

**COHOMOLOGY AND K -THEORY
OF APERIODIC TILINGS**

A Thesis
Presented to
The Academic Faculty

by

Jean P.X. Savinien

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in
Mathematics

School of Mathematics
Georgia Institute of Technology
August 2008

COHOMOLOGY AND K -THEORY OF APERIODIC TILINGS

Approved by:

Professor Jean Bellissard, Advisor
School of Mathematics
Georgia Institute of Technology
Atlanta GA

Professor Claude Schochet
Department of Mathematics
Wayne State University
Detroit MI

Professor Thang Le
School of Mathematics
Georgia Institute of Technology
Atlanta GA

Professor Stavros Garoufalidis
School of Mathematics
Georgia Institute of Technology
Atlanta GA

Professor Michael Loss
School of Mathematics
Georgia Institute of Technology
Atlanta GA

Date Approved: April 7 2008

Meinem Kuckuck

Du rufst mich aus dem Wald

Du kümmerst Dich um meinen Vogel

H D L

ACKNOWLEDGEMENTS

I was invited and funded by the Erwin Schrödinger Institute (Vienna, Austria) in the summer 2006 and by the Institut Henri Poincaré (Paris, France) in the spring 2007. It is a pleasure for me to thank those institutes.

I acknowledge financial support from my advisor, Jean Bellissard, who hired me as a Graduate Research Assistant on his NSF grants no. DMS-0300398 and no. DMS-0600956.

I acknowledge financial support from the *Centre de Physique Théorique* and the *Institut de Mathématiques de Luminy* (Marseille, France) in 2003. I would like to thank Pierre Fayet and Christoph Schweigert for helping me get a grant to start my PhD in France, and also Thomas Schucker, Anthony Wasserman, and Michael Puschnigg for helping me try to set a joint PhD program with Georgia Tech.

I would like to thank John Hunton and Claude Schochet for insightful discussions and very useful comments on my work. Also many thanks to Michael Loss, Thang Le and Stavros Garoufalidis for accepting to be in my committee.

I would like to thank Barbara Gentz-Bellissard for allowing me to use the facilities of the Humboldt University guesthouse for two visits to Berlin in the summer 2006.

I would like to express my warmest gratitude to my advisor Jean for carrying me in his luggage overseas, for his friendship, and his enthusiastic scientific support.

I finally would like to thank my roommates Torsten and Jean-Philippe for the good time we had together during these PhD years. Also many thanks to my good friends, in particular Alessandro my blues and limoncello buddy, John my NCG fellow, Thierry for introducing me to Eva (!), Nathan my math and meditation coach, Philippe for my “killer thanks” slide, and last but not least my Harley brother Bill!

TABLE OF CONTENTS

	DEDICATION	iii
	ACKNOWLEDGEMENTS	iv
	LIST OF FIGURES	vii
	SUMMARY	viii
I	PRELIMINARIES ON THE THEORY OF APERIODIC TILINGS	1
	1.1 Tilings and their hulls	1
	1.2 Delone sets and groupoid of the transversal	5
	1.3 Prototile space of a tiling	9
	1.4 The hull as an inverse limit of patch spaces	11
	1.5 The cut and projection method: a 1-dimensional example	17
II	TOPOLOGICAL INVARIANTS OF TILINGS	22
	2.1 Homotopy and homology of tiling spaces	22
	2.2 Introduction to Čech cohomology	23
	2.3 Cohomology of tilings	25
	2.3.1 Tiling Cohomologies	25
	2.3.2 The PV cohomology	27
	2.3.3 An example: PV cohomology in 1-dimension	33
	2.4 Isomorphism between Čech and PV cohomologies	40
	2.5 K -theory of tilings	45
III	APPLICATIONS	46
	3.1 Cohomology and deformation of tilings	46
	3.2 Homology and invariant measures on the hull	47
	3.3 K -theory and the gap labeling theorem	49
	3.4 Acoustic phonons in an aperiodic solid	50

IV	SPECTRAL SEQUENCES	59
4.1	Informal introduction to Spectral Sequences	59
4.1.1	The general idea of a spectral sequence	60
4.1.2	The graded case, example of homology and cohomology.	66
4.2	The Serre spectral sequence	74
4.2.1	Fibrations	75
4.2.2	Local coefficients systems	75
4.2.3	The spectral sequence	77
4.3	Exact couples	79
4.3.1	Exact couples for filtrations and cofiltrations	80
4.3.2	Direct limit of exact couples	83
4.3.3	Exact couples for the K -theory of a C^* -algebra	84
4.4	The Pimsner–Voiculescu spectral sequence	88
V	A SPECTRAL SEQUENCE FOR THE K -THEORY OF TILINGS	92
5.1	Historic background	92
5.2	The main theorem	97
5.3	Construction of the spectral sequence	101
VI	THE GROUPOID OF THE TRANSVERSAL	106
6.1	The simplicial category and simplicial objects	106
6.2	Nerve and classifying space of a small category	108
6.3	Classifying space of a group	109
6.4	Example: classifying space of \mathbb{Z}_2	111
6.5	Inverse limit of groupoids	113
6.6	Application to the groupoid of the transversal	115
6.7	The PV category of a tiling	121
	REFERENCES	132
	VITA	138

LIST OF FIGURES

1	Example of a prototile space	9
2	Example of a prototile map \mathfrak{p}_0	10
3	A 1-dimensional cut and project construction	17
4	Fibers over the prototile space	30
5	The PV local coefficient system	33
6	A 1-dimensional cut and project construction	34
7	Example of a tile's deformation	47
8	A 3-simplex in the classifying space of G	111

SUMMARY

Aperiodic tilings offer a rich mathematical diversity, from ergodic theory and dynamical system to algebraic topology and operator algebra, from algebraic geometry and number theory to noncommutative geometry. They also play an important role in the physics of aperiodic systems. A lot of progress has been made in recent years in describing topological properties of tilings. One approach is via the K -theory of their associated C^* -algebras.

The main part of this thesis is concerned with providing a tool to calculate the K -theory of a class of aperiodic tilings and relate it to its cohomology. We present in theorem 5.2.1 a spectral sequence that converges to the K -theory of the C^* -algebra of a tiling, and whose page-2 is given by a new tiling cohomology (defined in section 2.3.2), that we call PV cohomology in reference to the Pimsner–Voiculescu exact sequence [64]. It generalizes the cohomology of the base space of a fibration with local coefficients in the K -theory of its fiber (see sections 4.2.2 and 6.7). The spectral sequence can be seen as a generalization of the Serre spectral sequence to a class of spaces which are not fibered. We also prove in theorem 2.4.1 that the PV cohomology of a tiling is isomorphic to its integer Čech cohomology.

We present three secondary results as well: two formal ones related to the groupoid of a tiling, and an applied one about the physics of aperiodic solids. In section 3.4 we present a partial study of acoustic phonons in aperiodic solids and give an estimate on their energy spectrum. In chapter 6 we give a study of the groupoid of a tiling. We show in theorem 6.6 how it can be approximated by simpler ones associated with free categories generated by finite graphs. In section 6.7 we introduce the notion of a

PV category (definition 6.7.2) to give a general formalism for the PV local coefficient system that gives rise to the PV cohomology. We show in theorem 6.7.9 how to recover the groupoid of the tiling from its PV category.

The thesis is organized as follows.

Chapter 1 : preliminary definitions and results about aperiodic tilings.

Chapter 2 : review of topological invariants for tilings, definition of the PV cohomology (and an example of explicit calculation), and proof of the isomorphism with Čech cohomology.

Chapter 3 : review of some applications of topological invariants of tilings, and study of acoustic phonons in an aperiodic solid.

Chapter 4 : introduction and review of basics about spectral sequences (including examples), and some technical results about exact couples.

Chapter 5 : historical background, presentation of our spectral sequence, and construction.

Chapter 6 : study of the groupoid of a tiling, and formal properties of PV cohomology.

CHAPTER I

PRELIMINARIES ON THE THEORY OF APERIODIC TILINGS

We introduce here the preliminary definitions and results required for the work presented in this thesis. They are taken mostly from the references [7, 11] on tilings and Delone sets and the reader is referred to those papers for complete proofs. Let \mathbb{R}^d denote the usual Euclidean space of dimension d with euclidean norm $\|\cdot\|$. First of all we give the definition of a repetitive tiling of \mathbb{R}^d with finite local complexity, and then introduce the hull and transversal of such a tiling. The connection with Delone sets is briefly mentioned as well as the definition of the groupoid of the transversal of a Delone set. We then build prototile and patch spaces of a tiling and show how its hull can be approximated by those.

1.1 Tilings and their hulls

Definition 1.1.1 (i) *A tile of \mathbb{R}^d is a compact subset of \mathbb{R}^d which is homeomorphic to the unit ball.*

(ii) *A punctured tile is an ordered pair consisting of a tile and one of its points.*

(iii) *A tiling of \mathbb{R}^d is a covering of \mathbb{R}^d by a family of tiles whose interiors are pairwise disjoint. A tiling is said to be punctured if its tiles are punctured.*

(iv) *A prototile of a tiling is a translational equivalence class of tiles (including the puncture).*

(v) *The first corona of a tile in a tiling \mathcal{T} is the union of the tiles of \mathcal{T} intersecting*

it.

(vi) *A collared prototile of \mathcal{T} is the subclass of a prototile whose representatives have the same first corona up to translation.*

A collared prototile is a prototile where a local configuration of its representatives has been specified: each representative has the same neighboring tiles.

In the sequel it is implicitly assumed that tiles and tilings are punctured. All tiles are assumed to be finite Δ -complexes which are particular *CW*-complex structures (see section 2.3.2). They are also required to be compatible with the tiles of their first coronas, *i.e.* the intersection of any two tiles is itself a sub- Δ -complex of both tiles. In other words, tilings considered here are assumed to be Δ -complexes of \mathbb{R}^d .

Definition 1.1.2 *Let \mathcal{T} be a tiling of \mathbb{R}^d .*

(i) *A patch of \mathcal{T} is a finite union of neighboring tiles in \mathcal{T} that is homeomorphic to a ball. A patch is punctured by the puncture of one of the tiles that it contains. The radius of a patch is the radius of the smallest ball that contains it.*

(ii) *A pattern of \mathcal{T} is a translational equivalence class of patches of \mathcal{T} .*

(iii) *The first corona of a patch of \mathcal{T} is the union of the tiles of \mathcal{T} intersecting it.*

(iv) *A collared pattern of \mathcal{T} is the subclass of a pattern whose representatives have the same first corona up to translation.*

The following notation will be used in the sequel: prototiles and patterns will be written with a hat, for instance \hat{t} or \hat{p} , to distinguish them from their representatives. Often the following convention will be implicit: if \hat{t} is a prototile and \hat{p} a pattern, then t and p will denote their respective representatives that have their punctures at the origin $0_{\mathbb{R}^d}$.

The results in this paper are valid for the class of tilings that are *repetitive with finite local complexity*.

Definition 1.1.3 Let \mathcal{T} be a tiling of \mathbb{R}^d .

- (i) \mathcal{T} has Finite Local Complexity (FLC) if for any $R > 0$ the set of patterns of \mathcal{T} whose representatives have radius less than R is finite.
- (ii) \mathcal{T} is repetitive if for any patch of \mathcal{T} and every $\epsilon > 0$, there is an $R > 0$ such that for every x in \mathbb{R}^d there exists modulo an error ϵ with respect to the Hausdorff distance, a translated copy of this patch belonging to \mathcal{T} and contained in the ball $B(x, R)$.
- (iii) For any x in \mathbb{R}^d , let $\mathcal{T} + x = \{t + x : t \in \mathcal{T}\}$ denote its translation, then \mathcal{T} is aperiodic if there is no $x \neq 0$ in \mathbb{R}^d such that $\mathcal{T} + x = \mathcal{T}$.

For tilings with FLC the repetitivity condition (ii) above can be stated more precisely: A tiling \mathcal{T} with FLC is repetitive if given any patch there is an $R > 0$ such that for every x in \mathbb{R}^d there exists an exact copy of this patch in \mathcal{T} contained in the ball $B(x, R)$.

The class of repetitive tilings satisfying the FLC property is very rich and has been investigated for decades. It started in the 70's with the work of Penrose [63] and Meyer [60] and went on both from an abstract mathematical level and with a view towards applications, in particular to the physics of quasicrystals [52, 53]. It contains an important subclass of the class of *substitution tilings* that was reinvestigated in the 90's by Anderson and Putnam in [1], and also the class of tilings obtained by the *cut-and-projection* method, for which a comprehensive study by Hunton, Kellendonk and Forrest can be found in [32], and more generally it contains the whole class of *quasiperiodic tilings*, which are models for quasicrystals.

The Ammann aperiodic tilings are examples of substitution tilings (see [37] chapter 10, Ammann's original work does not appear in the literature). In addition other examples include the octogonal tiling as well as the famous Penrose "kite and darts"

[63] tilings which are both substitution and cut-and-projection tilings (see [37] chapter 10, and [68] chapter 4). However the so-called Pinwheel tiling (see [67] and [68] chapter 4) is a substitution tiling but does not satisfy the FLC property given here since prototiles are defined here as equivalence classes of tiles under only translations and not more general isometries of \mathbb{R}^d like rotations.

A topology has been proposed in [7] that applies to a large class of tilings (for which there exist an $r_0 > 0$ such that all tiles contain a ball of radius r_0). In the case considered here, where tiles are assumed to be finite *CW*-complexes, this topology can be adapted as follows.

Let \mathcal{F} be a family of tilings whose tiles contain a ball of a fixed radius $r_0 > 0$ and have compatible *CW*-complex structures (the intersection of any two tiles is a subcomplex of both). Given an open set O in \mathbb{R}^d with compact closure and an $\epsilon > 0$, a neighborhood of a tiling T in \mathcal{F} is given by

$$U_{O,\epsilon}(T) = \left\{ T' \in \mathcal{F} : \sup_{0 \leq k \leq d} h_k(O \cap T^k, O \cap T'^k) < \epsilon \right\},$$

where T^k and T'^k are the k -skeletons of T and T' respectively and h_k is the k -dimensional Hausdorff distance.

Let \mathcal{T} be a tiling of \mathbb{R}^d . The group \mathbb{R}^d acts on the set of all translates of \mathcal{T} , the action (translation) is denoted $T^a, a \in \mathbb{R}^d$: $T^a \mathcal{T} = \mathcal{T} + a = \{t + a : t \in \mathcal{T}\}$.

Definition 1.1.4 (i) *The hull of \mathcal{T} , denoted Ω , is the closure of $T^{\mathbb{R}^d} \mathcal{T}$.*

(ii) *The canonical transversal, denoted Ξ , is the subset of Ω consisting of tilings that have the puncture of one of their tiles at the origin $0_{\mathbb{R}^d}$.*

The hull of a tiling is seen as a dynamical system $(\Omega, \mathbb{R}^d, T)$ which, for the class of tilings considered here, has interesting properties that are now stated (see [11] section 2.3).

Theorem 1.1.5 [11, 52] *Let \mathcal{T} be a tiling of \mathbb{R}^d .*

- (i) *\mathcal{T} is repetitive if and only if the dynamical system of its hull $(\Omega, \mathbb{R}^d, \mathbb{T})$ is minimal.*
- (ii) *If \mathcal{T} has FLC, then its canonical transversal Ξ is totally disconnected.*
- (iii) *If \mathcal{T} is aperiodic, repetitive and has FLC, then Ξ is a Cantor set (perfect and totally disconnected).*

The minimality of the hull allows one to see any of its points as just a translate of \mathcal{T} .

Remark 1.1.6 A metric topology for tiling spaces has been used in the literature for historical reasons. Let \mathcal{T} be a repetitive tiling of \mathbb{R}^d with FLC. The orbit space of \mathcal{T} under translation by vectors of \mathbb{R}^d , $\mathbb{T}^{\mathbb{R}^d}\mathcal{T}$, is endowed with a metric as follows (see [11] section 2.3). For T and T' in $\mathbb{T}^{\mathbb{R}^d}\mathcal{T}$, let A denote the set of ε in $(0, 1)$ such that there exist x and x' in $B(0, \varepsilon)$ for which $\mathbb{T}^x T$ and $\mathbb{T}^{x'} T'$ agree on $B(0, \frac{1}{\varepsilon})$, *i.e.* their tiles whose punctures lie in the ball are matching, then

$$\delta(T, T') = \begin{cases} \inf A & \text{if } A \neq \emptyset, \\ 1 & \text{if } A = \emptyset. \end{cases}$$

Hence the diameter of $\mathbb{T}^{\mathbb{R}^d}\mathcal{T}$ is bounded by 1 and the action of \mathbb{R}^d is continuous.

For the class of repetitive tilings with FLC, the topology of the hull given in definition 1.1.4 is equivalent to this δ -metric topology [11].

1.2 Delone sets and groupoid of the transversal

The notions for tilings given in the previous section can be translated for the set of their punctures in terms of Delone sets.

Definition 1.2.1 *Let \mathcal{L} be a discrete subset of \mathbb{R}^d .*

- (i) Given $r > 0$, \mathcal{L} is r -uniformly discrete if any open ball of radius r in \mathbb{R}^d meets \mathcal{L} in at most one point.
- (ii) Given $R > 0$, \mathcal{L} is R -relatively dense if any open ball of radius R in \mathbb{R}^d meets \mathcal{L} in at least one point.
- (iii) \mathcal{L} is an (r, R) -Delone set if it is r -uniformly discrete and R -relatively dense.
- (iv) \mathcal{L} is repetitive if given any finite subset $p \subset \mathcal{L}$ and any $\epsilon > 0$, there is an $R > 0$ such that the intersection of \mathcal{L} with any closed ball of radius R contains a copy (translation) of p modulo an error of ϵ (w.r.t. the Hausdorff distance)
- (v) A patch of radius $R > 0$ of \mathcal{L} is a subset of \mathbb{R}^d of the form $(\mathcal{L} - x) \cap B(0, R)$, for some $x \in \mathcal{L}$. If for all $R > 0$ the set of its patches of radius R is finite then \mathcal{L} has finite local complexity.
- (vi) \mathcal{L} is aperiodic if there is no $x \neq 0$ in \mathbb{R}^d such that $\mathcal{L} - x = \mathcal{L}$.

Condition (iv) above is equivalent to saying that $\mathcal{L} - \mathcal{L}$, the set of vectors of \mathcal{L} , is discrete.

Given a punctured tiling \mathcal{T} if there exists $r, R > 0$ such that each of its tiles contains a ball a radius r and is contained in a ball of radius R , then its set of punctures $\mathcal{L}_{\mathcal{T}}$ is an (r, R) -Delone set, and it is repetitive and has FLC if and only if the tiling is repetitive and has FLC. Conversely, given a Delone set the Voronoi construction below gives a tiling, and they both share the same repetitivity or FLC properties.

Definition 1.2.2 Let \mathcal{L} be an (r, R) -Delone set of \mathbb{R}^d . The Voronoi tile at $x \in \mathcal{L}$, is defined by

$$T_x = \{y \in \mathbb{R}^d : \|y - x\| \leq \|y - x'\|, \forall x' \in \mathcal{L}\},$$

with puncture the point x . The Voronoi tiling \mathcal{V} associated with \mathcal{L} is the tiling of \mathbb{R}^d whose tiles are the Voronoi tiles of \mathcal{L} .

The tiles of the Voronoi tiling of a Delone set are (closed) convex polytopes that touch on common faces.

The hull and transversal of a Delone set are defined as follows [7].

Definition 1.2.3 *Let \mathcal{L} be a Delone set of \mathbb{R}^d . The hull Ω of \mathcal{L} is the closure in the weak- $*$ topology of the set of translations of the Radon measure that has a Dirac mass at each point of \mathcal{L}*

$$\Omega = \overline{\{\mathbb{T}^a \mu; a \in \mathbb{R}^d\}}^{w^*}, \quad \mu = \sum_{x \in \mathcal{L}} \delta_x,$$

and the transversal Ξ of \mathcal{L} is the subset

$$\Xi = \{\omega \in \Omega; \omega(\{0\}) = 1\}.$$

Given a general Delone set this topology of its hull is strictly coarser than the δ -metric topology for its Voronoi tiling given in remark 1.1.6, but if it is repetitive and has FLC then they are equivalent [11]. The weak- $*$ topology is here also equivalent to the local Hausdorff topology on the set $\mathbb{T}^{\mathbb{R}^d} \mathcal{L}$ of translates of \mathcal{L} : given an open set O in \mathbb{R}^d with compact closure and an $\epsilon > 0$, a neighborhood of ℓ is given by

$$U_{O,\epsilon}(\ell) = \{\ell' \in \mathbb{T}^{\mathbb{R}^d} \mathcal{L} : h_d(\ell \cap O, \ell' \cap O) < \epsilon\},$$

where h_d is the d -dimensional Hausdorff distance.

The hull of a Delone set is also seen as a dynamical system under the homeomorphic action of \mathbb{R}^d by translation. Just as in proposition 1.1.5, the hull and transversal of repetitive Delone sets with FLC have similar interesting properties.

Theorem 1.2.4 [7] *Let \mathcal{L} be a Delone set of \mathbb{R}^d .*

- (i) \mathcal{L} is repetitive if and only if the dynamical system of its hull $(\Omega, \mathbb{R}^d, \mathbb{T})$ is minimal.

(ii) If \mathcal{L} has FLC, then its transversal Ξ is totally disconnected.

(iii) If \mathcal{L} is aperiodic, repetitive and has FLC, then Ξ is a Cantor set (perfect and totally disconnected).

The two approaches of tilings or Delone sets which are repetitive and have FLC are thus equivalent. Such tilings give rise to Delone sets (their sets of punctures), and conversely such Delone sets give rise to tilings (their Voronoi tilings) and their respective hulls share the same properties.

Recall that a groupoid is a small category in which every morphism is invertible [26, 69].

Definition 1.2.5 *Let \mathcal{L} be a Delone set of \mathbb{R}^d . The groupoid of the transversal is the groupoid Γ whose set of objects is the transversal: $\Gamma^0 = \Xi$, and whose set of arrows is*

$$\Gamma^1 = \{(\xi, x) \in \Xi \times \mathbb{R}^d : \tau^{-x}\xi \in \Xi\}.$$

Given an arrow $\gamma = (\xi, x)$ in Γ^1 , its source is the object $s(\gamma) = \tau^{-x}\xi \in \Xi$ and its range the object $r(\gamma) = \xi \in \Xi$.

The Delone sets considered here are repetitive with FLC and the groupoids of their transversals are *étale*, i.e. given any arrow γ with range (or source) object x , there exists an open neighborhood O_x of x in Γ^0 and a homeomorphism $\varphi : O_x \rightarrow \Gamma^1$ that maps x to γ . This implies that the sets of arrows having any given object as a source or range are discrete. This follows because for such Delone sets their sets of vectors are discrete and given any fixed $l > 0$ their sets of vectors of length less than l are finite.

The groupoid of the transversal of a tiling is defined as the groupoid of the transversal of its Delone set of punctures. In chapter 6 we give a detailed study of some properties of such groupoids.

1.3 Prototile space of a tiling

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with FLC and assume its tiles are compatible finite CW -complexes (the intersection of two tiles is a subcomplex of both). A finite CW -complex \mathcal{B}_0 , called prototile space, is built out of its prototiles by gluing them along their boundaries according to all the local configurations of their representatives in \mathcal{T} , as illustrated by the figure below.

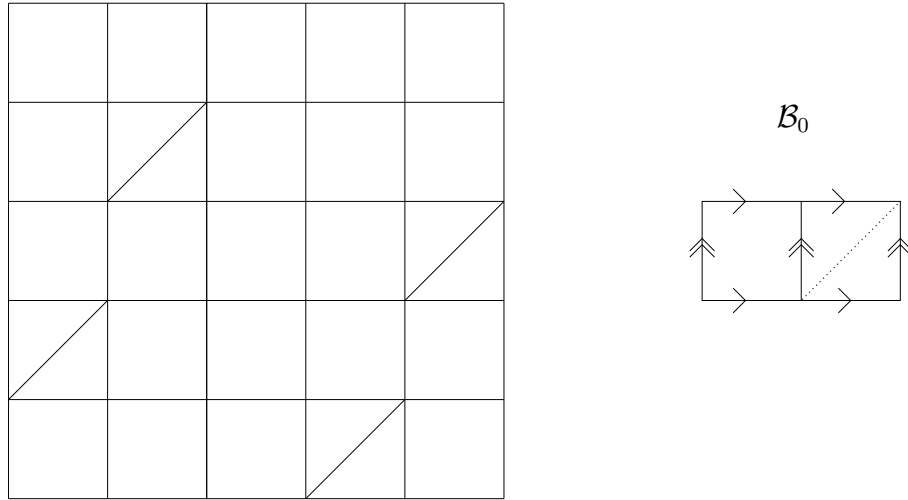


Figure 1: Example of a prototile space

Definition 1.3.1 Let $\hat{t}_j, j = 1, \dots, N_0$, be the prototiles of \mathcal{T} . Let t_j denote the representative of \hat{t}_j that has its puncture at the origin. The prototile space of \mathcal{T} , $\mathcal{B}_0(\mathcal{T})$, is the quotient CW -complex

$$\mathcal{B}_0(\mathcal{T}) = \coprod_{j=1}^{N_0} t_j / \sim,$$

where two n -cells $e_i^n \in t_i^n$ and $e_j^n \in t_j^n$ are identified if there exists $u_i, u_j \in \mathbb{R}^d$ for which $t_i + u_i$ and $t_j + u_j$ are tiles of \mathcal{T} such that $e_i^n + u_i$ and $e_j^n + u_j$ coincide on the intersection of their n -skeletons.

The collared prototile space of \mathcal{T} , $\mathcal{B}_0^c(\mathcal{T})$, is built similarly from the collared prototiles of \mathcal{T} : $t_i^c, i = 1, \dots, N_0^c$.

The images in $\mathcal{B}_0(\mathcal{T})$ or $\mathcal{B}_0^c(\mathcal{T})$ of the tiles $t_j^{(c)}$'s will be denoted τ_j and still be called tiles.

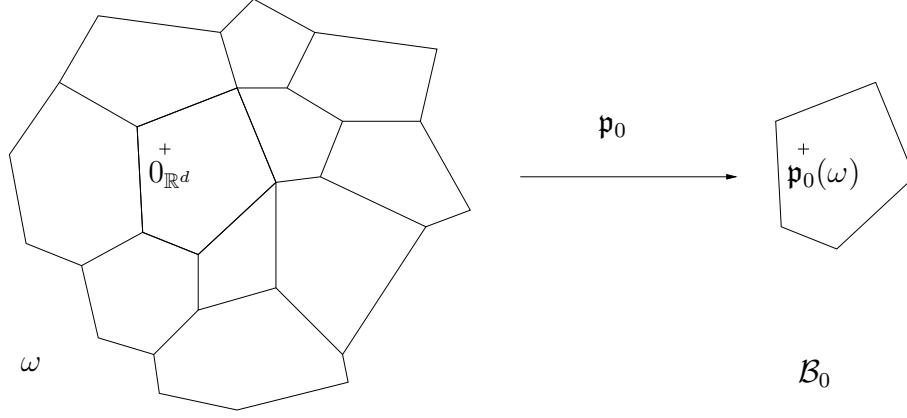


Figure 2: Example of a prototile map \mathfrak{p}_0

Proposition 1.3.2 *There is a continuous map $\mathfrak{p}_{0,\mathcal{T}}^{(c)} : \Omega \rightarrow \mathcal{B}_0^{(c)}(\mathcal{T})$ from the hull onto the (collared) prototile space.*

Proof. Let $\lambda_0^{(c)} : \coprod_{j=1}^{N_0^{(c)}} t_j \rightarrow \mathcal{B}_0^{(c)}(\mathcal{T})$ be the quotient map. And let $\rho_0^{(c)} : \Omega \times \mathbb{R}^d \rightarrow \coprod_{j=1}^{N_0^{(c)}} t_j^{(c)}$ be defined as follows. If x belongs to the intersection of k tiles $t^{\alpha_1}, \dots, t^{\alpha_k}$, in ω , with $t^{\alpha_l} = t_{j_l}^{(c)} + u_{\alpha_l}(\omega)$, $l = 1, \dots, k$, then $\rho_0^{(c)}(\omega, x) = \coprod_{l=1}^k x - u_{\alpha_l}(\omega)$ and lies in the disjoint union of the $t_{j_l}^{(c)}$'s.

The map $\mathfrak{p}_{0,\mathcal{T}}^{(c)}$ is defined as the composition: $\omega \mapsto \lambda_0^{(c)} \circ \rho_0^{(c)}(\omega, 0_{\mathbb{R}^d})$. The map $\rho_0^{(c)}(\cdot, 0_{\mathbb{R}^d})$ sends the origin of \mathbb{R}^d , that lies in some tiles of ω , to the corresponding tiles $t_j^{(c)}$'s at the corresponding positions.

In $\mathcal{B}_0^{(c)}(\mathcal{T})$, points on the boundaries of two tiles $\tau_j^{(c)}$ and $\tau_{j'}^{(c)}$ are identified if there are neighboring copies of the tiles $t_j^{(c)}$ and $t_{j'}^{(c)}$ somewhere in \mathcal{T} such that the two associated points match. This ensures that the map $\mathfrak{p}_{0,\mathcal{T}}^{(c)}$ is well defined, for if in \mathbb{R}^d tiled by ω , the origin $0_{\mathbb{R}^d}$ belongs to the boundaries of some tiles, then the corresponding points in $\coprod_{j=1}^{N_0^{(c)}} t_j^{(c)}$ given by $\rho_0^{(c)}(\omega, 0_{\mathbb{R}^d})$ are identified by $\lambda_0^{(c)}$.

Let x be a point in $\mathcal{B}_0^{(c)}(\mathcal{T})$, and O_x an open neighborhood of x . Say x belongs to the intersection of some tiles $\tau_{j_1}^{(c)}, \dots, \tau_{j_k}^{(c)}$. Let ω be a preimage of x : $\mathfrak{p}_{0,\mathcal{T}}^{(c)}(\omega) = x$. The preimage of O_x is the set of tilings ω' 's for which the origin lies in some neighborhood of tiles that are translates of $t_{j_1}^{(c)}, \dots, t_{j_k}^{(c)}$, and this defines a neighborhood of ω in the hull. Therefore $\mathfrak{p}_{0,\mathcal{T}}^{(c)}$ is continuous. \square

For simplicity, the prototile space $\mathcal{B}_0(\mathcal{T})$ is written \mathcal{B}_0 , and the map $\mathfrak{p}_{0,\mathcal{T}}$ is written \mathfrak{p}_0 .

The lift of the puncture of the tile τ_j in \mathcal{B}_0 , denoted $\Xi(\tau_j)$, is a subset of the transversal called the *acceptance zone* of the prototile \hat{t}_j . It consists of all the tilings that have the puncture of a representative of \hat{t}_j at the origin. The $\Xi(\tau_j)$'s for $j = 1, \dots, N_0$, form a clopen partition of the transversal, because any element of Ξ has the puncture of a unique tile at the origin which corresponds to a unique prototile. The $\Xi(\tau_j)$'s are thus Cantor sets like Ξ .

Remark 1.3.3 Although those results will not be used here, it has been proven in [11] that the prototile space \mathcal{B}_0 (as well as the patch spaces \mathcal{B}_p defined similarly in the next section, definition 1.4.1) has the structure of a flat oriented Riemannian branched manifold. Also, the hull Ω can be given a *lamination* structure as follows: the lifts of the interiors of the tiles $B_{0j} = \mathfrak{p}_0^{-1}(\tau_j^\circ)$ are boxes of the lamination which are homeomorphic to $\hat{t}_j^\circ \times \Xi(\tau_j)$ via the maps $(x, \xi) \mapsto \mathbb{T}^{-x}\xi$ which read as local charts. The leaves are homeomorphic to \mathbb{R}^d tiled by translates of \mathcal{T} , and Ξ is a transversal to the lamination.

1.4 The hull as an inverse limit of patch spaces

As in the previous section, let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with FLC, and assume its tiles are compatible finite CW-complexes (the intersection of two tiles is a subcomplex of both). Let $\mathcal{L}_{\mathcal{T}}$ denote the Delone set of punctures of \mathcal{T} .

It is repetitive and has FLC. Let $\mathcal{P}_{\mathcal{T}}$ denote the set of patterns of \mathcal{T} . As \mathcal{T} has finite local complexity (hence finitely many prototiles), the set $\mathcal{P}_{\mathcal{T}}$ is countable, and for any given $l > 0$ the set of patterns whose representatives have radius less than l is finite. A finite *CW*-complex \mathcal{B}_p , called a patch space, associated with a pattern \hat{p} in $\mathcal{P}_{\mathcal{T}}$ is built from the prototiles of an appropriate subtiling of \mathcal{T} , written \mathcal{T}_p below, in the same way that \mathcal{B}_0 was built from the prototiles of \mathcal{T} in definition 1.3.1. The construction goes as follows.

Let \hat{p} in $\mathcal{P}_{\mathcal{T}}$ be a pattern of \mathcal{T} .

- (i) Consider the sub-Delone set \mathcal{L}_p of $\mathcal{L}_{\mathcal{T}}$ consisting of punctures of all the representative patches in \mathcal{T} of \hat{p} . \mathcal{L}_p is repetitive and has FLC.
- (ii) The Voronoi tiling \mathcal{V}_p of \mathcal{L}_p is built and each point of $\mathcal{L}_{\mathcal{T}}$ is assigned to a unique tile of \mathcal{V}_p as explained below.
- (iii) Each tile v of \mathcal{V}_p is replaced by the patch p_v of \mathcal{T} made up of the tiles whose punctures have been assigned to v . This gives a repetitive tiling with FLC, \mathcal{T}_p , whose tiles are those patches p_v 's.
- (iv) \mathcal{B}_p is built out of the collared prototiles of \mathcal{T}_p , by gluing them along their boundaries according to the local configurations of their representatives in \mathcal{T}_p .

The second point needs clarifications since the tiles of \mathcal{V}_p are Voronoi tiles (convex polytopes, see definition 1.2.2) of \mathcal{L}_p and not patches of \mathcal{T} . If a point of $\mathcal{L}_{\mathcal{T}}$ (a puncture of a tile of \mathcal{T}) lies on the boundary of some (Voronoi) tiles of \mathcal{V}_p , a criterion for assigning it to a specific one is required. To do so, let u be a vector of \mathbb{R}^d that is not colinear to any of the faces of the tiles of \mathcal{V}_p (such a vector exists since \mathcal{V}_p has FLC, hence finitely many prototiles). A point x is said to be u -interior to a subset X of \mathbb{R}^d if there exist an $\epsilon > 0$ such that $x + \epsilon u$ belongs to the interior of X . Since u is not colinear to any of the faces of the tiles of \mathcal{V}_p , if a point x belongs to the

intersection of the boundaries of several (Voronoi) tiles of \mathcal{V}_p , it is u -interior to only one of them. This allows as claimed in (ii) to assign each point of $\mathcal{L}_{\mathcal{T}}$ to a unique tile of $\mathcal{V}_{\mathcal{T}}$. Now as explained in (iii), each Voronoi tile v can then be replaced by the patch of \mathcal{T} which is the union of the tiles of \mathcal{T} whose punctures are u -interior to v . Each patch p_v is considered a tile of \mathcal{T}_p and punctured by the puncture of the Voronoi tile v which is by construction the puncture of some representative patch of \hat{p} in \mathcal{T} . As patches of \mathcal{T} , the p_v 's are also compatible finite CW -complexes as they are made up of tiles of \mathcal{T} which are.

The prototiles of \mathcal{T}_p are actually patterns of \mathcal{T} , and thus \mathcal{T}_p is considered a subtiling of \mathcal{T} . From this remark it can be proven that there is a homeomorphism between the hull of \mathcal{T}_p and Ω , that conjugates the \mathbb{R}^d -action of their associated dynamical systems (see [11] section 2).

Definition 1.4.1 *The patch space \mathcal{B}_p is the collared prototile space of \mathcal{T}_p (definition 1.3.1): $\mathcal{B}_p = \mathcal{B}_0^c(\mathcal{T}_p)$.*

The images in \mathcal{B}_p of the patches p_j 's (tiles of \mathcal{T}_p) will be denoted π_j and still be called patches. The map $\mathfrak{p}_{0,\mathcal{T}_p}^c : \Omega \rightarrow \mathcal{B}_p$, built in proposition 1.3.2, is denoted \mathfrak{p}_p .

This construction of \mathcal{B}_p from \mathcal{T}_p is essentially the same as the construction of \mathcal{B}_0 from \mathcal{T} given in definition 1.3.1, the only difference being that collared prototiles of \mathcal{T}_p (collared patches of \mathcal{T}) are used here instead. Such patch spaces are said to *force their borders*. This condition was introduced by Kellendonk in [47] for substitution tilings and was required in order to be able to recover the hull as the inverse limit of such spaces. It was generalized in [11] for branched manifolds of repetitive tilings with FLC and coincides here with the above definition.

The map $f_p : \mathcal{B}_p \rightarrow \mathcal{B}_0$ defined by $f_p = \mathfrak{p}_0 \circ \mathfrak{p}_p^{-1}$ is surjective and continuous. It projects \mathcal{B}_p onto \mathcal{B}_0 in the obvious way: a point x in \mathcal{B}_p belongs to some patch π_j , hence to some tile, and f_p sends x to the corresponding point in the corresponding

tile $\tau_{j'}$. More precisely, if \tilde{x} in p_j is the point of \mathbb{R}^d corresponding to x in π_j , then \tilde{x} belongs to some tile, which is the translate of some $t_{j'}$ and $f_p(x)$ is the corresponding point in $\tau_{j'}$ in \mathcal{B}_0 . If x belongs to the boundaries of say $\pi_{j_1}, \dots, \pi_{j_k}$, in \mathcal{B}_p , then there are corresponding points $\tilde{x}_{j_1}, \dots, \tilde{x}_{j_k}$, in p_{j_1}, \dots, p_{j_k} , which are then on the boundaries of the copies of some tiles $t_{j'_1}, \dots, t_{j'_k}$. The boundaries of those tiles are identified by the map ρ_0 in the definition 1.3.2 of \mathfrak{p}_0 and $f_p(x)$ is the corresponding point on the common boundaries of the $\tau_{j'_1}, \dots, \tau_{j'_k}$.

Recall the convention stated after definition 1.1.2: if \hat{p} is a pattern, then p denotes its representative that has its puncture at the origin. Given two patterns \hat{p} and \hat{q} with $q \subset p$, the map $f_{qp} : \mathcal{B}_p \rightarrow \mathcal{B}_q$ defined by $f_{qp} = f_q^{-1} \circ f_p = \mathfrak{p}_q \circ \mathfrak{p}_p^{-1}$ is continuous and surjective. Given three patterns \hat{p}, \hat{q} and \hat{r} with $r \subset q \subset p$ the following composition rule holds: $f_{rq} \circ f_{qp} = f_{rp}$. Moreover given two arbitrary patterns \hat{q} and \hat{r} , there exists another pattern \hat{p} such that p contains q and r (since \mathcal{L}_q and \mathcal{L}_r are repetitive sub-Delone sets of $\mathcal{L}_{\mathcal{T}}$). Hence the index set of the maps f_p 's is a directed set and (\mathcal{B}_p, f_{qp}) is a projective system.

As shown in [11], the hull Ω can be recovered from the inverse limit $\varprojlim (\mathcal{B}_p, f_{qp})$ under some technical conditions that are for convenience taken here to be directly analogous to those given in [11] section 2.6. Namely: the patch spaces \mathcal{B}_p are required to *force their borders*, and only maps f_{qp} between patch spaces that are *zoomed out* of each other are allowed (definition 1.4.2 below). The first condition, as mentioned above, is fulfilled here by the very definition of patch spaces given above, because they are built out of *collared patterns* as can be checked from the more general definition of a branched manifold that forces its border given in [11] definition 2.43. The second condition is stated as follows.

Definition 1.4.2 *Given two patterns \hat{p}, \hat{q} in $\mathcal{P}_{\mathcal{T}}$, with $q \subset p$, \mathcal{B}_p is said to be zoomed out of \mathcal{B}_q if the following two conditions hold.*

- (i) For all $i \in \{1, \dots, N_p\}$, the patch p_i is the union of some copies of the patches q_j 's.
- (ii) For all $i \in \{1, \dots, N_p\}$, the patch p_i contains in its interior a copy of some patch q_j .

The first condition is equivalent to requiring that the tiles of \mathcal{L}_p are patches of \mathcal{L}_q .

Given a patch space \mathcal{B}_q it is always possible to build another patch space \mathcal{B}_p that is zoomed out of \mathcal{B}_q : it suffices to choose a pattern \hat{p} such that $p \supset q$ with a radius large enough. This can be done by induction for instance: building \mathcal{B}_p from patches of \mathcal{L}_q (which are patches of \mathcal{T}) the same way \mathcal{B}_q was built from patches of \mathcal{T} ; if the radius of q is large enough, then each patch of \mathcal{T} that \mathcal{B}_q is composed of will contain a tile of \mathcal{T} in its interior and so \mathcal{B}_q will be zoomed out from \mathcal{B}_0 .

Definition 1.4.3 A proper sequence of patch spaces of \mathcal{T} is a projective sequence $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ where, for all $l \geq 1$, \mathcal{B}_l is a patch space associated with a pattern \hat{p}_l of \mathcal{T} and $f_l = f_{p_{l-1}p_l}$, such that \mathcal{B}_l is zoomed out of \mathcal{B}_{l-1} , with the convention that $f_0 = f_{p_1}$ and \mathcal{B}_0 is the prototile space of \mathcal{T} .

Note that the first patch space in a proper sequence can be chosen to be the prototile space of the tiling (*i.e.* made of uncollared prototiles), all that matters for recovering the hull by inverse limit as shown in the next theorem, is that the next patch spaces are zoomed out of each other, and built out of collared patches (*i.e.* force their borders).

Theorem 1.4.4 The inverse limit of a proper sequence $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ of patch spaces of \mathcal{T} is homeomorphic to the hull of \mathcal{T} :

$$\Omega \cong \varprojlim (\mathcal{B}_l, f_l).$$

Proof. The homeomorphism is given by the map $\mathbf{p} : \Omega \rightarrow \varprojlim (\mathcal{B}_l, f_l)$, defined by $\mathbf{p}(\omega) = (\mathbf{p}_0(\omega), \mathbf{p}_1(\omega), \dots)$, with inverse $\mathbf{p}^{-1}(x_0, x_1, \dots) = \bigcap \{\mathbf{p}_l^{-1}(x_l), l \in \mathbb{N}\}$.

The map \mathbf{p} is surjective, because all the \mathbf{p}_l 's are. For the proof of injectivity, consider $\omega, \omega' \in \Omega$ with $\mathbf{p}(\omega) = \mathbf{p}(\omega')$. For simplicity the metric δ defined in remark 1.1.6 is used in this proof. Fix $\epsilon > 0$. To prove that $\delta(\omega, \omega') < \epsilon$ it suffices to show that the two tilings agree on a ball of radius $\frac{1}{\epsilon}$. For each l in \mathbb{N} , $\mathbf{p}_l(\omega) = \mathbf{p}_l(\omega')$ in some patch $\pi_{l,j}$ in \mathcal{B}_l . This means that the tilings agree on some translate of the patch $p_{l,j}$ that contains the origin. The definition of patch spaces made up from collared patterns (condition of *forcing the border* in [11]) implies that the tilings agree on the ball $B(0_{\mathbb{R}^d}, r_l)$, where r_l is the parameter of uniform discreteness of the Delone set \mathcal{T}_{p_l} . The assumption that the patch spaces are zoomed out of each other (condition (ii) in definition 1.4.2) implies that $r_l > r_{l-1} + r$, where r is the parameter of uniform discreteness of $\mathcal{L}_{\mathcal{T}}$. Hence $r_l > (l+1)r$, and choosing l bigger than the integer part of $\frac{1}{r\epsilon}$ concludes the proof of the injectivity of \mathbf{p} .

Condition (i) in definition 1.4.2 implies that for every $l \geq 1$, $\mathbf{p}_l^{-1}(x_l) \subset \mathbf{p}_{l-1}^{-1}(x_{l-1})$. The definition of \mathbf{p}^{-1} then makes sense by compactness of Ω because any finite intersection of some $\mathbf{p}_l^{-1}(x_l)$'s is non empty and closed.

A neighborhood of $x = (x_0, x_1, \dots)$ in $\varprojlim (\mathcal{B}_l, f_l)$ is given by $U_n(x) = \{y = (y_0, y_1, \dots) : y_i = x_i, i \leq n\}$ for some integer n . If ω is a preimage of x , $\mathbf{p}(\omega) = x$, then the preimage of $U_n(x)$ is given by all tilings ω' such that $\mathbf{p}_n(\omega') = \mathbf{p}_n(\omega)$, *i.e.* those tilings agree with ω on some patch p_n around the origin; they form then a neighborhood of ω in Ω . This proves that that \mathbf{p} and \mathbf{p}^{-1} are continuous. \square

Another important construction which allows one to recover the hull by inverse limit has been given by Gähler in an unpublished work as a generalization of the construction of Anderson and Putnam in [1]. Instead of gluing together patches to form the patch spaces \mathcal{B}_p 's, Gähler's construction consists in considering "multicollared protiles" and the graph that link them according to the local configurations of their

representatives in the tiling. This construction keeps track of the combinatorics of the patches that those “multicollared tiles” represent and thus is enough to recover the hull topologically, which is sufficient for topological concerns (for cohomology or K -theory in particular). The construction given here, and taken from [11] where those patch spaces are proven to be branched manifolds, takes more structure into account and the homeomorphism between the hull and the inverse limit of such branched manifolds which is built in [11] is not only a topological conjugacy but yields a conjugacy of the dynamical systems’ actions.

1.5 *The cut and projection method: a 1-dimensional example*

We describe here the cut and projection method for 1-dimensional tilings obtained by projection of a lattice in \mathbb{R}^2 . We also show how the transversal can be seen as the spectrum of a certain C^* -algebra associated with the tiling [7].

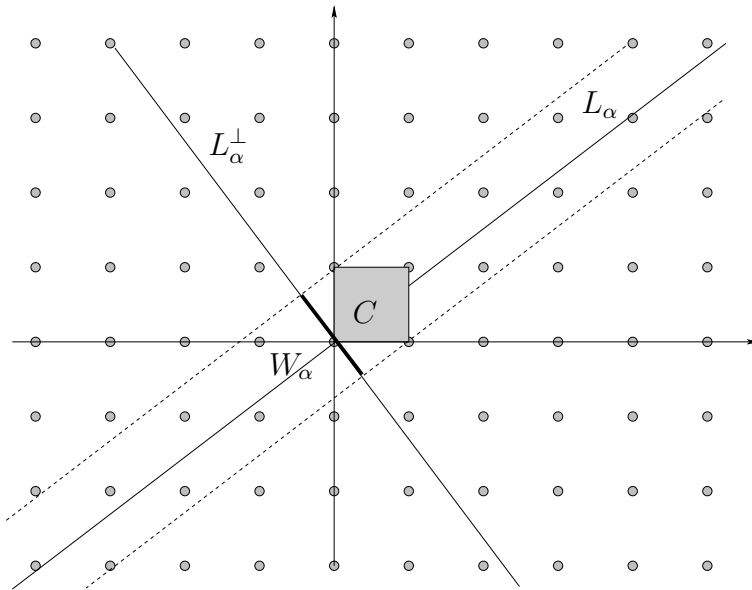


Figure 3: A 1-dimensional cut and project construction

Let \mathbb{R}^2 be the usual Euclidean plane with basis vectors e_1 and e_2 . Let \mathbb{Z}^2 be the lattice

of points of integer coordinates of the plane. Let $C = (0, 1] \times [0, 1)$ be the unit square “open on the top and left”. Fix an irrational number α in $\mathbb{R}_+ \setminus \mathbb{Q}$. Let L_α be line of slope α (through the origin) and $p_\alpha = \frac{1}{1+\alpha^2} \begin{bmatrix} \alpha^2 & \alpha \\ \alpha & 1 \end{bmatrix}$ the projection onto L_α . Let L_α^\perp be the orthogonal complement of L_α and $p_\alpha^\perp = 1 - p_\alpha = \frac{1}{1+\alpha^2} \begin{bmatrix} 1 & -\alpha \\ -\alpha & \alpha^2 \end{bmatrix}$. Let $W_\alpha = p_\alpha^\perp(C)$ be the projection of C onto L_α^\perp , and let $\Sigma = \{a \in \mathbb{Z}^2 : p_\alpha^\perp(a) \in W_\alpha\}$ be the set of points of \mathbb{Z}^2 that project to W_α . Therefore a lattice point $a = (a_1, a_2)$ in \mathbb{Z}^2 belongs to Σ if and only if $-\alpha \leq a_2 - a_1\alpha < 1$. The projection of Σ onto L_α defines a quasiperiodic tiling \mathcal{T}_α whose tiles’ edges are the projections of points of Σ . The tiling \mathcal{T}_α has two types of tiles that correspond to the projections of the vertical and horizontal of C . In particular \mathcal{T}_α is repetitive with FLC [7, 52, 53, 54].

The tiling corresponds to the unique bi-infinite path in Σ : $(a_n)_{n \in \mathbb{Z}}$ with a_0 the origin and $a_{n+1} = a_n + e_{i(n)}$ where $i(n)$ is 1 or 2. If we choose the closed unit square instead of C to project to L_α^\perp , then we have two more points in Σ , the projections of $\pm e_2$, and thus four such paths (determined by the choices of a_1 and a_{-1} as the projections of $\pm e_1$ and $\pm e_2$). Another choice for C that gives a unique path (therefore a unique tiling) is $[0, 1) \times (0, 1]$.

We give L_α^\perp an orientation towards the “northwest”, and identify it with the real line. Let us rescall it by $\sqrt{1 + \alpha^2}$, and identify W_α with the interval $[-\alpha, 1)$. Let Z_α be the projection of Σ onto W_α (with the rescalling just mentioned, this is $(1 + \alpha^2)p_\alpha^\perp(\Sigma)$): it is a dense subset of W_α . Let A_α be the C^* -algebra generated by the characteristic functions $\chi_{[x_1, x_2)}$ for $x_1 < x_2$ in Z_α , *i.e.* the closure for the uniform norm of linear combinations of such $\chi_{[x_1, x_2)}$ with coefficients in \mathbb{C} . We define the transversal Ξ_α as the spectrum of A_α : $\Xi_\alpha = \text{Sp}(A_\alpha)$. Therefore Ξ_α is compact (by Gelfand theorem) and totally disconnected (since A_α is generated by idempotents).

Lemma 1.5.1 *The C^* -algebra $B = C_0([-\alpha, 1))$ is a closed subalgebra of A_α*

Proof. Any continuous function f on $[-\alpha, 1)$ that vanishes at 1 can be expressed as a uniform limit of a sequence (f_n) in A_α : choose for instance $f_n = \sum_{i=1}^n f(x_i)\chi_{[x_{i-1}, x_i]}$ for $x_0 = -\alpha < x_1 < \dots < x_n = 1$ in Z_α . Hence B is a subalgebra of A_α .

If $f \in A_\alpha$ is the limit of a sequence (f_n) in B then f is continuous and $\lim_{1^-} f = \lim_{1^-} \lim_n f_n = \lim_n \lim_{1^-} f_n = 1$ because the convergence is for the uniform norm. So f belongs to B . Hence B is closed in A_α \square

Note that B is non unital for $[-\alpha, 1)$ is not compact.

By the duality between a locally compact Hausdorff space X and its Banach algebra of continuous functions vanishing at infinity $C_0(X)$, there is a continuous surjection $\pi : \text{Sp}(A_\alpha) = \Xi_\alpha \rightarrow \text{Sp}(B) = [-\alpha, 1)$. We can therefore see the transversal as a completion of $[-\alpha, 1)$ for a finer topology than the usual one. Let us describe this topology in more details.

Proposition 1.5.2 *For $x \in [-\alpha, 1) \setminus Z_\alpha$ let δ_x be the character $\delta_x(f) = f(x)$, and for $x \in Z_\alpha$ let $\delta_{x\pm}$ be the character $\delta_{x\pm}(f) = \lim_{h \downarrow 0} f(x \pm h)$. The spectrum of A_α is the set of characters δ_x and $\delta_{x\pm}$ (endowed with the weak-* topology).*

Proof. First recall that the spectrum of a C^* -algebra is a subspace of the unit ball in its dual endowed with the weak-* topology. Hence it is compact by Banach-Alaoglu's theorem.

Let $\xi \in \text{Sp}(A_\alpha)$. Consider a point $x_1 \in (-\alpha, 1)$ and define $\chi_1 = \chi_{[-\alpha, x_1]}$, and $\chi_2 = \chi_{[x_1, 1)}$. We have: $\chi_1 + \chi_2 = 1$ the constant function equal to 1 on $[-\alpha, 1)$, and $\xi(\chi_i), i = 1, 2$ is either 0 or 1. For all $f \in A_\alpha$, we can write $\xi(f) = \xi(\chi_1 f + \chi_2 f) = \xi(\chi_1)\xi(f) + \xi(\chi_2)\xi(f)$ and one of the last two term is 0, so that $\xi(f) = \xi(\chi_{i_1} f)$ where i_1 is 1 or 2. We can now pick up a point x_2 in the domain $[l_1, r_1)$ of χ_{i_1} and write $\chi_{i_1} = \chi_{[l_1, x_2]} + \chi_{[x_2, r_1)}$. Again ξ will be non zero on only one of those characteristic functions, that we call $\chi_{i_2} = \chi_{[l_2, r_2)}$, and we can write $\xi(f) = \xi(\chi_{i_2} f)$. We repeat this process inductively to get a sequence of points $x_i \in [l_i, r_i)$ that converges to some x .

If $x \notin Z_\alpha$ then ξ evaluates f at x , and if $x \in Z_\alpha$ it computes the left or right limit of f at x . \square

As a consequence, we can identify the transversal as the set $[-\alpha, 1)$ with a discontinuity at each $x \in Z_\alpha$. The completion consists in replacing each such x by two points x^- and x^+ , its left and right limits. Now a weak- $*$ open neighborhood of ξ in $\text{Sp}(A_\alpha)$ is given by a set of the form $U_{\xi, \varepsilon, f_1, \dots, f_n} = \{\eta : |\xi(f_i) - \eta(f_i)| < \varepsilon, i = 1, \dots, n\}$, for some $f_i \in A_\alpha$ and $\varepsilon > 0$. Let us choose only one function, and a generator of A_α , $f = \chi_{[x_1, x_2]}$ for $x_1 < x_2 \in Z_\alpha$, take $\varepsilon = 0$, and consider the character $\xi = \delta_x$ for $x \in (x_1, x_2) \setminus Z_\alpha$. The corresponding open neighborhood is easily seen to be the set of η such that $0 = |\eta(\chi_{[x_1, x_2]}) - \delta_x(\chi_{[x_1, x_2]})| = |\eta(\chi_{[x_1, x_2]}) - 1|$. And since η is some δ_y or δ_{y^\pm} the equality implies that y has to be in $[x_1, x_2)$. We thus see that the intervals $[x_1, x_2)$ of $[-\alpha, 1)$ with its usual topology, are open for the topology of the transversal (for which they read $[-x_1^+, x_2^-]$). They are also closed since their complements, $[-\alpha, x_1) \cup [x_2, 1)$, are the unions of two such sets. Those intervals form a countable basis of closed and open sets for the topology of the transversal. In addition, the transversal is perfect for this topology (every point is a point of accumulation). Hence it is a Cantor set (compact, perfect, and totally disconnected).

We identify the transversal with the unit circle endowed with a Cantor topology, \mathbb{S}_α^1 , by identifying $-\alpha^\pm$ and 1^\pm and rescaling the interval to one of length one. The Cantor circle \mathbb{S}_α^1 can also be defined directly as the spectrum of the C^* -algebra generated by the set $\{\chi_{[0, \frac{\alpha}{1+\alpha})} \circ T_\alpha^n : n \in \mathbb{Z}\}$ where $[0, \frac{\alpha}{1+\alpha})$ is the arc in \mathbb{S}^1 of length $\frac{\alpha}{1+\alpha}$ and T_α is the rotation by the angle $\frac{\alpha}{1+\alpha}$. A countable basis for the topology of \mathbb{S}_α^1 is given by the closed and open (clopen) intervals $i_{lm} = [l\frac{\alpha}{1+\alpha}, m\frac{\alpha}{1+\alpha}) \pmod 1$ for integers l, m .

In the special case when $\alpha = \sigma = \frac{\sqrt{5}-1}{2}$ is the gold mean, \mathcal{T}_σ is the Fibonacci tiling, and is also a substitution tiling with substitution rules $a \rightarrow ab$ and $b \rightarrow a$ [8, 82]. If α is quadratic irrational (a root of a polynomial of degree 2 with integer coefficients) then \mathcal{T}_α is also a substitution tiling [82].

We orient L_α in the direction of the first quadrant (northeast), and puncture the tiles by their left vertices for convenience (not their barycenters). The prototile space $\mathcal{B}_0 = \mathbb{S}^1 \vee \mathbb{S}^1$ is the wedge sum of two circles corresponding to the two prototiles with their vertices identified. The difference with section 2.3.2 is that the Ξ_Δ^n 's do not partition Ξ_Δ since there is only one vertex here, see definition 2.3.1. Consequently the canonical transversal and Δ -transversal are identical here. As explained earlier, the transversal Ξ_α is seen as the interval $[-\alpha, 1)$ on the line L_α^\perp . The interval $[-\alpha, 0)$ is the acceptance zone Ξ_a of the prototile a whose representatives are the projections of horizontal intervals of Σ , and $[0, 1)$ is the acceptance zone Ξ_b of the prototile whose representatives are the projections of vertical intervals of Σ .

CHAPTER II

TOPOLOGICAL INVARIANTS OF TILINGS

We describe here some topological invariants of interest for tilings, namely various cohomologies and K -theories. In particular we define a new tiling cohomology, the PV cohomology, and relate it to Čech cohomology. A link between cohomology and K -theory for tilings is given in chapter 5 where we present a spectral sequence which converges to the K -theory of a tiling with page-2 given by its PV cohomology.

2.1 Homotopy and homology of tiling spaces

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with FLC. Let Ω be its hull (definition 1.1.4).

Most homotopy and homology groups of Ω are of no interest for us. First the 0-th homology group $H_0(\Omega; \mathbb{Z})$ count the number of *paths components* of Ω . But a path component is an orbit and there are uncountably many of them. Hence

$$H_0(\Omega; \mathbb{Z}) \cong \mathbb{Z}^{\aleph_1}.$$

The homotopy groups $\pi_n(\Omega)$ and the homology groups $H_n(\Omega; \mathbb{Z})$, for $n > 0$ and $n \neq d$, probe a single path component. But each orbit is contractible so

$$\pi_n(\Omega) \cong H_n(\Omega; \mathbb{Z}) \cong 0.$$

The case $n = d$ is special, and we will discuss it briefly in section 3.2. Note that if \mathcal{T} is not FLC, then Ω might have some non contractible closed orbits, and therefore some non trivial homotopy groups.

On the other hand, for spaces such as Ω , the Čech cohomology is much better suited. First the 0-th cohomology group $\check{H}_0(\Omega; \mathbb{Z})$ counts the number of *connected*

components. But Ω is connected so

$$H^0(\Omega; \mathbb{Z}) \cong \mathbb{Z}.$$

Now the Čech cohomology is well-behaved under inverse limits: if $X = \varprojlim (X_n, g_n)$ then $\check{H}^*(X; \mathbb{Z}) \cong \varinjlim (\check{H}^*(X_n), g_n^*)$, see theorem 2.2.2, hence we have the following

Theorem 2.1.1 *Let (\mathcal{B}_n, f_n) be a proper sequence of patch spaces (definition 1.4.3).*

We have the isomorphism:

$$\check{H}^*(\Omega; \mathbb{Z}) \cong \varinjlim (\check{H}^*(\mathcal{B}_n), f_n^*).$$

2.2 Introduction to Čech cohomology

We review here the basic definitions and properties of Čech cohomology with integer coefficients. The reader is referred for instance to the classical book of Bott and Tu [19] (section 10 and 15 in particular) for the general theory of Čech cohomology with values in a sheaf of Abelian groups. Let X be a topological space and \mathcal{U} an open cover of X .

Definition 2.2.1 *The nerve of \mathcal{U} is the simplicial complex $N(\mathcal{U})$ defined as follows:*

- (i) $N(\mathcal{U})^0$ is the set of vertices $[\alpha]$ associated with each non empty \mathcal{U}_α in \mathcal{U} ,
- (ii) $N(\mathcal{U})^n$, for $n > 0$, is the set of n -simplices $[\alpha_0, \alpha_1, \dots, \alpha_n]$ associated with each non empty intersection $\mathcal{U}_{\alpha_0} \cap \mathcal{U}_{\alpha_1} \cap \dots \cap \mathcal{U}_{\alpha_n}$, and attached by their faces $\partial_i[\alpha_0, \alpha_1, \dots, \alpha_n]$, $0 \leq i \leq n$, to the corresponding $n-1$ -simplices $[\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_n]$, where the “hat” symbol over α_i indicates that this vertex is deleted from the sequence $\alpha_0, \dots, \alpha_n$.

The link with the formal definition of a nerve for a small category and its classifying space (see section 6.2) is as follows. Consider the small category $\mathcal{C}_{\mathcal{U}}$ obtained by

regarding the cover \mathcal{U} as a partially ordered set with relation that of set inclusion: its objects are the open sets in \mathcal{U} and its morphisms are given by inclusion: $i_{\alpha\beta} : U_\alpha \longrightarrow U_\beta$ if $U_\alpha \subset U_\beta$. The nerve of the cover \mathcal{U} is the geometric realization of the nerve of the category $\mathcal{C}_{\mathcal{U}}$, *i.e.* its classifying space.

The Čech cohomology of the covering \mathcal{U} is defined as the simplicial cohomology of its nerve

$$\check{H}^*(\mathcal{U}; \mathbb{Z}) := H_{\text{simplicial}}^*(N(\mathcal{U}); \mathbb{Z}).$$

If \mathcal{V} is a refinement of the covering \mathcal{U} , then there is a simplicial map $N(\mathcal{V}) \rightarrow N(\mathcal{U})$ and an induced canonical map on cohomology $\rho_{\mathcal{U}\mathcal{V}} : \check{H}^*(\mathcal{U}; \mathbb{Z}) \rightarrow \check{H}^*(\mathcal{V}; \mathbb{Z})$. The Čech cohomology of X is defined as the direct limit

$$\check{H}^*(X; \mathbb{Z}) := \varinjlim (\check{H}^*(\mathcal{U}; \mathbb{Z}), \rho_{\mathcal{U}\mathcal{V}}).$$

Note: the Čech cohomology of an open cover is usually defined with values in a presheaf (see for instance [19] section 10), but we will stick to this simpler version here.

In practice $\check{H}^*(X; \mathbb{Z})$ is often computable from finite calculation. A *good cover* \mathcal{U} of X is a cover of X such that every U_α and every non empty intersection $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ are contractible. For instance, a good cover of the circle \mathbb{S}^1 requires at least three open sets intersecting two by two. If \mathcal{U} is a good cover of X then the Čech cohomology of X is isomorphic to the Čech cohomology of \mathcal{U} :

$$\check{H}^*(X; \mathbb{Z}) \cong \check{H}^*(\mathcal{U}; \mathbb{Z}), \quad \text{for any good cover } \mathcal{U}.$$

If X is a simplicial complex, there exist a good cover \mathcal{U} whose nerve is homeomorphic to X : associated with each vertex $\alpha \in X^0$ let U_α be the union of α and the interior of all cells that have α as a vertex. The intersection $U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ is non empty if and only if the U_{α_i} share an $(n-1)$ -cell. Hence for a simplicial complex the Čech cohomology is isomorphic to the simplicial (hence singular and cellular) cohomology. This is also true for a Δ -complex, since it can be refined as a simplicial complex.

Theorem 2.2.2 *Assume that $X = \varprojlim (X_n, f_n)$ with each X_n a simplicial complex, and $f_n : X_n \rightarrow X_{n-1}$. Then*

$$\check{H}^*(X; \mathbb{Z}) \cong \varinjlim (\check{H}^*(X_n; \mathbb{Z}), f_n^*).$$

Proof. Let $\pi_n : X \rightarrow X_n$ be the projection onto the n -th coordinate in the inverse limit. Since $X \subset \prod_n X_n$ a basis for the open sets in X is given by preimages $\pi_n^{-1}(U_n)$ of open sets U_n in X_n . Let \mathcal{U}_0 be a good cover of X_0 , and for $n > 0$, let \mathcal{U}_n be a good cover of X_n that is a refinement of $f_n^{-1}(\mathcal{U}_{n-1})$. The preimage $\mathcal{V}_n = \pi_n^{-1}(\mathcal{U}_n)$ is a cover of X (although not a good cover), and one readily checks that $N(\mathcal{V}_n) \cong N(\mathcal{U}_n)$. Now $\check{H}^*(X) = \varinjlim \check{H}^*(\mathcal{V}_n)$, and $\check{H}^*(\mathcal{V}_n) = H_{\text{simplicial}}(N(\mathcal{V}_n)) \cong H_{\text{simplicial}}^*(N(\mathcal{U}_n)) = \check{H}^*(X_n)$, so that $\check{H}^*(X) \cong \varinjlim \check{H}^*(X_n)$. \square

2.3 Cohomology of tilings

We first review the various tiling cohomologies that have been used for tilings, and then introduce a new one: the PV cohomology. We give an explicit calculation of PV cohomology in section 2.3.3.

2.3.1 Tiling Cohomologies

Various cohomologies for tiling spaces have been used in the literature. For the class of tilings considered here (aperiodic and repetitive with FLC) they are all isomorphic. First, the Čech cohomology of the hull was introduced for instance in [1] for substitution tilings (and in full generality for tilings on Riemannian manifolds in [71]). Hulls of such tilings are obtained by inverse limit of finite CW -complexes, and their Čech cohomology is obtained by direct limit.

For repetitive tilings with FLC, if a lamination structure is given to the hull as in [11], the cohomology of the hull is defined by direct limit of the simplicial cohomologies of branched manifolds that approximate the hull by inverse limit (a generalisation of

theorem 1.4.4). In terms of the lamination structure on the hull, this cohomology is isomorphic to the transverse cohomology defined by Moore and Schochet in [61] for foliated spaces. This cohomology is isomorphic to the Čech cohomology of the hull, using the natural isomorphism between Čech and simplicial cohomologies that holds for such branched manifolds (which are CW -complexes) and passing to direct limit. Another useful cohomology, the Pattern-Equivariant (PE) cohomology, has been proposed by Kellendonk and Putnam in [49, 50] for real coefficients and then generalized to integer coefficients by Sadun [73]. This cohomology has been used for proving that the Ruelle-Sullivan map (associated with an ergodic invariant probability measure on the hull) from the Čech cohomology of the hull to the exterior algebra of the dual of \mathbb{R}^d is a ring homomorphism. Let \mathcal{T} be a repetitive tiling with FLC. Assume its tiles are compatible CW -complexes, so that it gives a CW -complex decomposition of \mathbb{R}^d . The group of PE n -cochains C_{PE}^n is a subgroup of the group of integer singular n -cochains of \mathbb{R}^d that satisfy the following property: an n -cochain φ is said to be PE if there exists a patch p of \mathcal{T} such that $\varphi(\sigma_1) = \varphi(\sigma_2)$, for 2 n -simplices σ_1, σ_2 with image cells e_1, e_2 , whenever there exists $x \in e_1, y \in e_2$ such that $\mathbf{p}_p(\mathbb{T}^{-x}\mathcal{T}) = \mathbf{p}_p(\mathbb{T}^{-y}\mathcal{T})$. The simplicial coboundary of a PE cochain is easily seen to be PE (possibly with respect to a patch of larger radius), and this defines the complex for integer PE cohomology. A PE cochain is a cochain that agree on points which have the same local environments in \mathcal{T} or equivalently is the pull back of a cochain on some patch space \mathcal{B}_p . Hence PE cohomology can be seen as the direct limit of the singular cohomologies of a proper sequence of patch spaces. Using the natural isomorphisms between cellular and Čech cohomologies that holds for those CW -complexes and taking direct limits, the integer PE cohomology turns out to be isomorphic to the Čech cohomology of the hull.

The PV cohomology described in the next section 2.3.2 is also isomorphic to the Čech cohomology of the hull (see theorem 2.4.1).

2.3.2 The PV cohomology

The definition of a Δ -complex structure is first recalled, following the presentation of Hatcher in his book on Algebraic Topology [38] section 2.1.

Given $n + 1$ points v_0, \dots, v_n , in $\mathbb{R}^m, m > n$, that are not collinear, let $[v_0, \dots, v_n]$ denote the n -simplex with vertices v_0, \dots, v_n . Let Δ^n denote the standard n -simplex

$$\Delta^n = \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \right\},$$

whose vertices are the unit vectors along the coordinate axes. An ordering of those vertices is specified and this allows to define a canonical linear homeomorphism between Δ^n and any other n -simplex $[v_0, \dots, v_n]$, that preserves the order of the vertices, namely, $(x_0, x_1, \dots, x_n) \mapsto \sum x_i v_i$.

If one of the $n + 1$ vertices of an n -simplex $[v_0, \dots, v_n]$ is deleted, then the remaining n vertices span an $(n - 1)$ -simplex, called a face of $[v_0, \dots, v_n]$. By convention the vertices of any subsimplex spanned by a subset of the vertices are ordered according to their order in the larger simplex.

The union of all the faces of Δ^n is the boundary of Δ , written $\partial\Delta^n$. The open simplex $\overset{\circ}{\Delta}^n$ is $\Delta^n \setminus \partial\Delta^n$, the interior of Δ^n .

A Δ -complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \rightarrow X$, with n depending on the index α , such that

- (i) The restriction $\sigma_\alpha|_{\overset{\circ}{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction.
- (ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \rightarrow X$. The face of Δ^n is identified with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.
- (iii) A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α .

A Δ -complex X can be built as a quotient space of a collection of disjoint simplices by identifying various subsimplices spanned by subsets of the vertices, where the identifications are performed using the canonical linear homeomorphism that preserves orderings of the simplices. It can be shown that X is a Hausdorff space. Then condition (iii) implies that each restriction $\sigma_\alpha|_{\overset{\circ}{\Delta}^n}$ is a homeomorphism onto its image, which is thus an open simplex in X . These open simplices $\sigma_\alpha(\overset{\circ}{\Delta}^n)$ are cells e_α^n of a CW -complex structure on X with the σ_α 's as characteristic maps.

It is now assumed that the tiles of \mathcal{T} are compatible finite Δ -complexes: the intersection of two tiles is a sub- Δ -complex of both. In addition each cell $e_\alpha^n = \sigma_\alpha(\overset{\circ}{\Delta}^n)$ of each tile is punctured, by say the image under σ_α of the barycenter of Δ^n . Hence \mathcal{T} can be seen as a Δ -complex decomposition of \mathbb{R}^d , and this Δ -complex structure gives a “refinement” of the tiling (each tile being decomposed into the union of the closures of the cells it contains). If \mathcal{T} is the Voronoi tiling of a Delone set its tiles can be split into d -simplices since they are polytopes: this gives a Δ -complex structures for which the maps σ_α are simply affine maps. The maps $\sigma_\alpha : \Delta^n \rightarrow \mathcal{B}_0$ of the Δ -complex structure of \mathcal{B}_0 , will be called the *characteristic maps* of the n -simplices on \mathcal{B}_0 represented by their image.

By construction (definition 1.3.1) as a quotient CW -complex, the prototile space \mathcal{B}_0 is a finite Δ -complex.

Definition 2.3.1 *The Δ -transversal, denoted Ξ_Δ , is the subset of Ω consisting of tilings that have the puncture of one of their cells (of one of their tiles) at the origin $0_{\mathbb{R}^d}$.*

Since \mathcal{T} has finitely many prototiles and they have finite CW -complex structures, the set of the punctures of the cells of the tiles of \mathcal{T} is a Delone set. This Delone set will be called the Δ -Delone set of \mathcal{T} , and denoted \mathcal{L}_Δ . The Δ -transversal is thus the canonical transversal of \mathcal{L}_Δ .

The Δ -transversal is not immediately related to the canonical transversal. Indeed the lift of the puncture of a cell does not belong to the canonical transversal in general, unless this puncture coincides with the puncture of the tile of \mathcal{B}_0 that contains that cell.

The Δ -transversal is the lift of the punctures of the cells in \mathcal{B}_0 . It is partitioned by the lift of the punctures of the n -cells, denoted Ξ_Δ^n which is the subset of Ω consisting of tilings that have the puncture of an n -cell at the origin. As for the canonical transversal, the Δ -transversal is a Cantor set, and the Ξ_Δ^n 's give a partition by clopen sets. The ring of continuous integer-valued functions on the Δ -transversal, $C(\Xi_\Delta, \mathbb{Z})$, is thus the direct sum of the $C(\Xi_\Delta^n, \mathbb{Z})$'s for $n = 0, \dots, d$.

Given the characteristic map σ of an n -simplex e of \mathcal{B}_0 , let $\Xi_\Delta(\sigma)$ denote the lift of the puncture of e , and χ_σ its characteristic function on Ξ_Δ . The subset $\Xi_\Delta(\sigma)$ is called the *acceptance zone* of σ . Since continuous integer-valued functions on a totally disconnected space are generated by characteristic functions of clopen sets, χ_σ belongs to $C(\Xi_\Delta, \mathbb{Z})$.

Consider the characteristic map $\sigma : \Delta^n \rightarrow \mathcal{B}_0$ of an n -simplex e of \mathcal{B}_0 , and τ a face of σ (*i.e.* the restriction of σ to a face of Δ^n) with associated simplex f (a face of e). The simplices e and f in \mathcal{B}_0 are contained in some tile τ_j . Viewing e and f as subsets of the tile t_j in \mathbb{R}^d , it is possible to define the vector $x_{\sigma\tau}$ that joins the puncture of f to the puncture of e . Notice that since \mathcal{B}_0 is a *flat* branched manifold (remark 1.3.3, and [11]), the vector $x_{\sigma\tau}$ is also well defined in \mathcal{B}_0 as a vector in a region containing the simplex e .

Definition 2.3.2 (i) Let σ and τ be characteristic maps of simplices in \mathcal{B}_0 . The operator $\theta_{\sigma\tau}$, on $C(\Xi_\Delta, \mathbb{Z})$, is defined by

$$\theta_{\sigma\tau} = \begin{cases} \chi_\sigma \mathbf{T}^{x_{\sigma\tau}} \chi_\tau & \text{if } \tau \subset \partial\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

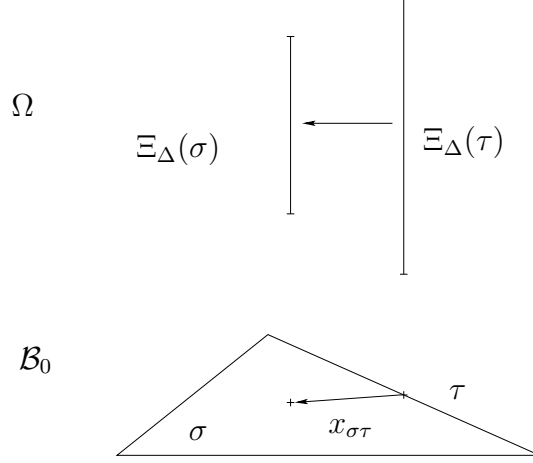


Figure 4: Fibers over the prototile space

where $\tau \subset \partial\sigma$ means that τ is a face of σ . Here the translation acts by $T^{x_{\sigma\tau}}f(\xi) = f(T^{-x_{\sigma\tau}}\xi)$ whenever $f \in C(\Xi_\Delta(\tau), \mathbb{Z})$.

(ii) The function ring of the transversal \mathcal{A}_{Ξ_Δ} is the ring (finitely) generated by the operators $\theta_{\sigma\tau}$ and their adjoints $\theta_{\sigma\tau}^* = \chi_\tau T^{-x_{\sigma\tau}} \chi_\sigma$ for $\tau \subset \partial\sigma$ and 0 otherwise, over all characteristic maps σ and τ of simplices in \mathcal{B}_0 .

The operators $\theta_{\sigma\tau}$'s satisfy the following properties:

$$\theta_{\sigma\tau} \theta_{\sigma\tau}^* = \chi_\sigma \quad \text{if } \tau \subset \partial\sigma, \quad (2.3.1a)$$

$$\sum_{\sigma: \partial\sigma \supset \tau} \theta_{\sigma\tau}^* \theta_{\sigma\tau} = \chi_\tau. \quad (2.3.1b)$$

The $\theta_{\sigma\tau}$'s are thus *partial isometries*. The ring \mathcal{A}_{Ξ_Δ} is unital and its unit is the characteristic function of the Δ -transversal $1_{\mathcal{A}_{\Xi_\Delta}} = \chi_{\Xi_\Delta}$. The ring of continuous integer-valued functions on the Δ -transversal $C(\Xi_\Delta, \mathbb{Z})$ is a left \mathcal{A}_{Ξ_Δ} -module.

Remark 2.3.3 It is important to notice that the operators $\theta_{\sigma\partial\sigma}$'s are in one-to-one correspondence with a set of arrows that generate Γ_Δ^1 , the arrows of the groupoid Γ_Δ of the Δ -transversal (groupoid of the transversal of the Delone set \mathcal{L}_Δ , see definition 1.2.5). The set of arrows Γ_Δ^1 is indeed in one-to-one correspondence with the set

of vectors joining points of \mathcal{L}_Δ , and it is generated by the set of vectors joining a point of \mathcal{L}_Δ to a “nearest neighbour” (a puncture of a face of the simplex of \mathcal{T} whose puncture is that point, or of a simplex containing it on its boundary). Each vector can be decomposed into a sum of such generating vectors, and each arrow can be decomposed into a composition of such generating arrows (corresponding to those generating vectors). This generating set is finite, since \mathcal{T} has FLC (in particular there are finitely many prototiles), and each of its vectors is an $x_{\sigma\partial_i\sigma}$ and thus corresponds to a unique operator $\theta_{\sigma\partial_i\sigma}$.

A “representation by partial isometries” of the set of generators of Γ_Δ^1 is given by

$$\theta(\gamma) = \chi_{\mathbf{p}_0 \circ \mathbf{p}_0^{-1}(s(\gamma))} \mathsf{T}(\gamma) \chi_{\mathbf{p}_0 \circ \mathbf{p}_0^{-1}(r(\gamma))},$$

where, if $\gamma = (\xi, x)$, then $\mathsf{T}(\gamma)$ is the translation operator T^x .

Let \mathcal{S}_0^n be the set of the characteristic maps $\sigma : \Delta^n \rightarrow \mathcal{B}_0$ of the n -simplices of the Δ -complex decomposition of \mathcal{B}_0 , and \mathcal{S}_0 the (disjoint) union of the \mathcal{S}_0^n 's. The group of simplicial n -chains on \mathcal{B}_0 , $C_{0,n}$, is the free abelian group with basis \mathcal{S}_0^n .

Definition 2.3.4 *The PV cohomology of the hull of \mathcal{T} is the homology of the complex $\{C_{PV}^n, d_{PV}^n\}$, where:*

(i) *the PV cochain groups are the groups of continuous integer valued functions on*

$$\Xi_\Delta^n : C_{PV}^n = C(\Xi_\Delta^n, \mathbb{Z}) \text{ for } n = 0, \dots, d,$$

(ii) *the PV differential, d_{PV} , is the element of \mathcal{A}_{Ξ_Δ} given by the sum over $n =$*

$1, \dots, d$, of the operators

$$d_{PV}^n : \begin{cases} C_{PV}^{n-1} \longrightarrow C_{PV}^n \\ d_{PV}^n = \sum_{\sigma \in \mathcal{S}_0^n} \sum_{i=0}^n (-1)^i \theta_{\sigma\partial_i\sigma} \end{cases} . \quad (2.3.2)$$

The “simplicial form” of d_{PV}^n makes it clear that $d_{PV}^{n+1} \circ d_{PV}^n = 0$ for $n = 1, \dots, d-1$. We shall call alternatively the PV cohomology of the hull of \mathcal{T} simply the *PV cohomology of \mathcal{T}* .

Remark 2.3.5 Thanks to the lamination structure on Ω described in remark 1.3.3 the map \mathfrak{p}_0

$$\begin{array}{ccc} \Xi_\Delta & \hookrightarrow & \Omega \\ & & \downarrow \mathfrak{p}_0 \\ & & \mathcal{B}_0 \end{array}$$

looks very much like a fibration of Ω with base space \mathcal{B}_0 and fiber Ξ_Δ . However because of the branched structure there are paths in \mathcal{B}_0 that cannot be lifted to any leaf of Ω . For example, if we can lift a loop γ , it gives rise to a path p_γ in some patch q of \mathcal{T} . Intuitively, the aperiodicity of \mathcal{T} prevents \mathfrak{p}_0 from being a fibration. Indeed if it were we could also lift γ^n for any arbitrary $n \in \mathbb{N}$, so we could find a patch q_n in \mathcal{T} that contains n copies of q put one after the other: this would force some form of periodicity for \mathcal{T} . Hence \mathfrak{p}_0 is not a fibration. Nevertheless the result in the present paper (theorem 5.2.1 in section 5.2) gives a spectral sequence analogous to the Serre spectral sequence for a fibration (see section 4.2).

Remark 2.3.6 This PV cohomology of \mathcal{T} is formally analogous to a cohomology with local coefficients (see section 4.2.2) but in a more general setting. Let G_x be the group $C(\mathfrak{p}_0^{-1}(x), \mathbb{Z})$ for x in \mathcal{B}_0 . As $\mathfrak{p}_0^{-1}(x)$ is a Cantor set, G_x is actually its K^0 -group: $G_x = K^0(\mathfrak{p}_0^{-1}(x))$ while its K^1 group is trivial. The family of groups $(G_x)_{x \in \mathcal{B}_0}$ is analogous to a local coefficient system, with the operators $\theta_{\sigma\tau}$'s in the role of group isomorphisms between the fibers, but they are not isomorphisms here (as they come from a groupoid). If σ in C_0^m is an n -chain, and φ in C_{PV}^{m-1} is a PV $(n-1)$ -cochain, then the differential

$$d_{PV}^m \varphi(\sigma) = \sum_{i=0}^n (-1)^i \theta_{\sigma \partial_i \sigma} \varphi(\partial_i \sigma),$$

takes the restrictions of φ to the faces $\partial_i \sigma$, which are elements of the groups $G_{x_{\partial_i \sigma}}$ (where $x_{\partial_i \sigma}$ is the puncture of the image simplex of $\partial_i \sigma$), and pull them to σ with the

operators $\theta_{\sigma\partial_i\sigma}$'s to get an element of the group G_{x_σ} (where x_σ is the puncture of the image simplex of σ).

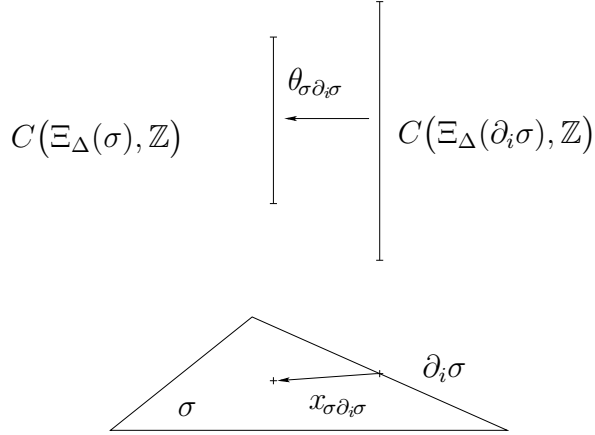


Figure 5: The PV local coefficient system

A rigorous formulation of the points in this remark as well as in the previous remark 2.3.3 will be given in section 6.7.

2.3.3 An example: PV cohomology in 1-dimension

This section presents the explicit calculation of PV cohomology of some 1-dimensional tilings obtained by “cut and projection”. This is to illustrate techniques that can be used to calculate PV cohomology, rather than showing its distinct features and differences compared to other cohomologies. Further examples in higher dimensions, and methods of calculation for PV cohomology will be investigated in a future work.

We consider the tiling \mathcal{T}_α described in section 1.5. We recall that it is obtained by projecting a strip of the lattice points Σ of \mathbb{Z}^2 parallel to a line L_α of irrational slope $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$.

The intersection of the strip with the orthogonal complement L_α^\perp is the “window” W_α , the projection of the unit cube “open on the top and left” $(0, 1] \times [0, 1)$ onto L_α^\perp . We give L_α^\perp and orientation towards the “northwest”, and identify W_α with the interval

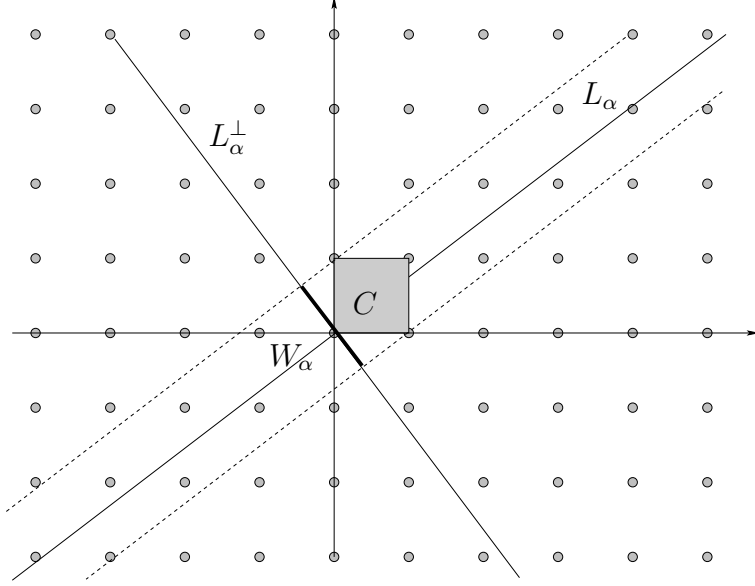


Figure 6: A 1-dimensional cut and project construction

$[-\alpha, 1)$. The transversal Ξ_α to the hull of \mathcal{T}_α is defined as the spectrum of a C^* -algebra associated with \mathcal{T}_α and can be viewed as a completion of the interval $[-\alpha, 1)$ for a finer topology than the usual one, where the intervals $[x_1, x_2)$, for $x_1 < x_2$ in the projection of Σ , are declared both open and closed (see proposition 1.5.2). With this topology, Ξ_α is a Cantor set.

We further map the interval $[-\alpha, 1)$ to the unit circle, and with the topology induced by Ξ_α , we get a Cantor circle \mathbb{S}_α^1 . It admits a countable basis of closed and open (clopen) sets $i_{lm} = [l\frac{\alpha}{1+\alpha}, m\frac{\alpha}{1+\alpha}) \bmod 1$ for integers l, m .

We orient L_α in the direction of the first quadrant (northeast), and puncture the tiles by their left vertices for convenience (not their barycenters). The prototile space $\mathcal{B}_0 = \mathbb{S}^1 \vee \mathbb{S}^1$ is the wedge sum of two circles corresponding to the two prototiles with their vertices identified. The difference with section 2.3.2 is that the Ξ_Δ^n 's do not partition Ξ_Δ since there is only one vertex here, see definition 2.3.1. Consequently the canonical transversal and Δ -transversal are identical here. As explained earlier, the transversal Ξ_α is seen as the interval $[-\alpha, 1)$ on the line L_α^\perp . The interval $[-\alpha, 0)$

is the acceptance zone Ξ_a of the prototile a whose representatives are the projections of horizontal intervals of Σ , and $[0, 1)$ is the acceptance zone Ξ_b of the prototile whose representatives are the projections of vertical intervals of Σ . The vectors $x_{\sigma\tau}$'s in the definitions of the operators $\theta_{\sigma\tau}$'s are here the projections of the canonical vectors of \mathbb{R}^2 onto L_α^\perp (in W_α). Once rescaled to the unit circle \mathbb{S}_α^1 , the operator \mathbb{T}^{x_a} becomes the rotation by angle $\frac{-\alpha}{1+\alpha}$, and \mathbb{T}^{x_b} the rotation by angle $\frac{1}{1+\alpha}$, and are thus equal. The PV complex reads here simply:

$$0 \longrightarrow C(\mathbb{S}_\alpha^1, \mathbb{Z}) \xrightarrow{d_{PV}=\text{id}-\theta_\alpha} C(\mathbb{S}_\alpha^1, \mathbb{Z}) \longrightarrow 0, \quad (2.3.3)$$

where θ_α is the rotation by $\frac{\alpha}{1+\alpha}$ on \mathbb{S}_α^1 , and is unitary.

Proposition 2.3.7 *The PV cohomology of (the hull Ω_α of) \mathcal{T}_α is given by*

$$\begin{cases} H_{PV}^0(\mathcal{B}_0; C(\mathbb{S}_\alpha^1, \mathbb{Z})) \cong \mathbb{Z} \\ H_{PV}^1(\mathcal{B}_0; C(\mathbb{S}_\alpha^1, \mathbb{Z})) \cong \mathbb{Z} \oplus \mathbb{Z} \end{cases}.$$

Proof. A function $f \in C(\mathbb{S}_\alpha^1, \mathbb{Z})$ reads as a finite sum $f = \sum n \chi_{I_n}$ where n is an integer and χ_{I_n} is the characteristic function of the clopen set $I_n = f^{-1}(n)$. The 0-th cohomology group is the set of invariant functions under θ_α . Each I_n is a finite disjoint union of base clopen sets i_{lm} 's. Given two base clopens $i_{lm} \subset I_n$ and $i_{l'm'} \subset I_{n'}$, $\theta_\alpha^{m-m'} i_{l'm'} = [(l' + m - m')\frac{\alpha}{1+\alpha}, m\frac{\alpha}{1+\alpha}) \bmod 1$, so that $i_{lm} \cap \theta_\alpha^{m-m'} i_{l'm'} \neq \emptyset$. Hence if f is invariant under θ_α , then n must equal n' and therefore f must be constant.

The calculation of the first cohomology group, the group of coinvariants under θ_α , relies upon the fact that any function $f \in C(\mathbb{S}_\alpha^1, \mathbb{Z})$ can be written as $n_f \chi_{i_{01}} + m_f \chi_{\mathbb{S}_\alpha^1}$ modulo $(1 - \theta_\alpha)C(\mathbb{S}_\alpha^1, \mathbb{Z})$, for some integers n_f, m_f , that are uniquely determined by the class of f . This technical point is tedious but elementary (it uses an encoding of the real numbers from the partial fraction decomposition of $\frac{\alpha}{1+\alpha}$), see lemma 2.3.9. \square

Remark 2.3.8 (i) Using integration (see lemma 2.3.10), $\int_{\mathbb{S}_\alpha^1}$, one gets a group isomorphism

$$H_{PV}^1(\mathcal{B}_0; C(\mathbb{S}_\alpha^1, \mathbb{Z})) \cong \mathbb{Z} \oplus \frac{\alpha}{1+\alpha} \mathbb{Z},$$

but PV cohomology by itself cannot distinguish between different values of α .

(ii) The above calculations yield also the K -groups of the tiling space:

$$\begin{cases} K^0(\Omega) \cong \check{H}^0(\Omega_\alpha; \mathbb{Z}) \cong H_{PV}^0(\mathcal{B}_0; C(\mathbb{S}_\alpha^1, \mathbb{Z})) \\ K^1(\Omega) \cong \check{H}^1(\Omega_\alpha; \mathbb{Z}) \cong H_{PV}^1(\mathcal{B}_0; C(\mathbb{S}_\alpha^1, \mathbb{Z})) \end{cases}$$

The isomorphism between Čech and PV cohomologies for hull of tilings is proven in theorem 2.4.1. The isomorphisms between K -theory and cohomology of the hull comes from the natural isomorphisms (Chern character) between K -theory and cohomology of CW -complexes of dimension less than 3 (a proof can be found in [1], proposition 6.2), and the fact that Ω_α is the inverse limit of such space (theorem 1.4.4).

(iii) With the previous remark, the PV complex (2.3.3) can be viewed as the Pimsner-Voiculescu exact sequence [64] for the K -theory of the C^* -algebra $C(\mathbb{S}_\alpha^1) \rtimes_{\theta_\alpha} \mathbb{Z}$:

$$\begin{array}{ccccc} K_0(C(\mathbb{S}_\alpha^1)) & \xrightarrow{\text{id}-\theta_\alpha} & K_0(C(\mathbb{S}_\alpha^1)) & \longrightarrow & K_0(C(\mathbb{S}_\alpha^1) \rtimes_{\theta_\alpha} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C(\mathbb{S}_\alpha^1) \rtimes_{\theta_\alpha} \mathbb{Z}) & \longleftarrow & K_1(C(\mathbb{S}_\alpha^1)) & \xleftarrow{\text{id}-\theta_\alpha} & K_1(\mathbb{S}_\alpha^1) \end{array} .$$

Indeed $C(\mathbb{S}_\alpha^1) \rtimes_{\theta_\alpha} \mathbb{Z}$ is the C^* -algebra of the groupoid of the transversal, which is Morita equivalent to the C^* -algebra of the hull of \mathcal{T}_α [9]. Also $K_0(C(\mathbb{S}_\alpha^1)) \cong C(\mathbb{S}_\alpha^1, \mathbb{Z})$ and $K_1(C(\mathbb{S}_\alpha^1)) \cong 0$ [12].

We now give the technical details for the calculation of $H_{PV}^1(\mathcal{B}_0; C(\mathbb{S}_\alpha^1, \mathbb{Z}))$. This group is the cokernel of $d_{PV} = \text{id} - \theta_\alpha$ in the complex (2.3.3). Let us denote it by G_α :

$$G_\alpha := \text{Coker } d_{PV} = C(\mathbb{S}_\alpha^1, \mathbb{Z}) / (\text{id} - \theta_\alpha)C(\mathbb{S}_\alpha^1, \mathbb{Z}) .$$

Let $\beta = \frac{\alpha}{1+\alpha}$, and let us also denote by χ_β the characteristic function of the arc $i_{01} = [0, \beta)$ in \mathbb{S}_α^1 , and by $\mathbf{1}$ the constant function equal to 1 on \mathbb{S}_α^1 .

Lemma 2.3.9 For each $f \in G_\alpha$ there exists two integers $(n_f, m_f) \in \mathbb{Z}^2$ such that $f = [n_f \chi_\beta + m_f \mathbf{1}]$ (where $[g]$ denotes the class in G_α of an element $g \in C(\mathbb{S}_\alpha^1, \mathbb{Z})$).

Proof. For $x \in \mathbb{R}$ we denote by $[x]$ and $\{x\}$ its integer and fractional parts respectively, and for $k \in \mathbb{Z}$ let $\chi_{k\beta}$ denote the characteristic function $\chi_{[0, k\beta \bmod 1]}$. Abusing notation we mean by $x \bmod 1$ the representative of the class $x + \mathbb{Z}$ that lies in the interval $[0, 1]$. Let the continuous fraction decomposition of β be given by the sequence of integers (b_n) , and let (β_n) be the sequence given by $\beta_0 = \beta, \beta_j = \{\frac{1}{\beta_{j-1}}\}, j \geq 1$, so that for all $j \geq 1$ we have $\beta_{j-1} = \frac{1}{b_j + \beta_j}$. Let χ_k be the characteristic function $\chi_{[0, \beta\beta_1 \cdots \beta_{k-1}]}$. The lemma will be proven using the following four claims.

Claim 1 Let I be a clopen set in \mathbb{S}_α^1 , then $\chi_I \sim \chi_{k\beta}$ in G_α for some integer $k \in \mathbb{Z}$.

First notice that if $f \in G_\alpha$ and $f_1 \in C(\mathbb{S}_\alpha^1, \mathbb{Z})$ is a representative of f , then $f_2 = \theta_\alpha f_1$ and $f_3 = \theta_\alpha^* f_1$ are also representatives of f . Now the clopen I is of the form $[n\beta \bmod 1, m\beta \bmod 1]$, and $\chi_I \sim \theta_\alpha^{*n} \chi_I = \chi_{\theta_\alpha^{*n} I}$. And we have $\theta_\alpha^{*n} I = [0, (m-n)\beta \bmod 1]$. Hence $\chi_I \sim \chi_{k\beta}$ with $k = m-n$.

Claim 2 For any integer $k \in \mathbb{Z}$ there exists a unique encoding of $k\beta \bmod 1$ as $k\beta = \varepsilon_k \sum_{j=1}^{n(k)} (-1)^{j-1} c_j \beta \beta_1 \cdots \beta_j \bmod 1$, where ε_k is the sign of k and with $n(k) \in \mathbb{N}$ and $c_j \in \{0, 1, \cdots, b_{j+1}\}, j = 1, \cdots, n(k)$.

We can encode uniquely any real number $x \in [0, 1]$ as $\sum_{j \geq 1} (-1)^{j-1} c_j \beta \beta_1 \cdots \beta_{j-1}$ with $c_j \in \{1, 2 \cdots b_j, \frac{1}{\beta_j}\}$ and the rule $c_j = \frac{1}{\beta_j} \Rightarrow c_{j+1} = 1$. In the case of multiples of β the previous series can be taken to be finite as we now show. The rational approximants of β are given by the fractions $\frac{p_j}{q_j}$ and the p_j 's and q_j 's are defined inductively as $p_0 = 0, p_1 = 1, q_0 = 1, q_1 = b_1$ and $p_{j+1} = b_{j+1}p_j + p_{j-1}, q_{j+1} = b_{j+1}q_j + q_{j-1}, j \geq 2$. Any 2×2

matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ gives a linear action on \mathbb{C}^2 and induces a rational action φ_A on \mathbb{CP}^1 as follows $A \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} a + bz \\ c + dz \end{bmatrix} \sim \begin{bmatrix} 1 \\ \varphi_A(z) \end{bmatrix}$. We have the property $\varphi_A \circ \varphi_B =$

φ_{AB} . Now let $A_n = \begin{bmatrix} b_n & 1 \\ 1 & 0 \end{bmatrix}$, we have $\varphi_{A_i}(z) = \frac{1}{a_i+z}$ and therefore we can express β

as follows $\beta = \varphi_{A_1}(\beta_1) = \varphi_{A_1} \circ \varphi_{A_2} \circ \cdots \circ \varphi_{A_j}(\beta_j)$. Writing $A_1 A_2 \cdots A_j = \begin{bmatrix} q_j & q_{j-1} \\ p_j & p_{j-1} \end{bmatrix}$

we see directly the induction formulae for the p_j 's and q_j 's given above, and we have

$\beta = \frac{p_j + b_j p_{j-1}}{q_j + b_j q_{j-1}}$ so that $\frac{p_j}{q_j}$ is the j -th rational approximant of β . Note that since

$\det(A_n) = -1$, $\det(A_1 A_2 \cdots A_j) = (-1)^j = q_j p_{j-1} - p_j q_{j-1}$, and Bézout's theorem

implies that p_j and q_j are coprime. We have $\beta - \frac{p_j}{q_j} = \frac{(-1)^j}{q_j(q_{j+1} + \beta_{j+1} q_j)} = \frac{(-1)^j \beta_j}{q_j(q_j + \beta_j q_{j-1})}$,

therefore $q_j \beta - p_j = \beta_j \frac{(-1)^j}{q_j + \beta_j q_{j-1}} = -\beta_j (q_{j-1} \beta - p_{j-1}) = \cdots = (-1)^j \beta_j \beta_{j-1} \cdots \beta_1 (q_0 \beta - p_0)$,

and so $q_j \beta = (-1)^j \beta \beta_1 \cdots \beta_j \pmod{1}$. Now given an integer $k \in \mathbb{Z}$, there exists

$n \in \mathbb{N}$ such that $q_n \leq |k| < q_{n+1}$, and we write the euclidean division of $|k|$ by q_n :

$|k| = d_n q_n + k_{n-1}$ with $k_{n-1} < q_n$ and $1 \leq d_n \leq b_{n+1}$ (since $q_{n+1} = b_{n+1} q_n + q_{n-1}$).

We now write the division of k_{n-1} by q_{n-1} and keep going on to get eventually $|k| =$

$d_n q_n + \cdots + d_1 q_1 + d_0 q_0$. Note that if $d_n = b_{n+1}$ then $k_{n-1} < q_{n-1}$ and so $d_{n-1} = 0$, and

we set $k_{n-1} = k_{n-2}$ and continue the process. We thus get that $0 \leq d_j \leq b_{j+1}$ and

the rule $d_j = b_{j+1} \Rightarrow d_{j-1} = 0$. Eventually with the identity $q_j \beta = (-1)^j \beta \beta_1 \cdots \beta_j$

$\pmod{1}$ we deduce that $|k| \beta = \sum_{j=0}^n (-1)^j d_j \beta \beta_1 \cdots \beta_j \pmod{1}$. We now let $c_j = d_{j+1}$

and $n(k) = n + 1$ to get the encoding $k \beta = \varepsilon_k \sum_{j=1}^{n(k)} (-1)^{j-1} c_j \beta \beta_1 \cdots \beta_{j-1} \pmod{1}$,

with $0 \leq c_j \leq b_j$ and the rule $c_j = b_j \Rightarrow c_{j-1} = 0$.

Claim 3 *With the encoding of $k \beta$ given in Claim 2 we have $\chi_{k \beta} \sim \varepsilon_k \sum_{j=1}^{n(k)} (-1)^{j-1} c_j \chi_j$ in G_α .*

We first show that for $i < j$ we have $\chi_{[0, \beta \beta_1 \cdots \beta_{i-1} \pm \beta \beta_1 \cdots \beta_{j-1}]} \sim \chi_i \pm \chi_j$. Indeed, $\chi_j \sim$

$\chi_{[\beta \beta_1 \cdots \beta_{i-1}, \beta \beta_1 \cdots \beta_{i-1} + \beta \beta_1 \cdots \beta_{j-1}]}$ by applying θ_α (respectively θ_α^*) n_i -times if i is odd (re-

spectively even) since $\beta \beta_1 \cdots \beta_{i-1} = (-1)^{i-1} n_i \beta \pmod{1}$ (see Claim 4), thus $\chi_i + \chi_j \sim$

$\chi_{[0, \beta \beta_1 \cdots \beta_{i-1} + \beta \beta_1 \cdots \beta_{j-1}]}$. Also $\chi_j \sim \chi_{[-\beta \beta_1 \cdots \beta_{j-1}, 0]} \sim \chi_{[\beta \beta_1 \cdots \beta_{i-1} - \beta \beta_1 \cdots \beta_{j-1}, \beta \beta_1 \cdots \beta_{i-1}]}$, and

thus $\chi_i - \chi_j \sim \chi_{[0, \beta \beta_1 \cdots \beta_{i-1} - \beta \beta_1 \cdots \beta_{j-1}]}$. We now prove the claim by induction on $n(k)$.

Assume the claim is true for all j for which $n(j) \leq n - 1$, and consider $k \beta \pmod{1} =$

$\varepsilon_k \sum_{j=1}^n (-1)^{j-1} c_j \beta \beta_1 \cdots \beta_{j-1}$ with $c_n \neq 0$. Let $\xi_{n-1} = \varepsilon_k \sum_{j=1}^{n-1} (-1)^j c_j \beta \beta_1 \cdots \beta_{j-1}$. By induction hypothesis we have $\varepsilon_k \sum_{j=1}^n (-1)^j c_j \chi_j \sim \chi_{[0, \xi_{n-1}]} + \varepsilon_k (-1)^{n-1} c_n \chi_n$. If $nk < 0$, then with the identification $c_n \chi_n \sim \chi_{[0, c_n \beta \beta_1 \cdots \beta_{n-1}]} \sim \chi_{[\xi_{n-1} - c_n \beta \beta_1 \cdots \beta_{n-1}, \xi_{n-1}]}$ we get that $\chi_{[0, \xi_{n-1}]} - c_n \chi_n \sim \chi_{[0, \xi_{n-1} - c_n \beta \beta_1 \cdots \beta_{n-1}]} = \chi_{k\beta}$, while if $nk > 0$ we use instead the identification $c_n \chi_n \sim \chi_{[\xi_{n-1}, \xi_{n-1} + c_n \beta \beta_1 \cdots \beta_{n-1}]}$ to get $\chi_{[0, \xi_{n-1}]} + c_n \chi_n \sim \chi_{[0, \xi_{n-1} + c_n \beta \beta_1 \cdots \beta_{n-1}]} = \chi_{k\beta}$.

Claim 4 For $j \in \mathbb{N}$, there exist $n_j, m_j \in \mathbb{Z}$ such that $\chi_j \sim n_j \chi_\beta + m_j \mathbf{1}$ in G_α , where the n_j 's and m_j 's are defined inductively by $n_1 = 1, n_2 = -b_1, m_1 = 0, m_2 = 1$ and $n_{j+1} = n_{j-1} + b_j n_j, m_{j+1} = m_{j-1} + b_j m_j$ for $j \geq 2$.

We first prove by induction that $\beta \beta_1 \cdots \beta_{j-1} = n_j \beta + m_j$. For $j = 1$ this reads $\beta = 1\beta + 0$ and for $j = 2$ we have $\beta \beta_1 = -b_1 \beta + 1$ (since $\beta = \frac{1}{b_1 + \beta_1}$) and this fixes the first two terms of the n_j 's and m_j 's. Now assume that for all $k \leq j$ there exist $n_k, m_k \in \mathbb{Z}$ such that $\beta \beta_1 \cdots \beta_{k-1} = n_k \beta + m_k$. Since $\beta_{j-1} = \frac{1}{b_j + \beta_j}$ we can write $\beta \beta_1 \cdots \beta_j = \beta \beta_1 \cdots \beta_{j-2} - b_j \beta \beta_1 \cdots \beta_{j-1}$ which is $(n_{j-1} - b_j n_j) \beta + (m_{j-1} - b_j m_j)$ by induction hypothesis, and this proves the identity. We now prove the claim similarly. For $j = 1$ this reads: $\chi_\beta = 1\chi_\beta + 0\mathbf{1}$, and for $j = 2$: $\chi_2 = \chi_{[0, \beta \beta_1]} \sim \chi_{[b_1 \beta, 1]} = -\chi_{[0, b_1 \beta]} + \chi_{[0, 1]} \sim -b_1 \chi_\beta + 1\mathbf{1}$. Now $\chi_{j+1} = \chi_{[0, \beta \beta_1 \cdots \beta_j]}$ which is equivalent to $\chi_{[b_j \beta \beta_1 \cdots \beta_{j-1}, \beta \beta_1 \cdots \beta_{j-2}]}$ and reads $-\chi_{[0, b_j \beta \beta_1 \cdots \beta_{j-1}]} + \chi_{[0, \beta \beta_1 \cdots \beta_{j-2}]} \sim -b_j \chi_j + \chi_{j-1}$, and by induction hypothesis this is $(n_{j-1} - b_j n_j) \chi_\alpha + (m_{j-1} - b_j m_j) \mathbf{1} = n_{j+1} \chi_\alpha + m_{j+1} \mathbf{1}$.

We now finish the proof of the lemma. Let $f \in G_\alpha$. We can write a representative $f_1 \in C(\mathbb{S}_\alpha^1, \mathbb{Z})$ of f as $f_1 = \sum_{i=1}^N l_i \chi_{I_i}$ where the l_i 's are integers and the I_i 's are disjoint clopens in \mathbb{S}_α^1 . By Claim 1 there are integers k_i 's such that $\chi_{I_i} \sim \chi_{k_i \beta}$ so that $f_1 \sim f_2 = \sum_{i=1}^N l_i \chi_{k_i \beta}$. By Claim 2, for each k_i there exists $n(k_i)$ and c_{ij} 's such that by Claim 3, $f_2 \sim f_3 = \sum_{i=1}^N \varepsilon_{k_i} \sum_{j=1}^{n(k_i)} (-1)^{j-1} c_{ij} \chi_j$. And finally by Claim 4, f_3 is equivalent to $f_4 = \sum_{i=1}^N \varepsilon_{k_i} \sum_{j=1}^{n(k_i)} (-1)^{j-1} c_{ij} (n_j \chi_\beta + m_j \mathbf{1})$, so let $n_f = \sum_{i=1}^N \varepsilon_{k_i} \sum_{j=1}^{n(k_i)} (-1)^{j-1} c_{ij} n_j$ and $m_f = \sum_{i=1}^N \varepsilon_{k_i} \sum_{j=1}^{n(k_i)} (-1)^{j-1} c_{ij} m_j$ to get that $f = [f_4]$ with $f_4 = n_f \chi_\beta + m_f \mathbf{1}$. \square

Lemma 2.3.10 *The map $\varphi : G_\alpha \rightarrow \mathbb{Z} \oplus \beta\mathbb{Z}$ given by integration over \mathbb{S}_α^1 $\varphi(f) = \int_{\mathbb{S}_\alpha^1} f = m_f + n_f\beta$ is a group isomorphism. So we have*

$$G_\alpha \cong \mathbb{Z} \oplus \beta\mathbb{Z}.$$

Proof. The first expression for φ is well defined on G_α because the integral of an invariant element $g \in (\text{id} - \theta_\alpha)C(\mathbb{S}_\alpha^1, \mathbb{Z})$ is zero, so two representatives in $C(\mathbb{S}_\alpha^1, \mathbb{Z})$ of an element of G_α have the same integral. The second expression is well defined by lemma 2.3.9. The map φ is clearly a group homomorphism and it is easily seen to be an isomorphism. Assume that we have two elements $f_1 = n_1\chi_\beta + m_1\mathbf{1}$ and $f_2 = n_2\chi_\beta + m_2\mathbf{1}$ in G_α with the same image under φ . Then $\int f_1 = \int f_2$ reads $(m_2 - m_1) + (n_2 - n_1)\beta = 0$ in the module $\mathbb{Z} \oplus \beta\mathbb{Z}$ (which is finitely generated with basis $\{1, \beta\}$ because $\beta = \frac{\alpha}{1+\alpha}$ is irrational as α is) therefore $n_2 - n_1 = m_2 - m_1 = 0$ so $f_1 = f_2$, which proves that φ is injective. Given $m + n\beta \in \mathbb{Z} \oplus \beta\mathbb{Z}$ let $f_{nm} = n\chi_\beta + m\mathbf{1}$ in G_α , we have $\varphi(f_{nm}) = m + n\beta$, thus φ is surjective.

□

2.4 Isomorphism between Čech and PV cohomologies

We prove here that the PV cohomology of a tiling is isomorphic to the Čech cohomology of its hull:

Theorem 2.4.1 *Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with FLC. Let Ω denote its hull, and Ξ_Δ its Δ -transversal. The PV cohomology of \mathcal{T} is isomorphic to the integer Čech cohomology of its hull:*

$$\check{H}^*(\Omega; \mathbb{Z}) \cong H_{PV}^*(\mathcal{B}_0; C(\Xi_\Delta, \mathbb{Z})).$$

The proof of theorem 2.4.1 follows from propositions 2.4.5 and 2.4.7 below. We first define a PV cohomology of \mathcal{B}_p , written $H_{PV}^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z}))$, and associated with the map $f_p : \mathcal{B}_p \rightarrow \mathcal{B}_0$ with “fiber” Σ_p . We prove that it is isomorphic to the simplicial

cohomology of \mathcal{B}_p in proposition 2.4.5, and then we establish in proposition 2.4.7 that the PV cohomology of the hull is isomorphic to the direct limit of the PV cohomologies of a proper sequence of patch spaces.

Let \mathcal{S}_p^n be the set of the characteristic maps $\sigma_p : \Delta^n \rightarrow \mathcal{B}_p$ of the n -simplices of the Δ -complex decomposition of \mathcal{B}_p , and \mathcal{S}_p the (disjoint) union of the \mathcal{S}_p^n 's. The group of simplicial n -chains on \mathcal{B}_p , $C_{p,n}$, is the free abelian group with basis \mathcal{S}_p^n .

Remark 2.4.2 The map $f_p : \mathcal{B}_p \rightarrow \mathcal{B}_0$ preserves the orientations of the simplices (the ordering of their vertices).

Given a simplex σ_p on \mathcal{B}_p , let $\Xi_{p,\Delta}(\sigma_p)$ denote the lift of the puncture of its image in \mathcal{B}_p and χ_{σ_p} its characteristic function. $\Xi_{p,\Delta}(\sigma_p)$ is called the *acceptance zone* of σ_p ; it is a clopen set in Ξ_Δ .

Lemma 2.4.3 *Given a simplex σ on \mathcal{B}_0 , its acceptance zone is partitioned by the acceptance zones of its preimages σ_p 's on \mathcal{B}_p*

$$\Xi_\Delta(\sigma) = \coprod_{\sigma_p \in f_{p\#}^{-1}(\sigma)} \Xi_{p,\Delta}(\sigma_p).$$

where $f_{p\#} : C_{p,n} \rightarrow C_{0,n}$ denotes the map induced by f_p on the simplicial chain groups.

Proof. The union of the $\Xi_{p,\Delta}(\sigma_p)$'s is equal to $\Xi_\Delta(\sigma)$, because each lift of the image simplex of a σ_p corresponds to the lift of the image simplex of σ that has a given local configuration (it is contained in some patch of \mathcal{T} or some tile of \mathcal{T}_p , see section 1.4). If σ_p and σ'_p are distinct in $f_{p\#}^{-1}(\sigma)$, then $\mathfrak{p}_p(\Xi_{p,\Delta}(\sigma_p)) \cap \mathfrak{p}_p(\Xi_{p,\Delta}(\sigma'_p))$ is empty, and thus so is $\Xi_{p,\Delta}(\sigma_p) \cap \Xi_{p,\Delta}(\sigma'_p)$. \square

The maps in $f_{p\#}^{-1}(\sigma)$ are thus in one-to-one correspondence with the set of atoms of the partition of $\Xi_\Delta(\sigma)$ by the $\Xi_{p,\Delta}(\sigma_p)$'s, and the union over σ in \mathcal{S}_0^n of the $f_{p\#}^{-1}(\sigma)$'s is just \mathcal{S}_p^n .

The simplicial n -cochain group C_p^n is $Hom(C_{p,n}, \mathbb{Z})$, the dual of the n -chain group $C_{p,n}$. It is represented faithfully on the group $C(\Xi_\Delta, \mathbb{Z})$ of continuous integer valued functions on the Δ -transversal by

$$\rho_{p,n} : \begin{cases} C_p^n & \longrightarrow & C(\Xi_\Delta^n, \mathbb{Z}) \\ \psi & \longmapsto & \sum_{\sigma_p \in \mathcal{S}_p^n} \psi(\sigma_p) \chi_{\sigma_p} \end{cases} . \quad (2.4.1)$$

The image of $\rho_{p,n}$ will be written $C(\Sigma_p^n, \mathbb{Z})$, to remind the reader that this consists of functions on the “discrete transversal” Σ_p^n which corresponds to the atoms of the partitions of the $\Xi_\Delta(\sigma)$ ’s by the $\Xi_{p,\Delta}(\sigma_p)$ ’s for $\sigma_p \in f_{p\#}^{-1}(\sigma)$. The representation $\rho_{p,n}$ is a group isomorphism onto its image $C(\Sigma_p^n, \mathbb{Z})$, its inverse is defined as follows: given $\varphi = \sum_{\sigma_p \in \mathcal{S}_p^n} \varphi_{\sigma_p} \chi_{\sigma_p}$, where φ_{σ_p} is an integer, $\rho_{p,n}^{-1}(\varphi)$ is the group homomorphism from $C_{p,n}$ to \mathbb{Z} whose value on the basis map σ_p is φ_{σ_p} .

Consider the characteristic map σ_p of an n -simplex e_p in \mathcal{B}_p . The simplex e_p is contained in some patch π_j . Viewing e_p as a subset of the patch p_j in \mathbb{R}^d , it is possible to define the vector $x_{\sigma_p \partial_i \sigma_p}$, for some i in $1, \dots, n$, that joins the puncture of the i -th face $\partial_i e_p$ (the simplex in \mathcal{B}_p whose characteristic map is $\partial_i \sigma_p$) to the puncture of e_p . As a consequence of remark 2.4.2, those vectors $x_{\sigma_p \partial_i \sigma_p}$ ’s are identical for all σ_p ’s in the preimage of the characteristic map σ of a simplex e on \mathcal{B}_0 , and equal to the vector $x_{\sigma \partial_i \sigma}$ which defines the operator $\theta_{\sigma \partial_i \sigma}$ in definition 2.3.2. By analogy, let $\theta_{\sigma_p \partial_i \sigma_p}$ be the operator $\chi_{\sigma_p} \mathbb{T}^{x_{\sigma_p \partial_i \sigma_p}} \chi_{\partial_i \sigma_p}$. With the relation $\mathbb{T}^a \chi_\Lambda = \chi_{\mathbb{T}^a \Lambda} \mathbb{T}^a$ for $\Lambda \subset \Omega, a \in \mathbb{R}^d$, and lemma 2.4.3 it is easily seen that $\theta_{\sigma \partial_i \sigma}$ is the sum of the $\theta_{\sigma_p \partial_i \sigma_p}$ ’s over all $\sigma_p \in f_{p\#}^{-1}(\sigma)$. Hence PV differential given in equation (2.3.2) can be written

$$d_{PV}^n = \sum_{\sigma_p \in \mathcal{S}_p^n} \sum_{i=0}^n (-1)^i \theta_{\sigma_p \partial_i \sigma_p} , \quad (2.4.2)$$

and is then well defined as a differential from $C(\Sigma_p^{n-1}, \mathbb{Z})$ to $C(\Sigma_p^n, \mathbb{Z})$.

Definition 2.4.4 Let $C_{PV}^n(p) = C(\Sigma_p^n, \mathbb{Z})$, for $n = 0, \dots, d$. The PV cohomology of the patch space \mathcal{B}_p , denoted $H_{PV}^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z}))$, is the homology of the complex $\{C_{PV}^n(p), d_{PV}^n\}$.

The notation $H_{PV}^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z}))$ requires some comments. The map $f_p : \mathcal{B}_p \rightarrow \mathcal{B}_0$ is a “branched covering” with discrete fibers, the \mathcal{S}_p^n ’s, that correspond to the “discrete transversals” Σ_p^n ’s:

$$\begin{array}{ccc} \Sigma_p & \hookrightarrow & \mathcal{B}_p \\ & & \downarrow \mathfrak{p}_p \\ & & \mathcal{B}_0 \end{array}$$

In analogy with remark 2.3.6, the PV cohomology of \mathcal{B}_p is analogous to a cohomology of the base space \mathcal{B}_0 with local coefficients in the K -theory of the “fiber” Σ_p .

Proposition 2.4.5 The PV cohomology of the patch space \mathcal{B}_p is isomorphic to its integer simplicial cohomology: $H_{PV}^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z})) \cong H^*(\mathcal{B}_p; \mathbb{Z})$.

Proof. We actually prove a stronger statement: the complexes $C_{PV}(p)^*$ and C_p^* are chain-equivalent. As mentioned earlier, $\rho_{p,n}$ in equation (2.4.1) defines a group isomorphism between the simplicial cochain group C_p^n and the PV cochain group $C_{PV}^n(p)$. Here they are both isomorphic to the direct sum $\mathbb{Z}^{|\mathcal{S}_p^n|}$, where $|\mathcal{S}_p^n|$ is the cardinality of \mathcal{S}_p^n , i.e. the number of n simplices on \mathcal{B}_p .

The differential of $\varphi \in C_{PV}^{n-1}(p)$, evaluated on an n -simplex σ_p , reads using (2.4.2) $d_{PV}^n \varphi(\sigma_p) = \sum_{\sigma_p \in \mathcal{S}_p^n} (d_{PV}^n \varphi)_{\sigma_p} \chi_{\sigma_p}$, with $(d_{PV}^n \varphi)_{\sigma_p} = \sum_{i=1}^n (-1)^i \varphi_{\partial_i \sigma_p}$. On the other hand the simplicial differential of $\psi \in C_p^{n-1}$ reads $\delta^n \psi(\sigma_p) = \sum_{i=1}^n (-1)^i \psi(\partial_i \sigma_p)$, and therefore $d_{PV}^n \circ \rho_{p,n-1} = \rho_{p,n} \circ \delta^n$, $n = 1, \dots, d$. Hence the $\rho_{p,n}$ ’s give a chain map and conjugate the differentials, and thus yield isomorphisms $\rho_{p,n}^*$ ’s between the n -th cohomology groups. \square

Lemma 2.4.6 *Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of \mathcal{T} . The following holds: $\Xi_\Delta \cong \varprojlim (\mathcal{S}_l, f_l)$, and $C(\Xi_\Delta, \mathbb{Z}) \cong \varinjlim (C(\Sigma_l, \mathbb{Z}), f^l)$, where f^l is the dual map to f_l .*

Proof. This is a straightforward consequence of theorem 1.4.4. \square

Proposition 2.4.7 *Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of \mathcal{T} . There is an isomorphism:*

$$H_{PV}^*(\mathcal{B}_0; C(\Xi_\Delta, \mathbb{Z})) \cong \varinjlim (H_{PV}^*(\mathcal{B}_0; C(\Sigma_l, \mathbb{Z})), f_l^*).$$

Proof. By the previous lemma 2.4.6 the cochain groups C_{PV}^n of the hull are the direct limits of the cochain groups $C_{PV}^n(l)$ of the patch spaces \mathcal{B}_l . Let $f_l^\# : C_{PV}^n(l) \rightarrow C_{PV}^n(l)$ denote the map induced by f_l on the PV cochain groups. Since the PV differential d_{PV} is the same for the complexes of each patch space \mathcal{B}_l , it suffices to check that the following diagram is commutative

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_{PV}^{n-1}(l) & \xrightarrow{d_P^n} & C_{PV}^n(l) & \longrightarrow & \dots \\ & & \downarrow f_l^\# & & \downarrow f_l^\# & & \\ \dots & \longrightarrow & C_{PV}^{n-1}(l+1) & \xrightarrow{d_P^n} & C_{PV}^n(l+1) & \longrightarrow & \dots \end{array}$$

which is straightforward using the relation $(f_l^\# \varphi)_{\partial_j \sigma_l} = \varphi_{f_l^\#(\partial_j \sigma_l)}$. \square

Proof of theorem 2.4.1. By theorem 2.2.2 the Čech cohomology of the hull is isomorphic to the direct limit of the Čech cohomologies of the patch spaces \mathcal{B}_i , $\check{H}^*(\Omega; \mathbb{Z}) \cong \varinjlim (\check{H}^*(\mathcal{B}_i; \mathbb{Z}), f_i^*)$. By the natural isomorphism between Čech and simplicial cohomologies for finite CW-complexes [79], $\check{H}^*(\mathcal{B}_i; \mathbb{Z}) \cong H^*(\mathcal{B}_i; \mathbb{Z})$, and by proposition 2.4.5, $H^*(\mathcal{B}_i; \mathbb{Z}) \cong H_{PV}^*(\mathcal{B}_0; C(\Sigma_i, \mathbb{Z}))$, so that $\check{H}^*(\mathcal{B}_i; \mathbb{Z}) \cong H_{PV}^*(\mathcal{B}_0; C(\Sigma_i, \mathbb{Z}))$. By proposition 2.4.7, the direct limit of the $H_{PV}^*(\mathcal{B}_0; C(\Sigma_l, \mathbb{Z}))$'s is the PV cohomology of the hull. Therefore the integer Čech cohomology of the hull is isomorphic to the PV cohomology of the hull: $\check{H}^*(\Omega; \mathbb{Z}) \cong H_{PV}^*(\mathcal{B}_0; C(\Xi_\Delta, \mathbb{Z}))$. \square

2.5 *K*-theory of tilings

Given a tiling \mathcal{T} which is aperiodic, repetitive, and has FLC, we can define its *K*-theory in the three following ways. First by the topological *K*-theory of the hull Ω of \mathcal{T} : $K^*(\Omega)$. Since $(\Omega, \mathbb{R}^d, \mathbb{T})$ is a dynamical system, *i.e.* we have a homeomorphic action of \mathbb{R}^d on Ω , we can consider the cross-product *C**-algebra $C(\Omega) \rtimes_{\mathbb{T}} \mathbb{R}^d$ which is the *C**-algebra associated with the tiling. We can now define the *K*-theory of \mathcal{T} as the *K*-theory of the *C**-algebra $C(\Omega) \rtimes_{\mathbb{T}} \mathbb{R}^d$: $K_*(C(\Omega) \rtimes_{\mathbb{T}} \mathbb{R}^d)$.

Those two possible definitions are equivalent here since by the Thom-Connes isomorphism [24] the cross-product *C**-algebra $C(\Omega) \rtimes_{\mathbb{T}} \mathbb{R}^d$ and the *C**-algebra $C(\Omega)$ of continuous functions on Ω have isomorphic *K*-groups (with a shift in dimension by d):

$$K_*(C(\Omega) \rtimes_{\mathbb{T}} \mathbb{R}^d) \cong K_{*+d}(C(\Omega)),$$

and since Ω is compact and Hausdorff, the *K*-theory of $C(\Omega)$ is isomorphic to the topological *K*-theory of Ω : $K_*(C(\Omega)) \cong K^*(\Omega)$. The third way is via the *C**-algebra of the groupoid Γ_{Ξ} of the transversal, which is Morita equivalent to the *C**-algebra of the tiling and thus has same *K*-theory.

Note however that those definitions would not be equivalent anymore if we change our definition of FLC for \mathcal{T} (definition 1.1.3) and allow a larger group G of isometries of \mathbb{R}^d to identify classes of tiles, and not just the group of translations. Indeed, the *C**-algebra associated with the tiling is in this case the cross product $C(\Omega) \rtimes G$ and there are no Thom-Connes isomorphism to relate its *K*-theory to the *K*-theory of $C(\Omega)$. Hence in this more general situation, the *K*-theory of the *C**-algebra of the tiling is not isomorphic to the topological *K*-theory of its hull.

CHAPTER III

APPLICATIONS

We present here some applications of topological invariants of tilings. The presentation of the first three sections is very informal: we explain the “physical ideas” and the heuristic behind results that have deeply motivated our study of topological invariants of tilings. In the last section we present a partial study of acoustic phonons in an aperiodic solid.

3.1 Cohomology and deformation of tilings

This section is an informal summary of the work of Clark and Sadun in [21]. Let \mathcal{T} be a repetitive tiling of \mathbb{R}^d with FLC. A deformation \mathcal{T}_f of \mathcal{T} is obtained by changing the edges of each tile of \mathcal{T} . Deforming a tile has to be done in a consistent way with its neighbors, hence each tile of \mathcal{T}_f is a deformation of a tile of \mathcal{T} that respects its local configuration. Since \mathcal{T} has FLC, it has finitely many prototile, and thus it suffices to define the deformation on the set of prototiles in a consistent way with all the local configurations of their representatives in \mathcal{T} . The prototile space \mathcal{B}_0 of \mathcal{T} (definition 1.3.1) is made from the prototiles by gluing them together according to the local configurations of their representatives in \mathcal{T} . Therefore the deformation \mathcal{T}_f of \mathcal{T} can be defined as a function $c_f : \mathcal{B}_0 \longrightarrow \mathbb{R}^d$ which assigns to each edge e in a tile in \mathcal{B}_0 a vector $c_f(e) \in \mathbb{R}^d$ which is the deformation of the edge e .

There are compatibility conditions: i) $\sum c_f(e_i) = 0$ for the edges e_i bounding a tile, ii) if the edges e_i are k -coplanar then so must the deformed edges $c_f(e_i)$ be, and iii) we forbid the deformation vectors to “cross each others”. Those conditions imply that c_f is a vector valued closed 1-cochain on \mathcal{B}_0 .

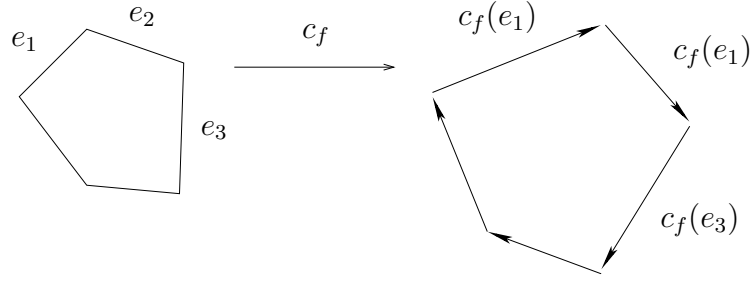


Figure 7: Example of a tile's deformation

The hull Ω of \mathcal{T} is homeomorphic to the inverse limit $\varprojlim(\mathcal{B}_n, f_n)$ of a proper sequence of patch spaces (see theorem 1.4.4). Let σ be the projection from $\Omega \cong \varprojlim(\mathcal{B}_n, f_n)$ to \mathcal{B}_0 . There is an induced map on cohomology:

$$\sigma^* : H^1(\mathcal{B}_0; \mathbb{R}^d) \longrightarrow \check{H}^1(\Omega; \mathbb{R}^d),$$

and Clark and Sadun showed that *the dynamical properties of \mathcal{T}_f only depends on $\sigma^*[c_f]$ in $\check{H}^1(\Omega; \mathbb{R}^d)$* . In particular, if $\sigma^*[c_f]$ is in an eigenspace of σ^* corresponding to an eigenvalue λ with $|\lambda| < 1$, then the dynamical systems of the hull Ω_f of \mathcal{T}_f and of Ω are conjugated.

3.2 Homology and invariant measures on the hull

We present here an informal summary of the work of Bellissard, Benedetti and Gambeaudo in [11] (sections 3, 4, and 5).

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with FLC. Let Ω be its hull and Ξ its canonical transversal. Let $\mathcal{M}(\Omega)$ be the set of real finite invariant measures on Ω : if $\mu \in \mathcal{M}(\Omega)$, $\mu(\Omega)$ is finite and for every measurable set A of Ω , we have $\mu(\mathbb{T}^x A) = \mu(A)$ for all x in \mathbb{R}^d . Let also $\mathcal{M}^m(\Omega)$, for $m > 0$, be the subset of $\mathcal{M}(\Omega)$ of measures μ such that $\mu(\Omega) = m$. The set $\mathcal{M}^m(\Omega)$ is a convex set with extremal points given by ergodic measures on Ω .

Let $\{\mathcal{B}_n, f_n\}$ be a proper sequence of patch spaces (definition 1.4.3), such that $\Omega \cong$

$\varprojlim(\mathcal{B}_n, f_n)$ (theorem 1.4.4), and let $\pi_n : \Omega \rightarrow \mathcal{B}_n$ be the projection onto the n -th component of the inverse limit. Let $d_i, i = 1, \dots, N_n$, be the domains of \mathcal{B}_n : the images in \mathcal{B}_n of the interiors of the patches p_i 's that it is made of (definition 1.4.1). Let x_i be image in \mathcal{B}_n of the puncture of p_i , and $\Xi_{n,i} = \pi_n^{-1}(x_i)$ the acceptance zone of the domain d_i (or patch p_i). As a consequence of lemma 2.4.3 the $\Xi_{n,i}, i = 1, \dots, N_n$, form a clopen partition \mathcal{P}_n of Ξ , and for $m > n$, \mathcal{P}_m is a refinement of \mathcal{P}_n .

Since Ξ is a Cantor set, a finite real *transverse measure* μ^t on Ξ is characterized by the countable data of real positive numbers for each clopen partition of Ξ , with obvious additive properties. Hence μ^t is uniquely determined by its values $\{\mu^t(\Xi_{n,i}) : n \in \mathbb{N}, 1 \leq i \leq N_n\}$. There is a one-to-one correspondance between finite invariant measures on Ω and transverse measures on Ξ that are invariant under the action of the groupoid Γ of the transversal (definition 1.2.5). We can express this correspondance as follows:

$$\mu(\pi_n^{-1}(d_i)) = \lambda_d(d_i) \mu^t(\Xi_{n,i}),$$

where $\lambda_d(d_i)$ is the d -dimensional Lebesgue measure of d_i (identified with the interior of the patch p_i in \mathbb{R}^d).

The numbers $\mu^t(\Xi_{n,i}), i = 1, \dots, N_n$, can be interpreted as a *positive weight* on \mathcal{B}_n which assigns a positive number to each domain d_i and respect the orientation of \mathcal{B}_n inherited from the patches p_i in \mathbb{R}^d . This defines a closed singular d -chain on \mathcal{B}_n whose homology class lies in a canonical positive cone of $H_d(\mathcal{B}_n; \mathbb{R})$.

Bellissard *et al.* proved that, by inverse limit $H_d(\Omega; \mathbb{R}) \cong \varprojlim(H_d(\mathcal{B}_n; \mathbb{R}), f_{n*})$, one gets a canonical positive cone in $H_d(\Omega; \mathbb{R})$ which is in one-to-one correspondance with $\mathcal{M}(\Omega)$, the set of finite invariant measures on Ω . Furthermore if $\dim H_d(\mathcal{B}_n; \mathbb{R}) = N$ for all n , then there are no more than N invariant ergodic measures on Ω , and if the $\|f_{n*}\|$'s are uniformly bounded then Ω has a unique invariant ergodic measure.

3.3 *K-theory and the gap labeling theorem*

Another important application of topological invariants of tilings is the gap labeling theorem, and I introduce its philosophy here. The problem was initiated in the early eighties by Bellissard in [44, 15] (see [9, 10, 11] for later developments). At that time the problem was to compute the spectrum of a Schrödinger operator H in an aperiodic potential. In section 5.1 where we describe the historical background for the topics in this thesis, we tell in particular the origin and the development of the gap labeling theorem.

An aperiodic solid \mathcal{S} can be modeled by a Delone set \mathcal{L} of \mathbb{R}^d whose points represent the atomic positions in \mathcal{S} at zero temperature. Equivalently, \mathcal{S} can be modeled by the Voronoi tiling \mathcal{T} of \mathcal{L} . A physical quantity of interest for \mathcal{S} is the electronic *density of states* $\mathcal{N}(E)$ of energy less than or equal to E . In order to calculate $\mathcal{N}(E)$ one needs a theory, *i.e.* a Hamiltonian H (Schrödinger operator) on $L^2(\mathcal{L})$, and to be able to solve its equations of motion. The art of the theoretical physicist is to find a theory which captures enough interesting physical properties of the system and yet is simple enough so that one can actually compute things with it! (in particular solve the equations of motion). The gap labeling theorem roughly speaking states that the choice of the theory (in a reasonable class) does not matter, only the topology of the hull of the tiling that models the system is relevant.

Let \mathcal{T} be an aperiodic tiling of \mathbb{R}^d whose set of punctures \mathcal{L} represents the atomic positions in an aperiodic solid \mathcal{S} . Let H be a Hamiltonian on $L^2(\mathcal{L})$ (an electronic theory for \mathcal{S}). The hull of Ω_H of H is a compactification of the set of translates of H (as H is typically an unbounded operator, the topology one chooses is the strong resolvent topology). There is an element h (self-adjoint) affiliated to the C^* -algebra $C^*(\Gamma)$ of the groupoid Γ of the transversal Ξ , and via the left regular representation π_ω of Γ , we have $\pi_\omega(h) = H_\omega$ for each element H_ω in Ω_H . The gap labeling theorem states that for as class \mathcal{C} of Hamiltonian (satisfying the “tight-binding condition”)

- (i) Ω_H is homeomorphic to Ω ,
- (ii) the spectral properties of the system depend only on the K -theory class of the spectral projections of h : $P_{h \leq E}$ (projection on “states of energy less than E ”),
- (iii) there is a pairing $K_0(C^*(\Gamma)) \times H_d(\Omega; \mathbb{R}) \longrightarrow \mathbb{R}_+$: given an invariant ergodic measure μ on Ω , it induces a trace τ_μ on $C^*(\Gamma)$ and we have:

$$\tau_{\mu*}[P_{h \leq E}] = \mathcal{N}(E),$$

where the measure μ comes from a homology class in $H_d(\Omega; \mathbb{R})$ as discussed in the previous section 3.2.

3.4 Acoustic phonons in an aperiodic solid

We give here a partial study of scalar vibration of atoms in an aperiodic solid \mathcal{S} , under the approximation of classical and harmonic motions. The atomic positions (at zero temperature) in \mathcal{S} are represented by points of a (r, R) -Delone set \mathcal{L} of \mathbb{R}^d (definition 1.2.1) which are the puncture of its Voronoi tiling \mathcal{T} (definition 1.2.2). Since \mathcal{L} is repetitive we can define its *density* ρ , which is the average number of points of \mathcal{L} per unit volume.

Let u_x denote the displacement of the atom at position $x \in \mathcal{L}$, and K_{xy} the spring constant between atoms at x and y in \mathcal{L} . Let us assume that K is symmetric and exponentially bounded:

$$K_{yx} = K_{xy}, \quad 0 \leq K_{xy} \leq K e^{-\|x-y\|}. \quad (3.4.1)$$

The equation of motion reads:

$$m_x \frac{d^2 u_x}{dt^2} = \sum_{y \in \mathcal{L}} K_{xy} (u_y - u_x)$$

where m_x is the mass of the atom at position $x \in \mathcal{L}$. The equation of the eigenmode of frequency ν reads:

$$(2\pi\nu)^2 v_x = \sum_{y \in \mathcal{L}} \frac{K_{xy}}{m_x} (v_x - v_y). \quad (3.4.2)$$

Let Ω be the hull of \mathcal{T} (definition 1.2.3). Let \mathcal{L}_ω denote the Delone set of punctures of $\omega \in \Omega$; Recall from sections 1.1 and 1.2 that Ω is compact, and that the group \mathbb{R}^d acts on Ω by homeomorphisms so that $(\Omega, \mathbb{R}^d, \mathbb{T})$ is a dynamical system. We now make the essential

Assumption: $(\Omega, \mathbb{R}^d, \mathbb{T})$ has a unique \mathbb{T} -invariant ergodic probability measure μ .

Recall from section 3.2 that μ is encoded by a homology class in a positive cone of the homology group $H_d(\Omega; \mathbb{R})$.

Let Ξ be the canonical transversal to Ω (definition 1.1.4) In terms of Delone sets, Ξ can be seen as the set of tilings whose Delone sets of punctures have a point at the origin. Let Γ be the groupoid of the transversal (definition 1.2.5). We recall that it is defined as

$$\Gamma = \{(\omega, x) \in \Xi \times \mathbb{R}^d : \mathbb{T}^{-x}\omega \in \Xi\}.$$

The set of objects $\Gamma^{(0)}$ is Ξ and given an arrow $\gamma = (\omega, x) \in \Gamma^{(1)}$ ω is its range and $\mathbb{T}^{-x}\omega$ its source. Recall from section 3.2 that the invariant ergodic measure μ has an associated transverse measure μ^t which is ergodic and invariant under Γ . We have an induced ergodic theorem on the transversal.

Theorem 3.4.1 *Let f be a continuous function of Ξ . For μ^t -almost every $\xi \in \Xi$ we have*

$$\frac{1}{\rho|\Lambda|} \sum_{x \in \mathcal{L}_\xi \cap \Lambda} f(\mathbb{T}^{-x}\xi) \xrightarrow{\Lambda \uparrow \mathbb{R}^d} \int_{\Xi} f(\omega) d\mu^t(\omega)$$

and the convergence is uniform with respect to the choice of the sets $\Lambda \subset \mathbb{R}^d$.

The function which assigns to each element $\omega \in \Gamma^{(0)}$ the counting measure on the set of arrows whose range is ω (range fiber) defines a transverse function on Γ . One can then build the groupoid C^* -algebra $C^*(\Gamma)$ [26, 69] of Γ by completion of its convolution algebra through the left regular representation π_ω of Γ on the continuous field of

Hilbert spaces $\mathcal{H}_\omega = \ell^2(\mathcal{L}_\omega)$. For $a \in C^*(\Gamma)$ and $v \in \mathcal{H}_\omega$ the representation reads:

$$(\pi_\omega(a)v)_x = \sum_{y \in \mathcal{L}_\omega} a(\mathbb{T}^{-x}\omega, y-x)v_y.$$

We can now rewrite the eigenmode equation (3.4.2) in \mathcal{H}_ω as

$$(2\pi\nu)^2 v = \pi_\omega(h)v, \quad (3.4.3)$$

where $h(\omega, x) = \delta_{x,0} \sum_{y \in \mathcal{L}} K(\omega, y) - (1 - \delta_{x,0})K(\omega, x)$ in $C^*(\Gamma)$, with $K(\omega, x) = K_{0x}/m_x$ in $C^*(\Gamma)$. The spring constant matrix K is a positive element of $C^*(\Gamma)$ that satisfies

$$0 \leq K(\omega, x) \leq K_+ e^{-\frac{|x|}{\lambda}}, \quad \forall (\omega, x) \in \Gamma.$$

as one can easily see from (3.4.1). The spring constant matrix is rewritten on \mathcal{H}_ω as

$$K(\omega) = \sum_{x \in \mathcal{L}_\omega} K(\omega, x)|x\rangle\langle x|,$$

where we have used the “bra-ket” notation of Dirac. If α and β belong to a Hilbert space V , then $|\alpha\rangle = \alpha$ and $\langle\beta| = \beta^*$ is the dual vector in V^* . If $\{e_i, i \in I\}$ is an orthonormal basis of V , let us write $|i\rangle = e_i$, and $\langle i| = e_i^*$ the dual vector defined by $e_i^*(e_j) = \delta_{ij}$, or in braket notation $\langle i|j\rangle = \delta_{ij}$. If we write β in this basis as $|\beta\rangle = \sum_{i \in I} \beta_i |i\rangle$, then its dual reads $\langle\beta| = \sum_{i \in I} \bar{\beta}_i \langle i|$. The inner product of α and β is written $\langle\alpha|\beta\rangle$ and reads $\sum_{i \in I} \bar{\beta}_i \alpha_i$. Now $|\alpha\rangle\langle\beta|$ stands for the tensor product $\alpha^* \otimes \beta$ in $V^* \otimes V$ and reads $\sum_{ij \in I} \bar{\alpha}_i \beta_j |i\rangle\langle j|$.

Let $H_\omega = \pi_\omega(h)$ be the positive and self-adjoint Hamiltonian on \mathcal{H}_ω given in matrix elements by

$$\langle\varphi, H_\omega\psi\rangle = \sum_{x,y \in \mathcal{L}_\omega} K(\mathbb{T}^{-x}\omega, y-x) (\bar{\varphi}_y - \bar{\varphi}_x) (\psi_y - \psi_x). \quad (3.4.4)$$

We define the *homogenized spring tensor* by

$$\mathcal{K} = \int_{\Xi} K(\omega) d\mu^t(\omega).$$

By theorem 3.4.1 we can write \mathcal{K} as the uniform limit of $\frac{1}{\rho|\Lambda|} \sum_{x \in \mathcal{L} \cap \Lambda} K(\Gamma^{-x}\omega)$ as $\Lambda \uparrow \mathbb{R}^d$. We define the homogenized operator as

$$H = \nabla^* \mathcal{K} \nabla, \quad (3.4.5)$$

on $H^1(\mathbb{R}^d)$. It is positive, self-adjoint, densely defined and unbounded on $L^2(\mathbb{R}^d)$. We see it as a ‘‘Laplacian’’, as it is given in matrix form by

$$\langle \psi, H\psi \rangle = K^{ij} \int_{\mathbb{R}^d} \partial_i \bar{\psi} \partial_j \psi$$

where we have used Einstein’s convention of summation over repeated indices.

We want to compare the spectra of H_ω and H (equations (3.4.4) and (3.4.5)). To do so, we map H_ω to $L^2(\mathbb{R}^d)$ and ‘‘renormalize’’ it. For a given $M > 0$, let L_M be the lattice $\sqrt{d}M \mathbb{Z}^d$, of cells $C_x^M = x + (-\frac{M}{\sqrt{d}}, \frac{M}{\sqrt{d}}]^d$, $x \in \sqrt{d}M \mathbb{Z}^d$. Let us denote by $B = B(0, r)$ and $B_x = B(x, r)$ for $x \in \mathcal{L}_\omega$ (open balls), by $Cv(x, y)$ the convex hull of $B_x \cup B_y$, and by $B_x^M = B(x, M + r)$ for $x \in L_M$.

Let $e \in C_c^\infty(B)$ be real positive, with $\|e\|_2 = 1$. We think of e as a smooth approximation of χ_B . For $x \in \mathcal{L}_\omega$ and $N \in \mathbb{N}_*$ we define

$$e_x(z) = e(z - x) \quad \text{and} \quad e_{\frac{x}{N}}^N(z) = N^{\frac{d}{2}} e_x(Nz) = N^{\frac{d}{2}} e(Nz - x).$$

The family $\left\{ e_{\frac{x}{N}}^N \right\}_{x \in \mathcal{L}_\omega}$ is *orthonormal* in $L^2(\mathbb{R}^d)$.

Let $\mathcal{J}_N : \mathcal{L}_\omega \rightarrow L^2(\mathbb{R}^d)$ be given by

$$\mathcal{J}_N = \sum_{x \in \mathcal{L}_\omega} |e_{\frac{x}{N}}^N\rangle \langle x| \quad \text{and} \quad \mathcal{J}_N^* = \sum_{x \in \mathcal{L}_\omega} |x\rangle \langle e_{\frac{x}{N}}^N|.$$

Notice that

$$\mathcal{J}_N^* \mathcal{J}_N = \mathbf{1}_{\mathcal{H}_\omega} \quad \text{and} \quad \mathcal{J}_N \mathcal{J}_N^* = \sum_{x \in \mathcal{L}_\omega} |e_{\frac{x}{N}}^N\rangle \langle e_{\frac{x}{N}}^N|$$

which is a projection of $L^2(\mathbb{R}^d)$. Hence \mathcal{J}_N is a *partial isometry* from \mathcal{H}_ω to $L^2(\mathbb{R}^d)$ with the following property (which can be deduced easily from the proof of theorem 3.4.3 below).

Proposition 3.4.2 *The projection $\frac{1}{\rho\|e\|_1^2}\mathcal{J}_N\mathcal{J}_N^*$ converges strongly to the identity on $L^2(\mathbb{R}^d)$ as $N \rightarrow \infty$.*

With the injection \mathcal{J}_N we now perform an averaging of H_ω on a cube of side length N . We define the scaling renormalization H_ω^N of H_ω as

$$H_\omega^N = \frac{N^2}{\rho\|e\|_1^2} \mathcal{J}_N H_\omega \mathcal{J}_N^*, \quad (3.4.6)$$

which is a positive, self-adjoint, and bounded operator on $L^2(\mathbb{R}^d)$.

Theorem 3.4.3 *(Estimation of $H_\omega^N - H$). There exist a constant $C > 0$ (depending only on e , K and \mathcal{L}_ω), and for every $\delta \in (0, 1)$ a constant $\epsilon_\delta^N \geq 0$ (depending only on K , \mathcal{L}_ω , N and δ) such that*

$$|H_\omega^N - H| \leq \epsilon_\delta^N (-\Delta) + \frac{C}{N^{1-\delta}} (-\Delta + \Delta^2) \quad (3.4.7)$$

with $\lim_{N \rightarrow \infty} \epsilon_\delta^N = 0$.

The conclusion is that when averaged over a large domain (N large) and at low energy (with respect to the Laplacian spectrum) the energy functional of atomic vibrations behaves like the homogenized operator $H = \nabla^* \mathcal{K} \nabla$.

We need two results for proving theorem 3.4.3: the transverse ergodic theorem 3.4.1 and Poincaré's inequalities for a ball. We recall Poincaré's inequalities for an open ball $B(x, r) \subset \mathbb{R}^d$ (see [30] section 5.8.1). Let $(f)_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r)} f$ denote the average of the function f over the ball $B(x, r)$ of Lebesgue measure $|B(x, r)|$.

Lemma 3.4.4 *(Poincaré's inequalities for a ball). Assume $1 \leq p \leq \infty$. Then there exists a constant $C_{d, p}$ depending only on d and p , such that*

$$\|f - (f)_{B(x, r)}\|_{L^p(B(x, r))} \leq C_{d, p} r \|\nabla f\|_{L^p(B(x, r))}$$

for each ball $B(x, r) \subset \mathbb{R}^d$ and each function $f \in W^{1, p}(B(x, r))$.

Proof of theorem 3.4.3. Unless otherwise stated, $\|f\|$ will denote the $L^2(\mathbb{R}^d)$ norm of f . Let $\varphi, \psi \in W^{2,2}(\mathbb{R}^d)$.

1. We define intermediate (self-adjoint and positive) operators on $W^{2,2}(\mathbb{R}^d)$

$$\begin{aligned}\langle \varphi, H_1^N \psi \rangle &= \frac{1}{\rho \|e\|_1^2} \sum_{x \in \mathcal{L}_\omega} K^{ij}(\mathbb{T}^{-x}\omega) \int \partial_i \bar{\psi} e^{\frac{x}{N}} \int e^{\frac{x}{N}} \partial_j \psi, \\ \langle \varphi, H_2^N \psi \rangle &= \frac{1}{\rho \|e\|_1^2} \sum_{x \in \mathcal{L}_\omega} K^{ij}(\mathbb{T}^{-x}\omega) \int e^{\frac{x}{N}} \partial_i \bar{\psi} \partial_j \psi \int e^{\frac{x}{N}}, \\ \langle \varphi, H_3^N \psi \rangle &= \sum_{x \in L_M} \left\{ \frac{1}{|C_x^M|} \sum_{a \in C_x^M \cap \mathcal{L}_\omega} K^{ij}(\mathbb{T}^{-a}\omega) \right\} \int_{\frac{C_x^M}{N}} \partial_i \bar{\psi} \partial_j \psi.\end{aligned}$$

We have

$$|H_\omega^N - H| \leq |H_\omega - H_1^N| + |H_1^N - H_2^N| + |H_2^N - H_3^N| + |H_3^N - H|. \quad (3.4.8)$$

2. **Claim:** $|H_\omega^N - H_1^N| \leq \frac{C}{N} (-\Delta + \Delta^2)$.

First

$$\langle \psi, H_\omega^N \psi \rangle = \frac{1}{\rho \|e\|_1^2} \sum_{x, y \in \mathcal{L}_\omega} K(\mathbb{T}^{-x}\omega, y-x) N^2 |\mathcal{J}_N^* \psi_x - \mathcal{J}_N^* \psi_y|^2,$$

with

$$\mathcal{J}_N^* \psi_x - \mathcal{J}_N^* \psi_y = \int d^d z e^{\frac{z}{N}} \left(\psi \left(z + \frac{y-x}{N} \right) - \psi(z) \right) = \alpha_{xy} + \beta_{xy},$$

where α_{xy}, β_{xy} are obtained by the second order Taylor formula for $\psi(z + \frac{y-x}{N}) - \psi(z)$, namely:

$$|\alpha_{xy}| = \left| \int e^{\frac{z}{N}} \frac{y-x}{N} \cdot \nabla \psi \right| \leq \frac{|y-x|}{N} \|\nabla \psi\|_{\frac{B_x}{N}},$$

and

$$\begin{aligned}|\beta_{xy}| &= \left| \left(\frac{y-x}{N} \right)^i \left(\frac{y-x}{N} \right)^j \int_0^1 dt (1-t) \int d^d z e^{\frac{z}{N}} \partial_i \partial_j \psi \left(z + t \frac{y-x}{N} \right) \right| \\ &\leq \frac{\|e\|_\infty}{N^2} |Cv(x, y)|^{\frac{1}{2}} |y-x|^2 \|\nabla^2 \psi\|_{\frac{Cv(x, y)}{N}}\end{aligned}$$

Now using that $||a|^2 - |b|^2| \leq |a-b|^2 + 2|b||a-b|$ we get

$$\begin{aligned}\frac{N^2}{\rho \|e\|_1^2} \left| |\mathcal{J}_N^* \psi_x - \mathcal{J}_N^* \psi_y|^2 - \left| \left\langle e^{\frac{x}{N}}, \frac{y-x}{N} \cdot \nabla \psi \right\rangle \right|^2 \right| &\leq \\ &\frac{C'(x, y)}{N^2} \|\nabla^2 \psi\|_{\frac{Cv(x, y)}{N}}^2 + \frac{C''(x, y)}{N} \|\nabla \psi\|_{\frac{B_x}{N}} \|\nabla^2 \psi\|_{\frac{Cv(x, y)}{N}}.\end{aligned}$$

Since $K(\omega, a) \leq K_+ e^{-\frac{|a|}{\lambda}}$, the sums over $x, y \in \mathcal{L}_\omega$ of the constants $C'(x, y)$ and $C''(x, y)$ converge. Note also that $\|\nabla^2 f\| \leq C\|\Delta f\|$, $\forall f \in W^{1,2}(\mathbb{R}^d)$ (as one can see using Fourier transform). We get

$$|\langle \psi, (H_\omega^N - H_1^N) \psi \rangle| \leq \frac{C}{N} \langle \psi, (-\Delta + \Delta^2) \psi \rangle,$$

which proves the claim.

3. Claim: $|H_1^N - H_2^N| \leq \frac{C}{N} (-\Delta + \Delta^2)$.

Set

$$\delta_{ij}^N = \int e^{\frac{x}{N}} \partial_i \bar{\psi} \int e^{\frac{x}{N}} \partial_j \psi - \int e^{\frac{x}{N}} \partial_i \bar{\psi} \partial_j \psi \int e^{\frac{x}{N}}.$$

We have

$$\begin{aligned} |K^{ij}(\mathbb{T}^{-x}\omega) \delta_{ij}^N| &\leq K^{ij}(\mathbb{T}^{-x}\omega) N^d \|e\|_\infty^2 \int d^d z \int d^d z' |\partial_i \bar{\psi}(z)| |\partial_j \psi(z) - \partial_j \psi(z')| \\ &\leq K_+ \|e\|_\infty^2 |B| 2C_{d,2} \frac{r}{N} \|\nabla \psi\|_{\frac{B_x}{N}} \|\nabla^2 \psi\|_{\frac{B_x}{N}} \end{aligned}$$

by inserting the average of $\partial_j \psi$ over the ball $\frac{B_x}{N}$ in the second term of the integral of the first inequality, and then applying Cauchy-Schwartz and Poincaré's inequalities (lemma 3.4.4) and using the symmetry of the tensor K : $K^{ij}(\omega, a) = K^{ji}(\omega, a)$. We then sum over $x \in \mathcal{L}_\omega$ to finish the proof of the claim.

4. Claim: For any $\delta \in (0, 1)$, $|H_2^N - H_3^N| \leq \frac{C}{N^{1-\delta}} (-\Delta + \Delta^2)$.

We write

$$\begin{aligned} \langle \varphi, H_2^N \psi \rangle &= \sum_{x \in L_M} \frac{1}{\rho} \sum_{a \in C_x^M \cap \mathcal{L}_\omega} K^{ij}(\mathbb{T}^{-a}\omega) N^{-d} \int \partial_i \bar{\psi} \partial_j \psi \frac{e_a(Nz) d^d(Nz)}{\|e\|_1}, \text{ and} \\ \langle \varphi, H_3^N \psi \rangle &= \sum_{x \in L_M} \left\{ \frac{1}{\rho} \sum_{a \in C_x^M \cap \mathcal{L}_\omega} K^{ij}(\mathbb{T}^{-a}\omega) \right\} N^{-d} \frac{1}{\left| \frac{C_x^M}{N} \right|} \int_{\frac{C_x^M}{N}} \partial_i \bar{\psi} \partial_j \psi. \end{aligned}$$

Notice that the two integrals in the above sums are normalized, and call δ_{ij}^N their difference. We insert the product $(\partial_i \bar{\psi})_{\frac{B_x^M}{N}} (\partial_j \psi)_{\frac{B_x^M}{N}}$ of the averages of $\partial_i \bar{\psi}$ and $\partial_j \psi$ over the ball $\frac{B_x^M}{N}$ in δ_{ij}^N , to get that $|\delta_{ij}^N| \leq \mu_{ij}(a) + \nu_{ij}(x)$ where

$$\begin{aligned} \mu_{ij}(a) &= \int \left(\partial_i \bar{\psi} \partial_j \psi - (\partial_i \bar{\psi})_{\frac{B_x^M}{N}} (\partial_j \psi)_{\frac{B_x^M}{N}} \right) \frac{e_a(Nz) d^d(Nz)}{\|e\|_1}, \text{ and} \\ \nu_{ij}(x) &= \frac{1}{\left| \frac{C_x^M}{N} \right|} \int_{\frac{C_x^M}{N}} \left(\partial_i \bar{\psi} \partial_j \psi - (\partial_i \bar{\psi})_{\frac{B_x^M}{N}} (\partial_j \psi)_{\frac{B_x^M}{N}} \right). \end{aligned}$$

We use the inequality

$$|fg - f_0g_0| \leq |f| |g - g_0| + |f - f_0| |g| + |f - f_0| |g - g_0|$$

and Cauchy-Schwartz and Poincaré's inequalities (lemma 3.4.4) to have

$$\begin{aligned} |\mu_{ij}(a)| &\leq C' N^d \frac{M}{N} \left(\|\partial_i \psi\|_{\frac{B_a}{N}} \|\nabla \partial_j \psi\|_{\frac{B_a^M}{N}} + \frac{M}{N} \|\nabla \partial_i \psi\|_{\frac{B_a^M}{N}} \|\nabla \partial_j \psi\|_{\frac{B_a^M}{N}} \right), \\ |\nu_{ij}(x)| &\leq C'' \frac{N^d}{|C_x^M|} \frac{M}{N} \left(\|\partial_i \psi\|_{\frac{B_x^M}{N}} \|\nabla \partial_j \psi\|_{\frac{B_x^M}{N}} + \frac{M}{N} \|\nabla \partial_i \psi\|_{\frac{B_x^M}{N}} \|\nabla \partial_j \psi\|_{\frac{B_x^M}{N}} \right). \end{aligned}$$

(We have symmetrized the indices i and j for simplicity in the above equations, but this does not matter because we will eventually contract μ and ν with the symmetric tensor K). Now

$$\begin{aligned} |\langle \psi, (H_2^N - H_3^N) \psi \rangle| &\leq \frac{C}{N} \sum_{x \in L_M} \sum_{a \in C_x^M \cap \mathcal{L}_\omega} N^{-d} K^{ij}(\mathbb{T}^{-a} \omega) (|\mu_{ij}(a)| + |\nu_{ij}(x)|) \\ &= E_\mu + E_\nu. \end{aligned}$$

The term E_ν is easy: first note that

$$N^{-d} K^{ij}(\mathbb{T}^{-a} \omega) |\nu_{ij}(x)| \leq \frac{C}{\rho |C_x^M|} \frac{M}{N} \left(\|\nabla \psi\|_{\frac{B_x^M}{N}} \|\nabla^2 \psi\|_{\frac{B_x^M}{N}} + \frac{M}{N} \|\nabla^2 \psi\|_{\frac{B_x^M}{N}}^2 \right),$$

and the term $\frac{1}{\rho |C_x^M|}$ balances the sum $\sum_{a \in C_x^M \cap \mathcal{L}_\omega}$ to give a term of order one, so that we get

$$E_\nu \leq C \frac{M}{N} \langle \psi, (-\Delta + \Delta^2) \psi \rangle.$$

For E_μ notice first that

$$N^{-d} K^{ij}(\mathbb{T}^{-a} \omega) |\mu_{ij}(a)| \leq C \frac{M}{N} \left(\|\nabla \psi\|_{\frac{B_a}{N}} \|\nabla^2 \psi\|_{\frac{B_a^M}{N}} + \frac{M}{N} \|\nabla^2 \psi\|_{\frac{B_a^M}{N}}^2 \right),$$

and in $\sum_{a \in C_x^M \cap \mathcal{L}_\omega}$ there is at most $|\frac{M}{r}|^d$ terms, so after summing over $x \in L_M$ we have

$$\begin{aligned} E_\mu &\leq C \frac{M}{N} \left(\sum_{x \in L_M} \|\nabla^2 \psi\|_{\frac{B_x^M}{N}}^2 \right)^{\frac{1}{2}} \left(\sum_{x \in L_M} \left(\sum_{a \in C_x^M \cap \mathcal{L}_\omega} \|\nabla \psi\|_{\frac{B_a}{N}} \right)^2 \right)^{\frac{1}{2}} + C \frac{M^{d+2}}{N^2} \|\nabla^2 \psi\|^2 \\ &\leq C \frac{M^{\frac{d}{2}+1}}{N} \langle \psi, (-\Delta + \Delta^2) \psi \rangle. \end{aligned}$$

We have used Cauchy-Schwartz inequality in the first line, and (convexity)

$$\left(\sum_{a \in C_x^M \cap \mathcal{L}_\omega} \|\nabla \psi\|_{\frac{B_a}{N}} \right)^2 \leq |C_x^M \cap \mathcal{L}_{T^{-x}\omega}| \sum_{a \in C_x^M \cap \mathcal{L}_\omega} \|\nabla \psi\|_{\frac{B_a}{N}}^2 \leq CM^d \|\nabla \psi\|_{\frac{C_x^M}{N}}^2$$

Now fix $\delta \in (0, 1)$, and let $M = N^{\frac{\delta}{\frac{d}{2}+1}}$ to complete the proof of the claim.

5. **Claim:** For any $\delta \in (0, 1)$, $|H_3^N - H| \leq \epsilon_\delta^N(\omega)(-\Delta)$, and $\lim_{N \rightarrow \infty} \epsilon_\delta^N = 0$.

We fix $\delta \in (0, 1)$ and let $M = N^{\frac{\delta}{\frac{d}{2}+1}}$. We have

$$\begin{aligned} |\langle \psi, (H_3^N - H) \psi \rangle| &\leq \sum_{x \in L_M} \left| K^{ij} - \frac{1}{\rho |C_x^M|} \sum_{a \in C_x^M \cap \mathcal{L}_\omega} K^{ij}(T^{-a}\omega) \right| \int_{\frac{C_x^M}{N}} \partial_i \bar{\psi} \partial_j \psi \\ &= \sup_{x \in L_M} \left| K^{ij} - \frac{1}{\rho |C_x^M|} \sum_{a \in C_x^M \cap \mathcal{L}_\omega} K^{ij}(T^{-a}\omega) \right| \int_{\mathbb{R}^d} \partial_i \bar{\psi} \partial_j \psi \\ &= \sup_{\omega \in \Xi} \left| K^{ij} - \frac{1}{\rho |C_0^M|} \sum_{a \in C_0^M \cap \mathcal{L}_\omega} K^{ij}(T^{-a}\omega) \right| \int_{\mathbb{R}^d} \partial_i \bar{\psi} \partial_j \psi \end{aligned}$$

The above sup is well defined as a function of M , ie of N and δ , since the hull Ω is uniquely ergodic. Let ϵ_δ^N be this sup, by theorem 3.4.1 we have

$$\lim_{N \rightarrow \infty} \epsilon_\delta^N = 0.$$

6. We put together the four claims in (3.4.8) eventually to complete the proof. \square

CHAPTER IV

SPECTRAL SEQUENCES

We introduce here the notion of spectral sequences. We first give an informal explanation of how the algebraic mechanism of a spectral sequence works. We give further details in the case of homology-, and cohomology-type spectral sequences associated with a bounded filtration. We next present the Serre spectral sequence for a fibration. We then give formal definitions and important properties of exact couples that will be needed in chapter 5. Finally we show some detailed applications to the K -theory of C^* -algebras, in particular we present the spectral sequence which generalizes the Pimsner–Voiculescu exact sequence in K -theory.

4.1 Informal introduction to Spectral Sequences

“There are many situations in algebraic topology where the relationship between certain homotopy, homology, or cohomology groups is expressed perfectly by an exact sequence. In other cases, however, the relationship may be more complicated and a more powerful algebraic tool is needed. In a wide variety of situations spectral sequences provide such a tool.” Allan Hatcher [39].

For example, the long exact sequences in homology or cohomology for a pair (X, Y) or a triple (X, Y, Z) generalize to a spectral sequence for an arbitrary increasing sequence of subspace $X_0 \subset X_1 \subset \cdots \subset X$ as we will show in section 4.1.2. Also, the Meyer–Vietoris long exact sequences for a decomposition $X = A \cup B$ generalize to spectral sequences for a cover of X by an arbitrary number of subsets.

A spectral sequence can be thought of as a book made of a sequence of *pages* (possibly infinitely many). Each page is an array of algebraic objects of an Abelian category:

groups, rings, modules, algebras, vector spaces... and is equipped with a *differential*: a morphism d between those objects, such that $d \circ d = 0$, so that they fit into *complexes*. The homology of those complexes are precisely the objects in the next page, and there is a new differential between them. We say that the spectral sequence *converges* if the iterated homologies become eventually trivial and the pages are eventually isomorphic (or surject onto the next ones). In the case of the spectral sequence generalizing the long exact sequence for a pair, or the Meyer-Vietoris long exact sequence, if it converges, then there is a procedure to recover the homology or cohomology of the space X from the last pages.

We first present the general idea of a spectral sequence and explain how the mechanics of the sequence works. We then restricts to the cases of homology and cohomology.

4.1.1 The general idea of a spectral sequence

This presentation is inspired by a talk given by J. Hunton [43].

Let \mathcal{C} and \mathcal{D} be two categories, $F : \mathcal{C} \rightarrow \mathcal{D}$ a functor, and X an object in \mathcal{C} . We wish to calculate $F(X)$. For monomorphisms $Y \hookrightarrow Z$ we assume that there exists objects (Y, Z) in \mathcal{C} and associated short exact sequences $0 \rightarrow Y \rightarrow Z \rightarrow (Z, Y) \rightarrow 0$, and a good notion of passage to long exact sequences in \mathcal{D} : $\cdots \rightarrow F(Y) \rightarrow F(Z) \rightarrow F(Z, Y) \rightarrow F(Y) \rightarrow \cdots$ (and the arrows are reversed if F is contravariant). We are thinking of functors graded over the integers. So the morphisms are changing degrees in the previous sequence. All this requires \mathcal{D} to be a *Abelian category* (in particular direct sums of objects, and kernels and cokernels of morphisms are well defined).

Example 4.1.1 (i) *Let \mathcal{C} be the category of C^* -algebras, \mathcal{D} the category of groups, and $F = K_*$ the K -theory functor. If I is a closed two sided ideal in A : $I \hookrightarrow A$, there is a short exact sequence $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$, and an induced long exact sequence in K -theory that we can write as the following exact triangle,*

called an exact couple:

$$\begin{array}{ccc}
 K_*(I) & \xrightarrow{\quad} & K_*(A) \\
 & \searrow & \swarrow \\
 & K_*(A/I) &
 \end{array}$$

(ii) Let \mathcal{C} be the category of CW-complexes, \mathcal{D} the category of groups, and $F = H_*$ the reduced homology functor. If Y is a subcomplex of Z : $Y \hookrightarrow Z$, there is a short exact sequence $0 \rightarrow Y \rightarrow Z \rightarrow Z/Y \rightarrow 0$, and an induced long exact sequence in homology written as the exact couple:

$$\begin{array}{ccc}
 H_*(Y) & \xrightarrow{\quad} & H_*(Z) \\
 & \searrow & \swarrow \\
 & H_*(Z/Y) &
 \end{array}$$

The goal is to calculate $F(X)$ from a *filtration* of X : we assume that there is a sequence of objects:

$$X_0 \hookrightarrow \dots \hookrightarrow X_{p-1} \hookrightarrow X_p \hookrightarrow X_{p+1} \hookrightarrow \dots \hookrightarrow X. \quad (4.1.1)$$

For example: X is a CW-complex and X_p is its p -skeleton. Let us also assume that $F(X_{p+1}, X_p)$ is known for all p , as well as $F(X_0)$ (supposed simple enough) and its image in $F(X)$: $F_0 = \text{Im}(F(X_0) \rightarrow F(X))$ (for the contravariant case, see section 4.1.2). Given this setup we have diagrams (induced filtration of $F(X)$):

$$F(X_0) \longrightarrow \dots \longrightarrow F(X_{p-1}) \xrightarrow{i_{p-1}} F(X_p) \xrightarrow{i_p} F(X_{p+1}) \longrightarrow \dots \longrightarrow F(X).$$

If the functor F is contravariant the arrows are reversed in the above induced filtration of $F(X)$.

But each of these arrows fit into long exact sequences that we can write together as

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & F(X_{p-1}) & \xrightarrow{i_{p-1}} & F(X_p) & \xrightarrow{i_p} & F(X_{p+1}) & \longrightarrow & \cdots \\
 & & \swarrow j_{p-2} & & \swarrow j_{p-1} & & \swarrow j_p & & \\
 & & & & F(X_p, X_{p-1}) & & F(X_{p+1}, X_p) & & \\
 & & \nwarrow k_{p-1} & & \nwarrow k_p & & \nwarrow k_{p+1} & & \\
 & & & & & & & &
 \end{array} \tag{4.1.2}$$

where all the triangles are exact (and typically the maps k_p 's change degree). A short hand notation for the above diagram is the exact couple

$$\begin{array}{ccc}
 \oplus_p F(X_p) & \xrightarrow{i} & \oplus_p F(X_p) \\
 & \swarrow k & \searrow j \\
 & \oplus_p F(X_p, X_{p-1}) &
 \end{array}$$

Let us first assume that all the i_p 's are inclusions. By exactness, this implies that $k_p = 0$, and j_p is onto for all p . Diagram (4.1.2) then reads:

$$\begin{array}{ccccccc}
 \cdots \hookrightarrow & F(X_{p-1}) & \xrightarrow{i_{p-1}} & F(X_p) & \xrightarrow{i_p} & F(X_{p+1}) & \hookrightarrow \cdots \\
 & \downarrow j_{p-2} & & \downarrow j_{p-1} & & \downarrow j_p & \\
 & F(X_{p-1}, X_{p-2}) & & F(X_p, X_{p-1}) & & F(X_{p+1}, X_p) &
 \end{array} \tag{4.1.3}$$

and we are left with solving inductively the extension problems given by the short exact sequences:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F(X_0) & \longrightarrow & F(X_1) & \longrightarrow & F(X_1, X_0) & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
 0 & \longrightarrow & F(X_{p-1}) & \longrightarrow & F(X_p) & \longrightarrow & F(X_p, X_{p-1}) & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & &
 \end{array}$$

Indeed, since we know $F(X_0)$ and $F(X_1, X_0)$, if we can solve the first exact sequence then it gives us $F(X_1)$. Next, since we know now $F(X_1)$ and $F(X_2, X_1)$, we can tackle the second exact sequence to deduce $F(X_2)$, and then use it to tackle the third exact sequence.... and so on and so forth.

The solutions to those extensions yields $\varprojlim F(X_p)$, and provided we can identify this inverse limit with $F(X)$, we have solved our initial problem.

Note however that those extension problems might not have a unique solution, as one can see with the following example:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 \longrightarrow 0 \\
 & & & & & & \uparrow \\
 & & & & & & \mathbb{Z} \oplus \mathbb{Z}_2 \\
 & & & & & & \downarrow \\
 & & & & & & \mathbb{Z}
 \end{array}$$

where the two exact sequences provide different extensions of \mathbb{Z}_2 by \mathbb{Z} .

Of course, in practice the i_p 's are not injective. The spectral sequence associated with the filtration (4.1.1) of X will process the data of all the long exact sequences written in diagram (4.1.2) step by step or *page by page*. If the spectral sequence *converges* then this process terminates eventually and yields the naive situation of diagram (4.1.3) where the i_p 's are injective so that one is left with a sequence of extensions. Let us describe now how this process works.

This process consists in “making” the i_p 's injective by killing elements of their kernels and transforming the groups to preserve exactness of the diagrams.

Let us look at diagram (4.1.2) and first consider an x^1 that belongs to the kernel of i_p in $F(X_p)$ but not to the image of i_{p-1} : $x^1 \in \text{Ker } i_p \setminus \text{Im } i_{p-1}$. By exactness there is a y^1 in $F(X_{p+1}, X_p)$ whose image is x^1 : $k_p(y) = x$. Since x^1 does not belong to the image of i_{p-1} then by exactness again there is a z^1 in $F(X_{p+1}, X_p)$ that is its image: $j_{p-1}(x^1) = y^1$. Let $d_p^1 = j_{p-1} \circ k_p$, it is a differential: $d_{p-1}^1 \circ d_p^1 = 0$ since $k_{p-1} \circ j_{p-1} = 0$

by exactness.

$$\begin{array}{ccc}
 & F(X_p) \ni x^1 & \xrightarrow{i_p} 0 \in F(X_{p+1}) \\
 & \swarrow j_{p-1} & \searrow k_p \\
 F(X_p, X_{p-1}) \ni y^1 & \xleftarrow{d_p^1} & z^1 \in F(X_{p+1}, X_p)
 \end{array}$$

The first page of the spectral sequence is given by the groups $E_p^1 = F(X_p, X_{p-1})$ and the differential d^1 .

We now eradicate such elements as x^1, y^1, z^1 by replacing (i) $F(X_p)$ by $i_{p-1}F(X_{p-1})$ and (ii) $F(X_{p+1}, X_p)$ by the homology group $H(F(X_{p+1}, X_p)) = \text{Ker } d_p^1 / \text{Im } d_{p+1}^1$. The replacements (i) kill elements like x^1 (in the kernel of i_p), while (ii) kill elements like y^1 (by moding out by the image of d^1) and z^1 (in the kernel of d^1). We get then *derived* triangles:

$$\begin{array}{ccccc}
 \cdots & \xrightarrow{i_{p-1}^2} & i_{p-1}(F(X_{p-1})) & \xrightarrow{i_p^2} & i_p(F(X_p)) \cdots \\
 & \swarrow j_{p-1}^2 & & \searrow k_p^2 & \swarrow j_p^2 \\
 H(F(X_p, X_{p-1})) & \xleftarrow{d_p^2} & & H(F(X_{p+1}, X_p)) & \xleftarrow{d_{p+1}^2} \cdots
 \end{array} \tag{4.1.4}$$

where i_p^2 is the restriction of i_p to $\text{Im } i_{p-1}$, j_p^2 the restriction of j_p to $\text{Im } i_p$, and $k_p^2 = k_{p*}$ is the induced map on homology. A very nice thing happens: the above triangles (4.1.4) are still exact! (see theorem 4.3.1). In particular we can define a differential $d^2 = j_2 \circ k_2$ ($d^2 \circ d^2 = 0$ by exactness). We define the second page of the spectral sequence by the groups $E_p^2 = \text{Ker } d_{p-1}^1 / \text{Im } d_p^1$ with the differential d^2 .

Looking at diagram (4.1.2), let us now consider an element x^2 in the kernel of i_p that belongs to the image of i_{p-1} but not to the image of $i_{p-1} \circ i_{p-2}$. This element is in the derived triangles (4.1.4) in the exact same situation as x^1 in the triangles (4.1.2), and gives rise to elements y^2 and z^2 similarly. Hence it suffices to take their derived

triangles to eradicate such elements as x^2, y^2, z^2 . And the derived triangles will give the third page of the spectral sequence: the groups $E_p^3 = \text{Ker } d_{p-1}^2 / \text{Im } d_p^2$ with the differential d^3 .

Inductively we see on diagram (4.1.2) that an element x^n in the kernel of i_p that belongs to the image of $i_{p-1} \circ \cdots \circ i_{p-n+1}$ but not to the image of $i_{p-1} \circ \cdots \circ i_{p-n+1} \circ i_{p-n}$ will be in the n -th derived triangles in the same situation as x^1 in the triangles (4.1.2), and therefore will be eradicated (as well as its associated elements y^n, z^n) in the $(n+1)$ -th derived triangles. And these triangles give the $(n+1)$ -th page of the spectral sequence $\{E^{n+1}, d^{n+1}\}$.

If this process terminates then we see that no x 's in the iterated images of the $F(X_p)$'s lie in the kernels of the i_p 's and thus we are left with the naive situation of (4.1.3) (where the i_p 's are inclusions) and a sequence of extension problems. Since the filtration (4.1.1) is bounded below, we see that when the process has converged, $F(X_p)$ has been replaced by $F_p = \text{Im}(F(X_p) \rightarrow F(X))$, its image in $F(X)$, and the groups $F(X_p, X_{p-1})$ have been replaced by the iterated homology groups which by exactness are the quotients F_p/F_{p-1} . (If the functor F is contravariant, then $F(X_p)$ has been replaced by $F^p = \text{Ker}(F(X) \rightarrow F(X_{p-1}))$, and $F(X_p, X_{p-1})$ by F^p/F^{p+1} , see section 4.1.2). Those quotients are the groups of the last page of the spectral sequence, denoted E^∞ . Hence the extension problems left to solve now are:

$$\begin{array}{ccccccc}
0 & \longrightarrow & F_0 & \longrightarrow & F_1 & \longrightarrow & F_1/F_0 \longrightarrow 0 \\
& & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & F_{p-1} & \longrightarrow & F_p & \longrightarrow & F_p/F_{p-1} \longrightarrow 0 \\
& & \vdots & & \vdots & & \vdots
\end{array}$$

where F_0 is assumed to be known, and the quotients F_p/F_{p-1} have been calculated.

Therefore we can recover $F(X)$ upon solving those extensions inductively. In fact, if the filtration (4.1.1) is not finite, we recover only the inverse limit $\varprojlim F(X_p)$. We say that $F(X)$ has been approximated by the *associated filtered object* $FX = \bigoplus_p F_p/F_{p-1}$ for the filtration of $F(X)$ induced by (4.1.1).

4.1.2 The graded case, example of homology and cohomology.

In the discussion of the previous paragraph we have explained the general mechanism of a spectral sequence. We now show how the grading of the functor F comes into the play.

Let us restrict for simplicity of the exposition to the case of homology and cohomology of topological spaces. Let X be a topological space, and assume it has a filtration by arbitrary subspaces:

$$X_0 \hookrightarrow \dots \hookrightarrow X_{p-1} \hookrightarrow X_p \hookrightarrow X_{p+1} \hookrightarrow \dots \hookrightarrow X. \quad (4.1.5)$$

Like in (4.1.1) the filtration is assumed to be bounded below: $X_p = \emptyset$ for $p < 0$. We further assume that the pairs (X, X_p) and (X_{p+1}, X_p) are p -connected for all $p \geq 0$.

The homology spectral sequence.

Like in example 4.1.1 (ii), let $F = H_*$ be the (say singular) homology functor on the category of topological spaces (with coefficients in a group G implicitly understood).

Remark 4.1.2 The assumptions that the pairs (X, X_p) and (X_{p+1}, X_p) are p -connected for all $p \geq 0$ imply that (see [38], lemma 2.34, section 2.2 p.137):

- (i) $\mathbf{H}_n(\mathbf{X}_p, \mathbf{X}_{p-1}) = \mathbf{0}$, for $p > n$, and
- (ii) the inclusion $X_p \hookrightarrow X$ induces an isomorphism on homology: $\mathbf{H}_n(\mathbf{X}_p) \cong \mathbf{H}_n(\mathbf{X})$ for $p > n$.

We replace n by $p+q$, and define the first page by $E_{p,q}^1 = H_{p+q}(X_p, X_{p-1})$ which is non zero for $p, q \geq 0$. Compared to the previous section where we only labeled elements

of a page by the degree (p) of the filtration (4.1.1), we now have another index (q) accounting for the degree of H . We write the first page on a two dimensional lattice with the group $E_{p,q}^1$ at coordinate (p, q) .

Set $D_{p,q}^1 = H_{p+q}(X_p)$, and let D^1 and E^1 be the direct sums of the $D_{p,q}^1$ and $E_{p,q}^1$ over $p, q \geq 0$ respectively. The long exact sequences in homology:

$$\cdots \longrightarrow H_{p+q}(X_{p-1}) \xrightarrow{i} H_{p+q}(X_p) \xrightarrow{j} H_{p+q}(X_p, X_{p-1}) \xrightarrow{k} H_{p+q-1}(X_{p-1}) \longrightarrow \cdots$$

fit into the exact couple that gives rise to the first page of the spectral sequence:

$$\begin{array}{ccc} D^1 & \xrightarrow{i_1^{(1,-1)}} & D^1 \\ & \swarrow k_1^{(-1,0)} & \searrow j_1^{(0,0)} \\ & E^1 & \end{array}$$

where the superscripts on the maps denote their bidegrees. Indeed, $i_1 : H_{p+q}(X_{p-1}) = D_{p-1,q+1}^1 \rightarrow H_{p+q}(X_p) = D_{p,q}^1$, $j_1 : H_{p+q}(X_p) = D_{p,q}^1 \rightarrow H_{p+q}(X_p, X_{p-1}) = E_{p,q}^1$, and $k_1 : H_{p+q}(X_p, X_{p-1}) = E_{p,q}^1 \rightarrow H_{p+q-1}(X_{p-1}) = D_{p-1,q}^1$. The differential $d_1 = j_1 \circ k_1$ therefore has bidegree $(0, 0) + (-1, 0) = (-1, 0)$, and goes horizontally 1 unit to the left on the first page:

$$E_{p-1,q}^1 \xleftarrow{d_1^{(-1,0)}} E_{p,q}^1$$

We now set $D_{p,q}^2 = i_1(D_{p-1,q+1}^1)$, and $E_{p,q}^2 = H(E_{p,q}^1, d^1) = \text{Ker}(d_1 : E_{p,q}^1 \rightarrow E_{p-1,q}^1) / \text{Im}(d_1 : E_{p+1,q}^1 \rightarrow E_{p,q}^1)$, and denote by D^2 and E^2 their direct sums over $p, q \geq 0$ respectively. The derived exact couple giving rise to the second page now reads:

$$\begin{array}{ccc} D^1 & \xrightarrow{i_2^{(1,-1)}} & D^2 \\ & \swarrow k_2^{(-1,0)} & \searrow j_2^{(-1,1)} \\ & E^2 & \end{array}$$

where $\mathbf{i}_2 = \mathbf{i}_1|_{\text{Im } i_1}$ is the restriction of i_1 to the image of i_1 , $\mathbf{j}_2 = \mathbf{j}_1|_{\text{Im } i_1}$ is the restriction of j_1 to the image of i_1 defined as $j_2(i_1(x)) = j_1(x) + d_1E^1$, and $\mathbf{k}_2 = (\mathbf{k}_1)_*$ is the induced map on homology: $k_2(e + d_1E^1) = k_1(e)$. As i_2 is the restriction of i_1 : $i_2 : D_{p,q}^2 = i_1(D_{p-1,q+1}^1) \rightarrow i_1 \circ i_1(D_{p-1,q+1}^1) = i_1(D_{p,q}^1) = D_{p+1,q-1}^2$. By definition $j_2(i_1(x)) = j_1(x) + d_1E^1$, and if $i_1(a) \in D_{p,q}^2 = i_1(D_{p-1,q+1}^1)$ then $a \in D_{p-1,q+1}^1$ so that $j_1(a) \in E_{p-1,q+1}^1$ and therefore its homology class $j_1(x) + d_1E^1$ belongs to $E_{p-1,q+1}^2$. Finally if $e^2 = e^1 + d^1E^1 \in E_{p,q}^2$ with $e^1 \in E_{p,q}^1$, then $k_2(e^2) = k_1(e^1) \in D_{p-1,q}^1$ and since $d_1e^1 = j_1 \circ k_1(e^1) = 0$ we have $k_1(e^1) \in \text{Ker } j_1 = \text{Im } i_1$ and thus $k_1(e^1) \in i_1(D_{p-2,q+1}^1) = D_{p-1,q}^2 \supset D_{p-1,q}^1$. The differential $d_2 = j_2 \circ k_2$ therefore has bidegree $(-2, 1)$, and goes diagonally 2 units to the left and 1 unit upward on the second page:

$$\begin{array}{ccc}
 E_{p-2,q+1}^2 & \cdot & \cdot \\
 & \swarrow d_2^{(-2,1)} & \\
 \cdot & \cdot & E_{p,q}^2
 \end{array}$$

We define inductively $D_{p,q}^r = i_{r-1}(D_{p-1,q+1}^{r-1})$, and $E_{p,q}^r = H(E_{p,q}^{r-1}, d_{r-1})$, and denote by D^r and E^r their direct sums over $p, q \geq 0$ respectively. The r -th derived exact couple giving rise to the r -th page is the derived pair of the $(r-1)$ -th derived pair and reads:

$$\begin{array}{ccc}
 D^r & \xrightarrow{i_r^{(1,-1)}} & D^r \\
 & \swarrow k_r^{(-1,0)} & \searrow j_r^{(1-r,r-1)} \\
 & & E^r
 \end{array}$$

The differential $d_r = j_r \circ k_r$ therefore has bidegree $(-r, r-1)$, and goes diagonally r units to the left and $r-1$ units upward on the r -th page:

$$\begin{array}{cccc}
 E_{p-r,q+r-1}^r & \cdot & \cdots & \cdot \\
 & \swarrow d_r^{(-r,r-1)} & & \\
 \vdots & \vdots & \cdots & \vdots \\
 \cdot & \cdot & \cdots & E_{p,q}^r
 \end{array}$$

Remark 4.1.3 (i) This spectral sequence is called a **first quadrant spectral sequence** because it has non trivial groups only at point of positive coordinates.

Indeed for $p, q < 0$ the groups $E_{p,q}^1$ are trivial and thus all the groups $E_{p,q}^r, r \geq 2$ on further pages are trivial too since they are quotient subgroups of $E_{p,q}^1$.

(ii) $\mathbf{D}_{p,q}^r = \mathbf{0}$ for $\mathbf{r} > \mathbf{p} - 1$. Indeed the filtration (4.1.5) is bounded below: $X_p = \emptyset$ for $p < 0$. And $D_{p,q}^r = i_{r-1}(D_{p-1,q+1}^{r-1}) = i_{r-1} \circ_{r-2}(D_{p-2,q+2}^{r-2}) = \dots = i_1^r(D_{p-r+1,q+r-1}^1)$ but $D_{p-r+1,q+r-1}^1 = H_{p+q}(X_{p-r+1})$ and $X_{p-r+1} = \emptyset$ for $p - r + 1 < 0$.

(iii) **For $\mathbf{r} > \mathbf{p} + \mathbf{q}$, $\mathbf{E}_{p,q}^r \cong \mathbf{E}_{p,q}^\infty$.** Indeed the groups $E_{p,q}^r$ that lie below the line $q = -p + r - 1$ remain unchanged on all further pages because they cannot be reached by any differential $d_s, s \geq r$ (d_s goes diagonally s units to the left and $s - 1$ units upward on the s -th page and therefore can only link those groups to 0 outside the first quadrant).

The long exact sequences on the above r -th derived pair read:

$$\dots \xrightarrow{j_r} E_{p-1+r,q+2-r}^r \xrightarrow{k_r} D_{p-2+r,q+2-r}^r \xrightarrow{i_r} D_{p-1+r,q+1-r}^r \xrightarrow{j_r} E_{p,q}^r \xrightarrow{k_r} D_{p-1,q}^r \xrightarrow{i_r} \dots \quad (4.1.6)$$

By remark 4.1.3 above, we have that if $r > q$ the left E^r -term is 0 by (i), if $r > p - 2$ the right D^r -term is 0 by (ii), and if $r > p + q$ the middle E^r -term is isomorphic to $E_{p,q}^\infty$. For $r > \max(p - 2, q, p + q)$ we therefore get the short exact sequence

$$0 \longrightarrow D_{p-2+r,q+2-r}^r \xrightarrow{i_r} D_{p-1+r,q+1-r}^r \xrightarrow{j_r} E_{p,q}^\infty \longrightarrow 0$$

and thus an isomorphism $E_{p,q}^\infty \cong D_{p-2+r,q+2-r}^r / i_r(D_{p-1+r,q+1-r}^r)$. Now $D_{p-2+r,q+2-r}^r$ is the iterated image $i_1^{r-1}(D_{p-1,q+1}^1)$ of $H_{p+q}(X_{p-1})$ in $H_{p+q}(X_{p+r-2})$. Similarly $i_r(D_{p-1+r,q+1-r}^r)$ is the iterated image $i_1^r(D_{p,q}^1)$ of $H_{p+q}(X_p)$ in $H_{p+q}(X_{p+r-1})$. Finally, by remark 4.1.2 (ii), for $r > q + 2$ the inclusions of X_{p+r-2} and X_{p+r-1} in X induce isomorphisms on

homology. Hence we have the isomorphism

$$E_{p,q}^\infty \cong F_{p+q}^p / F_{p+q}^{p-1}, \quad \text{where} \quad F_n^p = \text{Im} (H_n(X_p) \rightarrow H_n(X)) \quad (4.1.7)$$

and $E_{p,q}^r$ is also isomorphic to the above quotient for $r > \max(p-2, q+2, p+q)$.

The groups F_n^p give a filtration of $H_n(X)$ induced by the filtration (4.1.5) of X :

$$F_n^{0\zeta} \longrightarrow \dots \zeta \longrightarrow F_n^{p-1\zeta} \longrightarrow F_n^{p\zeta} \longrightarrow F_n^{p+1\zeta} \longrightarrow \dots \zeta \longrightarrow F_n^{n+1} = H_n(X).$$

The induced filtration is indeed finite since F_{n+1}^n is the image of $H_n(X_{n+1})$ in $H_n(X)$ which by remark 4.1.2 (ii) is isomorphic to $H_n(X)$.

The direct sum of the quotients $F^p H_n := F_n^p / F_n^{p-1}$ forms the *associated graded group* FH to the graded group $H(X) = \bigoplus_n H_n(X)$. We say that the spectral sequence $\{E^r, d_r\}$ converges to the homology of X , and write

$$E_{p,q}^1 \Rightarrow H_{p+q}(X_p),$$

and this means that the E -infinity page is isomorphic to the associated graded group HF as in the above equation (4.1.7).

If we know the associated graded group FH on the E -infinity page, then we can recover $H(X)$ by solving the $n+1$ extension problems inductively

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_n^0 & \longrightarrow & F_n^1 & \longrightarrow & F^1 H_n \longrightarrow 0 \\ & & \vdots & & \vdots & & \\ 0 & \longrightarrow & F_n^{p-1} & \longrightarrow & F_n^p & \longrightarrow & F^p H_n \longrightarrow 0 \\ & & \vdots & & \vdots & & \\ 0 & \longrightarrow & F_n^n & \longrightarrow & F_n^{n+1} & \longrightarrow & F^{n+1} H_n \longrightarrow 0 \end{array}$$

where F_n^0 is assumed to be known (supposed simple enough).

The cohomology spectral sequence.

Let now $F = H^*$ be the (say singular) cohomology functor on the category of topological spaces. The spectral sequence is easy to write down now that we have treated its homology version in some details.

The assumptions that the pairs (X, X_p) and (X_{p+1}, X_p) are p -connected for all $p \geq 0$ imply here as well, like in remark 4.1.2, that:

- (i) $\mathbf{H}^n(\mathbf{X}_p, \mathbf{X}_{p-1}) = \mathbf{0}$, for $p > n$, and
- (ii) the inclusion $X_p \hookrightarrow X$ induces an isomorphism on cohomology: $\mathbf{H}^n(\mathbf{X}) \cong \mathbf{H}^n(\mathbf{X}_p)$ for $p > n$.

We replace n by $p + q$, and define the first page by $E_1^{p,q} = H^{p+q}(X_p, X_{p-1})$ which is non zero for $p, q \geq 0$. Set $D_1^{p,q} = H^{p+q}(X_p)$, and let D_1 and E_1 be the direct sums of the $D_1^{p,q}$ and $E_1^{p,q}$ over $p, q \geq 0$ respectively. The long exact sequences in cohomology:

$$\dots \longrightarrow H^{p+q-1}(X_{p-1}) \xrightarrow{k} H^{p+q}(X_p, X_{p-1}) \xrightarrow{j} H^{p+q}(X_p) \xrightarrow{i} H^{p+q}(X_{p-1}) \longrightarrow \dots$$

fit into the exact couple that gives rise to the first page of the spectral sequence:

$$\begin{array}{ccc} D_1 & \xrightarrow{i_1^{(-1,1)}} & D_1 \\ & \swarrow j_1^{(0,0)} & \searrow k_1^{(1,0)} \\ & E_1 & \end{array}$$

where the upperscripts on the maps denote their bidegrees. Indeed, $i_1 : H^{p+q}(X_p) = D_1^{p,q} \rightarrow H^{p+q}(X_{p-1}) = D_1^{p-1,q+1}$, $j_1 : H^{p+q}(X_p, X_{p-1}) = E_1^{p,q} \rightarrow H^{p+q}(X_p) = D_1^{p,q}$, and $k_1 : H_{p+q-1}(X_{p-1}) = D_1^{p-1,q} \rightarrow H_{p+q}(X_p, X_{p-1}) = E_1^{p,q}$. The differential $d_1 = k_1 \circ j_1$ therefore has bidegree $(1, 0)$, and goes horizontally 1 unit to the right on the first page:

$$E_1^{p-1,q} \xrightarrow{d_1^{(1,0)}} E_1^{p,q}$$

We define inductively $D_r^{p,q} = i_{r-1}(D_{r-1}^{p+1,q-1}) \subset D_{r-1}^{p,q}$, and $E_r^{p,q} = H(E_{r-1}^{p,q}, d_{r-1})$, and the map $\mathbf{i}_r = \mathbf{i}_{r-1}|_{\text{Im } \mathbf{i}_{r-1}}$ as the restriction of i_{r-1} to the image of i_{r-1} , the map

$\mathbf{j}_r = (\mathbf{j}_{r-1})_*$ as the induced map on homology: $j_r(e + d_{r-1}E_{r-1}) = j_{r-1}(e)$, and the map $\mathbf{k}_r = \mathbf{k}_{r-1}|_{\text{Im } i_{r-1}}$ as the restriction of k_{r-1} to the image of i_{r-1} defined as $k_r(i_{r-1}(x)) = k_{r-1}(x) + d_{r-1}E_{r-1}$. Let us denote by D_r and E_r the direct sums over $p, q \geq 0$ of $D_r^{p,q}$ and $E_r^{p,q}$ respectively. The r -th derived exact couple giving rise to the r -th page reads:

$$\begin{array}{ccc}
 D_r & \xrightarrow{i_r^{(-1,1)}} & D_r \\
 & \swarrow j_r^{(0,0)} & \searrow k_r^{(-r,r-1)} \\
 & & E_r
 \end{array}$$

The differential $d_r = j_r \circ k_r$ therefore has bidegree $(-r, r-1)$, and goes diagonally r units to the right and $r-1$ units downward on the r -th page:

$$\begin{array}{ccccccc}
 E_{p,q}^2 & \cdot & \cdots & & \cdot & & \\
 & \vdots & \vdots & \searrow d_r^{(r,1-r)} & \vdots & & \\
 & \cdot & \cdot & \cdots & & & E_{p+r,q-r+1}^r
 \end{array}$$

Just like in remark 4.1.3 the spectral sequence has the following properties:

- (i) It is a *first quadrant spectral sequence*: **For all r , $\mathbf{E}_r^{p,q} = \mathbf{0}$ if $p, q < 0$.**
- (ii) **$\mathbf{D}_r^{p,q} \cong \mathbf{H}^{p+q}(\mathbf{X})$ for $r > q + 1$, and for all r , $\mathbf{D}_r^{p,q} = \mathbf{0}$ if $p, q < 0$.**
- (iii) **For $r > p + q$, $\mathbf{E}_r^{p,q} \cong \mathbf{E}_{p,q}^\infty$.**

The long exact sequences on the above r -th derived pair read:

$$\dots \xrightarrow{i_r} D_r^{p-r,q+r-1} \xrightarrow{k_r} E_r^{p,q} \xrightarrow{j_r} D_r^{p,q} \xrightarrow{i_r} D_r^{p-1,q+1} \xrightarrow{k_r} E_r^{p+r-1,q-r+2} \xrightarrow{k_r} \dots \quad (4.1.8)$$

By the above remarks, we have that if $r > q + 2$ the right E_r -term is 0 by (i), if $r > p$ the left D_r -term is 0 by (ii), and if $r > p + q$ the middle E_r -term is isomorphic to $E_{p,q}^\infty$

by (iii). For $r > \max(p, q + 2, p + q)$ we therefore get the short exact sequence

$$0 \longrightarrow E_{p,q}^\infty \xrightarrow{j_r} D_{p,q}^r \xrightarrow{i_r} D_{p-1,q+1}^r \longrightarrow 0$$

and thus an isomorphism $E_{p,q}^\infty \cong \text{Im}(j_r) \cong \text{Ker}(i_r : D_{p,q}^r \rightarrow D_{p-1,q+1}^r)$. Now $D_{p,q}^r$ is the iterated image $i_1^{r-1}(D_1^{p+r-1,q-r+1})$ of $H_{p+q}(X_{p+r-1})$ in $H_{p+q}(X_p)$. Similarly $i_r(D_{p-1,q+1}^r)$ is the iterated image $i_1^r(D_{p+r-2,q-r+2}^1)$ of $H_{p+q}(X_{p+r-2})$ in $H_{p+q}(X_{p-1})$. Finally, by (ii), for $r > q + 2$ the inclusions of X_{p+r-2} and X_{p+r-1} in X induce isomorphisms on cohomology. Hence the kernel of i_r can be rewritten as the quotient of the kernel of $H_{p+q}(X) \rightarrow H_{p+q}(X_{p-1})$ by the kernel of $H_{p+q}(X) \rightarrow H_{p+q}(X_p)$:

$$E_{p,q}^\infty \cong F_p^{p+q}/F_{p+1}^{p+q}, \quad \text{where} \quad F_p^n = \text{Ker}(H^n(X) \rightarrow H^n(X_{p-1})) \quad (4.1.9)$$

and $E_r^{p,q}$ is also isomorphic to the above quotient for $r > \max(p, q + 2, p + q)$.

The groups F_p^n give a filtration of $H^n(X)$ induced by the filtration (4.1.5) of X :

$$F_{n+1}^n \hookrightarrow \dots \hookrightarrow F_{p+1}^n \hookrightarrow F_p^n \hookrightarrow F_{p-1}^n \hookrightarrow \dots \hookrightarrow F_0^n = H_n(X).$$

The induced filtration is finite here aswell, since by (ii) $H^n(X_p) \cong H^n(X)$ for $p > n$ so that $F_p^n = 0, p > n + 1$.

The direct sum of the quotients $F_p H^n := F_p^n / F_{p+1}^n$ forms the *associated graded group* FH to the graded group $H(X) = \bigoplus_n H^n(X)$. We say that the spectral sequence $\{E_r, d_r\}$ converges to the cohomology of X , and write

$$E_1^{p,q} \Rightarrow H^{p+q}(X_p),$$

and this means that the E -infinity page is isomorphic to the associated graded group FH as in the above equation (4.1.9).

If we know the associated graded group FH on the E -infinity page, then we can

recover $H(X)$ by solving the $n + 1$ extension problems inductively

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F_{n+1}^n & \longrightarrow & F_n^n & \longrightarrow & F_n H^n \longrightarrow 0 \\
 & & \vdots & & & & \vdots \\
 0 & \longrightarrow & F_{p+1}^n & \longrightarrow & F_p^n & \longrightarrow & F_p H^n \longrightarrow 0 \\
 & & \vdots & & & & \vdots \\
 0 & \longrightarrow & F_1^n & \longrightarrow & F_0^n & \longrightarrow & F_0 H^n \longrightarrow 0
 \end{array}$$

where F_{n+1}^n is assumed to be known (supposed simple enough).

4.2 The Serre spectral sequence

We describe briefly in this section the Serre spectral sequences for the homology and cohomology for a fibration $F \hookrightarrow X \xrightarrow{p} B$. It was built by Serre in his PhD thesis [76], where he also developed the notion of fibration. A detailed construction of the spectral sequence can be found in [39].

The main purpose of this section is to make the analogy with our spectral sequence given in theorem 5.2.1 and the PV-cohomology transparent. We will not give any proofs here, but the interested reader will find precise references, mainly to [38, 39]. We also mention the Atiyah-Hirzebruch spectral sequence that we will use in the proof of theorem 5.2.1.

We first recall the basic definitions and properties of fibrations, and introduce local coefficient systems.

4.2.1 Fibrations

A map $X \xrightarrow{p} B$ is said to have the *homotopy lifting property* with respect to a space Y if, given a homotopy $f : Y \times [0, 1] \rightarrow B$ and a map $g : Y \rightarrow X$ such that $pg(y) = f(y, 0)$, there exists a homotopy $\tilde{f} : Y \times [0, 1] \rightarrow X$ such that $\tilde{f}(y, 0) = g(y)$ and $p\tilde{f} = f$.

$$\begin{array}{ccc}
 Y \times \{0\} & \xrightarrow{g} & X \\
 \downarrow & \nearrow \tilde{f} & \downarrow p \\
 Y \times [0, 1] & \xrightarrow{f} & B
 \end{array}$$

A map $X \xrightarrow{p} B$ that satisfies the homotopy lifting properties with respect to any space is called a *fibration*. It is called a *Serre fibration* if it satisfies the homotopy lifting properties with respect to any *CW*-complex.

We have the following important property (a proof of which can be found in [38] proposition 4.61 p. 405).

Proposition 4.2.1 *Let $X \xrightarrow{p} B$ be a fibration. The fibers $F_b = p^{-1}(b)$ over each path components of B are homotopy equivalent.*

It is customary to talk about “the fiber” F and to denote the fibration by $F \hookrightarrow X \xrightarrow{p} B$. If $\gamma : a \rightarrow b$ is a path in B , then it determines a homotopy equivalence L_γ between the fibers F_a and F_b .

4.2.2 Local coefficients systems

A *local system of groups* $\mathcal{G} = (G_b)$ over a space B is given by the data of: 1) a group G_b for each element $b \in B$ and 2) a group isomorphism $\theta_\gamma : G_a \rightarrow G_b$, for each path $\gamma : a \rightarrow b$ in B , which depends only on the homotopy class of γ and which satisfies an obvious transitivity condition: $\theta_{\gamma\gamma'} = \theta_\gamma\theta_{\gamma'}$. This notion was introduced by Steenrod in [80]. If B is path connected, then all the groups G_b are isomorphic

and the local system of group is uniquely determined by one of them, G_a , and the automorphisms defined in G_a by the fundamental group of B based at a , $\pi_1(B, a)$.

Remark 4.2.2 If B is path connected and its fundamental group acts trivially on the groups G_b then the G_b are canonically isomorphic to one of them, G_a , and one can see the local system of group as just the data of the group G_a over each point of B .

Example 4.2.3 If $F \hookrightarrow X \xrightarrow{p} B$ is a fibration over a path connected space B , then applying the homology functor to the fibers gives a local system of groups $\mathcal{H}_*(F) = (H_*(F_b))$. Indeed, as homotopy classes of paths in B give homotopy equivalences between the fibers over their end points, they therefore give isomorphisms between their homology groups. Similarly, cohomology and K -theory give local systems of groups $\mathcal{H}^*(F) = (H^*(F_b))$ and $\mathcal{K}^*(F) = (K^*(F_b))$.

Let B be a Δ -complex (see section 2.3.2) and $\mathcal{G} = (G_b)$ a local system of Abelian groups over B . If $\sigma : \Delta^n \rightarrow B$ is an n -simplex of B , let us denote by G_σ the group fiber over its barycenter (the image in B of the barycenter of Δ^n). We define the group $C_n(B; \mathcal{G})$ of n -chains with coefficients in \mathcal{G} as the group of formal linear combinations of n -simplices with coefficients in the groups over their barycenters: an n -chain is a finite formal sum $\sum g_i \sigma_i$ where σ_i is an n -simplex and $g_i \in G_{\sigma_i}$. The differential of an n -chain $g\sigma$ is given by

$$\partial(g\sigma) = \sum_{i=0}^n (-1)^i \theta_{\sigma, \partial_i \sigma}(g) \partial_i \sigma, \quad (4.2.1)$$

where $\partial_i \sigma$ denotes the $(n-1)$ -simplex that is the i -th face of σ , and $\theta_{\sigma, \partial_i \sigma} : G_\sigma \rightarrow G_{\partial_i \sigma}$ is the group isomorphism between their fibers. We define now for B its *homology with local coefficients* in the local system of groups \mathcal{G} as the homology of the complex of n -chain groups with local coefficients in \mathcal{G} and differential (4.2.1). We denote this homology by $H_*(B; \mathcal{G})$.

The *cohomology with local coefficients* in \mathcal{G} , $H^*(B; \mathcal{G})$, is defined as follows. The n -cochain group, $C^n(B; \mathcal{G})$, is the group of “local group homomorphisms” from $C_n(B; \mathcal{G})$ to \mathcal{G} : if φ is an n -cochain, and σ an n -simplex of B , then $\varphi(\sigma) \in G_\sigma$, and for an n -chain $\sum g_i \sigma_i$ we have $\varphi(\sum g_i \sigma_i) = \sum g_i \varphi(\sigma_i)$. The differential is defined by:

$$\delta\varphi(g\sigma) = \varphi(\partial(g\sigma)) = \sum_{i=0}^n (-1)^i \theta_{\sigma, \partial_i \sigma}(g) \varphi(\partial_i \sigma), \quad (4.2.2)$$

for an n -cochain φ and an $(n+1)$ -chain $g\sigma$.

By remark 4.2.2, if the fundamental group of B acts trivially on F , the local system of groups \mathcal{G} is trivial, *i.e.* the groups G_b are canonically isomorphic to one of them, G , and hence the homology and cohomology with local coefficients in \mathcal{G} are just the ordinary theories for B with coefficient in the group G .

As noted in remark 2.3.6 the PV cohomology of a tiling \mathcal{T} , can be seen as a generalization of a cohomology of the prototile space of \mathcal{T} , with local coefficients in the K -theory of the transversal to the hull of \mathcal{T} . The operators $\theta_{\sigma, \partial_i \sigma}$ are in this case only *partial isometries* and not isomorphisms. See section 6.7 for a general formulation of this point.

4.2.3 The spectral sequence

We give here a version of Serre spectral sequence for a fibration $F \hookrightarrow X \longrightarrow B$ over a path connected Δ -complex B . Let X_p denote the lift of the p -skeleton B_p of B . The pairs (X, X_p) and (X_{p+1}, X_p) are p -connected as lifts of the pairs (B, B_p) and (B_{p+1}, B_p) by the homotopy lifting property. We get a filtration of X : $X_0 \hookrightarrow X_1 \hookrightarrow \dots \hookrightarrow X_p \hookrightarrow \dots \hookrightarrow X$, and the induced filtrations of $H_*(X)$ and $H^*(X)$ give rise to spectral sequences converging to the homology and cohomology of X according to the construction shown in section 4.1.2.

Theorem 4.2.4 *Let $F \hookrightarrow X \longrightarrow B$ be a fibration over a path connected Δ -complex B . There is a spectral sequence $\{E_{p,q}^r, d_r\}$ with:*

- (i) $d_r : E_{p,q}^r \longrightarrow E_{p-r,q+r-1}^r$ and $E_{p,q}^{r+1} = \text{Ker } d_r / \text{Im } d_r$ the homology group at $E_{p,q}^r$,
- (ii) stable terms $E_{p,q}^\infty$ isomorphic to the successive quotients $F_{p+q}^p / F_{p+q}^{p-1}$ in a filtration $0 \subset F_{p+q}^0 \subset F_{p+q}^1 \cdots \subset F_{p+q}^{p+q} = H_{p+q}(X)$ of $H_{p+q}(X)$,
- (iii) $E_{p,q}^2 \cong H_p(B; \mathcal{H}_q(F))$.

The cohomology version reads:

Theorem 4.2.5 *Let $F \hookrightarrow X \longrightarrow B$ be a fibration over a path connected Δ -complex B . There is a spectral sequence $\{E_r^{p,q}, d_r\}$ with:*

- (i) $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ and $E_{r+1}^{p,q} = \text{Ker } d_r / \text{Im } d_r$ the homology group at $E_r^{p,q}$,
- (ii) stable terms $E_\infty^{p,q}$ isomorphic to the successive quotients $F_p^{p+q} / F_{p+1}^{p+q}$ in a filtration $0 \subset F_{p+q}^{p+q} \subset F_{p+q-1}^{p+q} \cdots \subset F_0^{p+q} = H^{p+q}(X)$ of $H^{p+q}(X)$,
- (iii) $E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F))$.

As noted at the end of section 4.2.2, by remark 4.2.2, if $\pi_1(B)$ acts trivially on F , then the second pages of the above spectral sequences are given by the ordinary homology or cohomology.

The above theorems are true in general for Serre fibrations over spaces of the homotopy type of a CW -complex, and for generalized homology or cohomology functors, like topological K -theory for instance (see [29] section 9.2 for instance). In the case of K -theory, and if we consider the trivial fibration $\cdot \hookrightarrow X \longrightarrow X$ of a space X over itself with fiber a point \cdot , then the corresponding spectral sequence for coefficients in \mathbb{Z} is known as the *Atiyah–Hirzebruch spectral sequence* [2].

An interesting property of the Atiyah–Hirzebruch spectral sequence is the following: the differential $d_r, r \geq 2$ is *torsion-valued* as we now show.

Proposition 4.2.6 *Let X be a space of the homotopy type of a finite CW complex. If $H^*(X; \mathbb{Z})$ is torsion free, then the Atiyah–Hirzebruch spectral sequence for the topological K-theory of X , $\{E_r^{p,q}, d_r\}$, collapses on page-2: $E_2^{p,q} = E_\infty^{p,q}$.*

Proof. Consider the version of the Atiyah–Hirzebruch spectral sequence over the rationals: it starts on page-2 with $\tilde{E}_2^{p,q} \cong H^*(X; \mathbb{Z}) \otimes \mathbb{Q}$ and converges to $K^*(X) \otimes \mathbb{Q}$. The ring $H^*(X; \mathbb{Z}) \otimes \mathbb{Q}$ is a finite dimensional vector space over \mathbb{Q} , since X has the homotopy type of a finite CW-complex. Let N be its total rank. The differentials \tilde{d}_r can be seen as linear maps on $H^*(X) \otimes \mathbb{Q}$ (since the groups \tilde{E}_r are subquotients of the $H^n(X) \otimes \mathbb{Q}$). If there are some non zero differentials in the spectral sequence, let $d_r, r \geq 2$ be the first one and s its rank. Then the total rank of \tilde{E}_r is N , and the total rank of $\tilde{E}_{r+1} = \text{Ker } \tilde{d}_r / \text{Im } \tilde{d}_r$ is $N - 2s$. Hence the total rank of \tilde{E}_∞ has to be less than or equal to $N - 2s$. But the chern character gives an isomorphism between $H^*(X; \mathbb{Z}) \otimes \mathbb{Q}$ and $K^*(X) \otimes \mathbb{Q}$, and therefore the total rank of \tilde{E}_∞ has to be N , a contradiction. This proves that the rational spectral sequence collapses on page-2.

Now if $H^*(X; \mathbb{Z})$ is torsion free, the map $i = (\otimes \mathbb{Q})^*$ induced in cohomology by the tensor product by \mathbb{Q} gives an inclusion $H^*(X; \mathbb{Z}) \xrightarrow{i} H^*(X; \mathbb{Z}) \otimes \mathbb{Q}$, and then an injective morphism between $\{E_r^{p,q}, d_r\}$ and its rational version $\{\tilde{E}_r^{p,q}, \tilde{d}_r\}$. Assume that there are non zero differentials $d_r, r \geq 2$: for some $x \in H^n(X; \mathbb{Z})$ we have $d_r(x) = y \neq 0$ in some $H^m(X; \mathbb{Z})$. But $\tilde{d}_r i(x) = i d_r(x) = i(y) \neq 0$, a contradiction. \square

4.3 Exact couples

We give here the basic definitions and properties of exact couples and their associated spectral sequences.

We also derived some technical results about exact couples. The formalism allows to state corollary 4.3.7 which leads to the isomorphism between the Schochet spectral sequence associated with the filtration of a C^* -algebra and the Schochet spectral

sequence associated with its corresponding cofiltration in theorem 4.3.11. The construction of a direct limit of spectral sequences is also recalled.

The original reference on exact couples is the work of Massey [56, 57]. The reader is also referred to [29, 59, 39] for further details.

4.3.1 Exact couples for filtrations and cofiltrations

An *exact couple* is a family $T = (D, E, i, j, k)$ where D and E are abelian groups and i, j, k are group homomorphisms making the following triangle exact.

$$\begin{array}{ccc}
 D & \xrightarrow{i} & D \\
 & \swarrow k & \searrow j \\
 & E &
 \end{array}
 \tag{4.3.1}$$

In more generality we could take elements in an Abelian category. Typically D and E are graded modules over a ring, and i, j, k module maps of various degrees. A morphism α between two exact couples T and T' is a pair of group homomorphisms (α_D, α_E) making the following diagram commutative

$$\begin{array}{ccccc}
 D & \xrightarrow{i} & D & & D \\
 & \searrow \alpha_D & & \searrow \alpha_D & \\
 & & D' & \xrightarrow{i'} & D' \\
 & \swarrow k & & \swarrow k' & \\
 & & E & & E' \\
 & & & \searrow \alpha_E & \\
 & & & &
 \end{array}
 \tag{4.3.2}$$

The composition map $d = j \circ k : E \rightarrow E$ is called the *differential* of the exact couple. Since (4.3.1) is exact, it follows that $d^2 = j \circ (k \circ j) \circ k = 0$. Let then $H_d(E)$ be the homology of the complex $E \xrightarrow{d} E$. Then the following fundamental results holds.

Theorem 4.3.1 (i) *There is a derived exact couple*

$$\begin{array}{ccc}
 D^{(1)} = i(D) & \xrightarrow{i^{(1)}} & D^{(1)} = i(D) \\
 & \swarrow k^{(1)} & \searrow j^{(1)} \\
 & E^{(1)} = H_d(E) &
 \end{array} \tag{4.3.3}$$

defined by $i^{(1)} = i|_{i(D)}$, $j^{(1)}(i(x)) = j(x) + \text{Im}(d)$ and $k^{(1)}(e + \text{Im}(d)) = k(e)$.

(ii) *The derivation $T \rightarrow T^{(1)}$ is a functor on the category of exact couples with morphisms of exact couples.*

Proof. The map $j^{(1)}$ is well defined: if $i(x) = i(x')$ then $x - x' \in \text{Ker } i = \text{Im } k$ so $x - x' = k(y)$ for some $y \in E$, and $j \circ k(e) = d(e) = j(x) - j(x')$ so that $j(x) + dE = j(x') + dE$. The map $k^{(1)}$ is well defined: if $e + dE = e' + dE$, then $e - e' = d(x)$ for some $x \in E$ and $k(e') = k(e) + k \circ d(x) = k(e) + k \circ j \circ k(x) = k(e)$ by exactness. Also, since $d(e) = 0$, $k(e) \in \text{Ker } j = \text{Im } i = D^{(1)}$.

We now prove the exactness of $T^{(1)}$. First:

$$\begin{aligned}
 \text{Ker } i^{(1)} &= \text{Im } i \cap \text{Ker } i = \text{Ker } j \cap \text{Im } k \\
 &= k(k^{-1}(\text{Ker } j)) = k(\text{Ker } d) = k^{(1)}(\text{Ker } d / \text{Im } d) \\
 &= \text{Im } k^{(1)}.
 \end{aligned}$$

Next, notice that $D^{(1)} = i(D) = D / \text{Ker } i$ so that

$$\begin{aligned}
 \text{Ker } j^{(1)} &= j^{-1}(\text{Im } d) / \text{Ker } i = j^{-1}(j(\text{Im } k)) / \text{Ker } i \\
 &= (\text{Im } k + \text{Ker } j) / \text{Ker } i = (\text{Ker } i + \text{Ker } j) / \text{Ker } i \\
 &= i(\text{Ker } j) = i(\text{Im } i) = \text{Im } i^{(1)}.
 \end{aligned}$$

Finally, $\text{Ker } k^{(1)} = \text{Ker } k / \text{Im } d = \text{Im } j / \text{Im } d = j(D) / \text{Im } d = \text{Im } j^{(1)}$ since $j \circ i = 0$ by exactness. \square

The n -th iterated derived couple of T is denoted $T^{(n)} = (D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)})$.

Remark 4.3.2 Point (ii) in theorem 4.3.3 implies that a morphism of exact couples $\alpha : T \rightarrow T'$ induces derived morphisms $\alpha^{(n)}$ between the derived couples: $\alpha_D^{(n)} = \alpha_D^{(n-1)}|_{D^{(n)}}$ is the restriction of $\alpha_D^{(n-1)}$ to $D^{(n)} = i^{(n-1)}(D^{(n-1)})$, and $\alpha_E^{(n)} = (\alpha_E^{(n-1)})_*$ is the induced map of $\alpha_E^{(n-1)}$ on homology. Also α_E conjugates the differentials, namely $\alpha_E^{(n)} \circ d^{(n)} = d'^{(n)} \circ \alpha_E^{(n)}$ for all n .

The *spectral sequences* considered here come from exact couples. Given an exact couple (E, D, i, j, k) , its associated spectral sequence is the family $(E_l, d_l)_{l \in \mathbb{N}}$ where $E_1 = E$ and $d_1 = d$, and for $l \geq 2$, $E_l = E^{(l-1)}$ is the derived E -term, and called the E_l -page or simply *page- l* of the spectral sequence, and $d_l = d^{(l-1)}$ is its differential. As noted above those E -terms are generally (bi)graded modules. A morphism of spectral sequence $\beta : (E_l, d_l)_{l \in \mathbb{N}} \rightarrow (E'_l, d'_l)_{l \in \mathbb{N}}$ is given by the α_E -morphism of the exact couples, and can be seen as a sequence of *chain maps* $\beta_l : E_l \rightarrow E'_l$ for each l , *i.e.* commute with the differentials: $d_l \beta_l = \beta_l d'_l$.

Definition 4.3.3 (i) An exact couple $T = (D, E, i, j, k)$ is said to be trivial if $E = 0$.

(ii) Two exact couples T and T' are said to be equivalent if there is a morphism $\alpha : T \rightarrow T'$ such that α_E is an isomorphism.

Remark 4.3.4 (i) If an exact couple is trivial, then it is of the form $(D, 0, \text{id}, 0, 0)$.

(ii) A trivial exact couple is identical to its derived couple.

(iii) An equivalence between two exact couples is equivalent to an isomorphism between their associated spectral sequences.

Definition 4.3.5 (i) An exact couple is said to converge whenever there is an $L \in \mathbb{N}$ such that for $l \geq L$ the l -th derived couples are trivial.

(ii) A spectral sequence is said to converge if its associated exact couple converges.

Given a morphism of exact couples $\alpha : T \rightarrow T'$, its kernel $\text{Ker } \alpha = (\text{Ker } \alpha_D, \text{Ker } \alpha_E, i, j, k)$ and image $\text{Im } \alpha = (\text{Im } \alpha_D, \text{Im } \alpha_D, i', j', k')$ are 3-periodic complexes (by commutativity of diagram (4.3.2)), but no longer exact couples in general.

Lemma 4.3.6 *Given an exact sequence of exact couples*

$$\cdots \rightarrow T_{m-1} \xrightarrow{\alpha_{m-1}} T_m \xrightarrow{\alpha_m} T_{m+1} \xrightarrow{\alpha_{m+1}} T_{m+2} \rightarrow \cdots,$$

that is $\text{Im } \alpha_{m-1} = \text{Ker } \alpha_m$ for all m , the following holds: if T_{m-1} and T_{m+2} are trivial, then T_m and T_{m+1} are equivalent.

Proof. Using Remark 4.3.4 the couples can be rewritten $T_{m-1} = (D_{m-1}, 0, \text{id}, 0, 0)$ and $T_{m+2} = (D_{m+2}, 0, \text{id}, 0, 0)$, and therefore $\alpha_{m-1} = (\alpha_{D_{m-1}}, 0)$, and $\alpha_{m+1} = (\alpha_{D_{m+1}}, 0)$. The exact sequence for the E terms then reads $\cdots \rightarrow 0 \rightarrow E_m \xrightarrow{\alpha_{E_m}} E_{m+1} \rightarrow 0 \rightarrow \cdots$, and therefore α_{E_m} is an isomorphism and gives an equivalence between T_m and T_{m+1} . \square

Corollary 4.3.7 *Given an exact triangle of exact couples, one of which is trivial, then the two others are equivalent.*

4.3.2 Direct limit of exact couples

Definition 4.3.8 *A direct system of exact couples $\{T_l, \alpha_{lm}\}_I$, is given by a directed set I , and a family of exact couples T_l 's and morphisms of exact couples $\alpha_{lm} : T_l \rightarrow T_m$ for $l \leq m$ (with α_{ll} the identity), such that given $l, m \in I$ there exists $n \in I$, $n \geq l, m$ with $\alpha_{ln} = \alpha_{mn} \circ \alpha_{lm}$.*

If T_l is written $(D_l, E_l, i_l, j_l, k_l)$, the definition of a direct system of associated spectral sequences is given similarly by keeping only the data of the E_l -terms, their differentials $j_l \circ k_l$ and the morphisms α_{E_l} .

Let R be a commutative ring. Recall that the direct limit of a directed system of R -modules, $\{M_l, \sigma_{lm}\}_I$, is given by the quotient of the direct product $\bigoplus_I M_l$ by the

R -module generated by all elements of the form $a_l - \sigma_{lm}(a_l)$ for $a_l \in M_l$, where each R -module M_l is viewed as a submodule of $\bigoplus_I A_l$.

Lemma 4.3.9 *Let $\{T_l, \alpha_{lm}\}_I$ be a direct system of exact couples, and write the couples as $T_l = (D_l, E_l, i_l, j_l, k_l)$, and the morphisms as $\alpha_{lm} = (\alpha_{D_{lm}}, \alpha_{E_{lm}})$. There is a direct limit exact couple $T = \varinjlim\{T_l, \alpha_{lm}\}$, given by $T = (D, E, i, j, k)$ with $D = \varinjlim\{D_l, \alpha_{D_{lm}}\}$, $E = \varinjlim\{E_l, \alpha_{E_{lm}}\}$, and i, j, k the maps induced by the i_l, j_l, k_l 's.*

Proof. The direct limits D and E are well-defined and it suffices to give the expressions of the module homomorphisms i, j, k and show that T is exact. Let d be in D , i.e. it is the class $[d_l]$ for some $d_l \in D_l$, then $i(d) = [i_l(d_l)]$. If $d = [d_m]$ for some $m \geq l$ then $d_m = \alpha_{D_{lm}}(d_l)$ and $[i_m(d_m)] = [i_m \circ \alpha_{D_{lm}}(d_l)] = [\alpha_{D_{lm}} \circ i_l(d_l)]$ because $\alpha_{D_{lm}}$ is a morphism of exact couple (commutativity of diagram (4.3.2)), and therefore $[i_m(d_m)] = [i_l(d_l)]$ and i is well-defined. Similarly for $d = [d_l]$ in D , $j(d) = [j_l(d_l)]$ in E , and for $e = [e_l]$ in E , $k(e) = [k_l(e_l)]$ in D , and are well-defined.

This proves also that T is a 3-periodic complex since the compositions $j \circ i$, $k \circ j$ and $i \circ k$ involves the compositions of the i_l, j_l, k_l , and are thus zero. Let $d \in \text{Ker } j$ and $d = [d_l]$, then $j(d) = [j_l(d_l)] = 0$ and by exactness of T_l there exists $d'_l \in D_l$ such that $d_l = i_l(d'_l)$, let then $d' = [d'_l]$ in D to have $d = [i_l(d'_l)] = i(d')$ and therefore $\text{Ker } j = \text{Im } i$. The other two relations $\text{Ker } i = \text{Im } k$ and $\text{Ker } k = \text{Im } j$ are proven similarly, and this shows that T is exact. \square

Corollary 4.3.10 *The result of lemma 4.3.9 for direct limit of exact couples also holds for direct limit of associated spectral sequences.*

4.3.3 Exact couples for the K -theory of a C^* -algebra

Let A be a C^* -algebra, and assume there is a finite filtration by closed two-sided ideals:

$$\{0\} = I_d \xrightarrow{i_d} I_{d-1} \xrightarrow{i_{d-1}} \cdots I_0 \xrightarrow{i_0} I_{-1} = A. \quad (4.3.4)$$

There is an associated cofiltration of A by the quotient C^* -algebras $F_p = A/I_p$:

$$A = F_d \xrightarrow{\rho_d} F_{d-1} \xrightarrow{\rho_{d-1}} \cdots F_0 \xrightarrow{\rho_0} F_{-1} = \{0\} . \quad (4.3.5)$$

Let $Q_p = I_{p-1}/I_p$ be the quotient C^* -algebra. There are short exact sequences:

$$0 \longrightarrow I_p \xrightarrow{i_p} I_{p-1} \xrightarrow{\pi_p} Q_p \longrightarrow 0 \quad (4.3.6a)$$

$$0 \longrightarrow Q_p \xrightarrow{j_p} F_p \xrightarrow{\rho_p} F_{p-1} \longrightarrow 0 \quad (4.3.6b)$$

$$0 \longrightarrow I_p \xrightarrow{l_p} A \xrightarrow{\sigma_p} F_p \longrightarrow 0 \quad (4.3.6c)$$

In (4.3.6a) i_p is the canonical inclusion, and π_p the quotient map $\pi_p(x) = x + I_p$.

In (4.3.6c) $l_p = i_p \circ i_{p-1} \circ \cdots \circ i_0$ is the canonical inclusion, and σ_p the quotient map $\sigma_p(x) = x + I_p$. And in (4.3.6b) j_p is the canonical inclusion $j_p(x + I_p) = l_{p-1}(x) + I_p$, and ρ_p the quotient map $\rho_p(x + I_p) = x + I_{p-1}$.

Associated with the short exact sequences of C^* -algebras (4.3.6a) and (4.3.6b) there are a long (6-term periodic) exact sequences in K -theory that can be written in exact couples:

$$T_I : \begin{array}{ccc} K(I) & \xrightarrow{i} & K(I) \\ \partial \swarrow & & \searrow \pi \\ & K(Q) & \end{array} \quad \text{with} \quad K(I) = \bigoplus_{p=-1}^d \bigoplus_{\varepsilon=0,1} K_\varepsilon(I_p) \quad (4.3.7a)$$

$$T_F : \begin{array}{ccc} K(F) & \xrightarrow{\rho} & K(F) \\ j \swarrow & & \searrow \partial \\ & K(Q) & \end{array} \quad \text{with} \quad K(F) = \bigoplus_{p=-1}^d \bigoplus_{\varepsilon=0,1} K_\varepsilon(F_p) \quad (4.3.7b)$$

with $K(Q) = \bigoplus_{p=-1}^d \bigoplus_{\varepsilon=0,1} K_\varepsilon(Q_p)$, and i, π, j and ρ the induced maps on K -theory.

Since the filtration (4.3.4) and the cofiltration (4.3.5) are finite, both exact couple converge to the K -theory of A .

Theorem 4.3.11 *The spectral sequence for the K-theory of a C^* -algebra A , associated with a filtration of A by ideals I_p as in (4.3.4), and the spectral sequence associated with its corresponding cofiltration by quotients $F_p = A/I_p$ as in (4.3.5), are isomorphic.*

Proof. By remark 4.3.4 (iii) it is sufficient to prove that the exact couples T_I (4.3.7a) and T_F (4.3.7b) are equivalent. By corollary 4.3.7 it is also sufficient to prove that T_I and T_F fit into an exact triangle with a trivial exact couple.

Let T_A be the the trivial exact couple $T_A = (K(A), 0, \text{id}, 0, 0)$. We prove that the short exact sequences (4.3.6c) induce the following exact triangle

$$\begin{array}{ccc}
 T_I & \xrightarrow{l} & T_A \\
 \partial \swarrow & & \searrow \sigma \\
 & T_F &
 \end{array} \tag{4.3.8}$$

The exactness of the triangle comes from the exactness of the K-theory functor (the short exact sequences (4.3.6c) induce long exact sequences in K-theory). We now verify that the applications l , σ and ∂ define morphisms between the exact couples T_I , T_A and T_F .

The commutativity of the following diagrams will be useful

$$\begin{array}{ccc}
 Q_p & \xrightarrow{j_p} & F_p & & F_p & \xrightarrow{\rho_p} & F_{p-1} \\
 \uparrow \pi_p & & \uparrow \sigma_p & \text{and} & \uparrow \sigma_p & & \uparrow \sigma_{p-1} \\
 I_{p-1} & \xrightarrow{i_{p-1}} & A & & A & \xrightarrow{\text{id}} & A
 \end{array}$$

which express the two identities

$$j_p \circ \pi_p = \sigma_p \circ l_{p-1}, \tag{4.3.9a}$$

$$\sigma_{p-1} = \rho_p \circ \sigma_p. \tag{4.3.9b}$$

Indeed, for $x \in I_{p-1}$, $j_p \circ \pi_p(x) = j_p(x + I_p) = l_{p-1}(x) + I_p$ which is equal to $\sigma_p \circ l_{p-1}(x)$, while for $x \in A$, $\rho_p \circ \sigma_p(x) = \rho_p(x + I_p) = x + I_{p-1}$ which is equal to $\sigma_{p-1}(x)$.

1. *Proof that $(l, 0) : T_I \rightarrow T_A$ is a morphism of exact couples, where the application 0 is the induced quotient map of the trivial short exact sequence*

$$0 \longrightarrow Q_p \xrightarrow{\text{id}} Q_p \xrightarrow{0} 0 \longrightarrow 0, \quad (4.3.10)$$

i.e. we prove that the following diagram is commutative (ε is 0 or 1):

$$\begin{array}{ccccccc} K_\varepsilon(I_p) & \xrightarrow{i_{p*}} & K_\varepsilon(I_{p-1}) & \xrightarrow{\pi_{p*}} & K_\varepsilon(Q_p) & \xrightarrow{\partial} & K_{\varepsilon+1}(I_p) \\ \downarrow l_{p*} & & \downarrow l_{p-1*} & & \downarrow 0 & & \downarrow l_{p*} \\ K_\varepsilon(A) & \xrightarrow{\text{id}_*} & K_\varepsilon(A) & \xrightarrow{0} & 0 & \xrightarrow{\partial} & K_{\varepsilon+1}(A) \end{array} \quad (4.3.11)$$

The middle square is commutative since the maps lead to 0. The commutativity of the left square comes from the functoriality of K-theory: since $l_p = l_{p-1} \circ i_p$, it follows that $\text{id}_* l_{p*} = l_{p*} = l_{p-1*} i_{p*}$. For the right square $l_{p*} = \text{id}_* l_{p*} = l_{p-1*} i_{p*}$ by commutativity of the left square, and thus $l_{p*} \partial = l_{p-1*} i_{p*} \partial = 0$ because $i_{p*} \partial = 0$ by exactness of the long exact sequence in K-theory induced by (4.3.6a).

2. *Proof that $(\sigma, \partial) : T_I \rightarrow T_A$ is a morphism of exact couples, where the application ∂ is the K-theory boundary map in the long exact sequence induced by (4.3.10), i.e. we prove that the following diagram is commutative (ε is 0 or 1):*

$$\begin{array}{ccccccc} K_\varepsilon(A) & \xrightarrow{\text{id}} & K_\varepsilon(A) & \xrightarrow{0} & 0 & \xrightarrow{\partial} & K_{\varepsilon+1}(A) \\ \downarrow \sigma_{p*} & & \downarrow \sigma_{p-1*} & & \downarrow \partial & & \downarrow \sigma_{p*} \\ K_\varepsilon(F_p) & \xrightarrow{\rho_{p*}} & K_\varepsilon(F_{p-1}) & \xrightarrow{\partial} & K_{\varepsilon+1}(Q_p) & \xrightarrow{j_{p*}} & K_{\varepsilon+1}(F_p) \end{array} \quad (4.3.12)$$

The right square is commutative since the maps start from 0. The commutativity of the left square holds by (4.3.9b) and functoriality of K-theory: $\sigma_{p-1*} \text{id}_* = \sigma_{p-1*} = \rho_{p*} \sigma_{p*}$. For the middle square we have by commutativity of the left square $\sigma_{p-1*} = \sigma_{p-1*} \text{id}_* = \rho_{p*} \sigma_{p*}$, and thus $\partial \sigma_{p-1*} = \partial \rho_{p*} \sigma_{p*} = 0$ because $\partial \rho_{p*} = 0$ by exactness of the long exact sequence in K-theory induced by (4.3.6b).

3. *Proof that $(\partial, \text{id}) : T_F \rightarrow T_I$ is a morphism of exact couples, where the application id is the induced map in K-theory of the identity map in (4.3.10), i.e. we prove that*

the following diagram is commutative (ε is 0 or 1):

$$\begin{array}{ccccccc}
K_\varepsilon(F_p) & \xrightarrow{\rho_{p^*}} & K_\varepsilon(F_{p-1}) & \xrightarrow{\partial} & K_{\varepsilon+1}(Q_p) & \xrightarrow{j_{p^*}} & K_{\varepsilon+1}(F_p) \\
\downarrow \partial & & \downarrow \partial & & \downarrow \text{id}_* & & \downarrow \partial \\
K_{\varepsilon+1}(I_p) & \xrightarrow{i_{p^*}} & K_{\varepsilon+1}(I_{p-1}) & \xrightarrow{\pi_{p-1^*}} & K_{\varepsilon+1}(Q_p) & \xrightarrow{\partial} & K_\varepsilon(I_p)
\end{array} \tag{4.3.13}$$

The proofs of the commutativity of each square go along the same lines: the images of the boundary operator ∂ can be taken to be the “same”. Given a short exact sequence $0 \rightarrow I \rightarrow A \rightarrow A/J \rightarrow 0$ where A is a C^* -algebra and J a closed two-sided ideal, the boundary map of an element $[x] \in K_\varepsilon(A/J)$ is computed via a lift $z \in A \otimes \mathcal{K}$ of x . For the left square, if $[x] \in K_\varepsilon(F_p)$, then $\partial[x]$ is computed via a lift $z \in K_\varepsilon(A)$ of x , *i.e.* $\sigma_{p^*}(z) = x$. But by (4.3.9b), $\sigma_{p-1^*}(z) = \rho_{p^*}\sigma_{p^*}(z) = \rho_{p^*}(x)$, thus z is also a lift of $\rho_{p^*}(x)$, and thus $\partial\rho_{p^*}[x]$ can be taken to be $i_{p^*}\partial[x]$ - the image of $\partial[x]$ in $K_{\varepsilon+1}(I_{p-1})$, which proves that the left square commutes. For the middle square, given $[x] \in K_\varepsilon(F_{p-1})$ if z is a lift of x to $A \otimes \mathcal{K}$, then $\sigma_{p^*}(z)$ is a lift of x to $F_p \otimes \mathcal{K}$ because $\rho_{p^*}\sigma_{p^*}(z) = \sigma_{p-1^*}(z) = x$ by (4.3.9b). And the induced projection π_{p^*} commutes with the boundary operator, and therefore the diagram commutes. Finally, for the right square, given $[x] \in K_{\varepsilon+1}(Q_p)$, if z is a lift of x to $I_{p-1} \otimes \mathcal{K}$, then $l_{p-1^*}(z)$ is a lift of $j_{p^*}(x)$ to $A \otimes \mathcal{K}$ because by (4.3.9a) $\sigma_{p^*}l_{p-1^*}(z) = j_{p^*}\pi_{p^*}(z) = j_{p^*}(x)$. Hence the right square commutes. \square

4.4 The Pimsner–Voiculescu spectral sequence

This spectral sequence was used and described already in [14] and is a special case of the Kasparov spectral sequence [46] for KK -theory.

Theorem 4.4.1 *Let \mathcal{A} be a C^* -algebra endowed with a \mathbb{Z}^d action α by $*$ -automorphisms.*

The PV complex is defined as $K_(\mathcal{A}) \otimes \Lambda^*\mathbb{Z}^d \xrightarrow{d_P} K_*(\mathcal{A}) \otimes \Lambda^*\mathbb{Z}^d$ with*

$$d_P = \sum_{i=1}^d (\alpha_{i^*} - \mathbf{1}) \otimes e_i \wedge,$$

where $\{e_1, \dots, e_d\}$ is the canonical basis of \mathbb{Z}^d , $\alpha_i = \alpha_{e_i}$ is the restriction of α to the i -th component of \mathbb{Z}^d , whereas $x \wedge$ is the exterior multiplication by $x \in \mathbb{Z}^d$.

There is a spectral sequence converging to the K -theory of \mathcal{A}

$$E_2^{rs} \Rightarrow K_{r+s+d}(\mathcal{A} \rtimes_{\alpha} \mathbb{Z}^d),$$

with page-2 isomorphic to the cohomology of the Pimsner complex.

Let \mathcal{A} be a C^* -algebra endowed with a \mathbb{Z}^d action α by $*$ -automorphisms. The Pimsner-Voiculescu spectral sequence of theorem 4.4.1 is built here. Its page-1 is identified with the PV complex $K_*(\mathcal{A}) \otimes \Lambda^* \mathbb{Z}^d \xrightarrow{d_P} K_*(\mathcal{A}) \otimes \Lambda^* \mathbb{Z}^d$ with differential $d = \sum_{i=1}^d (\alpha_{i,*} - \mathbf{1}) \otimes e_i \wedge$, where $\{e_1, \dots, e_d\}$ is the canonical basis of \mathbb{Z}^d , $\alpha_i = \alpha_{e_i}$ is the restriction of α to the i -th component of \mathbb{Z}^d , whereas $x \wedge$ is the exterior multiplication by $x \in \mathbb{Z}^d$.

Let $M_{\alpha}(\mathcal{A})$ denote the mapping torus of \mathcal{A} , namely the set of functions $f : [0, 1]^d \rightarrow \mathcal{A}$ such that $f(t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_d) = \alpha_i f(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_d)$. The C^* -algebras $\mathcal{A} \rtimes_{\alpha} \mathbb{Z}^d$ and $M_{\alpha}(\mathcal{A}) \rtimes \mathbb{R}^d$ are Morita equivalent [70], thus have the same K -theory, and by Thom-Connes isomorphism [24] $K_*(M_{\alpha}(\mathcal{A}) \rtimes \mathbb{R}^d) \cong K_{*-d}(M_{\alpha}(\mathcal{A}))$. Hence it suffices to build a spectral sequence for the K -theory of the mapping torus: $E_2^{rs} \Rightarrow K_{r+s}(M_{\alpha}(\mathcal{A}))$.

The mapping torus is filtered by the ideals I_s , $s = 0, \dots, d$ (with $I_{-1} = M_{\alpha}(\mathcal{A})$) of functions that vanish on the faces of dimension s (or s -faces) of the cube $[0, 1]^d$. Let $Q_s = I_{s-1}/I_s$ be the quotient C^* -algebra, which is isomorphic to $C_0((0, 1)^s) \otimes \mathcal{A} \otimes \Lambda^s \mathbb{C}^d$, or $S^s \mathcal{A} \otimes \Lambda^s \mathbb{C}^d$, where $S^s \mathcal{A}$ is the s -th suspension of \mathcal{A} . The quotient Q_s can be viewed as the set of continuous \mathcal{A} -valued functions on the s -faces and vanishing on the $(s-1)$ -faces of the cube. Indeed, thanks to the identifications by α in $M_{\alpha}(\mathcal{A})$, a function on the s -faces of the cube is uniquely determined by its values on the s -faces that contain the origin. The subsets of $\{1, \dots, d\}$ of the form $I = \{i_1, \dots, i_s\}$ with $i_1 < \dots < i_s$ allow to label the s -faces that contain the origin as the subsets

of the cube $\{t \in [0, 1]^d : t_i = 0, \forall i \notin I\}$, or equivalently by their normal vectors $e_I = e_{i_1} \wedge \cdots \wedge e_{i_s}$ in $\Lambda^s \mathbb{C}^d$.

The Pimsner-Voiculescu spectral sequence is the spectral sequence associated with the cofiltration of $M_\alpha(\mathcal{A})$ by the quotients $F_s = M_\alpha(\mathcal{A})/I_s$ which are isomorphic to the direct sums $\oplus (M_{\alpha_I}(\mathcal{A}) \otimes e_I)$ over all subsets $I = \{i_1, \dots, i_s\}$ of $\{1, \dots, d\}$ with $i_1 < \dots < i_s$, where α_I is the restriction of α to its components in I and $M_{\alpha_I}(\mathcal{A})$ the corresponding mapping torus. The quotient F_s can be viewed as the set of continuous \mathcal{A} -valued function on the faces of dimension s of the cube. Using the method of section ??, an exact couple is built out of the short exact sequences $0 \longrightarrow Q_s \xrightarrow{j_s} F_s \xrightarrow{\rho_s} F_{s-1} \longrightarrow 0$, where j_s is the inclusion, and the map ρ_s is given by $\rho_s(f_I \otimes e_I) = \sum_{i=1}^d \chi_I(i) ev_i(f_I) \otimes e_{I \setminus \{i\}}$ with χ_I the characteristic function of I and ev_i the evaluation at $t_i = 0$. The complex on page-1 of the associated spectral sequence then reads:

$$\begin{array}{ccccc}
K_{\varepsilon+s}(Q_s) & \xrightarrow{j_{s*}} & K_{\varepsilon+s}(F_s) & \xrightarrow{\partial} & K_{\varepsilon+1}(Q_{s+1}) \\
\downarrow \simeq & & & & \downarrow \simeq \\
K_\varepsilon(\mathcal{A}) \otimes \Lambda^s \mathbb{Z}^d & \xrightarrow{d_P} & & & K_\varepsilon(\mathcal{A}) \otimes \Lambda^{s+1} \mathbb{Z}^d
\end{array} \tag{4.4.1}$$

where $\varepsilon = 0$ or 1 . The above boundary map ∂ is calculated by taking a lift to $F_{s+1} \otimes \mathcal{K}$ of an element in $K_{\varepsilon+s}(F_s)$. In particular, functions on the s -faces, the f_I for $|I| = s$, have to be extended to faces of dimension $s+1$, the $f_{I \cup \{i\}}$ for $i \notin I$. Those extensions can be done separately for each $i \notin I$. Also, since the functions in the quotients Q_s and F_s are uniquely determined by their values on the s -faces containing the origin, $f_I \otimes e_I$ and its extension $f_{I \cup \{i\}} \otimes e_{I \cup \{i\}}$ have the same sign (inserting e_i at position i in $e_I = e_{i_1} \wedge \cdots \wedge e_{i_s}$). Hence it suffices to show that the differential corresponds to $(\alpha_{i_*} - 1) \otimes e_i \wedge$ for each $i \notin I$, which is a one dimensional problem. It therefore suffices to express the differential in the case of dimension 1.

In the case $d = 1$, the ideals I_s are $I_{-1} = M_\alpha(\mathcal{A})$, $I_0 = C_0((0, 1)) \otimes \mathcal{A} \cong S\mathcal{A}$ and $I_1 = \{0\}$, the quotients F_s are $F_{-1} = \{0\}$, $F_0 = \mathcal{A}$ and $F_1 = M_\alpha(\mathcal{A})$, and the quotients Q_s are $Q_0 = F_0 = \mathcal{A}$, $Q_1 = S\mathcal{A}$. The exact couple is built from the two exact sequences $0 \rightarrow Q_s \xrightarrow{j_s} F_s \xrightarrow{\rho_s} F_{s-1} \rightarrow 0$, for $s = 0$ and $s = 1$, namely $0 \rightarrow \mathcal{A} \xrightarrow{j_0=\text{id}} \mathcal{A} \rightarrow 0 \rightarrow 0$ and $0 \rightarrow S\mathcal{A} \xrightarrow{j_1} M_\alpha(\mathcal{A}) \xrightarrow{\rho_1} \mathcal{A} \rightarrow 0$. The differential on page-1 then reads:

$$\begin{array}{ccccc}
K_\varepsilon(\mathcal{A}) & \xrightarrow{\text{id}} & K_\varepsilon(\mathcal{A}) & \xrightarrow{\partial} & K_{\varepsilon+1}(S\mathcal{A}) \\
\uparrow = & & & & \uparrow \beta \\
K_\varepsilon(\mathcal{A}) & \xrightarrow{d_P} & & & K_\varepsilon(\mathcal{A})
\end{array} \tag{4.4.2}$$

where β is the Bott map, and $\varepsilon = 0$ or 1 . The reader is referred to [17] for the definition of the K -theory maps (Bott maps and boundary maps), and in particular to proposition 10.4.1 where this problem is treated briefly.

Consider the case $\varepsilon = 0$. Let $x = [p] - [p_l]$ in $K_0(\mathcal{A})$, where p is a projection in $M_n(\mathcal{A}^+)$, $p_l = \text{diag}(\mathbf{1}_k, 0)$ is a projection of rank $l \leq n$ and $p - p_l \in M_n(\mathcal{A})$. The map $h(s) = \text{diag}((1-s)p, s\alpha(p))$ is a lift of p to $M_{4n}(M_\alpha(\mathcal{A}))$, and $e^{2i\pi h(s)} = \text{diag}(e^{-2i\pi s p}, e^{2i\pi s \alpha(p)})$ which is homotopy to $e^{2i\pi((\alpha(p)-p))} = \beta((\alpha - \mathbf{1})p)$. The boundary of x : $\partial x = [e^{2i\pi h(s)}]$ is thus $\beta[(\alpha - \mathbf{1})p]$, and therefore $d_p = \alpha_* - \mathbf{1}$. The case $\varepsilon = 1$ is obtained immediately by suspension.

CHAPTER V

A SPECTRAL SEQUENCE FOR THE K -THEORY OF TILINGS

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with finite local complexity. We present a spectral sequence that converges to the K -theory of \mathcal{T} with page-2 given the PV cohomology of \mathcal{T} . As the PV cohomology of \mathcal{T} generalizes the cohomology of the base space of a fibration with local coefficients in the K -theory of its fiber, this spectral sequence is a generalization of the Serre spectral sequence (see section 4.2).

5.1 Historic background

Spectral sequences were introduced by Leray [55] during WWII, as a way to compute the cohomology of a sheaf. It was later put in the framework used today by Koszul [51]. One of the first applications of this method was performed by Borel and Serre [18]. Later, Serre, in his Ph. D. Thesis [76] defined the notion of (Serre) fibration and built a spectral sequence to compute their singular homology or cohomology. This more or less led him to calculate the rational homotopy of spheres.

In the early fifties, Hirzebruch made an important step in computing the Euler characteristic of various complex algebraic varieties and complex vector bundles over them [41]. He showed that this characteristic can be computed from the Chern classes of the tangent bundle and of the vector bundle through universal polynomials [40] which coincides with the Todd genus in the case of varieties. It allowed him to show that the Euler characteristic is additive for extensions namely if E, E', E'' are complex vector bundles and if $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ is an exact sequence, then

$\chi(E) = \chi(E') + \chi(E'')$. This additivity property led Grothendieck to define axiomatically an additive group characterizing this additivity relation, which he called the K -group [36]. It was soon realized by Atiyah and Hirzebruch [2] that the theory could be extended to topological spaces X and they defined the topological K -theory as a cohomology theory without the axiom of dimension. The K^0 -group is the set of stable equivalence classes of complex vector bundles over X , while K^1 (and more generally K^n) is the set of stable equivalence classes of complex vector bundles over the suspension (respectively the n -th suspension) of X . The Bott periodicity theorem reduces the number of groups to two only. Moreover, the Chern character was shown to define a natural map between the K -group and the integer Čech cohomology of X , and that it becomes an isomorphism when both groups are rationalized. In this seminal paper [2] Atiyah and Hirzebruch define a spectral sequence that will be used in the present work. It is a particular case of Serre spectral sequence for the trivial fibration of X by itself with fiber a point: it converges to the K -theory of X and its page-2 is isomorphic to the cohomology of X .

Almost immediately after this step, Atiyah and Singer extended the work of Hirzebruch to the *Index theorem* [4, 5, ?, 3, 6] for elliptic operators. Such an operator is defined between two vector bundles, it is unbounded, in general and, with the correct domains of definition defines a Fredholm operator. The index can be interpreted as an element of the K -group and is calculated through a formula which generalizes the results of Hirzebruch for algebraic varieties. Eventually, Atiyah and Singer extended the theory to the equivariant K -theory valid if a compact Lie group G acts on the vector bundle. If P is an elliptic operator commuting with G its index gives an element of the covariant K -theory. In a programmatic paper [78], Singer proposed various extensions to elliptic operators with coefficients depending on parameters. As an illustration of such a program, Coburn, Moyer and Singer [23] gave an index theorem for elliptic operators with almost periodic coefficients (see also [81]). Eventually the

index theorem became the cornerstone of Connes' program to build a *Noncommutative Geometry*. In a seminal paper [26] he defined the theory of *noncommutative integration* and showed that its first application was an index theorem for elliptic operators on a foliation (see also [25, 28]).

In the seventies, it was realized that the Atiyah-Hirzebruch K -theory could be expressed in algebraic terms through the C^* -algebra $\mathcal{C}(X)$ of continuous functions on the compact space X . The definition of the K -group requires then to consider matrix valued continuous functions $M_n(\mathcal{C}(X))$ instead for all n . The smallest C^* -algebra containing all of them is $\mathcal{C}(X) \otimes \mathcal{K}$, where \mathcal{K} denotes the C^* -algebra of compact operators on a separable Hilbert space. Therefore, since this later algebra is non commutative, all the construction could be used for any C^* -algebra. Then Kasparov [45, 46] defined the notion of KK -theory generalizing even more the K groups to correspondences between two C^* -algebras.

The problem investigated in this chapter is the latest development of a program that was initiated in the early eighties [44, 15] when the first version of the *gap labeling theorem* was proved (see [9, 10, 11] for later developments). At that time the problem was to compute the spectrum of a Schrödinger operator H in an aperiodic potential. Several examples were discovered of Schrödinger operators with a Cantor-like spectrum [42, 62]. Labeling the infinite number of gaps per unit length interval, was a challenge. It was realized that the K -theory class of the spectral projection on the spectrum below the gap was a proper way of doing so [15]. The first calculation was made on the Harper equation and gave an explanation for a result already obtained by Claro and Wannier [22], a result eventually used in the theory of the quantum Hall effect [83]. This problem was motivated by the need for a theory of aperiodic solids, in particular their electronic and transport properties. With the discovery of *quasicrystals* in 1984 [77], this question became crucial in Solid State Physics. The construction of the corresponding C^* -algebra became then the main issue and led to

the definition of the *hull* [9]. It was proved that the hull is a compact metrizable space Ω endowed with an action of \mathbb{R}^d via homeomorphisms. Then it was proved in [9, 10], that the resolvent of the Schrödinger operator H belongs to the C^* -algebra $\mathcal{A} = \mathcal{C}(\Omega) \rtimes \mathbb{R}^d$. With each \mathbb{R}^d -invariant probability measure μ on Ω is associated a trace τ_μ on this algebra. Following an argument described in [15], it was proved that a gap could be labeled by the value of the density of state, and that this value belongs to the image by the trace τ_μ of the group $K_0(\mathcal{A})$ (see section 3.3). During the eighties several results went on to compute the set of gap labels [65, 9]. In one dimension, detailed results could be proved (see [10] for a review). The most spectacular result was given in the case of a discretized Laplacian with a potential taking on finitely many values (called *letters*): in such a case the set of gap labels is the \mathbb{Z} -module generated by the occurrence probability of all possible finite words found in the sequence defined by the potential. If this sequence is given by a *substitution*, these occurrence probabilities can be computed explicitly in terms of the incidence matrix of the substitution and of the associated substitution induced on the set of words with two letters [66, 10]. The key property in proving such results was the use of the Pimsner-Voiculescu exact sequence [64]. Soon after, A. van Elst [84] extended these results to the case of 2D-potentials, using the same method.

In the French version [27] of his book on *Noncommutative geometry* [28], Connes showed how the general formalism he had developed could be illustrated with the special case of the Penrose tiling. Using the substitution rules for its construction, he introduced a C^* -algebra, which turns out to be AF, and computed its ordered K_0 -group, using the classification of AF-algebras obtained by Bratelli in 1972 [20]. This result was an inspiration for Kellendonk, who realized that, instead of looking at the inflation rule as a source of noncommutativity, it was actually better to consider the space translations of the tiling [47], in the spirit of the formalism developed in [15, 9, 10] for aperiodic solids. He extended this latter construction of the hull

to tilings and called this hull a *tiling space*. In this important paper, Kellendonk introduced the notion of *forcing the border* for a general tiling, which appears today as an important property for the calculation of the K-groups and cohomology of a tiling space. This work gave a strong motivation to prove the *Gap Labeling Theorem in higher dimension* and to compute the K-group of the hull. It was clear that the method used in one dimension, through the Pimsner-Voiculescu exact sequence could only be generalized through a spectral sequence. The first use of spectral sequences in computing the set of gap labels on the case of the 2D-octagonal tiling [13] was followed by a proof of the Gap Labeling Theorem for 3D-quasicrystals [14]. Finally, the work by Forrest and Hunton [31] used a classical spectral sequence to compute the full K-theory of the C^* -algebra for an action of \mathbb{Z}^d on the Cantor set. It made possible the computation of the K-theory and the cohomology of the hull for quasicrystals in two and three dimensions [32, 33]. Other examples of tilings followed later [72].

In addition to Kellendonk's work, several important contributions helped to build tools to prove the higher dimensional version of the Gap Labeling Theorem. Among the major contributions was the work of Lagarias [52, 53, 54], who introduced a geometric and combinatoric aspect of tiling through the notion of a Delone set. This concept was shown to be conceptually crucial in describing aperiodic solids [7]. Through the construction of Voronoi, Delone sets and tilings become equivalent concepts, allowing for various intuitive point of views to study such problems.

Another important step was performed in 1998 by Anderson and Putnam [1], who proposed to build the tiling space of a substitution tiling through a CW-complex built from the prototiles of a tiling (called prototile space here, see definition 1.3.1). The substitution induces a map from this CW-complex into itself and they showed that the inverse limit of such system becomes homeomorphic to the tiling space. A similar construction was proposed independently by Gambaudo and Martens in 1999 to describe dynamical systems. This latter case corresponds to 1D repetitive tilings

with finite local complexity, so that this latter construction goes beyond the substitution tilings. One interesting outcome of this work was a systematic construction of minimal dynamical systems, uniquely ergodic or not, and with positive entropy¹ (see [34]). Eventually, the Gambaudo-Martens construction led to the construction of the hull as an inverse limit of compact oriented branched flat Riemannian manifolds [11]² used in this thesis (although only their topological *CW*-complex structure is needed here). Equivalently, the hull can be seen as a *lamination* [35] of a *foliated space* [61]. The extension of this construction to include tilings without finite local complexity, such as the pinwheel model [68], was performed by Gambaudo and Benedetti [16] using the notion of solenoids (reintroduced by Williams in the seventies [85]).

5.2 *The main theorem*

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with finite local complexity (definition 1.1.3). The hull Ω is a compactification, with respect to an appropriate topology, of the family of translates of \mathcal{T} by vectors of \mathbb{R}^d (definition 1.1.4). The tiles of \mathcal{T} are given compatible Δ -complex decompositions (section 2.3.2), with each simplex punctured, and the Δ -transversal Ξ_Δ (definition 2.3.1) is the subset of Ω corresponding to translates of \mathcal{T} having the puncture of one of those simplices at the origin $0_{\mathbb{R}^d}$. The prototile space \mathcal{B}_0 (definition 1.3.1) is built out of the prototiles of \mathcal{T} (translational equivalence classes of tiles) by gluing them together according to the local configurations of their representatives in the tiling.

The hull is given a dynamical system structure via the natural action of the group \mathbb{R}^d on itself by translation [11]. The C^* -algebra of the hull is isomorphic to the crossed-product C^* -algebra $C(\Omega) \rtimes \mathbb{R}^d$.

¹This result circulated as a preprint in 1999 but was published only in 2006

²This paper was posted on [arXiv.com math.DS/0109062](https://arxiv.org/abs/math/0109062) in its earlier version in 2001 but was eventually published in a final form in 2006 only.

There is a map \mathfrak{p}_0 from the hull onto the prototile space (proposition 1.3.2)

$$\begin{array}{ccc} \Xi_\Delta & \hookrightarrow & \Omega \\ & & \downarrow \mathfrak{p}_0 \\ & & \mathcal{B}_0 \end{array}$$

which, thanks to a lamination structure on Ω (remark 1.3.3), resembles (although is *not*) a fibration with base space \mathcal{B}_0 and fiber Ξ_Δ (remark 2.3.5). The Pimsner–Voiculescu (PV) cohomology H_{PV}^* of the tiling (definition 2.3.4) is a cohomology of the base space \mathcal{B}_0 with “local coefficients” in the K -theory of the fiber Ξ_Δ (remark 2.3.6).

Theorem 5.2.1 *There is a spectral sequence that converges to the K -theory of the C^* -algebra of the hull*

$$E_2^{rs} \Rightarrow K_{r+s+d}(C(\Omega) \rtimes \mathbb{R}^d),$$

and whose page-2 is given by

$$E_2^{rs} \cong H_{PV}^r(\mathcal{B}_0; K^s(\Xi_\Delta)).$$

By an argument using the Thom–Connes isomorphism [24] the K -theory of $C(\Omega) \rtimes \mathbb{R}^d$ is isomorphic to the topological K -theory of Ω (with a shift in dimension by d), and the above theorem can thus be seen formally as a generalization of the Serre spectral sequence [76] for a certain class of laminations $\Xi_\Delta \hookrightarrow \Omega \rightarrow \mathcal{B}_0$, which are foliated spaces [61] but not fibrations.

This result brings a different point of view on a problem solved earlier by Hunton and Forrest in [31]. They built a spectral sequence for the K -theory of a crossed product C^* -algebra of a \mathbb{Z}^d -action on a Cantor set. Such an action exists always for tilings of finite local complexity, but it is by no means canonical. Indeed, thanks to a result of Sadun and Williams [74], the hull of a repetitive tiling with finite local complexity is homeomorphic to a fiber bundle over a torus with fiber the Cantor set.

These two results are sufficient to get the K -theory of the hull which, thanks to the Thom–Connes theorem [24], gives also the K -theory of the C^* -algebra of the tiling $\mathcal{C}(\Omega) \rtimes \mathbb{R}^d$. However, the construction of the hull through an inverse limit of branched manifolds, initiated by Anderson and Putnam [1] for the case of substitution tilings and generalized in [11] to all repetitive tilings with finite local complexity, suggests a different and more canonical construction. So far however, it is not yet efficient for practical calculations.

One dimensional repetitive tilings with finite local complexity are all Morita equivalent to a \mathbb{Z} -action on a Cantor set. The Pimsner–Voiculescu exact sequence [64] is then sufficient to compute the K -theory of the hull [10]. In the late nineties, before the paper by Forrest and Hunton was written, Mihai Pimsner suggested to Prof. Bellissard a method to generalize the theorem to \mathbb{Z}^d -actions. This spectral sequence was used and described already in [14] and is a special case of the Kasparov spectral sequence [46] for KK -theory. We recall it here for completeness and to justify naming H_{PV}^* after Pimsner and Voiculescu. Its construction is given in section 4.4.

Theorem 5.2.2 *Let \mathcal{A} be a C^* -algebra endowed with a \mathbb{Z}^d action α by $*$ -automorphisms.*

The PV complex is defined as $K_(\mathcal{A}) \otimes \Lambda^* \mathbb{Z}^d \xrightarrow{d_{PV}} K_*(\mathcal{A}) \otimes \Lambda^* \mathbb{Z}^d$ with*

$$d_{PV} = \sum_{i=1}^d (\alpha_{i*} - \mathbf{1}) \otimes e_i \wedge,$$

where $\{e_1, \dots, e_d\}$ is the canonical basis of \mathbb{Z}^d , $\alpha_i = \alpha_{e_i}$ is the restriction of α to the i -th component of \mathbb{Z}^d , whereas $x \wedge$ is the exterior multiplication by $x \in \mathbb{Z}^d$.

There is a spectral sequence converging to the K -theory of \mathcal{A}

$$E_2^{rs} \Rightarrow K_{r+s+d}(\mathcal{A} \rtimes_{\alpha} \mathbb{Z}^d),$$

with page-2 isomorphic to the cohomology of the PV complex.

A more topological expression of this theorem consists in replacing \mathbb{Z}^d by its classifying

space, the torus \mathbb{T}^d , with a *CW*-complex decomposition given by an oriented (open) d -cube and all of its (open) faces in any dimension. Then the PV complex can be proved to be isomorphic to the following complex: the cochains are given by covariant maps $\varphi(e) \in K_*(\mathcal{A})$, where e is a cell of \mathbb{T}^d and $\varphi(\bar{e}) = -\varphi(e)$ if \bar{e} is the face e with opposite orientation. The covariance means that if two cells e, e' differ by a translation $a \in \mathbb{Z}^d$ then $\varphi(e') = \alpha^a \varphi(e)$. The differential is the usual one, namely $d\varphi(e) = \sum_{e' \in \partial e} \varphi(e')$. If $H_{PV}^*(\mathbb{T}^d; K_*(\mathcal{A}))$ denotes the corresponding cohomology this gives

Corollary 5.2.3 *The PV cohomology group for the crossed product $\mathcal{A} \rtimes_{\alpha} \mathbb{Z}^d$ is isomorphic to $H_{PV}^*(\mathbb{T}^d; K_*(\mathcal{A}))$.*

The spectral sequence used by Forrest and Hunton in [31] in the case $\mathcal{A} = \mathcal{C}(X)$ where X is the Cantor set coincides with the PV spectral sequence. In this chapter we generalize this construction for tilings by replacing the classifying space \mathbb{T}^d , by the prototile space \mathcal{B}_0 .

The hull can also be built out of a *box decomposition* [11]. Namely a *box* is a local product of the transversal (which is a Cantor set) by a polyhedron in \mathbb{R}^d called the *base* of the box (see remark 1.3.3). Then the hull can be shown to be given by a finite number of such boxes together with identifications on the boundary of the bases. This is an extension of the mapping torus, for which there is only one box with base given by a *cube* [76]. Since \mathcal{A} in the present case is the space of continuous functions on the Cantor set X , $K_0(\mathcal{A})$ is isomorphic to $\mathcal{C}(X, \mathbb{Z})$ whereas $K_1(\mathcal{A}) = 0$ [12], leading to the present result. In Section 2.3.3 an example of an explicit calculation of the PV cohomology is proposed illustrating the way it can be used. Further examples, and methods of calculation of PV cohomology will be investigated in future research. However, as we showed in section 2.3.1, the PV cohomology is isomorphic to other cohomologies used so far on the hull, such as the Čech cohomology [1, 71], the group cohomology [31] or the pattern equivariant cohomology [49, 50, 73].

5.3 Construction of the spectral sequence

The proof of theorem 5.2.1 follows from the following theorem 5.3.1 and theorem 2.4.1 (isomorphism between PV and Čech cohomologies) that has been proven separately on section 2.4 for convenience.

It is important to notice that the first page of the spectral sequence of theorem 5.2.1 is the PV complex (definition 2.3.4) and therefore its second page is given by the PV cohomology. However the direct identification of the PV differential on the first page is highly technical and will not be presented here. The proof will use instead an approximation of the hull (theorem 1.4.4) as an inverse limit of patch spaces (definition 1.4.1), and a direct limit spectral sequence for the K -theory of the hull (theorem 5.3.1), with page-2 isomorphic to its Čech cohomology. Some abstract results on exact couples and spectral sequences are also needed, in particular some properties of the Atiyah–Hirzebruch spectral sequence [2]. For the convenience of the reader, those known results are grouped together and proven in a separate section, section 4.3.

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with FLC (definition 1.1.3), and assume its tiles are finite compatible Δ -complexes. Let Ω be its hull (definition 1.1.4) and Ξ_Δ its Δ -transversal (definition 2.3.1).

Theorem 5.3.1 *There is a spectral sequence that converges to the K -theory of the C^* -algebra of the hull*

$$E_2^{rs} \Rightarrow K_{r+s+d}(C(\Omega) \rtimes \mathbb{R}^d),$$

and whose page-2 is isomorphic to the integer Čech cohomology of the hull

$$E_2^{rs} \cong \begin{cases} \check{H}^r(\Omega; \mathbb{Z}) & s \text{ even,} \\ 0 & s \text{ odd.} \end{cases}$$

The proof of theorem 5.3.1 follows from propositions 5.3.2 and 5.3.3 below. First by Thom-Connes isomorphism [24], $K_{*+d}(C(\Omega) \rtimes \mathbb{R}^d) \cong K_*(C(\Omega))$, and therefore

it suffices to build a spectral sequence that converges to the K -theory of the C^* -algebra $C(\Omega)$. This is done by constructing a Schochet spectral sequence [75], $\{E_*^{r,s}\}$, associated with an appropriate filtration of $C(\Omega)$. It is shown in proposition 5.3.2 that this spectral sequence is the direct limit of spectral sequences, $\{E_*^{r,s}(l)\}$, for the K -theory of the C^* -algebras $C(\mathcal{B}_l)$ of continuous functions on the patch spaces of a proper sequence. Then $\{E_*^{r,s}(l)\}$ is shown to be isomorphic to the Atiyah-Hirzebruch spectral sequence [2] for the topological K -theory of \mathcal{B}_l in proposition 5.3.3.

For basic definitions, terminology and results on spectral sequences, the reader is referred to [59]. In chapter 4 we present an informal introduction to spectral sequences, as well as provide some technical results for exact couples (in section 4.3) that we will use here.

For $s = 0, \dots, d$, let $\Omega^s = \mathfrak{p}_0^{-1}(\mathcal{B}_0^s)$ be the lift of the s -skeleton of the prototile space, and let $I_s = C_0(\Omega \setminus \Omega^s)$. I_s is a closed two-sided ideal of $C(\Omega)$, it consists of functions that vanish on the faces of dimension s of the boxes of the hull (see remark 1.3.3). This gives a filtration of $C(\Omega)$:

$$\{0\} = I_d \hookrightarrow I_{d-1} \hookrightarrow \dots \hookrightarrow I_0 \hookrightarrow I_{-1} = C(\Omega). \quad (5.3.1)$$

Let $Q_s = I_{s-1}/I_s$, which is isomorphic to $C_0(\Omega^s \setminus \Omega^{s-1})$. Let $K(I)$ and $K(Q)$ be the respective direct sums of the $K_\epsilon(I_s)$ and $K_\epsilon(Q_s)$ over $\epsilon = 0, 1$, and $s = 0, \dots, d$. The short exact sequences $0 \rightarrow I_s \xrightarrow{i_s} I_{s-1} \xrightarrow{\pi_s} Q_s \rightarrow 0$, lead, through long exact sequences in K -theory, to the exact couple $\mathfrak{P} = (K(I), K(Q), i, \pi, \partial)$, where i and π are the induced maps and ∂ the boundary map in K -theory. Its associated Schochet spectral sequence [75], $\{E_*^{r,s}\}$, converges to the K -theory of $C(\Omega)$:

$$\begin{cases} E_1^{r,s} & \Rightarrow & K_{r+s}(C(\Omega)) \\ E_1^{r,s} & = & K_{r+s}(Q_s) \end{cases}. \quad (5.3.2)$$

Let \mathcal{B}_p be a patch space associated with a pattern \hat{p} of \mathcal{T} . Consider the filtration of

the C^* -algebra $C(\mathcal{B}_p)$ by the closed two sided-ideals $I_s(p) = C_0(\mathcal{B}_p \setminus \mathcal{B}_p^s)$:

$$\{0\} = I_d(p) \hookrightarrow I_{d-1}(p) \hookrightarrow \cdots \hookrightarrow I_0(p) \hookrightarrow I_{-1}(p) = C(\mathcal{B}_p). \quad (5.3.3)$$

Let $Q_s(p) = I_{s-1}(p)/I_s(p)$, which is isomorphic to $C_0(\mathcal{B}_p^s \setminus \mathcal{B}_p^{s-1})$. Let $K(I(p))$ and $K(Q(p))$ be the respective direct sums of the $K_\epsilon(I_s(p))$ and $K_\epsilon(Q_s(p))$ over $\epsilon = 0, 1$, and $s = 0, \dots, d$. The short sequences $0 \rightarrow I_s(p) \xrightarrow{i_{p,s}} I_{s-1}(p) \xrightarrow{\pi_{p,s}} Q_s(p) \rightarrow 0$, lead to the exact couple $\mathfrak{P}(p) = (K(I(p)), K(Q(p)), i_p, \pi_p, \partial)$, and its associated Schochet spectral sequence [75], $\{E_*^{r,s}(p)\}$, converges to the K -theory of $C(\mathcal{B}_p)$:

$$\begin{cases} E_1^{r,s}(p) \Rightarrow K_{r+s}(C(\mathcal{B}_p)) \\ E_1^{r,s}(p) = K_{r+s}(Q_s(p)) \end{cases}. \quad (5.3.4)$$

Proposition 5.3.2 *Given a proper sequence $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ of patch spaces of \mathcal{T} , the following holds:*

$$\{E_*^{r,s}\} \cong \varinjlim (\{E_*^{r,s}(l)\}, f_{l*}),$$

where $\{E_*^{r,s}(l)\}$ is the Schochet spectral sequence (5.3.4) for \mathcal{B}_l corresponding to the patch $p = p_l$.

Proof. The map $f_l : \mathcal{B}_l \rightarrow \mathcal{B}_{l+1}$ induces a morphism of exact couples from $f_{l*} : \mathfrak{P}(l) \rightarrow \mathfrak{P}(l+1)$. Indeed, consider the following diagram:

$$\begin{array}{ccccccccc} K_\epsilon(I_s(l)) & \xrightarrow{i_{l,s}} & K_\epsilon(I_{s-1}(l)) & \xrightarrow{\pi_{l,s}} & K_\epsilon(Q_s(l)) & \xrightarrow{\partial} & K_{\epsilon+1}(I_s(l)) \\ \downarrow f_{l*} & & \downarrow f_{l*} & & \downarrow f_{l*} & & \downarrow f_{l*} \\ K_\epsilon(I_s(l+1)) & \xrightarrow{i_{l+1,s}} & K_\epsilon(I_{s-1}(l+1)) & \xrightarrow{\pi_{l+1,s}} & K_\epsilon(Q_s(l+1)) & \xrightarrow{\partial} & K_{\epsilon+1}(I_s(l+1)) \end{array}$$

The left and middle squares are easily seen to be commutative. To check the commutativity of the right square, recall that given a short exact sequence $0 \rightarrow I \rightarrow A \rightarrow A/J \rightarrow 0$ where A is a C^* -algebra and J a closed two-sided ideal, the boundary map of an element $[x] \in K_\epsilon(A/J)$ is computed via a lift $z \in A \otimes \mathcal{K}$ of x . Let

$[x] \in K_\varepsilon(Q_s(l))$. If $z \in I_{s-1}(l) \otimes \mathcal{K}$ is a lift of x , then $f_{l*}z \in I_{s-1}(l+1) \otimes \mathcal{K}$ is a lift of $f_{l*}x$ and the commutativity of the right square follows.

As a consequence of theorem 1.4.4, $I \cong \varinjlim\{I(l), f_l^\#\}$ and $Q \cong \varinjlim\{Q(l), f_l^\#\}$, therefore $K(I) \cong \varinjlim\{K(I(l)), f_l^*\}$ and $K(Q) \cong \varinjlim\{K(Q(l)), f_l^*\}$. Hence $\{\mathfrak{P}(l), f_l^*\}$ is a direct system of exact couples and by lemma 4.3.9, $\mathfrak{P} \cong \varinjlim\{\mathfrak{P}(l), f_l^*\}$, and the same result on their associated spectral sequences follows by corollary 4.3.10. \square

Given a finite CW -complex X , the Atiyah-Hirzebruch spectral sequence [2] for the topological K -theory of X is a particular case of Serre spectral sequence [29, 59] for the trivial fibration of X by itself with fiber a point (see section 4.2.3):

$$\left\{ \begin{array}{l} E_{AH2}^{r,s} \Rightarrow K^{r+s}(X) \\ E_{AH2}^{r,s}(p) \cong H^r(X; K^s(\cdot)) \end{array} \right. , \quad (5.3.5)$$

where \cdot denotes a point, so $K^s(\cdot) \cong \mathbb{Z}$ if s is even, and 0 if s is odd.

It is defined on page-1 by $E_{AH2}^{r,s} = K^r(X^s, X^{s-1})$ in [2], and then proven that the page-2 is isomorphic to the cellular cohomology of X . If X is a locally compact Hausdorff space, the spectral sequence can be rewritten algebraically, using the isomorphisms $K^*(Y) \cong K_*(C_0(Y))$ and $K^*(Y, Z) \cong K_*(C_0(Y)/C_0(Z))$ for locally compact Hausdorff spaces Y, Z . Consider the Schochet spectral sequence [75] for the K -theory of the C^* -algebra $C_0(X)$ associated with its filtration by the ideals $I_s = C_0(X, X^s)$ of functions vanishing on the s -skeleton. Then the spectral sequence associated with the cofiltration of $C_0(X)$ by the ideals $F_s = C_0(X)/I_s \cong C_0(X^s)$ turns out to be this algebraic form of Atiyah-Hirzebruch spectral sequence.

Proposition 5.3.3 *The Schochet spectral sequence $\{E_*^{r,s}(p)\}$ for the K -theory of the C^* -algebra $C(\mathcal{B}_p)$ is isomorphic to the Atiyah-Hirzebruch spectral sequence for the topological K -theory of \mathcal{B}_p .*

Proof. By theorem 4.3.11 (section 4.3) the Schochet spectral sequences built from the filtration of $A(p) = C(\mathcal{B}_p)$ by the ideals $I_s(p)$, namely $\{E_*^{r,s}(p)\}$, and from the

cofiltration of $A(p)$ by the quotients $F_s(p) = A(p)/I_s(p) \cong C_0(\mathcal{B}_p^s)$, are isomorphic. As remarked above this last spectral sequence is nothing but the Atiyah-Hirzebruch spectral sequence. \square

Proof of theorem 5.3.1. Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of \mathcal{T} . By proposition 5.3.3, the page-2 of Schochet spectral sequence $\{E_*^{r,s}(l)\}$ is isomorphic to the simplicial cohomology of \mathcal{B}_l : $E_2^{r,s}(l) \cong \check{H}^r(\mathcal{B}_l; \mathbb{Z})$ for s even and 0 for s odd. By the natural isomorphism between simplicial and Čech cohomologies for CW -complexes [79], it follows that $E_2^{r,s}(l) \cong \check{H}^r(\mathcal{B}_l; \mathbb{Z})$ for s even and 0 for s odd. By theorem 2.2.2, $\check{H}^*(\Omega; \mathbb{Z}) \cong \varinjlim (\check{H}^*(\mathcal{B}_l; \mathbb{Z}), f_l^*)$, and therefore by proposition 5.3.2 the page-2 of Schochet spectral sequence for the K -theory of $C(\Omega)$ is isomorphic to the integer Čech cohomology of the hull: $E_2^{r,s} \cong \check{H}^r(\Omega; \mathbb{Z})$ for s even and 0 for s odd. \square

CHAPTER VI

THE GROUPOID OF THE TRANSVERSAL

We study here the groupoid of the transversal to the hull of a tiling (definition 1.2.5). We show how it can be approximated by simpler groupoids associated with free categories generated by finite graphs (theorems 6.6.3 and 6.6.5). Using this language from category theory we also give a general formalism for PV cohomology.

We first recall the construction of the classifying space for a small category. It is a simplicial complex and allows to define its (co)homology. We next recall the notion of inverse limit of groupoids. Then we build an approximation of the groupoid of the transversal and its cohomology groups. Finally we give a general framework for PV cohomology in terms of *PV categories* (definition 6.7.2).

6.1 *The simplicial category and simplicial objects*

The references for the first two sections are [28] (Appendix III.A) and [58] (chapter 16).

Recall that a *small category* \mathcal{C} is a category whose collections of objects $Obj(\mathcal{C})$ and morphisms $Mor(\mathcal{C})$ are sets.

Definition 6.1.1 *The simplicial category \mathcal{S} is the small category whose objects are the totally ordered finite sets*

$$Obj(\mathcal{S}) = \left\{ [n] = \{0 < 1 < 2 \cdots < n\} : n \in \mathbb{N} \right\}$$

and morphisms are the non decreasing maps.

The morphisms $\delta_i^n : [n-1] \rightarrow [n]$ and $\sigma_j^n : [n+1] \rightarrow [n]$ for $0 \leq i \leq n$ are given by

$$\delta_i^n(k) = \begin{cases} k & k < i \\ k + 1 & k \geq i \end{cases} \quad (6.1.1a)$$

$$\sigma_j^n(k) = \begin{cases} k & k \leq j \\ k - 1 & k > j \end{cases} \quad (6.1.1b)$$

and δ_i is the injection that misses i and σ_j the surjection such that $\sigma_j(j) = \sigma_j(j+1) = j$.

Proposition 6.1.2 *The morphisms δ_i^n and δ_j^n generate $\text{Mor}(\mathcal{S})$, which admits the following presentation*

$$\begin{aligned} \delta_j \delta_i &= \delta_i \delta_{j-1} \quad \text{for } i < j \\ \sigma_j \sigma_i &= \sigma_i \sigma_{j+1} \quad \text{for } i \leq j \\ \sigma_j \delta_i &= \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j, \\ 1_n & \text{if } i = j \text{ or } i = j + 1, \\ \delta_{i-1} \sigma_j & \text{for } i > j + 1. \end{cases} \end{aligned}$$

Given a category \mathcal{C} , a *simplicial object* in \mathcal{C} is a contravariant functor $X : \mathcal{S} \rightarrow \mathcal{C}$. Such a functor X is uniquely specified by the objects $X_n = X([n])$ and the morphisms $d_i^n = X(\delta_i^n) : X_n \rightarrow X_{n-1}$ (degeneracy maps) and $s_j^n = X(\sigma_j^n) : X_n \rightarrow X_{n+1}$ (face maps).

In the case where $\mathcal{C} = \mathbf{Set}$ is the category of sets X is called a *simplicial set*, and if $\mathcal{C} = \mathbf{Top}$ is the category of topological spaces X is called a *topological simplicial set*.

Let us consider the covariant functor $\Delta : \mathcal{S} \rightarrow \mathbf{Top}$, with

$$\Delta([n]) = \Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^n : t_i \geq 0, \sum t_i = 1\}$$

the standard n -simplex of \mathbb{R}^n and

$$\Delta(\delta_i^n) : \begin{cases} \Delta^{n-1} & \rightarrow \Delta^n \\ (t_0, t_1, \dots, t_{n-1}) & \mapsto (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_{n-1}) \end{cases}$$

the inclusion of Δ^{n-1} as the face opposite to vertex number i in Δ^n , and

$$\Delta(\sigma_j^n) : \begin{cases} \Delta^{n+1} & \rightarrow \Delta^n \\ (t_0, t_1 \cdots t_{n+1}) & \mapsto (t_0, \cdots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \cdots, t_{n+1}) \end{cases}$$

the restriction of Δ^{n+1} to its face opposite to vertex j .

Abusing notation the morphisms $\Delta(\delta_i^n)$ and $\Delta(\sigma_j^n)$ will respectively be written δ_i^n and σ_j^n for simplicity in the rest of these notes.

Definition 6.1.3 *Let X be a topological simplicial set, or a simplicial set with each X_n endowed with the discrete topology. The geometric realization $|X|$ of X is the quotient of the topological space $X \times_{\mathcal{S}} \Delta = \bigcup_{n \geq 0} X_n \times \Delta^n$ by the equivalence relation $(x, \alpha_* y) \sim (\alpha^* x, y)$ for all α in $Mor(\mathcal{S})$, where $\alpha_* = \Delta(\alpha)$ and $\alpha^* = X(\alpha)$.*

6.2 Nerve and classifying space of a small category

The nerve $N(\mathcal{C})$ of a small category \mathcal{C} is a simplicial set in \mathcal{C} that provides a combinatorial way of organizing the data of isomorphism classes of objects in \mathcal{C} . If $A \in Obj(\mathcal{C})$, $N(\mathcal{C})$ somehow encodes all objects isomorphic to A and keeps track of the various isomorphisms between all of these objects. This can become rather complicated, especially if the objects have many non-identity automorphisms. We define homotopy (and (co)homology) for simplicial sets, via their geometric realizations, and we can ask questions about the meaning of those various algebraic topology invariants. One hopes that the answers to such questions provide interesting information about the original category \mathcal{C} , or about related categories.

The notion of nerve is a direct generalization of the classical notion of classifying space of a discrete group; see section 6.3.

Definition 6.2.1 Let \mathcal{C} be a small category. The nerve $N(\mathcal{C})$ of \mathcal{C} is the simplicial set, i.e. the contravariant functor $N(\mathcal{C}) : \mathcal{S} \rightarrow \mathbf{Set}$, defined by

$$N(\mathcal{C})_0 = \text{Obj}(\mathcal{C})$$

$$N(\mathcal{C})_1 = \text{Mor}(\mathcal{C})$$

$$N(\mathcal{C})_n = \text{set of } n\text{-tuples } (f_1, \dots, f_n) \text{ of compatibles morphisms in } \mathcal{C} \text{ (} n \geq 2 \text{)}$$

and with face maps $d_i^n = N(\mathcal{C})(\delta_i^n)$ and degeneracy maps $s_j^n = N(\mathcal{C})(\sigma_j^n)$ given by

$$d_i^n(f_1, \dots, f_n) = \begin{cases} (f_2, \dots, f_n) & \text{for } i = 0 \\ (f_1, \dots, f_{i-1}, f_i f_{i+1}, \dots, f_n) & \text{for } 1 \leq i \leq n-1 \\ (f_1, \dots, f_{n-1}) & \text{for } i = n \end{cases}$$

$$s_j^n(f_1, \dots, f_n) = (f_1, \dots, f_j, 1, f_{j+1}, \dots, f_n)$$

This description of the nerve makes functoriality quite transparent; for example, a functor between small categories \mathcal{C} and \mathcal{D} induces a map of simplicial sets $N(\mathcal{C}) \rightarrow N(\mathcal{D})$. Moreover a natural transformation between two such functors induces a homotopy between the induced maps.

Definition 6.2.2 The classifying space of a small category \mathcal{C} is the geometric realization of its nerve.

We can now define the (co)homology of a small category via the simplicial (co)homology of its classifying space.

6.3 Classifying space of a group

The main reference for this section is Hatcher's book on Algebraic Topology [38], section 1.B.

Let G be a discrete group (i.e a group endowed with the discrete topology). The classifying space of G is a topological space BG whose fundamental group is G and whose higher homotopy groups are trivial: $\pi_1(BG) = G$ and $\pi_n(BG) = 0$, $n \geq 2$.

Let EG be the Δ -complex whose n -simplices are the ordered $(n+1)$ -tuples $[g_0, \dots, g_n]$ of elements of G . Such an n -simplex attaches to the $(n-1)$ -simplex $[g_0, \dots, \hat{g}_i, \dots, g_n]$ in the obvious way, just as a standard simplex attaches to its faces. (The notation \hat{g}_i means that this vertex is deleted.) The complex EG is contractible by the homotopy that slides a point $x \in [g_0, \dots, g_n]$ along the line segment in $[e, g_0, \dots, g_n]$ from x to the vertex $[e]$, where e is the identity element of G .

The group G acts on EG by left multiplication, an element $g \in G$ taking the simplex $[g_0, \dots, g_n]$ linearly onto the simplex $[gg_0, \dots, gg_n]$. Only the identity e takes any simplex to itself, so the action of G on EG is free and it is a covering space action. Hence the quotient map $EG \rightarrow EG/G$ is the universal cover of the orbit space $BG = EG/G$, and BG is a $K(G, 1)$ space.

Since G acts on EG by freely permuting simplices, BG inherits a Δ -complex structure from EG . The action of G on EG identifies all the vertices of EG , so BG has just one vertex. To describe the Δ -complex structure on BG explicitly, note first that every n -simplex of EG can be written uniquely in the following form

$$[g_0, g_0g_1, g_0g_1g_2, \dots, g_0g_1 \cdots g_n] = g_0[e, g_1, g_1g_2, \dots, g_1 \cdots g_n].$$

The image of the above simplex in BG may be denoted unambiguously by the symbol $[g_1|g_2|\cdots|g_n]$. In this notation the g_i 's and their ordered products can be used to label edges, viewing an edge label as the ratio between the two labels on the vertices at the endpoints of the edge, as indicated on the figure.

With this notation, the boundary of a simplex $[g_1|\cdots|g_n]$ of BG consists of the simplices $[g_2|\cdots|g_n]$, $[g_1|\cdots|g_{n-1}]$, and $[g_1|\cdots|g_i g_{i+1}|\cdots|g_n]$ for $i = 1, \dots, n-1$.

As noted in the previous section, we can define the (co)homology of a group as the simplicial (co)homology of its classifying space.

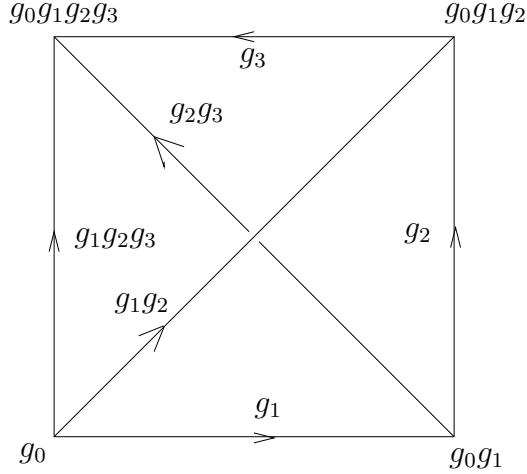


Figure 8: A 3-simplex in the classifying space of G

6.4 Example: classifying space of \mathbb{Z}_2

We consider the multiplicative group with two elements $\mathbb{Z}_2 = \{-1, 1\}$. We use the definition 6.2.2, and view \mathbb{Z}_2 as a small category with one object \cdot (which can be seen as the group itself $\{\mathbb{Z}_2\}$) and with morphisms the set of elements of the group.

Proposition 6.4.1 *The classifying space of the group \mathbb{Z}_2 is the infinite dimensional real projective space:*

$$B\mathbb{Z}_2 \cong \mathbb{RP}^\infty.$$

Proof. The nerve $N(\mathbb{Z}_2)$ consists of sets of n -tuples of elements of \mathbb{Z}_2 , that is

$$N(\mathbb{Z}_2)_0 = \cdot, \quad \text{and} \quad N(\mathbb{Z}_2)_n = \{(\epsilon_1, \dots, \epsilon_n) : \epsilon_i = \pm 1\} \quad \text{for } n \geq 1.$$

The 0-skeleton of $B\mathbb{Z}_2 = |N(\mathbb{Z}_2)|$ consists in just one vertex: $v = \cdot \times \Delta^0$.

The 1-skeleton has two 1-simplices: $|_\epsilon = \epsilon \times \Delta^1$ for $\epsilon = \pm 1$. The 1-simplex $|_1$ is degenerate, indeed

$$|_1 = 1 \times \Delta^1 = s_0^0(\cdot) \times \Delta^1 \quad \sim_\Delta \quad \cdot \times \sigma_0^0(\Delta^1) = \cdot \times \Delta^0 = v.$$

We now look at how the 1-simplex $|_{-1}$ is attached to the vertex v . The two end points of $|_{-1}$ are the two 0-simplices given by $-1 \times \delta_\epsilon^1(\Delta^0)$, for $\epsilon = \pm 1$, the face map

δ_ϵ^1 sending the vertex Δ^0 to one of the two endpoints of the 1-simplex Δ^1 . We have

$$-1 \times \delta_\epsilon^1(\Delta^0) \sim_\Delta d_\epsilon^1(-1) \times \Delta^0 = \cdot \times \Delta^0 = v,$$

and hence $|_{-1}$ is attached by both of its end points to v , and the 1-skeleton of $B\mathbb{Z}_2$ is just the circle \mathbb{S}^1 (it is *not* $\mathbb{R}P^1$).

The 2-skeleton has four 2-simplices $\Delta_{\epsilon_1\epsilon_2} = (\epsilon_1, \epsilon_2) \times \Delta^2$ for $\epsilon_1, \epsilon_2 = \pm 1$. But whenever one of the ϵ 's equals 1 the 2-simplex $\Delta_{\epsilon_1\epsilon_2}$ is degenerate. Indeed, we have

$$\begin{aligned} \Delta_{-11} &= (-1, 1) \times \Delta^2 = s_1^1(-1) \times \Delta^2 \sim_\Delta -1 \times \sigma_1^1(\Delta^2) = -1 \times \Delta^1 = |_{-1} \\ \Delta_{1-1} &= (1, -1) \times \Delta^2 = s_0^1(-1) \times \Delta^2 \sim_\Delta -1 \times \sigma_0^1(\Delta^2) = -1 \times \Delta^1 = |_{-1} \\ \Delta_{11} &= (1, 1) \times \Delta^2 = s_{0 \text{ or } 1}^1(1) \times \Delta^2 \sim_\Delta 1 \times \Delta^1 = s_0^0(\cdot) \times \Delta^1 \sim_\Delta \cdot \times \Delta^0 = v. \end{aligned}$$

We now look at how the 2-simplex Δ_{-1-1} is glued to the 1-skeleton. The three faces of Δ_{-1-1} are given by $(-1, -1) \times \delta_i^2(\Delta^1)$ for $i = 0, 1, 2$, the face map δ_i^2 sending the 1-simplex Δ^1 to the face of the 2-simplex Δ^2 opposite to the i -th vertex. We have

$$\begin{aligned} (-1, -1) \times \delta_i^2(\Delta^1) &= d_i^2((-1, -1)) \times \Delta^1 \sim_\Delta -1 \times \Delta_1 = |_{-1} \quad \text{for } i = 0, 2 \\ (-1, -1) \times \delta_1^2(\Delta^1) &= d_1^2((-1, -1)) \times \Delta^1 \sim_\Delta 1 \times \Delta_1 = s_0^0(\cdot) \times \Delta^1 \sim_\Delta v, \end{aligned}$$

and hence the face opposite to vertex $i = 1$ is contracted to the vertex v while the two faces opposite to vertices $i = 0$ and $i = 2$ are glued along $|_{-1}$ with opposite orientation. The 2-skeleton is therefore a 2-disk with diametrically opposite boundary points identified, *i.e.* $\mathbb{R}P^2$.

In general, the n -th skeleton has 2^n n -simplices but only a single non degenerate one, namely $\Delta^n = (-1, \dots, -1)_n \times \Delta^n$ (where the subscript in $(\dots)_n$ denotes the number of elements in the parenthesis) when all the ϵ 's are equal to -1 , and corresponds to the single n -simplex in the usual *CW*-complex decomposition of $\mathbb{R}P^\infty$. The $(n+1)$ -faces of Δ^n are given by $(-1, \dots, -1)_n \times \delta_i^n(\Delta^{n-1})$, the face map δ_i^n sending the

$(n-1)$ -simplex Δ^{n-1} to the face of Δ^n opposite to the i -th vertex. We have

$$\begin{aligned} (-1, \dots, -1)_n \times \delta_i^n(\Delta^{n-1}) &= d_i^n((-1, \dots, -1)_n) \times \Delta^{n-1} \\ &\sim_{\Delta} \begin{cases} (-1, \dots, -1)_{n-1} \times \Delta^{n-1} = \Delta^{n-1} & \text{for } i = 0 \text{ or } n, \\ (-1, \dots, -1, +1, -1 \dots, -1)_{n-1} \times \Delta^{n-1} & \text{for } 1 \leq i \leq n, \end{cases} \end{aligned}$$

where $+1$ is at position i in $(-1, \dots, -1, +1, -1 \dots, -1)_{n-1}$. Hence only the faces opposite to the vertices $i = 0$ and $i = n$ are non degenerate, and since they have opposite orientation, Δ^n is glued on Δ^{n-1} as an n -sphere by its two hemispheres (corresponding to those two faces) and its diametrically opposite boundary points are identified. By induction the n -th skeleton is therefore $\mathbb{R}P^n$. This proves that the classifying space of \mathbb{Z}_2 is the infinite dimensional projective space. \square

6.5 Inverse limit of groupoids

A *groupoid* Γ is a small category for which every morphism is an isomorphism. The standard notation and terminology are the following. The set of objects of Γ is denoted Γ^0 and the set of morphisms Γ^1 . Morphisms are usually called *arrows* of the groupoid. An object x in Γ^0 can be viewed as the identity morphism on 1_x on x . There are maps $r, s : \Gamma^1 \rightarrow \Gamma^0$, called the *range* and *source* maps that assign to each morphism its range and domain objects respectively. One says that two arrows γ and γ' are *composable* if the range of one equals the source of the other, for instance if $r(\gamma) = s(\gamma')$ then one can form the composition arrow $\gamma \circ \gamma'$ whose source is $s(\gamma)$ and range $r(\gamma')$. A *groupoid homomorphism* $\phi : \Gamma \rightarrow \Gamma'$ is a covariant functor, more precisely a map such that $\phi(1_x) = 1_{\phi(x)}$ and $\phi(\gamma_1 \circ \gamma_2) = \phi(\gamma_1) \circ \phi(\gamma_2)$. The class of groupoids together with groupoid homomorphisms forms the category **Gpd** of groupoids.

Lemma 6.5.1 *Product exists in the category **Gpd** of groupoids.*

Let I be an index set and $(\Gamma_i)_{i \in I}$ a family of groupoids. The product groupoid $\prod_{i \in I} \Gamma_i$ is the groupoid whose set of objects is the cartesian product $\prod_{i \in I} \Gamma_i^0$ and whose set of morphisms is the cartesian product $\prod_{i \in I} \Gamma_i^1$. Two arrows $(\gamma_i)_{i \in I}$ and $(\gamma'_i)_{i \in I}$ are composable in the product if and only if γ_i and γ'_i are composable for each $i \in I$.

Proof. It suffices to show that the product of two groupoids exists in **Gpd**. Let Γ_1 and Γ_2 be two groupoids. Let $\pi_i, i = 1, 2$ denote the canonical projection from $\Gamma_1 \times \Gamma_2$ to Γ_i . We show that $\Gamma_1 \times \Gamma_2$ is universally attracting in **Gpd**. Let Λ be a groupoid and suppose we are given a groupoid homomorphism $\rho_i : \Lambda \rightarrow \Gamma_i$ for each $i = 1, 2$. We have to show that there exists a unique groupoid homomorphism $f : \Lambda \rightarrow \Gamma_1 \times \Gamma_2$ such that $\rho_i = \pi_i \circ f$ for $i = 1, 2$, i.e. that the following diagram commutes.

$$\begin{array}{ccccc}
 & & \Lambda & & \\
 & \swarrow & \downarrow f & \searrow & \\
 & \rho_1 & & \rho_2 & \\
 \Gamma_1 & \xleftarrow{\pi_1} & \Gamma_1 \times \Gamma_2 & \xrightarrow{\pi_2} & \Gamma_2
 \end{array}$$

One readily checks that f is given by $f(\lambda) = (\rho_1(\lambda), \rho_2(\lambda))$. □

Corollary 6.5.2 *Inverse limit exists in the category **Gpd** of groupoids.*

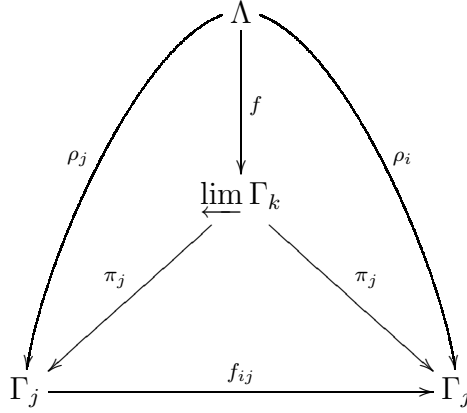
Let (Γ_i, f_{ij}) be an inverse system of objects and morphisms in **Gpd**, namely I is a directed partially ordered set, $(\Gamma_i)_{i \in I}$ is a family of groupoids and there is a family of groupoid homomorphisms $f_{ij} : \Gamma_j \rightarrow \Gamma_i$ for all $i \leq j$ such that f_{ii} is the identity on Γ_i and $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$. The inverse limit of the system (Γ_i, f_{ij}) is the subgroupoid of the product of the Γ_i 's given by

$$\varprojlim \Gamma_i = \left\{ (\gamma_i) \in \prod_{i \in I} \Gamma_i : \gamma_i = f_{ij}(\gamma_j) \text{ for all } i \leq j \right\}$$

with the canonical projections morphisms $\pi_i : \varprojlim \Gamma_i \rightarrow \Gamma_i$ on the i -th coordinate.

Proof. We prove that $\varprojlim \Gamma_i$ is universally attracting in **Gpd**. Let Λ be a groupoid and assume that we are given a groupoid homomorphism $\rho_i : \Lambda \rightarrow \Gamma_i$ for each $i \in I$ such

that $\rho_j = f_{ij}\rho_i$ for all $i \leq j$. We have to show that there exists a unique groupoid homomorphism $f : \Lambda \rightarrow \varprojlim \Gamma_k$ that makes the following diagram commute.



One readily checks that f is given by $f(\lambda) = (\rho_i(\lambda))_{i \in I}$. □

6.6 Application to the groupoid of the transversal

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with FLC. Every tile of \mathcal{T} is assumed to be a finite Δ -complex which is a particular (simplicial) CW -complex structure (see section 2.3.2). Every tile is also required to be compatible with the tiles that intersect it, *i.e.* the intersection of any two tiles is itself a sub- Δ -complex of both tiles. In other words, the tiling \mathcal{T} is assumed to be a Δ -complex of \mathbb{R}^d . In addition, every cell is punctured (by the image of the barycenter of the standard simplex that the cell is the image of).

Let Ω be the *hull* of \mathcal{T} (definition 1.1.4), and Ξ_Δ its Δ -*transversal* (the set of translates of \mathcal{T} which have the puncture of a cell at the origin $0_{\mathbb{R}^d}$, see definition 2.3.1). Let \mathcal{L}_Δ be the Delone set of \mathbb{R}^d of the punctures of the cells of \mathcal{T} (remark 2.3.3). We consider the groupoid Γ_Δ of the Δ -transversal, defined as follows (remark 2.3.3). The set of objects is the Δ -transversal: $\Gamma_\Delta^0 = \Xi_\Delta$, and the set of arrows is

$$\Gamma_\Delta^1 = \{(\xi, x) \in \Xi_\Delta \times \mathbb{R}^d : \tau^{-x}\xi \in \Xi_\Delta\}.$$

Given an arrow $\gamma = (\xi, x)$ in Γ_{Δ}^1 , its *source* is the object $s(\gamma) = \tau^{-x}\xi \in \Xi_{\Delta}$ and its *range* the object $r(\gamma) = \xi \in \Xi_{\Delta}$.

Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of \mathcal{T} (definition 1.4.3), that is a projective system where, for all $l \geq 1$, \mathcal{B}_l is a patch space associated with a pattern \hat{p}_l of \mathcal{T} and $f_l = f_{p_{l-1}p_l} : \mathcal{B}_l \rightarrow \mathcal{B}_{l-1}$, such that \mathcal{B}_l is zoomed out of \mathcal{B}_{l-1} , with the convention that $f_0 = f_{p_1}$ and \mathcal{B}_0 is the prototile space of \mathcal{T} . We recall from section 1.4 theorem 1.4.4, that the hull Ω is homeomorphic to the inverse limit of a proper sequence of patch spaces (theorem 1.4.4):

$$\Omega \cong \varprojlim (\mathcal{B}_l, f_l).$$

The two key ingredients used for constructing this homeomorphism are that in a proper sequence the patch spaces are *zoomed out* of each others (definition 1.4.2), and that the patch spaces *force their borders* (namely are built from collared patches of \mathcal{T} , as in definition 1.4.1).

The *graph of a patch space* \mathcal{B}_l is the graph whose vertices are the punctures of the cells of \mathcal{B}_l and whose edges are the line segments joining the puncture of an n -cell to those of its boundary $(n-1)$ -cells. Note that \mathcal{B}_l is a finite complex, and hence its graph is a finite graph. Let $\mathcal{G}_{\Delta l}$ be the free category generated by the graph of \mathcal{B}_l , namely the objects are the vertices of the graph, and the morphisms the paths in the graph (composition of morphisms is given by concatenation of paths). A path between two vertices can be traced both ways from one vertex to another, hence the morphisms are all invertible and thus $\mathcal{G}_{\Delta l}$ is a groupoid.

As \mathcal{B}_l is a *flat oriented branched manifolds* [11] its domains can be identified with subsets of \mathbb{R}^d and one can associate to each path p a vector u_p of \mathbb{R}^d by adding up the vectors corresponding to each edge line segments that p is made of. We consider a congruence relation R on the category $\mathcal{G}_{\Delta l}$: two paths $p_1, p_2 \in \text{Mor}(x, y)$ are $R_{x,y}$ -equivalent if $u_{p_1} = u_{p_2}$. This clearly defines an equivalence relation on

$Mor(x, y)$, and R is also a congruence for indeed if $p_1, p_2 \in Mor(x, y)$ are equivalent and $p'_1, p'_2 \in Mor(y, z)$ are equivalent then $p_1 p'_1$ and $p_2 p'_2$ are equivalent in $Mor(x, z)$ because $u_{p_1} = u_{p_2}$ and $u_{p'_1} = u_{p'_2}$ and thus $u_{p_1 p'_1} = u_{p_1} + u_{p'_1} = u_{p_2} + u_{p'_2} = u_{p_2 p'_2}$.

Definition 6.6.1 *The Δ -groupoid of a patch space \mathcal{B}_l , denoted $\Gamma_{\Delta l}$, is the quotient of $\mathcal{G}_{\Delta l}$ by the congruence relation R :*

$$\Gamma_{\Delta l} = \mathcal{G}_{\Delta l} / R.$$

Namely the objects are the vertices of the graph, and the morphisms the equivalence classes of paths in the graph. As for $\mathcal{G}_{\Delta l}$, every morphism is invertible and therefore $\Gamma_{\Delta l}$ is a groupoid. Indeed every path p between two vertices x and y gives rise to a backward path \bar{p} from y to x and one has $u_{\bar{p}} = -u_p$, hence if $\gamma_l = [p] \in Mor_{\mathcal{G}_{\Delta l}/R}(x, y)$ then $\gamma_l^{-1} = [\bar{p}] \in Mor_{\mathcal{G}_{\Delta l}/R}(y, x)$. To summarize, given an arrow $\gamma_l = [p]$ in $\Gamma_{\Delta l}^1$ its range and source objects are given by $r(\gamma_l) = r(p)$ and $s(\gamma_l) = s(p)$ respectively while its associated vector is $u_{\gamma_l} = u_p$ (and this does not depend on the choice of the representative path p).

We can also view the arrows of $\Gamma_{\Delta l}^1$ as homotopy classes of path in \mathcal{B}_l , and thus $\Gamma_{\Delta l}^1$ is a subgroupoid of the fundamental groupoid of \mathcal{B}_l . Hence we have

Proposition 6.6.2 *The groups of units of $\Gamma_{\Delta l}$, $\Gamma_{\Delta l}(x)$ for $x \in \Gamma_{\Delta l}^0$, are isomorphic to the fundamental groups of \mathcal{B}_l , $\pi_1(\mathcal{B}_l, x)$:*

$$\Gamma_{\Delta l}(x) \cong \pi_1(\mathcal{B}_l, x),$$

Consider the map $f_l : \mathcal{B}_l \rightarrow \mathcal{B}_{l-1}$ between two successive patch spaces in a proper sequence. As both patch spaces are flat branched manifolds, a path p in \mathcal{B}_l has the same associated vector than its image path $f_l(p)$ in \mathcal{B}_{l-1} : $u_p = u_{f_l(p)}$. It is easily seen from this that f_l induces a groupoid homomorphism $\Gamma_{\Delta l} \rightarrow \Gamma_{\Delta l-1}$, that we will still denote by f_l . We now state the main result of this chapter that gives an approximation of Γ_{Δ} by inverse limit of the $\Gamma_{\Delta l}$.

Theorem 6.6.3 *Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of \mathcal{T} . The groupoid Γ_Δ of the Δ -transversal Ξ_Δ is isomorphic to the inverse limit of the Δ -groupoids $\Gamma_{\Delta l}$ of the patch spaces \mathcal{B}_l :*

$$\Gamma_\Delta \cong \varprojlim \Gamma_{\Delta l}.$$

Proof. We recalled that the hull Ω is homeomorphic to the inverse limit of the patch spaces \mathcal{B}_l (theorem 1.4.4). An easy corollary is that the Δ -transversal Ξ_Δ is homeomorphic to the inverse limit of the set of punctures of the patch spaces \mathcal{B}_l 's. This implies immediately that the set of objects of the inverse limit of the $\Gamma_{\Delta l}$'s is isomorphic to the Δ -transversal and thus to the set of objects Γ_Δ^0 of Γ_Δ . The image in Γ_Δ^0 of an object x of the inverse limit groupoid will be denoted ξ_x . We are left with checking that the set of arrows of the inverse limit groupoid is isomorphic to the set of arrows of Γ_Δ . An arrow γ in the inverse limit groupoid is a sequence $(\gamma_l)_{l \in \mathbb{N}}$ of arrows γ_l in $\Gamma_{\Delta l}^1$ such that $\gamma_{l-1} = f_l(\gamma_l)$ for all $l \geq 1$. We first characterize the arrows in the inverse limit groupoid $\varprojlim \mathcal{G}_{\Delta l}$.

1. **Claim:** *If $p = (p_l)_{l \in \mathbb{N}}$ is an arrow in $\varprojlim \mathcal{G}_{\Delta l}$ then there exists an integer n_p for which p_l is a path entirely contained in a domain of \mathcal{B}_l for all $l \geq n_p$.*

Note first that if p_n is contained in a domain of \mathcal{B}_n , then for all $l \geq n$, p_l is contained in a domain of \mathcal{B}_l . For indeed, the sequence of patch spaces is proper, so \mathcal{B}_l is zoomed out of \mathcal{B}_{l-1} and thus the domains of \mathcal{B}_{l-1} embed into those of \mathcal{B}_l . Now if p_l is not contained in a domain of \mathcal{B}_l it must cross some branching points, but only finitely many as it is a finite path. Say p_l crosses a branching point connecting two patches π_i^l and π_j^l . Now π_i^l and π_j^l must correspond to patches which have translates that intersect in \mathcal{T} , otherwise the preimages of π_i^l and π_j^l would not touch in \mathcal{B}_m for $m > l$ large enough, and this would break the paths p_m . Hence as one looks further back in the inverse limit sequence $(p_l)_{l \in \mathbb{N}}$, the paths p_l cross less and less branching points, and there exist an integer n_p for which p_{n_p} is entirely contained in a domain of \mathcal{B}_{n_p} .

This proves Claim 1.

We can now easily characterize the arrows in the inverse limit groupoid $\varprojlim \Gamma_{\Delta l}$.

2. Claim: *If $\gamma = (\gamma_l)_{l \in \mathbb{N}}$ is an arrow in $\varprojlim \Gamma_{\Delta l}$ then there exists an integer n_γ for which γ_l can be represented as a couple $\gamma_l = (r(\gamma_l), u_\gamma)$ in $\Gamma_{\Delta l}^0 \times \mathbb{R}^d$ for all $l \geq n_\gamma$.*

The induced quotient map $q : \varprojlim \mathcal{G}_{\Delta l} \rightarrow \varprojlim \Gamma_{\Delta l}$ which sends $p = (p_l)_{l \in \mathbb{N}}$ to $\gamma = ([p_l])_{l \in \mathbb{N}}$ is surjective because each coordinate map $p_l \mapsto [p_l]$ is onto. Let γ be an arrow in $\varprojlim \Gamma_{\Delta l}$, and p an arrow in $\varprojlim \mathcal{G}_{\Delta l}$ with $q(p) = \gamma$. By Claim 1, there exists an integer n_p such that the paths p_l are entirely contained in domains of \mathcal{B}_l for $l \geq n_p$. For $l \geq n_p$, the arrow γ_l can thus be represented by the couple $\gamma_l = (r(\gamma_l), u_{p_l})$ in $\Gamma_{\Delta l}^0 \times \mathbb{R}^d$, with inverse $\gamma_l^{-1} = (s(\gamma_l), -u_{p_l})$. This representation of γ_l for $l \geq n_p$ depends on the choice of p only through the integer n_p , but the vector u_{p_l} is the same for all preimages of γ_l in $\mathcal{G}_{\Delta l}^1$, and for all $l \geq n_p$, and therefore only depends on γ . This proves Claim 2.

Now given an arrow γ in the inverse limit groupoid, let u_γ be the vector of \mathbb{R}^d joining the end points of γ_l for $l \geq n_\gamma$ as given by Claim 2. We now map the arrow γ in $\varprojlim \Gamma_l$ to the arrow $(\xi_{r(\gamma)}, u_\gamma)$ in Γ_Δ and this provides us with the desired groupoid isomorphism. \square

An immediate consequence on the cohomology of Γ_Δ (as the simplicial cohomology of its classifying space) follows by direct limit:

Corollary 6.6.4 *Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of \mathcal{T} . The cohomology of the groupoid Γ_Δ of the transversal Ξ_Δ is isomorphic to the direct limit of the cohomology of the Δ -groupoids $\Gamma_{\Delta l}$:*

$$H^*(\Gamma_\Delta; M) \cong \varinjlim H^*(\Gamma_{\Delta l}; M),$$

where M is a free module over a unital ring.

We now present a similar result for the groupoid Γ of the canonical transversal Ξ

(definitions 1.2.5 and 1.1.4). We endow every tile of \mathcal{T} with a canonical CW -complex structure consisting of a single d -dimensional cell which is its interior, and in a compatible way with its neighboring tiles: *i.e.* the intersection of any two tiles is a subcomplex of both. Each tile is assumed to be punctured (by an interior point). We recall that the canonical transversal Ξ is the set of translates of \mathcal{T} which have the puncture of a tile at the origin $0_{\mathbb{R}^d}$. If \mathcal{L} denote the Delone set of \mathbb{R}^d of the punctures of the tiles of \mathcal{T} , then the groupoid Γ of the canonical transversal, is the groupoid of the transversal to the hull of \mathcal{L} . Its set of objects is the canonical transversal: $\Gamma^0 = \Xi$, and its set of arrows is

$$\Gamma^1 = \{(\xi, x) \in \Xi \times \mathbb{R}^d : \mathsf{T}^{-x}\xi \in \Xi\}.$$

Given an arrow $\gamma = (\xi, x)$ in Γ^1 , its *source* is the object $s(\gamma) = \mathsf{T}^{-x}\xi \in \Xi$ and its *range* the object $r(\gamma) = \xi \in \Xi$.

The CW -complex structure of the tiles of \mathcal{T} allow to define the *Voronoi graph* of a patch space \mathcal{B}_l as follows: the vertices are the punctures of the tiles that \mathcal{B}_l is made of, and the edges are the line segments between the punctures of two tiles which share a face a codimension 1. One can now consider the free category \mathcal{G}_l generated by the Voronoi graph of \mathcal{B}_l , which as for $\mathcal{G}_{\Delta l}$ is a groupoid, and then the quotient category Γ_l where two paths are identified if they have the same end points and correspond to the same vector in \mathbb{R}^d . This quotient category, just like $\Gamma_{\Delta l}$ is a groupoid, and we call it the *canonical groupoid* of the patch space \mathcal{B}_l . Just as for theorem 6.6.3 and Corollary 6.6.4 we can prove similarly the following.

Theorem 6.6.5 *Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of \mathcal{T} . The groupoid Γ of the canonical transversal Ξ is isomorphic to the inverse limit of the canonical groupoids Γ_l of the patch spaces \mathcal{B}_l :*

$$\Gamma \cong \varprojlim \Gamma_l.$$

The cohomology of the groupoid Γ of the canonical transversal Ξ is isomorphic to the direct limit of the cohomology of the groupoids Γ_l :

$$H^*(\Gamma; M) \cong \varinjlim H^*(\Gamma_l; M),$$

where M is a free module over a unital ring.

6.7 The PV category of a tiling

We mentioned earlier (remark 2.3.6) that the PV cohomology of \mathcal{T} , $H_{PV}^*(\mathcal{B}_0; K^0(\Xi_\Delta))$, is a generalization of the cohomology of the base space of a fibration, with local coefficients in the K -theory of its fiber (see section 4.2.2). We show here that the structure of this “local coefficient system” can be formulated in terms of what we call the *PV category* of \mathcal{T} . We also show how the groupoid of the transversal can be recovered from this category.

Now let \mathcal{B}_l be a patch space of \mathcal{T} , and $\mathfrak{p}_l : \Omega \rightarrow \mathcal{B}_l$ the “projection” of the hull onto \mathcal{B}_l (see definition 1.4.1). For each $b \in \mathcal{B}_l$ the set $\mathfrak{p}_l^{-1}(b)$ is a Cantor set. Let $\mathcal{C}_{lb} = C(\mathfrak{p}_l^{-1}(b), \mathbb{Z})$ be the ring of continuous integer valued functions on $\mathfrak{p}_l^{-1}(b)$.

Recall that the set of punctures of the patch space \mathcal{B}_l is identified with the set of objects of its canonical groupoid Γ_l , and the set of punctures of its Δ -structure is identified with the set of objects of its Δ -groupoid $\Gamma_{\Delta l}$ (definition 6.6.1).

Recall from theorem 6.6.3 that for every arrow γ in $\Gamma_{\Delta l}^1$ there is a vector u_γ on \mathbb{R}^d joining its source to its range punctures along its associated path in the branched manifold \mathcal{B}_l . We consider the ring $C(\Xi_\Delta, \mathbb{Z}) = \oplus_x \mathcal{C}_{l,x}$ of continuous integer valued functions on the Δ -transversal, on which, for γ in $\Gamma_{\Delta l}^1$, we define the *partial isometry*

$$\theta_l(\gamma) = \chi_{\mathfrak{p}_l^{-1}s(\gamma)} \mathbb{T}^{u_\gamma} \chi_{\mathfrak{p}_l^{-1}r(\gamma)}, \quad (6.7.1)$$

where χ_Λ denotes the characteristic function of a subset Λ of Ξ_Δ and \mathbb{T} the translation operator. Such an operator acts as a partial isometry: $\theta_l(\gamma) : \mathcal{C}_{l,s(\gamma)} \rightarrow \mathcal{C}_{l,r(\gamma)}$. The

adjoint operator $\theta_l(\gamma)^*$ is given by $\theta_l(\gamma^{-1})$. And $\theta_l(\gamma)\theta_l(\gamma)^*$ is the characteristic function of the set $\Gamma^{u_\gamma}\mathfrak{p}_l^{-1}s(\gamma) \cap \mathfrak{p}_l^{-1}r(\gamma)$ and thus a projection in $\mathcal{C}_{l,r(\gamma)}$, while $\theta_l(\gamma)^*\theta_l(\gamma)$ is the characteristic function of the set $\mathfrak{p}_l^{-1}s(\gamma) \cap \Gamma^{-u_\gamma}\mathfrak{p}_l^{-1}r(\gamma)$ and thus a projection in $\mathcal{C}_{l,s(\gamma)}$.

Remark 6.7.1 The partial isometries $\theta_l(\gamma)$ give an immediate criteria for γ to have a lift in the inverse limit groupoid Γ_Δ . Indeed γ in $\Gamma_{\Delta l}^1$ lifts to an arrow in Γ_Δ if and only if $\theta_l(\gamma) \neq 0$. This allows to express the Δ -groupoid as a lift of $\Gamma_{\Delta l}$ with relations

$$\Gamma_\Delta = \{(\xi, u_\gamma) \in \Xi_\Delta \times \mathbb{R}^d : \gamma \in \Gamma_{\Delta l}^1, \xi \in \mathfrak{p}_l^{-1}s(\gamma), \theta_l(\gamma) \neq 0\}.$$

We now define the *PV category* \mathcal{C}_{PV} of the tiling \mathcal{T} . The objects of \mathcal{C}_{PV} are the rings $K^0(\mathfrak{p}_0^{-1}(x)) = C(\mathfrak{p}_0^{-1}(x), \mathbb{Z})$ for $x \in \Gamma_{\Delta 0}^0$ (*i.e.* the puncture of a simplex in \mathcal{B}_0). The morphisms of \mathcal{C}_{PV} are generated by the partial isometries $\theta_0(\gamma)$ in equation (6.7.1) and their (unique) adjoints $\theta_0^*(\gamma) = \theta_0(\gamma^{-1})$ for $\gamma \in \Gamma_{\Delta 0}^1$. We now give the formal properties of \mathcal{C}_{PV} , as a generalization of a system of local coefficients (see section 4.2.2).

Definition 6.7.2 *Let B be a CW-complex. A PV local coefficient system, or PV category, \mathcal{C} on B is a small inverse category, whose objects are unital rings indexed by elements of B , and whose morphisms are partial isometries indexed by homotopy classes of paths in B :*

(i) $Obj(\mathcal{C}) = (A_b)_{b \in B}$ where A_b is a unital ring,

(ii) $Mor(\mathcal{C}) = \langle \theta(\gamma), \theta^*(\gamma) \rangle$ has a (noncomutative) ring structure and is generated by partial isometries $A_a \xrightarrow{\theta(\gamma)} A_b$ for each homotopy class of path $a \xrightarrow{\gamma} b$ in B . We call the inverses adjoints and they are given by $\theta(\gamma)^* = \theta(\gamma^{-1})$, and $\theta(\gamma)\theta(\gamma)^*$ and $\theta(\gamma)^*\theta(\gamma)$ are idempotents in A_a and A_b respectively.

We can think of \mathcal{C}_{PV} as a lift of the groupoid $\Gamma_{\Delta 0}$. While all morphisms are invertible in $\Gamma_{\Delta 0}$ it is no longer the case in \mathcal{C}_{PV} , however the morphisms in \mathcal{C}_{PV} have more structure: they form a ring and are thus all composable.

The set of morphisms of \mathcal{C}_{PV} has an interesting structure: it is an *inverse semi group*. Indeed each morphism $\theta_0(\gamma)$ has a unique adjoint $\theta_0^*(\gamma)$ and satisfy $\theta_0(\gamma) = \theta_0(\gamma)\theta_0^*(\gamma)\theta_0(\gamma)$. The set of units is given by the operators $\theta_0^*(\gamma)\theta_0(\gamma)$ and $\theta_0(\gamma)\theta_0^*(\gamma)$. In view of (6.7.1) and the relation $\chi_{\Lambda}T^x = T^x\chi_{T^x\Lambda}$ for a clopen $\Lambda \subset \Xi_{\Delta}$, we can write any partial isometry $\theta_0(\gamma)$ as $\chi_{\Lambda}T^x\chi_{\Lambda'}$ with $T^x\Lambda = \Lambda'$. Hence the units $\theta_0^*(\gamma)\theta_0(\gamma)$ and $\theta_0(\gamma)\theta_0^*(\gamma)$ are the characteristic function $\chi_{\Lambda'}$ and χ_{Λ} respectively. This shows that the units are given by characteristic functions on clopen sets of the Δ -transversal. As a consequence, the groups of units in $Mor(\mathcal{C}_{PV})$ generate the objects of \mathcal{C}_{PV} as rings of integer valued functions on clopen sets of the Δ -transversal.

We can characterize the groups of units of $Mor(\mathcal{C}_{PV})$ as follows. We recall from definition 2.3.2 the partial isometry $\theta_{\sigma\tau} = \chi_{\sigma}T^{x_{\sigma\tau}}\chi_{\tau}$, with $x_{\sigma\tau} = x_{\sigma} - x_{\tau}$, which is non zero only if $\tau \in \partial\sigma$ or $\sigma \in \partial\tau$ and whose adjoint is $\theta_{\sigma\tau}^* = \theta_{\tau\sigma}$. Let us denote by $Mor(\mathcal{C}_{PV})^{(0)}$ the set of monomials (with coefficient 1) in the $\theta_{\sigma\tau}$ together with the adjoint and multiplication operations (this is an inverse semi group as we will see later). The units of $Mor(\mathcal{C}_{PV})$ are then given by elements of the form ww^* for $w \in Mor(\mathcal{C}_{PV})^{(0)}$. In analogy with the case of cut and projection tilings in 1-dimension studied in section 1.5, we can ask what is the relation between the Δ -transversal and the C^* -algebra generated by those units.

Theorem 6.7.3 *Let U_{Δ} be the C^* -algebra generated by the units in $Mor(\mathcal{C}_{PV})$:*

$$U_{\Delta} = C^*\{ww^* : w \in Mor(\mathcal{C}_{PV})^{(0)}\}.$$

The spectrum of U_{Δ} is homeomorphic to the Δ -transversal:

$$Sp(U_{\Delta}) \cong \Xi_{\Delta}.$$

Proof. First notice that $\text{Sp}(U_\Delta)$ is compact (by Gelfand's theorem) and totally disconnected (since U_Δ is generated by idempotents). Second, notice that the ring generated by the units ww^* is nothing but $C(\Xi_\Delta, \mathbb{Z})$, and hence U_Δ contains $C(\Xi_\Delta)$ since continuous complex valued functions on Ξ_Δ are uniform limits of linear combinations of characteristic functions on Ξ_Δ .

Let $\{\mathcal{B}_n\}$ be a proper sequence of patch spaces of \mathcal{T} (definition 1.4.3). For $n \geq 0$ let \mathcal{P}_n be the partition of Ξ_Δ by the acceptance zones of the simplices in \mathcal{B}_n . By lemma 2.4.3, \mathcal{P}_n is a refinement of \mathcal{P}_{n-1} , $n \geq 1$. Consider a character $\eta \in \text{Sp}(U_\Delta)$, and an element $a \in U_\Delta$. Let χ_{01} be the characteristic function of a clopen in \mathcal{P}_0 , and χ_{02} the characteristic function on its complement, so that $\chi_{01} + \chi_{02} = 1$ on Ξ_Δ . We have $\eta(a) = \eta(\chi_{01}a) + \eta(\chi_{02}a) = \eta(\chi_{01})\eta(a) + \eta(\chi_{02})\eta(a) = \eta(\chi_{i_0}a)$ for $\chi_{i_0} = \chi_{0i_0}$ with i_0 equals 1 or 2, since $\eta(\chi_{01})$ and $\eta(\chi_{02})$ are equal to 0 or 1. Let inductively $\chi_{n1} + \chi_{n2} = \chi_{i_{n-1}}$ be two characteristic functions of clopens in \mathcal{P}_n that partition the clopen of \mathcal{P}_{n-1} associated with $\chi_{i_{n-1}}$. We have $\eta(a) = \eta(\chi_{i_{n-1}}a) = \eta(\chi_{n1}a) + \eta(\chi_{n2}a) = \eta(\chi_{i_n}a)$ for $\chi_{i_n} = \chi_{ni_n}$ with i_n equals 1 or 2. Since $\varprojlim \mathcal{B}_n \cong \Omega$ (theorem 1.4.4) there exists $\xi \in \Xi_\Delta$ such that χ_{i_n} converges to the character δ_ξ defined by $\delta_\xi(a) = a(\xi)$. This proves that $\text{Sp}(U_\Delta)$ is to the set of δ_ξ for $\xi \in \Xi_\Delta$.

Let $\varphi : \Xi_\Delta \longrightarrow \text{Sp}(U_\Delta)$ be the isomorphism defined by $\varphi(\xi) = \delta_\xi$. Since U_Δ is generated by the $ww^* = \chi_{\Lambda w}$, the weak-* topology on $\text{Sp}(U_\Delta)$ is generated by the open sets $O_{\delta_\xi, \chi_\Lambda, \varepsilon} = \{\delta_{\xi'} : |\delta_{\xi'}(\chi_\Lambda) - \delta_\xi(\chi_\Lambda)| < \varepsilon\}$. But as $|\delta_{\xi'}(\chi_\Lambda) - \delta_\xi(\chi_\Lambda)| = |\chi_\Lambda(\xi') - \chi_\Lambda(\xi)|$ equals 0 or 1, we can take $\varepsilon = 0$ and only consider the open sets $O_{\delta_\xi, \chi_\Lambda, 0}$. Now if $\xi \notin \Lambda$, then $O_{\delta_\xi, \chi_\Lambda, 0} = \emptyset$, and if $\xi \in \Lambda$, then $O_{\delta_\xi, \chi_\Lambda, 0} = \{\delta_\eta : \eta \in \Lambda\}$. Hence $\varphi^{-1}(O_{\delta_\xi, \chi_\Lambda, 0})$ is empty of equals Λ . Hence φ is a homeomorphism. \square

We now show how to recover the groupoid Γ_Δ of the Δ -transversal from the morphisms of \mathcal{C}_{PV} . In [48] Kellendonk defined an inverse semi group $\mathcal{I}_\mathcal{T}$ associated with the tiling \mathcal{T} , showed how to recover the groupoid of the transversal from it, and built the C^* -algebra of the groupoid using representations of $\mathcal{I}_\mathcal{T}$ by partial isometries. We recall

his construction (and modify it slightly for convenience). Consider the translational equivalence classes of triples, $[p, t_1, t_2]$, where p is a patch of \mathcal{T} and t_1, t_2 , two of its tiles (allowing $t_1 = t_2$). There is a product $[p, t_1, t_2][p', t'_1, t'_2] = [p \cup p', t_1, t'_2]$ if $t_2 = t'_1$ and 0 otherwise, and an inverse $[p, t_1, t_2]^{-1} = [p, t_2, t_1]$. And there is a partial order $[p, t_1, t_2] \leq [p', t'_1, t'_2]$ if $t_1 = t'_1, t_2 = t'_2$, and there are representatives of the patches that satisfy $p \supset p'$ (Kellendonk actually considered the reverse order in [48]). The set of such triples together with this product and partial order defines the inverse semi group $\mathcal{I}_{\mathcal{T}}$. We can define a similar inverse semi group related to the Δ -decomposition of \mathcal{T} (instead of its tiles).

Definition 6.7.4 *Assume that the tiles of \mathcal{T} are compatible Δ -complexes (i.e. \mathcal{T} is a Δ -complex decomposition of \mathbb{R}^d). A Δ -patch of \mathcal{T} is the closure of a union of simplices of \mathcal{T} . A Δ -pattern of \mathcal{T} is a translational equivalence class of Δ -patches.*

Note that we do not require a Δ -patch to be connected.

Let σ_1 and σ_2 be two simplices in \mathcal{T} with punctures are at positions x_1 and x_2 . Let $\gamma_{\sigma_1\sigma_2}$ be the arrow in $\Gamma_{\Delta 0}$ joining the punctures of the images in \mathcal{B}_0 of σ_1 and σ_2 by a translation $x_{12} = x_1 - x_2$. The associated morphism of \mathcal{C}_{PV} is:

$$\theta_0(\gamma_{\sigma_1\sigma_2}) = \chi_{\mathfrak{p}_0^{-1} \circ \mathfrak{p}_0(x_1)} T^{x_{12}} \chi_{\mathfrak{p}_0^{-1} \circ \mathfrak{p}_0(x_2)}.$$

Let p_{Δ} be the Δ -patch made up of those two simplices, and \hat{p}_{Δ} its corresponding Δ -pattern. The characteristic function of the acceptance zone of \hat{p}_{Δ} is $\theta_0(\gamma_{\sigma_1\sigma_2})\theta_0(\gamma_{\sigma_1\sigma_2})^*$ if p_{Δ} is punctured by x_1 (or $\theta_0(\gamma_{\sigma_1\sigma_2})^*\theta_0(\gamma_{\sigma_1\sigma_2})$ if it is punctured by x_2). This is clear because this acceptance zone is the set of tilings in Ω with σ_1 at the origin and σ_2 at position $-x_{12}$: $\Xi(\sigma_1) \cup T^{-x_{12}}\Xi(\sigma_2)$ where $\Xi(\sigma_i) = \mathfrak{p}_0^{-1}(x_i), i = 1, 2$, which is exactly $\theta_0(\gamma_{\sigma_1\sigma_2})\theta_0(\gamma_{\sigma_1\sigma_2})^*$.

Lemma 6.7.5 *Let p_{Δ} be a Δ -patch of \mathcal{T} , made up of simplices $\sigma_i, i = 0, \dots, n$, and punctured by σ_0 , and \hat{p}_{Δ} its corresponding Δ -pattern. The characteristic function of*

the acceptance zone of \hat{p}_Δ is

$$\prod_{i=1}^n \theta_0(\gamma_{\sigma_0 \sigma_i}) \theta_0(\gamma_{\sigma_0 \sigma_i})^*.$$

Proof. Let p_i , for $i = 1, \dots, n$, be the Δ -patch that is the closure of $\sigma_0 \cup \sigma_i$. The acceptance zone of \hat{p}_Δ is the intersection of the acceptance zones of the Δ -patterns \hat{p}_i , for $i = 1, \dots, n$. \square

We easily see now that, as an inverse semi group, $Mor(\mathcal{C}_{PV})$ is isomorphic to the inverse semi group \mathcal{I}_Δ of doubly punctured Δ -patterns $[p_\Delta, \sigma_1, \sigma_2]$ analogous to the inverse semi group $\mathcal{I}_\mathcal{T}$ of Kellendonk. The product is given by $[p_\Delta, \sigma_1, \sigma_2][p'_\Delta, \sigma'_1, \sigma'_2] = [p_\Delta \cup p'_\Delta, \sigma_1, \sigma'_2]$ if $\sigma_2 = \sigma'_1$ and 0 otherwise, and the inverse by $[p_\Delta, \sigma_1, \sigma_2]^{-1} = [p_\Delta, \sigma_2, \sigma_1]$. And there is a partial order $[p_\Delta, \sigma_1, \sigma_2] \leq [p'_\Delta, \sigma'_1, \sigma'_2]$ if $\sigma_1 = \sigma'_1$, $\sigma_2 = \sigma'_2$, and there are representatives of the patches that satisfy $p_\Delta \supset p'_\Delta$. The isomorphism with $Mor(\mathcal{C}_{PV})$ is given by the map of inverse semi groups

$$[p_\Delta, \sigma_1, \sigma_2] \mapsto \chi_{\Lambda_1} T^{x_{\sigma_1 \sigma_2}} \chi_{\Lambda_2},$$

where Λ_1 and Λ_2 are the acceptance zones of the pattern \hat{p}_Δ punctured by σ_1 and σ_2 respectively, and $x_{\sigma_1 \sigma_2}$ is the vector joining their punctures (in any representative). The partial order on \mathcal{I}_Δ corresponds to the partial order on $Mor(\mathcal{C}_{PV})$: $\theta = \chi_{\Lambda_1} T^{x_{12}} \chi_{\Lambda_2} \leq \theta' = \chi_{\Lambda'_1} T^{x'_{12}} \chi_{\Lambda'_2}$ if $x_{12} = x'_{12}$, $\Lambda_1 \cap \Lambda'_1 \neq \emptyset$, $\Lambda_2 \cap \Lambda'_2 \neq \emptyset$, and we can write $\theta = u\theta'$ with u a unit (we can actually choose $\theta\theta^* = \chi_{\Lambda_1}$).

We recall from definition 2.3.2 the partial isometry $\theta_{\sigma\tau} = \chi_\sigma T^{x_{\sigma\tau}} \chi_\tau$, with $x_{\sigma\tau} = x_\sigma - x_\tau$, which is non zero only if $\tau \in \partial\sigma$ or $\sigma \in \partial\tau$ and whose adjoint is $\theta_{\sigma\tau}^* = \theta_{\tau\sigma}$.

Lemma 6.7.6 *Let σ_1 and σ_2 be two simplices of \mathcal{T} with punctures at x_1 and x_2 . Let p_{12} be the Δ -patch of \mathcal{T} that is the closure of $\sigma_1 \cup \sigma_2$ punctured by x_1 , and let \hat{p}_{12} be its corresponding Δ -pattern.*

Let P_{12} be the set of connected Δ -patterns \hat{p} such that: each representative p contains copies σ'_1 and σ'_2 of σ_1 and σ_2 at respective positions $x'_1 = x_1 - u_p$ and $x'_2 = x_2 - u_p$

(for some $u_p \in \mathbb{R}^d$) and contains all the simplices $\tau_i, i = 2, \dots, n_p - 1$ of \mathcal{T} which intersect the line segment from x'_1 to x'_2 .

(i) Let $\hat{p} \in P_{12}$. Let $\tau_1 = \sigma_1$ and $\tau_{n_p} = \sigma_2$. The characteristic function of the acceptance zone of \hat{p} is given by:

$$\chi_{\hat{p}} = \chi_{\Xi(\hat{p})} = \prod_{i=1}^{n_p-1} \theta_{\tau_i \tau_{i+1}}^* \prod_{i=1}^{n_p-1} \theta_{\tau_{n_p-i} \tau_{n_p-i+1}},$$

where we have used the same notation for the simplices τ_i and their classes in the prototile space \mathcal{B}_0 .

(ii) The characteristic function of the acceptance zone of \hat{p}_{12} is given by:

$$\chi_{\hat{p}_{12}} = \sum_{\hat{p} \in P_{12}} \chi_{\hat{p}}.$$

Proof. (i) Since a simplex is open, each point on the line segment from x'_1 to x'_2 intersects a unique simplex of \mathcal{T} , and therefore the sequence of simplices along the line, $\sigma'_1 = \tau_1, \tau_2, \dots, \tau_{n_p}, \tau_{n_p+1} = \sigma'_2$, is unique. Using the relation $T^x \chi_{\Lambda} = \chi_{T^x \Lambda} T^x$, the product $\theta_{\tau_1 \tau_2}^* \theta_{\tau_2 \tau_3}^*$ reads $\chi_{\tau_1} \chi_{T^{-x_{12}} \tau_2} \chi_{T^{-x_{13}} \tau_3} T^{x_{13}}$ where $x_{12} + x_{23} = x_1 - x_2 + x_2 - x_3 = x_{13}$. Hence the first product reads $\chi_{\tau_1} \chi_{T^{-x_{12}} \tau_2} \cdots \chi_{T^{-x_{1n_p}} \tau_{n_p}} T^{x_{1n_p}}$. Similarly the second product reads $T^{-x_{1n_p}} \chi_{\tau_1} \chi_{T^{-x_{12}} \tau_2} \cdots \chi_{T^{-x_{1n_p}} \tau_{n_p}}$, so that the right hand side of the equation reads $\chi_{\tau_1} \chi_{T^{-x_{12}} \tau_2} \cdots \chi_{T^{-x_{1n_p}} \tau_{n_p}}$. Now, by lemma 6.7.5, the acceptance zone of \hat{p} is the product over $i = 2, \dots, n_p$ of $\theta_0(\gamma_{\tau_1 \tau_i}) \theta_0(\gamma_{\tau_1 \tau_i})^* = \chi_{\tau_1} \chi_{T^{-x_{1i}} \tau_i}$ and therefore yields the same formula.

(ii) Since \mathcal{T} has FLC the set P_{12} is finite. By construction the patterns in P_{12} correspond to a unique sub-patterns of p_{12} and therefore give a partition of its acceptance zone. □

Given a patch space \mathcal{B}_n that is zoomed out of \mathcal{B}_0 (definitions 1.4.1 and 1.4.2) we can define partial isometries $\theta_{\sigma_n \tau_n}^{(n)}$ in a similar way that we defined $\theta_{\sigma \tau}$. (Those partial isometries appear in the expression of PV differential (2.4.2).) As noted in section

2.4, if σ_n, τ_n are preimages of σ, τ respectively, then we have $\theta_{\sigma_n \tau_n}^{(n)} = \chi_{\sigma_n} \mathbb{T}^{x_{\sigma\tau}} \chi_{\tau_n}$. In other words, if $\gamma_{\sigma_n \tau_n}$ is the arrow of Γ_{Δ_n} joining the puncture of τ_n to the puncture of σ_n (by the translation $x_{\sigma\tau}$), then $\theta_{\sigma_n \tau_n}^{(n)}$ equals $\theta_n(\gamma_{\sigma_n \tau_n})$ given by equation (6.7.1).

Proposition 6.7.7 *The set of partial isometries $\theta_{\sigma\tau}$ generates the ring (and inverse semi group) of morphisms of the PV category of \mathcal{T} . And if \mathcal{B}_n is a patch space of \mathcal{T} that is zoomed out of \mathcal{B}_0 , then the set of partial isometries $\theta_{\sigma_n \tau_n}^{(n)}$ also generates $Mor(\mathcal{C}_{PV})$.*

Proof. Consider a generator of $Mor(\mathcal{C}_{PV})$: $\theta_0(\gamma)$ for γ and arrow from σ_1 to σ_2 by a translation x in \mathcal{B}_0 . As in lemma 6.7.6 let p_{12} be the associated Δ -patch and P_{12} the corresponding set of connected Δ -patterns. As a consequence of the proof of lemma 6.7.6 we can write $\theta_0(\gamma)$ as the sum over P_{12} (by (ii)) of the products of $\theta_{\sigma\tau}$ (by (i)). Hence the $\theta_{\sigma\tau}$ and their adjoints generate $Mor(\mathcal{C}_{PV})$.

For the second statement it suffices to show that each $\theta_{\sigma\tau}$ is a combination of some $\theta_{\sigma_n \tau_n}^{(n)}$. This is immediate from lemma 2.4.3 because the acceptance zone of $\sigma \in \mathcal{B}_0$ is partitioned by the acceptance zones of its preimages $\sigma_n \in \mathcal{B}_n$: $\theta_{\sigma\tau}$ is the sum over the preimages of σ and τ of the corresponding $\theta_{\sigma_n \tau_n}^{(n)}$. \square

It is important to notice that the partial isometries $\theta_{\sigma\tau}$ or $\theta_{\sigma_n \tau_n}^{(n)}$ generate $Mor(\mathcal{C}_{PV})$ as a ring, but the inverse semi group they generate (*i.e.* without addition) are strict sub inverse semi groups. Let $\{\mathcal{B}_n, f_n\}$ be a proper sequence of patch spaces of \mathcal{T} (definition 1.4.3). Let $Mor(\mathcal{C}_{PV})^{(0)}$ be the inverse semi group generated by the $\theta_{\sigma\tau}$, that is the set of monomials (with coefficient 1) in the $\theta_{\sigma\tau}$ together with the adjoint and multiplication operations. And for $n \in \mathbb{N}$, let also $Mor(\mathcal{C}_{PV})^{(n)}$ be the inverse semi group generated by the $\theta_{\sigma_n \tau_n}^{(n)}$. As a consequence of lemma 6.7.7, for each $n \geq 0$, one can see $Mor(\mathcal{C}_{PV})$ as the ring of polynomials in the $\theta_{\sigma_n \tau_n}^{(n)}$, while $Mor(\mathcal{C}_{PV})^{(n)}$ can be seen as the set of monomials (with coefficient 1) in those variables. Also we have the inclusions $Mor(\mathcal{C}_{PV})^{(n)} \subsetneq Mor(\mathcal{C}_{PV})^{(m)}$ for all $n > m \geq 0$.

We now show how to recover the groupoid Γ_Δ from $Mor(\mathcal{C}_{PV})$. Consider the (point-wise) inverse semi group of decreasing sequences $a = (a_n)_{n \geq 0}$ in $Mor(\mathcal{C}_{PV})$, with $a_n \in Mor(\mathcal{C}_{PV})^{(n)}$. Consider the equivalence relation $a \sim b$ if for each $n \geq 0$ there exists $m \geq 0$ such that $b_m \leq a_n$, and $a_m \leq b_n$. We define \mathcal{A}_Δ to be the quotient inverse semi group:

$$\mathcal{A}_\Delta = \{a = (a_n)_{n \geq 0} : a_n \in Mor(\mathcal{C}_{PV})^{(n)}, a_n \leq a_{n-1}\} / \sim \quad (6.7.2)$$

with $a \sim b$ if $\forall n \geq 0, \exists m \geq 0, b_m \leq a_n$ and $a_m \leq b_n$.

Recall that a groupoid is an inverse semi group which has *cancellations*, that is for which the equality $ab = ac \neq 0$ implies that $b = c$.

Lemma 6.7.8 *The inverse semi group \mathcal{A}_Δ is a groupoid.*

Proof. We show that \mathcal{A}_Δ has cancellations. Consider three sequences a, b, c such that $ab = ac \neq 0$. We show that this implies $b = c$. We can write $a_n = \chi_{A_n} \mathbb{T}^x \chi_{A'_n}$ with $\mathbb{T}^x A'_n = A_n$, and similarly $b_n = \chi_{B_n} \mathbb{T}^y \chi_{B'_n}$ and $c_n = \chi_{C_n} \mathbb{T}^z \chi_{C'_n}$. Let $d_n = b_n b_n^* c_n = \chi_{D_n} \mathbb{T}^y \chi_{D'_n}$ where $D_n = B_n \cap C_n$, and thus d_n is also equal to $c_n c_n^* b_n$. Hence for all $n \geq 0, d_n \leq b_n, c_n$. This means that D_n is the acceptance zone of a Δ -patch $p(d_n)$, which contains the Δ -patches $p(b_n)$ and $p(c_n)$ whose acceptance zones are B_n and C_n respectively. Recall that the sequence of patch spaces $\{\mathcal{B}_n, f_n\}$ is proper: the patch spaces are zoomed out of each other (each patch of \mathcal{B}_n contains a patch of \mathcal{B}_{n-1} in its interior) and made of collared patches. Hence for $m > n$ the corresponding patches $p(b_m)$ and $p(c_m)$ contain the patches $p(b_n)$ and $p(c_n)$ respectively, and their radii grow with m . So for $m \geq n$ large enough both $p(b_m)$ and $p(c_m)$ contain $p(d_n)$, *i.e.* $b_m, c_m \leq d_n$. This proves that $b \sim d$ and $c \sim d$, and thus $b \sim c$. Therefore $b = c$ in \mathcal{A}_Δ . □

Theorem 6.7.9 *The groupoids \mathcal{A}_Δ and Γ_Δ are isomorphic.*

Proof. Let $[a] = [(a_n)_{n \geq 0}]$ be an element in \mathcal{A}_Δ . We can write a representative a as the sequence of $a_n = \chi_{A_n} \Gamma^{u_a} \chi_{A'_n}$, $n \geq 0$. For each n , $a_n a_n^*$ is the characteristic function of the clopen A_n in $\mathfrak{p}_n^{-1}(x_n)$ for some $x_n \in \mathcal{B}_n$. Let $b \sim a$ and write $b_n = \chi_{B_n} \Gamma^{u_b} \chi_{B'_n}$, $n \geq 0$, with B_n a clopen in $\mathfrak{p}_n^{-1}(y_n)$ for some $y_n \in \mathcal{B}_n$. For each n there is m such that $b_m \leq a_n$ and $a_m \leq b_n$, and this first implies that $u_a = u_b$. If $m \geq n$ then $b_m \leq a_n$ implies that $y_m \in \mathcal{B}_m$ projects to $x_n \in \mathcal{B}_n$ so $x_n = y_n$, while if $m < n$ then $a_m \leq b_n$ implies that x_m projects to y_n and so $y_n = x_n$. Hence any representative of $[a]$ determines the same vector $u \in \mathbb{R}^d$ and the same sequence $(x_n)_{n \geq 0}$ in $\varprojlim(\mathcal{B}_n, f_n)$. Let $\lambda : \varprojlim(\mathcal{B}_n, f_n) \rightarrow \Omega$ be the homeomorphism of theorem 1.4.4. Consider the map $\varphi : \mathcal{A}_\Delta \rightarrow \Gamma_\Delta$ given by $\varphi([a]) = (\lambda((x_n)_{n \geq 0}), u)$ if $u \neq 0$ and $\lambda((x_n)_{n \geq 0})$ if $u = 0$ (i.e. if $[a]$ is a unit).

We now show that φ is a groupoid homomorphism. Let $[a]$ and $[b]$ be two composable elements in \mathcal{A}_Δ . Since \mathcal{A}_Δ is a groupoid (lemma 6.7.8) we have $[a] = [a][b]^{-1}[b] = [ab^*b]$ and $[b] = [a][a]^{-1}[b] = [a^*ab]$. In other words given representatives a and b , we have $a \sim a'$ and $b \sim b'$ with $a'_n = a_n b_n^* b_n$ and $b'_n = a_n a_n^* b_n$. So we can choose the representatives a and b such that for all $n \geq 0$ we have $a_n = \chi_{A_n} \Gamma^u \chi_{B_n}$ and $b_n = \chi_{B_n} \Gamma^u \chi_{C_n}$ with $A_n \subset \mathfrak{p}_n^{-1}(x_n)$, $B_n = \Gamma^{-u} A_n \subset \mathfrak{p}_n^{-1}(y_n)$. Hence $\lambda((y_n)_{n \geq 0}) = \Gamma^{-u} \lambda((x_n)_{n \geq 0})$. We thus have $\varphi([a][b]) = \varphi([ab]) = (\lambda((x_n)_{n \geq 0}), u + v)$. On the other hand $\varphi([a])\varphi([b]) = (\lambda((x_n)_{n \geq 0}), u) (\lambda((y_n)_{n \geq 0}), v)$. But since $\lambda((y_n)_{n \geq 0}) = \Gamma^{-u} \lambda((x_n)_{n \geq 0})$ this last product is $(\lambda((x_n)_{n \geq 0}), u + v)$ and therefore we have $\varphi([a][b]) = \varphi([a])\varphi([b])$. We also clearly have $\varphi([a]^{-1}) = (\lambda((y_n)_{n \geq 0}), -u)$ which is $(\lambda((x_n)_{n \geq 0}), u)^{-1}$ and therefore $\varphi([a]^{-1}) = \varphi([a])^{-1}$.

We now show that φ is an isomorphism. Consider two distinct elements $[a]$ and $[b]$ in \mathcal{A}_Δ . For two representatives a and b write $a_n = \chi_{A_n} \Gamma^u \chi_{A'_n}$ and $b_n = \chi_{B_n} \Gamma^v \chi_{B'_n}$. Since $a \not\sim b$ there exists $m \geq 0$ for which $A_m \cap B_m = \emptyset$ and hence $A_n \cap B_n = \emptyset$ for all $n \geq m$. If $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ are the sequences in $\varprojlim(\mathcal{B}_n, f_n)$ associated with a and b respectively, we then have $x_n \neq y_n$ for all $n \geq m$ and thus $\lambda((x_n)_{n \geq 0}) \neq \lambda((y_n)_{n \geq 0})$

in Ξ_Δ so that $\varphi([a]) \neq \varphi([b])$. This proves that φ is injective. Now consider an arrow (ξ, u) in Γ_Δ^1 . Let $(x_n)_{n \geq 0} = \lambda^{-1}(\xi)$ and for all $n \geq 0$ let $a_n = \chi_{A_n} \Gamma^u \chi_{\Gamma^{-u} A_n}$ with $A_n = \mathfrak{p}_n^{-1}(x_n)$. Define $a = (a_n)_{n \geq 0}$, and we have $\varphi([a]) = (\xi, u)$. This proves that φ is surjective. The inverse of φ is defined as follows. Consider $[a] \in \mathcal{A}_\Delta$ and write again $a_n = \chi_{A_n} \Gamma^u \chi_{A'_n}$ for a representative a . Let $\varphi([a]) = (\xi, u)$ with $\xi = \lambda((x_n)_{n \geq 0})$. Define $\varphi^{-1}(\xi, u) = [\tilde{a}]$ with $\tilde{a}_n = \chi_{\tilde{A}_n} \Gamma^u \chi_{\tilde{A}'_n}$ with $\tilde{A}_n = \mathfrak{p}_n^{-1}(x_n) = \Gamma^u \tilde{A}'_n$. For all $n \geq 0$ we have $A_n \subset \tilde{A}_n$ and therefore $\tilde{a}_n \leq a_n$ or equivalently $\tilde{a}_n = \tilde{a}_n \tilde{a}_n^* a_n$. Hence we have $\tilde{a} = \tilde{a} \tilde{a}^* a$, so that $[\tilde{a}] = [\tilde{a} \tilde{a}^* a] = [\tilde{a}][\tilde{a}]^{-1}[a] = [a]$. This proves that the inverse is well defined: $\varphi^{-1} \circ \varphi([a]) = [a]$. \square

REFERENCES

- [1] ANDERSON, J. E. and PUTNAM, I. F., “Topological invariants for substitution tilings and their associated C^* -algebras,” *Ergodic Theory Dynam. Systems*, vol. 18, no. 3, pp. 509–537, 1998.
- [2] ATIYAH, M. F. and HIRZEBRUCH, F., “Vector bundles and homogeneous spaces,” in *Proc. Sympos. Pure Math., Vol. III*, pp. 7–38, Providence, R.I.: American Mathematical Society, 1961.
- [3] ATIYAH, M. F. and SEGAL, G. B., “The index of elliptic operators. II,” *Ann. of Math. (2)*, vol. 87, pp. 531–545, 1968.
- [4] ATIYAH, M. F. and SINGER, I. M., “The index of elliptic operators on compact manifolds,” *Bull. Amer. Math. Soc.*, vol. 69, pp. 422–433, 1963.
- [5] ATIYAH, M. F. and SINGER, I. M., “The index of elliptic operators. I,” *Ann. of Math. (2)*, vol. 87, pp. 484–530, 1968.
- [6] ATIYAH, M. F. and SINGER, I. M., “The index of elliptic operators. III,” *Ann. of Math. (2)*, vol. 87, pp. 546–604, 1968.
- [7] BELLISSARD, J., HERRMANN, D. J. L., and ZARROUATI, M., “Hulls of aperiodic solids and gap labeling theorems,” in *Directions in mathematical quasicrystals*, vol. 13 of *CRM Monogr. Ser.*, pp. 207–258, Providence, RI: Amer. Math. Soc., 2000.
- [8] BELLISSARD, J., IOCHUM, B., SCOPPOLA, E., and TESTARD, D., “Spectral properties of one-dimensional quasi-crystals,” *Comm. Math. Phys.*, vol. 125, no. 3, pp. 527–543, 1989.
- [9] BELLISSARD, J., “ K -theory of C^* -algebras in solid state physics,” in *Statistical mechanics and field theory: mathematical aspects (Groningen, 1985)*, vol. 257 of *Lecture Notes in Phys.*, pp. 99–156, Berlin: Springer, 1986.
- [10] BELLISSARD, J., “Gap labelling theorems for Schrödinger operators,” in *From number theory to physics (Les Houches, 1989)*, pp. 538–630, Berlin: Springer, 1992.
- [11] BELLISSARD, J., BENEDETTI, R., and GAMBAUDO, J.-M., “Spaces of tilings, finite telescopic approximations and gap-labeling,” *Comm. Math. Phys.*, vol. 261, no. 1, pp. 1–41, 2006.

- [12] BELLISSARD, J., BOVIER, A., and GHEZ, J.-M., “Gap labelling theorems for one-dimensional discrete Schrödinger operators,” *Rev. Math. Phys.*, vol. 4, no. 1, pp. 1–37, 1992.
- [13] BELLISSARD, J., CONTENSO, E., and LEGRAND, A., “ K -théorie des quasi-cristaux, image par la trace: le cas du réseau octogonal,” *C. R. Acad. Sci. Paris Sér. I Math.*, vol. 326, no. 2, pp. 197–200, 1998.
- [14] BELLISSARD, J., KELLENDONK, J., and LEGRAND, A., “Gap-labelling for three-dimensional aperiodic solids,” *C. R. Acad. Sci. Paris Sér. I Math.*, vol. 332, no. 6, pp. 521–525, 2001.
- [15] BELLISSARD, J., LIMA, R., and TESTARD, D., “Almost periodic Schrödinger operators,” in *Mathematics + physics. Vol. 1*, pp. 1–64, Singapore: World Sci. Publishing, 1985.
- [16] BENEDETTI, R. and GAMBAUDO, J.-M., “On the dynamics of \mathbb{G} -solenoids. Applications to Delone sets,” *Ergodic Theory Dynam. Systems*, vol. 23, no. 3, pp. 673–691, 2003.
- [17] BLACKADAR, B., *K-theory for operator algebras*, vol. 5 of *Mathematical Sciences Research Institute Publications*. Cambridge: Cambridge University Press, second ed., 1998.
- [18] BOREL, A. and SERRE, J.-P., “Impossibilité de fibrer un espace euclidien par des fibres compactes,” *C. R. Acad. Sci. Paris*, vol. 230, pp. 2258–2260, 1950.
- [19] BOTT, R. and TU, L. W., *Differential forms in algebraic topology*, vol. 82 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, 1982.
- [20] BRATTELI, O., “Inductive limits of finite dimensional C^* -algebras,” *Trans. Amer. Math. Soc.*, vol. 171, pp. 195–234, 1972.
- [21] CLARK, A. and SADUN, L., “When shape matters: deformations of tiling spaces,” *Ergodic Theory Dynam. Systems*, vol. 26, no. 1, pp. 69–86, 2006.
- [22] CLARO, F. and WANNIER, G., “Closure of bands for bloch electrons in a magnetic field,” *Phys. Stat. Sol. B*, vol. 88, pp. K147–K151, 1978.
- [23] COBURN, L. A., MOYER, R. D., and SINGER, I. M., “ C^* -algebras of almost periodic pseudo-differential operators,” *Acta Math.*, vol. 130, pp. 279–307, 1973.
- [24] CONNES, A., “An analogue of the Thom isomorphism for crossed products of a C^* -algebra by an action of \mathbf{R} ,” *Adv. in Math.*, vol. 39, no. 1, pp. 31–55, 1981.
- [25] CONNES, A., “A survey of foliations and operator algebras,” in *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, vol. 38 of *Proc. Sympos. Pure Math.*, pp. 521–628, Providence, R.I.: Amer. Math. Soc., 1982.

- [26] CONNES, A., “Sur la théorie non commutative de l’intégration,” in *Algèbres d’opérateurs (Sém., Les Plans-sur-Bex, 1978)*, vol. 725 of *Lecture Notes in Math.*, pp. 19–143, Berlin: Springer, 1979.
- [27] CONNES, A., *Géométrie non commutative*. Paris: InterEditions, 1990.
- [28] CONNES, A., *Noncommutative geometry*. San Diego, CA: Academic Press Inc., 1994.
- [29] DAVIS, J. F. and KIRK, P., *Lecture notes in algebraic topology*, vol. 35 of *Graduate Studies in Mathematics*. Providence, RI: American Mathematical Society, 2001.
- [30] EVANS, L. C., *Partial differential equations*, vol. 19 of *Graduate Studies in Mathematics*. Providence, RI: American Mathematical Society, 1998.
- [31] FORREST, A. H. and HUNTON, J., “The cohomology and K -theory of commuting homeomorphisms of the Cantor set,” *Ergodic Theory Dynam. Systems*, vol. 19, no. 3, pp. 611–625, 1999.
- [32] FORREST, A., HUNTON, J., and KELLENDONK, J., “Topological invariants for projection method patterns,” *Mem. Amer. Math. Soc.*, vol. 159, no. 758, pp. x+120, 2002.
- [33] GÄHLER, F., HUNTON, J., and KELLENDONK, J., “Torsion in tiling homology and cohomology.” At arXiv.com [math-ph/0505048](https://arxiv.org/abs/math-ph/0505048), 2005.
- [34] GAMBAUDO, J.-M. and MARTENS, M., “Algebraic topology for minimal Cantor sets,” *Ann. Henri Poincaré*, vol. 7, no. 3, pp. 423–446, 2006.
- [35] GHYS, É., “Laminations par surfaces de Riemann,” in *Dynamique et géométrie complexes (Lyon, 1997)*, vol. 8 of *Panor. Synthèses*, pp. ix, xi, 49–95, Paris: Soc. Math. France, 1999.
- [36] GROTHENDIECK, A., “La théorie des classes de Chern,” *Bull. Soc. Math. France*, vol. 86, pp. 137–154, 1958.
- [37] GRÜNBAUM, B. and SHEPHARD, G. C., *Tilings and patterns*. New York: W. H. Freeman and Company, 1987.
- [38] HATCHER, A., *Algebraic Topology*. Cambridge: Cambridge University Press, 2002. Available online at <http://www.math.cornell.edu/hatcher/#ATI>.
- [39] HATCHER, A., *Spectral Sequences in Algebraic Topology*. 2004. Available online at <http://www.math.cornell.edu/hatcher/#SSAT>.
- [40] HIRZEBRUCH, F., *Neue topologische Methoden in der algebraischen Geometrie*. Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 9, Berlin: Springer-Verlag, 1956.

- [41] HIRZEBRUCH, F., “Arithmetic genera and the theorem of Riemann-Roch for algebraic varieties,” *Proc. Nat. Acad. Sci. U. S. A.*, vol. 40, pp. 110–114, 1954.
- [42] HOFSTADTER, D., “Energy levels and wave functions of bloch electrons in a rational or irrational magnetic field,” *Phys. Rev. B*, vol. 14, pp. 2239–2249, 1976.
- [43] HUNTON, J., “An introduction to spectral sequences,” July 2005. talk delivered at Banff Research Station.
- [44] JOHNSON, R. and MOSER, J., “The rotation number for almost periodic potentials,” *Comm. Math. Phys.*, vol. 84, no. 3, pp. 403–438, 1982.
- [45] KASPAROV, G. G., “The operator K -functor and extensions of C^* -algebras,” *Izv. Akad. Nauk SSSR Ser. Mat.*, vol. 44, no. 3, pp. 571–636, 719, 1980.
- [46] KASPAROV, G. G., “Equivariant KK -theory and the Novikov conjecture,” *Invent. Math.*, vol. 91, no. 1, pp. 147–201, 1988.
- [47] KELLENDONK, J., “Noncommutative geometry of tilings and gap labelling,” *Rev. Math. Phys.*, vol. 7, no. 7, pp. 1133–1180, 1995.
- [48] KELLENDONK, J., “Integer groups of coinvariants associated to octagonal tilings,” in *Operator algebras and their applications (Waterloo, ON, 1994/1995)*, vol. 13 of *Fields Inst. Commun.*, pp. 155–169, Providence, RI: Amer. Math. Soc., 1997.
- [49] KELLENDONK, J., “Pattern-equivariant functions and cohomology,” *J. Phys. A*, vol. 36, no. 21, pp. 5765–5772, 2003.
- [50] KELLENDONK, J. and PUTNAM, I. F., “The Ruelle-Sullivan map for actions of \mathbb{R}^n ,” *Math. Ann.*, vol. 334, no. 3, pp. 693–711, 2006.
- [51] KOSZUL, J.-L., “Sur les opérateurs de dérivation dans un anneau,” *C. R. Acad. Sci. Paris*, vol. 225, pp. 217–219, 1947.
- [52] LAGARIAS, J. C., “Geometric models for quasicrystals I. Delone sets of finite type,” *Discrete Comput. Geom.*, vol. 21, no. 2, pp. 161–191, 1999.
- [53] LAGARIAS, J. C., “Geometric models for quasicrystals. II. Local rules under isometries,” *Discrete Comput. Geom.*, vol. 21, no. 3, pp. 345–372, 1999.
- [54] LAGARIAS, J. C. and PLEASANTS, P. A. B., “Repetitive Delone sets and quasicrystals,” *Ergodic Theory Dynam. Systems*, vol. 23, no. 3, pp. 831–867, 2003.
- [55] LERAY, J., “Structure de l’anneau d’homologie d’une représentation,” *C. R. Acad. Sci. Paris*, vol. 222, pp. 1419–1422, 1946.
- [56] MASSEY, W. S., “Exact couples in algebraic topology. I, II,” *Ann. of Math. (2)*, vol. 56, pp. 363–396, 1952.

- [57] MASSEY, W. S., “Exact couples in algebraic topology. III, IV, V,” *Ann. of Math. (2)*, vol. 57, pp. 248–286, 1953.
- [58] MAY, J. P., *A concise course in algebraic topology*. Chicago Lectures in Mathematics, Chicago, IL: University of Chicago Press, 1999.
- [59] MCCLEARY, J., *A user’s guide to spectral sequences*, vol. 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press, second ed., 2001.
- [60] MEYER, Y., *Algebraic numbers and harmonic analysis*. Amsterdam: North-Holland Publishing Co., 1972. North-Holland Mathematical Library, Vol. 2.
- [61] MOORE, C. C. and SCHOCHET, C., *Global analysis on foliated spaces*, vol. 9 of *Mathematical Sciences Research Institute Publications*. New York: Springer-Verlag, 1988. With appendices by S. Hurder, Moore, Schochet and Robert J. Zimmer.
- [62] MOSER, J., “An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum,” *Comment. Math. Helv.*, vol. 56, no. 2, pp. 198–224, 1981.
- [63] PENROSE, R., “Pentaplexity: a class of nonperiodic tilings of the plane,” *Math. Intelligencer*, vol. 2, no. 1, pp. 32–37, 1979/80.
- [64] PIMSNER, M. and VOICULESCU, D., “Exact sequences for K -groups and Ext-groups of certain cross-product C^* -algebras,” *J. Operator Theory*, vol. 4, no. 1, pp. 93–118, 1980.
- [65] PIMSNER, M. V., “Ranges of traces on K_0 of reduced crossed products by free groups,” in *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, vol. 1132 of *Lecture Notes in Math.*, pp. 374–408, Berlin: Springer, 1985.
- [66] QUEFFÉLEC, M., *Substitution dynamical systems—spectral analysis*, vol. 1294 of *Lecture Notes in Mathematics*. Berlin: Springer-Verlag, 1987.
- [67] RADIN, C., “The pinwheel tilings of the plane,” *Ann. of Math. (2)*, vol. 139, no. 3, pp. 661–702, 1994.
- [68] RADIN, C., *Miles of tiles*, vol. 1 of *Student Mathematical Library*. Providence, RI: American Mathematical Society, 1999.
- [69] RENAULT, J., *A groupoid approach to C^* -algebras*, vol. 793 of *Lecture Notes in Mathematics*. Berlin: Springer, 1980.
- [70] RIEFFEL, M. A., “Morita equivalence for operator algebras,” in *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, vol. 38 of *Proc. Sympos. Pure Math.*, pp. 285–298, Providence, R.I.: Amer. Math. Soc., 1982.

- [71] SADUN, L., “Tiling spaces are inverse limits,” *J. Math. Phys.*, vol. 44, no. 11, pp. 5410–5414, 2003.
- [72] SADUN, L., “Explicit computation of the cohomology for the chair tiling,” July 2005. talk delivered at Banff Research Station.
- [73] SADUN, L., “Pattern-equivariant cohomology with integer coefficients.” At arXiv.com math.DS/0602066, 2006.
- [74] SADUN, L. and WILLIAMS, R. F., “Tiling spaces are Cantor set fiber bundles,” *Ergodic Theory Dynam. Systems*, vol. 23, no. 1, pp. 307–316, 2003.
- [75] SCHOCHET, C., “Topological methods for C^* -algebras. I. Spectral sequences,” *Pacific J. Math.*, vol. 96, no. 1, pp. 193–211, 1981.
- [76] SERRE, J.-P., “Homologie singulière des espaces fibrés. Applications,” *Ann. of Math. (2)*, vol. 54, pp. 425–505, 1951.
- [77] SHECHTMAN, D., BLECH, I., GRATIAS, D., and CAHN, J., “Metallic phase with long range orientational order and no translational symmetry,” *Phys. Rev. Lett.*, vol. 53, pp. 1951–1953, 1984.
- [78] SINGER, I. M., “Future extensions of index theory and elliptic operators,” in *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, pp. 171–185. Ann. of Math. Studies, No. 70, Princeton, N.J.: Princeton Univ. Press, 1971.
- [79] SPANIER, E. H., *Algebraic topology*. New York: McGraw-Hill Book Co., 1966.
- [80] STEENROD, N. E., “Homology with local coefficients,” *Ann. of Math. (2)*, vol. 44, pp. 610–627, 1943.
- [81] ŠUBIN, M. A., “Spectral theory and the index of elliptic operators with almost-periodic coefficients,” *Uspekhi Mat. Nauk*, vol. 34, no. 2(206), pp. 95–135, 1979.
- [82] SÜTÖ, A., “The spectrum of a quasiperiodic Schrödinger operator,” *Comm. Math. Phys.*, vol. 111, no. 3, pp. 409–415, 1987.
- [83] THOULESS, D., KOHMOTO, M., NIGHTINGALE, M., and DEN NIJS, M., “Quantum hall conductance in two dimensional periodic potential,” *Phys. Rev. Lett.*, vol. 49, pp. 405–408, 1982.
- [84] VAN ELST, A., “Gap-labelling theorems for Schrödinger operators on the square and cubic lattice,” *Rev. Math. Phys.*, vol. 6, no. 2, pp. 319–342, 1994.
- [85] WILLIAMS, R. F., “Expanding attractors,” *Inst. Hautes Études Sci. Publ. Math.*, no. 43, pp. 169–203, 1974.

VITA

Jean P.X. Savinien
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160
Office: (404) 894-0614
Mobile: (404) 375-7112
savinien@math.gatech.edu

Education

Georgia Institute of Technology, Atlanta. PhD. in Mathematics expected 2008.

Ecole Normale Supérieure, Paris, France. M.S. in Theoretical Physics, (French *Diplôme d'Etudes Approfondies*) June 2002.

Université Paris Sud, Orsay, France M.S. in Fundamental Physics, June 2000; *Magistère*, June 2002.

Ecole Nationale des Ponts et Chaussées, Paris, France M.S. in Applied Mathematics and Engineering, June 2002.

Academic Positions

Graduate Research Assistant. Spring terms from 2003 to 2007, Summers 2004, 06, and 07.

Graduate Teaching Assistant. Other terms from Fall 2002 to current.

Awards

Junior Research Fellow, E. Schrödinger Institute, Vienna, Austria, Summer 2006.

Research Interests

Non commutative Geometry, Tiling Theory, Mathematical Physics, Physics of Aperiodic Solids.

Publications

[1] J. Savinien, J. Taron, R. Tarrach, “Triviality of GHZ Operators of Higher Spin”, *J. Phys. A: Math. Gen.* **33** (2000) L493-L495.

[2] J. Savinien, J. Bellissard, “A Spectral Sequence for the K -theory of Tiling Spaces”, E. Schrödinger Institute preprint 1919 (2007). Submitted to *Ergodic Theory and Dynamical Systems*. Available online at <http://arxiv.org/abs/0705.2483>.

Talks given

University of Kansas, Colloquium Lawrence, KS. “Aperiodic Tilings and K -theory” (50 min.), January 2008.

Aperiodic Order 2007 CIRM, Marseille, France. “A Spectral Sequence for the K -theory of Tiling Spaces” (40 min.), September 2007.

Georgia Institute of Technology, Noncommutative Geometry Seminar “Cohomology of Tilings” (50 min), October 2005.

Academic Services

Organizer of the Research Horizons Seminar From Fall 2005 to current. The School of Mathematics graduate students weekly seminar.

Research Experience

Ecole Normale Supérieure, Cachan, France. Spring 2002. Trainee at “Laboratoire de Photonique Quantique et Moléculaire”. Advisor: Professor Joseph Zyss.

University of Barcelona, Barcelona, Spain. Spring-Summer 2000. Trainee at “Instituto de Física Teórica” Advisor: Professor Rolf Tarrach.

Université Paris Sud, Orsay, France Summer 1999. Trainee at “Laboratoire de Physique Théorique et Modèles Statistiques”. Advisor: Professor Nicolas Pavloff.

Conferences

Aperiodic Order 2007 September 2007, CIRM, Marseille, France.

Groupoids and Stacks in Physics and Geometry March 2007, Institut Henri Poincaré, Paris, France.

Noncommutative Geometry and Operator Algebra NCGOA07 (May 2007), NCGOA05 (May 2005), NCGOA03 (May 2003), Vanderbilt University, Nashville, USA.

Analysis and Probability in Quantum Physics August 2006, PASI, Santiago, Chile.

Complex Quantum and Classical System and Effective Equations Summer 2006, E. Schrödinger Institute, Vienna, Austria.

Boltzman Centennial June 2006, E. Schrödinger Institute, Vienna, Austria.

Field Institute Summer School on Operator Algebras June 2005, University of Ottawa, ON Canada.

Noncommutative Geometry: from Quantum Physics to Motives April 2005, Vanderbilt University, Nashville, USA.

K-theory and Noncommutative Geometry May-July 2004, Institut Henri Poicaré, Paris, France.

Quantum Information Processing: Theory and Experiment July 2000, Benasque Center for Science, Benasque, Spain.

Teaching Experience

As a Graduate Teaching Assistant I have taught recitation classes. I have also been lead instructor, and I have had the responsibility of supervising graders, writing my own exams, determining grades and creating different activities for the class. Since 2002, I have taught the following classes:

Calculus III (Lead instructor in Fall 2005, and Fall 2007.) Calculus of functions of several variables and vector valued functions, multiple integration, vector analysis; taught from a traditional text.

Calculus II (Lead instructor in Summer 2005.) Differential and integral calculus, infinite and power series, basic differential equations, linear algebra; taught from a traditional text.