POLAR - LEGENDRE DUALITY IN CONVEX GEOMETRY AND GEOMETRIC FLOWS

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Edward C. White, Jr.
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POLAR - LEGENDRE DUALITY IN CONVEX GEOMETRY AND GEOMETRIC FLOWS

Approved by:

Dr. Evans Harrell, Advisor
School of Mathematics
Georgia Institute of Technology

Dr. Mohammad Ghomi
School of Mathematics
Georgia Institute of Technology

Dr. Guillermo Goldsztein
School of Mathematics
Georgia Institute of Technology

Date Approved: June 26, 2008

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$*$	Polar duality. A^*, for example, is the polar dual of A.
$g(x)$	The gauge function of a convex body.
∇	In 2 or 3 dimensions, the gradient.
∇	In higher dimensions, the Levi-Civita connection.
$\bar{\nabla}$	The gradient on the unit n-dimensional ball.
$h(u)$	The support function of a convex body.
h_θ, $h_{\theta\theta}$	The first and second derivatives of h, with respect to θ.
$K^\triangle$	The filled join of a convex body K.
$\bar{\triangle}$	The Laplacian on the unit sphere.
$\mathcal{L}$	The (Generalized) Legendre Transform.
$\phi(x)$	An immersion describing a convex body.
$\mathbf{u}$	A vector (any letter in boldface).
$\hat{\mathbf{u}}$	A unit normal vector to a the surface of a convex body.
$V(A)$	Volume of a convex body A.
$VP(A)$	Volume Product: $V(A) * V(A^*)$.

SUMMARY

This thesis examines the elegant theory of polar and Legendre duality, and its potential use in convex geometry and geometric analysis. As a starting point, it derives a theorem of polar - Legendre duality for all convex bodies, independent of the degree of smoothness or strict convexity of their boundaries. This theorem is captured in a commutative diagram that shows how it is possible to transform information about a convex body into information about its polar dual, and vice versa.

The notion of a geometric flow on a convex body inducing a flow on a polar dual is introduced, and equations for the original and the dual flow are worked out for a number of two dimensional flows. It is shown that in general these flows are not geometric, in the sense of being defined solely by local curvature, but the evolution equations, at least in two dimensions, have similarities to the inverse flows on the original convex bodies. This theory is extended, in a preliminary way, to three dimensions. Mahler's Conjecture in convex geometry is discussed in some detail, and polar-Legendre duality is used to re-examine the resolved two-dimensional problem. This leads to some ideas on ways to attack the still open three-dimensional conjecture.

As part of this work the ratio of the derivative of a support function to the support function, better known as the logarithmic derivative $\frac{h_\theta}{h}$, is examined analytically and geometrically. It is a ratio that is important in both two-dimensional and three-dimensional geometric flows. The logarithmic second derivative, $\frac{h_{\theta\theta}}{h}$, is important in Mahler's Conjecture.

CHAPTER 1

Introduction

Duality is one of the most powerful and elegant concepts in modern mathematics. It is powerful because it can be used to present two ways of approaching difficult problems, often with dramatic results. It is elegant because of the aesthetic appeal of the symmetry in ways of looking at complex problems. Mathematical duality, in various forms, appears prominently in optimization, mathematical physics and geometry. In geometry, duality can take on a compelling visual form. This is quite apparent in the case of polar duality of convex bodies. For every convex body in $\mathbb{R}^n$ there corresponds another convex body, its polar dual. The two are related in remarkable ways.

This thesis shall explore duality in depth, bringing in known but infrequently used results from the calculus of variations. The main emphasis is on the application of duality to two areas: (1) an open conjecture in convex geometry (Mahler's Conjecture) and (2) the study of geometric flows. Mahler's Conjecture postulates that there is a specific minimum bound for the product of the volume of a convex body A and the volume of its polar dual A^*. The notion of duality is clearly evident in this problem, and it was, in fact, the motivation for this thesis. Geometric flows are partial differential equations describing the evolution of surfaces such as the boundaries of convex bodies.

There is ongoing interest in the examination of different types of geometric flows and their properties, and it is here that duality may be of help.

The investigation of duality begins in Chapter 2, with a discussion of the support and gauge functions of a convex body, which leads naturally to an analysis of polar and Legendre duality. The combination of these two types of duality presents a complete picture of the relationship between a convex body and its polar dual. Chapter 3 opens the door for the first application of these concepts. Geometric flows are explored, particularly in the context of their representation by support functions.

This sets the stage for Chapter 4, which introduces the notion that the geometric flow of a convex body induces a flow (generally not geometric) on its polar dual. The logarithmic derivative $\frac{h_\theta}{h}$ turns out to be important in deriving the induced flow on the polar dual body. The background of Mahler's Conjecture is discussed in Chapter 5, along with a summary of known results and approaches that have been tried to date.

The role that a complete picture of polar duality may play in the conjecture is presented in Chapter 6. We derive the role of the logarithmic second derivative $\frac{h_{\theta\theta}}{h}$ in the conjecture. The Conclusion suggests an extension of earlier ideas to higher dimensions, and explores, in a preliminary way, possible relationships between geometric flows and Mahler's Conjecture.

CHAPTER 2

The Polar Dual, Support and Gauge Function

In this section we review some standard notions from convex geometry that are central to the rest of the thesis. There are a number of modern and classical sources for this material. Here we follow Webster [31], but also bring in ideas from Thompson [28], Giaquinta and Hildebrandt [9] and others.

2.1. **The polar dual**

A set $A \in \mathbb{R}^n$ is a convex body if it is a closed convex set in $\mathbb{R}^n$. The polar dual of A is described by the set

$$A^* = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} \leq 1, \text{ for all } \mathbf{a} \in \mathrm{A}\}.$$

The work of Webster [31] and Thompson [28] can be used to give an intuitive sense of what the polar dual of a convex body really is, at least in $\mathbb{R}^2$. If A contains the origin, then for a given point a_1 in A, the corresponding set $A^* = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{a_1} \cdot \mathbf{x} \leq 1\}$ is a half plane containing the origin. Taking successive points $\mathbf{a_1}, \mathbf{a_2}, \cdots \mathbf{a_i}$ in A produces an entire family of half planes

$$\mathbf{a_1^*} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a_1} \cdot \mathbf{x} \leq 1\}$$

$$\mathbf{a_2^*} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a_2} \cdot \mathbf{x} \leq 1\}$$

$$\dots$$

$$\mathbf{a_i^*} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a_i} \cdot \mathbf{x} \leq 1\}.$$

Then the polar dual A^* is the intersection of these half planes. In general, there is no intuitive way to visualize the polar dual of an arbitrary convex body – it is often necessary to work out the equation for the dual and graph the result. In a few specific cases there are rules of thumb: the polar dual of a point in $\mathbb{R}^2$ is all of $\mathbb{R}^2$; the polar dual of a circle (sphere) is another circle (sphere); and edges of a convex body become vertices in its polar dual, while vertices in a convex body become edges in its polar dual.

Two properties of the polar dual are easy to appreciate, and important to keep in mind. The first is that the polar dual of a convex body is itself a convex body. This follows from the observation that half planes are convex, and from a well known theorem that the intersection of convex bodies is itself a convex body. (For proofs, see [31]).

The second is that any change in the shape of the boundary of A will result in a change in the shape of the boundary of A^* – this follows from the definition of A^*.

2.2. **The support function**

Both the support and the gauge function turn out to be very useful. The support function, in particular, has many useful properties.

- Support functions are of great value in understanding polar and Legendre duality.

- The volumes of convex bodies can be computed from support functions, using an integral.

- Support functions can be used to define geometric flows.

- They give an easy way to compute geometric quantities such as the width of convex bodies.

- They can be expanded into spherical harmonics, enabling the construction of convex bodies.

A mathematically precise definition of the support function is as follows. If $A \subset R^n$ and $\mathbf{a} \in A$ the support function $h(\mathbf{u})$ is defined by:

$$(1) \qquad h(\mathbf{u}) = \sup\{\mathbf{u} \cdot \mathbf{a} : \mathbf{a} \in A,\ \mathbf{u} \in \mathbb{R}^n\}$$

As defined above, a unique support function exists for every convex body. While the equation above is the usual way to define the support function, it is also very often defined in terms of vectors on the unit sphere $\mathbb{S}^{n-1}$. These are two distinct representations of the support function.

2.2.1. *The intuitive support function*

An intuitive definition of the support function comes from considering tangent hyperplanes. We recall that a convex body is a closed, convex set, and let A be a convex body in $\mathbb{R}^3$.

If H is a hyperplane that is tangent to A, such that A lies completely on one side of it, then we call H a support hyperplane of A. At each point of a convex body A the support function (as a function of the unit normal vector to that point) can be viewed as the perpendicular distance from the support hyperplane at that point to the origin. This seemingly innocuous definition describes a powerful concept.

It will be helpful for later work to deepen this intuitive idea of the support function. Here we have closely followed Webster [31]. Suppose that $\mathbf{u} \in \mathbb{R}^n$. Then

$$H_\beta = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{u} \cdot \mathbf{x} = \beta\}, \tag{2}$$

where β is a real number, is the equation for a hyperplane with normal direction $\mathbf{u}$. We can view such a hyperplane as dividing two closed half spaces:

$$H_\beta^- = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{u} \cdot \mathbf{x} \leq \beta\} \tag{3}$$

$$H_\beta^+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{u} \cdot \mathbf{x} \geq \beta\} \tag{4}$$

Now let β vary. This will produce a family of hyperplanes orthogonal to $\mathbf{u}$. There will be two values, $\beta = a1$ and $\beta = a2$ for which H_β supports A, in the sense that H_β is tangent to points of A. This is illustrated below for $n = 2$. Note that the only hyperplane for which A lies in the lower half space is H_{a2}. So we can say that only $a2$ is such that $A \subseteq H_\beta^-$.

Now $A \subseteq H_\beta^-$ if and only if $\mathbf{u} \cdot \mathbf{a} \leq \beta$ for all $\mathbf{a} \in A$. Then specifically, $\sup\{\mathbf{u} \cdot \mathbf{a} : \mathbf{a} \in A\} \leq \beta$. If H_β supports A, then we have $\sup\{\mathbf{u} \cdot \mathbf{a} : \mathbf{a} \in A\} = \beta$. Recognizing that β can vary as a function of $\mathbf{u}$, we get the definition of the support function we had earlier, in Equation (1),

$$h(\mathbf{u}) = \sup\{\mathbf{u} \cdot \mathbf{a} : \mathbf{a} \in A,\ \mathbf{u} \in \mathbb{R}^n\}.$$

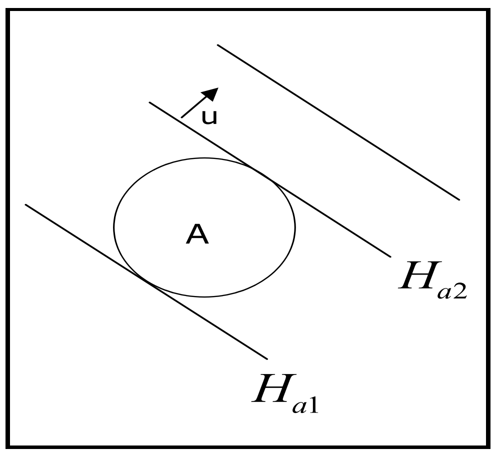

Figure 1. Support hyperplanes to a convex body A. [31], p.232

Note that the support function takes a vector $\mathbf{u} \in \mathbb{R}^n$ as its argument, and it returns a scalar. The support function depends upon the location of the origin, relative to the convex body. If the location of the origin is shifted, the changed, the support function changes. Specifically, if a convex body A with support function $h(\mathbf{u})_A$ contains the origin, and the origin is shifted to a new vector $\mathbf{a}$, then the new support function is $h'(\mathbf{u})_A = h(\mathbf{u})_A + \mathbf{a} \cdot \mathbf{u}$ [16].

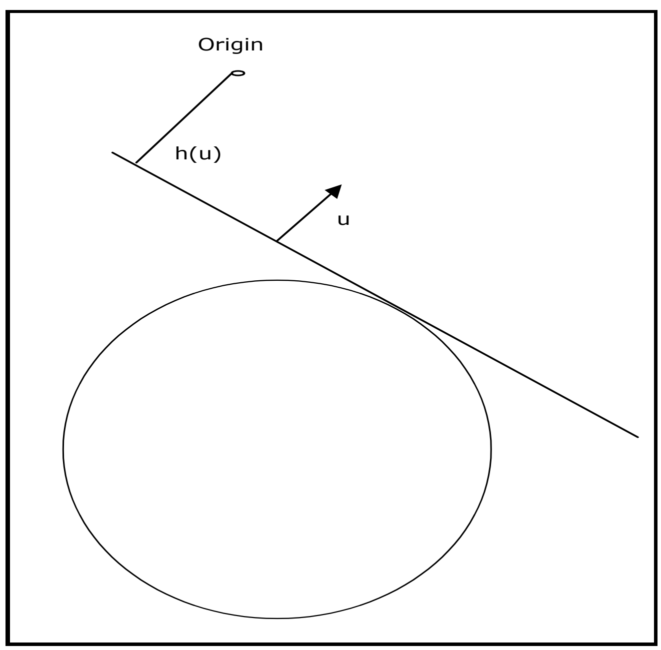

Figure 2. The support function. [31], p.232

In working with convex bodies, their volumes and their geometric flows, it is often more convenient to describe vectors relative to the unit sphere, as opposed to vectors in $\mathbb{R}^n$. It is helpful to have a definition of the support function that is consistent with this vantage point. This can be done by describing $h(\mathbf{u})$ as a function from the unit sphere $\mathbb{S}^{n-1}$ to $\mathbb{R}$, rather than as a function of $\mathbb{R}^n$ to $\mathbb{R}$. The argument $\mathbf{u}$ is then taken as a (unit) vector $\hat{\mathbf{u}}$ on $\mathbb{S}^{n-1}$, rather than as a vector on $\mathbb{R}^n$. The inner product $\hat{\mathbf{u}} \cdot \mathbf{a}$ can be viewed geometrically as a distance, and this gives us the heuristic definition of the support function that we had earlier – wherever the origin is located, the support function at a unit normal vector $\hat{\mathbf{u}}$ gives the (perpendicular) distance from the support hyperplane to a parallel line through the origin (if the origin lies in A). This is illustrated in the diagram below.

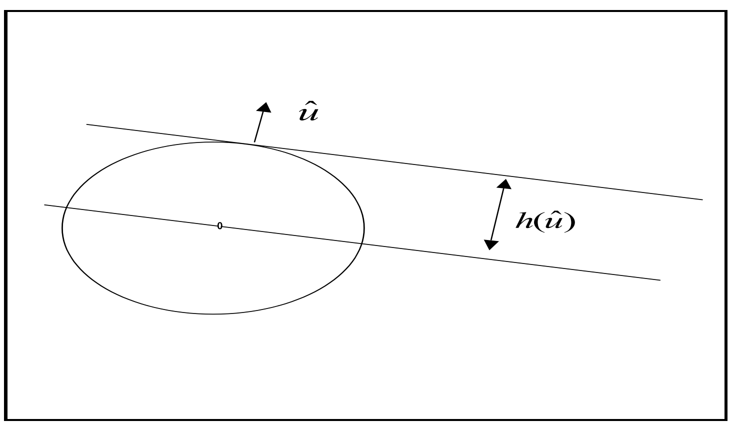

Figure 3. The support function as a distance.

Any support function on the unit sphere can be extended to a support function on all of $\mathbb{R}^n$. If $\hat{\mathbf{u}}$ is a vector on the unit sphere, we can represent the vectors in $\mathbb{R}^n$ by $\lambda\hat{\mathbf{u}}$, $\lambda \geq 0$. By homogeneity, $h(\lambda\hat{\mathbf{u}}) = \lambda h(\hat{\mathbf{u}})$, and this extends h to all $\mathbb{R}^n$.

2.2.2. *Representation of a convex body*

We can use the support function to completely describe a convex body. The boundary of a closed convex body A is the intersection of closed halfspaces, bounded by the hyperplanes that support it. Then A is written as [31]:

$$A = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{u} \cdot \mathbf{x} \leq h(\mathbf{u}),\ \mathbf{u} \in \mathbb{R}^n\}.$$

This can be used as an alternative to a parametric or other equation for a convex body. But it is not the only way in which we can represent convex body using a support function. It is sometimes useful to describe a surface as a smooth immersion, using its support function. Suppose that we are describing a surface by an immersion $\phi : \mathbb{S}^{n-1} \to \mathbb{R}^n$, and let $h : \mathbb{S}^{n-1} \to \mathbb{R}$ be its support function. Since h gives the perpendicular distance from the origin to the tangent plane at $\phi(\mathbf{z})$, $\mathbf{z} \in \mathbb{S}^{n-1}$, we can see that $\phi(\mathbf{z})$ has the following form [1]:

$$\phi(\mathbf{z}) = h(\mathbf{z})\mathbf{z} + \mathbf{a}(\mathbf{z}), \tag{5}$$

where $\mathbf{a}(\mathbf{z})$ is a vector tangent to $\mathbb{S}^{n-1}$ at $\mathbf{z}$, for each $\mathbf{z} \in \mathbb{S}^{n-1}$.

2.3. **The gauge function**

Conceptually, the gauge function can be thought of as a family of adjustable yardsticks, each extending in a different direction from the center of a convex body. Each of these yardsticks is scaled so that the measurement at the boundary of the convex body is equal to one. Given any vector as an input, the gauge function selects the appropriate yardstick and measures the distance of the vector from the convex body, relative to the boundary of the convex body. A more precise mathematical definition, as described in [31], follows. If $A \subset \mathbb{R}^n$ and $\lambda \geq 0$ is a real number, then the gauge function $g(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is defined by:

$$g(\mathbf{x}) = \inf\{\lambda \geq 0 : \mathbf{x} \in \lambda \mathrm{A},\ \mathbf{x} \in \mathbb{R}^{\mathrm{n}}\}. \tag{6}$$

2.3.1. *Characterizing the gauge function*

For a more visual description we use Figure 4, in which each g_i represents a copy of the convex body A, at some distance from the boundary of A, where g_1 is defined to be 1. The analogy of the yardstick and the convex body is readily apparent from the diagram. The gauge function, then, can be thought of as a type of distance metric. Giaquinta and Hildebrandt [9] gives a characterization of the family of possible gauge functions. A gauge function on $\mathbb{R}^n$ is a function $g : \mathbb{R}^n \to \mathbb{R}$ with the properties listed below.

1. If $\mathbf{x} \neq 0$ then $g(\mathbf{x}) > 0$. If $\mathbf{x} = 0$, then $g(\mathbf{0}) = \mathbf{0}$.
2. g is homogeneous, so that $g(\lambda \mathbf{x}) = \lambda \mathbf{x}$, for $\lambda \geq \mathbf{0}$.
3. g is a convex function, so that $g(\lambda x + \mu y) \leq \lambda g(x) + \mu g(y)$.

Every convex body has a gauge function with these properties. Giaquinta and Hildebrandt show [9] that every abstract gauge function is the gauge function of some convex body.

2.3.2. *Representation of a convex body*

From the definition of a gauge function it is intuitive that a convex body A can be described as the set of points where the gauge function is less than or equal to one. To prove this (following [31], p. 236), let $g(\mathbf{x})$ be a gauge function with $g(\mathbf{x}) \geq 0,\ \ \mathbf{x} \in A$. Then for each $\epsilon > 0,\ \ \mathbf{x} \in (g(\mathbf{x}) + \epsilon)\, A$, so that

$$\frac{\mathbf{x}}{g(\mathbf{x}) + \epsilon} \in A \tag{7}$$

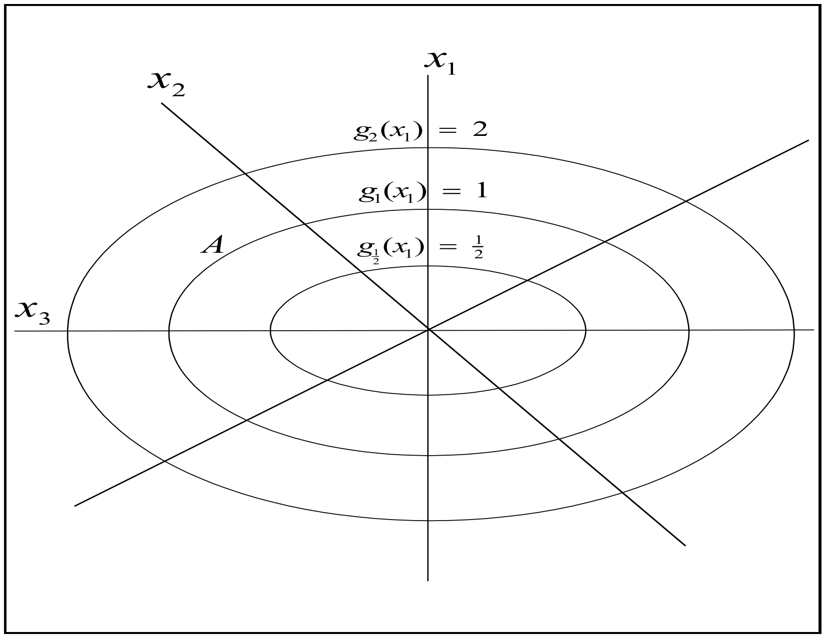

Figure 4. A representation of the gauge function of a convex body.

Let $\epsilon \to 0_+$. Then,

$$\frac{\mathbf{x}}{g(\mathbf{x})} \in A \tag{8}$$

If $0 < g(\mathbf{x}) \leq 1$, then $\mathbf{x} \in g(\mathbf{x})A \subset A$.

This establishes that $A = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 1\}$, and also that $\partial A = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) = 1\}$.

Note that the support function has the same properties as those described above, for the gauge function. Both have as input a vector quantity and as output a scalar quantity. As with the support function, we can consider the gauge function on the unit sphere by defining $g(\hat{\mathbf{u}}) : \mathbb{S}^n \to \mathbb{R}$ in place of $g(\mathbf{u}) : \mathbb{R}^n \to \mathbb{R}$.

If a convex set A contains the origin, then both the support and the gauge functions are positively homogeneous, convex functions, and both can be used to describe unique closed, convex sets.

Hence, we observe that for every convex set containing the origin, the support function is also a gauge function, and the gauge function is also a support function.

2.4. **Polar and Legendre duality**

There are some important relationships between the support and gauge functions of convex bodies and their polar duals. All of these relationships appear in the popular literature, but as they come from different fields of mathematics they generally are not brought together in any single place (one article that does bring these ideas together is [3]; it has a very different focus from this thesis).

We shall prove the following:

- Polar duality holds for support and gauge functions. This means that if one considers a support function on all of $\mathbb{R}^n$, then the support function h_A of a convex body A containing the origin is the gauge function g_{A^*} of its polar dual A^*. Similarly, the gauge function g_A of a convex body A is the support function h_{A^*} of its polar dual A^*.

- Generalized Legendre Transform duality holds for support and gauge functions. Giaquinta and Hildebrandt introduce a notion of the Generalized Legendre Transform, with which they analyze duality. They show that if the boundary of a convex body A is smooth and strictly convex, then the Generalized Legendre Transform of the gauge function of A is the gauge function of its polar dual A^*. Similarly, the Generalized Legendre Transform of the support function of a convex body A is the support function of the polar dual A^*.

Before proving these results, it is worthwhile to explore what they mean. We start with the notion of the Legendre Transform. There are several standard references for this transform. In [7] there is a treatment that is geared toward the study of partial differential equations, and we begin there. Let $G : \mathbb{R}^n \to \mathbb{R}$ be a function satisfying two criteria:

1. Convexity, i.e., the mapping $\mathbf{q} \to G(\mathbf{q})$ is convex, and

2. Superlinearity, i.e., $\lim_{|\mathbf{q}|\to\infty} \frac{\mathrm{G}(\mathbf{q})}{|\mathbf{q}|} = +\infty$.

Then the Legendre Transform of G is defined as [7]:

$$H = \mathcal{L}(G, \mathbf{p}) := \sup_{\mathbf{q}\in\mathbb{R}^n} \left(\mathbf{p}\cdot\mathbf{q} - G(\mathbf{q})\right), \;\; p \in \mathbb{R}^n.$$

One of the properties of the Legendre Transform is that it is its own inverse.

The transformation from $G(\mathbf{q})$ to $\mathcal{L}(G, \mathbf{p})$ implies a mapping from the variable $\mathbf{q}$ to the variable $\mathbf{p}$. Because the Legendre Transform is invertible, this needs to be an invertible mapping. In some of the situations that we will encounter, this mapping will not be invertible.

In the situations that we shall encounter we can correct this problem by defining what Giacquinta and Hildebrandt call "the generalized Legendre Transform". This is done by taking the Legendre Transform not of $G(\mathbf{q})$, but of $\frac{1}{2}G(\mathbf{q})^2$. Therefore the generalized Legendre Transform is [9]:

$$\frac{1}{2}H^2 = \mathcal{L}(\frac{1}{2}G(\mathbf{q})^2, \mathbf{p}) = \sup_{\mathbf{q}\in\mathbb{R}^n} \left(\mathbf{p}\cdot\mathbf{q} - \frac{1}{2}G(\mathbf{q})^2\right).$$

The generalized Legendre Transform plays an important role in duality.

Our claim that if a convex body under consideration contains the origin, then it is possible to establish the following operator commutation diagram. Following the nomenclature in the linear programming community, the original convex body is referred to as the primal, and the dual body is referred to as the dual. Generalized Legendre Duality is abbreviated G.L. The Legendre Transform is its own inverse, and the arrows could be drawn pointing in the opposite directions.

$$\begin{array}{ccccc} \textbf{Gauge Dual} & \xrightarrow{Polar} & \textbf{Support Primal} & \xrightarrow{Polar/G.L.} & \textbf{Gauge Primal} \\ \Big\Vert & & \Big\Vert & & \Big\Vert \\ \Big\downarrow & & {\scriptstyle G.L.}\Big\downarrow & & {\scriptstyle G.L.}\Big\downarrow \\ \Big\Vert & & \Big\Vert & & \Big\Vert \\ \textbf{Gauge Primal} & \xrightarrow{Polar} & \textbf{Support Dual} & \xrightarrow[Polar/G.L.]{} & \textbf{Gauge Dual} \end{array}$$

Figure 5. Polar - Legendre Duality Commutative Diagram

This diagram tells us that polar and Legendre duality together ensure that given either the support or the gauge function of either a convex body or its polar dual, we can derive the complete set of support and gauge functions for the convex body and its polar dual.

Hence, just once piece of information makes it possible to characterize both bodies, and to answer questions about the relationships between them.

2.4.1. *A theorem on polar duality*

The following theorem establishes some of the relationships in the operator commutation diagram presented earlier. This is a standard result in convex geometry, and the version presented here is from Webster[31].

Theorem 2.4.1 (Polar Duality) (Reference: [31]**, p. 238**)
Let that h and g respectively be the support and the gauge functions of a convex body $A \in \mathbb{R}^n$, containing the origin as an interior point. Then the support and the gauge functions of the polar dual A^* are respectively g and h.

Proof:
We start with a lemma, which states that any function meeting certain basic criteria is a gauge function of some convex body.

Lemma 2.4.1 (Reference: [31], p.237)
Let the function $f : \mathbb{R}^n \to R$ be a convex, non-negative, and positively homogeneous function, so that $f(\alpha\mathbf{x}) = |\alpha| f(\mathbf{x})$. Then f is the gauge function of a convex body A, defined by $A = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \leq 1\}$, containing the origin.

Proof (Lemma 2.4.1)

As a convex function defined on all of $\mathbb{R}^n$, f is continuous. Then A is closed and it contains the open set $A = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < 1\}$. This open set contains the origin. Note that A is the level set of a convex function, so it is a convex set. Accordingly, A is a closed, convex set containing the origin. Next, let g be the gauge function of A, so that

$$A = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 1\}.$$

The lemma will be established by showing that $g = f$. First consider $\mathbf{u} \in \mathbb{R}^n$ such that $g(\mathbf{u}) > 0$. Now g is positively homogeneous, and therefore $g(\mathbf{u}/g(\mathbf{u})) = 1$. Hence $\mathbf{u}/g(\mathbf{u}) \in A$. Also f is positively homogeneous. Therefore:

$$f(\mathbf{u}/g(\mathbf{u})) = \frac{f(\mathbf{u})}{g(\mathbf{u})} \leq 1, \text{ so that } \mathrm{f}(\mathbf{u}) \leq \mathrm{g}(\mathbf{u}).$$

Next, we deal with the case where $g(\mathbf{u}) = 0$. In this case, for all $\lambda > 0$, $\lambda\mathbf{u} \in A$. Therefore, $0 \leq f(\lambda\mathbf{u}) = \lambda f(\mathbf{u}) \leq 1$, and $f(\mathbf{u}) = 0$. We have therefore established that $f(\mathbf{u}) \leq g(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^n$. A reverse argument shows that $g(\mathbf{u}) \leq f(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^n$. Therefore, $f = g$, and f is the gauge function of A. We now return to the proof of the main part of Theorem 1.

If $h(\mathbf{u}) \leq 1$, then $\mathbf{u} \cdot \mathbf{a} \leq 1$ for all $\mathbf{a}$ in A, and $\mathbf{u} \in A^*$. Conversely, if $\mathbf{u} \in A^*$, then $\mathbf{u} \cdot \mathbf{a} \leq 1$ for all $\mathbf{a}$ in A. So $h(\mathbf{u}) \leq 1$. Therefore,

$$A^* = \{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \leq 1\}.$$

Since A contains the origin, h is non-negative. As h is a non-negative positively homogeneous function, it is (by virtue of Lemma 2.4.1) the gauge function of A^*. This shows that the support function of A is the gauge function of A^*. A similar argument can be used to establish that the support function of A^* is the gauge function of A.

□

This theorem makes it possible to define a mapping from the support function h of a convex body A to the gauge function g^* of its polar dual A^*, which we call the polar dual transformation.

Some care is needed in using this result, because there is a subtle point to keep in mind. When equating gauge functions to support functions and vice versa, it is important to remember that these functions are used in different ways on their respective convex bodies. So, while the support function of a convex body is the gauge function of its polar dual, the support function on the convex body will tend to be used differently from the gauge function on its polar dual. The table below, outlining the use of support and gauge functions to describe the boundaries of convex bodies, illustrates this. This need to pay close attention when moving from a convex body to its polar dual shall arise later in our work.

Table 1. A Subtlety In Polar Duality.

Gauge Function g of A	Support Function h of A	Support Function h^* of A^*
$\partial A = g^{-1}(1)$. No unit vector is needed.	$h(\mathbf{u}) = \inf\{(0, P_1\hat{\mathbf{u}}) : P$ outside A$\}$. ∂A is determined by the action of h on unit vectors.	Same function as g, but it is used as a support function, as in column 2, and not as a gauge function, as in column 1.

2.4.2. *A theorem on Legendre duality*

Legendre Transfom Duality is the notion that a special generalization of the Legendre Transform (the Generalized Legendre Transform) of the gauge function of a convex body is its support function. It follows from this that the inverse Generalized Legendre Transform of its support function (which is also a Generalized Legendre Transform) is its gauge function.

A detailed proof of Legendre Transform Duality appears in Giaquinta and Hildebrandt [9]. The proof gives a methodology that is useful for actually implementing Legendre Transform Duality, but it depends upon the smoothness and strict convexity (i.e., no flat sides) of the convex body in question.

As an alternative we present a proof constructed by the author's advisor Evans Harrell [11]. It is straightforward, and requires no smoothness or strict convexity assumptions.

Theorem 2.4.2 (Legendre Duality) (Reference: [11]**)**

Suppose that $g(\mathbf{x})$ is a gauge function of a convex body A in $\mathbb{R}^n$, so that $g : \mathbb{R}^n \to \mathbb{R}^+$ is a convex function, and $\alpha \geq 0 \Rightarrow g(\alpha\mathbf{x}) = \alpha g(\mathbf{x})$. Let $h(\mathbf{u})$ denote the support function of A. Then if $\mathcal{L}$ symbolizes the Legendre transform, we call $\mathcal{L}(\frac{1}{2}g^2)$ the generalized Legendre transform of g. Then:

$$\mathcal{L}\left(\frac{1}{2}g^2\right) = \frac{1}{2}(h(\mathbf{u})^2). \tag{9}$$

Proof:

We first show that $\mathcal{L}(\frac{1}{2}g^2) \leq \frac{1}{2}(h(\mathbf{u})^2)$.

Note that $\frac{1}{2}g^2$ is superlinear, and consequently for each $\mathbf{u}$ there exists a vector $\mathbf{z} = \mathbf{z}_*$ that maximizes $\mathbf{u} \cdot \mathbf{z}_* - \frac{1}{2}g^2(\mathbf{z}_*)$ making it possible to write the Generalized Legendre transform as

$$\mathcal{L}\left(\frac{1}{2}g^2\right)(\mathbf{u}) = r\mathbf{u} \cdot \zeta - \frac{1}{2}g^2(\mathbf{z}_*) = r(\mathbf{u} \cdot \zeta) - \frac{r^2}{2},$$

with $\mathbf{z}_* = r\zeta$, and $g(\zeta) = 1$.

Next we consider $\mathbf{u} \cdot \zeta$ as a number, and then write:

$$(\mathbf{u} \cdot \zeta)r - \frac{r^2}{2} \leq \max_{r \leq 0} \left((\mathbf{u} \cdot \zeta)r - \frac{r^2}{2} \right).$$

Now $ar - \frac{r^2}{2}$ is maximized when $r = a$, so that

$$ar - \frac{r^2}{2} \leq a^2 - \frac{a^2}{2} = \frac{a^2}{2}.$$

Taking $a = \mathbf{u} \cdot \zeta$, we get

$$\mathcal{L}\left(\frac{1}{2}g^2\right)(\mathbf{u}) \leq \frac{\mathbf{u} \cdot \zeta}{2}.$$

Since $g(\zeta) = 1$, we have

$$\frac{(\mathbf{u} \cdot \zeta)}{2} \leq \max_{g(\mathbf{x})=1} \frac{(\mathbf{u} \cdot \mathbf{x})}{2} = \frac{h^2(\mathbf{u})}{2}.$$

This establishes that

$$\mathcal{L}\left(\frac{1}{2}g^2\right) \leq \frac{1}{2}(h(\mathbf{u})^2). \tag{10}$$

Next, we show that $\mathcal{L}(\frac{1}{2}g^2) \geq \frac{1}{2}(h(\mathbf{u})^2)$. Since $\{x : g(\mathbf{x}) = 1\}$ is a compact set, for each $u \neq 0$ there exists a vector $\mathbf{x}_*$ such that

$$h(\mathbf{u}) := \sup_{g(\mathbf{x})=1} \mathbf{u} \cdot \mathbf{x} = \mathbf{u} \cdot \mathbf{x}_*.$$

Consider the Legendre transform:

$$\mathcal{L}\left(\frac{g^2}{2}\right)(\mathbf{u}) := \sup_{\mathbf{x}\in\mathbb{R}^n} \left(\mathbf{u} \cdot \mathbf{x} - \frac{g^2(\mathbf{x})}{2}\right).$$

Therefore, for the particular value $\mathbf{x} = \alpha\mathbf{x}_*$,

$$\mathcal{L}\left(\frac{g^2}{2}\right)(u) \geq \mathbf{u} \cdot \alpha\mathbf{x}_* - \frac{g^2(\alpha\mathbf{x}_*)}{2}$$

Using the definition of h and the homogeneity of g, this means that for any $\alpha > 0$,

$$\mathcal{L}\left(\frac{1}{2}g^2\right)(\mathbf{u}) \geq \alpha h(\mathbf{u}) - \frac{\alpha^2 g^2(\alpha\mathbf{x}_*)}{2} = \alpha h(\mathbf{u}) - \frac{\alpha^2}{2}$$

We now choose $\alpha = h(\mathbf{u})$ and get:

$$\mathcal{L}\left(\frac{1}{2}g^2\right)(\mathbf{u}) \geq \frac{1}{2}(h(\mathbf{u}))^2. \tag{11}$$

Combining the results of equations (10) and (11) gives our desired result:

$$\mathcal{L}\left(\frac{1}{2}g^2\right)(\mathbf{u}) = \frac{1}{2}(h(\mathbf{u}))^2.$$

□

It follows that the mapping $p : h(\mathbf{u}) \to g(\mathbf{x})$ corresponding to equation (9) is an invertible mapping on the set of gauge functions to itself, analogous to the geometric polar dual transform, discussed in section 2.4.1.

2.5. **Case study: the ellipse**

As a case study illustrating the relationship between support and gauge functions on convex bodies and their polar duals, we can use the ellipse in $\mathbb{R}^2$. The case of the ellipse is ideal, because it does not bury the main ideas in too many computational details, but it is also less of a special case than the circle. In Webster [31], ellipse computations similar to some of those presented here are featured in examples and exercises. The equation for an ellipse centered at the origin is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1,\ a > 0,\ b > 0. \tag{12}$$

We rewrite this as:

$$A = \{(x,y) : \left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) \le 1,\ a > 0,\ b > 0\}. \tag{13}$$

2.5.1. *The polar dual*

We have seen that in general the polar dual A^* of a convex body A is written as:

$$\{(u,v) : (x,y) \cdot (u,v) \le 1, \text{ for all (x,y)} \in \text{A}\}$$

Therefore, for the ellipse the following two sets of inequalities hold:

$$x\frac{x}{a^2} + y\frac{y}{b^2} \le 1, \quad a > 0,\ b > 0 \tag{14}$$

$$xu + yv \le 1 \tag{15}$$

By inspection, we see that this implies:

(16) $$u = \left(\frac{x}{a^2}\right), or \;\; x = ua^2$$

(17) $$v = \left(\frac{y}{b^2}\right), or \;\; y = ub^2$$

Using these expressions for x and y in equation (15), we get:

$$(au)^2 + (bv)^2 \leq 1$$

So the polar dual of A is the ellipse:

(18) $$A^* = \{(u, v) : (au)^2 + (bv)^2 \leq 1, \;\; a > 0, \;\; b > 0\}$$

Figure 6 is a sketch of an ellipse and its polar dual.

2.5.2. *Support function*

Next, we seek an expression for the support function of an ellipse. This follows from a straightforward computation. Recall that the support function is defined by:

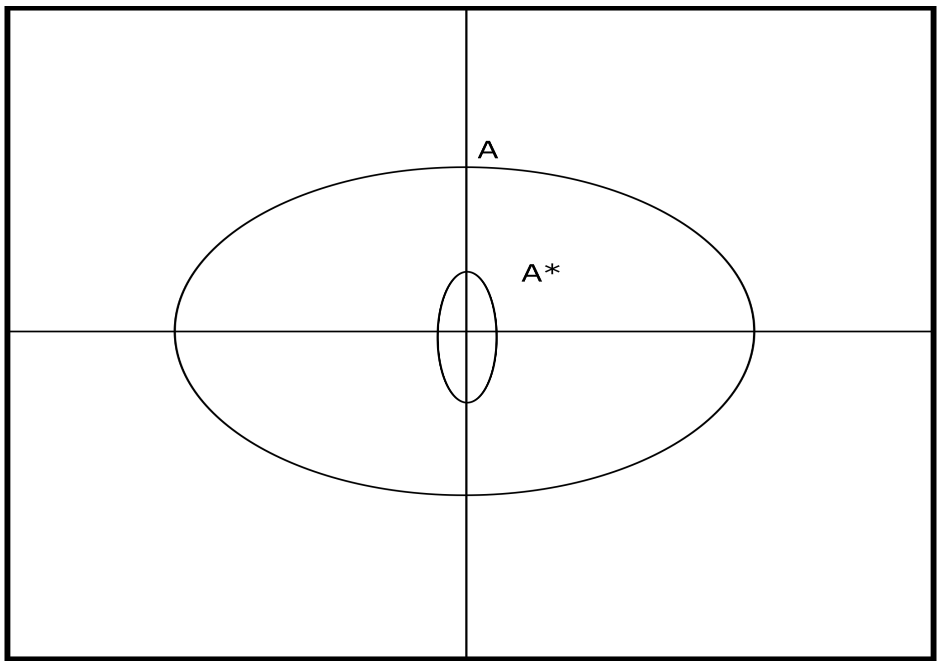

Figure 6. Representation of an ellipse A and its polar dual A^*

$$h(u,v) = \sup\{(u,v)\cdot(x,y) : (x,y) \in A\}$$

In this case we have:

$$\begin{aligned}
h(u,v) &= \sup\{(u,v)\cdot(x,y) : (x,y) \in A\} \\
&\leq \sup\{|(u,v)\cdot(x,y)| : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1\} \\
&= \sup\{|(u,v)\cdot\left(\sqrt{1-(y/b)^2}, \sqrt{1-(x/a)^2}\right)|\} \\
&\leq \sup\{|(u,v)\cdot(a,b)|\} = \\
&= \sup\sqrt{(ua)^2+(vb)^2} = \sqrt{(ua)^2+(vb)^2} = h(u,v)
\end{aligned}$$

So the $\leq$ expressions above are actually equalities, and we have an expression for the support function of A.

2.5.3. *Gauge function*

As we saw earlier, the general form of the equation for the gauge function, using components in place of vectors, is

$$g((u,v)) = \inf\{\lambda \geq 0 : (x,y) \in \lambda A\}.$$

Recall the equation for the ellipse, (13):

$$A = \{(x,y) : \left(\frac{x^2}{a^2}\right) + \left(\frac{y^2}{b^2}\right) \leq 1,\ a > 0,\ b > 0\}.$$

Now $(u,v) \in \lambda A$ if and only if

$$(u,v) \in \{(x,y) : \left(\frac{\lambda x}{a}\right)^2 + \left(\frac{\lambda y}{b}\right)^2 \leq \lambda^2\} \tag{19}$$

so that

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = (\inf \lambda)^2 \leq \lambda^2. \tag{20}$$

Therefore,

$$g(u,v) = \inf \lambda = \sqrt{(\inf \lambda)^2} = \sqrt{\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2} \tag{21}$$

2.5.4. *Support function of the dual*

We turn our attention to the ellipse A^* representing the polar dual of A, and seek its support function. Note that the support function of the dual A^* is equal to the gauge function of A. Recall the equation for the polar dual of the ellipse, (18).

$$A^* = \{(u,v) : (au)^2 + (bv)^2 \leq 1,\ a > 0,\ b > 0\}$$

The general form of its support function is given by:

$$h^*(u,v) = \sup\{(u,v)\cdot(x,y) : (x,y) \in A^*\}$$

The specific support function computation is as follows:

$$\begin{aligned}
h^*(u,v) &= \sup\{(u,v)\cdot(x,y) : (x,y) \in A^*\} \\
&\leq \sup\{|(u,v)\cdot(x,y)| : (ax)^2 + (by)^2 \leq 1\} \\
&= \sup\{|(u,v)\cdot\left(\frac{1}{a}\sqrt{1-(by)^2}, \frac{1}{b}\sqrt{1-(ax)^2}\right)|\} \\
&\leq \sup\{|(u,v)\cdot\left(\frac{1}{a}, \frac{1}{b}\right)|\} \\
&= \sqrt{\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2}
\end{aligned}$$

2.5.5. *Gauge function of the dual*

The general form of the gauge function in this case is:

$$g((u,v)) = \inf\{\lambda \geq 0 : x \in \lambda A\}$$

Now $(u,v) \in \lambda A^*$ if and only if

$$(u,v) \in \{(x,y) : (\lambda ax)^2 + (\lambda by)^2 \leq \lambda^2\}$$

which implies that

$$(ua)^2 + (vb)^2 = (\inf \lambda)^2 \leq \lambda^2$$

Therefore,

$$g^*(u,v) = \inf \lambda = \sqrt{(\inf \lambda)^2} = \sqrt{(ua)^2 + (vb)^2} \tag{22}$$

2.5.6. *Verification of polar duality*

We can now see from equations (19) and (22), that the support function of A is the gauge function of A^*. Specifically,

$$h(u,v) = \sqrt{(au)^2 + (bv)^2} = g^*(u,v).$$

This is consistent with Theorem 2.4.1. Moreover, equations (21) and (22) imply that the gauge function of A is the support function of A^*. Specifically,

$$g(u,v) = \sqrt{\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2}$$

This again is consistent with Theorem 2.4.1.

2.5.7. *Generalized Legendre Transform*

A slightly more complicated issue is that of the generalized Legendre Transforms of the support and gauge functions of A and A^*. We begin with the equation for the gauge function of an ellipse A:

$$g(x,y) = \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}.$$

Following the notation and the thought process in [9] we seek the generalized Legendre transform of g, which is the the function h such that $\frac{1}{2}h^2$ is the Legendre Transform of

$$Q(x,y) = \frac{1}{2}g^2(x,y).$$

This transform is:

$$\Phi(u,v) = \{(u,v)\cdot(x,y) - Q(x,y)\}, \;\; (x,y) = \Psi(u,v)$$

where $\Psi(u)$ is the mapping $\Psi(u) : x \mapsto (u, v) = Q_{(x,y)}(x, y)$.

Next, we calculate

$$\begin{aligned} (u, v) = Q_{(x,y)}(x, y) &= \nabla \left[\frac{1}{2} \left(\frac{x}{a} \right)^2 + \frac{1}{2} \left(\frac{y}{b} \right)^2 \right] \\ &= \left(\left(\frac{x}{a} \right) \frac{1}{a}, \left(\frac{y}{b} \right) \frac{1}{b} \right) \\ &= \left(\frac{x}{a^2}, \frac{y}{b^2} \right). \end{aligned}$$

In order to get x as a function of u and y as a function of v, set

$$\left(\frac{x}{a^2}, \frac{y}{b^2} \right) = (u, v)).$$

Then:

$\frac{x}{a^2} = u$, so that $x = a^2 u$ $\quad \frac{y}{b^2} = v$, so that $y = b^2 v$

We will use these x, y in the equation for Φ. Note that:

$$(u, v) \cdot (x, y) = (u, v) \cdot (a^2 u, b^2 v) = ua^2 + vb^2.$$

Also,

$$
\begin{aligned}
Q(x,y) &= \frac{1}{2}g^2(x,y) \\
&= \left[\frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{2}\left(\frac{y}{b}\right)^2\right] \\
&= \frac{1}{2}\left(\frac{a^4u^2}{a^2} + \frac{b^4v^2}{b^2}\right) \\
&= \frac{1}{2}\left(a^2u^2 + b^2v^2\right)
\end{aligned} \tag{23}
$$

Then,

$$\Phi(u,v) = u^2a^2 + v^2b^2 - \frac{1}{2}\left(u^2a^2 + v^2b^2\right) = \frac{1}{2}\left(u^2a^2 + v^2b^2\right)$$

Now:

$$\frac{1}{2}h(u,v)^2 = \Phi(u,v), so:$$

$$h(u,v) = \sqrt{a^2u^2 + b^2v^2}$$

This shows that the generalized Legendre transform of the gauge function of an ellipse A is the gauge function of the polar dual A^* of A. This is what we would expect based on the result of Theorem 2.4.2

The theory suggests that the inverse relations will hold as well. It should be possible, for example, to take the generalized Legendre transform of the support function of A and derive the support function of A^*. This is the next topic to be explored. We start with the equation for the support function of the ellipse,

$$h(x, y) = \sqrt{a^2x^2 + b^2y^2}.$$

As before, the Legendre Transform of

$$Q(x, y) = \frac{1}{2}h^2(x, y)$$

is used to get the Generalized Legendre Transform, this time of the support function.

The Legendre Transform of $Q(x, y)$ is:

$$\Phi(u, v) = \{(u, v) \cdot (x, y) - Q(x, y)\}, \mathrm{where(x, y) = \Psi(u, v)}.$$

We calculate:

$$(u, v) = Q_{(x,y)}(x, y) = \frac{1}{2}\nabla \left[a^2x^2 + b^2y^2\right] = \left(a^2x + b^2y\right).$$

Next, let

$$\left(a^2x, b^2y\right) = (u, v).$$

Then,

$$a^2x = u \Rightarrow x = \frac{u}{a^2}$$

$$b^2y = v \Rightarrow y = \frac{v}{b^2}$$

Therefore,

$$(u, v) \cdot (x, y) = \frac{u^2}{a^2} + \frac{v^2}{b^2}$$

Now,

$$\begin{aligned} Q(x, y) &= \frac{1}{2}\left(a^2x^2 + b^2y^2\right) \\ &= \frac{1}{2}\left(\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2\right) \end{aligned}$$

Thus we have:

$$\begin{aligned} \Phi(u, v) &= \left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 - \frac{1}{2}\left(\frac{u}{a}\right)^2 - \frac{1}{2}\left(\frac{v}{b}\right)^2 \\ &= \frac{1}{2}\left(\frac{u}{a}\right)^2 + \frac{1}{2}\left(\frac{v}{b}\right)^2 . \end{aligned}$$

Finally,

$$h(u,v) = \sqrt{\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2} \, .$$

Therefore, the Generalized Legendre Transform of the support function of an ellipse A is the support function of A^*, its polar dual, as would be expected from, Theorem 2.4.2. By polar duality, this problem is similar to that of showing that the Generalized Legendre transform of the gauge function of A^* is the gauge function of A.

CHAPTER 3

Support Functions and Geometric Flows

The treatment here closely follows that in Andrews [1]. Andrews develops ways to express geometric flows in order to make it easier to derive Harnack inequalities for them. At this stage we are not interested in Harnack inequalities, but the tools and techniques used in [1] are powerful, and central to the ideas we develop here.

3.1. **General results**

Let M^n represent an n-dimensional smooth, compact Riemannian manifold. To describe a geometric flow, we could imagine the manifold evolving in $\mathbb{R}^n$, in much the same way as the shape of the surface of a pond might "evolve" over a period of time as wind passes over it. Alternatively (and equivalently) we could imagine the flow as being a family of manifolds, embedded in $\mathbb{R}^n$, with each member of the family representing the manifold at a particular time. This vantage point turns out to be more convenient mathematically.

In this context, then, consider a mapping

$$\Phi : [0, T] \times M^n \to \mathbb{R}^n$$

describing a smoothly evolving family (over a finite time interval) of immersions. For convex bodies, each M^n could be identified with the unit sphere $\mathbb{S}^n$.

The evolution of this map can be described by a system of second order parabolic partial differential equations taking the form [1]:

$$\frac{\partial \Phi(x,t)}{\partial t} = -F(\mathcal{W}(\mathbf{x},\mathbf{t}), \mathbf{v}(\mathbf{x},\mathbf{t}))\, \mathbf{v}(\mathbf{x},\mathbf{t}), \tag{24}$$

where :

- F is the speed of motion of the hypersurfaces through $\mathbb{R}^{n+1}$;
- $v : M^n \times [0,T] \to \mathbb{S}^n$ is the unit normal to $\Phi_t(M^n)$ at $\Phi_t(x)$.
- $\mathcal{W}$ is the Weingarten map, related to the curvature of M^n.

The system of partial differential equations (24) is assumed to be invariant under translations in time and space, including under isometries of $\mathbb{R}^{n+1}$. As (24) is a system of parabolic partial differential equations satisfying the invariance conditions above, it can be considered a flow equation. Furthermore, as (24) is a flow equation that involves curvature (through F and $\mathcal{W}$), it can be considered a geometric flow equation. Next, we introduce some standard notions from Riemannian geometry. Detailed definitions of these terms can be found in the standard texts on the subject, including [17] and [23]. Here we shall work with more intuitive definitions.

We start with the idea of a tangent space. Let $\mathbf{x}$ be a point on the manifold M^n, and consider all the directions on M^n through which it is possible to pass through the point $\mathbf{x}$. Next, consider the set of vectors pointing in these directions. We call this set $T_x M^n$, the tangent space to a manifold at the point $\mathbf{x}$.

In $\mathbb{R}^3$, if M^n is the sphere $\mathbb{S}^2$, then the tangent plane to the point $\mathbf{x}$ on the sphere is the two dimensional tangent plane to the sphere at $\mathbf{x}$. We broaden this idea by considering all of the tangent spaces to all of the points of M^n. This is called the tangent bundle TM^n. Again using the analogy in $\mathbb{R}^3$, the tangent bundle to $\mathbb{S}^2$ is the set of all tangent planes to the sphere.

The notions of a tangent space and a tangent bundle can also be applied to an immersion $\phi(\mathbf{u})$ or $\phi(\mathbf{v})$ to get $T\phi(\mathbf{u})$ or $T\phi(\mathbf{v})$. In such a case it is possible to extend the usual notion of an inner product of derivatives of vectors to this more general setting by defining a function (specifically, a metric) from points on the tangent bundles to the real number line as follows:

$$g(\mathbf{u}, \mathbf{v}) = \langle T\phi(\mathbf{u}), T\phi(\mathbf{v}) \rangle \tag{25}$$

The ordinary vector derivative in $\mathbb{R}^3$ can also be extended to this more general setting, through the use of connections.

For any vector field $\mathbf{v}$, at some point $\mathbf{x}$, the connection on TM, written ∇, is the projection of the derivative of $\mathbf{v}$ onto the tangent space T_xM^n. It can be expressed as [1]

$$\nabla_{\mathbf{u}}\mathbf{v} = T_{\mathbf{x}^{-1}}\phi\left(\pi_{\mathbf{x}}\left(D_{T\phi(\mathbf{u})}T\phi(\mathbf{v})\right)\right), \tag{26}$$

for all $\mathbf{u}, \mathbf{v} \in T_{\mathbf{x}}M$, where $\pi_{\mathbf{x}}$ is the projection of $\mathbb{R}^{n+1}$ onto the image of the space $T_{\mathbf{x}}\phi$. An important concept in our derivation of a support function model of geometric flows is the Weingarten map, $\mathcal{W}$. Denote by

$$T\mathbf{v} : TM \to TS^n \subset \mathbb{R}^{n+1} \tag{27}$$

the derivative of the Gauss map. The Weingarten map measures the rate of change of the direction of a normal along a surface, and it is written:

$$\mathcal{W} : TM^n \to TM^n$$

$$\mathcal{W} = T\phi^{\mathbf{-1}} \circ T\mathbf{v}, \tag{28}$$

We recall that the eigenvalues of $\mathcal{W}$ at the point $\mathbf{x}$ are the principal curvatures κ_j of the convex bodies described by the immersion. We will also refer to the principal radii of curvature, $R_j = \frac{1}{\kappa_j}$.

We want to express the Weingarten map in terms of the support function, and to do this we begin with by defining the immersion in terms of the support function. By definition, the support function h of an immersion $\phi(\mathbf{z})$ is the normal distance from the tangent plane $T\phi(\mathbf{z})$ to the origin. If $\mathbf{a}(\mathbf{z})$ is a vector tangent to $\mathbb{S}^n$ at $\mathbf{z}$, for each $\mathbf{z} \in \mathbb{S}^n$, then we can write $\phi(\mathbf{z})$ as (see Figure 7):

$$\Phi(\mathbf{z}) = h(\mathbf{z})\mathbf{z} + \mathbf{a}(\mathbf{z}). \tag{29}$$

We seek to determine the form of $\mathbf{a}(\mathbf{z})$, and follow [1] in this regard. We can differentiate (29) in the direction $\mathbf{u}$ to get:

$$T\phi(\mathbf{u}) = D_{\mathbf{u}}\left(h(\mathbf{z})) + \mathbf{a}(\mathbf{z})\right) = \left(D_{\mathbf{u}}\mathbf{a}\right)\mathbf{z} + \left(D_{\mathbf{u}}\mathbf{z}\right)h + D_{\mathbf{u}}\mathbf{a} \tag{30}$$

The directional derivative $D_{\mathbf{u}}\mathbf{z}$ is the tangent to the vector $\mathbf{z}$ in the direction $\mathbf{u}$. Therefore, $(D_{\mathbf{u}}\mathbf{z}) = \mathbf{u}$. Then:

$$T\phi(\mathbf{u}) = \left(D_{\mathbf{u}}\mathbf{a}\right)\mathbf{z} + h\mathbf{u} + D_{\mathbf{u}}\mathbf{a}$$

Next, we address the $D_{\mathbf{u}}\mathbf{a}$ component. This discussion is based on [23]. Let the components of a vector field, covariant derivative and the directional derivative be denoted by X^i, $X^i_{;j}$, and $X^i_{,j} = \frac{\partial X^i}{\partial u_j}$, respectively. As usual, let Γ^i_{jk} represent the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{il} \left(g_{lj,k} + g_{lk,j} - g_{jk,l} \right),$$

where the terms in g are partial derivatives of the metric and its inverse. Then the components of the directional derivative are:

$$X^i_{;j} = X^j_i + \sum_k \Gamma^i_{jk} X^k.$$

In this case, on the unit sphere, several of the partial derivatives of the metrics are zero, and we are left with:

$$D_{\mathbf{u}}\mathbf{a} = \overline{\nabla}_{\mathbf{u}}\mathbf{a} - \overline{g}(\mathbf{u}, \mathbf{a})\mathbf{z}.$$

Assembling the terms together gives us the equation:

$$T\phi(\mathbf{u}) = \left(D_{\mathbf{u}} h + h\mathbf{u} + \overline{\nabla}_{\mathbf{u}}\mathbf{a} - \overline{g}(\mathbf{u}, \mathbf{a})\mathbf{z} \right).$$

Now the tangent spaces $T_{\mathbf{z}}\phi(T_s\mathbb{S}^n)$ and $T_{\mathbf{z}}\mathbb{S}^n$ are parallel to each other, so the component in the $\mathbf{z}$ direction on the right hand side of equation (30) must be zero.

Then

$$T\phi(\mathbf{u}) = h\mathbf{u} + \bar{\nabla}_{\mathbf{u}}\mathbf{a},$$

but this can only be true if

$$\mathbf{a}(\mathbf{z}) = \bar{\nabla}h.$$

Then equation (29) becomes

$$\phi(\mathbf{x}) = h(\mathbf{z})\mathbf{z} + \bar{\nabla}h \tag{31}$$

Next, recall the equation

$$\mathcal{W}(\mathbf{u}) = T\phi^{-1} \circ T\nu(\mathbf{u})$$

for the Weingarten map. In this case $T\nu = Id$, so the equation becomes:

$$\mathcal{W}^{-1} = T\phi$$

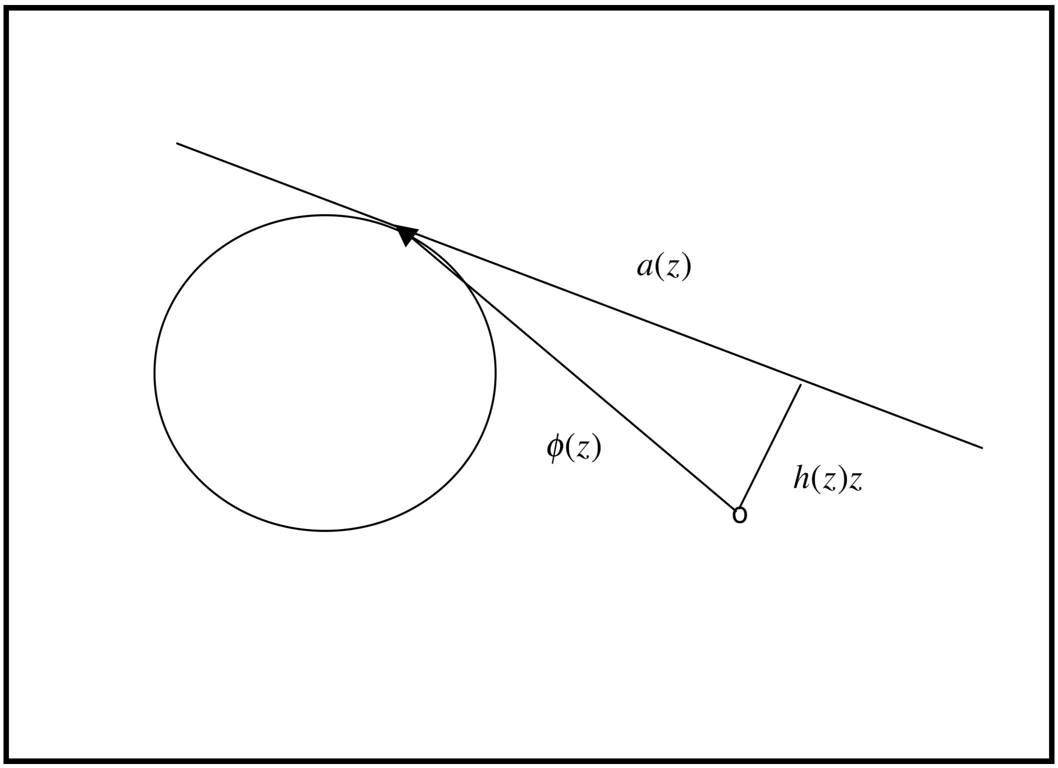

Figure 7. The immersion map and the support function.

Differentiating equation (31) then gives us:

$$\mathcal{W}^{-1}(\mathbf{u}) = \bar{\nabla}_{\mathbf{u}}(\bar{\nabla} h) + hId \tag{32}$$

This leads us to a main result for this section.

Theorem 3.1 (Reference: [1], **p.184)**

Let

$$\phi : M^n \times [0, T] \to \mathbb{R}^{n+1}$$

be a family of strictly convex immersions such that

$$\frac{\partial \phi(\mathbf{z}, t)}{\partial t} = -F\left(\mathcal{W}^{-1}[h(\mathbf{z}, t)], \mathbf{z}\right).$$

Then the support functions

$$h : \mathbb{S}^n \times [0, T] \to \mathbb{R}$$

satisfy the geometric evolution equations

$$\frac{\partial h}{\partial t} = \Phi\left(\mathcal{W}^{-1}[h(\mathbf{z}, t)], \mathbf{z}\right) \tag{33}$$

where $\mathcal{W}^{-1}$ is the inverse of the Weingarten map, and Φ has the form

$$\Phi(\mathcal{W}^{-1}, \mathbf{z}) = -F(\mathcal{W}, \mathbf{z}). \tag{34}$$

Proof:

Since $\phi(\mathbf{z}) = h(\mathbf{z})\mathbf{z} + \overline{\nabla} h$, we have

$$\frac{\partial \phi}{\partial t}(\mathbf{z}, \mathbf{t}) = \frac{\partial \mathbf{h}}{\partial t} + h\frac{\partial \mathbf{z}}{\partial t} + \frac{\partial \overline{\nabla h}}{\partial t} = -F(\mathcal{W}(\mathbf{z}, t), \nu(\mathbf{z}, t))\nu(\mathbf{z}, t).$$

Since $\mathbf{z}$ does not depend upon t, we have $h\frac{\partial \mathbf{z}}{\partial t} = 0$.

Therefore,

$\frac{\partial h}{\partial \mathbf{t}}$ must be a function of $\mathcal{W}$ and of $\mathbf{z}$, and we can write:

$$\frac{\partial h}{\partial t} = G(\mathcal{W}(\mathbf{z}, t), \nu(\mathbf{z}, t)\nu(\mathbf{z}, t).$$

To express the right hand side of the equation above in terms of a function of $\mathcal{W}^{-1}$, observe that there corresponds to $F(\mathcal{W})$ a dual function $\Phi(\mathcal{W}^{-1})$ such that $\Phi(\mathcal{W}^{-1}) = G(\mathcal{W})$.

$\square$.

We can now proceed to write geometric evolution equations in terms of the support function. Equations (32) and (33) give us an easy way to do this. However, the expression derived will not always be an elementary function of the support function, and in such cases more work may be needed to obtain a suitable form.

3.2. **Harmonic mean curvature flow**

This is one of the cases where the support function representation leads to a particularly simple form of the geometric flow. The harmonic mean curvature is the inverse of the trace of the inverse Weingarten map, and it can be written as:

$$\Phi = \left(tr\mathcal{W}^{-1}\right)^{-1} \tag{35}$$

Using equation (32) we get the following form for equation (35), where the R_j are the principle radii of curvature:

$$\Phi = \left[tr(\bar{\nabla}(\nabla h) + hI)\right]^{-1} = \frac{1}{\sum_j R_j} \tag{36}$$

Now $tr\bar{\nabla}_u \nabla s = \bar{\triangle}$, where $\bar{\triangle}$ is the Laplacian of h on the unit sphere, and $I(u) = n$. Then equation (36) becomes $\Phi = \left[\bar{\triangle} + ns\right]^{-1}$. Using equation (33), we can write the geometric evolution equation [1]:

$$\frac{\partial h}{\partial t} = \left[\bar{\triangle} h + nh\right]^{-1}. \tag{37}$$

3.3. **Inverse harmonic mean curvature flow**

The inverse mean harmonic curvature is the trace of the inverse of the Weingarten map. The curvature equation can be written as

$$\Phi = \left(tr\mathcal{W}^{-1}\right),$$

or alternatively,

$$\Phi = \left[tr\left(\bar{\nabla}\left(\nabla h\right) + hId\right)\right].$$

Following the earlier discussion of the harmonic mean curvature flow, we see that the inverse harmonic mean curvature flow in terms of the support function is [1]:

$$\frac{\partial h}{\partial t} = \left[\bar{\triangle} h + nh\right] = \sum_j R_j \tag{38}$$

3.4. **Mean curvature flow**

The mean curvature is the trace of the Weingarten map. The curvature equation is therefore:

$$\Phi = \left[tr \left(\bar{\nabla} \left(\nabla h \right) + hId \right)^{-1} \right] = \sum_j k_j$$

While this not generally one of the more complicated geometric flows, it does not have an immediate simplification in terms of the support function.

Thus, we write the geometric flow equation as:

$$\frac{\partial h}{\partial t} = \left[tr \left(\bar{\nabla} \left(\nabla h \right) + h \right)^{-1} \right].$$

3.5. **Gauss curvature flow**

The Gauss curvature is the inverse of the determinant of the inverse of the Weingarten map. Therefore, the curvature equation is:

$$\Phi = \left[\det \left(\bar{\nabla} \left(\nabla h \right) + h \right) \right].$$

Therefore, the geometric flow equation is:

$$\frac{\partial h}{\partial t} = \left[\det \left(\bar{\nabla} \left(\nabla h \right) + h \right) \right].$$

3.6. Affine normal flow

The affine normal flow has a similar form to the Gauss curvature flow. J. Loftin and M. Tsui [19] have shown that the affine normal flow, when written as a function of the support function restricted to a hyperplane, is:

$$\frac{\partial h}{\partial t} = -\left(\det(\bar{\nabla}\left(\nabla h\right)\right)^{\frac{-1}{n+2}}.$$

3.7. Ricci curvature flow

The Ricci curvature has an interesting expression in terms of the Weingarten map, which leads naturally to a support function representation of the Ricci flow. In a 1965 article N. Hicks [12] derived the following expression for the Ricci curvature, in terms of the Weingarten map of a hypersurface:

$$\Phi = \langle H\mathcal{W} - \mathcal{W}^2, \mathbf{u}\rangle,$$

where H is the mean curvature of a hyperspace ϕ, $\mathcal{W}$ is the Weingarten map, and $\mathbf{u}$ is a vector in ϕ. Using the expression for $\mathcal{W}$ that is implied by Equation (32) and the expression for the curvature flow in Equation (33), we get:

$$\frac{\partial h}{\partial t} = \langle H\left(\bar{\nabla}\left(\nabla h\right) + h\mathrm{Id}(\mathbf{u})\right)^{-1} - \left(\bar{\nabla}\left(\nabla h\right) + h\mathrm{Id}(\mathbf{u})\right)^{-2}, \mathbf{u}\rangle.$$

It is evident from this that any curvature flow that can be written in terms of the Weingarten map can be written in terms of its support function. This becomes important in investigating flows of polar dual body.

In fact, our next exercise is to extend the idea of a support function representation of a geometric flow to the dual space. The key idea is that a geometric flow on a convex body induces a distortion on its polar dual. To see this, reconsider equation (33), but written as a function of a unit normal vector, $\mathbf{u}$.

$$\frac{\partial h(\mathbf{u})}{\partial t} = \Phi\left(\mathcal{W}^{-1}[h(\mathbf{u})], \mathbf{u}\right)$$

We would like to be able to replace the support function $h(\mathbf{u})$ in the equation above with the gauge function $g(\mathbf{x})$ of the polar dual body, or equivalently, with the Legendre transform of the support function $h^*(\mathbf{u})$ of the polar body, which is $\mathcal{L}(h^*(\mathbf{x}))$. However, $h(\mathbf{u})$ is defined on the unit sphere, and for the duality between support and gauge functions to hold it is necessary to extend $h(\mathbf{u})$ to all $\mathbb{R}^n$. As we saw earlier, this can be done through homogeneity. We use denote the new vectors resulting from this extension by $\hat{\mathbf{x}}$ and $\hat{\mathbf{u}}$. This gives a new evolution equation:

$$\frac{\partial \mathcal{L}\left(h^*(\hat{\mathbf{x}})\right)}{\partial t} = \Phi\left(\mathcal{W}^{-1}[\mathcal{L}\left(h^*(\hat{\mathbf{x}})\right)], \hat{\mathbf{x}}\right) \tag{39}$$

Alternatively, this can be wriiten as:

$$\frac{\partial \mathcal{L}\left(h^*(\hat{\mathbf{x}})\right)}{\partial t} = \left[\bar{\nabla}_{\hat{\mathbf{x}}}\bar{\nabla}\left(\mathcal{L}(h^*)\right) + \mathcal{L}(h^*(\hat{\mathbf{x}}))(\mathrm{Id}(\hat{\mathbf{x}}))\right] \tag{40}$$

If the original immersion $\phi_{t=0}$ is strictly convex (i.e., convex with no flat edges) then the equation (40) will have the general form of a flow equation.

It might at first glance seem plausible to argue, then, that as the geometric flow described by equation (33) proceeds, another flow involving the polar dual body (equation (40)) takes place as well. However, we have to be careful on this point.

To use the Weingarten representation of the support function, the parameter $\mathbf{u}$ must be a unit normal vector. The corresponding parameter for the dual body, after we take the Legendre Transform, is $\mathbf{x}$. By the nature of the Legendre transform, $\mathbf{u}$ and $\mathbf{x}$ are closely related. But in general, if $\mathbf{u}$ is chosen to be a unit vector, then $\mathbf{x}$ is not unit vector. We shall address and solve this problem in the next chapter.

CHAPTER 4

GEOMETRIC FLOWS AND POLAR DUALITY

4.1. Background

Let A be a convex body in $\mathbb{R}^n$, with a support function $h(\mathbf{u})$ and a gauge function $g(\mathbf{x})$. If we consider support functions defined on all of $\mathbb{R}^n$, then A^* denotes the polar dual of A, with support function $h^*(\mathbf{x})$ and gauge function $g^*(\mathbf{u})$. By polar duality $g(\mathbf{x}) = h^*(\mathbf{x})$, and by Legendre duality, $\mathcal{L}\left(\frac{1}{2}h^2\right) = \frac{1}{2}g^2$.

If A is smooth and strictly convex, the two types of duality ensure the existence of a unique mapping from the points on the boundary of A (denoted as ∂A) to the points on the boundary of A^* (denoted as ∂A^*). The reason for this is easy to visualize in $\mathbb{R}^2$. On A, for each normal vector $\mathbf{u}$ there is one and only one vector $\mathbf{x}$ from the origin to ∂A, contacting the vector $\mathbf{u}$. Reversing the roles of $\mathbf{u}$ and $\mathbf{x}$ on A^*, we see that $\mathbf{x}$ and $\mathbf{u}$ correspond to a unique point on ∂A^*. The assumptions of the smoothness and strict convexity of A are required for this to be true. If A is not smooth and thus has corners, then there will be multiple normal directions at the corners, and there will not be a unique correspondence between ∂A and ∂A^*. On the other hand, if A is not strictly convex and therefore has a smooth flat edge, then many points $\mathbf{x}$ will correspond to one direction $\mathbf{u}$. Here, we assume that A is smooth and strictly convex, and in this section we work out some consequences of the unique correspondence.

Although the generalized Legendre Transform assigns a unique $\mathbf{x}$ to a particular direction $\mathbf{u}$ of h (when the gauge function is defined), that point $\mathbf{x}$ will not necessarily be a point on ∂A. Only when $\mathbf{u}$ is a unit vector does $h(\mathbf{u})$ correspond to the distance to a support plane at a particular point $\mathbf{x}$ on ∂A.

The solution to this problem is to rescale $\mathbf{x}$ to $\frac{\mathbf{x}}{r}$, where $r = |\mathbf{x}|$. Then if we choose $\hat{\mathbf{u}}$ to be a unit vector, we can use the homogeneity of g to write (following Giaquinta and Hildebrandt [9]):

$$h(\hat{\mathbf{u}}) = g(\mathbf{x}) = rg(\mathbf{x}/r) \tag{41}$$

Now if A is smooth and strictly convex, then for any $\mathbf{u}$ there is a unique $\mathbf{x}$ that minimizes $\mathbf{u} \cdot \mathbf{x} - \frac{1}{2}g^2(\mathbf{x})$, which, being a critical point, is [9]

$$\mathbf{x} = h(\mathbf{u})\nabla_{\mathbf{u}} h(\mathbf{u})$$

If $\mathbf{u}$ is chosen to be a unit vector then this equation becomes:

$$\mathbf{x} = h(\hat{\mathbf{u}})\nabla_{\hat{\mathbf{u}}} h(\hat{\mathbf{u}})$$

Then

$$r = |\mathbf{x}| = h(\hat{u}) \; |\nabla_{\hat{u}} h(\hat{u}) \; |,$$

so that

$$g\left(\frac{\mathbf{x}}{r}\right) = \frac{1}{|\nabla_{\hat{\mathbf{u}}} h(\hat{\mathbf{u}})|} \tag{42}$$

To use the Weingarten equation we need to separate the radial and parallel components of $\nabla_{\mathbf{u}} h$. The parallel components will be the same as those of the gradient on $\mathbb{S}^{n-1}$. Let $\rho = |\mathbf{u}|$. We get:

$$\nabla_{\mathbf{u}} h = \frac{\partial h(\mathbf{u})\hat{\mathbf{e}_\rho}}{\partial \rho} + \frac{1}{\rho}\nabla_{||} h(\mathbf{u}), \tag{43}$$

where $\nabla_{||}$ denotes the parallel component of the gradient.

By homogeneity we have:

$$h(\mathbf{u}) = \rho h\left(\frac{\mathbf{u}}{\rho}\right)$$

Therefore we get:

$$\nabla_{\mathbf{u}} h = h\left(\frac{\mathbf{u}}{\rho}\right)\hat{\mathbf{e}}_\rho + \nabla_{\mathbf{u}||} h\left(\frac{\mathbf{u}}{\rho}\right) \tag{44}$$

This equation can be used to give us a relationship between the gauge function and the support function of a convex body:

$$g\left(\frac{\mathbf{x}}{r}\right) = \frac{1}{\sqrt{\left(h(\frac{\mathbf{u}}{\rho})\right)^2 + \|\nabla_{\mathbf{u}||} h\left(\frac{\mathbf{u}}{\rho}\right)\|^2}} \tag{45}$$

4.2. **The case when n = 2**

We now turn to the situation in $\mathbb{R}^2$, where polar coordinates can be used. Our goal is to get an expression for the radius of curvature at a point of ∂A^* in terms of the radius of curvature at a point of ∂A. Let $g = g(r, \phi)$, and $h = h(\rho, \theta)$. A geometric description of the the angles θ and ϕ appears in the diagram below.

On the convex body A, θ is the angle between the unit normal to the convex body at some point on A, and the horizontal x-axis. On the convex body A^*, ϕ is the angle, as measured at the origin, between the normal vector to the body at some point, to the horizontal x-axis. We can now rewrite equation (44) in polar coordinates, to get:

$$\nabla_{\mathbf{u}} h = h(1, \theta)\hat{\mathbf{e}}_\rho + h_\theta \hat{\mathbf{e}}_\rho \tag{46}$$

where

$$\hat{e}_\rho \perp \hat{e}_\theta.$$

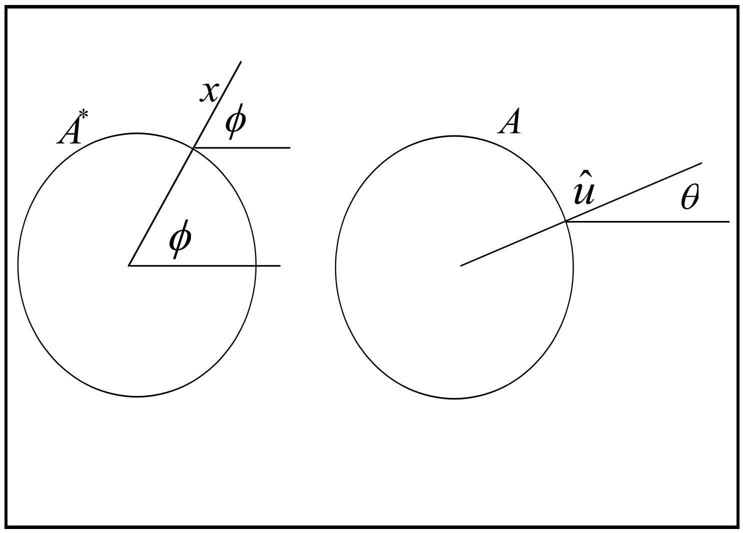

Figure 8. The relationship between ϕ and θ

The vector $\mathbf{x}$ can be written as:

$$\mathbf{x}(\mathbf{u}) = h(\mathbf{u})\nabla_{\mathbf{u}} h(\mathbf{u}) = h^2(1,\theta)\hat{\mathbf{e}}_\rho + h h_\theta(1,\theta)\hat{\mathbf{e}}_\theta, \tag{47}$$

where $\hat{\mathbf{e}}_\rho \perp \hat{\mathbf{e}}_\theta$.

We can write:

$$\|\mathbf{x}(\mathbf{u})\| = \sqrt{h^4(1,\theta) + h^2h_\theta^2} = h(1,\theta)\sqrt{h^2 + h_\theta^2} \tag{48}$$

By equation (47), we can see that:

$$\begin{aligned}
\|\mathbf{x}\|\cos(\phi) &= \mathbf{x}(\mathbf{u})\cdot\hat{\mathbf{e}}_{\mathbf{1}} \\
&= h^2(\hat{\mathbf{e}}_\rho\cdot\hat{\mathbf{e}}_{\mathbf{1}}) + hh_\theta(\hat{\mathbf{e}}_\rho\cdot\hat{\mathbf{e}}_{\mathbf{1}}) \\
&= h^2\cos(\theta) - hh_\theta\sin(\theta)
\end{aligned}$$

Then by equations (48) and (49) we get the following:

$$h(1,\theta)\ \ \cos(\phi)\ \ \sqrt{h^2 + h_\theta^2} = h^2\cos(\theta) - hh_\theta\sin(\theta)$$

$$cos(\phi)\sqrt{h^2 + h_\theta^2} = h\cos(\theta) - h_\theta\sin(\theta)$$

$$\cos(\phi) = \frac{h}{\sqrt{h^2 + h_\theta^2}}\cos(\theta) - \frac{h_\theta\sin(\theta)}{\sqrt{h^2 + h_\theta^2}} \tag{49}$$

Now let

$$\cos(\alpha) = \frac{h}{\sqrt{h^2 + h_\theta^2}}$$

$$\sin(\alpha) = \frac{h_\theta}{\sqrt{h^2 + h_\theta^2}}$$

From equation (49) and the trigonometric identity $\cos(\phi) = \cos(\theta + \alpha)$

$$\frac{h_\theta}{h} = \tan(\alpha)$$

Inverting the cosine in equation (49) gives us the following relationship between ϕ and θ:

(50) $$\phi = \theta + \arctan\left(\frac{h_\theta(1,\theta)}{h(1,\theta)}\right)$$

Moreover, the logarithmic derivative $\frac{h_\theta}{h}$ can be interpreted geometrically, and this is done in the diagram on the next page.

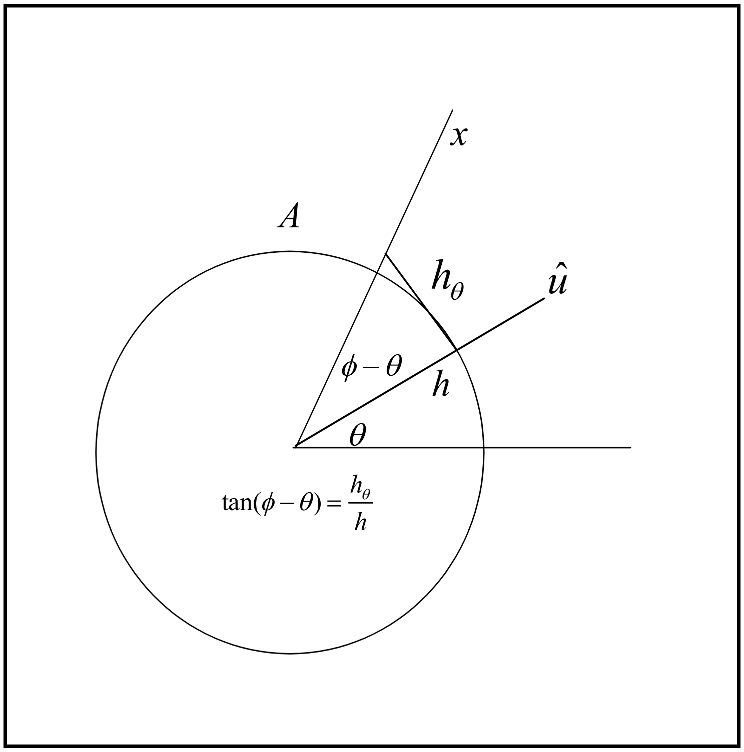

Figure 9. The geometry of the logarithmic derivative $\frac{h_\theta}{h}$.

The importance of the logarithmic derivative $\frac{h_\theta}{h}$ will become apparent shortly. Meanwhile, differentiating the expression (50) yields:

(51) $$d\phi = \left[1 + \frac{hh_{\theta\theta} - h_\theta^2}{h^2 + h_\theta^2}\right] d\theta = h\left(\frac{h_{\theta\theta} + h}{h^2 + h_\theta^2}\right) d\theta$$

The Weingarten equation in two dimensions using polar coordinates is $h_{\theta\theta} + h = R$, where R is the radius of curvature. Using the equations

$$h(\phi, \theta) = \rho(h, \theta)$$

and

$$\frac{\partial h}{\partial \rho(1, \theta)} = h$$

we can rewrite equation (45) in two dimensional polar coordinates to get:

$$g(1, \theta) = \frac{1}{\sqrt{h^2(1, \theta) + h_\theta^2(1, \theta)}},$$

and therefore,

$$g^2 = \frac{1}{h^2 + h_\theta^2}$$

With (51) this leads to:

(52) $$d\phi = hRg^2 d\theta$$

And by duality:

$$d\theta = h^* R^* g^{*2} d\phi = g R^* h^2 d\phi \tag{53}$$

By comparing (52) and (53) we see that

$$hRg^2 = \frac{1}{gR^* h^2} \iff h^3(1,\theta)R(\theta) = \frac{1}{g^3(1,\theta)R^*(\phi)} \tag{54}$$

Again recalling equation (45) in two dimensional polar coordinates, we have:

$$g(1,\theta) = \frac{1}{\sqrt{h^2(1,\theta) + h_\theta^2(1,\theta)}},$$

so that

$$h(1,\theta)g(1,\theta) = \frac{1}{\sqrt{1 + \left(\frac{h_\theta}{h}\right)^2}}$$

Using this, equation (54) and polar duality yields two results:

$$R^*(\phi) = \frac{1}{(hg)^3 R} = \left[1 + \left(\frac{h_\theta}{h}\right)^2\right]^{3/2} \frac{1}{R(\theta)} \tag{55}$$

$$R(\theta) = \left[1 + \left(\frac{g_\phi}{g}\right)^2\right]^{3/2} \frac{1}{R^*_\phi} \tag{56}$$

These equations are important, because they relate the radius of curvature of a convex body with that of its polar dual, making it possible to compare geometric flow parameters, and volumes. In the case of geometric flows, we can use equations (33) and (34) with $\Phi(\mathcal{W}, \mathbf{z}) = -F(\mathcal{W}, \mathbf{z}) = R(\theta)$ to write:

$$\frac{\partial h}{\partial t} = R(\theta) \tag{57}$$

$$\frac{\partial h^*}{\partial t} = \left[1 + \left(\frac{h_\theta}{h}\right)^2\right]^{3/2} \frac{1}{R(\theta)} \tag{58}$$

This pair of equations tell us the relationship between the geometric flow on a two-dimensional convex body and the distortion on its polar dual. A geometric flow on the boundary ∂A of a convex body A as defined by equation (57) induces a distortion on the boundary ∂A^* of its polar dual A^* defined by equation (58). In general, the distortion on the polar dual will not be a geometric flow (i.e., a flow defined locally in terms of the curvature). The exceptional case when it will be is when $h_\theta \equiv 0$, in which case the convex body and its dual are both circles.

The table below summarizes various types of flows, in $\mathbb{R}^2$ for convex bodies and their polar duals. In general, the flow of the polar dual of a body in $\mathbb{R}^2$ is, up to a factor of

$$\left[1 + \left(\frac{h_\theta}{h}\right)^2\right]^{3/2} .$$

the inverse of the geometric flow on the original body. Hence, the logarithmic derivative of h helps determine the polar dual flow. The entries under the title "Geometric Flow" refer to flows on the original body, and the complete flow equation would be $\frac{\partial h}{\partial t}$ = Table Entry. The entries under the title "Dual Distortion" refer to distortions on the polar dual of the original body, and the complete distortion equation would be $\frac{\partial h^*}{\partial t}$ = Table Entry. For enhanced readability, we have written

$$\mathcal{R}(h, \theta) = \left[1 + \left(\frac{h_\theta}{h}\right)^2\right]^{3/2} .$$

Now $\mathcal{R}(h, \theta)$ can be thought of as a "curvature modification factor", which determines the amount by which the flow of the polar dual differs from the inverse of the flow on the original convex body, as a result of the difference in geometry between the original convex body and the polar dual.

Table 2 can be used with the support function of the ellipse to directly write the equation for the harmonic mean curvature flow of the ellipse and of its polar dual, in polar coordinates:

$$\frac{\partial h}{\partial t} = \left(\bar{\triangle} \sqrt{u^2 a^2 cos^2\theta + u^2 b^2 sin^2\theta} + n\sqrt{u^2 a^2 cos^2\theta + u^2 b^2 sin^2\theta} \right)^{-1} \tag{59}$$

$$\frac{\partial h^*}{\partial t} = \left(1 + \frac{cos\phi sin\phi (b^2 - a^2)}{u(a^2 cos^2\phi + b^2 sin^2\phi)^{3/2}} \right)^{3/2} \left[\frac{\partial h}{\partial t} \right]^{-1}. \tag{60}$$

Table 2. Geometric Flows and their Dual Flows

Type of Curvature Flow	Geometric Flow	Dual Distortion
Harmonic Mean	$\left[\bar{\triangle}h + 2h\right]^{-1}$	$R\left[\bar{\triangle}h + 2h\right]$
Inverse Harmonic Mean	$\left[\bar{\triangle}h + 2h\right]$	$R\left[\bar{\triangle}h + 2h\right]^{-1}$
Mean	$\left[tr\left(\bar{\nabla}(\nabla h) + hId\right)^{-1}\right]$	$R\left[tr\left(\bar{\nabla}(\nabla h) + hId\right)^{-1}\right]^{-1}$
Gauss	$\left[det\left(\bar{\nabla}(\nabla h) + hId\right)\right]$	$R\left[det\left(\bar{\nabla}(\nabla h) + hId\right)\right]^{-1}$
Affine Normal	$-\left[det(\bar{\nabla}(\nabla h)\right]^{-\left(\frac{1}{4}\right)}$	$-R\left[det(\bar{\nabla}(\nabla h)\right]^{-4}$
Ricci	$\langle H\left(\bar{\nabla}(\nabla h) + hId\right)^{-1}$ $- H\left(\bar{\nabla}(\nabla h) + hId\right)^{-2}, \mathbf{u}\rangle$	$R\langle H\left(\bar{\nabla}(\nabla h) + hId\right)^{-1}$ $- H\left(\bar{\nabla}(\nabla h) + hId\right)^{-2}, \mathbf{u}\rangle^{-1}$

Note: If the quantities in the table are expressed in polar coordinates, then the variables in the original convex body would include θ, while the variables in the dual body would include ϕ.

4.3. **The case when** $n = 3$

What about the situation in three dimensions? It is possible to do spherical coordinate calculations that are similar to the polar coordinate computations in the two-dimensional case. However, in three dimensions the calculations do not lead to a simple expression for the curvature, as in (52).

To see this, consider equation (44), where $\frac{1}{\rho}\nabla_{||}h(\mathbf{u})$ is the angular component of the gradient, and now consists of two terms rather than one:

$$\frac{1}{\rho}\nabla_{||}h(\mathbf{u}) = \frac{1}{\rho}\frac{\partial h(\mathbf{u})}{\partial \phi}\hat{\mathbf{e}}_{\phi} + \frac{1}{\rho \sin\phi}\frac{\partial h(\mathbf{u})}{\partial \theta}\hat{\mathbf{e}}_{\theta}. \tag{61}$$

Note that (similar to the two dimensional case),

$$h(\rho, \theta, \phi) = \rho(h, \theta, \phi),$$

so that

$$\frac{\partial h}{\partial \rho}(1, \theta, \phi) = h.$$

Define gauge and support functions:

$$g = g(r, \omega, \tau)$$

$$h = h(\rho, \theta, \phi)$$

Our goal is to relate ω to θ, and τ to ϕ, with the hope of extracting useful information about the relationship between the curvature at a point on the boundary ∂A of a convex body A and the curvature at a point on the boundary ∂A^* of its polar dual A^*.

We can write:

$$\nabla_{\mathbf{u}} h(\mathbf{u}) = h(1, \theta, \rho) + \frac{1}{\sin\theta} h_\theta \hat{\mathbf{e}}_\theta + h_\phi \hat{\mathbf{e}}_\phi.$$

$$x(u) = h(u)\nabla_u h(u) = h^2 \hat{e}_\rho + \frac{hh_\theta}{\sin\phi}\hat{e}_\theta + hh_\rho \hat{e}_\phi$$

$$\begin{aligned} |x(u)| &= \sqrt{h^4 + \frac{h^2 h_\theta^2}{(\sin(\phi))^2 + h^2 h_\phi^2}} \\ &= h\sqrt{h^2 + \frac{h_\theta^2}{(\sin(\phi))^2 + h_\phi^2}} \end{aligned}$$

Now ω is the angle between the vector x and the x_1 axis. We have:

$$\begin{aligned}
|x(\mathbf{u})|\cos(\omega) &= x(\mathbf{u})\cdot\hat{e}_1 \\
&= h^2(\hat{e}_\rho\cdot\hat{e}_1) + \frac{hh_\theta}{\sin(\phi)}(\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{e}}_\mathbf{1}) + hh_\phi(\hat{\mathbf{e}}_\phi\cdot\hat{\mathbf{e}}_\mathbf{1}) \\
&= h^2\cos(\theta)\sin(\phi) - \frac{hh_\theta}{\sin(\phi)}\sin(\phi) + hh_\phi\cos(\theta)\cos(\phi)
\end{aligned}$$

Thus:

$$\sqrt{h^2 + \frac{h_\theta^2}{\sin_\phi^2} + h_\phi^2}\ \cos(\omega) = h^2\cos(\theta)\sin(\phi) - hh_\theta + hh_\phi\cos(\theta)\cos(\phi)$$

Note that :

$$g(x/r) = \frac{1}{\sqrt{h^2(u/\rho) + \left|\nabla_{u||}h(u/\rho)\right|^2}}$$

Then:

$$\omega = \cos^{-1}\left[\frac{h^2\cos(\theta)\sin(\phi) - hh_\theta + hh_\phi\cos(\theta)\cos(\phi)}{g}\right]$$

Differentiating ω with respect to θ yields the following equation:

$$\frac{d\omega}{d\theta} = \frac{\left(\sqrt{h^2 + \frac{h_\theta^2}{\sin(\theta)^2} + h_\phi^2}\right)(2hh_\theta\cos(\theta)\sin(\phi) - h^2\sin(\theta)\sin(\phi) - hh_{\theta\theta} - (h_\theta)^2)}{\sqrt{g(x/r)^2 - (h^2\cos(\theta)\sin(\phi) - hh_\theta + hh_\phi\cos(\theta)\cos(\phi))^2}}$$

$$+ \frac{\left(\sqrt{h^2 + \frac{h_\theta^2}{\sin(\theta)^2} + h_\phi^2}\right)(h_\theta h_\phi\cos(\theta)\cos(\phi) - hh_\phi\sin(\theta)\cos(\phi))}{\sqrt{g(x/r)^2 - (h^2\cos(\theta)\sin(\phi) - hh_\theta + hh_\phi\cos(\theta)\cos(\phi))^2}}$$

$$+ \frac{-(h^2\cos(\theta)\sin(\phi))\left(\sqrt{h^2 + \frac{h_\theta^2}{\sin(\theta)^2} + h_\phi^2}\right)(2hh_\theta + \frac{2h_\theta h_{\theta\theta}}{\sin(\theta)^2} + 2h_\phi h_\theta)}{\sqrt{g(x/r)^2 - (h^2\cos(\theta) sin(\phi) - hh_\theta + hh_\phi\cos(\theta)\cos(\phi))^2}}$$

$$+ \frac{-(hh_\theta)\left(\sqrt{h^2 + \frac{h_\theta^2}{\sin(\theta)^2} + h_\phi^2}\right)(2hh_\theta + \frac{2h_\theta h_{\theta\theta}}{sin(\theta)^2} + 2h_\phi h_\theta)}{\sqrt{g(x/r)^2 - (h^2\cos(\theta)\sin(\phi) - hh_\theta + hh_\phi\cos(\theta)\cos(\phi))^2}}$$

$$+ \frac{((hh_\phi\cos(\theta)\cos(\phi))\left(\sqrt{h^2 + \frac{h_\theta^2}{\sin(\theta)^2} + h_\phi^2}\right)(2hh_\theta + \frac{2h_\theta h_{\theta\theta}}{\sin(\theta)^2} + 2h_\phi h_\theta)}{\sqrt{g(x/r)^2 - (h^2\cos(\theta)\sin(\phi) - hh_\theta + hh_\phi\cos(\theta)\cos(\phi))^2}}$$

We can do the same analysis with τ, the angle between x and the x_2 axis.

$$
\begin{aligned}
|x(\mathbf{u})|cos(\tau) &= x(\mathbf{u}) \cdot \hat{\mathbf{e}}_{\mathbf{2}} \\
&= h^2(\hat{\mathbf{e}}_\rho \cdot \hat{\mathbf{e}}_{\mathbf{2}}) + \frac{hh_\theta}{\sin(\phi)}(\hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_{\mathbf{2}}) + hh_\phi(\hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_{\mathbf{2}}) \\
&= h^2 \sin(\theta) \sin(\phi) + \frac{hh_\theta}{sin(\phi)} \cos(\theta) + hh_\phi \sin(\theta) \cos(\phi)
\end{aligned}
$$

Thus:

$$
\sqrt{h^2 + \frac{h_\theta^2}{\sin(\phi)^2} + h_\phi^2} \cos(\tau) = h^2 \sin(\theta) sin(\phi) + \frac{hh_\theta}{\sin(\phi)} \cos(\theta) + hh_\phi \sin(\theta) \cos(\phi)
$$

Then:

$$
\tau = \cos^{-1}\left(\frac{h^2 \sin(\theta) sin(\phi) + \frac{hh_\theta}{\sin(\phi)} \cos(\theta) + hh_\phi \sin(\theta) \cos(\phi)}{g}\right)
$$

Differentiating τ with respect to ϕ yields the following equation:

$$\frac{d\tau}{d\phi} = \frac{\left(\sqrt{h^2 + \frac{h_\theta^2}{sin(\theta)^2} + h_\phi^2}\right)\left(\frac{\sin_\phi(h_\phi h_\theta + hh_{\theta\phi} - hh_\theta\cos(\phi))}{(\sin(\phi)^2)}\right)\cos(\theta))}{\sqrt{g(x/r)^2 - (h^2\sin(\theta)\sin(\phi) + \frac{hh_\theta}{\sin(\phi)}\cos(\theta) + hh_\phi\sin\theta\cos(\phi))^2}}$$

$$+ \frac{\left(\sqrt{h^2 + \frac{h_\theta^2}{sin(\theta)^2} + h_\phi^2}\right)(h_\phi)^2\sin(\theta)\cos(\phi) + hh_{\phi\phi}\sin(\theta)\cos(\phi)}{\sqrt{g(x/r)^2 - (h^2\sin(\theta)\sin(\phi) + \frac{hh_\theta}{\sin(\phi)}\cos(\theta) + hh_\phi\sin\theta\cos(\phi))^2}}$$

$$- \frac{hh_\phi\sin(\theta)\sin(\phi)}{\sqrt{g(x/r)^2 - (h^2\sin(\theta)\sin(\phi) + \frac{hh_\theta}{\sin(\phi)}\cos(\theta) + hh_\phi\sin\theta\cos(\phi))^2}}$$

$$- \frac{(h^2\sin(\theta)\sin(\phi))(h^2 + \frac{h_\theta^2}{(\sin(\theta)^2} + h_\phi^2)^{-1/2}(2hh_\theta + \frac{2h_\theta h_{\theta\phi}}{\sin(\theta)^2} - 2h_\phi h_{\phi\phi})}{\sqrt{g(x/r)^2 - (h^2\sin(\theta)\sin(\phi) + \frac{hh_\theta}{\sin(\phi)}\cos(\theta) + hh_\phi\sin\theta\cos(\phi))^2}}$$

$$- \frac{\frac{hh_\theta}{sin(\phi)}cos(\theta)(h^2 + \frac{h_\theta^2}{(\sin(\theta)^2)} + h_\phi^2)^{-1/2}(2hh_\theta + \frac{2h_\theta h_{\theta\phi}}{sin(\theta)^2} - 2h_\phi h_{\phi\phi})}{\sqrt{g(x/r)^2 - (h^2\sin(\theta)\sin(\phi) + \frac{hh_\theta}{\sin(\phi)}\cos(\theta) + hh_\phi\sin\theta\cos(\phi))^2}}$$

$$- \frac{hh_\phi\sin(\theta)\cos(\phi)(h^2 + \frac{h_\theta^2}{(\sin(\theta)^2} + h_\phi^2)^{-1/2}(2hh_\theta + \frac{2h_\theta h_{\theta\phi}}{\sin(\theta)^2} - 2h_\phi h_{\phi\phi})}{\sqrt{g(x/r)^2 - (h^2\sin(\theta)\sin(\phi) + \frac{hh_\theta}{\sin(\phi)}\cos(\theta) + hh_\phi\sin\theta\cos(\phi))^2}}$$

The Weingarten Equation in three dimensions, using spherical coordinates, is:

$$R_1(\theta,\phi) + R_2(\theta,\phi) = \frac{1}{\sin(\theta)}\,\frac{\partial}{\partial\theta}\left(h_\theta \sin(\theta)\right) + h_{\phi\phi} + 2h$$

Suppose that we now form the sum $d\omega + d\tau$. It is evident that the key components of the Weingarten Equation, $h_\theta, h_{\theta\theta}$, and $h_{\phi\phi}$ all appear in this sum. But without additional information there is no immediately apparent way to isolate $R_1 + R_2$ in the sum.

However, it is possible to determine from equations (45) and (61) that in three dimensions

(62)
$$g\left(\frac{\mathbf{x}}{r}\right) = \frac{1}{\sqrt{(h(\mathbf{u}/\rho))^2 + |\nabla_\| h(\mathbf{u}/\rho)|^2}},$$

$$= \frac{1}{\sqrt{(h(\mathbf{u}/\rho))^2 + (h_\theta(\mathbf{u}/\rho))^2 + \frac{(h_\phi(\mathbf{u}/\rho))^2}{\sin^2\theta}}},$$

$$= \frac{1}{h(\mathbf{u}/\rho)\sqrt{1 + \left(\frac{h_\theta(\mathbf{u}/\rho)}{h(\mathbf{u}/\rho)}\right)^2 + \left(\frac{h_\phi(\mathbf{u}/\rho)}{\sin(\theta)h(\mathbf{u}/\rho)}\right)^2}}$$

so that the relationship between the gauge function and the support function of a convex body depends upon the logarithmic derivatives of the support function. Therefore, in any dimension the distortion induced on the polar dual of a convex body by a geometric flow of the original body is entirely determined by the logarithmic derivatives of the support function of the original body.

CHAPTER 5

Mahler's Conjecture

5.1. **The conjecture**

Mahler's Conjecture dates back to the work of Kurt Mahler (1903-1988), who worked primarily in number theory. There were, in fact, several conjectures by Mahler outside of geometry (some of which have been proven). These are also referred to as conjectures of Mahler from time to time. The open conjecture we are exploring here dates back to a convex geometry paper written in 1939 [21].

The usual statement of Mahler's Conjecture is based on the notions of a convex body, its polar dual, and the volumes of each. In its simplest form, Mahler's Conjecture states that product of the volume of a convex body and the volume of its polar dual (which is called the Mahler Volume, or alternatively the volume product) has a definitive lower bound [27], [28]. This is known to be true in two dimensions. It is conjectured that, if A is a convex body in $\mathbb{R}^n$ for $n \geq 3$,

$$\mathrm{VolumeProduct} = \mathrm{VP(A)} = \mathrm{V(A)V(A^*)} \geq \frac{4^{\mathrm{n}}}{\Gamma(\mathrm{n}+1)}. \tag{63}$$

An important property of the volume product is that it is invariant under affine transformations of a convex body.

There is also an upper bound on the volume product, which is known to correspond to the product of the volume of the n-sphere and the volume of its polar dual (which is also an n-sphere). The specific upper bound is:

$$VP(A) = V(A)V(A^*) \leq \frac{\Gamma(3/2)^{2n}4^n}{\Gamma(n/2+1)^2}. \tag{64}$$

This inequality is call the Blaschke - Santalò inequality. The specific statement of the theorem is that if A is a convex body, then its volume product is described by $V(A)V(A^*) \leq \omega_n^2$, where ω_n is the volume of the n-sphere, with equality occurring when A is an ellipsoid [18], is called the Blaschke - Santalò Theorem. Hence, the inverse problem to Mahler's Conjecture is completely understood.

The conjectured lower bound for the volume product, $\frac{4^n}{\Gamma(n+1)}$, corresponds to the n-cube (a n-dimensional convex body with faces having four edges each), the n-octagon (an n-dimensional convex body with faces having eight edges each) , and their polar duals (when $n = 3$, the polar dual of the 3-cube is the 3-octagon).

If we take the Blaschke-Santalò Theorem and Mahler's Conjecture together, they state that the volume product of any convex body lies between the volume product of the n-cube and the volume product of the n-sphere. There are two cases of the conjecture, both of which are open.

The first is the general case, where A is any convex body. This is the case described by Equation (63). The second is the special case where A is a centrally symmetric convex body (a convex body is called centrally symmetric if there is some vector by which A can be translated so that $-t \in A$ when t is in A, [8]). In theory, the centrally symmetric case should be easier to solve than the general case. In the centrally symmetric case Mahler's Conjecture has the following form, and many of the results known to date partially prove this version of the conjecture.

$$VP(A) := V(A)V(A^*) \geq \frac{4^n}{n!}. \tag{65}$$

5.2. **The intuitive geometric interpretation**

How, then, to geometrically interpret Mahler's conjecture? There are at least two different interpretations. One is a classic interpretation that shows how the volume product actually is a volume. The second is a novel approach that assigns a intuitive geometric meaning to the volume product that is completely unrelated to volume. Both of these are basic, intuitive interpretations rather than rigorous statements about Mahler's conjecture. Their purpose is to give a visual sense of what the conjecture is about, and they do not motivate the more rigorous discussion that follows. Accordingly, our discussion of these intuitive interpretations is intentionally brief.

Volume Approach. An argument by Thompson [28] outlines the classical interpretation. If $A \in \mathbb{R}^n$, then we can consider the polar dual A^* to be in a space which we call $\mathbb{R}^{n*}$. it is possible to create a new space from the (Cartesian) product $\mathbb{R}^n \times \mathbb{R}^{n*}$. In this case the convex body $A^n \times A^{n*} \in \mathbb{R}^n \times \mathbb{R}^{n*}$ has volume $VP(A)$. The conjecture is that this volume is bounded below, as in equation (63).

Pointedness Approach. Tao [27] notes that if Mahler's Conjecture is true, the volume product is essentially a measure of the degree of pointedness of a convex body, where the pointedness refers to the presence of sharp corners on the body. In this context, the unit n-sphere, with the highest volume product, is the least pointed, and the unit n-cubes and n-octagons, with the lowest volume product, are most pointed.

Why is Mahler's Conjecture important?

- It has been an open problem for a long time. Mahler's Conjecture is easy to state, but it has defied resolution for about 68 years. The resolution of problems of this type can sometimes reveal new mathematical directions and insight.

- It could establish a new metric. If the conjecture is true, then the volume product could be used as a new measure of the degree to which convex bodies have sharp corners.

- The inverse problem is important. The Blaschke-Santalo problem has proven to be of high importance in other areas of mathematics, most notably affine geometry. Mahler's Conjecture may similarly turn out to be of interest in other fields of geometry.

- It has a relationship to wavelets. Keith Ball has shown [4] that one case of Mahler's Conjecture is equivalent to a problem regarding the scaling equation in the theory of wavelets.

- It has potential implications for the theory of duality, and hence for optimization. Polar duality has a close relationship to linear optimization, and any resolution of the conjecture that sheds new light on duality may have relevance to optimization.

5.3. **Partial results known to date**

There are two categories of partial results on Mahler's Conjecture. In the first category, there are cases in which the conjecture has been proven, corresponding to specific known convex bodies. An example of this is the n-sphere – its volume product is known (by the Blaschke - Santalò inequality), and it is larger than that of the n-cube. So we can say that the n-sphere satisfies Mahler's Conjecture. In fact, for any convex body or set of bodies for which the volume product is known, it has been possible to establish a partial Mahler's Conjecture result in this category. A counteraxample to Mahler's Conjecture would be a specific convex body with a volume product smaller than that of the n-cube. The challenge of this category of results is to establish the conjecture for increasingly larger sets of convex bodies, until no convex body is excluded.

In the second category of partial results, there are theorems that provide lower bound estimates which hold for all convex bodies, but which are lower than the volume of the n-cube. One basic result of this type is the claim that the volume product is greater than or equal to zero.

Since the volumes of all convex bodies are greater than zero, their volume products are greater than zero, and the statement is true for all convex bodies. But it clearly is not as strong a statement as the one claiming that the volume product (of any convex body) is greater than or equal to the volume of the n-cube. Hence, our basic result is only a partial Mahler's Conjecture result. The challenge with this category of results is to find better partial results (i.e., ones with volume products above zero, and as close as possible to that of the n-cube).

Results in this category are useful in that they do apply to all convex bodies, but they are vacuous when applied to any specific convex body whose volume product is already known. For example, if we take our basic result and apply it to the n-cube, we get the statement that the volume product of the n-cube is greater than zero. But this was already known by direct computation, and the basic result has provided no new information.

The interesting point about Mahler's Conjecture is the wide variety of mathematical tools that have been used to investigate it, ranging from elementary methods to highly sophisticated tools. It is worth investigating some of the methods that have been used to attack the problem.

Two of the known results are particularly important in terms of the mathematical development, because they represent the best known results for each of the two types discussed earlier.

5.3.1. *Result 1: Zonoids*

In 1988 Gordon, Meyer and Reisner [10] proved that Mahler's Conjecture holds for a class of convex bodies known as zonoids. This proof was preceded by one of Reisner, in 1986, which used more complicated methods. Historically, Reisner's 1986 proof was the first to establish that zonoids satisfy Mahler's Conjecture, but the 1988 proof establishes the result using only classical analysis and the geometry of zonoids.

The zonoid result is important, because it establishes that there is a circumstance in which Mahler's Conjecture is true in its entirety, rather than just up to a factor depending upon n. Closely following [10], we begin with some definitions.

A zonotope is a convex body in $\mathbb{R}^n$ which is made up of a finite Minkowski sum of line segments. A zonoid is a symmetric, convex body which is a limit (in the Haussdorff metric) of zonotopes. The intuitive way to think of a zonoid, then, is as a symmetric convex body that can be derived from a sequence of bodies made up of line segments. Because zonoids are symmetric, the version of Mahler's conjecture that is relevant here is the symmetric version, in Equation (65).

Let A be convex body in $\mathbb{R}^n$, and let A^* be its polar dual, with norm $\|\|_{A^*}$, so that if

$$A^* = \{\mathbf{y} \in \mathbb{R}^n : |\langle \mathbf{x}, \mathbf{y} \rangle| \leq 1, \text{ for all } \mathbf{x} \in \mathrm{A}\},$$

then

$$|\langle \mathbf{y} \rangle|_A^* = \max\{\langle \mathbf{x}, \mathbf{y} \rangle : \mathbf{x} \in A\}.$$

Next, we define a hyperplane and the projection of the convex body onto it. Choose a vector $\mathbf{x}$ on the $n-1$ dimensional unit sphere. Let $H(\mathbf{x})$ be a hyperplane through the origin, orthogonal to $\mathbf{x}$. $P_{\mathbf{x}}^{\perp}$ will denote the orthogonal projection onto $H(\mathbf{x})$. If we let $A(\mathbf{x}) = A \cap H(\mathbf{x})$, then we see that

$$A(\mathbf{x}) \subset P_{\mathbf{x}^\perp} = \{\mathbf{z} \in H(\mathbf{x}) : \mathbf{z} + \lambda \mathbf{x} \in A, \text{ for some } \lambda \in \mathbb{R}\}.$$

Furthermore, if $\mathbf{x}$ is a point on the $n-1$ dimensional unit sphere, S^{n-1}, then it can be shown that $\left(P_{\mathbf{x}}^{\perp}\right)^{*} = A^{*}(\mathbf{x})$.

Let μ represent the unique positive even Borel measure on S^{n-1}.

Mathematically, there are two key properties of a zonoid [10]. First, if A is a zonoid, then:

$$A = \frac{1}{2}\int_{S^{n-1}} \{\alpha\mathbf{x} : -1 \geq \mathbf{x} \leq 1\}\, d\mu(\mathbf{x})$$

where the integration is over discrete measures, made up of Minkowski sums of segments in $\mathbb{R}^n$, which approximate μ.

Second, the norm associated with the polar dual of the zonoid, $\|\|_{A^*}$ has the form [10]:

$$\|\mathbf{y}\|_{A^*} := \frac{1}{2}\int_{S^{n-1}} |\mathbf{x}\cdot\mathbf{y}|d\mu(\mathbf{x}).$$

With these definitions, we can establish two lemmas.

Lemma 5.3.1 [10]
If A is a zonoid in $\mathbb{R}^n$ and its supporting positive Borel measure is μ, then there is a point $x_0 \in \mathbb{S}_{n-1}$ for which the following volume inequality holds:

$$(n+1)V(A)\int_{A^*} \|\langle x_0, y_0\rangle\| d\mathbf{y} \geq 2V(A^*)V(P_{x_0})^{\perp}A. \tag{66}$$

Proof:

This result is a special case of the more general volume equation:

$$(n+1)V(A)\int_{\mathbb{S}^{n-1}}\left[\int_{A^*}|\mathbf{x}\cdot\mathbf{y}|d\mathbf{y}\right]d\mu(\mathbf{x}) = 2V(A^*)\int_{\mathbb{S}^{n-1}}V((P_{x_0})^{\perp})d\mu(\mathbf{x}).$$

This equation follows from applying Fubini's Theorem to the integral on the left hand side, to get:

$$\int_{\mathbb{S}^{n-1}}\left[\int_{A^*}|\mathbf{x}\cdot\mathbf{y}|d\mathbf{y}\right]d\mu(\mathbf{x}) = \frac{2n}{n+1}2V(A^*),$$

and then noting that, as a result of the volume equation for a zonoid,

$$V(A) = \frac{1}{n}\int_{\mathbb{S}^{n-1}}V(P_{x_0})^{\perp}d\mu(\mathbf{x}).$$

Lemma 5.3.2 [10]

For a symmetric convex body B in $\mathbb{R}^n$,

(67) $$\int_B|\mathbf{x}\cdot\mathbf{y}|d\mathbf{y} \leq \frac{n}{2(n+1)}\frac{V(B)^2}{V(B(\mathbf{x}))}.$$

Theorem 5.3.1 [10]

If A is a zonoid in $\mathbb{R}^n$, then $VP(A) \geq \frac{4^n}{n!}$.

Proof:

The proof theorem proceeds by induction on n. It is easily seen to be true for $n = 1$. Then if A is a zonoid in $\mathbb{R}^n$, equations (66) and (67) show that

$$(68) \qquad 2V(P_{x_0})^{\perp}v(A^*) \leq (n+1)V(A)\int_{A^*} |x_0 \cdot y_0| dy \leq \frac{nV(A)V(A^*)}{2V(A^*(x_0))}$$

Earlier we saw that $(P_{\mathbf{x}^{\perp}})^* = A$, so $(A^*(x_0))^*$ is the same as $P_{x_0}^{\perp}A$. Equation (68) then establishes that $VP(A) \geq (4/n)VP(P_{x_0}^{\perp}A)$. As $P_{x_0}^{\perp}A$ is a zonoid in $\mathbb{R}^{(n-1)}$, we can use induction on n to find that

$$VP(P_{x_0}^{\perp}A) \geq \frac{4^{n-1}}{(n-1)!}.$$

From this it is evident that $VP(A) \geq \frac{4^n}{n!}$, which establishes the main theorem.

□

5.3.2. *Result 2: True up to a factor of* 2^{-n}

The best result obtained so far for all convex bodies is one by Kuperberg [15], stating that for all symmetric convex bodies A, with $\mathbb{B}^n$ designating the n-dimensional Euclidean ball,

$$VP(A) \geq 2^{-n} V(B_n).$$

The proof of this is detailed, and here will present just a summary of the methodology in [15]. We start by defining the subsets

$$K^{\pm} = \{(\mathbf{x}, \mathbf{y}) \in K \times K^{*} : \mathbf{x} \cdot \mathbf{y} = \pm 1\}$$

of the hyperboloids

$$H^{\pm} = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \cdot \mathbf{y} = \pm 1\}.$$

Next, we define the filled join of two sets, following Kuperberg [15]. If A and B are two sets in $\mathbb{R}^n$, then their geometric join is the union of the line segments that connect any two points in A and B. If the geometric join of A and B is a closed manifold of codimension 1, then the compact region of $\mathbb{R}^n$ that it encloses is known as the filled join between A and B.

Let $K^{\triangle}$ designate the filled join of the subsets (of hyperboloids) K^+ and K^-.

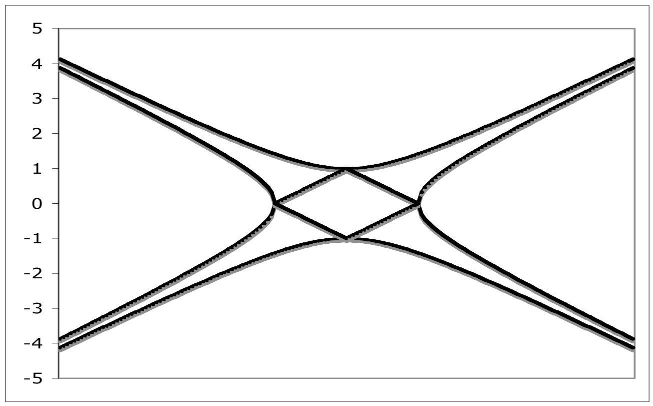

Figure 10. Filled join, nested in between hyperbola.

In the illustration, the filled join is represented by the diamond shaped box in the middle of the diagram. The volume of $K^{\triangle}$ can be expressed as:

$$V(K^{\triangle}) = \frac{(a-1)!(b-1)!}{(a+b)!} \int_{(\mathbf{x},\mathbf{y}) \in K^{+} \times K^{-}} \mathbf{x} \wedge \mathbf{y} d\mathbf{x}^{\wedge a-1} \wedge d\mathbf{y}^{\wedge b-1}$$

The lower bound for this volume can be shown to be proportional to a form of Gauss linking integral (an integral over two possibly knotted curves in $\mathbb{R}^3$). This can be used to find the minimum volume bound for $K^{\triangle}$. The volume product can then be compared to the volume of $K^{\triangle}$ to establish the result.

5.3.3. *A table of known results*

The table below summarizes a number of the known results regarding Mahler's Conjecture. Some refer to all convex bodies, while others refer to specific subsets of the set of all convex bodies. The results have generally been derived using very different methods. The extensive web log by Tao [27] has been used in compiling this table.

Table 3. Known Mahler's Conjecture Results

Result VP $\geq$	Due To	Convex Bodies	Date	Reference
$\frac{4^n}{\Gamma(n+1)}$	Mahler	All, n = 2, incomplete	1939	[21]
$\frac{n^{-n/2}\Gamma\left(\frac{3}{2}\right)^{2n}4^n}{\Gamma(n/2+1)}$	John	All	1948	[13]
$\frac{4}{n!}$	Reisner	Zonoids	1986	[24]
$\frac{4^n}{\Gamma(n+1)}$	Saint-Raymond	Some convex bodies	1987	[25]
$\frac{C^{-n}\Gamma(\frac{3}{2})^{2n}4^n}{\Gamma(n/2+1)}$	Bourgain, Milman	All convex bodies	1987	[5]
$\frac{4}{n!}$	Gordon, Meyer, Reisner	Zonoids	1988	[10]
$\frac{4^n}{\Gamma(n+1)}$	Mayer	All, n = 2, Equality case	1991	[22]
$\frac{2^n\Gamma\left(\frac{3}{2}\right)^{2n}4^n}{\binom{2n}{n}\Gamma(n/2+1)}$	Kuperberg	All	1992	[14]
$\frac{4^n}{\Gamma(n+1)}$	Lopez, Reisner	Certain Polytopes	1998	[20]
$\left(\frac{2^n(n!)^2}{(2n)!}\right)\frac{\Gamma\left(\frac{3}{2}\right)^{2n}4^n}{\Gamma(n/2+1)}$	Kuperberg	All	2006	[15]

CHAPTER 6

Duality and Mahler's Conjecture

The curvature and the support function of a convex body lead naturally to an expression for its volume. We are interested in this in order to compute a form of the volume product that may give clues to a lower bound. Using the Divergence Theorem, we get the following derivation for the volume of a convex body:

$$\mathrm{Vol(A)} = \int_{\mathrm{R}} 1 \cdot \mathrm{d^n}\mathbf{x} = \int_{\mathrm{R}} \frac{\nabla(\mathbf{x})}{\mathrm{h}} \mathrm{d^n}\mathbf{x} = \frac{1}{\mathrm{n}} \int_{\mathrm{R}} \mathbf{x} \cdot \hat{\mathrm{n}} \mathrm{dS} \tag{69}$$

Next we change variables and integrate over the Gauss sphere to get:

$$\mathrm{Vol(A)} = \frac{1}{\mathrm{n}} \int_{\mathrm{S^{n-1}}} \mathrm{h}(\mathbf{u}) \mathrm{d}\omega,$$

where, by an equation of Gauss, $d\omega = \prod_L kds$

Then,

$$\mathrm{Vol(A)} = \frac{1}{\mathrm{n}} \int_{\mathrm{S^{n-1}}} \mathrm{h}(\mathbf{u}) \prod \mathrm{R_L}(\mathbf{u}) \mathrm{d}\omega \tag{70}$$

In the case where $n = 2$ we get:

$$\mathrm{Vol(A)} = \frac{1}{2}\int_0^{2\pi}(\mathrm{h}(\theta)\mathrm{R}(\theta))\mathrm{d}\theta \tag{71}$$

$$= \frac{1}{2}\int_0^{2\pi}(h^2 + hh_{\theta\theta})d\theta$$

For the volume of the polar dual we recall the equations (52) and (55), i.e.

$$d\phi = hRg^2 d\theta$$

$$R^*(\phi) = \frac{1}{(hg)^3 R}$$

to obtain:

$$\mathrm{Vol(A^*)} = \frac{1}{2}\int_0^{2\pi}\mathrm{h}^*(\phi)\mathrm{R}^*(\phi)\mathrm{d}\phi \tag{72}$$

$$= \frac{1}{2}\int_0^{2\pi} g(\phi)R^*(\phi)d\phi$$

$$= \frac{1}{2}\int_0^{2\pi} hR^*Rg^3 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{h^2}d\theta$$

Multiplying the two volumes together to get the volume product yields:

$$V(A)V(A^*) = \left[\frac{1}{2}\int_0^{2\pi}(h^2 + hh_{\theta\theta})d\theta\right]\left[\frac{1}{2}\int_0^{2\pi}\frac{1}{h^2Rg}d\theta\right], \tag{73}$$

or alternatively,

$$V(A)V(A^*) = \left[\frac{1}{2}\int_0^{2\pi} hRd\theta\right]\left[\frac{1}{2}\int_0^{2\pi}\frac{1}{h^2}d\theta\right], \tag{74}$$

By the Cauchy-Schwarz inequality, then,

$$V(A)V(A^*) \geq \frac{1}{4}\left(\int_0^{2\pi} R^{1/2}h^{-1/2}(\theta)d\theta\right)^2. \tag{75}$$

Equivalently, we can write this inequality as:

(76) $$V(A)V(A^*) \geq \frac{1}{4}\left(\int_0^{2\pi} \sqrt{1 + \left(\frac{h_{\theta\theta}}{h}\right)}\ d\theta\right)^2.$$

The second form shows the dependance of the volume product on the logarithmic (second) derivative of the support function, $\frac{h_{\theta\theta}}{h}$, which is just the derivative of the logarithmic derivative $\frac{h_\theta}{h}$ encountered earlier.

6.1. **Extending beyond two dimensions**

In two dimensions Mahler's Conjecture is already known to be true. However, the volume equations derived above provide insight that may be extendible to higher dimensions. As with geometric flows, the logarithmic derivatives of the support function are important in two dimensions. The convex bodies with the larger logarithmic derivatives of support functions are, for a given R, the ones with the smallest volume products. Those with the smaller logarithmic derivatives have correspondingly larger volume products. Some of these ideas extend to three and higher dimensions as well. In the Conclusion we present some equations that can be of use in investigating higher dimensional versions of Mahler's Conjecture.

CHAPTER 7

Conclusion

At least in the two dimensional case, it appears as though duality can add to the understanding of geometric flows, and of Mahler's conjecture, with the logarithmic second derivative $\frac{h_{\theta\theta}}{h}$ playing an important role. In the case of geometric flows we saw some basic results in three dimensions. A question that arises is whether these results extend to higher dimensions.

To address the first question, note that Equation (43) was formulated in such a way to be applicable to all dimensions. Calculating the parallel gradient of this equation gives:

$$\begin{aligned}\nabla_{\mathbf{x}||} g\left(\frac{\mathbf{x}}{r}\right) &= -\frac{1}{2}\left(h^2\left(\frac{\mathbf{u}}{\rho}\right) + \left|\nabla_{\mathbf{u}||} h\left(\frac{\mathbf{u}}{\rho}\right)\right|^2\right)^{-3/2} \\ &\cdot \nabla_{\mathbf{x}||}\left(h^2\left(\frac{\mathbf{u}}{\rho}\right) + \left|\nabla_{\mathbf{u}||} h\left(\frac{\mathbf{u}}{\rho}\right)\right|^2\right) \\ &= -\frac{1}{2}\left(g\left(\frac{\mathbf{x}}{r}\right)\right)^3 J\nabla_{\mathbf{u}||}\left(h^2\left(\frac{\mathbf{u}}{\rho}\right) + \left|\nabla_{\mathbf{u}||} h\left(\frac{\mathbf{u}}{\rho}\right)^2\right|\right),\end{aligned}$$

where J is the Jacobi matrix of the transformation

$$J = \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) |_{\text{parallel components}}$$

Therefore,

$$\nabla_{\mathbf{x}||} g \left(\frac{\mathbf{x}}{r} \right) = \frac{\left(hJ\nabla_{\mathbf{u}||} h + J\nabla_{\mathbf{u}||} h (\nabla_{\mathbf{u}||} h)^2 \right)}{g^3 \left(\frac{\mathbf{x}}{r} \right)},$$

and

$$\frac{1}{4} \nabla_{\mathbf{x}||} g^4 \left(\frac{\mathbf{x}}{r} \right) = - \left(\sum_j R_j - (n - 2/h) \right) J\nabla_{\mathbf{u}||} h \left(\frac{\mathbf{u}}{\rho} \right)$$

Applying this equation to the dual body and using polar duality:

$$\frac{1}{4} \nabla_{\mathbf{u}||} h^4 \left(\frac{\mathbf{u}}{\rho} \right) = - \left(\sum_j R_j^* - (n - 2/g) \right) J\nabla_{\mathbf{x}||} g \left(\frac{\mathbf{x}}{r} \right)$$

Solving for n:

$$\begin{aligned} n &= \left[\frac{1}{4} \nabla_{\mathbf{x}||} g^4 \left(\frac{\mathbf{x}}{r} \right) \right] \left[J\nabla_{\mathbf{u}||} h \left(\frac{\mathbf{u}}{\rho} \right) \right]^{-1} + \sum_j R_j + \frac{2}{h} \\ &= \left[\frac{1}{4} \nabla_{\mathbf{u}||} h^4 \left(\frac{\mathbf{u}}{\rho} \right) \right] \left[J\nabla_{\mathbf{x}||} g \left(\frac{\mathbf{x}}{r} \right) \right]^{-1} + \sum_j R_j^* + \frac{2}{g} \end{aligned}$$

Finally, it is possible to obtain expressions for the sums of the curvature tensor components for the initial convex body and its polar dual:

$$
\begin{aligned}
\sum_j R_j &= -\left[\frac{1}{4}\nabla_{\mathbf{x}||} g^4\left(\frac{\mathbf{x}}{r}\right)\right]\left[J\nabla_{\mathbf{u}||} h\left(\frac{\mathbf{u}}{\rho}\right)\right]^{-1} \\
&+ \left[\frac{1}{4}\nabla_{\mathbf{u}||} h^4\left(\frac{\mathbf{u}}{\rho}\right)\right]\left[J\nabla_{\mathbf{x}||} g\left(\frac{\mathbf{x}}{r}\right)\right]^{-1} \\
&+ 2\left(\frac{1}{g}-\frac{1}{h}\right)+\sum_j R_j^*
\end{aligned}
$$

$$
\begin{aligned}
\sum_j R_j^* &= -\left[\frac{1}{4}\nabla_{\mathbf{u}||} h^4\left(\frac{\mathbf{u}}{\rho}\right)\right]\left[J\nabla_{\mathbf{x}||} h\left(\frac{\mathbf{x}}{r}\right)\right]^{-1} \\
&+ \left[\frac{1}{4}\nabla_{\mathbf{x}||} g^4\left(\frac{\mathbf{x}}{r}\right)\right]\left[J\nabla_{\mathbf{u}||} h\left(\frac{\mathbf{u}}{\rho}\right)\right]^{-1} \\
&+ 2\left(\frac{1}{h}-\frac{1}{g}\right)+\sum_j R_j
\end{aligned}
$$

Here it is evident that the curvature equations depend on the curvature flow of the dual body, and on various derivatives of support and gauge functions, including the parallel component connection on fourth powers of the support and gauge functions of the original body, and the inverse of the Jacobian map of the connection on the support and gauge functions. As a result, these derivatives figure prominently in higher dimensional geometric flows and in the higher dimension variant of Mahler's Conjecture.

In looking at possible directions for future work it is natural to ask whether there is any connection between geometric flows and Mahler's Conjecture. In [2] Ben Andrews an affine normal geometric flow to re-prove the Blaschke-Santalò Theorem, which is essentially the inverse problem to Mahler's Conjecture. There are other examples in which geometric flow have been used to establish geometric inequalities; see [29], [30] and [26].

However, Mahler's Conjecture appears to be a fundamentally more difficult problem, and no extension of previous work on flows and geometric inequalities to the problem is apparent. Our work here shows that the logarithmic derivative $\frac{h_\theta}{h}$ is important in both geometric flows and Mahler's Conjecture, although in different ways and in different forms. In dimensions greater than 2, other derivatives of support and gauge functions also figure prominently. These quantities may lie at the root of a deep connection between geometric flows and Mahler's Conjecture.

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School of Mathematics,, Georgia Institute of Technology., Atlanta, GA 30332-0161,