number of points for $n \geq N'$ then for $n \geq \max\{N, N'\}$, $\mathrm{dist}(x, p) < \epsilon$ for all $x \in B(0, R_0) \cap \mathcal{L}_n$. This gives that the Hausdorff distance $\rho_H(p, B(0, R_0) \cap \mathcal{L}_n) < \epsilon$ for $n \geq \max\{N, N'\}$ and that $\mathcal{L}_n \cap B(0, R_0)$ converges to $\mathcal{L}_\omega \cap B(0, R_0)$ in the Hausdorff metric.

Now assume that for any open ball $B(0, R_0)$ with $\mathcal{L}_\omega \cap B(0, R_0) = \mathcal{L}_\omega \cap \overline{B(0, R_0)}$, $\mathcal{L}_n \cap B(0, R_0)$ converges to $\mathcal{L}_\omega \cap B(0, R_0)$ in the Hausdorff metric. Let $f \in \mathcal{C}_c(\mathbb{R}^d)$ and let $\epsilon > 0$ be given. Since $f$ has support in a compact set, it is possible to choose $R_0$ large enough so that $f$ is 0 outside of $B(0, R_0)$. Because $\mathcal{L}_\omega$ is $r$-discrete, it is possible to make sure $R_0$ is such that $\mathcal{L}_\omega \cap B(0, R_0) = \mathcal{L}_\omega \cap \overline{B(0, R_0)}$. Now $f$ is continuous on $\overline{B(0, R_0)}$ so it is uniformly continuous here. Thus, if $p = B(0, R_0 - \epsilon) \cap \mathcal{L}_\omega$, then $0 < \delta < r/2$ can be chosen so that if $||x - x'|| \leq \delta$ then $||f(x) - f(x')|| < \epsilon/|p|$. Since $\mathcal{L}_\omega$ and $\mathcal{L}_n$ are $r$-discrete, there are only a finite number of points in the ball of radius $R_0$. If $\rho_H(p, B(0, R_0) \cap \mathcal{L}_n) < \delta$ then since $\delta < r/2$ then to each $y \in p$ there is a unique $x_y \in B(0, R_0) \cap \mathcal{L}_n$ with $||x_y - y|| < \delta$. Furthermore, this correspondence is one to one since $\operatorname{dist}(x, p) < \delta$ for all $x \in B(0, R_0) \cap \mathcal{L}_n$. Thus,

$$|\nu^{\mathcal{L}_n}(f) - \omega(f)| = |\sum_{y \in p} f(x_y) - f(y)| \leq \sum_{y \in p} |f(x_y) - f(y)| < \sum_{y \in p} \frac{\epsilon}{|p|} = \epsilon$$

for all $n$ such that $\rho_H(p, B(0, R_0) \cap \mathcal{L}_n) < \delta$. Thus, $\nu^{\mathcal{L}_n}(f) \to \omega(f)$. □

Because of the way it was defined, $\Omega$ carries an $\mathbb{R}^d$ action. As is common in dynamics, it is important to find a transversal to this action.

**Definition 44** *The* canonical transversal*, $\Xi$ is defined to be the set $\{\omega \in \Omega : 0 \in \mathcal{L}_\omega\} \subset \Omega$.*

**Proposition 17** *If $\mathcal{L}$ is an $(r, R)$-Delone set of finite type, then $\partial\mathcal{T}$ is homeomorphic to $\Xi$.*

**Proof:** Let $F : \Xi \to \partial\mathcal{T}$ be defined as follows. Let $\omega \in \Xi$. Then $\mathcal{L}_\omega$ is an $(r, R)$-Delone set and in general $\omega = \lim_{n\to\infty} \nu^{\mathcal{L}+a_n}$. If $p_k = B(0, r_k) \cap (\mathcal{L} - a_n)$, then $p_k \in \mathcal{P}_k$ by definition and it can be assumed that $r_k$ satisfies the hypotheses of lemma 9. Thus $\lim_{n\to\infty} p_k = B(0, r_k) \cap \mathcal{L}_\omega$. Since $\mathcal{P}_k$ is finite, then this limit is eventually constant and $B(0, r_k) \cap \mathcal{L}_\omega \in \mathcal{P}_k$. So if $p_i = \mathcal{L}_\omega \cap B(0, r_i)$, then $p_i \in \mathcal{P}_i$. It is obvious that for $m > n$, $\pi_{m,n}(p_m) = p_n$. Thus $p_0, p_1, \ldots$ is an infinite sequence of patches. Since each vertex in $\mathcal{T}$ is a patch, then this an infinite sequence of vertices. Thus, if $F(\omega) = \{p_i\}_{i=0}^{\infty}$, then $F : \Xi \to \partial\mathcal{T}$.

To prove injectivity, suppose $F(\omega) = F(\omega') = \{p_i\}_{i=0}^{\infty}$. Then $\mathcal{L}_\omega$ and $\mathcal{L}_{\omega'}$ agree on every ball centered at the origin. So for any ball of radius $R_0$, the Hausdorff distance, $\rho_H$, between $B(0, R_0) \cap \mathcal{L}_\omega$ and $B(0, R_0) \cap \mathcal{L}_{\omega'}$ is zero. Thus for any $f \in \mathcal{C}_c(\mathbb{R}^d)$, $\omega(f) = \omega'(f)$. Therefore, $\omega = \omega'$ since $\mathfrak{M}(\mathbb{R}^d)$ is Hausdorff in the weak $*$-topology.

To prove surjectivity, let $\{p_i\}_{i=0}^{\infty} \in \partial\mathcal{T}_{\mathcal{L}}$. By definition, each patch $p_n$ comes as a subset of some Delone set $\mathcal{L} - a_n$. Since $\Omega$ is compact $\{\nu^{\mathcal{L}-a_n}\}_{n=0}^{\infty}$ has a convergent subsequence $\{\nu^{\mathcal{L}-a_j}\}_{j=0}^{\infty}$ with $\lim_{j\to\infty} \nu^{\mathcal{L}-a_j} = \omega$ for some $\omega \in \Omega$. Because of the previous lemma, $\omega \in \Xi$ (i.e. $\Xi$ is closed and thus compact). Moreover, $\omega \cap B(0, r_n) = p_n$ since $(\mathcal{L} - a_j) \cap B(0, r_n) = p_n$ for $j \geq n$. So $F(\omega) = \{p_i\}_{i=0}^{\infty}$.

To show that $F$ is continuous, let $p \in \mathcal{P}_n \subset \mathcal{V}$. Then $F^{-1}([p]) = \{\omega \in \Xi : B(0, r_n) \cap \mathcal{L}_\omega = p\}$. Now since $\mathcal{P}_n$ is finite, there exists $\epsilon$ so that $\rho_H(p, p') > \epsilon$ for all $p' \in \mathcal{P}_n$ with $p' \neq p$. Let $h_p^\epsilon$ and $h$ be as defined in the proof of the previous lemma. Then $F^{-1}([p]) = (h_p^\epsilon)^{-1}(|p|) \cap h^{-1}(|p|)$. This is true since if $\omega$ is such that $B(0, r_n) \cap \mathcal{L}_\omega \in (h_p^\epsilon)^{-1}(|p|)$ then $p \subset B(0, r_n) \cap \mathcal{L}_\omega$. But if $B(0, r_n) \cap \mathcal{L}_\omega \in h^{-1}(|p|)$ then $|p| \geq |B(0, r_n) \cap \mathcal{L}_\omega|$ and so $p = B(0, r_n) \cap \mathcal{L}_\omega$. Since $F^{-1}([p]) = (h_p^\epsilon)^{-1}(|p|) \cap h^{-1}(|p|)$ is open in the weak $*$-topology then $F$ is continuous. Consequently, $F$ is a continuous bijective map from a compact to a Hausdorff space and thus a homeomorphism. $\square$

In general, $\Xi$ is not a Cantor set.

**Definition 45** *An $(r, R)$-Delone set $\mathcal{L}$ is* repetitive *if for any finite subset $p \subset \mathcal{L}$ and $0 < \delta < r$ then there exists $R'$ such that any ball $B(x, R')$ (for $x \in \mathbb{R}^d$) contains a finite subset $p'$ and some $a_x \in \mathbb{R}^d$ with the Hausdorff distance between $p$ and $p' - a_x$ less than $\epsilon$.*

**Corollary 3** *If $\mathcal{L}$ is a repetitive, aperiodic $(r, R)$-Delone set of finite type, then $\Xi$ is* *a Cantor set.*

**Proof:** Let $p \in \mathcal{P}_n \subset \mathcal{V}$ and let $\delta > 0$ be such that the Hausdorff distance $\rho_H(p', p) > \delta$ for any other $p' \in \mathcal{P}_n$ with $p \neq p'$. Let $\nu^{\mathcal{L}-a} \in [p]$. Since $\mathcal{L}$ is repetitive, then there exists an $R'$ and a $p' \subset B(x, R')$ with $\rho_H(p, p' - a_x) < \delta/4$. By making $x$ large enough, it can guaranteed that $a \notin p'$ and thus that $p \neq p'$. Since $0 \in p$, then there exists $y \in p' - a$ with $||y|| < \delta/4$. This gives that $0 \in p' - a_x - y$ and $\rho_H(p, p' - a_x - y) < \delta/2$ and consequently $p = p' - a_x - y$. So $(\mathcal{L} - a_x - y) \cap B(0, r_n) = p$ and since $\mathcal{L}$ is aperiodic then $\mathcal{L} - a_x - y \neq \mathcal{L} - a$. Thus, $\nu^{\mathcal{L}-a_x-y}, \nu^{\mathcal{L}-a} \in [p]$ and $p$ must have at least two children. Thus $\mathcal{T}$ is a Cantorian tree and $\partial\mathcal{T} \simeq \Xi$ is a Cantor set. $\square$

Since every vertex has a descendant with at least two children, using edge reduction it is possible to obtain a reduced, rooted Cantorian tree $\mathcal{T}'$ such that $\partial\mathcal{T} = \partial\mathcal{T}'$. Let $\epsilon : \mathcal{V} \to \mathbb{R}^+$ be defined by $\epsilon(p) = 1/r_n$ if $p \in \mathcal{P}_n$. Then $\epsilon$ is a weight function on $\partial\mathcal{T}'$ and thus there is a corresponding regular ultrametric $d$ on $\Xi$. For $p, p' \in \Xi$, $d(p, p') = 1/r_n$ if $p$ and $p'$ agree on a ball of radius $r_n$ but disagree on a ball of radius $r_{n+1}$.

### 5.3.2 The Fibonacci Tiling

In this section, the Fibonacci Tiling is studied. For more information on the Fibonacci tiling, the reader can consult [31]. The Fibonacci Tiling gives a specific example of a Cantor set that is the transversal of a repetitive, aperiodic Delone set of finite type. In fact, tilings and Delone sets are closely related and the construction of the Delone set in this section is an instance of this close relationship.

The Fibonacci tiling is a covering of $\mathbb{R}$ by finite intervals given by the Fibonacci sequence, $\overline{\omega}$. This sequence is determined by the following substitution $\phi$ on the alphabet $\{a, b\}$:

$$\phi(a) = ab, \quad \phi(b) = a \ .$$

For example,

$$\phi^0(b) = b,\ \phi^1(b) = a,\ \phi^2(b) = ab,\ \phi^3(b) = aba,\ \phi^4(b) = abaab,\ \phi^5(b) = abaababa, \text{ etc.}$$

It is easy to see that $\phi^k(b)$ is a word with length equal to the kth Fibonacci number, $\mathrm{Fib}_k$. Recall that the Fibonacci numbers are determined by the recurrence relation,

$$\mathrm{Fib}_n := \left\{ \begin{array}{cl} 0 & \text{if } n = -1 \\ 1 & \text{if } n = 0 \\ \mathrm{Fib}_{n-2} + \mathrm{Fib}_{n-1} & \text{if } n \geq 1 \end{array} \right\}$$

The substitution $\phi$ has a fixed point, that is a word $\overline{\omega}$ of infinite length such that $\phi(\overline{\omega}) = \overline{\omega}$, and this fixed point is called the *Fibonacci sequence*. It also turns out that this substitution is *Sturmian* - it has exactly $n+1$ subwords of length $n$ (see [31] Corollary 5.4.10). To get a tiling of $\mathbb{R}$, let $b$ correspond to the interval of length one and $a$ by an abuse of notation to the interval of length $\phi = (1+\sqrt{5})/2$ (i.e. the golden ratio). Therefore, there are two tiles, one of length 1 and one of length $\phi$. It is then easily shown that the size of $\phi^k(b)$ is $\phi^k$. It is then possible to get a larger and larger tiling of $\mathbb{R}$ by taking $\phi^n(b)$ to the right of the origin and $\phi^n(a)$ to the left of the origin.

In order to get a Delone set $\mathcal{L}$, put a point at the left endpoint of each tile. This gives a set $\mathcal{L}$ that is $(1/2, 2\phi)$-Delone. $\mathcal{L}$ is finite type since there are only two different intervals. If $\mathcal{L}$ was periodic, then the number of subwords of a given length must be bounded. Thus $\mathcal{L}$ is aperiodic since it is Sturmian. To see that $\mathcal{L}$ is repetitive, notice that any subword occurs in $\phi^k(b)$ for some $k$. Since $\phi^k(\overline{\omega}) = \overline{\omega}$, then this subword occurs in any word of length $\mathrm{Fib}_{k+1}$ and thus $\mathcal{L}$ is repetitive. Consequently, its hull $\Xi$ is a Cantor set and is given as the boundary of the tree of patches, $\mathcal{T}$.

Because there are only two different intervals in the tiling then the metric $d$ from the previous section is equivalent to a simpler metric $d_V$. This metric is defined as follows. Given $\mathcal{L}_1, \mathcal{L}_2 \in \Xi$, if the largest subword around the origin that they agree on has length $n$, then $d_V(\mathcal{L}_1, \mathcal{L}_2) = 1/n$. For the Fibonacci tiling, since any subword $w \subset \overline{\omega}$ then $\operatorname{diam}(w) = \phi n_a + n_b$ where $n_a$ is the number of $a$'s in $w$ and $n_b$ is the number of $b$'s in $w$. Thus for a word $w$ of length $N$, $N \leq \operatorname{diam}(w) \leq \phi N$. Consequently,

$$\frac{2d_V(\mathcal{L}_1, \mathcal{L}_2)}{\phi} \leq d(\mathcal{L}_1, \mathcal{L}_2) \leq 2d_V(\mathcal{L}_1, \mathcal{L}_2)$$

for any $\mathcal{L}_1, \mathcal{L}_2 \in \Xi$ and the metrics are equivalent.

It is now possible to compute the zeta function of the Fibonacci tiling by flushing out the details of the tree structure of $\mathcal{T}$. Since the vertices are given by patches and since every patch is a subword of $\overline{\omega}$ of even length, then it is necessary to understand which subwords of even length of $\overline{\omega}$ have more than one child. That is if $w$ is a subword of length $n$ how many subwords $w'$ of length $n+2$ are there such that $w = w_1' w w_2'$ with $w_i' \in \{a, b\}$. In the study of the combinatorics of words, such words are called *special words* (see [31], [12]). For a Sturmian sequence, because there are exactly $n+1$ subwords of size $n$, then the number of special words of size $n$ is equal to $n+1-n = 1$. Since the present sequence is biinfinite, there are two types of special words: words that are left special (i.e. have two different extensions to the left) and words that are right special. By [12], there is one left special word of each length and one right special word of each length. The exact nature of these special words is given by the proposition below, but first we need two lemmas.

**Lemma 10** *Let $w$ be the first $n$ letters of $\phi^k(a)$. If $w$ ends in $b$, then there exists $w'$ such that $w = \phi(w')$.*

**Proof:** The proof will be by induction on the length of $w$. For $w = ab$, $w = \phi(a)$. Now assume that for any $w$ that is the first $m$ letters of $\phi^k(a)$ with $m < n$ and ends in $b$ that $w = \phi(w')$. Let $w$ be the first $n$ letters of $\phi^k(a)$ and suppose $w$ ends in $b$. Then

$w = w_1 \cdots w_{n-2}ab$. If $w_{n-2} = b$, then $w_1 \cdots w_{n-2} = \phi(w')$ by the inductive hypothesis and therefore $w = \phi(w'a)$. If $w_{n-2} = a$, then $w_{n-3} = b$ and $w_1 \cdots w_{n-3} = \phi(w')$ by the inductive hypothesis. Thus, $w = \phi(w'ba)$. □

**Lemma 11** *Let $w$ be the first $n$ letters of $\phi^k(a)$. $w$ is a palindrome if and only if $n = Fib_j - 2$ for some $j$.*

**Proof:** We first prove that if $n = \mathrm{Fib}_j - 2$ for some $j$, then if $w$ is the first $n$ letters of $\phi^j(a)$ it is a palindrome. The proof is by induction. To begin, we see that $\phi^2(a) = aba$. Thus $w = a$ and is certainly a palindrome. We now proceed by induction on $j$. That is we assume that for $j < J$, if $n = \mathrm{Fib}_j - 2$ then the word of length $n$ is a palindrome. But $\phi^J(a) = \phi^{J-1}(ab) = \phi^{J-1}(a)\phi^{J-2}(a) = \phi^{J-2}(a)\phi^{J-3}(a)\phi^{J-2}(a)$. So $\phi^J(a) = w'w_1w_2w''w_3w_4w'w_1w_2$ where $w'$ and $w''$ are palindromes. This gives that $w = w'w_1w_2w''w_3w_4w'$. But $\phi^k(a)$ either ends in $ab$ or $ba$. Now if $\phi^k(a)$ ends in $ab$, then $\phi^{k+1}$ ends in $ba$. Similarly, if $\phi^k(a)$ ends in $ba$, then $\phi^{k+1}(a)$ ends in $ab$. So without loss of generality, we can assume $w = w'abw''baw'$ and we see that indeed $w$ is a palindrome.

We now prove that if $w$ is a palindrome then $w = \phi(w')a$ where $w'$ is also a palindrome. Since $w$ starts with $aba$ then it must end in $aba$. So we see that $w = w''a$ where $w''$ ends in $b$. But then $w'' = \phi(w')$. We now suppose that $w'$ is not a palindrome. Then either $w' = w_1 \cdots w_n b \cdots a w_n \cdots w_1$ or $w' = w_1 \cdots w_n a \cdots b w_n \cdots w_1$. In the first case,

$$w = \phi(w_1) \cdots \phi(w_n) a \cdots ab\phi(w_n) \cdots \phi(w_1)a.$$

But then since $w$ is a palindrome, the letter following the $a$ must be a $b$. This cannot happen since it would mean the substitution of some letter starts with $b$. In the latter case,

$$w = \phi(w_1) \cdots \phi(w_n) ab \cdots a\phi(w_n) \cdots \phi(w_1)a.$$

But this contradicts the fact that $w$ is a palindrome. Consequently, $w'$ must be a palindrome. To finish the proof of the proposition, we realize that therefore

$$w = \phi^N(a)\phi^{N-1}(a)\cdots\phi(a)a$$

for some $N$. But then the length of $w$ is

$$S_N = 1 + 2 + 3 + 5 + \cdots + \mathrm{Fib}_N = \sum_{i=0}^{N} \mathrm{Fib}_i$$

Since $S_0 = 1 = \mathrm{Fib}_2 - 2$ and $S_N = S_{N-1} + \mathrm{Fib}_N$, then $S_N = \mathrm{Fib}_{N+2} - 2$. $\square$

With these two lemmas in hand, we can finally get an answer to which subwords of $\overline{\omega}$ have more than one child.

**Proposition 18** *A word $w$ has more than one child if and only if $w$ is the first $n$ letters of $\phi^k(a)$ for some $k$ or if $w$ is the reverse of such a word. Moreover if $w$ is a palindrome, then $w$ has three children. If $w$ is not a palindrome, then $w$ has two children and the reverse of $w$ has two children.*

**Proof:** Suppose $w$ is the first $n$ letters of $\phi^k(a)$ with $n < \mathrm{Fib}_k$ and let $w_{n+1}$ be the $n+1$ letter. Now $\phi^{k+2}(a) = \phi^{k+1}(a)\phi^k(a)$ and $\phi^{k+3}(a) = \phi^{k+2}(a)\phi^k(a)\phi^{k-1}(a)$. Since $\phi^{k+1}(a)$ and $\phi^{k+2}(a)$ end in different letters, we know that $aww_{n+1}$ and $bww_{n+1}$ are subwords of $\phi^{k+3}(a)$. Thus $w$ has two children. Moreover, since $w$ is contained in some palindrome then $w'$, the reverse of $w$, is also a subword. By the same reasoning, $w_{n+1}w'a$ and $w_{n+1}w'b$ are both subwords and therefore, $w'$ has two children.

Since our substitution is Sturmian, we know that there are $n+1$ words of length $n$. Thus since our tree only contains words of odd length, there are exactly two extra vertices added at each level of the tree. If $w \neq w'$ then $w$ and $w'$ are the only vertices with more than one child. If $w = w'$, then $aww_{n+1}, bww_{n+1}, w_{n+1}wa$, and $w_{n+1}wb$ are all subwords. However, two of these coincide and we see that $w$ has three children and is therefore the only vertex at this level with more than one child. $\square$

It is then possible to draw the unreduced tree.

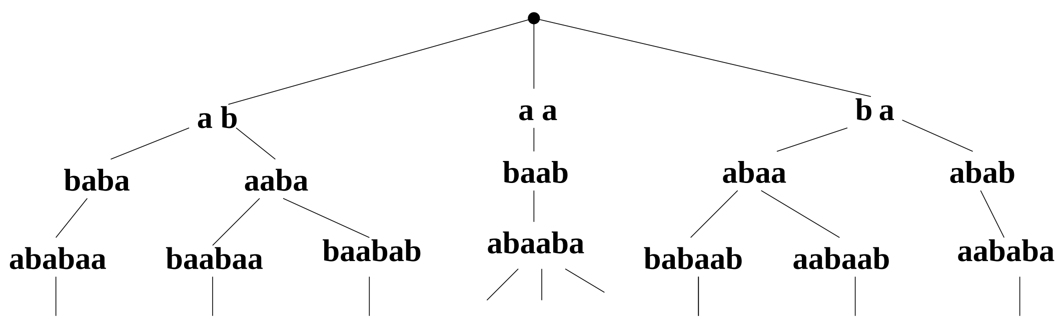


**Figure 3:** The Unreduced Tree of the Fibonacci Tiling

We can now study $\zeta(s)$ for the Fibonacci tiling and we do so with the metric $d_V$. This is not such a restriction since $d_V$ and $d$ are equivalent and therefore will have the same singularities. By the previous proposition, we know that

$$\begin{aligned}\zeta(s) &= 2\sum_{n=1}^{\infty}\frac{1}{(2n)^s} - \sum_{\mathrm{Fib}_k \text{ even}}\frac{1}{(\mathrm{Fib}_k-2)^s}\\ &= 2\frac{1}{2^s}\sum_{n=1}^{\infty}\frac{1}{n^s} - \sum_{\mathrm{Fib}_k \text{ even}}\frac{1}{(\mathrm{Fib}_k-2)^s}\end{aligned}$$

Since for $n$ even,

$$\begin{aligned}\mathrm{Fib}_n &= \mathrm{Fib}_{n-1}+\mathrm{Fib}_{n-2} = \cdots = \sum_{k=0}^{n/2}\frac{(n/2)!}{(n/2-k)!k!}\mathrm{Fib}_k\\ &\geq \sum_{k=0}^{n/2}\frac{(n/2)!}{(n/2-k)!k!} = 2^{n/2}\end{aligned}$$

we know that the second sum converges for $\Re(s) > 0$. But we know that the first term is just a multiple of the Riemann zeta function which is analytic for $\Re(s) > 0$ except for a simple pole at $s = 1$ with residue equal to 1. Thus $\zeta(s)$ has abscissa of convergence $s_0 = 1$ and we see that the upper box dimension of the transversal $\Xi$ of the Fibonacci tiling is 1. Since the Fibonacci sequence is Sturmian, this shows that the upper box dimension is equal to the complexity of the Fibonacci tiling.

This computation also reveals a somewhat startling insight. The zeta function has no complex poles in the positive real axis. In [49], the author suggests that a fractal be defined to be a set whose zeta function has at least one nonreal complex pole with positive real part. Therefore, the transversal of the Fibonacci tiling despite being a Cantor set would not be considered a fractal according to this definition. This is most likely a result of the fact that the metric was taken from the combinatorics of the tiling.

Now that the $\zeta$-function has been computed it should be possible to compute the measure and then the Laplacian. This is the subject of future research.

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