DETERMINATION OF OPTIMUM REJECT ALLOWANCES IN MANUFACTURING

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Sang Hoon Chang

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DETERMINATION OF OPTIMUM REJECT

ALLOWANCES IN MANUFACTURING

Approved:

Dr. H. M. Wadsworth

Prof. E. C. Franklin

Dr. E. R. Immel

Date Approved by Chairman: May 20, 1963

FOREWORD

The writer was introduced to the reject allowance problem when he took a course, I.E. 306, Production Control, in Winter Quarter, 1962, at Georgia Institute of Technology. The instructor of this course, Professor E. C. Franklin, gave the writer the initial encouragement to investigate the possibility of any improvement in the method of solving this problem, and this study is the outgrowth of the resulting investigation.

Throughout the period of this research Dr. H. M. Wadsworth, the thesis advisor, has given continuous encouragement as well as much of his time for the consultations and criticisms to help improve the contents and the presentation of the thesis material. Dr. E. R. Immel, of the School of Mathematics, has given some helpful suggestions with regard to the mathematical phase of the problem.

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Finally, the writer wishes to take this opportunity to express his gratitude and affection to the late Mr. Kendall Weisiger, the former chairman of the Atlanta Rotory Educational Foundation, who initially made it possible for the writer to come to study in the United States.

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SUMMARY

This research is devoted to the task of developing a reject allowance model which may be conveniently applied to determine the optimum reject allowance in a manufacturing situation which has the following characteristics. First, the item to be produced is a custom-order type. Second, the producer and the consumer agree upon a 100% quality inspection plan. Third, the specific order quantity must be produced without an overage or shortage allowance. Fourth, reliable estimates of the production and cost parameters are available. Fifth, the process is a sequence of Bernoulli trials.

In this manufacturing situation, the expected total relevant cost of production is expressed as

$$E_n(Z) = C_1 \sum_{x=y+1}^{n} p_n(x) + C_o \sum_{x=0}^{y-1} (y-x)p_n(x) + C_s \sum_{x=0}^{n} xp_n(x)$$

where

- y: reject allowance quantity.
- n: run size, $n = r + y$.
- x: the number of defective units resulted in the run.
- p: process fraction defective.
- C_1: the shortage lump sum loss. This is the total loss associated with a shortage which includes the resetup cost for a make-up run when $x > y$.
- C_o: the overage unit cost: the standard unit cost minus the scrap value per good unit produced in excess of the order quantity.

C_s: the spoilage unit cost: the standard unit cost minus the average scrap value per defective unit resulted in the run.

$p_n(x)$: the probability of having x defectives when n is started in a production run. When the binomial probability law is applied,

$$p_n(x) = b(x;n,p) = \binom{n}{x} p^x(1-p)^{n-x}$$

Z: the cost function.

$E_n(Z)$: the expected total relevant cost of production when the run size is n.

The optimum reject allowance is that corresponds to the following conditions.

$$E_n(Z) \leq E_{n-1}(Z)$$

$$E_n(Z) \leq E_{n+1}(Z)$$

The exact model for the optimum reject allowance is derived as

$$\frac{p_n(y+1) - \sum_{x=0}^{y+1} \left[p_n(x) - p_{n+1}(x)\right]}{\sum_{x=0}^{y} p_n(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y+1} (y+1-x)\left[p_n(x) - p_{n+1}(x)\right]} \leq \frac{C_o}{C_1}$$

$$\leq \frac{p_{n-1}(y) - \sum_{x=0}^{y} \left[p_{n-1}(x) - p_n(x)\right]}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y} (y-x)\left[p_{n-1}(x) - p_n(x)\right]}$$

From the exact model, an approximate model is derived as

$$\frac{p_n(y+1)}{\sum_{x=0}^{y} p_n(x) + \frac{C_s}{C_o} p} \leq \frac{C_o}{C_1} \leq \frac{p_{n-1}(y)}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p}$$

The exact model is capable of precisely predicting the optimum reject allowance for all values of p, n, C_1, C_o, and C_s, whenever the optimum solution exists with respect to the economic objective as defined in this problem. The approximate model produces, in general, a compatible solution to that obtained by the exact model. The approximation improves with smaller p, larger r, larger $\frac{C_s}{C_o}$, and smaller $\frac{C_o}{C_1}$. The exact model requires a substantial amount of calculations, but the approximate model is relatively simple and convenient to use in applications. Once acquainted with the procedure of using the model, it would take only two or three minutes of computations to obtain a solution by the approximate model.

For practical solutions of the reject allowance problem, the approximate model is recommended, because it gives a relatively reliable solution and is convenient to use. An exception to this rule might be the situation arising when the cost parameters have relatively large economic values.

The models are developed for the specific manufacturing situation which may be an ideal and hypothetical case. However, approximately similar situations do exist in industry. In those situations, the underlying techniques and the models presented in this thesis may be useful in obtaining the proper solutions.

CHAPTER I

INTRODUCTION

The General Problem

This study is an investigation of the problem of determining the proper reject allowance for a manufacturing operation. A typical example of the problem shall be described first. Suppose a manufacturer receives an order from a laboratory for, say, 20 units of machine parts to be used for an experimental instrument. The order specifies exactly 20 units, no more nor less. The manufacturer estimates the process fraction defective at 0.05, and must decide how many units he should start in production. Assume, at the end of the run, the finished products are to be inspected 100% according to the quality requirements. If less than 20 good units are produced, the deficient quantity must be replenished by make-up runs. If more than 20 good units are produced, the surplus units would have to be scrapped. There is a risk of both shortage and overage losses no matter what starting quantity is used.

In general, the reject allowance problem is limited to a certain type of manufacturing situation which has the following characteristics.

a. The item to be produced is a custom-order type, which is not usually stocked in inventory.

b. The producer and the consumer agree upon a 100% quality inspection plan.

c. The specific order quantity must be produced without overage or shortage allowances: i.e., either overage or shortage create substantial economic loss to the producer.

Background

In recent years, solutions to the reject allowance problem have been proposed by others in the Journal of Industrial Engineering and other publications. Research toward a solution at Cornell University was sponsored by the National Science Foundation [7]. The methods of solution found in the literature during the course of this research may be classified as the exact and approximate methods.

The exact methods usually require laborious computational procedures for obtaining the solution. Sometimes, the economic gain expected by utilizing the method may be small, and the excessive effort required in computations may result in the loss of any net gain realized by the solution. On the other hand, the approximate methods are relatively convenient to use, but are often not reliable in application. As a result of this, there is a need for developing a method which is both reliable and convenient to use in practice.

The Specific Problem

There are a few factors which must be investigated carefully to develop an optimum reject allowance model. First, the determination of the optimum reject allowance requires a probabilistic approach since the reject occurrence is a random phenomenon. Second, there are four parameters which need to be estimated before we can start working on the problem; these parameters are the process fraction defective p, the shortage lump sum cost C_1, the overage unit cost C_o, and the spoilage unit cost C_s. We assume, for the purpose of this research, reliable estimates of these parameters are available. This assumption is important, because the effectiveness of the model depends on the reliability of the estimates of these parameters. In addition to these four parameters, the order quantity r has an effect on the amount of the allowance to be provided.

The Study Procedure

A reject allowance model which can exactly predict the optimum allowance shall be derived first in this study. The effectiveness of this model is evaluated in terms of the expected total relevant cost of production. From the exact model, an approximate model shall be derived next, and the degree of approximation errors considered. The question of "how does the model behave for various values of p, r, C_1, C_o, and C_s?" shall be discussed in Chapter III and IV.

CHAPTER II

LITERATURE SURVEY

In recent years a few analytical methods for determining an optimum reject allowance have been proposed by Bowman and others. The earliest relevant publication known to this author was by Bowman [1] in 1955. Bowman assumes that a long run historical record is available which shows the number of units started and the number of good units finished in past production runs. The ratio of start to good is calculated for each production run. From the large aggregate of these ratios, the relative frequency is calculated for convenient ratio intervals, and a cumulative frequency distribution is obtained. Since the cumulative frequency distribution is shown as the function of the ratio of start to good, this distribution is good for all ranges of run sizes. Bowman assumes that the ratio of start to good is invariant with the run size, and develops a marginal cost analysis for determining the optimum reject allowance.

A similar approach, which requires an empirical frequency distribution, is also presented by Schlaifer [12, Chapter 8]. The advantage of utilizing the empirical frequency distribution, as seen in the approach by Bowman and Schlaifer, is the convenience of needing only one distribution for all ranges of run sizes; whereas, a different distribution would be needed for a different run size, if, for example, the binomial distribution would be used instead. The disadvantage of the approach is that it is usually not practical to assume the availability of long run historical records for production runs. The compilation of the frequency distribution also would incur additional clerical expenses.

Under certain conditions, it would be more practical to assume that the fraction defective p is constant regardless of the run size, and apply the binomial probability distribution to depict the reject occurrences in this problem. Based on this assumption, Bowman and Fetter [2] presented a second method in which the Poisson approximation to the binomial distribution was used for computing the optimum reject allowance. The difficulty in utilizing the binomial distribution in this problem is, as already mentioned, that different distributions are needed for different run sizes. The binomial probability, $b(x;n,p)$, where x is the reject quantity realized when n units are started with the fraction defective p, has a distinct distribution for each n, and n varies according to variation in the reject allowance quantity.

The approach to the problem by Bowman and Fetter [2] is that of balancing the expected incremental cost and the expected incremental gain. They define the expected incremental cost as the increase in expected overage loss due to the additional unit in the allowance, and the incremental gain as the reduction in expected shortage loss. They say that the optimum reject allowance balances the expected incremental cost and the expected incremental gain.

Their formula is

$$\frac{\sum_{x=0}^{y} p_{n+1}(x)}{\sum_{x=0}^{y-1} p_n(x)} = \frac{C_1 + C_2}{C_1}$$

where C_1 and C_2 are the shortage lump sum loss and overage unit loss respectively, $p_{n+1}(x)$ is the Poisson approximation to the binomial distribution $b(x;n+1, p)$, $p_n(x)$ is the Poisson approximation to $b(x;n, p)$, x is the reject quantity and y is the optimum reject allowance. The limitations of their formula are twofold: First, it often fails to predict the proper allowance. It seems that their expression for the expected incremental cost is mathematically inaccurate, which may account for the failure mentioned. Second, it is often inconvenient to use this formula with the probability table. Both of these limitations shall be considered and an improved formula will be presented in this thesis.

Llewellyn [10] presents a procedure in which the Poisson approximation to the binomial distribution is used. The formula is

$$\sum_{x=y+1}^{n} p_n(x) - \sum_{x=y+2}^{n} p_{n+1}(x) < \frac{C_2}{C_1}$$

The symbols have the same meaning as before. Llewellyn's formula is convenient to use, and it can predict the proper optimum allowances for processes which have small fraction defectives and the cost ratio $C_2/C_1 < 0.01$. However, if the cost ratio $C_2/C_1 > 0.10$ and or p is large, his formula fails in many cases. He considers only the "potential high losses" which seems to over-simplify the situation for large cost ratios and large fraction defectives. In both studies, by Bowman and Fetter [2], and by Llewellyn [10], the expected increase in scrap loss due to the additional increase in the reject allowance is not mentioned; however, when the fraction defective is large and order quantity is small, the increase in the reject allowance causes a significant increase in the expected spoilage loss, which cannot be ignored when the unit cost is high.

Bryant [3] gives an example of the reject allowance problem. He sets up an equation expressing the expected total cost as the sum of the setup cost, the total unit cost, and the expected shortage cost. The Poisson or the Normal approximation to the binomial distribution is suggested for solving the equation. He suggests a step by step trial and error procedure beginning with the selection of some arbitrary number for the reject allowance. Franklin [5] gives an illustration of such a solution. Bryant's equation is mathematically correct; however, it would require a time-consuming procedure before a solution can be determined. Furthermore, his equation is appropriate only for the case when $C_o = C_s$.

Goode and Saltzman [6] presented an exact method using the binomial distribution. Later, they [7] presented a modified method. They set up an equation of the expected total relevant cost in which the subsequent shortage cost is also included. The subsequent shortage cost is incurred when a shortage is realized in the second make-up run, in the third make-up run, and so forth. The limitation of Goode and Saltzman's procedure is that it takes a prohibitive amount of computational effort without the aid of an electronic computer. Their procedure may be simplified by ignoring the subsequent shortage cost in the equation. In the illustrative case examples in [6] and [7], compatible solutions may be obtained when the subsequent shortage cost is completely ignored.

Levitan [9] of I.B.M. Corporation has developed a sequential algorithm with a rigorous mathematical treatment. His method is developed primarily for the use of an electronic computer. The feasibility of the use of the electronic computer in the reject allowance problem is not considered in this thesis.

To summarize, there are two general requirements inherent in the reject allowance problem. First, the model must precisely predict the proper optimum reject allowance within the limitations of assumptions made. Second, the use of the model must be convenient to apply. The procedures by Bowman and Fetter [2] and Llewellyn [10] are relatively convenient in application, but are often unreliable when p, $\frac{C_o}{C_1}$, and $\frac{C_s}{C_o}$ are large. On the other hand, the procedures by Goode and Saltzman [6] [7] are reliable, but are inconvenient to use, especially when r and p are large. This research is devoted to the task of developing a model which is both reliable and convenient, and is flexible for reasonably large values of p, r, $\frac{C_o}{C_1}$, and $\frac{C_s}{C_o}$.

CHAPTER III

THE OPTIMUM REJECT ALLOWANCE MODEL

The Reject Probability Distribution

The reject probability distribution may be obtained in two different ways with two different assumptions. First, the empirical frequency distribution may be used with the assumption that the ratio of start to good is invariant with the run size. Bowman [1] and Schlaifer [14] have presented optimum reject allowance models using this approach. The advantages and the disadvantages of this type of approach are discussed briefly in Chapter II.

The second method is to approximate the probability distribution by a theoretical distribution; for example by the binomial distribution which assumes that the fraction defective is constant regardless of the run size. This type of approach will be used in this thesis.

The fraction defective may be considered constant if the process satisfies two conditions: first, the conditions under which each unit is produced must be independent, and second, the process must be stable. If, for example, tool wear and slight changes in machine settings increase the fraction defective occurring in the later portion of a run, the process would not be stable. For another example, if the lot to lot variation in machinability of raw material is sufficient to cause the reject quantity to vary from lot to lot, the trials would not be independent. When it is known that the process is stable and the conditions under which each unit is produced are independent for a production run, it may be assumed that we are dealing with a sequence of Bernoulli trials. This assumption enables us to use the binomial distribution to compute the reject probability for the production run.

The Minimum Expected Total Relevant Cost of Production

Suppose an order quantity r is to be produced in a production run for which there is a known process fraction defective p. The problem is to determine an optimum reject allowance quantity y to be added to r. The starting quantity, $n=r+y$, would then be such that the expected total relevant production costs would be a minimum. We assume that 100% quality inspection is given at the conclusion of the run, and this inspection is 100% effective. As the result of the inspection, x defective units are known to be produced among the n units started. Therefore, $n-x$ units are deliverable to the customer. According to the contract, if $n-x < r$, the deficient quantity should be replenished by the make-up run; on the other hand, if $n-x > r$, the customer will not pay the full price for the units in excess of r. Equation (1) gives, for this situation, the expected total relevant cost of production when n is the starting quantity. The meaning of the symbols used in this discussion is shown in Appendix II, Glossary of Symbols.

$$E_n(z) = C_1 \sum_{x=y+1}^{n} p_n(x) + C_o \sum_{x=0}^{y} (y-x)\, p_n(x) + C_s \sum_{x=0}^{n} x p_n(x) \qquad (1)$$

In the expression above, the first term is the expected shortage cost. The shortage cost is assumed to be a lump sum loss which is independent of the number of units short. The second term is the expected overage cost, which is proportional to the number of excess units produced. If no reject allowance is provided, no overage cost would be incurred. The third term is the expected spoilage cost. The expression

$$\sum_{x=0}^{n} x \, p_n(x)$$

is the expectation of x, given a discrete probability function $p_n(x)$. If $p_n(x)$ is the binomial function, then

$$\sum_{x=0}^{n} x \, p_n(x) = np$$

Substituting the relationship above into (1),

$$E_n(Z) = C_1 \sum_{x=y+1}^{n} p_n(x) + C_o \sum_{x=0}^{y} (y-x) \, p_n(x) + C_s \, np \qquad (2)$$

The expected total relevant cost when n-1 is the starting quantity and y-1 is the reject allowance is

$$E_{n-1}(Z) = C_1 \sum_{x=y}^{n-1} p_{n-1}(x) + C_o \sum_{x=0}^{y-1} (y-1-x) \, p_{n-1}(x) + C_s(n-1)p \qquad (3)$$

The expected total relavant cost when n+1 is the starting quantity and y+1 is the reject allowance is

$$E_{n+1}(Z) = C_1 \sum_{x=y+2}^{n+1} p_{n+1}(x) + C_o \sum_{x=0}^{y+1} (y+1-x)p_{n+1}(x) + C_s(n+1)p \qquad (4)$$

If $E_n(Z)$ is the minimum expected total relevant cost with respect to n, then the following relationship exists.

$$E_n(Z) \leq E_{n-1}(Z) \tag{5}$$

$$E_n(Z) \leq E_{n+1}(Z)$$

<u>The Optimum Reject Allowance Model</u>

From (2), (3), (4), and (5), we shall derive the reject allowance model. First, consider

$$E_n(Z) \leq E_{n-1}(Z) \tag{5.1}$$

Substituting (2) and (3) into (5.1),

$$C_1 \sum_{x=y+1}^{n} p_n(x) + C_o \sum_{x=0}^{y} (y-x)\, p_n(x) + C_s np$$

$$\leq C_1 \sum_{x=y}^{n-1} p_{n-1}(x) + C_o \sum_{x=0}^{y-1} (y-1-x) p_{n-1}(x) + C_s(n-1)p$$

$$C_o \left[\sum_{x=0}^{y} (y-x)\, p_n(x) - \sum_{x=0}^{y-1} (y-1-x) p_{n-1}(x) \right] + C_s p$$

$$\leq C_1 \left[\sum_{x=y}^{n-1} p_{n-1}(x) - \sum_{x=y+1}^{n} p_n(x) \right] \tag{5.1.1}$$

$$C_o \left\{ \sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y-1} (y-x) \left[p_{n-1}(x) - p_n(x) \right] \right\}$$

$$\leq C_1 \left\{ p_{n-1}(y) - \sum_{x=0}^{y} \left[p_{n-1}(x) - p_n(x) \right] \right\}$$

$$\frac{C_o}{C_1} \leq \frac{p_{n-1}(y) - \sum_{x=0}^{y} \left[p_{n-1}(x) - p_n(x)\right]}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y} (y-x) \left[p_{n-1}(x) - p_n(x)\right]} = A \qquad (6)$$

From (5.1.1) we can also obtain the following expression.

$$C_o \left\{ \sum_{x=0}^{y-1} p_n(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y-1} (y-1-x) \left[p_{n-1}(x) - p_n(x)\right] \right\}$$

$$\leq C_1 \left\{ p_n(y) - \sum_{x=0}^{y-1} \left[p_{n-1}(x) - p_n(x)\right] \right\}$$

$$\frac{C_o}{C_1} \leq \frac{p_n(y) - \sum_{x=0}^{y-1} \left[p_{n-1}(x) - p_n(x)\right]}{\sum_{x=0}^{y-1} p_n(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y-1} (y-1-x) \left[p_{n-1}(x) - p_n(x)\right]} = C \qquad (6.1)$$

The ratio C on the left side of (6.1) is equivalent to A on the left side of (6). This is illustrated in Example 1.

Example 1. Consider a situation where $r = 2$, $p = 0.05$, $C_s = C_o$, $y = 4$, and $p_n(x) = b(x;n,p)$. The ratios on the left sides of (6) and (6.1) are as follows. The sample calculation is shown in Appendix 1.

$$A = \frac{p_{n-1}(y) - \sum_{x=0}^{y} \left[p_{n-1}(x) - p_n(x)\right]}{\sum_{x=0}^{y-1} p_{n-1}(x) + p - \sum_{x=0}^{y} (y-x) \left[p_{n-1}(x) - p_n(x)\right]}$$

$$= \frac{0.0209 - 0.0011}{0.9742 + 0.05 - 0.0489} = \frac{0.0198}{0.9753} \doteq 0.0203$$

$$C = \frac{p_n(y) - \sum_{x=0}^{y-1} \left[p_{n-1}(x) - p_n(x)\right]}{\sum_{x=0}^{y-1} p_n(x) + p - \sum_{x=0}^{y-1} (y-1-x) \left[p_{n-1}(x) - p_n(x)\right]}$$

$$= \frac{0.0238 - 0.0040}{0.9702 + 0.05 - 0.0447} = \frac{0.0198}{0.9755} \doteq 0.0203$$

Next consider

$$E_n(Z) \leq E_{n+1}(Z) \tag{5.2}$$

Substituting (2) and (4) into (5.2),

$$C_1 \sum_{x=y+1}^{n} p_n(x) + C_o \sum_{x=0}^{y} (y-x)\, p_n(x) + C_s np$$

$$\leq C_1 \sum_{x=y+2}^{n+1} p_{n+1}(x) + C_o \sum_{x=0}^{y+1} (y+1-x) p_{n+1}(x) + C_s(n+1)p$$

$$C_1\left[\sum_{x=y+1}^{n} p_n(x) - \sum_{x=y+2}^{n+1} p_{n+1}(x)\right] \leq C_o\left[\sum_{x=0}^{y+1}(y+1-x)p_{n+1} - \sum_{x=0}^{y}(y-x)p_n(x)\right] + C_s p \tag{5.2.1}$$

$$C_1 \left\{ p_n(y+1) - \sum_{x=0}^{y+1} \left[p_n(x) - p_{n+1}(x) \right] \right\} \le C_o \left\{ \sum_{x=0}^{y} p_n(x) \right.$$

$$\left. + \frac{C_s}{C_o} p - \sum_{x=0}^{y+1} (y+1-x) \left[p_n(x) - p_{n+1}(x) \right] \right\}$$

$$B = \frac{p_n(y+1) - \sum_{x=0}^{y+1} \left[p_n(x) - p_{n+1}(x) \right]}{\sum_{x=0}^{y} p_n(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y+1} (y+1-x) \left[p_n(x) - p_{n+1}(x) \right]} \le \frac{C_o}{C_1} \qquad (7)$$

From (5.2.1) an alternative expression of (7) is obtained.

$$\frac{p_{n+1}(y+1) - \sum_{x=0}^{y} \left[p_n(x) - p_{n+1}(x) \right]}{\sum_{x=0}^{y} p_{n+1}(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y} (y-x) \left[p_n(x) - p_{n+1}(x) \right]} \le \frac{C_o}{C_1} \qquad (7.1)$$

Combining (6) and (7),

$$B \le \frac{C_o}{C_1} \le A \qquad (8)$$

where

$$A = \frac{p_{n-1}(y) - \sum_{x=0}^{y} \left[p_{n-1}(x) - p_n(x) \right]}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y} (y-x) \left[p_{n-1}(x) - p_n(x) \right]}$$

$$B = \frac{p_n(y+1) - \sum_{x=0}^{y+1}\left[p_n(x) - p_{n+1}(x)\right]}{\sum_{x=0}^{y} p_n(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y+1}(y+1-x)\left[p_n(x) - p_{n+1}(x)\right]}$$

Inequality (8) is the optimum reject allowance model. Since, (6.1) and (7.1) are equivalent expressions to (6) and (7) respectively, we can obtain an expression equivalent to (8) by combining (6.1) and (7.1). However, we will use inequality (8) as our exact model. The reason for this choice will be explained later when the approximation model is derived.

This model was constructed based on the assumptions that C_1, C_o, C_s, and p are known and constant, and the reject probability follows the binomial distribution. The reject allowance quantity y satisfying (8) is the optimum reject allowance. Furthermore, when y is the optimum reject allowance, $n = r + y$ is the optimum starting quantity. The use of the model is illustrated in the following example.

Example 2 Assume the following data are known, and we want to determine the optimum y.

$$r = 20,\ p = 0.05,\ C_s = C_o,\ C_o = 0.05C_1\ .$$

A and B are calculated according to inequation (8) and tabulated in Table 1, below. For similar sample calculations see Appendix 1.

Table 1

y	n	A	B
1	21	0.917	0.262
2	22	0.262	0.0725
3	23	0.0725	0.0204
4	24	0.0204	0.0046
5	25	0.0046	

It should be noted on Table 1 that A for y is B for y - 1. In other words, there is no need to calculate B separately from A. Using the model in (8), we next find the value of y which satisfies

$$B \leq 0.05 \leq A$$

In Table 1, at y = 3,

$$A = 0.0725, \text{ and } B = 0.0204 .$$

Therefore, the optimum reject allowance is 3.

The validity of the result obtained above may be illustrated by showing that y = 3 satisfies the condition in (5).

$$E_{23}(Z) = C_o\left[\frac{C_1}{C_o}\sum_{x=4}^{n} p_{23}(x) + \sum_{x=0}^{3}(3-x)\; p_{23}(x) + (23)(0.05)\right]$$

$$= C_o\left[(20)(0.0258) + 1.8816 + 1.15\right] \doteq 3.45\; C_o$$

$$E_{22}(Z) = C_o\left[(20)(0.0948) + 1.0116 + 1.10\right] \doteq 4.00\; C_o$$

$$E_{24}(Z) = C_o\left[(20)(0.0060) + 2.8071 + 1.20\right] \doteq 4.13\; C_o$$

$$\therefore\; E_{23}(Z) < E_{22}(Z)$$

$$E_{23}(Z) < E_{24}(Z)$$

The Monotone Decreasing Characteristic of the Ratio

In Table 1 we observe that A and B are monotone decreasing with respect to y. As long as the ratios are monotone decreasing, the model of (8) can be applied in determining the reject allowance. This is so, because the condition in (5) is based on the fundamental assumption that as n is increased, the shortage loss will be expected to decrease and the overage and scrap unit losses increase; therefore, A and B should be monotone decreasing to satisfy this condition. However, the ratios are not always monotone decreasing with respect to y, for $y \geq 0$. Consider the next example.

Example 3 From the following data, Table 2 is constructed.

$$r = 20,\ p = 0.20,\ C_s = C_o,\ \frac{C_o}{C_1} \leq 0.446$$

Table 2

y	n	A
1	21	0.220
2	22	0.393
3	23	0.446
4	24	0.397
5	25	0.276
6	26	0.185
7	27	0.136
8	28	0.075
9	29	0.044
10	30	0.026
11	31	0.013
12	32	0.007

In Table 2 we observe that the ratio is increasing for y when $0 < y \leq 3$, and decreasing for y when $3 < y$. The model of (8) is not applicable in the region where the ratio is increasing, when $n \leq 23$ in this example. If the ratio $\frac{C_1}{C_o}$ is greater than 0.446, the inequality

$$B \leq \frac{C_o}{C_1} \leq A$$

cannot be met. Then a question arises "What is the decision rule when $0.446 < \frac{C_o}{C_1}$?" In this situation, the rate of reduction in the expected shortage lump sum loss is smaller than the rate of increase in the expected overage and scrap unit losses as the allowances are added. Therefore, the proper reject allowance should be zero. A ratio this high, however, is probably a rare case in a physical situation.

CHAPTER IV

APPLICATION OF THE MODEL

The Approximate Model

The optimum reject allowance model of (8) requires relatively lengthy computations. A simpler model which would give a close approximation to the inequation (8), and is more convenient to use in practical applications shall be discussed. Consider the approximations A' or A" of ratio A (Eq. 6),

$$A' = \frac{p_{n-1}(y)}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o}\, p} \qquad A'' = \frac{p_{n-1}(y)}{\sum_{x=0}^{y-1} p_n(y)}$$

Also consider the approximations C' and C" of ratio C (Eq. 6.1). Recall that the ratios A and C are equivalent to each other.

$$C' = \frac{p_n(y)}{\sum_{x=0}^{y-1} p_n(x) + \frac{C_s}{C_o}\, p} \qquad C'' = \frac{p_n(y)}{\sum_{x=0}^{y-1} p_n(y)}$$

We must decide which of the four expressions, A', A", C', and C" gives the closest approximation to the true ratio A or C. This shall be considered in the following example.

Example 4 Using the same data as in Example 1, the approximate ratios are computed as below.

$$A' = \frac{p_{n-1}(y)}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p} = \frac{0.0209}{0.9742 + .05} = \frac{0.0209}{1.0242} \doteq 0.0204$$

$$A'' = \frac{p_{n-1}(y)}{\sum_{x=0}^{y-1} p_{n-1}(x)} = \frac{0.0209}{0.9742} \doteq 0.0215$$

$$C' = \frac{p_n(y)}{\sum_{x=0}^{y-1} p_n(x) + \frac{C_s}{C_o} p} = \frac{0.0238}{0.9702 + 0.05} = \frac{0.0238}{1.0202} \doteq 0.0233$$

$$C'' = \frac{p_n(y)}{\sum_{x=0}^{y-1} p_n(x)} = \frac{0.0238}{0.9702} \doteq 0.0247$$

Recall from Example 1

$$A = B \doteq 0.0203$$

The ratios are calculated for other values of y and tabulated in Table 3.

Table 3

y	n	A=C	A'	A"	C'	C"
1	21	0.917	0.920	1.05	0.965	1.09
2	22	0.262	0.259	0.276	0.276	0.295
3	23	0.0725	0.0762	0.0802	0.0835	0.0885
4	24	0.0204	0.0204	0.0215	0.0233	0.0247
5	25	0.0046	0.0048	0.0050	0.0057	0.0060

In Example 4 we observe that among the four approximate ratios considered, A' uniformly gives the closest approximation to the true ratio A. In general, the approximation by A" or C" would accompany relatively large errors of approximation. When p is large or $\frac{C_s}{C_o}$ p is large, the error incurred by eliminating $\frac{C_s}{C_o}$ p in the denominator of the ratio would become significant. Since we assumed that $\frac{C_s}{C_o}$ p are known and constant, the degree of computational convenience in using A" or C" is not much better than using A' or C'. For this reason, we eliminate A" and C" from further consideration, and focus our attention in selecting either A" or C" to be an appropriate approximation of A or C. Recall that A is equivalent to C, but A' is not equivalent to C'.

We need to determine which is smaller, $|A-A'|$ or $|A-C'|$. In Example 4, we observe that

$$|A-A'| < |A-C'|$$

for all y. In order to study the effect that the large fraction defective may have on the approximations, the ratios A, A', and C' are calculated for p = .10, p = .20, p = .30, and p = .40, and shown in Table 4. We observe in Table 4 that in the monotone decreasing region of the ratio,

$$|A-A'| < |A-C'| .$$

There seems to be no indication that C' might be a better approximation than A' in the region of our interest. Therefore, it may be reasonable to say that, if a small degree of approximation error is tolerable, A' gives reasonably close approximations to A for all applicable ranges of y and p.

Table 4

Ratios A, A' and C'

$$A = \frac{p_{n-1}(y) - \sum_{x=0}^{y} \left[p_{n-1}(x) - p_n(x)\right]}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{p} (y-x) \left[p_{n-1}(x) - p_n(x)\right]}$$

$$A' = \frac{p_{n-1}(y)}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p} \qquad C' = \frac{p_n(y)}{\sum_{x=0}^{y-1} p_n(x) + \frac{C_s}{C_o} p}$$

$n = r + y, \qquad r = 20, \qquad C_s = C_o$

p	y	n	A	A'	C'
0.15	1	21	0.917	0.920	0.965
	2	22	0.262	0.259	0.276
	3	23	0.0725	0.0762	0.0835
	4	24	0.0204	0.0204	0.0233
	5	25	0.0046	0.0048	0.0057
0.10	1	21	1.21	1.22	1.26
	2	22	0.610	0.610	0.640
	3	23	0.300	0.303	0.365
	4	24	0.133	0.132	0.146
	5	25	0.0547	0.0565	0.0633
	6	26	0.0222	0.0222	0.0264
	7	27	0.0081	0.0082	0.0100

(Continued)

Table 4 (Continued)

p	y	n	A	A'	C'
0.20	3*	23	0.446	0.500	0.490
	4	24	0.397	0.411	0.422
	5	25	0.276	0.296	0.316
	6	26	0.185	0.200	0.218
	7	27	0.136	0.128	0.143
	8	28	0.075	0.078	0.090
	9	29	0.044	0.046	0.054
	10	30	0.026	0.026	0.032
	11	31	0.013	0.014	0.017
	12	32	0.007	0.007	0.009
0.30	6*	26	0.236	0.298	0.290
	7	27	0.231	0.275	0.279
	8	28	0.198	0.233	0.245
	9	29	0.162	0.183	0.202
	10	30	0.137	0.144	0.160
	11	31	0.101	0.107	0.122
	12	32	0.069	0.077	0.089
	13	33	0.048	0.054	0.064
	14	34	0.033	0.036	0.047
	15	35	0.022	0.024	0.030
	16	36	0.015	0.016	0.020
	17	37	0.009	0.010	0.012
	18	38	0.006	0.006	0.008
0.40	12*	32	0.139	0.186	0.190
	13	33	0.123	0.165	0.174
	14	34	0.102	0.141	0.169
	15	35	0.088	0.117	0.147
	16	36	0.073	0.096	0.110
	17	37	0.065	0.077	0.090
	18	38	0.050	0.061	0.071
	19	39	0.038	0.047	0.057
	20	40	0.026	0.036	0.043
	21	41	0.022	0.026	0.033
	22	42	0.017	0.019	0.025
	23	43	0.011	0.014	0.019
	24	44	0.008	0.010	0.613
	25	45	0.006	0.007	0.009

*The ratios for smaller y are not shown here, because they do not satisfy the condition of (5). See Example 3.

Furthermore, Table 4 illustrates that the approximation becomes better for smaller p. When p is fixed, the approximation is better for larger y.

By analogy, as A' is selected to be the approximation of A, so B' may represent B as the closest approximation, where

$$B' = \frac{p_n(y+1)}{\sum_{x=0}^{y} p_n(x) + \frac{C_s}{C_o} p}$$

and we have the approximate reject allowance model in (9).

$$B' = \frac{p_n(y+1)}{\sum_{x=0}^{y} p_n(x) + \frac{C_s}{C_o} p} \leq \frac{C_o}{C_1} \leq \frac{p_{n-1}(y)}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p} = A' \qquad (9)$$

The Use of the Approximate Model

Example 5   Suppose the optimum reject allowance is to be determined from the following data.

$$r = 20, \qquad p = 0.05, \qquad C_s = C_o \qquad C_1 = 20\ C_o$$

Using the relationship in (9), an approximate solution is obtained by observation of the corresponding section in Table 4. From Table 4, $A' = 0.0762$ at $y = 3$, and $B' = 0.0204$ at $y = 3$. Substituting these ratios into (9),

$$0.0204 < \frac{C_o}{C_1} < 0.0762$$

Since $\frac{C_o}{C_1}$ = 0.05 satisfies the relationship above, y = 3 is the optimum reject allowance in this case. It is to be noted that the ratio B' at y is the ratio A' at y + 1.

It is not necessary to construct a table such as Table 4 to obtain the solution when the approximate model is used. A simple manipulation on a binomial probability table, such as that published by Harvard University Press (8), is sufficient. Once acquainted with the procedure, it would require only two or three minutes to obtain the solution. Next, we shall consider a situation where the approximate solution would differ from the exact solution.

Example 6 Suppose the data are the same as in Example 5 with one exception: $\frac{C_o}{C_1}$ = 0.075. From Table 4, we observe that the exact solution to be y = 2, whereas the approximate solution is y = 3. The expected total relevant cost for y = 3, and y = 2 are as follows.

$$E_{23}(Z) = C_o\left[\frac{C_1}{C_o}\sum_{x=4}^{n} p_{23}(x) + \sum_{x=0}^{3} (3-x)\, p_{23}(x) + (23)(0.05)\right]$$

$$= C_o\left[\frac{1}{0.075}(0.0258) + 1.8816 + 1.15\right] \doteq 3.38\ C_o$$

$$E_{22}(Z) = C_o\left[\frac{1}{0.075}(0.0948) + 1.0116 + 1.10\right] \doteq 3.37\ C_o$$

The expected total relevant cost is comparable as shown above, and both y = 3 and y = 2 are optimum solutions. Hence, in this case, the "seemingly apparent" discrepancy in the solutions obtained by (8) and by (9) is insignificant with regard to the economic objective of the problem.

In Table 4, we observe that the approximation error is large for large fraction defectives. This is due to the relative increase in the magnitude of the denominator of A' as p becomes large. When the magnitude of the denominator of A' is large, the effect of neglecting the last term in the denominator of A becomes less significant. For the same reason, the approximation error becomes small when the ratio $\frac{C_s}{C_o}$ becomes large, which indicates that the effect of the spoilage unit loss may become significant when $\frac{C_s}{C_o}$ p is large.

The Poisson Approximation

The binomial probability table which has been used in this study is Tables of the Cumulative Binomial Probability Distribution by Harvard University Press [8]. There are two major limitations in using this table. First, it gives the cumulative probability only. Hence, we need to calculate the individual probability to be used in the numerator of the ratios A and A'. Second, it does not give all values of n. For example, in the interval of n = 500 and n = 1,000, the Table gives the probability for n = 500, 550, 600, ..., in steps of 50 units. Therefore, interpolation is required to find the probability for any n not shown in the Table. The Table gives the probability for every n when $n \leq 50$. In the interval of n = 50 and n = 100, it gives n = 50, 52, 54, ..., in steps of two units. When n is larger than 100, this table is not convenient to use.

As the binomial probability distribution may be approximated by the Poisson probability distribution when $p \leq .10$ or $np < 5$ as shown in

Duncan [4], we may use the Poisson probability table in this situation. Poisson's Exponential Binomial Limit by Molina [11] gives two tables. In his Table I, the individual probability is given, and in Table II, the cumulative probability. Molina gives the probability for $np \leq 100$ and for all x in this region. Hence, Molina's table is more convenient to use than the table published by Harvard University [8]. The disadvantage of using Molina's table is its introduction of further approximation errors.

Table 5 shows the ratios A' as calculated by using the Poisson approximation. The comparison with the ratios as calculated by using the binomial distribution indicates that, if small errors of approximation are tolerable, the use of the Poisson approximation may be satisfactory in calculating the ratios when $p \leq .10$ or $np < 5$.

The Ratio $\frac{C_o}{C_1}$

In using models (8) and (9), we assume that the parameters p, r, C_1, C_o, and C_s are known, and our problem is to determine the optimum y. We have already discussed the effect of p, r, and the ratio $\frac{C_s}{C_o}$ on the solution. Now we shall consider the ratio $\frac{C_o}{C_1}$.

In Table 4, we observe two characteristics of the ratio $\frac{C_o}{C_1}$. First, when p is small, a slight shift in the ratio $\frac{C_o}{C_1}$ has a negligible effect on the selection of y; however, when p is large, a slight shift in the ratio $\frac{C_o}{C_1}$ has a large effect on y. Second, when p is small, the reject allowance may be provided even for large ratio of $\frac{C_o}{C_1}$; however, when p is large, the problem cannot be considered for large ratios of $\frac{C_o}{C_1}$. For example, when p is 0.05, $\frac{C_o}{C_1}$ can vary from zero to about

Table 5

Poisson Approximation to the Binomial

Distribution in Computing A'

$$A' = \frac{p_{n-1}(y)}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p}$$

$n = r + y$ $r = 20$

$C_s = C_o$

p	2	n	(n-1)p	A Binomial	A' Binomial	A' Poisson
0.05	1	21	1.00	0.917*	0.920*	0.88
	2	22	1.05	0.262	0.259	0.25
	3	23	1.10	0.073	0.076	0.073
	4	24	1.15	0.020	0.020	0.022
	5	25	1.20	0.005	0.005	0.006
0.10	1	21	2.0	1.21	1.22	1.17
	2	22	2.1	0.610	0.610	0.59
	3	23	2.2	0.300	0.303	0.29
	4	24	2.3	0.933	0.132	0.13
	5	25	2.4	0.055	0.057	0.060
	6	26	2.5	0.022	0.022	0.026
	7	27	2.6	0.008	0.008	0.011

*These ratios are copied from Table 4, and shown here for comparison with the Poisson approximation.

0.90; however, when p is 0.40, $\frac{C_o}{C_1}$ can vary from zero to only about 0.13. The second characteristic has been partially explained in Example 3.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The reject allowance problem which is treated in this thesis is applicable only to a certain type of manufacturing situations whose characteristics are listed as follows.

1. The item in production is a custom-order type, which is usually not stocked in inventory.

2. The producer and the consumer agree upon a 100% quality inspection plan.

3. The specific order quantity must be prdduced without the overage or shortage allowance, i.e. both overage and shortage create substantial economic loss to the producer.

4. Reliable estimates of the parameters p, C_1, C_o, and C_s are available.

5. The process is a sequence of Bernoulli trials.

In this manufacturing situation, the expected total relevant cost of production is expressed as

$$E_n(Z) = C_1 \sum_{x=y+1}^{n} p_n(x) + C_o \sum_{x=0}^{y-1} (y-x)\, p_n(x) + C_s \sum_{x=0}^{n} x\, p_n(x) \qquad (1)$$

If y is the optimum reject allowance, and correspondingly n the optimum starting quantity, then the optimality is defined if the following conditions are satisfied.

$$E_n(Z) \leq E_{n-1}(Z) \tag{5}$$

$$E_n(Z) \leq E_{n+1}(Z)$$

Under the assumptions described above, the optimum reject allowance model has been derived as

$$B \leq \frac{C_o}{C_1} \leq A \tag{8}$$

where

$$A = \frac{p_{n-1}(y) - \sum_{x=0}^{y} \left[p_{n-1}(x) - p_n(x)\right]}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y} (y-x) \left[p_{n-1}(x) - p_n(x)\right]}$$

$$B = \frac{p_n(y+1) - \sum_{x=0}^{y+1} \left[p_n(x) - p_{n+1}(x)\right]}{\sum_{x=0}^{y} p_n(x) + \frac{C_s}{C_o} p - \sum_{x=0}^{y+1} (y+1-x) \left[p_n(x) - p_{n+1}(x)\right]}$$

$$A \text{ at } y \equiv B \text{ at } y-1$$

The model shown in (8) is the exact model. From (8), an approximate model is derived as

$$B' \leq \frac{C_o}{C_1} \leq A' \tag{9}$$

$$A' = \frac{p_{n-1}(y)}{\sum_{x=0}^{y-1} p_{n-1}(x) + \frac{C_s}{C_o} p}$$

$$B' = \frac{p_n(y+1)}{\sum_{x=0}^{y} p_n(x) + \frac{C_s}{C_o} p}$$

$$A' \text{ at } y \equiv B' \text{ at } y - 1$$

The two models above may be used for all values of p, n, C_1, C_o, and C_s, as long as the condition in (5) is satisfied by the solution. This means that the models are applicable only in the region where the ratios A and B, or A' and B', are monotone decreasing. When the ratios A and B or A' and B' are not monotone decreasing, the addition of the reject allowance is not appropriate with respect to the economic objective defined in this problem. Consistent with the assumptions made in this study, the exact model in (8) is capable of predicting the optimum reject allowance. In general, the approximate solution by the use of the model in (9) produces a compatible solution to that obtained by (8). The approximation improves with smaller p, larger r, larger $\frac{C_s}{C_o}$, and smaller $\frac{C_o}{C_1}$. The exact model requires a substantial amount of calculations, but the approximate model is relatively simple and convenient to use in applications.

Recommendations

The approximate model in (9) is recommended for practical solutions of the reject allowance problem, because it gives a relatively reliable solution and is convenient to use. An exception to this rule might be the situation arising when the cost parameters C_1, C_o, and C_s have relatively large economic values. In this case the lengthy calculations required in (8) to find an exact solution may be justified.

A binomial probability table is needed to carry out the calculations. The table by Harvard University [8] may be used when $n \leq 100$. In the situation where $p \leq .10$ or $np < 5$, it is more convenient to use Molina's table [11] for the Poisson approximation to the binomial distribution. Molina's table gives both the individual terms and the cumulative terms, which makes it especially suitable for the use with the models. If $n > 100$ and $p > .10$ or $np > 5$, the binomial table by Harvard University Press [8] may be used with necessary interpolations. In such situation, the normal approximation to the binomial may be used instead. An example of the method of using the normal approximation is explained in Franklin [5].

The models are developed for the specific manufacturing situation described in this thesis. This situation is an ideal and hypothetical case. However, the writer believes that approximately similar situations do exist in industry. In those situations, the underlying techniques and the models presented in this thesis may be useful in obtaining the proper solutions. The type of industry which seems to have the closest resemblance to this situation is that using job-shop manufacturing methods.

Sometimes an under or over shipment allowance is provided in the order contract. The consumer then agrees to accept the shipment of finished product as long as the shortage or overage is within a specified limit, say 2% of the order quantity, as is the normal sales policy at the Small Motor Division, Westinghouse Corporation. In this situation, another model should be developed for determining the optimum reject allowance. Another point of interest for future study is the development of a method for determining the size of this shipment allowance. The optimum shipment allowance with respect to the economic objective of the problem may be determined using a similar study procedure to that used in this thesis.

APPENDICES

APPENDIX I

SAMPLE CALCULATION FOR EXAMPLE 1

x	$\sum_{x}^{23} p_{23}(x)$*	$\sum_{0}^{x-1} p_{23}(x)$	$p_{23}(x)$	$\sum_{x}^{24} p_{24}(x)$*	$\sum_{0}^{x-1} p_{24}(x)$	$p_{24}(x)$	$p_{23}(x)-p_{24}(x)$
0	1.0000	0.0000	0.3074	1.0000	0.0000	0.2920	0.0154
1	0.6926	0.3074	0.3720	0.7080	0.2920	0.3688	0.0032
2	0.3206	0.6794	0.2154	0.3392	0.6608	0.2233	-0.0079
3	0.1052	0.8948	0.0794	0.1152	0.8841	0.0861	-0.0067
4	0.0258	0.9742	0.0209	0.0298	0.9702	0.0238	-0.0029
5	0.0049	0.9951	0.0041	0.0060	0.9940	0.0050	-0.0009
6	0.0008	0.9992	0.0007	0.0010	0.9990	0.0009	-0.0002
7	0.0001	0.9979	0.0001	0.0001	0.9999	0.0001	0.0000
8	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000

$$p_{n-1}(y) = p_{23}(4) = 0.0209$$

$$\sum_{x=0}^{y} \left[p_{n-1}(x) - p_n(x)\right] = \sum_{x=0}^{4} \left[p_{23}(x) - p_{24}(x)\right] = 0.0154 + 0.0032 - 0.0079 - 0.0067 - 0.0029 = 0.0011$$

$$\sum_{x=0}^{y-1} p_{n-1}(x) = \sum_{x=0}^{3} p_{23}(x) = 0.9742$$

$$\sum_{x=0}^{y} (y-x)\left[p_{n-1}(x) - p_n(x)\right] = \sum_{x=0}^{4} (4-x)\left[p_{23}(x) - p_{24}(x)\right]$$

$$= (4)(-0.0154) + 3(0.0032) - 2(0.0032) - (1)(0.0067)$$

$$= 0.0489$$

*From Harvard University Press [8].

$$A = \frac{p_{n-1}(y) - \sum_{x=0}^{y} \left[p_{n-1}(x) - p_n(x)\right]}{\sum_{x=0}^{y-1} p_{n-1}(x) + p - \sum_{x=0}^{y} (y-x) \left[p_{n-1}(x) - p_n(x)\right]}$$

$$= \frac{p_{23}(4) - \sum_{x=0}^{4} \left[p_{23}(x) - p_{24}(x)\right]}{\sum_{x=0}^{3} p_{23}(x) + p - \sum_{x=0}^{4} (4-x) \left[p_{23}(x) - p_{24}(x)\right]}$$

$$= \frac{0.0209 - 0.0011}{0.9753 + 0.05 - 0.0489} = \frac{0.0198}{0.9753} \doteq 0.0203$$

APPENDIX II

GLOSSARY OF SYMBOLS

r: order quantity.

y: reject allowance quantity.

n: run size, $n = r + y$.

x: the number of defective units resulted in the run.

p: process fraction defective.

C_1: the shortage lump sum loss. This is the total loss associated with a shortage which includes the resetup cost for a make-up run when $x > y$.

C_o: the overage unit cost: the standard unit cost minus the scrap value per good unit produced in excess of the order quantity.

C_s: the spoilage unit cost: the standard unit cost minus the average scrap value per defective unit resulted in the run.

C_2: the cost parameter when $C_o = C_s$.

$p_n(x)$: the probability of having x defectives when n is started in a production run. When the binomial probability law is applied,

$$p_n(x) = b(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

Z: the cost function.

$E_n(Z)$: the expected total relevant cost of production when the run size is n.

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