APPROXIMATING THE CIRCUMFERENCE OF 3-CONNECTED CLAW-FREE GRAPHS

A Thesis
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Mathematics

Georgia Institute of Technology
December 2008

APPROXIMATING THE CIRCUMFERENCE OF 3-CONNECTED CLAW-FREE GRAPHS

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ACKNOWLEDGEMENTS

I want to thank my committee, my family, and my close friends for their patience and support. Members of my committee, thank you for your time, understanding, and helpful comments. Dr. Yu, thank you so much for all the advice over the years and the time and effort you put in to ultimately help me graduate. Stephen and Mitch, thank you for being such good friends and for pealing me away from my work on occasion. Mom, Dad, Shannon, Paul, and Chris, thank you so much for your love, support, and patience. This was a very long and arduous task and I could not have done it without everyone's help. Thank you.

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SUMMARY

Jackson and Wormald [16] show that every 3-connected $K_{1,d}$-free graph, on n vertices, contains a cycle of length at least $\frac{1}{2}n^{\gamma_d}$ where $\gamma_d = (\log_2 6 + 2\log_2(2d+1))^{-1}$. For $d = 3$, $\gamma_d \sim 0.122$. Improving this bound, we prove that if G is a 3-connected claw-free graph on $n \geq 6$ vertices, then there exists a cycle C in G such that $|E(C)| \geq \alpha n^{\gamma} + 5$, where $\gamma = \log_3 2$ and $\alpha \geq 1/7$ is a constant.

To do this, we instead prove a stronger theorem that requires the cycle to contain two specified edges. We then use Tutte decomposition to partition the graph and then use the inductive hypothesis of our theorem to find paths or cycles in the different parts of the decomposition.

CHAPTER I

INTRODUCTION

Jackson and Wormald [16] show that every 3-connected $K_{1,d}$-free graph, on n vertices, contains a cycle of length at least $\frac{1}{2}n^{\gamma_d}$ where $\gamma_d = (\log_2 6 + 2\log_2(2d+1))^{-1}$. For $d = 3$, $\gamma_d \sim 0.122$. In this thesis, we improve this bound to $\alpha n^{\gamma} + 5$, where $\gamma = \log_3 2 \sim 0.631$ and $\alpha \geq 1/7$ is a constant.

The methods used in this thesis, if used more exhaustively, may have more profound an impact. We may be able to improve the exponent in this bound further (perhaps as high as $\log_6 4 \sim 0.774$). However, if we can obtain an exponent greater than $\log_2(1+\sqrt{5})-1 \sim 0.69$, we could then extend our result to also improve the current best bound for 3-connected cubic graphs by Jackson [13]. This will be discussed more thoroughly in the historical background section of the introduction.

In section 1.1, we introduce the notation needed. In many regards it is similar to the notation found in Diestel's text on Graph Theory [9]. As a result, readers familiar with this notation may choose to peruse the earlier part of this section. However, the latter half of this section includes less standard concepts, such as Tutte decomposition. In particular, the notation for Tutte decomposition tends to vary from author to author depending on its specific use. We borrow the notation of [16], which is fairly standard, and then develop it further to better serve our purposes.

In section 1.2, we discuss the history of our problem – the search for long cycles in 3-connected graphs. We conclude this section with the statement of the main theorem.

In section 1.3, we discuss the organization of the rest of the thesis.

1.1 Notation

A *graph* G is defined by a pair of sets $V(G), E(G)$ such that the elements of $E(G)$ are the 2-element subsets of $V(G)$. For notational simplicity, we simply write $e = uv = vu$. We refer to the elements of $V(G)$ as the *vertices* of G and the elements of $E(G)$ as the *edges* of G. If $|V(G)| = n$, then G is said to be of *order* n. In defining a graph G, we will often combine $V(G)$ and $E(G)$ into a single set, where the size of each element makes it clear whether it is an edge or a vertex. For example $G = \{x, y, xy\}$ has $V(G) = \{x, y\}$ and $E(G) = \{xy\}$.

$x, y \in V(G)$ are said to be *adjacent* if $xy \in E(G)$. For an edge $e = uv \in E(G)$, let $V(e) = \{u, v\}$. $e, f \in E(G)$ such that $e \neq f$ are said to be *adjacent* if $V(e) \cap V(f) \neq \emptyset$. $v \in V(G)$ is said to be *incident* with $e \in E(G)$ if $v \in V(e)$. The *degree* of a vertex v in G is the number of edges incident with v. A graph G is said to be *cubic* if for all $v \in V(G)$, the degree of v is 3. We denote $N_G(v) = \{x \in V(G) : vx \in E(G)\}$ (or simply $N(v)$) as the *set of neighbors* of v in G.

If G, G' are graphs, we say G' is a *subgraph* of G (i.e. $G' \subseteq G$) if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. If $G' \subseteq G$ and $\{xy \in E(G) : x, y \in V(G')\} = E(G')$, then G' is an *induced subgraph* of G. Alternately, we say $V(G')$ *induces* G' in G.

A *path* is a non-empty graph of the form $V(P) = \{x_0, ..., x_k\}, E(P) = \{x_0x_1, x_1x_2, ..., x_{k-1}x_k\}$ where all the x_i are distinct. Alternately we write $P = x_0x_1...x_k$. x_0 and x_k are referred to as the *ends* of a path $P = x_0...x_k$. If $P = x_0...x_k$ is a path, then $P \cup x_{k-1}x_k$ is called a *cycle* when $k \geq 2$. The *length* of a cycle or path is the size of its edge set. The maximum length of a cycle in a graph G is referred to as the *circumference* of G.

A non-empty graph G is said to be *connected* if for any two vertices $u, v \in V(G)$, there exists a path in G from u to v. A maximal connected subgraph of G is called a *component* of G. We say a graph G is *k-connected* if $|V(G)| \geq k + 1$ and for any $S \subseteq V(G)$ with $|S| \leq k - 1$, $G - S$ is connected. In a graph G, if $S \subseteq V(G)$ and

$G - S$ is not connected, then S is said to be a *cut* of G. Further, if $|S| = k$, then S is said to be a *k-cut*.

A *claw* in a graph is an induced subgraph isomorphic to $K_{1,3}$. A graph G with no claw is said to be *claw-free*.

Let $S \subseteq E(G)$. Let $V(S) = \cup_{e \in S} V(e)$. When we refer to the subgraph *induced by the set of edges* S, we mean the subgraph of G induced by $V(S)$. In particular, let $e, f \in E(G)$. If $V(e) \cup V(f)$ is a 3-cut of G, then $\{e, f\}$ is said to *induce a 3-cut* G. Note that if $\{e, f\}$ induces a 3-cut in G, then e and f are adjacent.

Let H be a subgraph of a graph G. Then we define an *H-bridge* of G as a subgraph of G which is induced by the edges in a component of $G - H$ and by the edges of G from that component to H.

Next we establish the preliminary notation needed for Tutte decomposition. We start by borrowing the notation of Jackson and Wormald [16], which establishes Tutte decomposition for any 2-connected graph. Note that this decomposition, as its name suggests, is originally due to Tutte [27]. Tarjan [12] later published an $O(|V(G)| + |E(G)|)$ algorithm to perform the Tutte decomposition. Once we establish this notation for general 2-connected graphs, we then define our own slightly different notation – which combines the structures defined in Jackson and Wormald's paper. Note that much of the notation for a general 2-connected graph will only be used as an intermediate in defining the terms we use in the thesis and hence will not be used outside the introduction.

Let G be a 2-connected graph and $\{x, y\}$ be a 2-cut of G. Let $G_1, ..., G_k$ be the components of $G - \{x, y\}$, with $k \geq 2$. For $i = 1...k$, let H_i be the subgraph of G induced by $V(G_i) \cup \{x, y\}$, but with the edge xy removed (if it exists). We refer to H_i as *$\{x, y\}$-components* of G. If $xy \in E(G)$, we refer to $\{x, y, xy\}$ as the *trivial $\{x, y\}$-component* of G. For any subgraph H of G, let $H' = G - (H - \{x, y\})$. We say that H_i is *excisable* if H_i is nontrivial and either H_i or H_i' is 2-connected. If for some i, H_i is excisable, then $\{x, y\}$ is called a *hinge* of G and H_i is called a *hinge component* of G. We say a hinge $\{x, y\}$ is of Type I if there are exactly two $\{x, y\}$-components of G; we say the hinge is of Type II otherwise. Note that any hinge will necessarily be of Type I if the graph G is claw-free.

We now add "fake edges" to our graphs, which we keep separate from the original edges. For each hinge $\{x, y\}$, add one such fake edge with ends x and y for each excisable $\{x, y\}$-component and in so doing define the augmented graph G^a. Let H be an excisable $\{x, y\}$-component of G associated with the fake edge e. Define the augmented graph D by adding the fake edge e to H and define the augmented graph D' by adding the fake edge e to H'. D and D' are called the *cleavage graphs* of G at e. We define *cleavage units* as the minimal cleavage graphs obtained by recursively finding the cleavage graphs of cleavage graphs. The significance of cleavage units is that they are fundamental structures within the graph. Cleavage units do not have hinges and every fake edge of G^a belongs to exactly two cleavage units. Most importantly, each cleavage unit is either 3-connected, a cycle of length at least 3, or a multiple edge.

Given this structure, we now define the notation we use in the thesis. If two cleavage units share the same fake edge e, we define *combining* them as taking their union and deleting the fake edge e. Let a *pre-3-block* of G be a maximal subgraph of G^a up to combination of cleavage units such that 3-connected cleavage units may only be combined with multiple edge cleavage units. In other words, each pre-3-block is either a single 3-connected cleavage unit combined with any number of adjacent multiple edges or any number of adjacent cycles combined with any number of adjacent multiple edges.

If x, y are the ends of a fake edge in a pre-3-block of G, then we call $\{x, y\}$ a *special 2-cut* of G. If B^a is a pre-3-block of G, we define a *3-block* of G from B^a by no longer distinguishing between "real" and "fake" edges and then replacing all multiple edges with a single edge. If B is a 3-block of G with edge $xy \in E(B)$ such that $xy \notin E(G)$, then we refer to xy as a *virtual edge*. Define any vertex in a 3-block as *internal* if it is not part of a special 2-cut of G. Note that all 3-blocks of G are either 3-connected or a union of cycles. Further, note that special 2-cuts are merely the intersection of adjacent 3-blocks and are a subset of the hinges of G. We define a *chain of cycles* to be a 3-block that is the union of cycles. In particular, we define a *chain of triangles* to be a 3-block that is the union of triangles.

(1.1.1) Theorem. *If G is a 2-connected graph, then Tutte's decomposition will partition G into 3-blocks along its special 2-cuts. Each 3-block is either a chain of cycles or is 3-connected.*

Let G be a 3-connected graph with a vertex a such that $G-a$ is not 3-connected. By taking the Tutte decomposition of $G-a$, we mean the process of finding the special 2-cuts and 3-blocks of the graph $G-a$, as well as the virtual edges in those 3-blocks. From this entire discussion, we primarily use the terms "special 2-cut, 3-block, virtual edge, and Tutte decomposition" in the actual thesis. The other terms are only briefly used when discussing the structure of a Tutte decomposition in Chapter 2, section 2.

1.2 History and motivation

Finding long cycles in graphs has a long history in the field of graph theory. In 1931, Whitney [28] proved that 4-connected planar triangulations are Hamiltonian. Tutte [26] later proved that every 4-connected planar graph contains a Hamilton cycle. Since then, mathematicians have sought to characterize other classes of graphs which are Hamiltonian, find long cycles and paths in certain classes of graphs, find cycles and paths with very specific structural properties. Faudree, Flandrin, and Ryjacek [10] provide an excellent survey of these topics. Since there are a multitude of such results, we strive to limit our survey to 3-connected and 4-connected graphs, cubic graphs, and claw-free graphs.

Whitney's theorem showed that 4-connected planar triangulations were 4-face-colorable. In that light, there was hope for a simple proof of the 4-color theorem if one could show that every planar cubic graph was Hamiltonian – which was conjectured to be true in 1880 by Tait [21] for 3-connected cubic graphs in general. However, Tutte demonstrated that neither was the case and provided a 3-connected planar cubic graph as counterexample [25].

There have been more recent developments in the study of 4-connected planar graphs. Building on Tutte's technique, Thomassen [23] proved that in any 4-connected planar graph there is a Hamilton path between any given pair of distinct vertices (i.e. Hamiltonian-connected). Thomas and Yu [22] proved that the deletion of any two vertices from a 4-connected planar graph results in a Hamiltonian graph.

Short of finding a Hamiltonian cycle, merely finding a long cycle in a graph is both difficult and of significant interest for practical and theoretical reasons. The concept of visiting as many vertices as possible without having to retrace ones steps is related to the travelling salesman problem. Real world problems that can be translated into this context directly benefit from sure knowledge of a long cycle. Further, finding the longest cycle in a graph is in general very difficult. In fact, simply approximating its length to a constant factor is known to be NP-hard [17]. We turn our attention to results bounding the length of a longest cycle in graphs with certain structural properties.

We first consider 3-connected planar graphs on n vertices. Barnette [1] showed that the circumference of such a graph is at least $c\sqrt{\log n}$. This bound was improved by Clark [8] to $\exp(\frac{1}{6}\sqrt{\ln n})$. Then Jackson and Wormald [14] were to first to show a polynomial bound cn^{γ} ($\gamma \sim 0.2$). Chen and Yu [5] later improve this bound to

$n^{\log_3 2}$ (note that $\log_3 2 \sim 0.63$). Recently, Chen, Gao, Yu, and Zang [6] proved that if G is a 3-connected graph on n vertices with maximum degree $d \geq 4$, then G has a cycle of length $\Omega(n^{\log_{d-1} 2})$.

The original motivation for this thesis is improving the bound for the circumference of 3-connected cubic graphs and hence we now shift our attention to cubic graphs. Barnette [1] showed that planar 3-connected cubic graphs have circumference at least $3(\log_2 n) - 10$. Bondy and Entringer [2] then showed that 2-connected cubic graphs have circumference at least $4(\log_2 n) - 4(\log_2 \log_2 n) - 20$. Lang and Walther [18] showed this was best possible for 2-connected cubic graphs. Bondy and Simonovits [3] then showed that 3-connected cubic graphs have circumference at least $\exp(c\sqrt{\ln n})$. Further they constructed an infinite family of 3-connected cubic graphs with largest cycle of length at most n^γ, $\gamma = \log_9 8 \sim 0.94$. If G is a 3-connected cubic graph and $e_1, e_2 \in E(G)$, then Jackson [13] proves that there is a cycle C in G such that $e_1, e_2 \in C$ and $|E(C)| \geq n^\gamma + 1$ where $\gamma = \log_2(1 + \sqrt{5}) - 1 \sim 0.69$. This is currently the best known bound. More recently, Feder, Motwani, and Subi [11] find a polynomial time algorithm for finding a long cycle in a 3-connected cubic graph, albeit for the weaker bound $\Omega(n^{\log_3 2})$.

We sought to improve the bound for 3-connected cubic graphs by improving the bound for 3-connected $K_{1,3}$-free graphs. We will discuss the technique in the next paragraph, but for now we turn our attention to results on 3-connected claw-free graphs. Jackson and Wormald [16] prove that if G is a 3-connected $K_{1,d}$-free graph with n vertices, then G has a cycle of length at least $\frac{1}{2}n^\gamma$, where $\gamma = (\log_2 6 + 2\log_2(2d+1))^{-1}$. Note that for $d = 3$, $\gamma \sim 0.122$ in the above bound. This is previously the best known bound. As Jackson and Wormald's proof is not specifically designed for claw-free graphs, the bound likely has room for improvement. It is worth noting that it is an open conjecture of Matthews and Sumner [19] that any 4-connected claw-free graph is Hamiltonian and it is an open conjecture of Thomassen [24] that every 4-connected line graph is Hamiltonian. These two conjectures are equivalent [20].

As the original motivation for this research was to improve the bound for the circumference of 3-connected cubic graphs, we briefly describe how we would have done so. Consider any vertex v of degree 3 with neighbors v_1, v_2, v_3. Define the following operation: replace v with a triangle defined by the three new vertices v_1', v_2', v_3' and add the edges $v_i v_i'$ for $i = 1, ..., 3$. By successively performing this operation to all vertices of a 3-connected cubic graph G, we obtain a 3-connected claw-free graph G'. If we find a cycle C' in G', we can contract all the triangles created by the operation back to their original vertices and hence find a cycle C in G that is proportional in length to C'. Thus if we can find a polynomial bound for the circumference of 3-connected claw-free graphs with the exponent of the leading term $\gamma > \log_2(1 + \sqrt{5}) - 1 \sim 0.69$, we could improve the bound for 3-connected cubic graphs.

It is worth noting other results related to our problem, but which unfortunately do not aid in our proof. Bondy and Locke [4] prove that if there is a path L of length l in a 3-connected graph G, then G contains a cycle C which contains at least $\frac{2}{3}l$ edges of L. Note, if there was a result which found a path of length cn^γ in a 3-connected claw-free graph, then we could use Bondy and Locke's result to immediately find a cycle of comparable length. Unfortunately no such path result exists. The other result is Chudnovsky and Seymour's [7] recent characterization of claw-free graphs. This is a very powerful structural theorem, but uses line graphs as the building blocks of its decomposition. As a result, it is not entirely useful in our approach. For our purposes, we use the claw-free structure to simplify the Tutte decomposition and then only use the claw-freeness in very local settings, such as proving a certain edge must exist or to study the neighborhood of a specific vertex. Further, we use induction to avoid characterizing the structure of our 3-blocks when possible. Though considering the full characterization of certain 3-blocks may be useful for finding very specific paths, we have not needed to define the structure of a 3-block to such an extent, thusfar.

In this thesis we find a polynomial bound for the circumference of 3-connected claw-free graphs with the exponent of the leading term $\gamma = \log_3 2 \sim 0.63$.

(1.2.1) Theorem. *If G is a 3-connected claw-free graph on $n \geq 6$ vertices, then there exists a cycle C in G such that $|E(C)| \geq \alpha n^{\gamma} + 5$, where $\gamma = \log_3 2$ and $\alpha \geq 1/7$ is a constant.*

Thus we improve Jackson and Wormald's bound for the circumference of 3-connected claw-free graphs. We do not improve Jackson's bound for the circumference of 3-connected cubic graphs. However, we believe that with a more exhaustive use of the techniques in this thesis, we will eventually be able to improve the bound for 3-connected cubic graphs.

To prove Theorem (1.2.1), we instead prove an even stronger result.

(1.2.2) Theorem. *Let G be a 3-connected claw-free graph on $n \geq 6$ vertices and let $e, f \in E(G)$ such that $\{e, f\}$ does not induce a 3-cut. Then there exists a cycle C in G such that $e, f \in C$ and $|E(C)| \geq \alpha n^{\gamma} + 5$, where $\gamma = \log_3 2$ and $\alpha \geq 1/7$ is a constant.*

1.3 Organization

The thesis is organized as follows. Chapter 2 is dedicated to the proof of more basic results used in the proof of Theorem (1.2.2). Chapter 3 is dedicated to more complicated results used in the proof of Theorem (1.2.2). The proofs in Chapter 3 often involve many cases and may at times use more than one Tutte decomposition. Chapter 4 then invokes the results of previous chapters in order to prove

Theorem (1.2.2) and subsequently discussed future work as well as applications to 3-connected cubic graphs.

The proof of Theorem (1.2.2), involves two steps. First, we define a certain path $Z_G(e)$ in our graph G (say from a_1 to a_2) which contains the special edge e. Second, we find a sufficiently long path in the rest of the graph $G - (Z_G(e) - \{a_1, a_2\})$ from a_1 to a_2 which contains the other special edge f. Together, these paths will give the desired cycle for Theorem (1.2.2). The majority of Chapter 2 and Chapter 3 will involve developing the theoretical framework and machinery needed to ultimately prove the existence of this second path, given the first path.

Chapter 2 focuses on the more basic results.

As the proof of Theorem (1.2.2) is inductive, in section 2.1, we prove a result on graphs of order ≤ 6, which proves the base case. We also prove another result for graphs of order ≤ 6. We then prove useful properties of the convex function $f(x) = x^\gamma$.

In section 2.2, we describe the Tutte decomposition of a special type of 2-connected claw-free graph into 3-blocks. In particular, we consider the Tutte decomposition of $G - a$, where G is a 3-connected claw-free graph, $a \in V(G)$, and $G - a$ is not 3-connected. We prove that the 2-cuts of $G - a$ form a linear structure. We go on to characterize the 3-blocks of $G - a$ which are not 3-connected. We also prove that slight changes to the structure of a 3-connected 3-block results in a graph which satisfies the hypotheses of the main theorem. In later proofs, invoking the inductive hypothesis in such a modified graph will allow us to find paths or cycles in the original 3-block.

In section 2.3, we prove several results for finding paths and cycles through 3-blocks which are not 3-connected. As the structure of these 3-blocks is very restricted, these results are intuitively obvious. However, as they are needed several times in more complicated proofs, we do go through the effort of formally recording these results.

This concludes Chapter 2.

In Chapter 3 we continue to prove results for finding paths and cycles in 3-blocks – however, the proofs in this Chapter are substantially more complicated.

In section 3.1, we prove results for finding paths and cycles through 3-blocks which are 3-connected. As a proof technique, we begin to use the inductive hypothesis of Theorem (1.2.2). We go on to prove results for finding paths and cycles through multiple consecutive 3-blocks (regardless of whether they are 3-connected or not). However, for the purposes of proving the main theorem, we need to do more than just find paths in each of the 3-blocks – we need the paths to have their ends agree in order to connect them together. Thus we will need to find very specific types of paths in certain 3-blocks. The results in this section suffice in most situations – but not all.

In section 3.2, we prove another result for a very specific type of path. This proof uses more than one Tutte decomposition and is rather lengthy. We present a preliminary result that simplifies the analysis.

In section 3.3, we prove the final two results needed for the proof of Theorem (1.2.2). Though the path $Z_G(e)$ has not yet been defined, the hypotheses of these last two lemmas assume a structure that would result from deleting all but the ends of $Z_G(e)$ from the graph. As a result, these two lemmas are precisely what is needed for the proof of the main theorem. Though they are conceptually simple, these two lemmas are very technical and hence require long proofs with extreme attention to detail. This is by far the longest section in the thesis.

In Chapter 4, we first define the path $Z_G(e)$ and then invoke results from previous Chapters (primarily the two lemmas in section 3.3), to finish the proof. We then discuss future work as well as applications to 3-connected cubic graphs.

CHAPTER II

BASIC RESULTS

2.1 Base cases and basic inequalities

In this section we prove the base cases for the main theorem.

(2.1.1) Lemma. *Let G be a 3-connected claw-free graph on $n \leq 6$ vertices and let $e, f \in E(G)$ such that $\{e, f\}$ does not induce a 3-cut. Then G has a Hamilton cycle which contains e and f.*

Proof. Since G is 3-connected, there is a cycle C in G such that $\{e, f\} \subseteq E(C)$. We choose such C that $|C|$ is maximum. Let P_1, P_2 denote the components of $C - \{e, f\}$, each of which is a path. We may assume that $|V(C)| < |V(G)|$, as otherwise C is the desired Hamilton cycle. In particular, $|V(C)| \leq 5$, and there is a vertex $v \in V(G) - V(C)$.

Since G is 3-connected, there exist three paths Q_1, Q_2, Q_3 from v to $v_1, v_2, v_3 \in V(C)$, respectively, such that $V(Q_i \cap Q_j) = \{v\}$ for $\{i, j\} \subseteq \{1, 2, 3\}$ and $V(Q_i \cap C) = \{v_i\}$ for $i \in \{1, 2, 3\}$. Without loss of generality, we may assume that $v_1, v_2 \in P_1$ such that e, v_1, v_2, f occur on C in this cyclic order.

Note that $(C - V(P_1(v_1, v_2))) \cup Q_1 \cup Q_2$ is a cycle in G containing both e and f. So by the choice of C, $|P_1[v_1, v_2]| \geq |Q_1 \cup Q_2| \geq 3$. Thus $3 \leq |P_1| \leq 4$, and so $1 \leq |P_2| \leq 2$.

We may assume $|P_2| = 1$. For, suppose $|P_2| = 2$. Then $|P_1| = 3$, since $|C| \leq 5$. Let u_1, u_2 denote the vertices of P_2 with u_1, u_2 incident to e, f, respectively, and let w be the vertex of P_1 other than v_1 and v_2. If $wv \in E(G)$, then the cycle $u_1v_1vwv_2u_2u_1$ contradicts the choice of C. So $wv \notin E(G)$. Since G is 3-connected, we may therefore assume without loss of generality that $u_1w \in E(G)$.

If $u_2v \in E(G)$ then the cycle $u_1v_1vu_2v_2wu_1$ contradicts the choice of C. So $u_2v \notin E(G)$. Then since G is claw-free, we must have $u_2w \in E(G)$. Hence the cycle $u_1v_1vv_2u_2wu_1$ contradicts the choice of C.

So let u denote the unique vertex in P_2.

Suppose $|P_1| = 3$. Let w denote the vertex of P_1 other than v_1 and v_2. Then since $|P_1[v_1, v_2] \geq 3$, $w \in P_1(v_1, v_2)$. Since $\{e, f\}$ does not induce a 3-cut in G, v and w are contained in a component of $G - \{u, v_1, v_2\}$. Hence, as $|V(G)| \leq 6$, $vw \in E(G)$ or there is a sixth vertex of G, say x, such that $vx, wx \in E(G)$. In the former case, the cycle uv_1wvv_2u contradicts the choice of C; and in the latter case, the cycle uv_1wxvv_2u contradicts the choice of C.

Now assume $|P_1| = 4$. Let w_1, w_2 denote the vertices of P_1 other than v_1 and v_2. First, consider that case $w_1, w_2 \in P_1(v_1, v_2)$. We may assume without loss of generality that u, v_1, w_1, w_2, v_2 occur on C in the cyclic order listed. Since $|V(G)| \leq 6$ and $\{e, f\}$ does not induce a 3-cut in G, we may assume by symmetry between w_1 and w_2 that $vw_1 \in E(G)$. Then the cycle $uv_2w_2w_1vv_1u$ contradicts the choice of C. Therefore, we may assume that u, w_1, v_1, w_2, v_2 occur on C in this cyclic order. Since G is claw-free, $\{v_1, w_1, w_2, v\}$ does not induce a claw. If $w_1w_2 \in E(G)$ then the cycle $uw_1w_2v_1vv_2u$ contradicts the choice of C; if $w_1v \in E(G)$ then the cycle $uw_1vv_1w_2v_2u$ contradicts the choice of C; and if $vw_2 \in E(G)$ then $uw_1v_1vw_2v_2u$ contradicts the choice of C. □

(2.1.2) Lemma. *Let G be a 3-connected claw-free graph of order $n \leq 6$, a_1, a_2 nonadjacent vertices of G, and $f \in E(G)$. Then G' has a path P from a_1 to a_2 such that $f \in P$, $|E(P)| \geq n - 2$.*

Proof. Take a longest path P in G from a_1 to a_2 such that $f \in E(P)$. If $|E(P)| \geq n - 2$, done. Hence there exists $v \in V(G) - V(P)$. Since G is 3-connected, there exist three independent paths from v to $v_1, v_2, v_3 \in V(P)$, where a_1, v_1, v_2, v_3, a_2

are on P in order. Without loss of generality, we may assume $f \notin v_1Pv_2$. Thus $|E(v_1Pv_2)| \geq 2$, otherwise we contradict the maximality of P. Let u be a vertex between v_1 and v_2 on P. Thus $|E(P)| \geq 3$. Thus we may assume $n = 6$ and $P = a_1uv_2a_2$. Note that $v_2a_2 = f$. If $uv \in E(G)$, then $a_1vuv_2a_2$ contradicts the maximality of P.

Let z be the sixth vertex in G. Suppose $uz \in E(G)$. If $za_1 \in E(G)$, then $a_1zuv_2a_2$ contradicts the maximality of P. If $zv_2 \in E(G)$, then $a_1uzv_2a_2$ contradicts the maximality of P. If $zv \in E(G)$, then $a_1vzuv_2a_2$ contradicts the maximality of P. As G is 3-connected, we may assume $uz \notin E(G)$.

Suppose $a_1z \in E(G)$. As G is claw-free, $\{a_1, v, z, u\}$ is not a claw and hence $zv \in E(G)$. Then $a_1zvv_2a_2$ contradicts the maximality of P. Thus we may assume $a_1z \notin E(G)$.

$\{zv, zv_2, za_2, ua_2, a_1v_2\} \subseteq E(G)$. Thus $a_1vzv_2a_2$ contradicts the maximality of P. □

(2.1.3) Lemma. *Let $n_1, ..., n_k$ be real numbers in the interval $[0, 1]$ such that $k \geq 3$, $\sum_{i=1}^{k} n_i = 1$, $n_i \geq n_k$ $\forall i = 1, ..., k-1$. Then $\sum_{i=1}^{k-1} n_i^{\gamma} \geq 1$ for $\gamma = \log_k(k-1)$.*

Proof. Let $f = \sum_{i=1}^{k-1} n_i^{\gamma}$. Let $F(f, \lambda) = f - \lambda((\sum_{i=1}^{k} n_i) - 1)$. Set $\frac{\partial F}{\partial n_j} = \gamma n_j^{\gamma-1} - \lambda = 0$ for $j < k$, and $\frac{\partial F}{\partial n_k} = -\lambda = 0$.

Thus $\lambda = 0$ and hence $n_j = 0$ for $j < k$ give rise to a critical point, which is not in the feasible region. Therefore, the minimum occurs at the boundary. We have no restriction on n_k, but we do have the other restriction that $n_j \geq n_{j+1}$.

Since $\sum_{i=1}^{k} n_i = 1$, if $n_i = 1$ for some $i \neq k$, then $n_j = 0$ for $j \neq i$ and hence $f = 1$. So we may assume $n_i < 1$ for all $i = 1, ..., k$. Hence the boundary of the feasible region is when $n_i = n_k$ for some $i \neq k$. Without loss of generality, $n_{k-1} = n_k$. Iterating, we find that the minimum of f may be obtained when $n_2 = ... = n_k$.

Thus, let $g = n_1^\gamma + (k-2)n_k^\gamma$ and $n_1 = 1-(k-1)n_k^\gamma$. Then $g(n_k) = [1-(k-1)n_k]^\gamma + (k-2)n_k^\gamma$, and $g'(n_k) = -(k-1)\gamma[1-(k-1)n_k]^{\gamma-1} + (k-2)\gamma n_k^{\gamma-1}$.

It is easy to see that $g'(n_k) = 0$ has a unique solution. It is also easy to see that $g''(n_k) < 0$. Since $n_k \geq 0$ and $n_k \leq 1/k$, $g(n_k)$ achieves global minimum at 0 or $1/k$. Note that $g(0) = 1^\gamma + 0 = 1 \geq 1$, and $g(1/k) = [1-(k-1)/k]^\gamma + (k-2)(1/k)^\gamma = (1/k)^\gamma + (k-2)(1/k)^\gamma = (k-1)(1/k)^\gamma$. But $k^\gamma = k-1$ as $\gamma = \log_k(k-1)$. Thus $g(1/k) = 1$. And hence $g \geq 1$. As g and f have the same minimum, $f \geq 1$. □

2.2 Structure of a decomposition

In our proof, we will look for specific vertices that when deleted, will keep the graph 3-connected. However, when no such vertex exists, deleting a vertex will make the graph 2-connected, but not 3-connected, and we will then look at its decomposition. With this in mind, we first prove several important structural properties granted to such a decomposition by claw-freeness and the original graph's 3-connectivity.

(2.2.1) Lemma. *Let G be a 3-connected claw-free graph and $a \in V(G)$ such that $G-a$ is not 3-connected. Let $\{b,c\}$ be a 2-cut of $G-a$. Then $G-\{a,b,c\}$ has exactly two components.*

Proof. For contradiction, assume $G-\{a,b,c\}$ has at least 3 components. Let C_1, C_2, C_3 be three such distinct components of $G-\{a,b,c\}$. b must have a neighbor in each of these components, else $G-a$ is not 2-connected. Let $c_i \in V(C_i)$, $i = 1,2,3$, be neighbors of b. However, $\{b, c_1, c_2, c_3\}$ induce a claw in G, a contradiction.

□

Let G be a 3-connected claw-free graph and $a \in V(G)$ such that $G-a$ is not 3-connected. Let $\{b,c\}$ be a 2-cut of $G-a$. When we look at the Tutte decomposition of $G-a$, for the cleavage units, we see a collection of 3-connected graphs, cycles, and multiple edges. However, Lemma (2.2.1) implies that any multiple edge will have exactly two fictitious edges and none of these will be of interest to us (hence why we define 3-blocks to ignore multiple edges). Note that if two cycles C_1 and C_2 are cleavage units in the decomposition of $G - a$ such that $|V(C_1 \cap C_2)| = 2$, then C_1 and C_2 are in the same 3-block in the decomposition of $G - a$ and the vertices in $C_1 \cap C_2$ are adjacent in G.

To determine the structure of the decomposition of $G-a$, it will be temporarily useful to define it in terms of a graph D. As G is 3-connected and claw-free, we will see that D is a path. Let the vertices of D be the 3-blocks of the decomposition of $G - a$. Let two vertices of D be adjacent iff their corresponding 3-blocks share both of the vertices of a 2-cut in $G - a$. In a sense, they are connected through that 2-cut and the edge in D corresponds to that 2-cut in $G-a$. Note that a 2-cut of $G - a$ that corresponds to an edge in D is a *special 2-cut*. Lastly, consider a fixed 3-block of the decomposition of $G - a$. Recall that we define any vertex in that 3-block as *internal* if it is not part of a special 2-cut of $G - a$.

We now study the structure of the graph D. First, we show D is a tree. Clearly, D must be connected, by definition of the decomposition of $G - a$. Assume for contradiction that D has a cycle. Pick any edge $e \in E(D)$ that is in that cycle. $D-e$ will remain connected. Let e correspond to the special 2-cut $\{b, c\}$ in $G-a$. Since $D - e$ is connected, this implies that $(G - a) - \{b, c\}$ is also connected. However, $\{b, c\}$ is a 2-cut in $G - a$ and deleting those vertices will disconnect the graph, a contradiction. Thus D is a tree.

In fact, D must be a path. The leaves of D correspond to 3-connected graphs or chains of cycles, and hence must have at least three vertices. Fix one such leaf, say L, and let $\{b, c\}$ be the special 2-cut in $G - a$ corresponding to the only edge in D that is incident with L. Note that L has at least one internal vertex. If a is not adjacent to any internal vertices of L, then $G - \{b, c\}$ would be disconnected

– contradicting the fact that G is 3-connected. Thus for every leaf in D, a must have at least one neighbor that is an internal vertex of that leaf. If there are three leaves in D, then a and its internal neighbors, one from each of those leaves, would induce a claw in G – contradicting the claw-freeness of G. Thus D has at most two leaves, and hence, D is a path.

Furthermore, since D is a path and a must be adjacent to vertices internal to both 3-blocks corresponding to the ends of the path (as the original graph was 3-connected). However, a cannot have neighbors internal to any other 3-block , otherwise G would have a claw.

Lastly, note that there is the possibility that D is just a single vertex. However, in that case, the only 3-blocks of in the decomposition of $G - a$ must be a chain of cycles as $G - a$ was assumed to be not 3-connected.

Now suppose $|D| \geq 2$. In general, we call the 3-blocks corresponding to the leaves in D as the *extreme 3-blocks* of $G - a$ and all other 3-blocks are referred to as *middle 3-blocks*. However, due to the simple structure of D, we assign an orientation (left to right) to D for a more intuitive notation. One extreme 3-block is the "leftmost" 3-block and the other is the "rightmost" 3-block. As D is a path, there is a well defined order from left to right between both edges and vertices. Hence it should be clear what is meant by left or right of a given 3-block or a given special 2-cut in the decomposition. Further, this analysis applies equally well if we defined D in terms of 2-cuts, not just special 2-cuts. Thus chains of cycles also have a linear structure and this left to right orientation extends in general to all 2-cuts, not just the special 2-cuts.

However, there may be confusion by what is meant as left or right of a particular vertex in a 2-cut – and thus clarification is required. For $b \in V(G-a)$, let $N_{G-a}(b)$ be the neighbors of b in $G - a$. We seek to define $L_{G-a}(b)$ and $R_{G-a}(b)$, subsets of $N_{G-a}(b)$ that are the vertices "left" and "right" of b in the decomposition of $G-a$.

If b is in only one 2-cut $\{b, c\}$ of $G - a$ such that $\{b, c\} = V(C_l \cap C_r)$, where C_l and C_r are the two components of $(G - a) - \{b, c\}$, left and right (respectively) of $\{b, c\}$ in the decomposition of $G - a$, define $L_{G-a}(b) = N_{G-\{a,c\}}(b) \cap V(C_l)$ and define $R_{G-a}(b) = N_{G-\{a,c\}}(b) \cap V(C_r)$. Note that these sets do not include the vertex c.

Because of the linear structure of the 2-cuts in $G - a$, there is an ordering from "left" to "right" on the 2-cuts of $G - a$ that contain b. Let $\{b, c_l\}$ be the leftmost and let $\{b, c_r\}$ be the rightmost. Let C_l be the component left of $\{b, c_l\}$ in $(G-a)-\{b, c_l\}$, and let C_r be the component right of $\{b, c_r\}$ in $(G-a)-\{b, c_r\}$. We define $L_{G-a}(b) = N_{G-\{a,c_l\}}(b) \cap V(C_l)$ and define $R_{G-a}(b) = N_{G-\{a,c_r\}}(b) \cap V(C_r)$.

We now prove that $L_{G-a}(b)$ and $R_{G-a}(b)$ are both cliques in G. This structural result will be extremely useful in the following section.

(2.2.2) Lemma. *Let G be a 3-connected claw-free graph, $a \in V(G)$, and $\{b, c\}$ a 2-cut of $G - a$. Fix an orientation from left to right on the decomposition of $G - a$. Then*

(1) $N_{G-\{a,c\}}(b)$ induces two disjoint cliques in G, one on $L_{G-a}(b)$ and the other on $R_{G-a}(b)$,

(2) and if c is adjacent to b in G, then c is adjacent in G to all $L_{G-a}(b)$ or all of $R_{G-a}(b)$

Proof. By definition, $L_{G-a}(b)$ and $R_{G-a}(b)$ are both not empty and partition $N_{G-\{a,c\}}(b)$. Assume for contradiction that $l_1, l_2 \in L_{G-a}(b)$ where $l_1 l_2 \notin E(G)$. Let $r \in R_{G-a}(b)$. Then $\{b, l_1, l_2, r\}$ induces a claw in G, a contradiction. Thus $L_{G-a}(b)$ induces a clique in G. Similarly $R_{G-a}(b)$ induces a clique in G. By construction, neither clique has edges to the other in G and hence $N_{G-\{a,c\}}(b)$ induces two disjoint cliques in G.

Assume that b and c are adjacent in G. Now assume for contradiction that there exist vertices $l \in L_{G-a}(b)$ and $r \in R_{G-a}(b)$, where neither is adjacent to c in G. Then $\{b, l, r, c\}$ induces a claw in G, a contradiction. □

The significance of Lemma (2.2.2) will become apparent when one of the 3-blocks containing $\{b, c\}$ is 3-connected, in which case, the neighbors (except possibly c) of b in that 3-block induces a clique. Thus we will be able to add a new vertex adjacent to b, and as long as it is also adjacent to c, we will be able add some structure to this small graph, but still preserve claw-freeness.

We also define the vertices left and right of a. In the decomposition of $G - a$, let L and R be the leftmost and rightmost, respectively, 3-blocks. (If the decomposition of $G - a$ is not a chain of cycles, let L and R be the leftmost and rightmost cycles, respectively.) Let $\{a_L, b_L\}$ and $\{a_R, b_R\}$ be the 2-cuts of $G - a$ contained in L and R respectively. If $N_G(a) \cap L$ induces a clique, then let $L_{G-a}(a) = N_G(a) \cap L$. If $N_G(a) \cap L$ does not induce a clique, but $(N_G(a) \cap L) - a_L$ does induce a clique, then let $L_{G-a}(a) = (N_G(a) \cap L) - a_L$. If $N_G(a) \cap L$ does not induce a clique, but $(N_G(a) \cap L) - b_L$ does induce a clique, then let $L_{G-a}(a) = (N_G(a) \cap L) - b_L$. Otherwise, let $L_{G-a}(a) = (N_G(a) \cap L) - \{a_L, b_L\}$. Define $R_{G-a}(a)$ similarly. Intuitively, $L_{G-a}(a)$ is the largest clique in L all of whose vertices are adjacent to a in G.

Note that we have now defined $L_{G-a}(b)$ and $R_{G-a}(b)$ for any type of vertex $b \in V(G)$ except those that are internal vertices of a 3-connected 3-block. But this is not a problem as the notion of left or right inside a 3-connected 3-block simply does not make sense.

Since ultimately we will be searching for paths or cycles within these two types of 3-blocks, it will be helpful to study their structure.

We first study chains of cycles, which have a very restricted structure. It is important to note that the definition of a chain of cycles also allows an arbitrary orientation from left to right on the cycles. Since we imposed such an orientation on the entire decomposition, it is natural to extend that orientation to each 3-block that is a chain of cycles.

(2.2.3) Lemma. *Let G be a 3-connected claw-free graph, $a \in V(G)$, and M be a chain of cycles in the decomposition of $G - a$. Let $b \in V(M)$. Then*

(1) b belongs to no more than 3 cycles in M, and

(2) if b belongs to a special 2-cut in the decomposition of $G - a$, then b belongs to no more than 2 cycles in M.

Proof. Assume first that b does not belong to any special 2-cut in the decomposition of $G-a$. Suppose that (1) fails. Then we may label the cycles that contain b from left to right as $C_1, ..., C_m$, $m \geq 4$. Let $c_0, ..., c_m$ be the neighbors of b in M in order from left to right such that $c_i \in C_i \cap C_{i+1}$ for $1 \leq i \leq m-1$, $c_0 \in C_1 - C_2$, and $c_m \in C_m - C_{m-1}$. Note that $bc_2 \in E(G)$, since $\{b, c_2\} = V(C_2) \cap V(C_3)$. Further $bc_0 \in E(G)$, as $\{b, c_0\}$ is not a 2-cut of $G - a$ (and hence this cannot be a virtual edge). Similarly, $bc_m \in E(G)$. Since $m \geq 4$, $\{b, c_0, c_2, c_m\}$ induce a claw in G, a contradiction.

Now assume that b belongs to a special 2-cut $\{b, c_0\}$ in the decomposition of $G-a$. By symmetry of $G-a$, assume that $\{b, c_0\}$ is on the left of M and note that the 3-block left of $\{b, c_0\}$ is 3-connected. Now assume (2) fails. Then b belongs to $m \geq 3$ cycles in M. As above, we enumerate the neighbors of b in these cycles from left to right as $c_1, ..., c_m$. Further, let $c_l \in L_{G-a}(b)$. As $m \geq 3$, $\{b, c_l, c_1, c_m\}$ induce a claw in G, a contradiction. □

The following lemma gives a description of the structure of a chain of cycles. Figure 2.2.1 gives an example of a chain of triangles (which will be fully described in Lemma (2.2.5)) and exactly depicts the graphs referred to as *square, square with a triangle, square with a triangle on opposite sides*

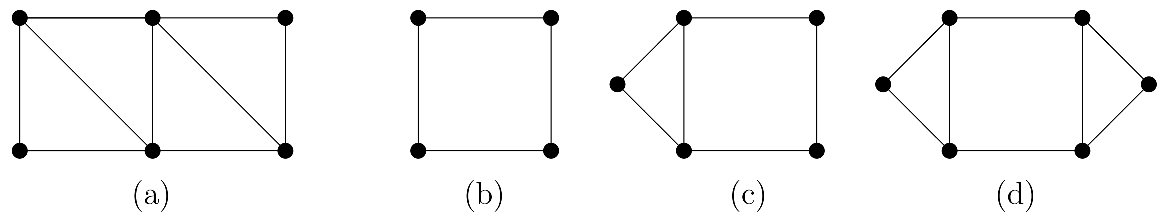

Figure 2.2.1: (a) Example of a chain of triangles (b) Square (c) Square with one triangle (d) Square with a triangle on opposite sides

(2.2.4) Lemma. *Let G be a 3-connected claw-free graph with $a \in V(G)$. Let M be a chain of cycles in the decomposition of $G - a$. Then $G - a$ is a cycle of length 5, or the following holds:*

(1) M is either a chain of triangles, a square, a square and a triangle, or a square with a triangle on opposite sides,

(2) if M is a square with a triangle on opposite sides then M is the only 3-block in the decomposition; and

(3) if M is a square with a single triangle then M is an extreme 3-block and all neighbors of the triangle are in $M + a$ and include a.

Proof. We may assume that the decomposition of $G - a$ either has at least two 3-blocks or is a chain of at least two cycles; otherwise we can prove $G - a$ is a cycle of length 5.

First, assume that some cycle C in the chain M is of length greater than 4. Without loss of generality, we may assume that there is a cycle in M right of C or there is a 3-block right of C in the decomposition of $G - a$. Suppose there is either another cycle in M or a 3-block that is left of C in the decomposition of $G - a$. Note that at most 4 vertices in C have degree greater than 2 in $G - a$, while the remaining vertices in C have degree 2 in $G - a$. Let x be a vertex in C of degree 2 in $G - a$. As G is 3-connected, x must be incident to a in G. Let $l \in L_{G-a}(a)$

and $r \in R_{G-a}(a)$. Clearly $\{a, x, l, r\}$ induce a claw in G, a contradiction. So we may assume that M is the leftmost 3-block and C is the leftmost cycle. Exactly two adjacent vertices in C have degree greater than 2 in $G - a$ and the remaining vertices have degree 2 in $G - a$. So there are at least 3 vertices in C of degree 2 in $G - a$, and there is a pair $\{x, y\}$ of such vertices which are not adjacent in C. As G is 3-connected, both x and y must be incident to a in G. Lastly, let $r \in R_{G-a}(a)$. Clearly, $\{a, x, y, r\}$ induce a claw in G, a contradiction.

Thus we may assume that M is a chain of cycles of length at most 4, and of of which, say $C = b_1c_1c_2b_2b_1$, is of length 4, as otherwise, M is a chain of triangles, and (1) holds.

Consider the possibility of a cycle C_l in M that is left of C and has two vertices in common with C. Without loss of generality, let $\{b_1, c_1\}$ be the two vertices common to C and C_l. If C_l is a square, say $C_l = xyc_1b_1x$, then $\{c_1, b_1, c_2, y\}$ induces a claw in G. So C_l is a triangle, say $C_l = xb_1c_1x$. Importantly, xb_1 or xc_1 must be an edge in G, thus say $xb_1 \in E(G)$. Then, xc_1 must also be an edge in G, else $\{b_1, x, c_1, b_2\}$ induces a claw in G. If there exists $l \in L_{G-a}(b_1) - C_l$ then $\{b_1, l, c_1, b_2\}$ induces a claw in G – a contradiction. Hence $L_{G-a}(b_1) = \{x\}$. Similarly, $L_{G-a}(c_1) = \{x\}$. Thus, if C is not the leftmost cycle in M, then there is only a single triangle left of C in M, and M is the leftmost 3-block in the decomposition of $G - a$.

By symmetry, if C is not the rightmost cycle in M, then there is only a single triangle right of C in M and M is the rightmost 3-block in the decomposition of $G - a$. This restricts the structure of M to the few cases outlined in the statement of the Lemma. □

We now turn to the structure of a chain of triangles.

(2.2.5) Lemma. *Let G be a 3-connected claw-free graph with $a \in V(G)$, let*

M be a 3-block in the decomposition of $G - a$ with $|V(M)| = m$, and assume that M is a chain of triangles. Then the vertices of M may be labelled as $x_1, ..., x_{\lfloor \frac{m}{2} \rfloor}, y_1, ..., y_{\lceil \frac{m}{2} \rceil}$ such that

(1) $E(M) = \{x_i x_{i+1}, x_i y_{i+1}, y_i x_i, y_i y_{i+1} : 1 \leq i \leq \lfloor \frac{m}{2} \rfloor - 1\} \cup \{y_{\lfloor \frac{m}{2} \rfloor} x_{\lfloor \frac{m}{2} \rfloor}, x_{\lfloor \frac{m}{2} \rfloor} y_{\lceil \frac{m}{2} \rceil}, y_{\lfloor \frac{m}{2} \rfloor} y_{\lceil \frac{m}{2} \rceil}\}$, and

(2) if $\{b, c\} \subseteq V(M)$ is a special 2-cut of G then $\{b, c\} = \{x_1, y_1\}$ $\{b, c\} = \{x_{\lfloor \frac{m}{2} \rfloor}, y_{\lceil \frac{m}{2} \rceil}\}$, and $M - \{x_1 y_1, x_{\lfloor \frac{m}{2} \rfloor} y_{\lceil \frac{m}{2} \rceil}\} \subseteq G$.

Proof. If $|V(M)| \in \{3, 4\}$, this claim is trivial; so assume $|V(M)| > 4$. Thus M has at least 3 triangles.

We fix an orientation on the decomposition of $G - a$. As M contains at least 3 triangles and by Lemma (2.2.3), the vertices of the leftmost triangle must have precisely degrees $2, 3, 4$ in M. Label these as y_1, x_1, y_2 respectively. Thus the leftmost triangle is defined by $\{y_1, x_1, y_2\}$. Let x_2 be the vertex that defines the next triangle $\{x_1, y_2, x_2\}$ in M.

From now on, alternate in subscript between x and y as the new vertex that defines the next triangle. By definition, one vertex in the most recently labelled triangle cannot be in any more cycles, due to Lemma (2.2.3). Thus at each step, the next triangle must contain the two remaining vertices of the current triangle, and we have a unique (up to orientation) labelling of the vertices of M.

Note that the only pairs of vertices which may have been special 2-cuts in $G - a$ correspond to $\{x_1, y_1\}$ and $\{x_{\lfloor \frac{m}{2} \rfloor}, y_{\lceil \frac{m}{2} \rceil}\}$ under this labelling. Thus the edges induced by these pairs are the only edges which are possibly not in G. □

We now turn our attention to the structure of a 3-connected 3-block.

(2.2.6) Lemma. *Let G be a 3-connected claw-free graph, $a \in V(G)$, M be a 3-connected 3-block in the decomposition of $G - a$. Let $\{b_1, c_1\}$ denote a special*

2-cut of $G - a$ contained in M such that the decomposition of $M - c_1$ has at least two 3-blocks and b_1 is in the leftmost 3-block, and let $\{b', c'\}$ denote an arbitrary special 2-cut in the decomposition of $M - c_1$. Then

(1) b_1 is the only internal vertex in the leftmost 3-block in the decomposition of $M - c_1$,

(2) $L_{G-a}(a) \nsubseteq \{b', c'\}$, and

(3) if $\{b_2, c_2\} \subseteq V(M)$ is the special 2-cut in the decomposition of $G - a$ other than $\{b_1, c_1\}$, then $\{b_2, c_2\} \neq \{b', c'\}$, and if $|\{b_2, c_2\} \cap \{b', c'\}| = 1$, then $\{b_2, c_2\} \cup \{b', c'\}$ is in a 3-block in the decomposition of $M - c_1$ that is either a square or a chain of two triangles.

Proof. Note that c_1 must have neighbors that are internal vertices in the leftmost and rightmost 3-blocks in the decomposition of $M - c_1$, say L and R, respectively; for otherwise M would have a 2-cut.

To prove (1), we assume for contradiction that there exists $x \in L_{M-c_1}(c_1)$ such that $x \neq b_1$ and x is an internal vertex of L. Let $y \in R_{M-c_1}(c_1)$ be an internal vertex of R. Note that as M is 3-connected and G is claw-free, we have $xc_1, yc_1 \in E(G)$. Since $\{b_1, c_1\}$ is a special 2-cut in the decomposition of $G - a$, there exists $z \in N_G(c_1)$ such that $z \notin V(M)$. Thus $\{c_1, x, y, z\}$ induce a claw in G, a contradiction. So such x does not exist. Hence, b_1 must be an internal vertex of the leftmost 3-block in the decomposition of $M - c_1$, otherwise M would not be 3-connected. So (1) holds.

If $L_{G-a}(a) \nsubseteq V(M)$, then $L_{G-a}(a) \nsubseteq \{b', c'\}$. So we may assume $L_{G-a}(a) \subseteq V(M)$. If $|L_{G-a}(a)| \geq 3$, then clearly $L_{G-a}(a) \nsubseteq \{b', c'\}$. So assume $|L_{G-a}(a)| \leq 2$ and assume further that $L_{G-a}(a) \subseteq \{b', c'\}$. Let $d \in L_{G-a}(a)$. Let $x \in L_{M-c_1}(d)$ and let $y \in R_{M-c_1}(d)$. Note that $x, y \notin L_{G-a}(a)$ since we assume $L_{G-a}(a) \subseteq \{b', c'\}$.

Thus $\{d, x, y, a\}$ induces a claw in G, a contradiction. Thus $L_{G-a}(a) \nsubseteq \{b', c'\}$, and we have (2).

To prove (3), assume $\{b_2, c_2\}$ is another special 2-cut in M in the decomposition of $G - a$. Assume further that $\{b_2, c_2\} \cap \{b', c'\} \neq \emptyset$ and that without loss of generality that $b_2 = b'$. Let $x \in L_{M-c_1}(b')$ and let $y \in R_{M-c_1}(b')$. Since $\{b_2, c_2\}$ is a special 2-cut in the decomposition of $G - a$, there exists $z \in N_G(b')$ such that $z \notin V(M)$. If both $x, y \neq c_2$, then $\{b', x, y, z\}$ induces a claw in G, a contradiction. So $c_2 \in \{x, y\}$ for any such choice of x and y. Hence $\{b', c'\} \neq \{b_2, c_2\}$, and we may assume without loss of generality that $x = c_2$ and that $\{x\} = L_{M-c_1}(b')$. So the 3-block M' in the decomposition of $M - c_1$ that is immediately left of $\{b', c'\}$ is a chain of cycles. Thus by Lemma (2.2.4) and by claw-freeness at b_1, c_2, the M' is either a square or a chain of at most two triangles. □

We also want a path that goes from left to right through each 3-connected 3-block, say M. In order to do that, we add a vertex that is adjacent to the vertices in some clique of M and a vertex adjacent to another clique of M, and then add an edge between the two new vertices. If we can find a cycle in this new graph using the edge between the two new vertices, then deleting the new vertices from the cycle results in the desired path through M. Since we want to use induction, we have to prove that this new graph still satisfies all the requirements of the main theorem and is smaller than G itself.

We now prove the specific result we need.

(2.2.7) Lemma. *Let G be a 3-connected claw-free graph, $a \in V(G)$, and M be a 3-connected 3-block in the decomposition of $G - a$. Let S_1 and S_2 denote two cliques in M such that neither is contained in the other. Assume for each $i \in \{1, 2\}$ and for each new vertex x not in M, $M \cup \{x, xy : y \in V(S_i)\}$ is claw-free.*

(1) Suppose $|S_i| \geq 2$ for both i, and let $\overline{M} = M \cup \{x_1, x_2, x_1x_2, x_1y, x_2z : y \in$

$V(S_1), z \in V(S_2)\}$, where x_1, x_2 are new vertices not in M. Then $\overline{M}$ is 3-connected and claw-free, and for any $e \in E(M)$, $\{e, x_1x_2\}$ does not induce a 3-cut in $\overline{M}$.

(2) Suppose $\max\{|S_1|, |S_2|\} \geq 2$, and let $\overline{M} = M \cup \{x, xy : y \in V(S_1 \cup S_2)\}$. Then $\overline{M}$ is 3-connected and claw-free, and for any $e \in E(M)$ and any $s \in S_1 \cup S_2$, $\{e, xs\}$ does not induce a 3-cut in $\overline{M}$.

(3) Suppose $V(S_1) = \{s_1\}$ and let $\overline{M} = M \cup \{s_1y : y \in V(S_2)\}$. Then $\overline{M}$ is 3-connected and claw-free, and for any $e \in E(M)$ and $s_2 \in V(S_2)$, $\{e, s_1s_2\}$ does not induce a 3-cut in $\overline{M}$.

Proof. Since M is 3-connected, the graph $\overline{M}$ in (3) is 3-connected. For (1) and (2), since M is 3-connected and by the requirement on the size of S_i, no 2-cut of $\overline{M}$ contains the new vertex. Hence since neither of S_1, S_2 is properly contained in the other, the graph $\overline{M}$ in (1) and (2) is also 3-connected.

Since $M \cup \{x, xy : y \in V(S_i)\}$ is claw-free for both i and for any new vertex x, we see that the graph $\overline{M}$ in (1), (2) and (3) is claw-free.

Now (1) holds, since e and x_1x_2 are not incident, and so cannot induce a 3-cut in $\overline{M}$. Also (2) holds, since $\overline{M} - x$ is 3-connected, and so $\{e, xs_1\}$ does not induce a 3-cut in $\overline{M}$.

To prove (3), it suffices to show that $M \cong K_4$ or $\overline{M} - b$ is 3-connected. Note first that $\overline{M} - b = M - b$. Since $M \cup \{x, xy : y \in V(S_i)\}$ is claw-free, $N_M(b)$ induces a clique in M. Thus for any pair of vertices that do not include b, any path between them that contains b can be modified to use the edge, say e, from the vertex x immediately before b to the vertex y immediately after b in the path. Since such a modified path uses no new vertices, the deletion of b does not lower the connectivity of M; unless in any three internally disjoint paths in M between x and y include both xy and xby, and $M - \{b, xy\}$ has a cut vertex separating x

from y. Note in the exceptional case, since $N_M(b)$ is a clique, $M \cong K_4$. Therefore, if $M \ncong K_4$ then $M - b$ is 3-connected, and hence $\overline{M} - b$ is 3-connected. □

As a consequence, we have the following

(2.2.8) Lemma. *Let $n \geq 7$ be an integer and assume the assertion of Theorem (1.2.2) holds for graphs of order $< n$. Let M be a 3-connected claw-free graph and let $m = |V(M)| < n$. Let $\{b_1, c_1\} \subseteq V(M)$ such that $b_1c_1 \in E(M)$ and $N_M(b_1) - c_1$ and $N_M(c_1) - b_1$ each induce a clique in M. Let $e = b_1c_1$ and let $f \in E(M - \{b_1, c_1\})$. If $m \geq 6$ then there exists a cycle C in M such that $\{e, f\} \subseteq E(C)$ and $|C| \geq \alpha m^\gamma + 5$.*

Proof. By assumption, M is 3-connected, claw-free, and $|V(M)| < n$. Since e and f do not share a vertex, they cannot induce a 3-cut. Suppose $m \geq 6$. Since we assume Theorem (1.2.2) holds for graphs with less than n vertices, there is a cycle C in M such that $\{e, f\} \subseteq E(C)$ and $|C| \geq \alpha m^\gamma + 5$. □

2.3 Chains of cycles

When trying to create a long cycle in G, we will often construct the cycle by connecting cycles or paths that we found in individual 3-blocks of the decomposition of $G - a$. The key point is that depending on the situation, sometimes it will be necessary to find a single path through such a 3-block and sometimes it will be necessary to find a cycle.

Finding the cycles we want in these 3-blocks will be relatively easy and thus we begin our analysis with them as a warm up. The ultimate goal of this section is to then take cycles in a string of adjacent 3-blocks and combine them into a cycle going through all of those 3-blocks.

We dedicate this section to the proof of a number of useful lemmas about paths and cycles in a chain of cycles. We will use the structural results in the previous section.

Note that a path from set A to set B may use vertices from $A \cup B$ as internal vertices.

(2.3.1) Lemma. *Let G be a 3-connected claw-free graph, $a \in V(G)$, M be a 3-block in the decomposition of $G - a$. Assume that M is a chain of triangles whose vertices are labelled as in Lemma (2.2.5), and let e be an arbitrary edge of M. Then there exists a path P in M from $\{x_1, y_1\}$ to $\{x_{\lfloor \frac{m}{2} \rfloor}, y_{\lceil \frac{m}{2} \rceil}\}$ such that $e \in P$, $P - e \subseteq G$, $|V(P) \cap \{x_1, y_1\}| = 1$ unless $e = x_1y_1$, and the following holds:*

(1) If $m = 3$ then $|E(P)| = 1$ when $e \notin \{x_1y_1, x_{\lfloor \frac{m}{2} \rfloor}y_{\lceil \frac{m}{2} \rceil}\}$, and $|E(P)| = 2$ when $e \in \{x_1y_1, x_{\lfloor \frac{m}{2} \rfloor}y_{\lceil \frac{m}{2} \rceil}\}$.

(2) If $m \geq 4$ then $|E(P)| = m - 3$ when $e \notin \{x_1y_1, x_{\lfloor \frac{m}{2} \rfloor}y_{\lceil \frac{m}{2} \rceil}\}$, and $|E(P)| = m - 2$ when $e \in \{x_1y_1, x_{\lfloor \frac{m}{2} \rfloor}y_{\lceil \frac{m}{2} \rceil}\}$.

Proof. Consider $m = 3$. If $e \in \{x_1y_1, x_1y_2\}$ then $\{y_1y_2, e\}$ induces the desired path for (1). If $e \notin \{x_1y_1, x_1y_2\}$ then the edge e induces the desired path for (1).

Thus we may assume $m \geq 4$. Let Q be the path induced by $\{x_iy_i, x_iy_{i+1} :$ for all $i\}$, and let P' be obtained from Q by removing both ends of Q. Then P' is a path of length $m - 3$.

If $e \in P'$, then P' is the desired path for (2). If $e \in \{x_1y_1, x_{\lfloor \frac{m}{2} \rfloor}y_{\lceil \frac{m}{2} \rceil}\}$, then simply add e and its incident vertex to P' and this new path is the desired path (of length $m - 2$) for (2). If $e = x_ix_{i+1}$, let $P := (P' - \{x_iy_{i+1}, x_{i+1}y_{i+1}\}) \cup e$. If $e = y_1y_2$, let $P := (P' - x_1) \cup \{y_1, e\}$. If $i \neq 1$, $e = y_iy_{i+1}$ and x_{i+1} is a vertex in the graph, let $P := (P' - \{x_iy_i, x_iy_{i+1}\}) \cup e$. If $i \neq 1$, $e = y_iy_{i+1}$ and x_{i+1} is not a vertex in the graph, then m is odd and $y_{i+1} = y_{\lceil \frac{m}{2} \rceil}$, and let $P := (P' - x_{\lfloor \frac{m}{2} \rfloor}) \cup \{e, y_{\lceil \frac{m}{2} \rceil}\}$. In all cases, P is the desired path (of length $m - 3$) for (2). □

The next result finds a cycle in a chain of cycles M that contains all vertices of degree 2. In particular, such a cycle will contain any edge in M whose ends are the vertices of a special 2-cut.

(2.3.2) Lemma. *Let G be a 3-connected claw-free graph, $a \in V(G)$, M be a 3-block in the decomposition of $G - a$ such that M is a chain of cycles. Then M has a Hamilton cycle that contains all vertices of degree 2 in M.*

Proof. By Lemma (2.2.4) either M is a square, a square and a triangle, a square with a triangle on both sides, or a chain of triangles. In any case, by simply deleting all edges e of M such that $V(e)$ is a 2-cut of M, we obtain the desired Hamilton cycle in M. □

We continue with a lemma that find paths through a chain of cycles containing a specific edge.

(2.3.3) Lemma. *Let G be a 3-connected claw-free graph, $a \in V(G)$, and M be an extreme 3-block in the decomposition of $G - a$ (without loss of generality, the leftmost 3-block) such that M is a chain of cycles and $m = |V(M)|$. Let $\{b_1, c_1\}$ denote the special 2-cut of $G - a$ contained in M, and let $e \in E(M)$ be arbitrary. Then there exists a path P in M from $\{b_1, c_1\}$ to $L_{G-a}(a)$ such that $e \in E(P)$, $P - e \subseteq G$, and the following holds:*

(1) If $m \leq 4$ then $|E(P)| \geq \left\lfloor \frac{m}{2} \right\rfloor$ when $e \neq b_1c_1$, and $|E(P)| \geq \left\lfloor \frac{m}{2} \right\rfloor + 1$ when $e = b_1c_1$.

(2) If $m = 5$ and M is not a chain of triangles, then $|E(P)| = 3$ when $e \neq b_1c_1$, and $|E(P)| = 4$ when $e = b_1c_1$.

(3) If $m \geq 5$ and M is a chain of triangles, then $|E(P)| \geq m - 3$ when $e \neq b_1c_1$, and $|E(P)| \geq m - 2$ when $e = b_1c_1$.

Moreover, if $m \geq 4$, $|E(P)| \geq \alpha M^\gamma + 1$.

Proof. By Lemma (2.2.4) M is either a square, a square and a triangle, or a chain of triangles.

Suppose M contains a square, say xyb_1c_1x. First, assume M is a square. Then $\{x, y\} \subseteq L_{G-a}(a)$. If $e \neq b_1c_1$ then c_1xy or b_1yx gives the desired path for (1), and if $e = b_1c_1$ then c_1b_1yx gives the desired path for (1). Thus, we may assume M is a square and a triangle. Let z be the vertex in the triangle that is not in the square; then $z \in L_{G-a}(a)$. If $e = b_1c_1$ then c_1b_1yxz gives the desired path for (2). If $e \neq b_1c_1$ then c_1xyz or b_1yxz is the desired path for (2).

Thus we may assume M is a chain of triangles. Let the vertices of M be labelled as in Lemma (2.2.5), and without loss of generality let $\{b_1, c_1\} = \{x_{\lfloor \frac{m}{2} \rfloor}, y_{\lceil \frac{m}{2} \rceil}\}$. Then $y_1 \in L_{G-a}(a)$.

Suppose $m \geq 5$. Note that $G' := (G - y_1) + ay_2$ is 3-connected and claw-free, and we may view $M - y_1$ (which is chain of triangles) as the the decomposition of $G' - a$. So we can apply Lemma (2.3.1) to $M - y_1$ and find a path P' in $M - y_1$ from $\{b_1, c_1\}$ to x_1 such that $|E(P)| \geq (m-1) - 3$ if $e \neq b_1c_1$, and $|E(P)| \geq (m-1) - 2$ if $e = b_1c_1$. Now $P := P' \cup \{y_1, x_1y_1\}$ gives the desired path for (3).

Now assume $m = 4$. Then $\{b_1, c_1\} = \{x_2, y_2\}$. If $e \neq b_1c_1$ then $y_2x_1y_1$ is the desired path for (1), and if $e = b_1c_1$ then $x_2y_2x_1y_1$ is the desired path for (1).

Finally, consider $m = 3$. Then $\{b_1, c_1\} = \{x_1, y_2\}$. If $e \neq b_1c_1$ then y_2y_1 is the desired path for (1), and if $e = b_1c_1$ then $x_2y_2y_1$ is the desired path for (1). □

In order to combine various paths in 3-blocks, we also need two results about a path in a chain of cycles from left to right and avoiding a specific vertex.

(2.3.4) Lemma. *Let G be a 3-connected claw-free graph, $a \in V(G)$, and M be a middle 3-block in the decomposition of $G - a$ such that M is a chain of cycles and $m = |V(M)|$. Let $\{b_1, c_1\}$ and $\{b_2, c_2\}$ denote the special 2-cuts of $G - a$ contained in M. Then there exists a path P in M from b_1 to $\{b_2, c_2\}$ such that $c_1 \notin P$, $P \subseteq G$, and the following holds:*

(1) If $m = 3$, then $|E(P)| = 0$ when $b_1 \in \{b_2, c_2\}$, and $|E(P)| = 1$ when $b_1 \notin$

$\{b_2, c_2\}$.

(2) *If* $m = 4$ *then* $|E(P)| = 1$.

(3) *If* $m \geq 5$ *then* $|E(P)| \geq m - 3$.

Moreover, $|E(P)| \geq \alpha m^\gamma$ *when* $m \geq 4$, *and* $|E(P)| \geq \alpha m^\gamma + 1$ *when* $m \geq 5$.

Proof. By Lemma (2.2.4) M is either a square, or a chain of triangles. Note that if $m = 4$ then $\{b_1, c_1\} \cap \{b_2, c_2\} \neq \emptyset$ (as M is a middle block). So if $m = 4$ and it is trivial to construct the desired path for (2). Thus we may assume $m \neq 4$, and hence M is a chain of triangles.

Suppose $m = 3$. As there are only three vertices total, $|\{b_1, c_1\} \cap \{b_2, c_2\}| = 1$. If $b_1 \in \{b_2, c_2\}$ then b_1 is the desired path for (1). If $b_1 \notin \{b_2, c_2\}$ then without loss of generality assume $c_1 = c_2$, and so $b_1 b_2$ is the desired path for (1).

Thus we may assume $m \geq 5$. Let $\{b, c\}$ denote the neighborhood of $\{b_1, c_1\}$ in M, such that $bb_1, cc_1, b_1 c \in E(M)$. Note that $M'' := M - \{b_1, c_1\}$ is a chain of triangles as $m \geq 5$. So applying induction, we may find a path P' in $M' - c$ from b to $\{b_2, c_2\}$ such that $c \notin P'$, $P' \subseteq G$, and $|E(P')| \geq (m - 2) - 3 = m - 5$. Now $P := P' \cup \{c, b_1, bc, b_1 c\}$ gives the desired path for (3). □

(2.3.5) Lemma. *Let* G *be a 3-connected claw-free graph,* $a \in V(G)$, *and* M *be an extreme 3-block in the decomposition of* $G - a$ *(without loss of generality, the leftmost 3-block) such that* M *is a chain of cycles and* $m = |V(M)|$. *Let* $\{b_1, c_1\}$ *denote the special 2-cut of* $G - a$ *contained in* M. *Then there exists a path* P *in* M *from* b_1 *to* $L_{G-a}(a)$ *such that* $c_1 \notin P$, $P \subseteq G$, *and the following hold:*

(1) *If* $m \leq 4$, *then* $|E(P)| \geq \left\lfloor \frac{m}{2} \right\rfloor$.

(2) *If* $m \geq 5$, *then* $|E(P)| \geq \alpha m^\gamma + 2$

Proof. By Lemma (2.2.4) M is either a square, or a square with a triangle, or a chain of triangles.

Suppose M contains a square. Then b_1c_1 must be contained in the square. So let $b_1c_1xyb_1$ be the square. If M is a square, then $x, y \in L_{G-a}(a)$, and b_1yx is the desired path for (1). So assume M is a square with a triangle. Let $xyzx$ be that triangle. Then $z \in L_{G-a}(a)$, and b_1yxz is the desired path for (2).

Thus we may assume M is a chain of triangles. Suppose $m = 4$. Let b_1c_1y be one the triangles, and let yxc_1y or yxb_1y denote the other triangle in M. Then $x \in L_{G-a}(a)$, and in either case, b_1yx is as desired for (1).

Suppose $m = 3$. Let b_1c_1x denote the triangle in M. Clearly, $x \in L_{G-a}(a)$, and b_1x is as desired for (1).

Thus we may assume $m \geq 5$. Let $\{b, c\}$ denote the neighborhood of $\{b_1, c_1\}$ in M, such that $bb_1, cc_1, b_1c \in E(M)$. Note that $M'' := M - \{b_1, c_1\}$ is a chain of triangles as $m \geq 5$. So applying induction, we may find a path P' in $M' - c$ from b to $L_{G-a}(a)$ such that $c \notin P'$, $P' \subseteq G$, and $|E(P')| \geq \alpha(m-3)^\gamma + 1$. Now $P := P' \cup \{c, b_1, bc, b_1c\}$ gives the desired path for (3). □

CHAPTER III

ADVANCED RESULTS

3.1 Basic paths and cycles through 3-connected 3-blocks

Again, let G be a 3-connected claw-free graph and $a \in V(G)$. In this section we instead concentrate on finding paths through the 3-connected 3-blocks in the decomposition of $G - a$. As in the previous section on chains of cycles, we intend to find paths that go through several consecutive 3-blocks in the decomposition of $G - a$. However, the proofs of those results will at times be substantially more complicated. The source of the complication is due to the nature of the paths in each of the individual 3-blocks. It will be important for our path to contain one special edge e that is somewhere in $G - a$. By construction, we will be able to require that e lie entirely inside one 3-block in the decomposition. Thus we will first find a path through the 3-block that actually contains e. However, when we seek to extend this path through the next 3-block, new requirements arise. We no longer need to go through a special edge e, but we instead need the path to start at the exact same vertex that the previous path ends in (and avoid the other vertex in the shared special 2-cut). Thus we will need to prove separate results for each of these two scenarios.

We begin our analysis with 3-blocks that actually contain the special edge e. Note that there are several different length results highlighted in the following Lemmas. For clarity, we briefly provide an outline for the proofs to motivate the differences. To a given 3-connected 3-block M, we will add a small number of vertices and edges and then use the inductive hypothesis of Theorem (1.2.2) to find a cycle. We then delete the vertices and edges we added to find the desired

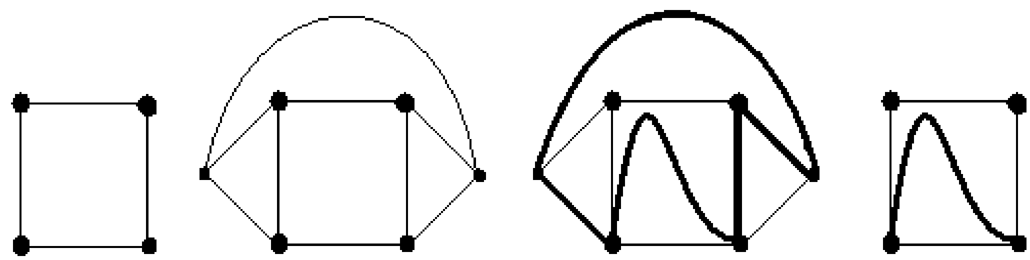

Figure 3.1.1: Representation of a 3-connected 3-block to illustrate the inductive technique used throughout this section. Note that this figure is greatly simplified and only important vertices are drawn. (a) Representation of a 3-connected 3-block M (b) Modified graph M' (c) Using the inductive hypothesis to find a path in M' (d) Obtain a path in the original 3-block M

path. However, we do not want this path to contain any virtual edges (except perhaps the special edge e). Fortunately, if the path contains any such virtual edges, they will have to be at its ends, and thus we can remove them by simply shortening the path by one or two edges. If we shorten the path in this manner, it also restricts the structure of the path on the side that was shortened. The different length results and their associated structural restrictions stem from whether or not virtual edges were removed from the path in this process.

(3.1.1) Lemma. *Let $n \geq 7$ and assume the assertion of Theorem (1.2.2) holds for graphs of order $< n$. Let G be a 3-connected claw-free graph of order at most n, $a \in V(G)$, and M be a middle 3-connected 3-block in the decomposition of $G - a$ with $m = |V(M)|$. Let $\{b_1, c_1\}$ and $\{b_2, c_2\}$ be the special 2-cuts of $G - a$ contained in M. Let e be an arbitrary edge in M. Then there exists a path P in M from $\{b_1, c_1\}$ to $\{b_2, c_2\}$ such that $e \in P$, $P - e \subseteq G$, and the following hold:*

(1) $|E(P)| \geq \alpha(m + 2)^\gamma$,

(2) $|E(P)| \geq \alpha(m+2)^\gamma + 1$, *unless for both i, $|V(P) \cap \{c_i, b_i\}| = 1$ and $c_i b_i \notin G$,*

and

(3) $|E(P)| \geq \alpha(m+2)^\gamma + 2$, *unless for some* i, $|V(P) \cap \{c_i, b_i\}| = 1$ *and* $c_ib_i \notin G$.

Proof. Since M is not an extreme 3-block, besides a there are at least 2 other vertices in G that are not in M. Thus $4 \leq m \leq n-3$.

Let $\overline{M} = M \cup \{x_i, x_ib_i, x_ic_i : i = 1, 2\}$, where x_1 and x_2 are new vertices not in M. Let $f = x_1x_2$. By Lemma (2.2.7)(1), $\overline{M}$ is 3-connected, claw-free, and $\{e, f\}$ does not induce a 3-cut in $\overline{M}$. Since $6 \leq |\overline{M}| \leq n-1$, it follows from assumption that there is a cycle C in $\overline{M}$ such that $e, f \in C$ and $|C| \geq \alpha(m+2)^\gamma + 5$. Then $P = C - \{x_1, x_2\}$ is a path in M such that $|E(P)| = |C| - 3$; however, P may also contain b_1c_1 or b_2c_2 which may be virtual edges. Note that if P contains either, then they are at the ends of P by construction. Without loss of generality, we may assume the ends of P are c_i if $c_ib_i \in P$. Let $P' = P - \{c_i : c_ib_i \in E(P)$ and $c_ib_i \notin E(G)\}$. Then $|E(P')| \geq |E(P)| - 2 \geq \alpha(m+2)^\gamma$. Note that for each $i \in \{1, 2\}$ for which we did not remove c_i from P, the bound for $|E(P')|$ improves by 1. □

Next we find such paths in a 3-connected extreme 3-block. The major difference is that on only one side of the 3-block there will be a special 2-cut as before. However, the other side will merely be the neighbors of a – which may be a single vertex, or a clique. Lastly, due to the nature of our induction, we cannot directly consider the case where $m = n-2$.

(3.1.2) Lemma. *Let* $n \geq 7$ *and assume the assertion of Theorem (1.2.2) holds for graphs of order* $< n$. *Let* G *be a 3-connected claw-free graph of order at most* n, $a \in V(G)$, M *be an extreme 3-connected 3-block in the decomposition of* $G - a$ *(without loss of generality, the leftmost 3-block) with* $m = |V(M)| < n-2$. *Let* $\{b_1, c_1\}$ *be the special 2-cut of* $G - a$ *contained in* M, *and let* $e \in E(M)$

be arbitrary. Then there exists path P in M from $\{b_1, c_1\}$ to $L_{G-a}(a)$ such that $e \in E(P)$, $P - e \subseteq G$, and the following holds:

(1) if $m = 4$ then $|E(P)| \geq 2$, and $|E(P)| = 3$ if $b_1c_1 \in G$;

(2) if $m \geq 5$ and $|L_{G-a}(a)| = 1$ then $|E(P)| \geq \alpha(m+1)^\gamma + 2$, and $|E(P)| \geq \alpha(m+1)^\gamma + 3$ unless $|V(P) \cap \{b_1, c_1\}| = 1$ and $b_1c_1 \notin E(G)$;

(3) if $m \geq 5$ and $|L_{G-a}(a)| \geq 2$ then $|E(P)| \geq \alpha(m+2)^\gamma + 1$, and $|E(P)| \geq \alpha(m+2)^\gamma + 2$ unless $|V(P) \cap \{b_1, c_1\}| = 1$ and $b_1c_1 \notin E(G)$.

Moreover, in all cases above, $|E(P)| \geq \alpha(m+2)^\gamma + 1$.

Proof. Assume $m = 4$. Let $\{b_2, c_2\}$ be the other two vertices of M. As $|L_{G-a}(a) \cap V(M - \{b_1, c_1\})| \geq 1$, we may assume without loss of generality that $b_2 \in L_{G-a}(a)$. If $b_1c_1 \in G$ then the path P in (1) (of length 3) can be easily found. So assume $b_1c_1 \notin G$. Then the path P can be found of length 3, unless $e = b_2c_2$ in which case P has length 2.

Thus we may assume $m \geq 5$. Next we need to consider two cases based on $|L_{G-a}|$.

Case 1. $|L_{G-a}(a)| = 1$.

Let d be the unique vertex in $L_{G-a}(a)$. Let $\overline{M} = M \cup \{x, xd, xb_1, xc_1\}$, where x is a new vertex not in M. By Lemma (2.2.7)(2), $\overline{M}$ is 3-connected, claw-free, and $\{e, xd\}$ does not induce a 3-cut in $\overline{M}$.

As $m \geq 5$, we may use the inductive hypothesis of Theorem (1.2.2) to find a cycle C in $\overline{M}$ such that $\{e, xd\} \subseteq E(C)$ and $|C| \geq \alpha(m+1)^\gamma + 5$. Then $P = C - x$ is a path in M and $|E(P)| = |C| - 2$; however, P may also contain b_1c_1 which need not be in $E(G)$. If $b_1c_1 \in P$, then we may assume without loss of generality that c_1 is an end of P. Let $P' = P - \{c_1 : c_1b_1 \in E(P) - E(G)\}$. Then

$|E(P')| \geq |E(P)| - 1 \geq \alpha(m+1)^\gamma + 2$. Moreover, if $c_1b_1 \in P$ then the bound for $|E(P')|$ improves by 1.

Case 2. $|L_{G-a}(a)| \geq 2$.

Let $\overline{M} = M \cup \{x_1, x_2, x_1b_1, x_1c_1, x_2y : y \in L_{G-a}(a)\}$. By Lemma (2.2.7)(1), $\overline{M}$ is 3-connected, claw-free, and $\{e, x_1x_2\}$ does not induce a 3-cut. As $n-2 > m \geq 5$, we may use the inductive hypothesis of Theorem (1.2.2) to find a cycle C in $\overline{M}$ such that $\{e, x_1x_2\} \subseteq E(C)$ and $|C| \geq \alpha(m+2)^\gamma + 5$. (Note that this is the place we need $m < n-2$.) Then $P = C - \{x_1, x_2\}$ is a path in M and $|E(P)| = |C| - 3$. Let $P' = P - \{c_1 : c_1b_1 \in E(P) - E(G)\}$. Then $|E(P')| \geq |E(P)| - 1 \geq \alpha(m+2)^\gamma + 1$. Moreover, if $c_1b_1 \in E(G)$, the bound for $|E(P')|$ improves by 1. □

We now consider a string of 3-blocks from the decomposition of $G - a$ and attempt to find a long cycle going through all of them.

(3.1.3) Lemma. *Let $n \geq 7$ be an integer and assume the assertion of Theorem (1.2.2) holds for graphs of order $< n$. Let G be a 3-connected claw-free graph of order at most n, $a \in V(G)$, and $M_1, ..., M_k$ ($k \geq 2$) be consecutive 3-blocks (without loss of generality, from left to right) in the decomposition of $G - a$ with $m = |V(\cup_{i=1}^k M_i)|$. Let $e \in E(M_1)$ such that $V(e) \neq V(M_1 \cap M_2)$ and $V(e)$ is not a cut of M_1, and let $f \in E(M_k)$ such that $V(f) \neq V(M_{k-1} \cap M_k)$ and $V(f)$ is not a cut of M_k.*

(1) If $m = 5$ then there is a Hamilton cycle C in $\cup_{i=1}^k M_i$ such that $\{e, f\} \subseteq E(C)$ and $C - \{e, f\} \subseteq G$.

(2) If $m \geq 6$ then there is a cycle C in $\cup_{i=1}^k M_i$ such that $\{e, f\} \subseteq E(C)$, $C - \{e, f\} \subseteq G$, and $|E(C)| \geq \alpha m^\gamma + 5$.

Proof. Let $m_i = |V(M_i)|$ for $i = 1, \ldots, k$, and let $\{c_i, b_i\} = V(M_i \cap M_{i+1})$ for $i = 1, \ldots, k-1$. Let $e = c_0b_0$ and let $f = c_kb_k$. Using Lemma (2.1.1) (when M_i is

3-connected and $m_i \le 6$) or Lemma (2.2.8) (when M_i is 3-connected and $m_i \ge 6$) or Lemma (2.3.2) (when M_i is a chain of cycles), we can find a cycle C_i in M_i such that $\{c_{i-1}b_{i-1}, c_ib_i\} \subseteq E(C_i)$, $C_i - \{c_{i-1}b_{i-1}, c_ib_i\} \subseteq G$, C_i is a Hamilton cycle in M_i if $m_i \le 5$, and $|C_i| \ge \alpha m_i^\gamma + 5$ if $m \ge 6$.

Let $C = (\cup_{i=1}^k C_i) - \{c_ib_i : 1 \le i \le k-1\}$. Clearly, C is a cycle, $\{e, f\} \subseteq E(C)$, and $C - \{e, f\} \subseteq G$. It remains to determine the length of C.

First, assume $m = 5$. As $k \ge 2$, M_1 and M_2 cannot both be 3-connected. Hence without loss of generality, $M_1 = K_4$ and M_2 is a triangle. Thus C_1 and C_2 are Hamilton cycles in M_1 and M_2 respectively, and C is a Hamilton cycle in $G - a$.

Now assume $m \ge 6$, which implies $m \ge \alpha m^\gamma + 5$. Note that $|C| = (\sum_{i=1}^k (|C_i| - 2)) + 2$. If for all i, C_i is Hamilton cycle in M_i, then C is Hamilton cycle in $G - a$, and hence $|C) \ge \alpha m_i^\gamma + 5$. Thus at least one C_i has $|C_i| - 2 \ge \alpha m_i^\gamma + 5 - 2 = \alpha m_i^\gamma + 3 \ge \alpha m_i^\gamma$. Note further that $m_i - 2 \ge \alpha m_i^\gamma$ as $m_i \ge 3$, for all i. Thus $|C| \ge \alpha \sum_{i=1}^k m_i^\gamma + 3 + 2 \ge \alpha m^\gamma + 5$. □

Next we need to prove results about paths that avoid a specific vertex. However, the proofs of these results will be somewhat more complicated. In fact, we will need to prove two results together by induction, one for when M is an extreme 3-block and one for when it is a middle 3-block.

(3.1.4) Lemma. *Let $n \ge 7$ and assume the assertion of Theorem (1.2.2) holds for graphs of order $< n$. Let G be a 3-connected claw-free graph of order at most n, $a \in V(G)$, and M be a 3-connected 3-block in the decomposition of $G - a$ with $m = |V(M)| \ge 5$.*

(1) *Assume M is a middle 3-block in the decomposition of $G - a$, and let $\{b_1, c_1\}$ and $\{b_2, c_2\}$ be the special 2-cuts of $G - a$ contained in M. Then there exists a path P in M from b_1 to $\{b_2, c_2\}$ such that $c_1 \notin P$, $P \subseteq G$, and $|E(P)| \ge \alpha m^\gamma + 1$.*

(2) *Assume M is an extreme 3-block in the decomposition of $G-a$ (without loss of generality, the leftmost), and let $\{b_1, c_1\}$ be the special 2-cut of $G-a$ contained in M. Then there exists a path P in M from b_1 to $L_{G-a}(a)$ such that $c_1 \notin P$, $P \subseteq G$, and $|E(P)| \geq \alpha m^\gamma + 2$.*

Proof. Let $S = \{b_2, c_2\}$ for (1), and $S = L_{G-a}(a)$. Note that for (1), $S \cap \{b_1, c_1\} = \emptyset$, and for (2), $S - \{b_1, c_1\} \neq \emptyset$.

First, we show (1) holds when $m = 5, 6$. Since M is a middle 3-block, $\{b_1, c_1\} \cap \{b_2, c_2\} = \emptyset$. If there exists an $x \in V(M) - \{b_1, c_1, b_2, c_2\}$ such that $xb_1, xv \in E(G)$ for some $v \in \{b_2, c_2\}$, then b_1xv gives the desired path for (1). So we may assume such x does not exist. Then $m = 6$, and let $x, y \in V(M) - \{b_1, c_1, b_2, c_2\}$ such that $xb_1 \in E(G)$. Then $\{xy, xb_1, xc_1, yb_2, yc_2\} \subseteq E(M)$. Now b_1xyb_2 is the desired path for (1).

Next, we prove (2) for $m = 5, 6$. Let $b \in L_{G-a}(a) - \{b_1, c_1\}$. Since $M - c_1$ is 2-connected, there exist internally disjoint paths Q_1, Q_2 from b_1 to b, and we may assume $|E(Q_1)| \geq |E(Q_2)|$, and subject to this $|V(Q_1 \cup Q_2)|$ is maximum. So $|E(Q_1)| \geq 2$. In fact we may assume $|E(Q_1)| = 2$ as otherwise Q_1 gives the desired P. Let u be the internal vertex of Q_1. Suppose $|E(Q_2)| = 2$ then let v denote the internal vertex of Q_2. Since $N_M(b_1) - c_1$ induces a clique in M, $uv \in E(M)$. Hence b_1uvb gives the desired P. So we may assume $|E(Q_2)| = 1$. Since $m \geq 5$, let $x \in V(M) - \{b_1, c_1, b, u\}$. Then there are paths R_1, R_2 from x to $r_1, r_2 \in V(Q_1 \cup Q_2)$ internally disjoint from $Q_1 \cup Q_2$ such that $V(R_1 \cap R_2) = \{x\}$. By the maximality of $|V(Q_1 \cup Q_2)|$, we must have $u \in \{r_1, r_2\}$. Then $Q_1 \cup R$ contains the desired path P.

For induction consider some $m \geq 7$ and assume that the assertion of Lemma (3.1.4) is true for 3-connected 3-blocks of order $< m$.

Consider now a 3-connected 3-block M such that $|V(M)| = m$. Note that $m \leq n-2$ as in both (1) and (2) it is clear that there are at least two 3-blocks in the decomposition of $G - a$. For (1), M is a middle 3-block of the decomposition of $G - a$, and we define $S = \{b_2, c_2\}$. For (2), M is the leftmost 3-block of the decomposition of $G - a$, so we define $S = L_{G-a}(a)$.

Claim 1. We may assume $M - c_1$ is not 3-connected.

Suppose $M - c_1$ is 3-connected. Let $\overline{M} = (M - c_1) + \{x, xb_1, xs : s \in S\}$, where x is a new vertex. Let e be an arbitrary edge of $M - c_1$. By Lemma (2.2.7)(1), $\overline{M}$ is 3-connected and claw-free, and $\{e, xb_1\}$ does not induce a 3-cut. Lastly, $|V(\overline{M})| = m - 1 + 1 = m$ and $7 \leq m < n$; thus we can use Theorem (1.2.2) to find a cycle C in $\overline{M}$ such that $\{e, xb_1\} \subseteq E(C)$ and $|C| \geq \alpha m^\gamma + 5$. Then $C - x$ is a path in $M - c_1$ from b_1 to S, and if $b_2c_2 \in E(C - x)$ then it would be at one end of the path. Note that this is the only edge in $M - c_1$ which need not be an edge of G. Without loss of generality, we may assume c_2 is an end of $C - x$. Thus if $b_2c_2 \in E(M - c_1) - E(G)$, we define $P = (C - x) - c_2$; otherwise define $P = C - x$. In either case, $|E(P)| \geq |C| - 3 \geq \alpha m^\gamma + 2$. So both (1) and (2) hold.

Claim 2. We may assume that the decomposition of $M - c_1$ has at least two 3-blocks.

For, suppose the decomposition of $M - c_1$ has exactly one 3-block. Then by Claim 1, $M - c_1$ is a chain of cycles. By Lemma (2.2.4) and since $m \geq 7$, $M - c_1$ is either a chain of triangles or a square with a triangle on opposite sides. Further, $N_M(c_1) - b_1$ is a clique by Lemma (2.2.2). Thus in either case, b_1 must be one of the two vertices of degree 2 in $M - c_1$.

Consider first the case where $M - c_1$ is a square with a triangle on opposite sides. Let $z_1z_2z_3z_4z_1$ be the square in $M - c_1$ and let $z_1z_2b_1z_1$ and $z_3z_4dz_3$ be the triangles in $M - c_1$. Choose an $s \in S$. By symmetry, we may assume that $s \in \{z_1, z_3, d\}$. If $s = z_1$, then $b_2c_2 \in \{z_1b_1, z_1z_2, z_1z_4\}$, and so, $b_1z_2z_3dz_4z_1$ or $b_1z_2z_3dz_4$ gives the desired path for (1) or (2). If $s = z_3$, then $b_2c_2 \in \{z_3d, z_3z_4, z_3z_2\}$, and so, $b_1z_2z_1z_4dz_3$ or $b_1z_2z_1z_4z_3$ gives the desired path for (1) or (2). If $s = d$ then

$b_2c_2 \in \{dz_3, dz_4\}$, and so, $b_1z_1z_2z_3z_4d$ or $b_1z_2z_1z_4z_3d$ gives the desired path for (1) or (2).

Thus we may assume that $M - c_1$ is a chain of triangles. As in Lemma (2.2.5), label the vertices of $M - c_1$ as $x_1, ..., x_{\lfloor \frac{m-1}{2} \rfloor}, y_1, ..., y_{\lceil \frac{m-1}{2} \rceil}$, where $b_1 = y_1$.

Assume for some $i > 1$ that $y_i \in S$. Then $b_2c_2 \in \{y_ix_{i-1}, y_ix_i, y_iy_{i-1}, y_iy_{i+1}\}$. If $i < \lceil \frac{m-1}{2} \rceil$ or if $i = \lceil \frac{m-1}{2} \rceil$ and $\lceil \frac{m-1}{2} \rceil = \lfloor \frac{m-1}{2} \rfloor$, then let $P = y_1x_1...y_{i-1}x_{i-1}x_i...x_{\lfloor \frac{m-1}{2} \rfloor}y_{\lceil \frac{m-1}{2} \rceil}...y_i$. If $i = \lceil \frac{m-1}{2} \rceil$ and $\lceil \frac{m-1}{2} \rceil > \lfloor \frac{m-1}{2} \rfloor$, then let $P = y_1x_1...y_{i-1}x_{i-1}y_i$. We see that P or $P - y_i$ gives the desired path for (1) and (2).

Next assume for some i that $x_i \in S$. Then $b_2c_2 \in \{x_ix_{i-1}, x_iy_i, x_iy_{i+1}, x_ix_{i+1}\}$. Let $P = y_1x_1...y_{i-1}x_{i-1}y_i...y_{\lceil \frac{m-1}{2} \rceil}x_{\lfloor \frac{m-1}{2} \rfloor}...x_i$. Then P or $P - x_i$ gives the desired path for (1) and (2).

By Claim 2, let $k \geq 2$ and let $M_1, ..., M_k$ be the 3-blocks from left to right in the decomposition of $M - c_1$. Note that since $b_1c_1 \in E(M)$, b_1 must be in M_1 or M_k. Since the assignment of orientation was arbitrary, we may assume $b_1 \in M_1$. Let $m_i = |V(M_i)|$. Then $m_i < n - 2$.

For $1 \leq i \leq k-1$, let $S_i = V(M_i) \cap V(M_{i+1})$, and let e_i denote the virtual edge between the vertices of S_i. Let $S_0 = \{b_1\}$. As S induces a clique in M and as G is claw-free, it follows from Lemma (2.2.6) that there is only one M_i which contains all of S. Let j be the index such that $S \subseteq V(M_j)$. We will construct a path in $\cup_{i=1}^{k} M_i$ from S_0 to S that is of the desired length by finding a path in M_j, a path in $\cup_{i=1}^{j-1} M_i$, and a path in $\cup_{i=j+1}^{k} M_i$.

Claim 3. There exists a path P_j in M_j from $s_{j-1} \in S_{j-1}$ to S such that

(a) if $j < k$ then $e_j \in P_j$ and $P_j - e_j \subseteq G$,

(b) if $j > 1$ then $|E(P_j)| \geq \alpha(m_j - 2)^\gamma + 1$, and

(c) if $j = 1$ then $|E(P_j)| \geq \alpha(m_j - 3)^\gamma + 2$.

To prove Claim 3, let $e \in E(M_j)$ be arbitrary if $j = k$; otherwise, let $e = e_j$.

First, assume $m_j \leq 4$. It is clear that we can find the path P_j of length 2, and hence, satisfying (a) and (b). To prove (c), let $j = 1$. If $m_1 = 4$, then M_1 is either K_4 or a chain of two triangles. In either case it is trivial to construct the desired path from b_1 to S containing e of length $3 \geq 2 + \alpha m_1^\gamma$, and (c) holds. So assume $m_1 = 3$. Let $V(M_1) = \{b_1, x, y\}$ where $\{x, y\} = S_1$. Since $b_1 \neq S$, either x or $y \in S$. Without loss of generality, assume $x \in S$. Thus it is trivial to construct the desired path from b_1 to x containing e of length $2 \geq 2 + \alpha(m_1 - 3)^\gamma$, and (c) holds.

Now assume $m_j \geq 5$. For $j > 1$, let $M^* = M_j \cup \{x, xy : y \in S_{j-1} \cup S_j\}$, where x is a new vertex and $S_j = S$ if $j = k$. By Lemma (2.2.7)(2), M^* is 3-connected and claw-free. Moreover, M_j is the only 3-block in the decomposition of $M^* - c_1$. Let e be an arbitrary edge of M_j. The (a) and (b) follows by applying Lemmas (2.3.3) and (3.1.2) (with S_j, M_j, S_{j+1} and $M^* - x$ playing roles of $\{b_1, c_1\}$, M, $L_{G-a}(a)$ and $G - a$, respectively).

So we may assume $j = 1$. If M_1 is a chain of cycles, then M_1 is a chain of triangles or a square with a triangle. Hence it is easy to find the path P_1 so that $|E(P_1)| \geq m_1 - 2 \geq \alpha m_1^\gamma + 2$, and (c) holds. So we may further assume that M_1 is 3-connected.

Suppose $|S| \geq 2$. Let $M^* = M_1 \cup \{z, zb_1, zs : s \in S\}$ and let $m^* = |V(M^*)|$. Then $m^* = m_1 + 1 < n$. By Lemma (2.2.7)(2), M^* is 3-connected and claw-free and $m^* < n$, and$\{zb_1, e\}$ does not induce a 3-cut in M^*. Thus by the inductive hypothesis of the main theorem, we can find a cycle C in M^* containing $\{zb_1, e\}$ of length at least $\alpha(m_1 + 1)^\gamma + 5$. Let $P_1 = C - z$ (and if $|S| = 2$, the edge incident to both vertices of S is not an edge of G, and P_1 contains that edge, then remove it as well), gives the desired path of length at least $\alpha m_1^\gamma + 2$, and (c) holds.

So we may assume $|S| = 1$. Let $M^* = M_1 \cup \{b_1 s : s \in S\}$ and let $m^* =$

$|V(M^*)| = m_1$ (which is less than n). By Lemma (2.2.7)(3), M^* is 3-connected and claw-free, and $\{b_1s, e\}$ does not induce a 3-cut in M^*. Thus by the inductive hypothesis of the main theorem, we can find a cycle C in M^* containing $\{b_1s, e\}$ of length at least $\alpha(m^*)^\gamma + 5$. Let $P_1 = C - b_1s$, which gives the desired path for (c).

Let $m_l = |V(\cup_{i=1}^{j-1} M_i)|$.

Claim 4. If $j \geq 2$, there exists a path P_L in $\cup_{i=1}^{j-1} M_i$ from b_1 to s_{j-1} such that

(a) $S_{j-1} - \{s_{j-1}\} \not\subseteq P_L$, $P_L \subseteq G$, $|E(P_L)| \geq \alpha(\sum_{i=1}^{j-1} m_i)^\gamma$, and

(b) $|E(P_L)| \geq \alpha m_l^\gamma + 1$ unless $j = 2$ and $m_1 = 3$.

First, we find the path P_i in M_i from $s_i \in S_i$ to $s_{i-1} \in S_{i-1}$ for $i = j-1, j-2, \ldots, 1$ in that order such that $s_1 = b_1$, $S_i \cap V(P_i) = \{s_i\}$, $P_i \subseteq G$, $|E(P_i)| \geq \alpha(m_i - 4)^\gamma + 1$ when $2 \leq i \leq j-1$ and $m_i \neq 3$, $|E(P_1)| \geq \alpha m_1^\gamma + 2$ when $m_1 \geq 5$, $|E(P_1)| \geq 2$ when $m_1 = 4$, and $|E(P_1)| \geq 1$ if $m_1 = 3$. If M_i is a chain of cycles, we find P_i by Lemmas (2.3.4) and (2.3.5). If M_i is 3-connected and $5 \leq |V(M_i)| < m$ we use the inductive hypothesis of Lemma (3.1.4)(1) to find our path of length at least $\alpha m_i^\gamma + 1$. If M_1 is 3-connected and $5 \leq |V(M_i)| < m$ we use the inductive hypothesis of Lemma (3.1.4)(2) to find our path of length at least $\alpha m_1^\gamma + 2$. If $M_i \cong K_4$, then trivially there is such a path of length at least 1 (or 2 if $i = 1$). If $m_1 = 3$, then clearly we can find P_1 so that $|E(P_1)| = 1$.

Let $P_L := \bigcup_{i=1}^{j-1} P_i$. Then P_L is a path from s_j to b_1, $S_{j-1} - \{s_{j-1}\} \not\subseteq P_L$, and $P_L \subseteq G$. It remains to prove the lower bound on $|E(P_L)|$. Clearly, we may assume $j \geq 3$.

We may assume that $j \geq 4$ or $m_2 \neq 3$. For, suppose $j = 3$ and $m_2 = 3$. Then M_2 is a triangle and M_1 is 3-connected. If $m_1 \geq 5$, then $|E(P_L)| = |E(P_1)| + |E(P_2)| \geq \alpha m_1^\gamma + 2 \geq \alpha(\sum_{i=1}^{j-1} m_i)^\gamma + 1$. So assume $M_1 \cong K_4$. Then $|E(P_L)| = |E(P_1)| + |E(P_2)| \geq 2 \geq \alpha(\sum_{i=1}^{j-1} m_i)^\gamma + 1$.

Therefore, if M_i is a triangle for some $2 \leq i \leq j-1$ then $j \geq 4$ and M_{i-1} or M_{i+1} is 3-connected. Hence, by combining at most two triangles with one 3-connected 3-block, we conclude that $\sum_{i=2} j-1|E(P_i)| \geq \alpha(m_l - m_1 + 2)^\gamma$.

We may assume $m_1 \leq 4$. Otherwise, $m_1 \geq 5$. Then $|E(P_1)| \geq \alpha m_1^\gamma + 2$. Hence $|E(P_L)| = \sum_{i=1}^{j-1} |E(P_i)| \geq \alpha m_1^\gamma + 2 + \alpha(m_l - m_1 + 2)^\gamma \geq \alpha m_l^\gamma + 2$.

We may further assume $m_1 = 3$. Otherwise, $m_1 = 4$. Recall that by Lemma (2.2.6), b_1 is the only vertex in $L_{M-c_1}(c_1)$ that is an internal in M_1. So M_1 cannot be a square, and hence must be K_4 or a chain of two triangles. It is trivial to construct the P_1 such that $|E(P_1)| = 2$. This implies $|E(P_L)| \geq 2 + \alpha(\sum_{i=2}^{j-1} m_i)^\gamma \geq \alpha(\sum_{i=1}^{j-1} m_i)^\gamma + 1$.

So M_1 is a triangle, and M_2 is 3-connected. We may assume $j \geq 3$; otherwise $j = 2$, $P_L = P_1$, and $|E(P_L)| \geq 1 \geq \alpha m_l^\gamma$, and Claim 5 holds.

Now let $\{x, y\} = S_1$. Note that $xb_1, yb_1 \in E(G)$. Further, b_1 has a neighbor z in G which is not a vertex in M. Since $\{b_1, x, y, z\}$ does not induce a claw in G, $xy \in E(G)$. So by applying Lemma (3.1.1), we can find P_2 so that $|E(P_2)| \geq \alpha(m_2 + 2)^\gamma + 1$. Thus $|E(P_L)| \geq 1 + \alpha(m_2 + 2)^\gamma + 1 \geq \alpha m^\gamma + 2$.

Let $m_r = |V(\cup_{i=j+1}^k M_i)|$.

Claim 5. If $j < k$ there exists a path P_R in $\cup_{i=j+1}^k M_i$ between the vertices of S_j such that $P_R \subseteq G$ and $|E(P_R)| \geq \alpha m_r^\gamma + 1$.

Let e_{j+1} denote the edge of M_{j+1} incident with both vertices of S_{j+1}. If $j+1 \neq k$, then by Lemma (3.1.3), there is a cycle C in $\cup_{i=j+1}^k M_i$ such that $e_{j+1} \in C$, $C - e_{j+1} \subseteq G$, and $|C| \geq \alpha(m_r)^\gamma + 4$, and $P_R := C - e_{j+1}$ is the desired path for Claim 5.

We may thus assume $j + 1 = k$. If $m_k \leq 5$, by Lemma (2.1.1) (when M_k is 3-connected) and Lemma (2.3.2) (when M_k is a chain of cycles) there is a Hamilton cycle C in M_k such that $e_{j+1} \in C$, $C - e_{j+1} \subseteq G$, and $|C| \geq 3 \geq \alpha m_k^\gamma + 2$, and $P_R := C - e_{j+1}$ is the desired path for Claim 5.

So assume $m_k \geq 6$. Then by Lemmas (2.2.8) (when M_k is 3-connected) and (2.3.2) (when M_k is a chain of cycles), there is a cycle C in M_k such that $e_{j+1} \in C$, $C - e_{j+1} \subseteq G$, and $|C| \geq 3 \geq \alpha m_k^\gamma + 2$. Then again, $P_R := C - e_{j+1}$ is the desired path for Claim 5.

Let $P := P_L \cup P_j \cup P_R$. Then P is a path in M from b_1 to S such that $c_1 \notin P$ and $P \subseteq G$.

If $j = 1$, then $|E(P)| = E(P_1)| + |E(P_R)| \geq \alpha(m_1 - 3)^\gamma + 2 + \alpha m_r^\gamma + 1 \geq \alpha m^\gamma + 2$, as desired for (1) and (2).

If $1 < j < k$, then $|E(P)| = |E(P_L)| + |E(P_j)| + |E(P_R)| \geq \alpha m_l^\gamma + \alpha(m_j - 2)^\gamma + 1 + \alpha m_r^\gamma + 1 \geq \alpha m^\gamma + 2$, as desired for (1) and (2).

So we may assume $j = k$. By (b) of Claim 3, $|E(P_k)| \geq \alpha(m_k - 2)^\gamma + 1$.

If $j \neq 2$ or $m_1 \neq 3$, then by Claim 4, $|E(P_L)| \geq \alpha m_l^\gamma + 1$ Then $|E(P)| \geq \alpha m_i l^\gamma + 1 + \alpha(m_k - 2)^\gamma + 1 \geq \alpha m^\gamma + 2$, as desired for (1) and (2).

So we may assume $j = 2$ and $m_1 = 3$. Then M_k is an extreme 3-connected 3-block in $M - c_1$. If $m_k = 4$ then $M_k \cong K_4$ and we can find the path P_k so that $|E(P_k)| \geq 2$; and so $|E(P)| \geq 3 \geq \alpha m^\gamma + 2$, and both (1) and (2) holds. So assume $m_k \geq 5$. Applying Lemma (3.1.2) (with M, c_1, M_k playing the roles of G, a, M, respectively), we find path P_k so that $|E(P_k)| \geq \alpha(m_k + 1)^\gamma + 1 = \alpha m^\gamma + 1$. Now $|E(P)| \geq \alpha m^\gamma + 2$, as desired. □

Next, we extend the above result to allow us to continue our path through not just one adjacent 3-block, but several consecutive 3-blocks instead.

(3.1.5) Lemma. *Let $n \geq 7$ and assume the assertion of Theorem (1.2.2) holds for graphs of order $< n$. Let G be a 3-connected claw-free graph of order at most n, $a \in V(G)$, and $M_1, ..., M_k$ ($k \geq 2$) be consecutive 3-blocks in the decomposition of $G - a$ such that M_1 and M_k are both middle 3-blocks, and M_1 is not a triangle when $k = 1$. Let $m = |V(\cup_{i=1}^k M_i)|$, $\{b_0, c_0\} \subseteq V(M_1)$ be the special 2-cut of $G - a$*

with $\{b_0, c_0\} \neq V(M_1) \cap V(M_2)$*, and* $\{b_k, c_k\} \subseteq V(M_k)$ *be the special 2-cut of* $G - a$ *with* $\{b_k, c_k\} \neq V(M_{k-1} \cap M_k)$*. Then there is a path* P *in* $\cup_{i=1}^k M_i$ *from* b_0 *to* $\{b_k, c_k\}$ *such that* $c_0 \notin P$*,* $P \subseteq G$*. Further,*

1. *If* M_1 *and* M_k *are triangles, then* $|E(P)| \geq \alpha(m-2)^\gamma + 1$*.*

2. *If exactly one of* M_1 *and* M_k *is a triangle, then* $|E(P)| \geq \alpha(m-1)^\gamma + 1$*.*

3. *Otherwise,* $|E(P)| \geq \alpha m^\gamma + 1$*.*

Proof. We find a path P_i in each M_i, in the order $i = 1, \ldots, k$, so that $\cup_{i=1}^k P_i$ gives the desired path P. Let $m_i = |V(M_i)|$ for $i = 1, \ldots, k$, and let $S_i = \{b_i, c_i\} = V(M_i) \cap V(M_{i+1})$ for $i = 1, \ldots, k-1$. Let $S_0 = \{b_0, c_0\}, S_k = \{b_k, c_k\}$. We proceed by induction on k.

Suppose $k = 2$. Consider first the case where M_2 is a triangle. Thus M_1 is 3-connected. If $m_1 \geq 5$, then by Lemma (3.1.4)(1), we find P_1 in M_1 from b_0 to S_1 (say b_1) such that $c_0 \notin P_1$, $E(P_1) \subseteq E(G)$, $b_1c_1 \notin P_1$, $|E(P_1)| \geq \alpha m_1^\gamma + 1$. Trivially we find a path P_2 in M_2 from b_1 to S_2 (say b_2) such that $c_1 \notin P_2$, $E(P_2) \subseteq E(G)$, $b_2c_2 \notin P_1$, $|E(P_2)| \geq 0$. Thus $P := P_1 \cup P_2$ gives the desired path for the lemma. If $M_1 \cong K_4$, then we find P the desired path for the lemma directly. Thus we may assume M_2 is not a triangle.

Suppose M_2 is a chain of cycles. Thus M_1 is 3-connected. By direct construction or Lemma (3.1.4)(1), we find a path P_1 in M_1 from b_0 to S_1 (say b_1) such that $c_0 \notin P_1$, $E(P_1) \subseteq E(G)$, $b_1c_1 \notin P_1$, $|E(P_1)| \geq \alpha(m_1 - 4)^\gamma + 1$. As $m_2 \geq 4$, by Lemma (2.3.4), we find a path P_2 in M_2 from b_1 to S_2 (say b_2) such that $c_1 \notin P_2$, $E(P_2) \subseteq E(G)$, $b_2c_2 \notin P_2$, $|E(P_2)| \geq m_2 - 3$. $P := P_1 \cup P_2$ gives the desired path for the lemma. Thus when $k = 2$, we may assume that M_2 is 3-connected. With a very similar argument, we show that we may assume that M_1 is 3-connected.

Suppose M_1 is a triangle. If $M_2 \cong K_4$, then we find P the desired path for the lemma directly. Trivially we find a path P_1 in M_1 from b_0 to S_1 (say b_1) such that $c_0 \notin P_1$, $E(P_1) \subseteq E(G)$, $b_1c_1 \notin P_1$, $|E(P_1)| \geq 0$. By Lemma (3.1.4)(1), we find P_2 in M_2 from b_1 to S_2 (say b_2) such that $c_1 \notin P_2$, $E(P_2) \subseteq E(G)$, $b_2c_2 \notin P_2$, $|E(P_2)| \geq \alpha m_2^\gamma + 1$. $P := P_1 \cup P_2$ gives the desired path for the lemma. Thus when $k = 2$, we may assume that M_1 is not a triangle.

Suppose that M_1 is a chain of cycles. As $m_1 \geq 4$, by Lemma (2.3.4), we find a path P_1 in M_1 from b_0 to S_1 (say b_1) such that $c_0 \notin P_1$, $E(P_1) \subseteq E(G)$, $b_1c_1 \notin P_1$, $|E(P_1)| \geq m_1 - 3$. By direct construction or Lemma (3.1.4)(1), we find a path P_1 in M_1 from b_1 to S_2 (say b_2) such that $c_1 \notin P_2$, $E(P_2) \subseteq E(G)$, $b_2c_2 \notin P_2$, $|E(P_2)| \geq \alpha(m_2 - 4)^\gamma + 1$. $P := P_1 \cup P_2$ gives the desired path for the lemma. Thus when $k = 2$, we may assume that M_1 is 3-connected.

Thus when $k = 2$, we may assume that M_1 and M_2 are 3-connected. By direct construction or Lemma (3.1.4)(1), we find a path P_1 in M_1 from b_0 to S_1 (say b_1) such that $c_0 \notin P_1$, $E(P_1) \subseteq E(G)$, $b_1c_1 \notin P_1$, $|E(P_1)| \geq \alpha(m_1 - 4)^\gamma + 1$. By direct construction or Lemma (3.1.4)(1), we find a path P_2 in M_2 from b_1 to S_2 (say b_2) such that $c_1 \notin P_2$, $E(P_2) \subseteq E(G)$, $b_2c_2 \notin P_2$, $|E(P_2)| \geq \alpha(m_2 - 4)^\gamma + 1$. $P := P_1 \cup P_2$ gives the desired path for the lemma.

Thus proves the lemma for $k = 2$, the base case of our induction.

Now for induction consider some $k \geq 3$ where the statement of the lemma is true for $j < k$.

Let $\overline{m} = |V(\cup_{i=1}^{k-1})| = m - m_k + 2$. By the inductive hypothesis, there is a path $\overline{P}$ from b_0 to S_{k-1} (say b_{k-1}) such that $c_0 \notin \overline{P}$, $E(\overline{P}) \subseteq E(G)$, $b_ic_i \notin \overline{P}$. However, $|E(\overline{P})|$ depends on how many of M_1 and M_k are triangles.

Suppose M_k is a triangle. Trivially we find a path P_k in M_k from b_{k-1} to S_k (say b_k) such that $c_{k-1} \notin P_k$, $E(P_k) \subseteq E(G)$, $b_kc_k \notin P_k$, $|E(P_k)| \geq 0$. Note that M_{k-1} is not a triangle. If M_1 is a triangle, then by induction, $|E(\overline{P})| \geq \alpha(\overline{m} - 1)^\gamma + 1$. $P := \overline{P} \cup P_2$ gives the desired path for the lemma. If M_1 is not a triangle, then by induction, $|E(\overline{P})| \geq \alpha(\overline{m})^\gamma + 1$. $P := \overline{P} \cup P_k$ gives the desired path for the lemma.

Thus we may assume M_k is not a triangle.

Regardless of the structure of M_1 and M_{k-1}, $|E(\overline{P})| \geq \alpha(\overline{m}-2)^\gamma + 1$. As $m_k \geq 4$, by Lemma (2.3.4), we find a path P_k in M_k from b_{k-1} to S_k (say b_k) such that $c_{k-1} \notin P_k$, $E(P_k) \subseteq E(G)$, $b_k c_k \notin P_k$, $|E(P_k)| \geq m_k - 3$. $P := \overline{P} \cup P_k$ gives the desired path for the lemma. □

(3.1.6) Lemma. *Let $n \geq 7$ and assume the assertion of Theorem (1.2.2) holds for graphs of order $< n$. Let G be a 3-connected claw-free graph of order at most n, $a \in V(G)$, and $M_1, ..., M_k$ ($k \geq 2$) be consecutive 3-blocks (without loss of generality, from left to right) in the decomposition of $G - a$ such that M_1 is a middle 3-block and M_k is an extreme 3-block (in this case, the rightmost 3-block). Let $m = |V(\cup_{i=1}^k M_i)|$, and let $\{b_0, c_0\} \subseteq v(M_1)$ be the special 2-cut of $G - a$ such that $\{b_0, c_0\} \neq V(M_1 \cap M_2)$. Then there is a path P in $\cup_{i=1}^k M_i$ from b_0 to $R_{G-a}(a)$ such that $c_0 \notin P$, $P \subseteq G$. Further,*

1. *If M_1 and M_k are triangles, then $|E(P)| \geq \alpha(m-2)^\gamma + 2$.*
2. *If exactly one of M_1 and M_k is a triangle, then $|E(P)| \geq \alpha(m-1)^\gamma + 2$.*
3. *Otherwise, $|E(P)| \geq \alpha m^\gamma + 2$.*

Proof. We find a path P_i in each M_i, in the order $i = 1, \ldots, k$, so that $\cup_{i=1}^k P_i$ gives the desired path P. To this end, Lemma (3.1.5) is extremely useful. Let $m_i = |V(M_i)|$ for $i = 1, \ldots, k$. Let $S_i = \{b_i, c_i\} = V(M_i) \cap V(M_{i+1})$ for $i = 1, \ldots, k-1$. Let $S_0 = \{b_0, c_0\}$, let $S_k = R_{G-a}(a)$.

If $k > 2$, then by Lemma (3.1.5) we find a path $\overline{P}$ from b_0 to $\{b_{k-1}, c_{k-1}\}$ such that $c_0 \notin \overline{P}$, $E(\overline{P}) \subseteq E(G)$, $b_{k-1}c_{k-1} \notin \overline{P}$. However, $|E(\overline{P})|$ depends on how many of M_1 and M_{k-1} are triangles.

Suppose M_k is a triangle. Trivially we find a path P_k in M_k from b_{k-1} to S_k such that $c_{k-1} \notin P_k$, $E(P_k) \subseteq E(G)$, $|E(P_k)| = 1$. Note that M_{k-1} is not a triangle.

If M_1 is a triangle, then by Lemma (3.1.5)(2), $|E(\overline{P})| \geq \alpha((m-1)-1)^\gamma + 1$. $P = \overline{P} + P_k$ gives the desired path for the lemma. Thus we may assume M_1 is not a triangle. If $k > 2$, then by Lemma (3.1.5)(3), $|E(\overline{P})| \geq \alpha((m)-1)^\gamma + 1$. $P := \overline{P} \cup P_k$ gives the desired path for the lemma. Thus we may assume $k = 2$. Thus M_1 is 3-connected. If $M_1 \cong K_4$, then we find P the desired path for the lemma directly. By Lemma (3.1.4)(1), we find P_1 in M_1 from b_0 to S_1 (say b_1) such that $c_0 \notin P_1$, $E(P_1) \subseteq E(G)$, $b_1c_1 \notin P_1$, $|E(P_1)| \geq \alpha m_1^\gamma + 1$. We trivially find P_2 as before. $P := P_1 \cup P_2$ gives the desired path for the lemma. Thus we may assume M_k is not a triangle.

Regardless of the structure of M_1 and M_{k-1}, if $k > 2$ then $|E(\overline{P})| \geq \alpha(m - (m_k - 2) - 2)^\gamma + 1$. By direct construction or Lemma (3.1.4)(2), we find a path P_k in M_k from b_{k-1} to S_k such that $c_{k-1} \notin P_k$, $E(P_k) \subseteq E(G)$, $|E(P_k)| \geq \alpha(max\{0, m_k - 5\})^\gamma + 2$. $P := \overline{P} \cup P_2$ gives the desired path for the lemma. □

3.2 An advanced path through a 3-connected 3-block

This section contains a more advanced path result. The first lemma is used to aid in the proof of the second result. The second lemma, along with most of the previous lemmas throughout this Chapter and Chapter 2, are then used in the proof of the last two lemmas in the following section. Those last two lemmas are cited directly in the proof of the main theorem and make that proof nearly immediate.

(3.2.1) Lemma. *Let $n \geq 7$ and assume that Theorem (1.2.2) holds for graphs with $< n$ vertices. Let G be a 3-connected graph of order at most n, $a \in V(G)$, and $M_1 \ldots M_k$ ($k \geq 1$) be consecutive 3-blocks in the decomposition of $G - a$. Let $c_1d_1 \in E(M_1)$ and $S \subseteq V(M_1)$ such that $\{c_1, d_1\} \neq V(M_1 \cap M_2) \neq S \neq \{c_1, d_1\}$, $|S| \leq 2$, and $M_1 + \{z, zy : y \in S\}$ is claw-free (for a new vertex z). Let $c_2d_2, r_2s_2 \in E(M_2)$ such that $\{c_2, d_2\} \neq V(M_{k-1} \cap M_k) \neq \{r_2, s_2\} \neq \{c_2, d_2\}$. Let $m = |V(\cup_{i=1}^k M_i)|$.*

Moreover, if $k = 1$ then $c_1d_1 \notin \{c_2d_2, r_2s_2\}$ and $S \notin \{\{c_2, d_2\}, \{r_2, s_2\}\}$. Then there exist paths Q_i, $i = 1, 2, 3, 4$, in $\cup_{i=1}^{k} M_i$ such that $|E(Q_i)| \geq \alpha m^\gamma$ and

(1) Q_1 is from S to $\{c_1, d_1\}$, $c_2d_2 \in Q_1$, and $Q_1 - \{c_2d_2, r_2s_2\} \subseteq G$,

(2) Q_2 is from S to $\{c_1, d_1\}$, $r_2s_2 \in Q_2$, and $Q_2 - \{c_2d_2, r_2s_2\} \subseteq G$,

(3) Q_3 is from S to $\{c_2, d_2\}$, $c_1d_1 \in Q_3$, and $Q_3 - \{c_1d_1, r_2s_2\} \subseteq G$, and

(4) Q_4 is from S to $\{c_2, d_2\}$, $r_2s_2 \in Q_3$, and $Q_3 - \{c_1d_1, r_2s_2\} \subseteq G$.

Proof. Let $m_i = |V(M_i)|$ for $i = 1, \ldots, k$, and let $S_i = V(M_i \cap M_{i+1})$ for $i = 1, \ldots, k-1$. Let $e = c_2d_2$ if $k = 1$; otherwise, let e be the virtual edge with $V(e) = S_1$.

First, we use Lemmas (3.1.1) and (3.1.2) or Lemmas (2.3.1) and (2.3.3) (for chain of cycles) to find a path P_1 in M_1 from S to $\{c_1, d_1\}$ such that $e \in P_1$, $P_1 - e \subseteq G$, and $|E(P_1)| \geq \alpha m_1^\gamma$. If $k = 1$, we have (1). So assume $k \geq 2$. Apply Lemma (2.2.8), we find a cycle C in $\cup_{i=2}^{k} M_i$ through e, c_2d_2 such that $C - \{e, c_2d_2, r_2s_2\} \subseteq G$, and $|C| \geq \alpha(m-(m_1-2)-3)^\gamma+3$. Now $Q_1 := (P_1-e)\cup(C-e)$ gives the desired path.

The proof of (2) is the same by exchanging the roles of c_2d_2 and r_2s_2 in the above proof.

To prove (3), we use Lemmas (3.1.1) and (3.1.2) or Lemmas (2.3.1) and (2.3.3) (for chain of cycles) to find a path P_1 in M_1 from S to S_1 such that $c_1d_1 \in P_1$, $P_1 - c_1d_1 \subseteq G$, and $|E(P_1)| \geq \alpha m_1^\gamma$. By Lemmas (3.1.6) and (3.1.4)(2), we may find a path P_2 in $\cup_{i=2}^{k} M_i$ from the end of P_1 in S_1, say s_1, to $\{c_2, d_2\}$ such that $S_1 - \{s_1\} \not\subseteq P_2$, $P_2 \subseteq G$, and $|E(P_2)| \geq \alpha(m-(m_1-2)-3)^\gamma+1$. Now $Q_3 := P_1 \cup P_2$ gives the desired path.

To prove (4), we apply Lemmas (3.1.1) and (3.1.2) or Lemmas (2.3.1) and (2.3.3) (for chain of cycles) to find a path P_k in M_k from $\{c_2, d_2\}$ to S_{k-1} (with

$S_0 = S$) such that $r_2s_2 \in P_k$, $P_k - r_2s_2 \subseteq G$, and $|E(P_k)| \geq \alpha m_k^\gamma$. If $k = 1$, we are done. If $k \geq 2$ then apply Lemmas (3.1.6) and (3.1.4)(2), we find a path P_1 from the end of P_k in S_{k-1}, say s_{k-1}, to $\{c_2, d_2\}$ such that $S_{k-1} - s_{k-1} \not\subseteq P_1$, $P_1 \subseteq G$, and $|E(P_1)| \geq \alpha(m - (m_k - 2) - 3)^\gamma + 1$. Now $Q_4 := P_1 \cup P_k$ gives the desired path. □

The following result uses "double decomposition." In other words, this proof will require us to take a 3-block from the initial decomposition and decompose it further.

(3.2.2) Lemma. *Let $n \geq 7$ and assume Theorem (1.2.2) holds for graphs with $< n$ vertices. Let M be a 3-connected claw-free graph on $m \geq 6$ vertices, where $m < n$, let $\{x, y\} \subseteq V(M)$ such that $xy \in E(M)$, and $N_M(y) - x$ and $N_M(x) - y$ each induce a clique in M, and let $z \in V(M) - \{x, y\}$. Then there is a path P in M from y to z such that $x \notin P$ and $|E(P)| \geq \alpha m^\gamma + 2$.*

Proof. As usual we delete a vertex and decompose the resulting graph. As the structure near x and y is fairly restricted, we choose instead to delete z. Thus consider $M - z$.

Claim 1. We may assume $M - z$ is not 3-connected, and the decomposition of $M - z$ has at least two 3-blocks.

First, assume $M - z$ is 3-connected. As $M - z$ has $m - 1 \geq 5$ vertices, we may apply Lemma (3.1.4)(2) to find a path Q in M from y to some $z' \in N_M(z)$ such that $x \notin Q$ and $|E(Q)| \geq \alpha(m-1)^\gamma + 2$. Let $P := Q \cup \{z, zz'\}$. Then P is the desired path since $|E(P)| \geq \alpha(m-1)^\gamma + 3 \geq \alpha m^\gamma + 2$.

Thus we may assume that $M - z$ is not 3-connected. Now suppose there is only one 3-block in the decomposition of $M - z$. Then the decomposition of $M - z$ must be a chain of cycles, and hence the virtual edges all correspond to edges in

M. Since $m \geq 6$, $M - z$ is a chain of triangles (with at least three triangles), or a square with one triangle, or a square with two triangles.

Since $xy \in E(M)$ and both $N_M(x) - y$ and $N_M(y) - x$ induce cliques in M, $\{x, y\}$ is contained in a unique cycle, say C, in the decomposition of $M - z$.

Suppose C is not the leftmost or rightmost cycle in the decomposition. Then the decomposition of $M - z$ is a chain of exactly three triangles, or a square with two triangles. In this case, there are exactly two vertices in $M - z$ with degree 2, both are adjacent to z. It is easy to see that $(M - z) - x$ has a Hamilton path Q from y to z', one of these degree 2 vertices in $M - z$. Now $P := Q \cup \{z, zz'\}$ is the desired path, since $|E(P)| \geq 4 \geq \alpha m^\gamma + 2$.

So by symmetry we may assume that C is the leftmost cycle in the decomposition of $M - z$. Then it is easy to see that $(M - z) - x$ contains Hamilton path Q from y to z', where z' is a vertex with degree 2 in $M - z$. Then $|E(Q)| \geq m - 3$. Note that $z'z \in E(G)$. Let $P := Q \cup \{z, zz'\}$. Then P is the desired path since $|E(P)| = m - 2 \geq \alpha m^\gamma + 2$ for $m \geq 6$.

Thus, let $M_1, ..., M_k$, $k \geq 2$, be consecutive 3-blocks (from left to right) in the decomposition of $M - z$. Let $m_i = |V(M_i)|$ for $i = 1, \ldots, k$, and $\{b_i, c_i\} = V(M_i) \cap V(M_{i+1})$ for $i = 1, \ldots, k - 1$.

Claim 2. We may assume that $\{x, y\} \cap \{b_i, c_i\} = \emptyset$ for $1 \leq i \leq k - 1$.

Note that $\{x, y\} \neq \{b_i, c_i\}$ for $1 \leq i \leq k - 1$, since $N_M(x) - y$ and $N_M(y) - x$ each induce a clique in M. Now suppose $x \in \{b_{j-1}, c_{j-1}\}$ for some $j \geq 2$; the case $y \in \{b_{j-1}, c_{j-1}\}$ will be symmetric, and we only point out when difference occurs. By symmetry, we may assume $y \in M_j$.

Since $N_M(x) - y$ induces a clique in M, y is the only neighbor of x in $M_j - \{b_{j-1}, c_{j-1}\}$. Hence, M_j is chain of cycles. Since $N_M(y) - x$ induces a clique in M, M_j is either a chain of at most two triangles, or a square, or a square and a single triangle. Also note that $j = k$ unless M_j is a single triangle or single square.

Let $z_1 \in \{b_{j-1}, c_{j-1}\} - \{x\}$. By Lemma (3.1.6) (for $j-1 \geq 2$) or Lemma (3.1.4) (for $j-1=1$), there is a path P_1 in $\cup_{i=1}^{j-1} M_i$ from z_1 to some $z' \in L_{M-z}(z)$ such that $x \notin P_1$, $P_1 \subseteq G$, and $|E(P_1)| \geq \alpha m_l^\gamma + 1$, where $m_l = |V(\cup_{i=1}^{j-1} M_i)|$.

Suppose $j = k$. Note that there is a path Q in $M_j - x$ from z_1 to y such that $Q \subseteq G$ and $|E(Q)| \geq 1$. Let $P := (P_1 \cup Q) \cup \{z, zz'\}$. Then P is the desired path since $|E(P)| \geq \alpha m_l^\gamma + 1 + 2 \geq \alpha m^\gamma + 2$.

Hence we may assume $j < k$. Then M_j is a single triangle or single square, and $\{b_j, c_j\} = \{y, z_2\}$, where $z_2 = z_1$ or xyz_2z_1x is a square. Let $m_r = |V(\cup_{i=j+1}^{k} M_i)|$. By Lemmas (3.1.3) and (2.2.8), there exists a cycle C in $\cup_{i=j+1}^{k} M_i$ such that $yz_2 \in C$, $C - yz_2 \subseteq G$, $|E(C)| \geq \alpha m_r^\gamma + 3$. Let P be obtained from $P_1 \cup (C - yz_1)$ by adding $\{z, zz'\}$ and possibly z_1z_2. Then P is the desired path, since $|E(P)| \geq \alpha m_l^\gamma + 1 + \alpha m_r^\gamma + 3 + 1 \geq \alpha m^\gamma + 2$.

Note that if $y \in \{b_{j-1}, c_{j-1}\}$ and $x \in M_j$, we essentially switch the left-right orientation in the above argument, and obtain the same result.

Let x and y be in M_j. By Claim 2, x and y are internal vertices in M_j. The remainder of the analysis consists of considering the type and location of M_j. Without loss of generality $j \leq k/2$ as the ordering from left to right was arbitrary.

Claim 3. We may assume $j \geq 2$.

Suppose $j = 1$.

Case 1. M_j is 3-connected.

By Lemma (3.1.4)(2) ($\{x, y\}$ may be viewed as a special 2-cut in a decomposition of some $H - a$ and $\{b_1, c_1\}$ may be viewed as $L_{H-a}(a)$), there is a path P_1 in M_1 from y to $\{b_1, c_1\}$ such that $x \notin P_1$, $P_1 \subseteq M$, and $|E(P_1)| \geq \alpha m_1^\gamma + 2$. Without loss of generality assume P_1 ends at b_1.

Let $m_0 = |V(\cup_{i=2}^{k} M_i)|$. If $k > 2$, then by Lemma (3.1.6), there is a path P_2 from b_1 to some $z' \in R_{M-z}(z)$ such that $c_1 \notin P_2$, $P_2 \subseteq M$, and $|E(P_2)| \geq \alpha m_0^\gamma + 1$. Let $P := (P_1 \cup P_2) \cup \{z, zz'\}$. Then $|E(P)| \geq \alpha m_1^\gamma + 1 + \alpha m_0^\gamma + 1 + 1 \geq \alpha m^\gamma + 3$, and P is the desired path.

So assume $k = 2$. If $m_0 \geq 5$ then by Lemma (3.1.4)(2) and (2.3.5) we can find a path P_2 from b_1 to some $z' \in R_{M-z}(z)$ such that $c_1 \notin P_2$, $P_2 \subseteq M$, and $|E(P_2)| \geq \alpha m_0^\gamma + 2$. If $M_2 \cong K_4$, or M_2 is a square or a chain of at most two triangles, then we find such a path P_2 of length $1 \geq \alpha m_0^\gamma$. Let $P := (P_1 \cup P_2) \cup \{z, zz'\}$. Then $|E(P)| \geq \alpha m_1^\gamma + 1 + \alpha m_0^\gamma + 1 \geq \alpha m^\gamma + 2$.

Case 2. M_j is a chain of cycles.

Since x, y are internal vertices of M_1 and there is no claw centered at z, we see that x, y must be contained in the leftmost cycle in M_j as a chain of cycles. Let bc denote the edge of leftmost cycle such that $\{b, c\}$ is a 2-cut of $M - z$. Then $\{b, c\} \neq \{x, y\}$, since $N_M(x) - y$ and $N_M(y) - x$ each induce a clique in M.

We claim that there is a path P_1 in M_1 from y to $\{b_1, c_1\}$ such that $x \notin P_1$, $P_1 \subseteq M$, $|E(P_1)| \geq m_1 - 3$, and $|E(P_1)| = m_1 - 2$ when $m_1 = 3$ (i.e., M_1 is a triangle). This is straightforward to check, since M_1 is a square, or a square with a triangle, or a chain of triangles. In particular, we have $|E(P_1)| \geq \max\{1, m_1 - 3\}$.

Without loss of generality, we may assume that P_1 ends at b_1. Let $m_r = |V(\cup_{i=j+1}^k M_i)|$. By Lemma (3.1.4)(2) (if $j + 1 = k$) or by Lemma (3.1.6) (if $j + 1 < k$), there is a path P_2 from b_1 to some $z' \in R_{M-z}(z)$ such that $c_1 \notin P_2$, $P_2 \subseteq M$, and $|E(P_2)| \geq \alpha m_r^\gamma + 1$. Let $P := (P_1 \cup P_2) \cup \{z, zz'\}$. Then $|E(P)| \geq \max\{1, m_1 - 3\} + \alpha m_r^\gamma + 1 + 1 \geq \alpha m^\gamma + 2$ (since $\alpha \geq 1/7$).

Claim 4. $j < k - 1$, $\{b_{j-1}, c_{j-1}\} \cap \{b_j, c_j\} = \emptyset$, $|V(M_j)| \geq 6$, and we may assume that M_j is 3-connected.

By Claim 3 and since $j \leq k/2$, we see that $k \geq 4$ and $j < k - 1$. Hence M_j is a middle 3-block in the decomposition of $M - z$. Therefore, since M_j has internal vertices (by Claim 2) and M is claw-free, $\{b_{j-1}, c_{j-1}\} \cap \{b_j, c_j\} = \emptyset$. So $|V(M_j)| \geq 6$.

For, suppose M_j is a chain of cycles. By Claim 3, M_j is a middle block in the decomposition of $M - z$, M_j is a square or a chain of triangles. Since $N_M(x) - y$ and $N_M(y) - x$ each induce a clique in M, M_j is either a square or a triangle, contradicting $|V(M_j)| \geq 6$.

Let $m_l = |V(\cup_{i=1}^{j-1} M_i)|$ and let $m_r = |V(\cup_{i=j+1}^{k} M_i)|$. Note that $m_l + m_j + m_r = m + 3$. We now further decompose M_j by deleting x and consider the possible structures of $M_j - x$.

Claim 5. We may assume that the decomposition of $M_j - x$ has at least two 3-blocks.

Suppose otherwise that the decomposition of $M_j - x$ has a unique 3-block. Then is must be 3-connected or a chain of 3-blocks.

Case 1. $M_j - x$ is 3-connected.

By Lemma (3.1.2), there is a path P_j in $M_j - x$ from $\{b_{j-1}, c_{j-1}\}$ to y such that $b_j c_j \in P_j$, $P_j - b_j c_j \subseteq M$, and $|E(P_j)| \geq \alpha(m_j + 2)^\gamma + 1$. Without loss of generality, assume P_j ends in b_{j-1}.

By Lemmas (3.1.6), (3.1.4)(2) and (2.3.5), there is a path P_l in $\cup_{i=1}^{j-1} M_i$ from b_{j-1} to some $z' \in L_{M-z}(z)$ such that $c_{j-1} \notin P_l$, $P_l \subseteq M$, $|E(P_l) \geq \alpha m_l^\gamma$.

Since $j < k - 1$, it follows from Lemma (3.1.3) that there is a cycle C_r in $\cup_{i=j+1}^{k} M_i$ such that $b_j c_j$, $C_r - b_j c_j \subseteq M$, and $|C_r| \geq \alpha(m_r - 3)^\gamma + 3$.

Let $P := (P_l \cup (P_j - b_j c_j) \cup (C_r - b_j c_j)) \cup \{z, zz'\}$. Then $|E(P)| \geq \alpha m_l^\gamma + \alpha(m_j + 2)^\gamma + \alpha(m_r - 3)^\gamma + 2 + 1 \geq \alpha m^\gamma + 3$, and so P is the desired path from y to z in $M - x$.

Case 2. The decomposition of $M_j - x$ is a chain of cycles.

Then since $|V(M_j)| \geq 6$, we see that $M_j - x$ is a square with a triangle, or a square with two opposite triangles, or a chain of at least three triangles. Since $N_{M_j}(b_{j-1}) - c_{j-1}$, $N_{M_j}(c_{j-1}) - b_{j-1}$, $N_{M_j}(b_j) - c_j$, and $N_{M_j}(c_j) - b_j$ all induce cliques in M, $\{b_{j-1}, c_{j-1}\}$ (respectively, $\{b_j, c_j\}$) cannot be shared by two cycles in the decomposition of $M_j - x$.

We want a path P_j from y to $\{b_{j-1}, c_{j-1}\}$ such that $b_jc_j \in P_j$, $P_j - b_jc_j \subseteq M$, and $|E(P_j)| \geq \alpha m_j^\gamma + 1$. If this P_j exists, then assume P_j ends at b_{j-1}, by the same argument as in Case 1, we can find P_l and C_r so that $P := (P_l \cup (P_j - b_jc_j) \cup (C_r - b_jc_j)) \cup \{z, zz'\}$ is the desired path.

We now find the path P_j. Suppose that $M_j - x$ is a chain of triangles. In fact there are at least four triangles in this chain; as otherwise, since $\{b_{j-1}, c_{j-1}\} \cap \{b_j, c_j\} = \emptyset$ and $N_M(y) - x$ induces a clique in M, we would force a claw in M centered at one of $\{b_{j-1}, c_{j-1}, b_j, c_j\}$. Moreover, for the same reason (and with an appropriate orientation), $b_{j-1}c_{j-1}$ and b_jc_j belong to the leftmost and rightmost triangles, respectively. Then it is easy to see that there is a path P_j from y to $\{b_{j-1}, c_{j-1}\}$ such that $b_jc_j \in P_j$, $P_j - b_jc_j \subseteq M$, and $|E(P_j)| \geq m_j - 3 \geq \alpha m_j^\gamma + 1$.

Hence, we may assume that there is a square, say $s_1s_2t_2t_1s_1$, in the decomposition of $M_j - x$. Let $s_1s_2s_3s_1$ be a triangle in the decomposition of $M_j - x$; and if there is a second triangle in the decomposition of M_j then let $t_1t_2t_3t_1$ be that one.

First consider where both $b_{j-1}c_{j-1}$ and b_jc_j are edges in the square. Then without loss of generality we may assume $b_{j-1}c_{j-1} = s_1t_1$ and $b_jc_j = s_2t_2$. Note that $y = s_3$ or $y = t_3$. It is now easy to see that the path P_j can be found so that $|E(P_j)| \geq m_j - 3 \geq \alpha m_j^\gamma + 1$.

Now assume exactly one of $b_{j-1}c_{j-1}$ and b_jc_j is in the square. Without loss of generality we may assume $\{b_{j-1}c_{j-1}, b_jc_j\} = \{s_2t_2, s_3s_1\}$. Then $y = t_3$ or $y = t_1$. Again, it is easy to see that the path P_j can be found so that $|E(P_j)| \geq m_j - 3 \geq \alpha m_j^\gamma + 1$.

So we may assume that neither $b_{j-1}c_{j-1}$ nor b_jc_j is in the square, and so they are in different triangles. Again, it is easy to see that the path P_j can be found so that $|E(P_j)| \geq m_j - 3 \geq \alpha m_j^\gamma + 1$.

By Claim 5, let $H_1, ..., H_h$, $h \geq 2$, be consecutive 3-blocks (from left to right) in the decomposition of $M_j - x$. Let $h_i = |V(H_i)|$. Let $\{r_i, s_i\} = V(H_i \cap H_{i+1})$.

As $xy \in E(M_j)$, we may assume y is in H_1. Note further that as x's neighbors in M_j are y and a clique, y is the only internal vertex of H_1 that is a neighbor of x in M_j. Let $b_{j-1}c_{j-1} \in H_{g_1}$ and $b_jc_j \in H_{g_2}$, and without loss of of generality, assume $g_1 \leq g_2$.

There are five major parts to this structure: $\cup_{i=1}^{j-1} M_i$, $\cup_{i=j+1}^{k} M_i$, $\cup_{i=g_1}^{g_2} H_i$, $a_{j-1}b_{j-1}$, a_jb_j, $\cup_{i=1}^{g_1-1} H_i$, and $\cup_{i=g_2+1}^{h} H_i$. Note that it is possible for one of the last two parts to be empty. This structure is graphically depicted in Figure 3.2.1

We now proceed to find a path or cycle in each of the five parts, so that when combined appropriately, gives the desired path. Let $h_l = |V(\cup_{i=1}^{g_1-1} H_i)|$ and let $h_r = |V(\cup_{i=g_2+1}^{h} H_i)|$.

First, there is a path P_{h_l} in $\cup_{i=1}^{g_1-1} H_i$ from y to any given $s \in \{r_{g_1-1}, s_{g_1-1}\}$ such that $\{r_{g_1-1}, s_{g_1-1}\} - \{s\} \not\subseteq P_{h_l}$, $P_{h_l} \subseteq M$, and $|E(P_{h_l})| \geq \alpha h_l^{\gamma}$. This is clear if $h_l \leq 4$. So assume $h_l \geq 5$. Then the existence of P_{h_l} follows from Lemmas (3.1.6) and (3.1.4)(2).

By Lemmas (3.1.6) and (3.1.4)(2), there is a path P_{m_l} in $\cup_{i=1}^{j-1} M_i$ from a given $t \in \{b_{j-1}, c_{j-1}\}$ to some $z' \in L_{M-z}(z)$ such that $\{b_{j-1}, c_{j-1}\} - \{t\} \not\subseteq P_{m_l}$, $P_{m_l} \subseteq M$, and $|E(P_{m_l})| \geq \alpha(m_l - 3)^{\gamma} + 1$. Similarly, there is a path P_{m_r} in $\cup_{i=j+1}^{k} M_i$ from a given $t \in \{b_j, c_j\}$ to some $z' \in R_{M-z}(z)$ such that $\{b_j, c_j\} - \{t\} \not\subseteq P_{m_r}$, $P_{m_r} \subseteq M$, and $|E(P_{m_r})| \geq \alpha(m_r - 3)^{\gamma} + 1$.

Since $j < k-1$, it follows from Lemma (3.1.3) there is a cycle C_{m_r} in $\cup_{i=j+1}^{k} M_i$ such that $b_jc_j \in C_{m_r}$, $C_{m_r} - b_jc_j \subseteq M$, and $|C_{m_r}| \geq \alpha(m_r - 3)^{\gamma} + 3$. Similarly, there is a cycle C_{m_l} in $\cup_{i=1}^{j-1} M_i$ such that $b_{j-1}c_{j-1} \in C_{m_l}$, $C_{m_l} - b_{j-1}c_{j-1} \subseteq M$, and $|C_{m_l}| \geq \alpha(m_l - 3)^{\gamma} + 3$. This follows from Lemma (3.1.3) if $m_r \geq 5$ and $g_1 > 2$; otherwise it follows from Lemmas (2.2.8) and (2.3.2).

There is a cycle C_{h_r} in $\cup_{i=g_2+1}^{h} H_i$ such that $r_{g_2}s_{g_2} \in C_{h_r}$, $C_{h_r} - r_{g_2}s_{g_2} \subseteq M$, and $|C_{h_r}| \geq \alpha(h_r - 3)^{\gamma} + 3$. This follows from Lemma (3.1.3) if $h_r \geq 5$ and $g_1 < h - 1$; otherwise it follows from Lemmas (2.2.8) and (2.3.2).

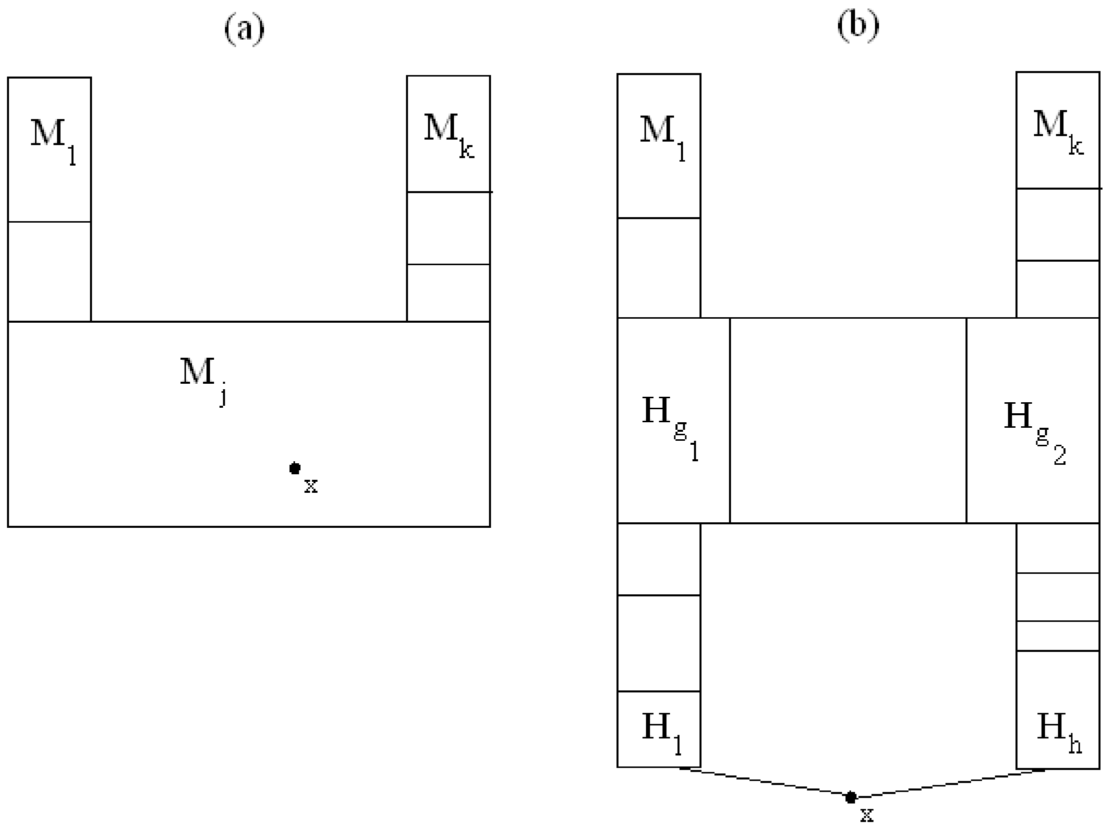

Figure 3.2.1: Representation of the decomposition of $M - z$ and the double decomposition of $(M - z) - x$. Note this is a very simplified representation and only the vertex x is drawn. Each enclosed region represents a 3-block and only some of the 3-blocks are labelled. (a) Representation of $M - z$. There are 3 major parts. (b) Another representation of $M - z$, indicative of the structure of $(M - z) - x$. There are 5 major parts.

Next we apply Lemma (3.2.1) with S as $\{y\}$ or $\{r_{g_1-1}, s_{g_1-1}\}$, c_1d_1 as $b_{j-1}c_{j-1}$, c_2d_2 as b_jc_j, r_1s_1 as $r_{g_1-1}s_{g_1-1}$, and r_2s_2 as $r_{g_2}s_{g_2}$. Let $h_0 = |V(\cup_{i=g_1}^{g_2} H_i)|$.

First, by Lemma (3.2.1)(1), there is a path Q in $\cup_{i=g_1}^{g_2} H_i$ from y or $\{r_{g_1-1}, s_{g_1-1}\}$ to $\{b_{j-1}, c_{j-1}\}$ such that $b_jc_j \in Q_1$, $Q_1 - \{b_jc_j, r_{g_2}s_{g_2}\} \subseteq M$, and $|E(Q_1)| \geq \alpha h_0^\gamma$. We choose s and t so that they are the ends of Q, and let $P := (Q - b_jc_j) \cup P_{m_l} \cup P_{h_l} \cup (C_{m_r} - b_jc_j) \cup \{z, zz'\}$. (If necessary, replace $r_{g_2}s_{g_2}$ by a path in $\cup_{i=g_2+1}^{h} H_i$.) Thus P is a path in $M - x$ from y to z, and $|E(P)| \geq \alpha h_0^\gamma + \alpha(m_l - 3)^\gamma + 3 + \alpha h_l^\gamma + \alpha(m_r - 3)^\gamma + 3 - 2 + 1 \geq \alpha(h_0^\gamma + m_l^\gamma + h_l^\gamma + (m_r - 3)^\gamma) + 3$.

Second, by Lemma (3.2.1)(2), we find a path Q_2 in $\cup_{i=g_1}^{g_2} H_i$ from y or $\{r_{g_1-1}, s_{g_1-1}\}$ to $\{b_{j-1}, c_{j-1}\}$ such that $r_{g_2}s_{g_2} \in Q_2$, $Q - b_jc_j \subseteq M$, and $|E(Q_2)| \geq \alpha h_0^\gamma$. We choose s and t to be the ends of Q_2, and let $P := (Q - r_{g_2}s_{g_2}) \cup P_{m_l} \cup P_{h_l} \cup (C_{h_r} - r_{g_2}s_{g_2}) \cup \{z, zz'\}$ (and if necessary, replace b_jc_j by a path in $\cup_{i=j+1}^{k} M_i$.) Then P is a path in $M - x$ from y to z, and $|E(P)| \geq \alpha h_0^\gamma + \alpha(m_l - 3)^\gamma + 3 + \alpha h_l^\gamma + \alpha(h_r - 3)^\gamma + 3 - 2 + 1 \geq \alpha(h_0^\gamma + m_l^\gamma + h_l^\gamma + (h_r - 3)^\gamma) + 3$.

Third, by exchanging the roles of m_l and m_r and applying Lemma (3.2.1)(3), we find a path from y to z such that $x \notin P$ and $|E(P)| \geq (h_0^\gamma + (m_r - 3)^\gamma + h_l^\gamma + (h_r - 3)^\gamma) + 4$. (Note that P leaves from $\cup_{i=j+1}^{k} M_i$ to z instead of through $\cup_{i=1}^{j-1} M_i$.)

Fourth, we find a path Q in $\cup_{i=g_1}^{g_2} H_i$ from y or $\{r_{g_1-1}, s_{g_1-1}\}$ to $\{b_{j-1}, c_{j-1}\}$ such that $b_jc_j, r_{g_2}s_{g_2} \in Q_4$ and $P_{g_1} \subseteq M$. Note that this path always exists, and we have no guarantee about its length. We choose s and t to be the ends of Q, and let $P := (Q - \{b_jc_j, r_{g_2}s_{g_2}\}) \cup P_{m_l} \cup P_{h_l} \cup (C_{m_r} - b_jc_j) \cup (C_{h_r} - r_{g_2}s_{g_2}) \cup \{z, zz'\}$. Thus $|E(P)| \geq \alpha(m_l - 3)^\gamma + 3 + \alpha h_l^\gamma + \alpha(m_r - 3)^\gamma + 3 + \alpha(h_r - 3)^\gamma + 3 - 2 + 1 \geq \alpha(m_l^\gamma + h_l^\gamma + (m_r - 3)^\gamma + (h_r - 3)^\gamma) + 4$.

Given these options, note that our path always has αh_l^γ. However, among the values in $\{h_{g_1}^\gamma, m_l^\gamma, m_r^\gamma, h_r^\gamma\}$, each of the four possibilities of P is missing exactly one different (and each path is missing a different one). Since we may choose the path of the greatest length, let $\{B_1, B_2, B_3, B_4\} = \{h_{g_1}, m_l, m_r, h_r\}$ such that $B_i \geq B_{i+1}$.

We may conclude $|E(P)| \geq \alpha(h_l^\gamma + B_1^\gamma + B_2^\gamma + B_3^\gamma) + 2$.

Thus by Lemma (2.1.3), we have that $B_1^\gamma + B_2^\gamma + B_3^\gamma \geq (B_1 + B_2 + B_3 + B_4)^\gamma$. Thus $|P| \geq |P| \geq \alpha(h_l^\gamma + (h_{g_1} + m_l + m_r + h_r)^\gamma) + 2 \geq \alpha m_j^\gamma + 2$. □

3.3 Final two path results

The following two lemmas are similar and are used directly in the proof of the main theorem. In the proof of the main theorem, we will delete a path containing the edge e from the graph G. In the following two lemmas, we consider the subgraph that results from that deletion. a_1 and a_2 play the role of the ends of the path we delete and e plays the role of f, the remaining edge we need our cycle to contain.

These lemmas are similar in spirit, as they differ only in the location of the edge e. Furthermore, though the proofs of these lemmas are long, similar concepts are used repeatedly throughout. Small, yet important structural differences will require different combinations of these concepts.

(3.3.1) Lemma. *Let $n \geq 7$ and assume the assertion of Theorem (1.2.2) holds for graphs of order $< n$. Let G be a 3-connected claw-free graph of order n, $\{a_1, a_2\} \subseteq V(G)$ such that neither $G - a_1$ nor $G - a_2$ is 3-connected. Let $e \in E(G - \{a_1, a_2\})$. Then there is a path P in $G - \{a_1, a_2\}$ from $N(a_1)$ to $N(a_2)$ such that $e \in E(P)$ and $|E(P)| \geq \alpha(n + 2)^\gamma + 2$.*

Proof. We proceed by induction on n, but will need to make a distinction between two cases, depending on the location of the edge e. Before making this distinction, we consider the base case where $n = 7$.

Let P' be a longest path in G from a_1 to a_2 such that $e \in P'$. Assume for contradiction that $|E(P')| \leq 4$. Hence there exists $v \in V(G) - V(P')$. Since G is 3-connected, there exist three independent paths from v to $v_1, v_2, v_3 \in V(P')$, where a_1, v_1, v_2, v_3, a_2 are on P' in order. Without loss of generality, we may assume

$e \notin v_1P'v_2$. Thus $|E(v_1Pv_2)| \geq 2$, otherwise we contradict the maximality of P'. Let u be a vertex between v_1 and v_2 on P'. Thus $|E(P')| \geq 4$. Thus we may assume that $P' = a_1uv_2v_3a_2$ and that $v_2v_3 = e$. If $uv \in E(G)$, then $a_1vuv_2v_3a_2$ contradicts the maximality of P'. If $va_2 \in E(G)$, then $a_1uv_2v_3va_2$ contradicts the maximality of P'. If $ua_2 \in E(G)$, then $a_1vv_3v_2ua_2$ contradicts the maximality of P'. Thus as G is 3-connected, $\{uv_3, a_2v_2, a_1a_2\} \subseteq E(G)$. However, $\{a_1, u, v, a_2\}$ would induce a claw in G – a contradiction. Thus we may assume $|E(P')| \geq 5$. Then $P := P' - \{a_1, a_2\}$ is the desired path for the lemma. Thus we may assume $n \geq 8$.

We now establish the structure of $G - a_1$ through Tutte decomposition. For $k \geq 1$, let $M_1, ..., M_k$ be the consecutive 3-blocks in the decomposition of $G - a_1$ (without loss of generality, from left to right). Let $m_i = |V(M_i)|$ and let $S_i = \{b_i, c_i\} = V(M_i \cap M_{i+1})$. Note that for all i, $\{b_i, c_i\}$ is a special 2-cut of $G - a_1$. Thus $b_ic_i \in E(M_i) \cap E(M_{i+1})$. Let $S_0 = N_{M_1}(a_1)$ and let $S_k = N_{M_k}(a_1)$.

We now distinguish two cases, based on the location of the edge e. We first consider the case where there is a 3-block M of the decomposition of $G - a_1$ where $a_2, e \in M$. The second case is where no such M exists.

Case I. There exists 3-block M in the decomposition of $G - a_1$ where $a_2, e \in M$.

Let j be the index of the 3-block containing a_2 and e.

Structurally there are three different sections to this graph. The sections are M_j, the 3-blocks left of M_j, and the 3-blocks right of M_j. Hence we first label these sections. Let $N_1 = \cup_{i=1}^{j-1} M_i$, $N_2 = \cup_{i=j+1}^{k} M_i$. Note that some of $\{N_1, N_2\}$ may be empty. Let $n_1 = |V(N_1)|, n_2 = |V(N_2)|$.

Claim 1. We may assume that M_j is 3-connected.

Otherwise, M_j is a chain of cycles. Since e is not incident to a_1 or a_2, it is easy to find a either a path P_j from S_{j-1} (say b_{j-1}) to a_2 such that $e \in P_j$, if $j < k$ then $b_jc_j \in P_j$ and $|E(P_j)| \geq \alpha(max\{0, m_j - 4\})^\gamma + 2$ or a path P'_j from S_j (say b_j) to a_2

such that $e \in P'_j$, if $j > k$ then $b_{j-1}c_{j-1} \in P'_j$ and $|E(P'_j)| \geq \alpha(max\{0, m_j-4\})^\gamma+2$. Without loss of generality, we may assume that we can find such a path P_j.

If N_2 is not empty, then by Lemmas (3.1.3) and (2.2.8) we find a cycle C_2 in N_2 such that $b_jc_j \in C_2$, $E(C_2 - b_jc_j) \subseteq E(G)$, and $|E(C_2)| \geq \alpha(n_2-4)^\gamma + 4$. If N_1 is not empty then it contains at least one 3-connected 3-block and hence by Lemmas (3.1.6) and (3.1.4)(2), we find a path P_1 in N_1 from b_{j-1} to S_0, such that $c_{j-1} \notin P_1$, $E(P_1) \subseteq E(G)$, $|E(P_1)| \geq \alpha(n_1-4)^\gamma + 2$.

If N_1 and N_2 are not empty, $P := (P_1 + (P_j - b_jc_j) + (C_2 - b_jc_j)) - a_2$ gives the desired path for the lemma. If N_1 is empty but N_2 is not empty, $P = ((P_j - b_jc_j) + (C_2 - b_jc_j)) - a_2$ gives the desired path for the lemma. If N_1 is not empty but N_2 is empty, $P = (P_1 + P_j) - a_2$ gives the desired path for the lemma. If both N_1 and N_2 are empty, $G - a_1$ is a chain of triangles and hence $|E(P_j)| \geq \alpha(m_j)^\gamma + 4$ and hence $P := P_j - a_2$ is the desired path for the lemma.

This proves Claim 1.

Claim 2. We may assume that a_2 is not in a special 2-cut of $G - a_1$.

Otherwise, without loss of generality, we may assume that N_1 is not empty and $a_2 \in S_{j-1}$. How we proceed depends on the relative sizes of $\{n_1, m_j, n_2\}$.

Let $t := \min\{n_1, m_j, n_2\}$.

We may assume $t \neq n_2$. Suppose $t = n_2$. Thus $t \geq 0$. Without loss of generality, we may assume $a_2 = c_{j-1}$. By direct verification or Lemma (2.2.8), we find a cycle C_j in M_j such that $e, b_{j-1}c_{j-1} \in C_j$, $E(C_j - b_jc_j - b_{j-1}c_{j-1} \subseteq E(G)$ and $|E(C_j)| \geq \alpha(m_j-4)^\gamma+4$. If N_2 is not empty, $e \neq b_jc_j$, and $b_jc_j \in C_j$, we replace b_jc_j in C_j with a path in N_2. As N_1 is not empty, then by Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b_{j-1} to S_0, such that $c_{j-1} \notin P_1$, $E(P_1) \subseteq E(G)$, $|E(P_1)| \geq \alpha(n_1-3)^\gamma + 1$. Now it is easy to verify that (since n_2 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup (C_j - b_{j-1}c_{j-1})) - a_2$ is the desired path for the lemma.

We may assume $t \neq n_1$. Suppose $t = n_1$. Thus $t \geq 3$ and hence $n_2 \geq 4$. First assume $n_2 = 4$. Then $n_1 = 3$. By direct verification or Lemma (2.2.8), we find a cycle C_j in M_j such that $e, b_{j-1}c_{j-1} \in C_j$, $E(C_j - b_jc_j - b_{j-1}c_{j-1}) \subseteq E(G)$, and $|E(C_j)| \geq \alpha(m_j - 4)^\gamma + 4$. We trivially find a path P_1 in N_1 from b_{j-1} to S_0 such that $c_{j-1} \notin P_1$, $E(P_1) \subseteq E(G)$, $|E(P_1)| = 1$. $P := (P_1 + (C_j - b_{j-1}c_{j-1})) - a_2$ is the desired path for the lemma (if necessary, replace b_jc_j with a path in N_2). Thus we may assume $n_2 \geq 5$. Without loss of generality, we may assume $a_2 = c_{j-1}$.

If $m_j \leq 6$, we directly find a path P' in M_j from b_{j-1} to S_j (say b_j) such that $e \in P'$, $c_{j-1} \notin P'$, $E(P') \subseteq E(G)$, and $|E(P')| \geq 1$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_2 in N_2 from b_j to S_k, such that $c_j \notin P_2$, $E(P_2) \subseteq E(G)$, $|E(P_2)| \geq \alpha(n_2 - 4)^\gamma + 2$. Trivially we find a path P_1 in N_1 from b_{j-1} to a_2 such that $E(P_1) \subseteq E(G)$, $|E(P_1)| \geq 2$. Now it is easy to verify that (since n_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P' \cup P_2) - a_2$ is the desired path for the lemma. Thus we may assume $m_j \geq 7$.

We find a path P' in M_j from S_{j-1} (say b') to S_j (say b_j) such that $e \in P'$, $E(P') \subseteq E(G)$, if $b' \neq a_2$ then $a_2 \notin P'$, and $|E(P')| \geq \alpha(m_j + 1)^\gamma + 2$. Without loss of generality, assume e is not incident to b_j. Suppose $M_j - b_{j-1}$ is 3-connected. Then $M_j - b_{j-1}c_{j-1}$ is 3-connected. Let $M' := M_j \cup \{z_1, z_1b_{j-1}, z_1b_j, z_1c_j\}$. M' is 3-connected and claw-free. Thus by the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1b_{j-1} \in C'$, $|E(C')| \geq \alpha(m_j + 1)^\gamma + 5$. C' either contains the desired path P' or contains a path which can be trivially modified by removing a_2 to obtain the desired path P'. Suppose instead that $M_j - b_j$ is 3-connected. Then $M_j - b_jc_j$ is 3-connected. Let $M' := M_j \cup \{z_1, z_1b_j, z_1b_{j-1}, z_1c_{j-1}\}$. M' is 3-connected and claw-free. Thus by the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1b_j \in C'$, $|E(C')| \geq \alpha(m_j + 1)^\gamma + 5$. C' contains the desired path P'. Lastly, suppose $M_j - b_j$ and $M_j - b_{j-1}$ are not

3-connected. By the inductive hypothesis of Lemma (3.3.1) we find a path P_j in $M_j - b_j - b_{j-1}$ from $N(b_j)$ (say b'_j) to $N(b_{j-1})$ (say b'_{j-1}) such that $e \in E(P_j)$ and $|E(P_j)| \geq \alpha(m_j + 2)^\gamma + 2$. We can then trivially extend P_j to obtain the desired path P'. In any case, we find the desired path P'.

By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_2 in N_2 from b_j to S_k, such that $c_j \notin P_2$, $E(P_2) \subseteq E(G)$, $|E(P_2)| \geq \alpha(n_2 - 4)^\gamma + 2$. Trivially we find a path P_1 in N_1 from b' to a_2 such that $E(P_1) \subseteq E(G)$, $|E(P_1)| \geq 0$. Now it is easy to verify that (since n_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P' \cup P_2) - a_2$ is the desired path for the lemma.

So $t = m_j$. Thus $t \geq 4$, and $n_1, n_2 \geq 5$. Without loss of generality, we may assume $a_2 = c_{j-1}$. Trivially, we find a path P_j in M_j from b_{j-1} to S_j (say b_j) such that $e \in P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq 1$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_2 in N_2 from b_j to S_k, such that $c_j \notin P_2$, $E(P_2) \subseteq E(G)$, $|E(P_2)| \geq \alpha(n_2 - 4)^\gamma + 2$. By Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_1 in N_1 such that $b_{j-1}c_{j-1} \in C_1$ and $|E(C_1)| \geq \alpha n_1^\gamma + 2$. Now it is easy to verify that (since m_j can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_j \cup (C_1 - b_{j-1}c_{j-1}) \cup P_2) - a_2$ is the desired path for the lemma.

This proves Claim 2.

As $G - a_1$ is not 3-connected and as M_j is 3-connected by Claim 1, the decomposition of $G - a_1$ is not one 3-block. Without loss of generality, assume N_1 is not empty.

First we consider the case where $m_j = 4$. By claim 2, N_2 must be empty. Thus it trivial to construct a path P_j in M_j from a_2 to S_{j-1} (say b_{j-1})) such that $e \in P_j$, $|E(P_j)| = 3$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b_{j-1} to S_0 such that $c_{j-1} \notin P_1$, $|E(P_1)| \geq \alpha(n_1 - 3)^\gamma + 1$. $P := (P_1 \cup P_j) - a_2$ is the desired path for the lemma. Thus we may assume $m_j > 4$.

Let $t := \min\{n_1, m_j, n_2\}$.

Claim 3. We may assume $t \neq n_i$, for $i = 1, 2$.

Otherwise we may assume $t = n_i$ for some i. By symmetry assume $t = n_2$.

If $e = b_{j-1}c_{j-1}$, let $b' = b_{j-1}$. Otherwise, let $b' \in S_{j-1}$ such that b' is not incident to e. Let $c' \in S_{j-1}$ such that $c' \neq b'$. If $e \neq b_{j-1}c_{j-1}$, let $e' = e$. If $e = b_{j-1}c_{j-1}$, let $e' \in E(M_j - b')$ such that $e' \in E(G)$.

We seek to find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $e \in P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha(max\{0, m_j - 6\})^\gamma + 3$ and if N_2 is not empty $e \neq b_jc_j$ and $b_jc_j \in P_j$ then we replace b_jc_j with a path in N_2. If $m_j \leq 6$, it is easy to verify directly that such a path P_j exists. If $m_j \geq 7$ and $M_j - b'$ is not 3-connected, then by the inductive hypothesis of Lemma (3.3.1), we find a path P'_j in $M_j - b' - a_2$ from $N(b')$ to $N(a_2)$ such that $e' \in P'_j$, $|E(P'_j)| \geq \alpha(m_j + 2)^\gamma + 2$. This path P'_j can trivially be extended into the desired path P_j. Thus we may assume $m_j \geq 7$ and $M_j - b'$ is 3-connected. If $e = b_{j-1}c_{j-1}$, then by Lemma (3.2.2) we find path P'_j in M_j from b' to a_2 such that $c' \notin P'_j$, $|E(P'_j)| \geq \alpha m_j^\gamma + 2$. $P_j = P'_j + \{c_{j-1}, b_{j-1}c_{j-1}\}$ gives the desired path P_j. Thus we may assume $e' = e$. At this point, it suffices to find a path P'_j in $M_j - b'$ from $N(b')$ to a_2 such that $e \in P'_j$, $|E(P'_j)| \geq \alpha(max\{0, m_j - 6\})^\gamma + 2$. If there exists v in $M_j - b'$ that is adjacent to b', $v \neq a_2$ and not incident to e, and $(M_j - b') - v$ is 3-connected, then we can continue iterating the deletion of such vertices v and have the same problem on a smaller graph. Thus consider M'_j, the subgraph of $M_j - b'$ obtained by deleting a maximal sequence of such vertices v and let b^* be the last such vertex v deleted. Let $m'_j = |V(M'_j)|$. As M'_j is 3-connected, $N(b^*) \cap M_j \subseteq V(e) \cup \{a_2\}$. By the above, we can find a path P^*_j in M'_j from $V(e)$ to a_2 such that $e \in P^*_j$, $E(P^*_j) \subseteq E(G)$, $|E(P^*_j)| \geq \alpha(max\{0, m'_j - 6\})^\gamma + 3$. Thus we naturally extend P^*_j to obtain our path P_j as desired.

By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from c_{j-1} to S_0 such that $b_{j-1} \notin P_1$, $|E(P_1)| \geq \alpha(n_1 - 3)^\gamma + 1$. Now it is easy to verify that (since n_2 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P_j) - a_2$ gives the desired path for the lemma.

This proves Claim 3.

By Claim 3, $t = m_j$. Thus $t \geq 5$ and $n_i \geq 6$ for $i = 1, 2$.

Consider $M'_j = M_j + \{z, zb_{j-1}, zc_{j-1}, za_2\}$. Clearly M'_j is 3-connected and $\{za_2, e, b_jc_j\}$ is not a 3-edge cut of M'_j. Hence there is a cycle C' in M'_j containing za_2, e, b_jc_j. C' contains a path P' in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $e, b_jc_j \in P'$ and $|E(P')| \geq 2$.

By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b_{j-1} to S_0 such that $c_{j-1} \notin P_1$, $|E(P_1)| \geq \alpha(n_1 - 2)^\gamma + 2$. By Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_2 in N_2 such that $b_jc_j \in C_2$ and $|E(C_2)| \geq \alpha n_2^\gamma + 5$. Now it is easy to verify that (since m_j can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_j \cup (C_2 - b_jc_j) \cup P_1) - a_2$ is the desired path for the lemma.

This proves Case I.

Case II. There does not exist a 3-block M in the decomposition of $G - a_1$ such that $a_2, e \in M$.

Let j_1 be the index of a 3-block containing the edge e and let j_2 be the index of a 3-block containing a_2. Note that it is possible for a_2 or e to be contained in multiple 3-blocks. However, $j_1 \neq j_2$. Thus we assume $j_1 < j_2$. Given these constraints, then choose j_1 then j_2 to be as small as possible.

Structurally there are five different sections to this graph; though, we will combine some of them together in the analysis that follows. The sections are M_{j_1}, M_{j_2}, the 3-blocks between M_{j_1} and M_{j_2}, the 3-blocks left of M_{j_1}, and the 3-blocks right of M_{j_2}. Hence we first label these sections. Let $N_1 = \cup_{i=1}^{j_1-1} M_i$, $N_2 = \cup_{i=j_2+1}^{k} M_i$, $N^* = \cup_{i=j_1+1}^{j_2-1} M_i$. Note that some of $\{N_1, N_2, N^*\}$ may be empty. Let $n_1 = |V(N_1)|, n_2 = |V(N_2)|, n^* = |V(N^*)|$.

Based on the relative sizes of $\{n_1, n_2, m_{j_1}, m_{j_2}, n^*\}$, we will choose to construct our path in different ways. We proceed to prove several claims that will substantially restrict the structure of G.

Claim 1. We may assume M_{j_2} is not 3-connected.

Otherwise, we may assume M_{j_2} is 3-connected. Let $N_1^* := \cup_{i=1}^{j_2-1} M_i$. Let $n_1^* = |V(N_1^*)|$. How we proceed depends on the relative sizes of $\{n_1^*, m_{j_2}, n_2\}$.

Let $t := \min\{n_1^*, m_{j_2}, n_2\}$.

We may assume $t \neq n_1^*$. Suppose $t = n_1^*$. Thus $t \geq 3$. Without loss of generality, we may assume $b_{j_2} \neq a_2$.

We find a path P_{j_2} in M_{j_2} from a_2 to b_{j_2} such that $b_{j_2-1}c_{j_2-1} \in P_{j_2}$, $|E(P_{j_2})| \geq \alpha(max\{0, m_{j_2} - 6\})^\gamma + 3$. If $m_{j_2} \leq 6$, it is easy to verify the existence of such a path P_{j_2} directly. By choice of j_2 to be minimal, $a_2 \notin S_{j_2-1}$. If $a_2 = c_{j_2}$, then by direct construction or Lemma (2.2.8) we find a cycle C_{j_2} in M_{j_2} such that $b_{j_2-1}c_{j_2-1}, b_{j_2}c_{j_2} \in C_{j_2}$, $|E(C_{j_2})| \geq \alpha(m_{j_2} - 4)^\gamma + 4$; $P_{j_2} = C_{j_2} - b_{j_2}c_{j_2}$ gives the desired path. Thus we may assume $a_2 \neq c_{j_2}$. We assume for the hypothesis of Claim 1 that M_{j_2} is 3-connected. Further, $M_{j_2} - a_2$ is not 3-connected as $G - a_2$ is not 3-connected. If $M_{j_2} - b_{j_2}$ is not 3-connected, then by Lemma (3.3.1), we find a path P'_{j_2} in $M_{j_2} - b_{j_2} - a_2$ from $N(b_{j_2})$ to $N(a_2)$ such that $b_{j_2-1}c_{j_2-1} \in P'_{j_2}$, $|E(P'_{j_2})| \geq \alpha(m_{j_2} + 2)^\gamma + 2$. We trivially extend P'_{j_2} to the desired path P_{j_2} (note that as $a_2 \neq c_{j_2}$, either vertex in S_{j_2} can be labelled b_{j_2}). Thus we may assume $M_{j_2} - b_{j_2}$ is 3-connected. Thus $M_{j_2} - b_{j_2}c_{j_2}$ is 3-connected. Let $M' := (M_{j_2} - b_{j_2}c_{j_2}) \cup \{z_1, z_1b_{j_2}\} \cup \{z_1u : u \in S_{j_2-1}\}$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $z_1b_{j_2} \in C'$, $|E(C')| \geq \alpha(m_{j_2} + 1)^\gamma + 5$. C' contains the desired path P_{j_2}.

Trivially, we find a path $P_{n_1^*}$ in N_1^* from b_{j_2-1} to c_{j_2-1} such that $e \in P_{n_1^*}$, $b_{j_2-1}c_{j_2-1} \notin P_{n_1^*}$, $E(P_{n_1^*}) \subseteq E(G)$, and $|E(P_{n_1^*})| \geq 2$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_{n_2} in N_2 from b_{j_2} to S_k, such that $c_{j_2} \notin P_{n_2}$,

$|E(P_{n_2})| \geq \alpha(n_2-3)^{\gamma}+1$. Now it is easy to verify that (since n_1^* can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_{j_2} - \{b_{j_2-1}c_{j_2-1}, a_2\}) \cup P_{n_1^*} \cup P_{n_2}$ is the desired path for the lemma. Thus we may assume $t \neq n_1^*$.

We may assume $t \neq n_2$. Suppose $t = n_2$. First, we find a path $P_{n_1^*}$ in N_1^* from S_0 to S_{j_2-1} (say b_{j_2-1}), such that $e \in P_{n_1^*}$, if $e \neq b_{j_2-1}c_{j_2-1}$ then $b_{j_2-1}c_{j_2-1} \notin P_{n_1^*}$ and $c_{j_2-1} \notin P_{n_1^*}$, $E(P_{n_1^*}) \subseteq E(G)$, and $|E(P_{n_1^*})| \geq \alpha(n_1^* - 3)^{\gamma} + 1$. If $n_1^* = 3$, finding $P_{n_1^*}$ is trivial. Let $M' = N_1^* \cup \{z_1, z_2, z_1z_2, z_2b_{j_2-1}, z_2c_{j_2-1}\} \cup \{z_1u : u \in S_0\}$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle $C_{n_1^*}$ in M' such that $e, z_1z_2 \in M'$, $|E(C_{n_1^*})| \geq \alpha(n_1^* + 2)^{\gamma} + 5$. Either $C_{n_1^*}$ contains the desired path $P_{n_1^*}$, or $C_{n_1^*}$ contains a path which can be trivially modified by either deleting b_{j_2-1} (if $b_{j_2-1}c_{j_2-1} \in C'$ and $e \neq b_{j_2-1}c_{j_2-1}$) or by removing c_{j_2-1} (otherwise) that then is the desired path $P_{n_1^*}$.

We then find a path P_{j_2} in M_{j_2} from b_{j_2-1} to a_2 such that $b_{j_2-1}c_{j_2-1} \notin P_{j_2}$ and $|E(P_{j_2})| \geq \alpha m_{j_2}^{\gamma} + 2$. If $m_{j_2} \leq 5$, it is trivial to find such a path P_{j_2} such that $|E(P_{j_2})| \geq 3$. If $m_{j_2} \geq 6$, we find P_{j_2} by Lemma (3.2.2). In any case, if $b_{j_2}c_{j_2} \in P_{j_2}$, we replace it with a path in N_2. Now it is easy to verify that (since n_2 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_{j_2} - a_2) \cup P_{n_1^*}$ is the desired path for the lemma. Thus we may assume $t \neq n_2$.

So we may assume $t = m_{j_2}$. Thus $t \geq 4$. Note that $a_2 \notin S_{j_2-1}$ by minimality of j_2. First, we find a path $P_{n_1^*}$ in N_1^* exactly as above from S_0 to S_{j_2-1} (say b_{j_2-1}), such that $e \in P_{n_1^*}$, $b_{j_2-1}c_{j_2-1} \notin P_{n_1^*}$, $c_{j_2-1} \notin P_{n_1^*}$, $|E(P_{n_1^*})| \geq \alpha(n_1^* - 3)^{\gamma} + 1$. Next we find a path P_{j_2} in M_{j_2} from b_{j_2-1} to a_2 such that $b_{j_2}c_{j_2} \in P_{j_2}$, $|E(P_{j_2})| \geq 2$. Note that the existence of a path with that structure implies it has at least 2 edges. By Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_{j_2}c_{j_2} \in C_{n_2}$ and $|E(C_{n_2})| \geq \alpha n_2^{\gamma} + 2$. Now it is easy to verify that (since m_{j_2} can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P_{n_1^*} \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma.

This proves Claim 1.

Claim 2. We may assume that a_2 is adjacent to a vertex in $v \in S_{j_2-1}$ such that the degree of v in M_{j_2} is 2.

Otherwise a_2 is not adjacent to any vertices in S_{j_2-1} of degree 2 in M_{j_2}. Let $N_1' := \cup_{i=1}^{j_1} M_i$. Let $n_1' = |V(N_1')|$. Let $N_1^* := \cup_{i=1}^{j_2-1} M_i$. Let $n_1^* = |V(N_1^*)|$.

First we show that we may assume that N^* is empty. Otherwise, suppose that N^* is not empty. As M_{j_2} is a chain of cycles, N^* contains at least one 3-connected 3-block. Let $M' := N_1' \cup \{z_1, z_2, z_1z_2, z_2b_{j_1}, z_2c_{j_1}\} \cup \{z_1u : u \in S_0\}$. If $|S_0| = 1$, let $z_1 = z_2$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e \in C'$, if $|S_0| > 1$ then $z_1z_2 \in C'$, if $|S_0| = 1$ then $z_1u \in C'$ where $u \in S_0$, and $|E(C')| \geq \alpha(n_1'+1)^\gamma + 5$. C' contains a path P' in N_1' from S_0 to S_{j_1} (say b_{j_1}) such that $e \in P'$, $E(P') \subseteq E(G)$, and $|E(P')| \geq \alpha(n_1'+1)^\gamma + 1$. By Lemmas (3.1.5) and (3.1.4)(1), we find a path P_{n^*} in N^* from b_{j_1} to S_{j_2-1} (say b_{j_2-1}), such that $c_{j_1} \notin P_{n^*}$, $|E(P_{n^*})| \geq \alpha(n^* - 2)^\gamma + 1$.

It is trivial to find a path P_{j_2} in M_{j_2} from b_{j_2-1} to a_2 such that $c_{j_2-1} \notin P_{j_2}$, if N_2 is not empty then $b_{j_2}c_{j_2} \in P_{j_2}$, $|E(P_{j_2})| = m_{j_2} - 2$. If N_2 is empty, note that $m_{j_2} \geq 4$ and hence $P := P' \cup P_{n^*} \cup (P_{j_2} - a_2)$ gives the desired path for the lemma. If N_2 is not empty, then by Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_{j_2}c_{j_2} \in C_{n_2}$ and $|E(C_{n_2})| \geq \alpha n_2^\gamma + 2$. $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P' \cup P_{n^*} \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma. Thus we may assume N^* is empty. Thus M_{j_1} is 3-connected.

If $m_{j_1} \leq 6$ and $j_1 > 1$, it is easy to find a path P_{j_1} in M_{j_1} from S_{j_1} (say b_{j_1}) to S_{j_1-1} (say b_{j_1-1}) such that $e \in P_{j_1}$, $b_{j_1-1}c_{j_1-1} \notin P_{j_1}$, and $|E(P_{j_1})| \geq 2$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b_{j_1-1} to S_0 such that $c_{j_1-1} \notin P_1$, $|E(P_1)| \geq \alpha(n_1 - 3)^\gamma + 1$. If N_2 is empty, note that $m_{j_2} \geq 4$ and hence $P := P_{j_1} \cup P_1 \cup (P_{j_2} - a_2)$ is the desired path for the lemma. If N_2 is not empty, then $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P_{j_1} \cup P_1 \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma.

If $m_{j_1} \leq 6$ and $j_1 = 1$, it is trivial to find a path P_{j_1} in M_{j_1} from S_{j_1} (say b_{j_1}) to S_0 such that $e \in P_{j_1}$ and $|E(P_{j_1})| \geq 3$. If N_2 is empty, note that $m_{j_2} \geq 4$ and hence $P := P_{j_1} \cup (P_{j_2} - a_2)$ gives the desired path for the lemma. If N_2 is not empty, then $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P_{j_1} \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma.

Thus we may assume $m_{j_1} \geq 7$.

Next assume that $j_1 = 1$ and $|S_0| = 1$. Let $u \in S_0$. Let $M' := M_1 \cup \{z_1, z_1u, z_1b_1, z_1c_1\}$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1u \in E(C')$ and $|E(C')| \geq \alpha(m_1 + 1)^\gamma + 5$. C' contains a path P' in M_1 from S_0 to S_{j_1} (say b_{j_1}) such that $e \in P'$, $|E(P')| \geq \alpha(m_{j_1} + 1)^\gamma + 2$. Hence either $P := P' \cup (P_{j_2} - a_2)$ or $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P' \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma.

Thus we may assume that either $j_1 > 1$ or that $|S_0| > 1$. In either case, $|S_{j_1-1}| > 1$. Let $b' \in S_{j_1-1}$ such that e is not incident to b'. Without loss of generality, assume e is not incident to b_{j_1}.

Suppose $M_{j_1} - b'$ is not 3-connected and $M_{j_1} - b_{j_1}$ is not 3-connected. Then by the induction hypothesis of Lemma (3.3.1) we find a path P'_{j_1} in $M_{j_1} - b' - b_{j_1}$ from $N(b')$ to $N(b_{j_1})$ such that $e \in P'_{j_1}$, $|E(P'_{j_1})| \geq \alpha(m_{j_1} + 2)^\gamma + 2$. We trivially extend P'_{j_1} to find a path P_{j_1} in M_{j_1} from S_{j_1-1} (say b^*) to b_{j_1} such that $e \in P_{j_1}$, $E(P_{j_1}) \subseteq E(G)$, $|E(P_{j_1})| \geq \alpha(m_{j_1} + 2)^\gamma + 2$. If $j_1 = 1$ and N_2 is empty, then $P := P_{j_1} \cup (P_{j_2} - a_2)$ gives the desired path for the lemma. If $j_1 = 1$ and N_2 is not empty, then $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P_{j_1} \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma. Thus we may assume $j_1 > 1$. Let $c^* \in S_{j_1-1}$ such that $c^* \neq b^*$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b^* to S_0 such that $c^* \notin P_1$, $|E(P_1)| \geq \alpha(n_1 - 3)^\gamma + 1$. If N_2 is empty, then $P := P_1 \cup P_{j_1} \cup (P_{j_2} - a_2)$ is the desired path for the lemma. If N_2 is not empty, then $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P_1 \cup P_{j_1} \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma.

Suppose $M_{j_1} - b_{j_1}$ is 3-connected. Thus $M_{j_1} - b_{j_1}c_{j_1}$ is 3-connected. Let $M' := (M_{j_1} - b_{j_1}c_{j_1}) \cup \{z_1, z_1b_{j_1}\} \cup \{z_1u : u \in S_{j_1}\}$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1b_{j_1} \in E(C')$ and $|E(C')| \geq \alpha(m_{j_1} + 1)^\gamma + 5$. If $j_1 = 1$, C' contains a path P' in M' from S_0 to b_{j_1} such that $e \in P'$, $|E(P')| \geq \alpha(m_{j_1} + 1)^\gamma + 3$. Hence either $P := P' \cup (P_{j_2} - a_2)$ or $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P' \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma. Thus we may assume $j_1 > 1$. In this case, C' contains a path P' in M' from S_{j_1-1} (say b_{j_1-1}) to b_{j_1} such that $e \in P'$, $b_{j_1-1}c_{j_1-1} \notin P'$, $|E(P')| \geq \alpha(m_{j_1} + 1)^\gamma + 2$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b_{j_1-1} to S_0 such that $c_{j_1-1} \notin P_1$, $|E(P_1)| \geq \alpha(n_1 - 3)^\gamma + 1$. Thus either $P = P_1 \cup P' \cup P_{j_2} - a_2)$ or $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P_1 \cup P' \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma.

Thus we may assume $M_{j_1} - b_{j_1}$ is not 3-connected, but $M_{j_1} - b'$ is 3-connected.

Suppose $j_1 > 1$. Without loss of generality, assume $b' = b_{j_1-1}$. Thus $M_{j_1} - b_{j_1-1}c_{j_1-1}$ is 3-connected. Let $M' = (M_{j_1} - b_{j_1-1}c_{j_1-1}) \cup \{z_1, z_1b_{j_1-1}\} \cup \{z_1u : u \in S_{j_1-1}\}$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1b_{j_1-1} \in E(C')$ and $|E(C')| \geq \alpha(m_{j_1} + 1)^\gamma + 5$. C' contains a path P' in M' from b_{j_1-1} to S_{j_1} (say b_{j_1}) such that $e \in P'$, $|E(P')| \geq \alpha(m_{j_1} + 1)^\gamma + 2$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b_{j_1-1} to S_0 such that $c_{j_1-1} \notin P_1$, $|E(P_1)| \geq \alpha(n_1 - 3)^\gamma + 1$. Thus either $P := P_1 \cup P' \cup P_{j_2} - a_2)$ or $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P_1 \cup P' \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma.

Thus we may assume $j_1 = 1$ and that $M_1 - b'$ is 3-connected. Suppose $|S_0| = 2$.

Then let $M' := (M_1 - b') \cup \{z_1, z_1b_1, z_1c_1, z_1u\}$, where $u \in (S_0 - b')$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1u \in C'$, $b' \notin C'$, and $|E(C')| \geq \alpha(m_1+1)^\gamma + 5$. C' contains a path P' in M_1 from u to S_j (say b_j) such that $e \in P'$, $b' \notin P'$, $E(P') \subseteq E(G)$, and $|E(P')| \geq \alpha(m_1+1)^\gamma + 2$. Thus either $P := P' \cup P_{j_2} - a_2)$ or $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup P' \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma.

Thus we may assume $|S_0| > 2$. Let $M' := (M_1 - b') \cup \{z_1, z_2, z_1z_2, z_2b_1, z_2c_1\} \cup \{z_1u : u \in S_0 - b'\}$, where $u \in (S_0 - b')$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1z_2 \in C'$, $b' \notin C'$, and $|E(C')| \geq \alpha(m_1+2)^\gamma + 5$. C' contains a path P' in M_1 from $(S_0 - b')$ (say b^*) to S_j (say b_j) such that $e \in P'$, $b' \notin P'$, $E(P') \subseteq E(G)$, and $|E(P')| \geq \alpha(m_1+2)^\gamma + 1$. Thus either $P := (P' \cup \{b', b^*b'\}) \cup P_{j_2} - a_2)$ or $P := (P_{j_2} - \{b_{j_2}c_{j_2}, a_2\}) \cup (P' \cup \{b', b^*b'\}) \cup (C_{n_2} - b_{j_2}c_{j_2})$ is the desired path for the lemma.

This proves Claim 2.

Next we consider one very special case that would significantly complicate all further analysis if not considered separately. The proof of the following claim is similar, though different to that of Claim 2.

Claim 3. We may assume there exists $v \in M_k$, $v \in N(a_1)$, $v \neq a_2$.

Otherwise, a_2 is the only neighbor of a_1 in M_k. Thus N_2 is empty and $M_{j_2} = M_k$ is a triangle. Note $b_{j_2-1}c_{j_2-1} \in E(G)$ as G is claw-free.

Let $N_1^* := \cup_{i=1}^{j_1} M_i$. Let $n_1^* = |V(N_1^*)|$.

If $n_1^* = 3$, $|E(P_{n_1^*})| = 2$ (note that $|S_0| \geq 2$). Further, N^* must contain at least one 3-block that is 3-connected. If $n^* \leq 5$, it is trivial to find a path P_{n^*} in N^* from b_{j_1} to S_{j_2-1} (say b_{j_2-1}), such that $c_{j_1} \notin P_{n^*}$, $|E(P_{n^*})| \geq 1$. $P = P_{n_1^*} + P_{n^*}$ is the desired path for the lemma. If $n^* \geq 6$, by Lemmas (3.1.5) and (3.1.4)(1), we find a path P_{n^*} in N^* from b_{j_1} to S_{j_2-1} (say b_{j_2-1}), such that $c_{j_1} \notin P_{n^*}$, $|E(P_{n^*})| \geq$

$\alpha(n^* - 2)^\gamma + 1$. $P := P_{n_1^*} \cup P_{n^*}$ is the desired path for the lemma. Thus we may assume $n_1^* \geq 4$.

We next show that we may assume N^* is empty. Otherwise $n^* \geq 4$.

First, we find a path $P_{n_1^*}$ in N_1^* from S_0 to S_{j_1} (say b_{j_1}), such that $e \in P_{n_1^*}$, if $e \neq b_{j_1}c_{j_1}$ then $b_{j_1}c_{j_1} \notin P_{n_1^*}$, if $e \neq b_{j_1}c_{j_1}$ then $c_{j_1} \notin P_{n_1^*}$, $|E(P_{n_1^*})| \geq \alpha(n_1^* + 2)^\gamma + 1$. Let $M' := N_1^* \cup \{z_1, z_2, z_1z_2, b_{j_1}c_{j_1}\} \cup \{z_1u : u \in S_0\} \cup \{z_2u : u \in S_{j_1}\}$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1z_2 \in M'$, $|E(C')| \geq \alpha(n_1^* + 2)^\gamma + 5$. Either C' contains the desired path $P_{n_1^*}$, or C' contains a path which can be trivially modified by either deleting b_{j_1} (if $b_{j_1}c_{j_1} \in C'$ and $e \neq b_{j_1}c_{j_1}$) or by removing c_{j_1} (otherwise) that then is the desired path $P_{n_1^*}$.

If $n^* \leq 5$, it is trivial to find a path P_{n^*} in N^* from b_{j_1} to S_{j_2-1} (say b_{j_2-1}), such that $c_{j_1} \notin P_{n^*}$, $|E(P_{n^*})| \geq 2$. $P := P_{n_1^*} \cup P_{n^*}$ is the desired path for the lemma. If $n^* \geq 6$, by Lemmas (3.1.5) and (3.1.4)(1), we find a path P_{n^*} in N^* from b_{j_1} to S_{j_2-1} (say b_{j_2-1}), such that $c_{j_1} \notin P_{n^*}$, $|E(P_{n^*})| \geq \alpha(n^* - 2)^\gamma + 1$. Note that as $n_1^* \geq 4$ and $n^* \geq 4$, $(n_1^* + 2)^\gamma + (n^* - 2)^\gamma \geq (n + 2)^\gamma$. Thus $P := P_{n_1^*} \cup P_{n^*}$ is the desired path for the lemma.

This proves that we may assume N^* is empty in Claim 3.

If $m_{j_1} \leq 6$ and $j_1 > 1$, it is trivial to find a path P_{j_1} in M_{j_1} from S_{j_1} (say b_{j_1}) to S_{j_1-1} (say b_{j_1-1}) such that $e \in P_{j_1}$, $b_{j_1-1}c_{j_1-1} \notin P_{j_1}$, and $|E(P_{j_1})| \geq 2$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b_{j_1-1} to S_0 such that $c_{j_1-1} \notin P_1$, $|E(P_1)| \geq \alpha(n_1 - 3)^\gamma + 1$. Thus $P := P_1 \cup P_{j_1}$ is the desired path for the lemma.

If $m_{j_1} \leq 6$ and $j_1 = 1$, it is trivial to find a path P_{j_1} in M_{j_1} from S_{j_1} (say b_{j_1}) to S_0 such that $e \in P_{j_1}$ and $|E(P_{j_1})| \geq 3$. Thus $P := P_{j_1}$ is the desired path for the lemma.

Thus we may assume $m_{j_1} \geq 7$.

Note that by the hypothesis of this lemma, $e \neq b_{j_1}c_{j_1}$. If $j_1 = 1$ and as $|S_0| \geq 2$, let $b' \in S_0$ such that b' is not incident to e. If $j_1 > 1$, by choice of j_1, $e \neq b_{j_1-1}c_{j_1-1}$. Thus if $j_1 > 1$, let $b' \in S_{j_1-1}$ such that b' is not incident to e. Without loss of generality, assume b_{j_1} is not incident to e.

Suppose $M_{j_1} - b'$ is not 3-connected and $M_{j_1} - b_{j_1}$ is not 3-connected. Then by the induction hypothesis of Lemma (3.3.1) we find a path P'_{j_1} in $M_{j_1} - b' - b_{j_1}$ from $N(b')$ to $N(b_{j_1})$ such that $e \in P'_{j_1}$, $|E(P'_{j_1})| \geq \alpha(m_{j_1}+2)^\gamma + 2$. By extending P'_{j_1} to b_{j_1} and perhaps extending it to b', we find a path P_{j_1} in M_{j_1} from S_{j_1-1} (say b^*) to b_{j_1} such that $e \in P_{j_1}$, $E(P_{j_1}) \subseteq E(G)$, $|E(P_{j_1})| \geq \alpha(m_{j_1}+2)^\gamma + 3$. If $j_1 = 1$, then $P := P_{j_1}$ gives the desired path for the lemma. Thus we may assume $j_1 > 1$. Let $c^* \in S_{j_1-1}$ such that $c^* \neq b^*$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b^* to S_0 such that $c^* \notin P_1$, $|E(P_1)| \geq \alpha(n_1-3)^\gamma + 1$. Thus $P := P_1 \cup P_{j_1}$ is the desired path for the lemma.

Suppose $M_{j_1} - b_{j_1}$ is 3-connected. Thus $M_{j_1} - b_{j_1}c_{j_1}$ is 3-connected. Let $M' := (M_{j_1} - b_{j_1}c_{j_1}) \cup \{z_1, z_1b_{j_1}\} \cup \{z_1u : u \in S_{j_1}\}$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1b_{j_1} \in E(C')$ and $|E(C')| \geq \alpha(m_{j_1}+1)^\gamma + 5$. If $j_1 = 1$, C' contains a path P' in M' from S_0 to b_{j_1} such that $e \in P'$, $|E(P')| \geq \alpha(m_{j_1}+1)^\gamma + 3$. $P := P'$ gives the desired path for the lemma. Thus we may assume $j_1 > 1$. In this case, C' contains a path P' in M' from S_{j_1-1} (say b_{j_1-1}) to b_{j_1} such that $e \in P'$, $b_{j_1-1}c_{j_1-1} \notin P'$, $|E(P')| \geq \alpha(m_{j_1}+1)^\gamma + 2$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b_{j_1-1} to S_0 such that $c_{j_1-1} \notin P_1$, $|E(P_1)| \geq \alpha(n_1-3)^\gamma + 1$. Thus $P := P_1 \cup P'$ is the desired path for the lemma.

Thus we may assume $M_{j_1} - b_{j_1}$ is not 3-connected, but $M_{j_1} - b'$ is 3-connected.

Suppose $j_1 > 1$. Without loss of generality, assume $b' = b_{j_1-1}$. Thus $M_{j_1} - b_{j_1-1}c_{j_1-1}$ is 3-connected. Let $M' = (M_{j_1} - b_{j_1-1}c_{j_1-1}) \cup \{z_1, z_1b_{j_1-1}\} \cup \{z_1u : u \in S_{j_1-1}\}$. M' is 3-connected and claw-free. By the inductive hypothesis of

Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1b_{j_1-1} \in E(C')$ and $|E(C')| \geq \alpha(m_{j_1}+1)^\gamma + 5$. C' contains a path P' in M' from b_{j_1-1} to S_{j_1} (say b_{j_1}) such that $e \in P'$, $|E(P')| \geq \alpha(m_{j_1}+1)^\gamma + 3$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_1 in N_1 from b_{j_1-1} to S_0 such that $c_{j_1-1} \notin P_1$, $|E(P_1)| \geq \alpha(n_1-3)^\gamma + 1$. Thus $P := P_1 \cup P'$ is the desired path for the lemma.

Thus we may assume $j_1 = 1$ and that $M_1 - b'$ is 3-connected. Suppose $|S_0| = 2$. Then let $M' := (M_1 - b') \cup \{z_1, z_1b_1, z_1c_1, z_1u\}$, where $u \in (S_0 - b')$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1u \in C'$, $b' \notin C'$, and $|E(C')| \geq \alpha(m_1+1)^\gamma + 5$. C' contains a path P' in M_1 from u to S_j (say b_j) such that $e \in P'$, $b' \notin P'$, $E(P') \subseteq E(G)$, and $|E(P')| \geq \alpha(m_1+1)^\gamma + 3$. Thus $P := P'$ is the desired path for the lemma.

Thus we may assume $|S_0| > 2$. Let $M' := (M_1 - b') \cup \{z_1, z_2, z_1z_2, z_2b_1, z_2c_1\} \cup \{z_1u : u \in S_0 - b'\}$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C' in M' such that $e, z_1z_2 \in C'$, $b' \notin C'$, and $|E(C')| \geq \alpha(m_1+2)^\gamma + 5$. C' contains a path P' in M_1 from $(S_0 - b')$ (say b^*) to S_j (say b_j) such that $e \in P'$, $b' \notin P'$, $E(P') \subseteq E(G)$, and $|E(P')| \geq \alpha(m_1+2)^\gamma + 2$. Thus $P := P' \cup \{b', b^*b'\}$ is the desired path for the lemma.

This proves Claim 3.

Now that we've proven Claim 3, we can always assume there is a neighbor of a_1 in M_k other than a_2. Without this result, if a_2 were in M_k, a potential path could not have an end in M_k that is a neighbor of a_1.

Claim 4. We may assume $n^* = 0$.

Otherwise, $n^* \neq 0$. Let $N_1^* := \cup_{i=1}^{j_1} M_i$. Let $n_1^* = |V(N_1^*)|$. Let $N_2^* := \cup_{i=j_2}^{k} M_i$. Let $n_2^* = |V(N_2^*)|$. How we proceed depends on the relative sizes of $\{n_1^*, n^*, n_2^*\}$.

Suppose $t = n_1^*$. Thus $t \geq 3$.

First, we find a path P_{j_2} in $M_{j_2}^*$ from S_{j_2} (say b_{j_2}) to a_2, such that $b_{j_2-1}c_{j_2-1} \in$

P_{j_2}, if $j_2 < k$ then $b_{j_2}c_{j_2} \notin P_{j_2}$, $|E(P_{j_2})| \geq \alpha(m_{j_2} - 3)^\gamma + 2$. As M_{j_2} is a chain of cycles and by Claim 2, it is easy to find such a path P_{j_2}. As N^* is not empty, by Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_{n^*} in N^* such that $b_{j_1}c_{j_1}, b_{j_2-1}c_{j_2-1} \in C_{n^*}$ and $|E(C_{n^*})| \geq \alpha(n^* - 4)^\gamma + 2$. It is trivial to find a cycle $C_{n_1^*}$ in N_1^* such that $e, b_{j_1}c_{j_1} \in C_{n_1^*}$, $E(C_{n_1^*} - b_{j_1}c_{j_1}) \subseteq E(G)$, and $|E(C_{n_1^*})| \geq 3$. If N_2 is empty, then it is easy to verify that (since n_1^* can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_{j_2} - b_{j_2-1}c_{j_2-1} - a_2) \cup (C_{n^*} - b_{j_2-1}c_{j_2-1} - b_{j_1}c_{j_1}) \cup (C_{n_1^*} - b_{j_1}c_{j_1})$ gives the desired path for the lemma. If N_2 is not empty, then by Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_{n_2} in N_2 from b_{j_2} to S_k such that $c_{j_2} \notin P_{n_2}$, $|E(P_{n_2})| \geq \alpha(n_2 - 3)^\gamma + 1$. Now it is easy to verify that (since n_1^* can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_{j_2} - \{b_{j_2-1}c_{j_2-1}, a_2\}) \cup (C_{n_1^*} - b_{j_2-1}c_{j_2-1}) \cup P_{n_2}$ is the desired path for the lemma.

Suppose $t = n_2^*$. Thus $t \geq 3$.

First, we find one of two types of paths in N_1^*: a path $P_{n_1^*}$ in N_1^* from S_0 to S_{j_1} (say b_{j_1}), such that $e \in P_{n_1^*}$, $c_{j_1} \notin P_{n_1^*}$, $|E(P_{n_1^*})| \geq \alpha(n_1^* - 3)^\gamma + 1$, and if $n_1^* > 3$ then $|E(P_{n_1^*})| \geq \alpha(n_1^* + 2)^\gamma + 1$; or a path $P'_{n_1^*}$ in N_1^* from S_0 to S_{j_1} (say b_{j_1}), such that $e \in P'_{n_1^*}$, $c_{j_1} \in P'_{n_1^*}$, if $e \neq b_{j_1}c_{j_1}$ then $b_{j_1}c_{j_1} \notin P'_{n_1^*}$, $|E(P'_{n_1^*})| \geq \alpha(n_1^* - 3)^\gamma + 2$. If $n_1^* = 3$, finding $P_{n_1^*}$ is trivial. Let $M' := N_1^* \cup \{z_1, z_2, z_1z_2, b_{j_1}c_{j_1}\} \cup \{z_1u : u \in S_0\} \cup \{z_2u : u \in S_{j_1}\}$. M' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle $C_{n_1^*}$ in M' such that $e, z_1z_2 \in M'$, $|E(C_{n_1^*})| \geq \alpha(n_1^* + 2)^\gamma + 5$. $C_{n_1^*}$ contains the desired path $P_{n_1^*}$ or $P'_{n_1^*}$.

Next we find a path P_{n^*} in N^* from b_{j_1} to S_{j_2-1} (say b_{j_2-1}) such that $c_{j_1} \notin P_{n^*}$, $b_{j_2-1}c_{j_2-1} \notin P_{n^*}$, $|E(P_{n^*})| \geq \alpha(n^* - 4)^\gamma + 1$ and if $n^* \geq 6$ then $|E(P_{n^*})| \geq \alpha(n^* - 2)^\gamma + 1$. If $n^* \leq 5$, this can be verified directly. If $n^* \geq 6$, by Lemmas (3.1.5) and (3.1.4)(1), we find such a path P_{n^*} where $|E(P_{n^*})| \geq \alpha(n^* - 2)^\gamma + 1$.

Trivially we find a path $P_{n_2^*}$ in N_2^* from b_{j_2-1} to a_2 such that $c_{j_2-1} \notin P_{n_2^*}$,

$|E(P_{n_2^*})| \geq 1$.

If we found a path $P'_{n_1^*}$, then $P := P'_{n_1^*} \cup P_{n^*} \cup (P_{n_2^*} - a_2)$ gives the desired path for the lemma. Thus we may assume that we found $P_{n_1^*}$ in N_1^*. In particular, this implies $e \neq b_{j_1}c_{j_1}$. To improve our bounds, we consider two cases separately.

Consider where $n^* \leq 5$. It is trivial to find a path P' in N^* from b_{j_1} to S_{j_2-1} (say b_{j_2-1}) such that $b_{j_1}c_{j_1}, b_{j_2-1}c_{j_2-1} \notin P'$, and $|E(P')| \geq 3$. Thus $P := P'_{n_1^*} \cup P' \cup (P_{n_2^*} - a_2)$ gives the desired path for the lemma. Thus we may assume $n^* \geq 6$.

Consider where $n_1^* = 3$. It is trivial to find a cycle C_1 in N_1^* such that $e, b_{j_1}c_{j_1} \in C_1$, $|E(C_1)| = 3$. As $n^* \geq 6$, by Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_{n^*} in N^* such that $b_{j_1}c_{j_1}, b_{j_2-1}c_{j_2-1} \in C_{n^*}$ and $|E(C_{n^*})| \geq \alpha(n^*)^\gamma + 5$. Trivially we find a path P_2 in N_2^* from S_k to a_2 such that $b_{j_2-1}c_{j_2-1} \in P_2$, $|E(P_2)| \geq 1$. Thus $P := (C_1 - b_{j_1}c_{j_1}) \cup (C_{n^*} - \{b_{j_1}c_{j_1}, b_{j_2-1}c_{j_2-1}\}) \cup (P_2 - a_2)$ is the desired path for the lemma. Thus we may assume $n_1^* \geq 4$.

Thus in particular, $|E(P_{n_1^*})| \geq \alpha(n_1^* + 2)^\gamma + 1$ and $|E(P_{n^*})| \geq \alpha(n^* - 2)^\gamma + 1$. Now it is easy to verify that (since n_2^* can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := P_{n_1^*} \cup P_{n^*} \cup (P_2 - a_2)$ is the desired path for the lemma.

So we may assume $t \neq n_i$, $i = 1, 2$. Hence $t = n^*$ and $t \geq 4$.

In this case we will find paths in N_1^* and N_2^* separately and then just connect them with a path of any length in N^*.

We find a path $P_{n_1^*}$ in N_1^* from S_0 to S_{j_1} (say b_{j_1}), such that $e \in P_{n_1^*}$, if $e \neq b_{j_1}c_{j_1}$ then $b_{j_1}c_{j_1} \notin P_{n_1^*}$, $E(P_{n_1^*}) \subseteq E(G)$, $|E(P_{n_1^*})| \geq \alpha(n_1^* - 3)^\gamma + 1$ If $n_1^* = 3$, it is trivial to find $P_{n_1^*}$. Let $M'_1 := N_1^* \cup \{z_1, z_2, z_1z_2, b_{j_1}c_{j_1}\} \cup \{z_1u : u \in S_0\} \cup \{z_2u : u \in S_{j_1}\}$. M'_1 is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle $C_{n_1^*}$ in M'_1 such that $e, z_1z_2 \in M'_1$, $|E(C_{n_1^*})| \geq \alpha(n_1^* + 2)^\gamma + 5$. $C_{n_1^*}$ contains the desired path $P_{n_1^*}$.

Next we find a path $P_{n_2^*}$ in N_2^* from S_{j_2-1} (say b_{j_2-1}) to a_2 such that $b_{j_2-1}c_{j_2-1} \notin$

$P_{n_2^*}$, $E(P_{n_2^*}) \subseteq E(G)$, and $|E(P_{n_2^*})| \geq \alpha(n_2^* - 4)^\gamma + 2$. Note that as $t \geq 4$, $n_2^* \neq 3$. Thus if $n_2 = 0$, it is trivial to construct $P_{n_2^*}$ directly as M_{j_2} is a chain of cycles. It is trivial to construct a path P_{j_2} in M_{j_2} from S_{j_2-1} (say b_{j_2-1}) to a_2 such that $b_{j_2}c_{j_2} \in P_{j_2}$, $b_{j_2-1}c_{j_2-1} \notin P_{j_2}$, and $|E(P_{j_2})| \geq \alpha(max\{0, m_{j_2} - 4\})^\gamma + 2$. By Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_{j_2}c_{j_2} \in C_{n_2}$ and $|E(C_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 4$. $P_{n_2^*} := (P_{j_2} - b_{j_2}c_{j_2}) \cup (C_{n_2} - b_{j_2}c_{j_2})$. Hence we can always find the desired path $P_{n_2^*}$.

Lastly, we trivially find a path P_{n^*} in N^* from b_{j_1} to b_{j_2-1} such that $c_{j_1}, c_{j_2-1} \notin P_{n^*}$, $|E(P_{n^*})| \geq 1$. Now it is easy to verify that (since n^* can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := P_{n_1^*} \cup P_{n^*} \cup (P_{n_2^*} - a_2)$ is the desired path for the lemma.

This proves Claim 4.

This implies that M_{j_1} is 3-connected and hence $m_{j_2} \geq 4$. Further, this implies $e \neq b_{j_1}c_{j_1}$.

Claim 5. We may assume $n_1 = 0$.

Otherwise, $n_1 \neq 0$. Let $N_2^* := \cup_{i=j_2}^k M_i$. Let $n_2^* = |V(N_2^*)|$. How we proceed depends on the relative sizes of $\{n_1, m_{j_1}, n_2^*\}$.

Suppose $t = n_1$. Thus $t \geq 3$.

By direct construction (when $m_{j_1} \leq 5$) or Lemma(2.2.8) (otherwise), we find a cycle C_{j_1} in M_{j_1} such that $b_{j_1}c_{j_1}, e \in C_{j_1}$ and $|E(C_{j_1})| \geq \alpha(m_{j_1} - 4)^\gamma + 4$. If $b_{j_1-1}c_{j_1-1}$ is in C_{j_1}, we replace it with a path in N_1.

Next we find a path $P_{n_2^*}$ in N_2^* from S_{j_2-1} (say b_{j_2-1}) to a_2 such that $b_{j_2-1}c_{j_2-1} \notin P_{n_2^*}$, $E(P_{n_2^*}) \subseteq E(G)$, and $|E(P_{n_2^*})| \geq \alpha(n_2^* - 3)^\gamma + 1$. Thus if $n_2 = 0$, it is trivial to construct $P_{n_2^*}$ directly as M_{j_2} is a chain of cycles. It is trivial to construct a path P_{j_2} in M_{j_2} from S_{j_2-1} (say b_{j_2-1}) to a_2 such that $b_{j_2}c_{j_2} \in P_{j_2}$, $b_{j_2-1}c_{j_2-1} \notin P_{j_2}$, and $|E(P_{j_2})| \geq \alpha(m_{j_2} - 3)^\gamma + 1$. By Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_{j_2}c_{j_2} \in C_{n_2}$ and $|E(C_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 4$.

$P_{n_2^*} := (P_{j_2} - b_{j_2}c_{j_2}) \cup (C_{n_2} - b_{j_2}c_{j_2})$. Hence we can always find the desired path $P_{n_2^*}$.

Now it is easy to verify that (since n_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (C_{j_1} - b_{j_1}c_{j_1}) \cup (P_{n_2^*} - a_2)$ is the desired path for the lemma.

Suppose $t = n_2^*$. Thus $t \geq 3$.

Consider first the case where $n_1 \leq 4$. As we can assume $t \neq n_1$, $n_1 = 4$ and $n_2^* = 3$. We find a cycle C_{n^*} in N^* such that $e, b_{j_2-1}c_{j_2-1} \in C_{n^*}$, $|E(C_{n^*})| \geq \alpha(max\{0, n^* - 4\})^\gamma + 4$. If $n^* \leq 5$, it is trivial to find such a cycle of length at least 4. If $n^* \geq 6$, we find it by using the inductive hypothesis of Theorem (1.2.2). $P := C_{n^*} - b_{j_2-1}c_{j_2-1}$ is the desired path for the lemma.

Thus we may assume $n_1 \geq 5$.

We find a path P_{n^*} in N^* from S_{j_1} (say b_{j_1}) to S_{j_2-1} (say b_{j_2-1}) such that $e \in P_{n^*}$, $b_{j_1}c_{j_1}, b_{j_2-1}c_{j_2-1} \notin P_{n^*}$, $|E(P_{n^*})| \geq \alpha(n^*+2)^\gamma$. This can be verified directly for $n^* \leq 5$ and is a consequence of Lemma (3.1.1) for $n^* \geq 6$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_{n_1} in N_1 from b_{j_1} to S_0 such that $c_{j_1} \notin P_{n_1}$, $|E(P_{n_1})| \geq \alpha(n_1 - 2)^\gamma + 2$. We trivially find a path $P_{n_2^*}$ in N_2^* from b_{j_2-1} to a_2 such that $c_{j_2-1} \notin P_{n_2^*}$, $E(P_{n_2^*}) \subseteq E(G)$, and $|E(P_{n_2^*})| \geq 1$. Now it is easy to verify that (since n_2^* can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := P_{n_1} \cup P_{n^*} \cup (P_{n_2^*} - a_2)$ gives the desired path for the lemma.

Thus we may assume $t \neq n_i$, $i = 1, 2$. Hence $t = n^*$ and $t \geq 4$.

We find a path $P_{n_2^*}$ in N_2^* from S_{j_2-1} (say b_{j_2-1}) to a_2 such that $b_{j_2-1}c_{j_2-1} \notin P_{n_2^*}$, $E(P_{n_2^*}) \subseteq E(G)$, and $|E(P_{n_2^*})| \geq \alpha(n_2^* - 4)^\gamma + 2$. Thus if $n_2 = 0$, it is trivial to construct $P_{n_2^*}$ directly as M_{j_2} is a chain of cycles. It is trivial to construct a path P_{j_2} in M_{j_2} from S_{j_2-1} (say b_{j_2-1}) to a_2 such that $b_{j_2}c_{j_2} \in P_{j_2}$, $b_{j_2-1}c_{j_2-1} \notin P_{j_2}$, and $|E(P_{j_2})| \geq \alpha(m_{j_2} - 3)^\gamma + 1$. By Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_{j_2}c_{j_2} \in C_{n_2}$ and $|E(C_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 4$.

$P_{n_2^*} := (P_{j_2} - b_{j_2}c_{j_2}) \cup (C_{n_2} - b_{j_2}c_{j_2})$. Hence we can always find the desired path $P_{n_2^*}$.

We then trivially find a path P_{n^*} in N^* from b_{j_2-1} to S_{j_1} (say b_{j_1}) such that $e \in P_{n^*}$, $c_{j_2-1} \notin P_{n^*}$, $b_{j_1}c_{j_1} \notin P_{n^*}$, $|E(P_{n^*})| \geq 1$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_{n_1} in N_1 from b_{j_1} to S_0 such that $c_{j_1} \notin P_{n_1}$, $|E(P_{n_1})| \geq \alpha(n_1 - 4)^\gamma + 2$. Now it is easy to verify that (since n^* can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := P_{n_1} \cup P_{n^*} \cup (P_{n_2^*} - a_2)$ is the desired path for the lemma.

This proves Claim 5.

Let $N_2^* := \cup_{i=j_2}^k M_i$. Let $n_2^* = |V(N_2^*)|$.

We find a path $P_{n_2^*}$ in N_2^* from S_k to a_2 such that $b_{j_2-1}c_{j_2-1} \in P_{n_2^*}$, $E(P_{n_2^*}) \subseteq E(G)$, and $|E(P_{n_2^*})| \geq \alpha(n_2^* - 3)^\gamma + 1$. Thus if $n_2 = 0$, it is trivial to construct $P_{n_2^*}$ directly as M_{j_2} is a chain of cycles. It is trivial to construct a path P_{j_2} in M_{j_2} from S_{j_2} (say b_{j_2-1}) to a_2 such that $b_{j_2-1}c_{j_2-1} \in P_{j_2}$, $b_{j_2}c_{j_2} \notin P_{j_2}$, and $|E(P_{j_2})| \geq \alpha(m_{j_2} - 3)^\gamma + 1$. By Lemmas (3.1.6), (2.3.5), and (3.1.4)(2), we find a path P_{n_2} in N_2 from b_{j_2} to S_k such that $c_{j_2} \notin P_{n_2}$, $|E(P_{n_2})| \geq \alpha(n_1 - 4)^\gamma + 2$.

By Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_{j_2}c_{j_2} \in C_{n_2}$ and $|E(C_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 4$. $P_{n_2^*} := (P_{j_2} - b_{j_2}c_{j_2}) \cup (C_{n_2} - b_{j_2}c_{j_2})$. Hence we can always find the desired path $P_{n_2^*}$.

By direct construction (when $m_{j_1} \leq 5$) or Lemma(2.2.8) (otherwise), we find a cycle C_{j_1} in M_{j_1} such that $b_{j_1}c_{j_1}, e \in C_{j_1}$ and $|E(C_{j_1})| \geq \alpha(m_{j_1} - 4)^\gamma + 4$. $P := (C_{j_1} - b_{j_1}c_{j_1}) \cup (P_{n_2^*} - a_2)$ is the desired path for the lemma.

This proves Case II and hence the lemma. □

(3.3.2) Lemma. *Let $n \geq 7$ and assume the assertion of Theorem (1.2.2) holds for graphs of order $< n$. Let G be a 3-connected claw-free graph of order n, $\{a_1, a_2\} \subseteq V(G)$ such that neither $G - a_1$ nor $G - a_2$ is 3-connected. Let $e = aa_1 \in E(G - a_2)$*

such that $\{a_1, a_2, a\}$ *are not a 3-cut of* G*. Then there is a path* P *in* $G - \{a_1, a_2\}$ *from* a *to* $N(a_2)$ *such that* $|E(P)| \geq \alpha(n+2)^\gamma + 2$*.*

Proof. We now establish the structure of $G - a_1$ through Tutte decomposition. For $k \geq 1$, let $M_1, ..., M_k$ be the consecutive 3-blocks in the decomposition of $G - a_1$ (without loss of generality, from left to right) such that $a \in M_1$. Let $m_i = |V(M_i)|$ and let $S_i = \{b_i, c_i\} = V(M_i \cap M_{i+1})$. Note that for all i, $\{b_i, c_i\}$ is a special 2-cut of $G - a$. Thus $b_i c_i \in E(M_i), E(M_{i+1})$. Let $S_0 = N_{M_1}(a_1)$ and let $S_k = N_{M_k}(a_1)$. Let j be the minimum index of a 3-block containing a_2. Note that it is possible for a_2 or a to be contained in multiple 3-blocks.

Structurally there are four different sections to this graph; though, we will combine some of them together in the analysis that follows. The sections are M_1, M_j, the 3-blocks between M_1 and M_j, and the 3-blocks right of M_j. Hence we first label these sections. If $j \leq 2$, we say N_1 is empty. Otherwise, let $N_1 := \cup_{i=2}^{j-1} M_i$. If $j = k$, we say N_2 is empty. Otherwise, let $N_2 := \cup_{i=j+1}^{k} M_i$. Let $n_1 = |V(N_1)|, n_2 = |V(N_2)|$.

If $k = 1$, $G - a_1$ is a chain of cycles. As $\{a, a_1, a_2\}$ do not form a 3-cut of G, it is easy to find a path P' in $G - a_1$ from a to a_2 such that $|E(P')| \geq \alpha(m_1 - 6)^\gamma + 4$. $P = P' - a_2$ gives the desired path for the lemma. Thus we may assume $k \geq 2$.

Claim 1. We may assume $j > 1$.

Otherwise $j = 1$ and hence $a_2, a \in M_1$. How we proceed depends on $|\{a, a_2\} \cap S_1|$. As $\{a, a_1, a_2\}$ are not a 3-cut of G, $|\{a, a_2\} \cap S_1| \leq 1$.

Consider first the case where $m_1 \leq 4$. If M_1 is 3-connected, then we trivially find a path P_1 in M_1 from a to a_2 such that $b_1 c_1 \in P_1$, $|E(P_1)| = 3$. By Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_1 c_1 \in C_{n_2}$, $E(C_{n_2} - b_1 c_1) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2 - 3)^\gamma + 3$. $P := ((P_1 - b_1 c_1) \cup (C_{n_2} - b_1 c_1)) - a_2$ is the desired path for the lemma. Thus we may assume M_1 is a chain of cycles. We trivially find a path P_1 in M_1 from a to a_2 such that $b_1 c_1 \in P_1$, $|E(P_1)| = 2$.

As M_1 is a chain of cycles, N_2 is not a chain of cycles and hence $n_2 \geq 4$. Thus by Lemmas (3.1.3) and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_1c_1 \in C_{n_2}$, $E(C_{n_2} - b_1c_1) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 4$. $P := ((P_1 - b_1c_1) \cup (C_{n_2} - b_1c_1)) - a_2$ is the desired path for the lemma.

Thus we may assume $m_1 \geq 5$.

How we proceed depends on the value of $|\{a, a_2\} \cap S_1|$.

Case 1. $|\{a, a_2\} \cap S_1| = 1$.

Let $\{a', a'_2\} = \{a, a_2\}$ such that $a' \in S_1$ and $a'_2 \notin S_1$. Without loss of generality, assume $b_1 = a'$. We find a path P_1 in M_1 from c_1 to a'_2 such that $a' \notin P_1$, $|E(P_1)| \geq \alpha m_1^\gamma + 2$. Recall that $\{a, a_1, a_2\}$ do not form a 3-cut in G. If $m_1 = 5$, it is easy to find such a path P_1. Thus we may assume $m_1 \geq 6$ and we find P_1 by Lemma (3.2.2).

By Lemmas (3.1.3), (2.3.2), and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_1c_1 \in C_{n_2}$, $E(C_{n_2} - b_1c_1) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2 - 3)^\gamma + 3$. $P := (P_1 \cup (C_{n_2} - b_1c_1)) - a_2$ is the desired path for the lemma.

This proves Case 1.

Case 2. $|\{a, a_2\} \cap S_1| = 0$.

We find a path P_1 in M_1 from a to a_2 such that $b_1c_1 \in P_1$, $|E(P_1)| \geq \alpha(max\{0, m_1 - 6\})^\gamma + 3$. If $m_1 \leq 6$, it is easy to verify the existence of such a path P_1 directly. If M_1 is a chain of cycles, it is trivial to construct such a path P_1. Thus we may assume M_1 is 3-connected. Further, $M_1 - a_2$ is not 3-connected as $G - a_2$ is not 3-connected. If $M_1 - a$ is not 3-connected, then by Lemma (3.3.1), we find a path P'_1 in $M_1 - a - a_2$ from $N(a)$ to $N(a_2)$ such that $b_1c_1 \in P'_1$, $|E(P'_1)| \geq \alpha(m_1 + 2)^\gamma + 2$. We trivially extend P'_1 to the desired path P_1. Thus we may assume $M_1 - a$ is 3-connected. We find a maximal path P' in M_1 from a to some vertex $a' \in M_j$ such that $a_2, b_1, c_1 \notin P'$, $E(P') \subseteq E(G)$, $M_1 - V(P')$ is 3-connected. Let $M'_1 = M_1 - V(P' - a')$ and let $m'_1 = |V(M'_1)|$. Note that it is possible that $M_1' = M_1$. If $a'a_2 \in E(M_j')$ then we find a cycle C_1' in M_1' such that $b_1c_1, a'a_2 \in C_1'$ such that $|E(C_1')| \geq \alpha(m_1' - 4)^\gamma + 4$. $P_1 := P' \cup (C_1' - a'a_2)$ gives the desired path. Thus we may assume $a'a_2 \notin M_1'$. Since $M_1' - a'$ is 3-connected and since $N(b_1) - c_1$ and $N(c_1) - b_1$ are cliques, we may assume there exists $a^* \in M_1'$ such that $a'a^* \in E(G)$ and $a^* \notin \{a_2, b_1, c_1\}$. By choice of P', $M_1' - a' - a^*$ is not 3-connected. Thus by direct construction or Lemma (3.3.1), we find a path P_1' in $(M_1' - a') - a^* - a_2$ from $N(a^*)$ to $N(a_2)$ such that $b_1c_1 \in P_1'$, $|E(P_1')| \geq \alpha(max\{0, m_1' - 6\})^\gamma + 1$. We trivially extend P_1' to obtain a path P_1^* in $(M_1' - a') - a_2$ from a^* to a_2 such that $b_1c_1 \in P_1^*$, $|E(P_1^*)| \geq \alpha(m_1')^\gamma + 2$. $P_1 := P' \cup P_1^* \cup a'a^*$ gives the desired path.

By Lemmas (3.1.3), (2.3.2), and (2.2.8), we find a cycle C_{n_2} in N_2 such that $b_1c_1 \in C_{n_2}$ and $|E(C_{n_2})| \geq \alpha(n_2 - 3)^\gamma + 3$. $P := ((P_1 - b_1c_1) \cup (C_{n_2} - b_1c_1)) - a_2$ is the desired path for the lemma.

This proves Case 2 and hence Claim 1.

Claim 2. We may assume M_j is 3-connected.

Otherwise, we may assume M_j is a chain of cycles. Let $N_2' := \cup_{i=j}^k M_i$. Let $n_2' = |V(N_2')|$. How we proceed depends on the relative sizes of $\{m_1, n_1, n_2'\}$ and is further complicated by the location of a, a_2.

Let $t = min\{m_1, n_1, n_2'\}$.

Suppose $t = n_1$. Thus $t \geq 0$.

We consider three cases.

Case 1. N_1 is empty and $n_2' = 3$.

If $a \in S_1$, then by direct construction or Lemma (2.2.8), we find a cycle C_1 in M_1 such that $b_1c_1 \in C_1$, $|E(C_1)| \geq \alpha(m_1 - 4)^\gamma + 4$. $P = (C_1 - b_1c_1)$ gives the desired path for the lemma.

Thus we may assume $a \notin S_1$. If $m_1 \leq 7$ we easily find a path P_1 in M_1 from a to S_1 (say b_1) such that $b_1c_1 \notin P_1$, $|E(P_1)| \geq 3$. $P = P_1$ gives the desired path for the lemma. Thus we may assume $m_1 \geq 8$.

Next we find a path P_1 in M_1 from a to S_1 (say b_1) such that $b_1c_1 \notin P_1$, $|E(P_1)| \geq \alpha m_1^\gamma + 3$. In order to find a such a path, we first need to modify M_1 slightly. Let P_a and P_b be two disjoint paths in M_1 such that P_a is a path from a to some vertex a^*, P_b is a path from b_1 to some vertex b^*, $E(P_a) \subseteq E(G)$, $E(P_b) \subseteq E(G)$, $M_1 - (P_a - a^*) - (P_b - b^*)$ is 3-connected, $M_1 - (P_a) - (P_b - b^*)$ is not 3-connected, $M_1 - (P_a - a^*) - (P_b)$ is not 3-connected. Note that the trivial paths a and b_1 necessarily satisfy all but the final two connectivity requirements. Hence paths P_a, P_b exist and we pick any such pair. Let $M_1' := M_1 - (P_a - a^*) - (P_b - b^*)$. Let $m_1' = |V(M_1')|$. Let $d = |E(P_a)| + |E(P_b)|$. Next we find a path P_1' in M_1' from a^* to b^* such that $E(P_1') \subseteq E(G)$. If $m_1' \leq 6$, it is trivial to find such P_1' where $|E(P_1')| \geq 2$. Further, if $m_1' \leq 6$, $d \geq 2$ and hence we can trivially extend P_1' to obtain P_1 as desired. Thus we may assume $m_1' \geq 7$. By Lemma (3.3.1) we find a path P_1' in $M_1' - a^* - b^*$ from $N(a^*)$ to $N(b^*)$ such that $E(P_1') \subseteq E(G)$, $|E(P_1')| \geq \alpha(m_1 + 2)^\gamma + 2$. Trivially extend P_1' to a^*, b^* and then through P_a and P_b to obtain P_1, as desired. Thus we always find a path P_1 as desired. $P := P_1$ gives the desired path for the lemma.

This proves Case 1.

Case 2. N_1 is empty and $n_2' > 3$.

If $a \in S_1$, then by direct construction or Lemma (2.2.8), we find a cycle C_1 in M_1 such that $b_1c_1 \in C_1$, $|E(C_1)| \geq \alpha(m_1 - 4)^\gamma + 4$. Without loss of generality, assume $a \neq c_1$. As $\{a, a_1, a_2\}$ do not form a 3-cut in G, it is trivial to find a path P_j in M_j from c_1 to a_2 such that $b_1 \notin P_j$, if $j < k$ then $b_jc_j \in P_j$, and $|E(P_j)| \geq \alpha(m_j - 3)^\gamma + 1$. If $j < k$, by Lemmas (3.1.3) and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$, $E(C_{n_2} - b_jc_j) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2 - 3)^\gamma + 3$. If $j < k$, then $P := ((C_1 - b_1c_1) \cup (P_j - b_jc_j) \cup (C_{n_2} - b_jc_j)) - a_2$ is the desired path for the lemma. If $j = k$, then $P := ((C_1 - b_1c_1) \cup P_j) - a_2$ is the desired path for the lemma.

Thus we may assume $a \notin S_1$. We first find a path P_2 in N_2' from S_{j-1} (say b_{j-1}) to a_2 such that $c_{j-1} \notin P_2$, $E(P_2) \subseteq E(G)$, $|E(P_2)| \geq \alpha(n_2'-4)^\gamma+2$. We find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $c_{j-1} \notin P_j$, if $j < k$ then $b_jc_j \in P_j$, $|E(P_j)| \geq \alpha(m_j-3)^\gamma+1$. If $j = k$, as $n_2' > 3$, $|E(P_j)| \geq \alpha(m_j-4)^\gamma+2$ and hence $P_2 := P_j$ as desired. If $j < k$, by Lemmas (3.1.3) and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$, $E(C_{n_2} - b_jc_j) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2-3)^\gamma+3$. $P_2 := (P_j - b_jc_j) \cup (C_{n_2} - b_jc_j)$ as desired.

Next we find a path P_1 in M_1 from b_{j-1} to a such that $c_{j-1} \notin P_1$, $|E(P_1)| \geq \alpha(m_1-4)^\gamma+2$. If $m_1 \leq 5$, we directly construct P_1. If $m_1 \geq 6$, we find P_1 by Lemma (3.2.2). $P := (P_1 \cup P_2) - a_2$ is the desired path for the lemma.

This proves Case 2.

Case 3. N_1 is not empty.

We first find a path P_2 in N_2' from S_{j-1} (say b_{j-1}) to a_2 such that $c_{j-1} \notin P_2$, $E(P_2) \subseteq E(G)$, $|E(P_2)| \geq \alpha(n_2'-3)^\gamma+1$. We find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $c_{j-1} \notin P_j$, if $j < k$ then $b_jc_j \in P_j$, $|E(P_j)| \geq \alpha(m_j-3)^\gamma+1$. If $j = k$, $P_2 := P_j$ as desired. If $j < k$, by Lemmas (3.1.3) and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$, $E(C_{n_2} - b_jc_j) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2-3)^\gamma+3$. $P_2 := (P_j - b_jc_j) \cup (C_{n_2} - b_jc_j)$ as desired.

If $a \in S_1$, then by direct construction or Lemma (2.2.8), we find a cycle C_1 in M_1 such that $b_1c_1 \in C_1$, $|E(C_1)| \geq \alpha(m_1-4)^\gamma+4$. Without loss of generality, assume $a \neq c_1$. As $\{a, a_1, a_2\}$ do not form a 3-cut in G, it is trivial to find a path P_{n_1} in N_1 from c_1 to b_{j-1} such that $b_1, c_{j-1} \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, and $|E(P_{n_1})| \geq 1$. Now it is easy to verify that (since n_1 can be recovered by taking 2 largest out of 3) $P := ((C_1 - b_1c_1) \cup P_{n_1} \cup P_2) - a_2$ gives the desired path for the lemma.

Thus we may assume $a \notin S_1$. By direct construction or by Lemma (3.2.2), we find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| \geq \alpha(m_1-3)^\gamma+1$. As N_1 is not empty, it contains a 3-connected 3-block. Thus it is trivial to find a path P_{n_1} in N_1 from b_1 to b_{j-1} such that $c_{j-1} \notin P_{n_1}$, $b_1c_1 \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, and $|E(P_{n_1})| \geq 2$. Now it is easy to verify that (since n_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := ((P_1 \cup P_{n_1} \cup P_2) - a_2$ is the desired path for the lemma.

This proves Case 3 and hence we may assume $t \neq n_1$.

Suppose $t = m_1$. Thus $t \geq 3$.

We consider three cases.

Case 1. $n_2' = 3$.

Thus $t = 3$ and hence $m_1 = 3$. If $n_1 \leq 5$, then it is trivial to find a path P in $G - a_1 - a_2$ from a to $N(a_2)$ such that $|E(P)| \geq 3$, which is as desired by the lemma. Thus we may assume $n_1 \geq 6$. If $a \in S_1$, we find a path P_1 in M_1 from a to c_1 such that $b_1c_1 \notin P_1$, $|E(P_1)| = 2$. By Lemmas (3.1.5) and (3.1.4)(1), we find a path P_{n_1} in N_1 from b_1 to S_{j-1} (say b_{j-1}), such that $b_1 \notin P_{n_1}$, $b_{j-1}c_{j-1} \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, and $|E(P_{n_1})| \geq \alpha(n_1)^\gamma + 1$ (as both M_2, M_{j-1} are 3-connected). $P := P_1 \cup P_{n_1}$ is the desired path for the lemma. Hence we may assume $a \notin P_1$.

We want to find a path P_{n_1} in N_1 from S_1 (say b_1) to S_{j-1} (say b_{j-1}) such that $E(P_{n_1}) \subseteq E(G)$ and $|E(P_{n_1})| \geq \alpha(n_1)^\gamma + 2$. Depending on the structure of N_1, we find P_{n_1} in a variety of ways. If $m_2 \leq 5$, then it is trivial to find path P_2 in M_2 from S_1 (say b_1) to S_2 (say b_2) such that $E(P_2) \subseteq E(G)$, $|E(P_2)| \geq 3$. As $n_1 \geq 6$, $N_1 - (M_2 - S_2)$ contains at least one 3-connected 3-block. Thus by Lemmas (3.1.5) and (3.1.4)(1), we find a path P_{n_1}' in $N_1 - (M_2 - S_2)$ from b_2 to S_{j-1} (say b_{j-1}), such that $c_2 \notin P_{n_1}'$, $b_{j-1}c_{j-1} \notin P_{n_1}'$, $|E(P_{n_1}')| \geq \alpha(n_1 - (m_2-2)-1)^\gamma + 1$. $P_{n_1} = P_2 \cup P_{n_1}'$ is the desired path. Thus we may assume $m_2 \geq 6$. We find a path P_2 in M_2 from S_1 (say b_1) to S_2 (say b_2) such that $E(P_2) \subseteq E(G)$, $|E(P_2)| \geq \alpha m_2^\gamma + 2$. If $M_2 - b_1$ is not 3-connected and $M_2 - b_2$ is not 3-connected, then by Lemma (3.3.1), we find a path P_2' in $M_2 - b_1 - b_2$ from $N(b_1)$ to $N(b_2)$ such that $|E(P_2')| \geq \alpha(m_2+2)^\gamma + 2$. We trivially extend P_2' to obtain P_2 as desired. Thus we may assume without loss of generality that $M_2 - b_1$ is 3-connected. Thus $M_2 - b_1c_1$ is 3-connected. Let $M_2' := (M_2 - b_1c_1) \cup \{z_1, z_1b_1, z_1b_2, z_1c_2\}$. M_2' is 3-connected and claw-free. By the inductive hypothesis of Theorem (1.2.2), we find a cycle C_2' in M_2' which contains the desired path P_2. Thus in any case, we find the desired path P_2. If $N_1 = M_2$, $P_{n_1} := P_2$ is the desired path. Thus we may assume $j > 2$ and hence N_1 contains a 3-connected 3-block other than M_2. Thus by Lemmas (3.1.5) and (3.1.4)(1), we find a path P_{n_1}' in $N_1 - (M_2 - S_2)$ from b_2 to S_{j-1} (say b_{j-1}), such that $c_2 \notin P_{n_1}'$, $b_{j-1}c_{j-1} \notin P_{n_1}'$, $|E(P_{n_1}')| \geq \alpha(n_1 - (m_2 - 2) - 1)^\gamma + 1$. $P_{n_1} := P_2 \cup P_{n_1}'$ is the desired path. Hence in any case, we find the desired path P_{n_1}.

Trivially, we find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| = 1$. $P := P_1 \cup P_{n_1}$ is the desired path for the lemma.

This proves Case 1.

Case 2. $n_2' = 4$.

Thus $t =\leq 4$ and hence $m_1 \leq 4$. If $n_1 \leq 5$, then it is trivial to find a path P in $G - a_1 - a_2$ from a to $N(a_2)$ such that $|E(P)| \geq 3$, which is as desired by the lemma. Thus we may assume $n_1 \geq 6$.

If $a \in S_1$, we find a path P_1 in M_1 from a to c_1 such that $b_1c_1 \notin P_1$, $|E(P_1)| \geq 2$. By Lemmas (3.1.5) and (3.1.4)(1), we find a path P_{n_1} in N_1 from b_1 to S_{j-1} (say b_{j-1}), such that $b_1 \notin P_{n_1}$, $b_{j-1}c_{j-1} \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, and $|E(P_{n_1})| \geq \alpha(n_1 - 1)^\gamma + 1$ (as M_{j-1} is 3-connected). We then trivially find a path P_{n_2}' in N_2' from b_{j-1} to a_2 such that $c_{j-1} \notin P_{n_2}'$, $E(P_{n_2}') \subseteq E(G)$, $|E(P_{n_2}')| \geq 1$. $P := (P_1 \cup P_{n_1} \cup P_{n_2}') - a_2$ is the desired path for the lemma. Hence we may assume $a \notin P_1$.

If $M_1 \cong K_4$, then by Lemmas (3.1.5) and (3.1.4)(1), we find a path P_{n_1} in N_1 from S_1 (say b_1) to S_{j-1} (say b_{j-1}) such that $E(P_{n_1}) \subseteq E(G)$ and $|E(P_{n_1})| \geq \alpha(n_1 - 1)^\gamma + 1$. Trivially we find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$,

$|E(P_1)| = 2$. Trivially we find a path P'_{n_2} in N'_2 from b_{j-1} to a_2 such that $c_{j-1} \notin P'_{n_2}$, $|E(P'_{n_2})| \geq 1$. $P := (P_1 \cup P_{n_1} \cup P'_{n_2}) - a_2$ is the desired path for the lemma. Thus we may assume M_1 is a chain of cycles.

Then exactly as in Case 1, we find a path P_{n_1} in N_1 from S_1 (say b_1) to S_{j-1} (say b_{j-1}) such that $E(P_{n_1}) \subseteq E(G)$ and $|E(P_{n_1})| \geq \alpha(n_1)^\gamma + 2$. Trivially we find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| \geq 1$. Trivially we find a path P'_{n_2} in N'_2 from b_{j-1} to a_2 such that $c_{j-1} \notin P'_{n_2}$, $|E(P'_{n_2})| \geq 1$. $P := (P_1 \cup P_{n_1} \cup P'_{n_2}) - a_2$ is the desired path for the lemma.

This proves Case 2.

Case 3. $n'_2 \geq 5$.

First we find a path P'_{n_2} in N'_2 from S_{j-1} (say b_{j-1}) to a_2 such that $c_{j-1} \notin P'_{n_2}$, $E(P'_{n_2} \subseteq E(G)$, $|E(P'_{n_2})| \geq \alpha(n'_2 - 5)^\gamma + 3$. Assume $n_2 = 0$. As $n'_2 \geq 5$ and as M_j is a chain of cycles, it is trivial to find P'_{n_2} directly. Thus we may assume N_2 is not empty. It is trivial to find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $b_jc_j \in P_j$, $c_{j-1} \notin P_j$, $|E(P_j)| \geq \alpha(m_j - 3)^\gamma + 1$. By Lemmas (3.1.3) and (2.2.8) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$ and $|E(C_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 4$. $P'_{n_2} := (P_j - b_jc_j) \cup (C_{n_2} - b_jc_j)$ is the desired path.

By direct construction or by Lemmas (3.1.5) and (3.1.4)(1), we find a path P_{n_1} in N_1 from b_{j-1} to S_1 (say b_1), such that $c_{j-1} \notin P_{n_1}$, $b_1c_1 \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, and $|E(P_{n_1})| \geq \alpha(max\{0, n_1 - 5\})^\gamma + 1$. Trivially, we find a path P_1 in M_1 from b_1 to a such that $b_1c_1 \notin P_1$, $|E(P_1)| \geq 0$. Now it is easy to verify that (since m_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := ((P_1 \cup P_{n_1} \cup P'_{n_2}) - a_2$ is the desired path for the lemma.

This proves Case 3 and hence we may assume $t \neq m_1$.

Suppose $t = n'_2$. Thus $t \geq 3$.

If $n'_2 = 3$, then by the arguments provided above for $t = m_1$ where $n'_2 = 3$, we find the desired path for the lemma. Hence we may assume $n'_2 \geq 4$ and hence that

$t \geq 4$.

If $n_1 \leq 5$, then it is trivial to find a path P in $G - a_1 - a_2$ from a to $N(a_2)$ such that $|E(P)| \geq 4$, which is as desired by the lemma. Thus we may assume $n_1 \geq 6$.

First we find a path P_1 in M_1 from S_1 (say b_1) to a such that $E(P_1) \subseteq E(G)$, $|E(P_1)| \geq \alpha(m_1 - 4)^\gamma + 2$. If $m_1 \leq 5$, it is easy to construct such a path directly. If $a \in S_1$ and $m_1 \geq 6$, by Lemmas (2.2.8) and (2.3.2) we find a cycle C_1 in M_1 such that $b_1c_1 \in C_1$, $|E(C_1)| \geq \alpha m_1^\gamma + 5$. $P_1 := C_1 - b_1c_1$ gives the desired path. If $a \notin S_1$ and $m_1 \geq 6$, then by direct construction (when M_1 is a chain of triangles) or Lemma (3.2.2) we find the desired path P_1.

Next we find a path P_{n_1} in N_1 from b_1 to S_{j-1} (say b_{j-1}) such that $c_1 \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, and $|E(P_{n_1})| \geq \alpha(n_1 - 1)^\gamma + 1$. Depending on the structure of N_1, we find P_{n_1} in a variety of ways. If N_1 contains multiple 3-blocks, we find P_{n_1} by Lemma (3.1.5). Thus we may assume $N_1 = M_2$ and further M_2 is 3-connected. Thus we find P_{n_1} by Lemma (3.1.4)(1).

Trivially we find a path P'_{n_2} in N'_2 from b_{j-1} to a_2 such that $c_{j-1} \notin P'_{n_2}$, $E(P'_{n_2} \subseteq E(G)$, and $|E(P'_{n_2})| \geq 1$. Now it is easy to verify that (since n'_2 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P_{n_1} \cup P'_{n_2}) - a_2$ is the desired path for the lemma.

This proves Claim 2.

Thus we may now assume that M_j is 3-connected. The arguments that follow are similar to those of Claim 2. The major difference, of course, is going to be in how we find our path in M_j.

Claim 3. If $j < k$, we may assume $a_2 \notin S_j$.

Otherwise, we may assume that $j < k$ and that $a_2 \in S_j$. How we proceed depends on the relative sizes of $\{m_1, m_j, n_2\}$. Note that the size of n_1 will be of little consequence.

Let $t = min\{m_1, m_j, n_2\}$.

Suppose $t = m_j$. Thus $t \geq 4$.

First we find a path P_1 in M_1 from S_1 (say b_1) to a such that $E(P_1) \subseteq E(G)$, $|E(P_1)| \geq \alpha(m_1 - 4)^\gamma + 2$. If $m_1 \leq 5$, it is easy to construct such a path directly. If $a \in S_1$ and $m_1 \geq 6$, by Lemmas (2.2.8) and (2.3.2) we find a cycle C_1 in M_1 such that $b_1c_1 \in C_1$, $|E(C_1)| \geq \alpha m_1^\gamma + 5$. $P_1 := C_1 - b_1c_1$ is the desired path. If $a \notin S_1$ and $m_1 \geq 6$, then by direct construction (when M_1 is a chain of triangles) or Lemma (3.2.2) we find the desired path P_1.

Next, by Lemmas (3.1.5), (3.1.4)(1), (2.3.4) we find a path P_{n_1} in N_1 from b_1 to S_{j-1} (say b_{j-1}) such that $c_1 \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$ and $|E(P_{n_1})| \geq \alpha(max\{0, n_1 - 3\})^\gamma$. Note that if $n_1 \leq 3$, P_{n_1} may be the trivial path of length 0.

We find a path P_{n_2} in N_2 from S_j (say b_j) to a such that $E(P_{n_2}) \subseteq E(G)$, $|E(P_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 3$. By Lemmas (3.1.3), (2.2.8), (2.3.2) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$, $E(C_{n_2} - b_jc_j) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 4$. $P_{n_2} := (C_{n_2} - b_jc_j)$ is the desired path.

Lastly, as M_j is 3-connected, we trivially find a path P_j in M_j from b_j to b_{j-1} such that $c_j, c_{j-1} \notin P_j$, $|E(P_j)| \geq 1$. Now it is easy to verify that (since m_j can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P_{n_1} \cup P_j \cup P_{n_2}) - a_2$ is the desired path for the lemma.

Thus we may assume $t \neq m_j$.

Suppose $t = m_1$. Thus $t \geq 3$.

Let $N_1' = \cup_{i=2}^{j} M_i$. Let $n_1' = |V(N_1')|$.

We consider two cases.

Case 1. $n_2 \geq 4$.

We find a path P_{n_2} in N_2 from S_j (say b_j) to a_2 such that $E(P_{n_2}) \subseteq E(G)$, $|E(P_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 3$. By Lemmas (3.1.3), (2.2.8), (2.3.2) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$, $E(C_{n_2} - b_jc_j) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 4$. $P_{n_2} := (C_{n_2} - b_jc_j)$ is the desired path.

Next we find a path $P^*_{n_1}$ in N'_1 from b_j to S_1 (say b_1) such that $c_j \notin P^*_{n_1}$, $E(P^*_{n_1}) \subseteq E(G)$, if a is not an end of $P^*_{n_1}$ then $a \notin P^*_{n_1}$, $|E(P^*_{n_1})| \geq \alpha(n'_1 - 4)^\gamma + 1$. First consider the case where N_1 is empty or a triangle. If $m_j \leq 5$, we find $P^*_{n_1}$ by direct construction. By Lemma (3.1.4)(1), we find a path P_j in M_j from b_j to S_{j-1} (say b_{j-1}) such that $c_j \notin P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha m_j^\gamma + 1$. If a is an end of P_j or if $a \notin P_j$, P_j can be trivially extended to obtain $P^*_{n_1}$ as desired. Otherwise we may modify P_j to remove a and then extend it obtain $P^*_{n_1}$. Thus we may assume N_1 is not empty and is not a triangle. By direct construction or by Lemma (3.1.4)(1), we find a path P_j in M_j from b_j to S_{j-1} (say b_{j-1}) such that $c_j \notin P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha(m_j - 4)^\gamma + 1$. We find a path P_{n_1} in N_1 from b_{j-1} to S_1 (say b_1) such that $c_{j-1} \notin P_{n_1}$, if a is not an end of P_{n_1} then $a \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$ and $|E(P_{n_1})| \geq \alpha(n_1 - 4)^\gamma + 1$. By Lemmas (3.1.5), (3.1.4)(1), (2.3.4) we find a path P'_{n_1} in N_1 from b_{j-1} to S_1 (say b_1) such that $c_{j-1} \notin P'_{n_1}$, $E(P'_{n_1}) \subseteq E(G)$ and $|E(P'_{n_1})| \geq \alpha(n_1 - 4)^\gamma$. If a is an end of P'_{n_1} or if $a \notin P'_{n_1}$, $P_{n_1} = P'_{n_1}$ as desired. Otherwise we may modify P'_{n_1} to remove a and hence obtain P_{n_1}. $P^*_{n_1} := P_{n_1} \cup P_j$, as desired.

Trivially, we find a path P_1 in M_1 from b_1 to a such that $|E(P_1)| \geq 0$, if $c_1 \in P^*_{n_1}$ then $c_1 \notin P_1$. Note that by construction of $P^*_{n_1}$, if $c_1 \in P^*_{n_1}$ then $a \neq c_1$. Now it is easy to verify that (since m_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P^*_{n_1} \cup P_{n_2}) - a_2$ is the desired path for the lemma.

This proves Case 1.

Case 2. $n_2 = 3$.

Thus $m_1 = 3$. If $n'_1 \leq 5$, then it is trivial to find a path P in $G - a_1 - a_2$ from a to $N(a_2)$ such that $|E(P)| \geq 3$, which is as desired by the lemma. Thus we may assume $n'_1 \geq 6$.

Suppose $a \notin S_1$. Without loss of generality, assume $b_j \neq a_2$. By Lemmas (3.1.4)(1) and (3.1.5), we find a path P'_{n_1} in N'_1 from b_j to S_1 (say b_1) such that $c_j \notin P'_{n_1}$, $E(P'_{n_1}) \subseteq E(G)$, $|E(P'_{n_1})| \geq \alpha(n'_1)^\gamma + 1$. We trivially find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| = 1$. We trivially find a path P'_{n_2} in N'_2 from b_j to a_2 such that $b_jc_j \notin P'_{n_2}$, $|E(P'_{n_2})| = 2$. $P := (P_1 \cup P'_{n_1} \cup P'_{n_2}) - a_2$ is the desired path for the lemma. Thus we may assume $a \in S_1$.

Suppose N'_1 contains more than one 3-block. Then $j \neq 2$, M_2 and M_j are both 3-connected. By Lemma (3.1.4)(1), we find a path P_j in M_j from b_j to S_{j-1} (say b_{j-1}) such that $c_j \notin P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha(m_j - 4)^\gamma + 1$. Next we find a path P' in N_1 from b_{j-1} to S_1 (say b_1) such that $c_{j-1} \notin P'$, $E(P') \subseteq E(G)$, if a is not an end of P' then $a \notin P'$, $|E(P')| \geq \alpha(n_1 - 4)^\gamma + 1$. By Lemmas (3.1.4)(1) and (3.1.5), we find a path P_{n_1} in N_1 from b_{j-1} to S_1 (say b_1) such that $c_{j-1} \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, $|E(P_{n_1})| \geq \alpha(n_1 - 4)^\gamma + 1$. If a is an end of P_{n_1} or if $a \notin P_{n_1}$, then $P' = P_{n_1}$ as desired. Otherwise we may modify P_{n_1} to remove a and hence obtain P'. We trivially find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| \geq 0$. We trivially find a path P_{n_2} in N_2 from b_j to a_2 such that $b_jc_j \notin P_{n_2}$, $E(P_{n_2}) \subseteq E(G)$, and $|E(P'_{n_2})| = 2$. $P := (P_1 \cup P' \cup P_j \cup P_{n_2}) - a_2$ is the desired path for the lemma. Thus we may assume $j = 2$.

Without loss of generality, assume that $a \neq b_1$ and that $a_2 \neq b_j$.

Suppose $M_2 - c_j$ is 3-connected. Thus $M_2 - b_jc_j$ is 3-connected. Let $M'_2 := (M_2 - b_jc_j) \cup \{z_1, z_1b_j\} \cup \{z_1u : u \in S_1\}$. M'_2 is 3-connected and claw-free. Thus by the inductive hypothesis of Theorem (1.2.2), we find a cycle C'_2 in M'_2 such that $z_1b_j \in C'_2$, $|E(C'_2)| \geq \alpha(m_2 + 1)^\gamma + 5$. C'_2 contains a path P_2 in $M_2 - b_jc_j$ from b_j to S_1 (say b^*) such that $E(P_2) \subseteq E(G)$, if a is not an end of P_2 then $a \notin P_2$, $|E(P_2)| \geq \alpha(m_2+1)^\gamma+2$. We trivially find a path P_1 in M_1 from b^* to a such that $V(P_1 \cap P_2) = \{b^*\}$, $|E(P_1)| \geq 0$. We trivially find a path P_{n_2} in N_2 from b_j to a_2 such that $b_jc_j \notin P_{n_2}$, $E(P_{n_2}) \subseteq E(G)$, and $|E(P_{n_2})| = 2$. $P := (P_1 \cup P_2 \cup P_{n_2}) - a_2$ is the desired path for the lemma. Thus we may assume $M_2 - c_j$ is not 3-connected.

Suppose $M_2 - c_1$ is 3-connected. By a symmetric argument (interchanging the roles of b_j and b_1 above), we find the desired path for the lemma. Thus we may assume $M_2 - c_1$ is not 3-connected.

Thus $M_2 - c_1$ and $M_2 - c_j$ are not 3-connected. If $m_2 = 6$, then it is trivial to construct the desired path P such that $|E(P)| \geq 5$. By Lemma (3.3.1) we find a path P_2 in $M_2 - c_1 - c_j$ from $N(c_1)$ (say d_1) to $N(c_j)$ (say d_j) such that $|E(P_2)| \geq \alpha(m_2 + 2)^\gamma + 2$. If $d_1 \neq b_1$, let $P_1 = d_1c_1$. Otherwise, we let P_1 be the path in M_1 from d_1 to a such that $b_1c_1 \notin P_1$, $|E(P_1)| = 2$. If $d_j \neq b_j$, let $P_{n_2} = d_jc_j$. Otherwise, we let P_{n_2} be the path in N_2 from d_j to a_2 such that $b_jc_j \notin P_{n_2}$, $E(P_{n_2}) \subseteq E(G)$, and $|E(P_{n_2})| = 2$. $P := (P_1 \cup P_2 \cup P_{n_2}) - a_2$ is the desired path for the lemma.

This proves Case 2 and hence that we may assume $t \neq m_1$.

Suppose $t = n_2'$. Thus $t \geq 3$.

Thus we may assume $m_1 \geq 4$. Let $N_1' = \cup_{i=2}^{j} M_i$. Let $n_1' = |V(N_1')|$.

We consider two cases.

Case 1. $a \in S_1$.

We find a path P_1 from S_1 (say b_1) to a such that $E(P_1) \subseteq E(G)$, $|E(P_1)| \geq \alpha(m_1 - 4)^\gamma + 3$. By Lemmas (2.2.8), (2.3.2) we find a cycle C_1 in M_1 such that $b_1c_1 \in C_1$ and $|E(C_1)| \geq \alpha(m_1 - 4)^\gamma + 4$. $P_1 := (C_1 - b_1c_1)$ is the desired path.

Next we find a path $P^*_{n_1}$ in N_1' from b_1 to S_j (say b_j) such that $c_1 \notin P^*_{n_1}$, $E(P^*_{n_1}) \subseteq E(G)$, if a_2 is not an end of $P^*_{n_1}$ then $a_2 \notin P^*_{n_1}$, $|E(P^*_{n_1})| \geq \alpha(n_1' - 4)^\gamma + 1$. First consider the case where N_1 is empty or a triangle. If $m_j \leq 5$, we find $P^*_{n_1}$ by direct construction. By Lemma (3.1.4)(1), we find a path P_j in M_j from b_j to S_{j-1} (say b_{j-1}) such that $c_j \notin P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha m_j^\gamma + 1$. If a_2 is an end of P_j or if $a_2 \notin P_j$, P_j can be trivially extended to obtain $P^*_{n_1}$ as desired. Otherwise we may modify P_j to remove a_2 and then extend it obtain $P^*_{n_1}$. Thus we may assume N_1 is not empty and is not a triangle. By Lemmas (3.1.5), (3.1.4)(1), (2.3.4), we find a path P_{n_1} in N_1 from b_1 to S_{j-1} (say b_{j-1}) such that $c_1 \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$

and $|E(P_{n_1})| \geq \alpha(n_1-4)^\gamma+1$. Next we find a path P_j in M_j from b_{j-1} to S_j (say b_j) such that $c_{j-1} \notin P_j$, if a_2 is not an end of P_j then $a_2 \notin P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha(m_j-4)^\gamma+1$. By direct construction or by Lemma (3.1.4)(1), we find a path P'_j in M_j from b_{j-1} to S_j (say b_j) such that $c_{j-1} \notin P'_j$, $E(P'_j) \subseteq E(G)$, $|E(P'_j)| \geq \alpha(m_j-4)^\gamma+1$. If a_2 is an end of P'_j or if $a_2 \notin P'_j$, $P_j := P'_j$ as desired. Otherwise we may modify P'_j to remove a_2 and hence obtain P_j. $P^*_{n_1} := P_{n_1} \cup P_j$, as desired.

Trivially, we find a path P_{n_2} in N_2 from b_j to a_2 such that $|E(P_{n_2})| \geq 0$, if $c_j \in P^*_{n_1}$ then $c_1 \notin P_{n_2}$, and $E(P_{n_2}) \subseteq E(G)$. Note that by construction of $P^*_{n_1}$, if $c_j \in P^*_{n_1}$ then $a_2 \neq c_j$. Now it is easy to verify that (since n_2 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P^*_{n_1} \cup P_{n_2}) - a_2$ is the desired path for the lemma.

This proves Case 1.

Case 2. $a \notin S_1$.

We find a path P_{n_2} in N_2 from S_j (say b_j) to a_2 such that $E(P_{n_2}) \subseteq E(G)$, $|E(P_{n_2})| \geq \alpha(n'_2-3)^\gamma+2$. By Lemmas (3.1.3), (2.2.8), (2.3.2) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$, $E(C_{n_2}-b_jc_j) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2-3)^\gamma+3$. $P_{n_2} := (C_{n_2}-b_jc_j)$ is the desired path.

Exactly as in Case 1 of when we assumed $t = m_1$ above, we find a path $P^*_{n_1}$ in N'_1 from b_j to S_1 (say b_1) such that $c_j \notin P^*_{n_1}$, $E(P^*_{n_1}) \subseteq E(G)$, if a is not an end of $P^*_{n_1}$ then $a \notin P^*_{n_1}$, $|E(P^*_{n_1})| \geq \alpha(n'_1-4)^\gamma+1$.

By direct construction or Lemma (3.1.4)(2) we find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| \geq \alpha(m_1-4)^\gamma+1$. $P := (P_1 \cup P^*_{n_1} \cup P_{n_2}) - a_2$ is the desired path for the lemma.

This proves Claim 3.

Note that we may assume $m_j \geq 5$.

Claim 4. We may assume $a \notin S_1$.

Otherwise, we may assume $a \in S_1$.

Let $N_2' = \cup_{i=j}^{k} M_i$. Let $n_2' = |V(N_2')|$. How we proceed depends on the relative sizes of $\{m_1, n_1, n_2'\}$.

Let $t = min\{m_1, n_1, n_2'\}$.

Suppose $t = n_1$. Thus $t \geq 0$.

We consider two cases.

Case 1. $n_1 \geq 5$ or $N_1 \cong K_4$.

Thus $t \geq 4$. Without loss of generality, assume $b_1 \neq a$.

We find a path P_1 in M_1 from b_1 to a such that $b_1c_1 \notin P_1$, $|E(P_1)| \geq \alpha(m_1)^\gamma + 2$. By Lemmas (2.2.8) and (2.3.2) we find a cycle C_1 in M_1 such that $b_1c_1 \in C_1$ and $|E(C_1)| \geq \alpha(m_1 - 4)^\gamma + 4$. $P_1 := (C_1 - b_1c_1)$ is the desired path.

We find a path P'_{n_2} in N_2' from S_{j-1} (say b_{j-1}) to a_2 such that $E(P'_{n_2}) \subseteq E(G)$, $|E(P'_{n_2})| \geq \alpha(n_2')^\gamma + 2$. If $n_2 = 0$, then we find P'_{n_2} by direct construction or by Lemma (3.2.2). Thus we may assume $n_2 \neq 0$. Thus we find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $b_jc_j \in P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha(m_j+2)^\gamma+2$. If $m_j \leq 6$, we construct P_j directly. Otherwise $m_j \geq 7$. If $M_j - b_{j-1}$ is not 3-connected, then by Lemma (3.3.1), we find a path P_j' in $M_j - b_{j-1} - a_2$ from $N(b_{j-1})$ to $N(a_2)$ such that $b_jc_j \in P_j'$, $|E(P_j')| \geq \alpha(m_j + 2)^\gamma + 2$. We trivially extend P_j' to the desired path P_j. Thus we may assume $M_j - b_{j-1}$ is 3-connected. We find a maximal path P' in M_j from b_{j-1} to some vertex $b' \in M_j$ such that $a_2, b_j, c_j \notin P'$, $E(P') \subseteq E(G)$, $M_j - V(P')$ is 3-connected. Let $M_j' = M_j - V(P' - b')$ and let $m_j' = |V(M_j')|$. Note that it is possible that $M_j' = M_j$. If $b'a_2 \in E(M_j')$ then we find a cycle C_j' in M_j' such that $b_jc_j, b'a_2 \in C_j'$ such that $|E(C_j')| \geq \alpha(m_j' - 4)^\gamma + 4$. $P_j := P' \cup (C_j' - b'a_2)$ is the desired path. Thus we may assume $b'a_2 \notin M_j'$. Since $M_j' - b'$ is 3-connected and since $N(b_j) - c_j$ and $N(c_j) - b_j$ are cliques, we may assume there exists $b^* \in M_j'$ such that $b'b^* \in E(G)$ and $b^* \notin \{a_2, b_j, c_j\}$. By choice of P', $M_j' - b' - b^*$ is not 3-connected. Thus by direct construction or Lemma (3.3.1), we find a path P'_j in $(M'_j - b') - b^* - a_2$ from $N(b^*)$ to $N(a_2)$ such that $b_jc_j \in P'_j$, $|E(P'_j)| \geq \alpha(max\{0, m'_j - 6\})^\gamma + 1$. We trivially extend P'_j to obtain a path P^*_j in $(M'_j - b')$ from b^* to a_2 such that $b_jc_j \in P^*_j$, $|E(P^*_j)| \geq \alpha(m'_j)^\gamma + 2$. $P_j := P' \cup P^*_j \cup b'b^*$ is the desired path. By Lemmas (3.1.3), (2.2.8), (2.3.2) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$, $E(C_{n_2} - b_jc_j) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2 - 3)^\gamma + 3$. $P'_{n_2} := (P_j - b_jc_j) \cup (C_{n_2} - b_jc_j)$ is the desired path.

Trivially we find a path P_{n_1} in N_1 from b_1 to b_{j-1} such that $c_1, c_{j-1} \notin N_1$, $E(P_{n_1}) \subseteq E(G)$, $|E(P_{n_1})| \geq 1$. Now it is easy to verify that (since n_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P_{n_1} \cup P'_{n_2}) - a_2$ is the desired path for the lemma.

This proves Case 1.

Case 2. $n_1 \leq 4$ and $N_1 \ncong K_4$.

How we proceed depends on the relative sizes of $\{m_1, m_j, n_2\}$.

Let $t' = min\{m_1, m_j, n'_2\}$.

Suppose $t' = n_2$. Thus $t' \geq 0$. Without loss of generality, assume $b_1 \neq a$.

We find a path P_1 in M_1 from b_1 to a such that $b_1c_1 \notin P_1$, $|E(P_1)| \geq \alpha(m_1 - 3)^\gamma + 2$. By Lemmas (2.2.8) and (2.3.2) we find a cycle C_1 in M_1 such that $b_1c_1 \in C_1$ and $|E(C_1)| \geq \alpha(m_1 - 3)^\gamma + 3$. $P_1 := (C_1 - b_1c_1)$ is the desired path.

We then find a path P_{n_1} in N_1 from b_1 to S_{j-1} (say b_{j-1}) such that $c_1, c_{j-1} \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, $|E(P_{n_1})| \geq 0$.

We find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $c_{j-1} \notin P_j$, $|E(P_j)| \geq \alpha(m_j)^\gamma + 2$. If $m_j \leq 5$, we construct P_j directly. Otherwise $m_j \geq 6$ and we find P_j by Lemma (3.2.2). If $j < k$ and $b_jc_j \in P_j$, replace this edge with a path in N_2. Now it is easy to verify that (since n_2 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P_{n_1} \cup P_j) - a_2$ is the desired path for the lemma.

Hence we may assume that $t' \neq n_2$.

Suppose $t' = m_j$. Thus $t' \geq 5$. Without loss of generality, assume $b_1 \neq a$.

We find a path P_1 in M_1 from b_1 to a such that $b_1c_1 \notin P_1$, $|E(P_1)| \geq \alpha(m_1 - 5)^\gamma + 4$. By Lemmas (2.2.8) and (2.3.2) we find a cycle C_1 in M_1 such that $b_1c_1 \in C_1$ and $|E(C_1)| \geq \alpha(m_1 - 5)^\gamma + 5$. $P_1 := (C_1 - b_1c_1)$ is the desired path.

We then find a path P_{n_1} in N_1 from b_1 to S_{j-1} (say b_{j-1}) such that $c_1, c_{j-1} \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, $|E(P_{n_1})| \geq 0$.

Trivially, we find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $b_jc_j \in P_j$, $c_{j-1} \notin P_j$, $|E(P_j)| \geq 3$.

As $t' \neq n_2$, $n_2 \geq 6$. Thus by Lemmas (3.1.3), (2.2.8), (2.3.2) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$ and $|E(C_{n_2})| \geq \alpha(n_2)^\gamma + 5$. Now it is easy to verify that (since m_j can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P_{n_1} \cup (P_j - b_jc_j) \cup (C_{n_2} - b_jc_j)) - a_2$ is the desired path for the lemma.

Hence we may assume that $t' \neq m_j$.

Suppose $t' = m_1$. Thus $t' \geq 3$.

We find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $b_jc_j \in P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha(m_j + 2)^\gamma + 2$. If $m_j \leq 6$, we construct P_j directly. Otherwise $m_j \geq 7$. If $M_j - b_{j-1}$ is not 3-connected, then by Lemma (3.3.1), we find a path P'_j in $M_j - b_{j-1} - a_2$ from $N(b_{j-1})$ to $N(a_2)$ such that $b_jc_j \in P'_j$, $|E(P'_j)| \geq \alpha(m_j + 2)^\gamma + 2$. We trivially extend P'_j to the desired path P_j. Thus we may assume $M_j - b_{j-1}$ is 3-connected. We find a maximal path P' in M_j from b_{j-1} to some vertex $b' \in M_j$ such that $a_2, b_j, c_j \notin P'$, $E(P') \subseteq E(G)$, $M_j - V(P')$ is 3-connected. Let $M'_j = M_j - V(P' - b')$ and let $m'_j = |V(M'_j)|$. Note that it is possible that $M'_j = M_j$. If $b'a_2 \in E(M'_j)$ then we find a cycle C'_j in M'_j such that $b_jc_j, b'a_2 \in C'_j$ such that $|E(C'_j)| \geq \alpha(m'_j - 4)^\gamma + 4$. $P_j = P' + (C'_j - b'a_2)$ gives the desired path. Thus we may assume $b'a_2 \notin M'_j$. Since $M'_j - b'$ is 3-connected and since $N(b_j) - c_j$ and $N(c_j) - b_j$ are cliques, we may assume there exists $b^* \in M'_j$ such that $b'b^* \in E(G)$ and $b^* \notin \{a_2, b_j, c_j\}$. By choice of P',

$M_j' - b' - b^*$ is not 3-connected. Thus by direct construction or Lemma (3.3.1), we find a path P_j' in $(M_j' - b') - b^* - a_2$ from $N(b^*)$ to $N(a_2)$ such that $b_jc_j \in P_j'$, $|E(P_j')| \geq \alpha(max\{0, m_j' - 6\})^\gamma + 1$. We trivially extend P_j' to obtain a path P_j^* in $(M_j' - b')$ from b^* to a_2 such that $b_jc_j \in P_j^*$, $|E(P_j^*)| \geq \alpha(m_j')^\gamma + 2$. $P_j := P' \cup P_j^* \cup b'b^*$ is the desired path.

By Lemmas (3.1.3), (2.2.8), (2.3.2) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$ and $|E(C_{n_2})| \geq \alpha(n_2 - 4)^\gamma + 4$.

If $a = c_{j-1}$ and $b_{j-1}c_{j-1} \in P_j$, let $P^* := P_j - b_{j-1}$. Now it is easy to verify that (since m_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := ((P_j - b_jc_j) \cup (C_{n_2} - b_jc_j)) - a_2$ is the desired path for the lemma. If $a = c_{j-1}$, $a \in P_j$ and $b_{j-1}c_{j-1} \notin P_j$, then we can trivially modify P_j to remove a and obtain a path P^* in M_j from b_{j-1} to a_2 such that $b_jc_j \in P^*$, $E(P^*) \subseteq E(G)$, $|E(P^*)| \geq \alpha(m_j+2)^\gamma + 1$. We then trivially find a path P_1' in $M_1 \cup N_1$ from b_1 to a such that $E(P_1') \subseteq E(G)$. Now it is easy to verify that (since m_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1' \cup (P^* - b_jc_j) \cup (C_{n_2} - b_jc_j)) - a_2$ is the desired path for the lemma. If $a = b_{j-1}$ then it is easy to verify that (since m_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_j - b_jc_j) \cup (C_{n_2} - b_jc_j)) - a_2$ is the desired path for the lemma. Thus we may assume $a \notin S_{j-1}$ We then trivially find a path P_1' in $M_1 \cup N_1$ from b_1 to a such that $E(P_1') \subseteq E(G)$, $c_{j-1} \notin P_1'$. Now it is easy to verify that (since m_1 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1' \cup (P_j - b_jc_j) \cup (C_{n_2} - b_jc_j)) - a_2$ is the desired path for the lemma.

Hence we may assume $t' \neq m_1$, which proves Case 2 and hence that we may assume $t \neq n_1$.

Suppose $t = m_1$. Thus $t \geq 3$.

Thus $n_1 \geq 4$ and hence $a \notin S_{j-1}$.

Exactly as above in Case 1 where we supposed $t = n_1$, we find a path P_{n_2}' in

N_2' from S_{j-1} (say b_{j-1}) to a_2 such that $E(P_{n_2}') \subseteq E(G)$, $|E(P_{n_2}')| \geq \alpha(n_2')^\gamma + 2$.

If $m_1 = 3$, then M_2 is 3-connected. We find a path $P_{n_1}^*$ in N_1 from b_{j-1} to S_1 (say b_1) such that $c_{j-1} \notin P_{n_1}^*$, if a is not an end of $P_{n_1}^*$ then $a \notin P_{n_1}^*$, $E(P_{n_1}^*) \subseteq E(G)$, $|E(P_{n_1})| \geq \alpha(n_1 - 4)^\gamma + 1$. By Lemmas (3.1.4)(1) and (3.1.5), we find a path P_{n_1} in N_1 from b_{j-1} to S_1 (say b_1) such that $c_{j-1} \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, $|E(P_{n_1})| \geq \alpha(n_1 - 4)^\gamma + 1$. If a is an end of P_{n_1} or if $a \notin P_{n_1}$, then $P_{n_1}^* = P_{n_1}$ as desired. Otherwise we may modify P_{n_1} to remove a in order to obtain $P_{n_1}^*$ as desired. We then trivially find a path P_1 in M_1 from b_1 to a such that $E(P_1) \subseteq E(G)$ and if $c_1 \in P_{n_1}^*$ then $c_1 \notin P_1$. $P := (P_1 \cup P_{n_1}^* \cup P_{n_2}') - a_2$ is the desired path for the lemma.

Thus we may assume $m_1 \geq 4$. Thus $n_1 \geq 5$. We find a path $P_{n_1}^*$ in N_1 from b_{j-1} to S_1 (say b_1) such that $c_{j-1} \notin P_{n_1}^*$, if a is not an end of $P_{n_1}^*$ then $a \notin P_{n_1}^*$, $E(P_{n_1}^*) \subseteq E(G)$, $|E(P_{n_1})| \geq \alpha(n_1)^\gamma + 1$. By Lemmas (3.1.4)(1), (2.3.4), and (3.1.5), we find a path P_{n_1} in N_1 from b_{j-1} to S_1 (say b_1) such that $c_{j-1} \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, $|E(P_{n_1})| \geq \alpha(n_1)^\gamma + 1$. If a is an end of P_{n_1} or if $a \notin P_{n_1}$, then $P_{n_1}^* := P_{n_1}$ as desired. Otherwise we may modify P_{n_1} to remove a in order to obtain $P_{n_1}^*$ as desired. We then trivially find a path P_1 in M_1 from b_1 to a such that $E(P_1) \subseteq E(G)$ and if $c_1 \in P_{n_1}^*$ then $c_1 \notin P_1$. $P := (P_1 \cup P_{n_1}^* \cup P_{n_2}') - a_2$ is the desired path for the lemma.

Hence we may assume $t \neq m_1$.

Suppose $t = n_2'$. Thus $t \geq 5$. Without loss of generality, assume $b_1 \neq a$.

We find a path P_1 in M_1 from b_1 to a such that $b_1c_1 \notin P_1$, $|E(P_1)| \geq \alpha(m_1)^\gamma + 2$. By Lemmas (2.2.8) and (2.3.2) we find a cycle C_1 in M_1 such that $b_1c_1 \in C_1$ and $|E(C_1)| \geq \alpha(m_1 - 5)^\gamma + 5$. $P_1 = (C_1 - b_1c_1)$ is the desired path. By Lemmas (3.1.4)(1), (2.3.4), and (3.1.5), we find a path P_{n_1} in N_1 from b_1 to S_{j-1} (say b_{j-1}) such that $c_1 \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, $|E(P_{n_1})| \geq \alpha(n_1)^\gamma + 1$. We then trivially find a path P_{n_2}' in N_2' from b_{j-1} to a such that $c_{j_1} \notin P_{n_2}'$, $E(P_{n_2}') \subseteq E(G)$.

Now it is easy to verify that (since n'_2 can be recovered by taking 2 largest out of 3 by Lemma (2.1.3)) $P := (P_1 \cup P_{n_1} \cup P'_{n_2}) - a_2$ is the desired path for the lemma.

This proves Claim 4.

Thus we may assume $a \notin S_1$.

Claim 5. We may assume $n_1 = 0$.

Otherwise $n_1 \geq 3$. Let $N'_2 = \cup_{i=j}^{k} M_i$. Let $n'_2 = |V(N'_2)|$. We consider two cases.

Case 1. $n_1 \geq 4$.

Exactly as above in Claim 4, Case 1 of where where we supposed $t = n_1$, we find a path P'_{n_2} in N'_2 from S_{j-1} (say b_{j-1}) to a_2 such that $E(P'_{n_2}) \subseteq E(G)$, $|E(P'_{n_2})| \geq \alpha(n'_2)^\gamma + 2$. By Lemmas (3.1.4)(1), (2.3.4), and (3.1.5), we find a path P_{n_1} in N_1 from b_{j-1} to S_1 (say b_1) such that $c_{j-1} \notin P_{n_1}$, $E(P_{n_1}) \subseteq E(G)$, $|E(P_{n_1})| \geq \alpha(n_1 - 4)^\gamma + 1$. By direct construction or Lemma (3.2.2), we find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| \geq \alpha(max\{0, m_1 - 4\})^\gamma + 1$. $P := (P_1 \cup P_{n_1} \cup P'_{n_2}) - a_2$ is the desired path for the lemma.

Case 2. $n_1 = 3$.

If $n_2 = 0$, then without loss of generality, assume $c_{j-1} = c_1$. By direct construction or by Lemma (3.2.2) we find a path P_j in M_j from b_{j-1} to a_2 such that $c_{j-1} \notin P_j$, $|E(P_j)| \geq \alpha(m_j - 5)^\gamma + 2$. By direct construction or Lemma (3.2.2), we find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| \geq \alpha m_1^\gamma + 1$. $P := (P'_1 \cup P_j \cup b_1 b_{j-1}) - a_2$ is the desired path for the lemma.

Thus we may assume $n_2 \neq 0$.

We find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $b_j c_j \in P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha(m_j + 2)^\gamma + 2$. If $m_j \leq 6$, we construct P_j directly. Otherwise $m_j \geq 7$. If $M_j - b_{j-1}$ is not 3-connected, then by Lemma (3.3.1), we find a path P'_j in $M_j - b_{j-1} - a_2$ from $N(b_{j-1})$ to $N(a_2)$ such that $b_j c_j \in P'_j$, $|E(P'_j)| \geq \alpha(m_j + 2)^\gamma + 2$. We trivially extend P'_j to the desired path P_j. Thus we may assume $M_j - b_{j-1}$ is 3-connected. We find a maximal path P' in M_j from b_{j-1} to some vertex $b' \in M_j$ such that $a_2, b_j, c_j \notin P'$, $E(P') \subseteq E(G)$, $M_j - V(P')$ is 3-connected. Let $M'_j = M_j - V(P' - b')$ and let $m'_j = |V(M'_j)|$. Note that it is possible that $M'_j = M_j$. If $b'a_2 \in E(M'_j)$ then we find a cycle C'_j in M'_j such that $b_jc_j, b'a_2 \in C'_j$ such that $|E(C'_j)| \geq \alpha(m'_j - 4)^\gamma + 4$. $P_j := P' \cup (C'_j - b'a_2)$ is the desired path. Thus we may assume $b'a_2 \notin M'_j$. Since $M'_j - b'$ is 3-connected and since $N(b_j) - c_j$ and $N(c_j) - b_j$ are cliques, we may assume there exists $b^* \in M'_j$ such that $b'b^* \in E(G)$ and $b^* \notin \{a_2, b_j, c_j\}$. By choice of P', $M'_j - b' - b^*$ is not 3-connected. Thus by direct construction or Lemma (3.3.1), we find a path P'_j in $(M'_j - b') - b^* - a_2$ from $N(b^*)$ to $N(a_2)$ such that $b_jc_j \in P'_j$, $|E(P'_j)| \geq \alpha(max\{0, m'_j - 6\})^\gamma + 1$. We trivially extend P'_j to obtain a path P^*_j in $(M'_j - b')$ from b^* to a_2 such that $b_jc_j \in P^*_j$, $|E(P^*_j)| \geq \alpha(m'_j)^\gamma + 2$. $P_j := P' \cup P^*_j \cup b'b^*$ is the desired path.

By Lemmas (3.1.3), (2.2.8), (2.3.2) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$, $E(C_{n_2} - b_jc_j) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2 - 3)^\gamma + 3$.

By direct construction or Lemma (3.2.2), we find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| \geq \alpha m_1^\gamma + 1$. $P := (P'_1 \cup (P_j - b_jc_j) \cup (C_{n_2} - b_jc_j) \cup b_1b_{j-1}) - a_2$ is the desired path for the lemma.

This proves Claim 5.

Claim 6. We may assume that $n_2 = 0$.

Otherwise $n_2 \neq 0$.

Exactly as above in Claim 5, Case 2, where we assume $n_2 \neq 0$, we find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $b_jc_j \in P_j$, $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha(m_j + 2)^\gamma + 2$. By Lemmas (3.1.3), (2.2.8), (2.3.2) we find a cycle C_{n_2} in N_2 such that $b_jc_j \in C_{n_2}$, $E(C_{n_2} - b_jc_j) \subseteq E(G)$, and $|E(C_{n_2})| \geq \alpha(n_2 - 3)^\gamma + 3$. By direct construction or Lemma (3.2.2), we find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| \geq \alpha(max\{0, m_1 - 4\})^\gamma + 1$. $P := (P'_1 \cup (P_j - b_jc_j) \cup (C_{n_2} - b_jc_j) \cup b_1b_{j-1}) - a_2$ is the desired path for the lemma.

This proves Claim 6.

Thus we may assume that $k = 2$, M_2 is 3-connected, and $a, a_2 \notin S_1$. We now directly prove the lemma.

We find a path P_j in M_j from S_{j-1} (say b_{j-1}) to a_2 such that $E(P_j) \subseteq E(G)$, $|E(P_j)| \geq \alpha(max\{0, m_j - 6\})^\gamma + 3$. If $m_j \leq 6$, we construct P_j directly. Otherwise $m_j \geq 7$. We find a maximal path P' in M_j from b_{j-1} to some vertex $b' \in M_j$ such that $a_2 \notin P'$, $E(P') \subseteq E(G)$, $M_j - V(P')$ is 3-connected. Let $M'_j := M_j - V(P' - b')$ and let $m'_j = |V(M'_j)|$. Note that it is possible that $M'_j = M_j$. If $b'a_2 \in E(M'_j)$ then we find a cycle C'_j in M'_j such that $b_jc_j, b'a_2 \in C'_j$ such that $|E(C'_j)| \geq \alpha(m'_j - 4)^\gamma + 4$ and if $m'_j > 4$ then $|E(C'_j)| \geq \alpha(m'_j - 5)^\gamma + 5$. Note that if $m'_j = 4$, then $|E(P')| \geq 1$. Thus in any case, $P_j = P' + (C'_j - b'a_2)$ gives the desired path. Thus we may assume $b'a_2 \notin M'_j$. We may assume there exists $b^* \in M'_j$ such that $b'b^* \in E(G)$ and $b^* \neq a_2$. By choice of P', $M'_j - b' - b^*$ is not 3-connected. Thus by direct construction or Lemma (3.3.1), we find a path P'_j in $(M'_j - b') - b^* - a_2$ from $N(b^*)$ to $N(a_2)$ such that $b_jc_j \in P'_j$, $|E(P'_j)| \geq \alpha(max\{0, m'_j - 6\})^\gamma + 1$. We trivially extend P'_j to obtain a path P^*_j in $M'_j - b'$ from b^* to a_2 such that $b_jc_j \in P^*_j$, $|E(P^*_j)| \geq \alpha(m'_j)^\gamma + 2$. $P_j := P' \cup P^*_j \cup b'b^*$ is the desired path.

By direct construction or Lemma (3.2.2), we find a path P_1 in M_1 from b_1 to a such that $c_1 \notin P_1$, $|E(P_1)| \geq \alpha(max\{0, m_1 - 4\})^\gamma + 1$. $P := (P_1 \cup P_j) - a_2$ is the desired path for the lemma. □

CHAPTER IV

CONCLUSION

4.1 Proof of Theorem (1.2.2)

The proof is by induction and then by simple application of previously proven lemma. We prove the base case $n = 6$ by Lemma (2.1.1). Thus we may assume $n \geq 7$.

First we define a path $Z_G(e)$ as follows. Let $e = x_0y_0$, and let $Z_G(e) := x_r \dots x_0y_0 \dots y_s$ be a maximal path in G such that

(1) if $V(f) \cap V(Z) \neq \emptyset$ then $f \in E(Z)$, or $V(f) \cap V(Z) = \{x_r\}$, or $V(f) \cap V(Z) = \{y_s\}$

(2) for $0 \leq i \leq r-1$ and $0 \leq j \leq s-1$, $G - (\{x_i, \dots, x_0\} \cup \{y_0, \dots, y_j\}$ is 3-connected

(3) neither $G - V(Z - x_r)$ nor $G - V(Z - y_s)$ is 3-connected

(4) if $V(f)$ is a 2-cut in $G - V(Z - y_s)$ (respectively, $G - V(Z - x_r)$) then $y_s \notin V(f)$ (respectively, $x_r \notin V(f)$)

Let $G' = G - (Z_G(e) - x_r - y_s)$. Let $n' = |V(G')|$.

First we show that we can either directly construct the desired cycle for the theorem or that such a path $Z_G(e)$ exists. Let $Z'_G(e) = x_r ... x_0y_0 ... y_s$ be a maximal path which satisfies (1), (2), (4). Note that x_0y_0 satisfies all these conditions, and hence $Z'_G(e)$ exists. It suffices to show that one can construct a path which satisfies (1) $-$ (4) from the path $Z'_G(e)$, or a cycle which satisfies the Theorem.

If $Z'_G(e)$ satisfies (3), then $Z_G(e) = Z'_G(e)$. Thus we may assume, without loss of generality, that $G - V(Z' - y_s)$ is 3-connected. If f is incident to x_r, let

$f = x_{r+1}x_r$ and $Z'_G(e) \cup \{x_{r+1}, x_{r+1}x_r\}$ would satisfy (1), (2), (4) – contradicting the maximality of $Z'_G(e)$. Thus f is not incident to x_r.

Consider instead where f is incident to y_s. If $G - V(Z' - x_r)$ is 3-connected, then we similarly contradict the maximality of $Z'_G(e)$. Hence we may assume that $G - V(Z' - x_r)$ is not 3-connected. Let $f = y_s y'$. Let $X = N_{G-V(Z'-y_s)}(x_r) - \{y_s, y'\}$. $|X| \geq 1$. Note that $Z'_G(e) \cup \{x_{r+1}, x_{r+1}x_r\}$, for any $x_{r+1} \in X$, satisfies (1) and (2). Consider the Tutte decomposition of $G - V(Z' - x_r)$, impose an orientation from left to right on the 3-blocks. Let $Y \subseteq V(G - V(Z' - x_r))$ such that for any $y \in Y$, $\{y', y\}$ are a 2-cut in $G - V(Z' - x_r)$. If there exists $x_{r+1} \in X$ such that $x_{r+1} \notin Y$, then $Z'_G(e) \cup \{x_{r+1}, x_{r+1}x_r\}$ satisfies (4) and hence contradicts the maximality of $Z'_G(e)$. Thus we may assume $X \subseteq Y$. Note that this implies that y' is in a 2-cut of $G - V(Z' - x_r)$. First we consider the simple case where the decomposition of $G - V(Z' - x_r)$ is a single 3-block, namely, a chain of cycles. It is easy to see that there exists $x' \in X$ such that there is a path P' in $G - V(Z)$ from x' to y' which contains all but at most 1 vertex of $G - V(Z)$. As $n \geq 7$, $C := P' \cup Z'_G(e) \cup \{x'x_r, y'y_s\}$ is the desired cycle for the Theorem. Thus we may assume that the decomposition of $G - V(Z' - x_r)$ is not a single 3-block. Let M and M' be the leftmost and rightmost 3-blocks respectively in this decomposition containing y'. Recall that we may assume y' is in a 2-cut. By (2), we may assume $M \neq M'$ and that y' is in a special 2-cut. Note that $|Y| \leq 2$. Since we may assume that $X \subseteq Y$, $|\{x_r y_s, x_r y'\} \cap E(G)| \geq 1$.

Suppose $x_r y' \in E(G)$. Let $m \in M$ and $m' \in M'$ be internal vertices in their respective 3-blocks that are adjacent to y'. As G is claw-free, $\{y', x_r, m, m'\}$ does not induce a claw and hence without loss of generality $x_r m \in E(G)$. Then $Z'_G(e) \cup \{m, x_r m\}$ satisfies (4) and hence contradicts the maximality of $Z'_G(e)$. Thus we may assume $x_r y' \notin E(G)$. Thus we may assume $x_r y_s \in E(G)$. Let M_1 and M_k be the two extreme 3-blocks in the decomposition of $G - V(Z' - x_r)$. Let $m_1 \in M_1$

and $m_k \in M_k$ be internal vertices in their respectively 3-blocks that are adjacent to y_s. As G is claw-free, $\{y_s, x_r, m_1, m_k\}$ does not induce a claw and hence without loss of generality $x_r m_1 \in E(G)$. Then $Z'_G(e) \cup \{m, x_r m_1\}$ satisfies (4) and hence contradicts the maximality of $Z'_G(e)$. Thus we may assume f is not incident to y_s.

Thus we may assume f is not incident to either x_r or y_s. If x_r has a neighbor x_{r+1} in $G - V(Z')$ such that $x_{r+1} \notin \{y_s\} \cup V(f)$, then it is easy to see that $Z'_G(e) \cup \{x_{r+1}, x_{r+1}x_r\}$ would satisfy (1), (2), (4) – contradicting the maximality of $Z'_G(e)$. Hence we may assume $N_{G-V(Z')}(x_r) = \{y_s\} \cup V(f)$. Let $x_{r+1} \in V(f)$. Let $Z^*_G(e) = Z'_G(e) \cup \{x_{r+1}, x_{r+1}x_r\}$. Clearly $Z^*_G(e)$ satisfies (1), (2). As the degree of x_r in $G - V(Z' - y_s)$ is 3, $G - V(Z' - x_r)$ is not 3-connected and $\{x_r\} \cup V(f)$ is in an extreme chain of cycles in the decomposition of $G - V(Z' - x_r)$. Furthermore, as $G - V(Z' - y_s)$ is 3-connected, this chain of cycles containing $\{x_r\} \cup V(f)$ is a single triangle and $V(f)$ is a special 2-cut in the decomposition of $G - V(Z' - x_r)$. Consequently, $\{y_s\} \cup V(f)$ is not a 3-cut of $G - V(Z' - y_s)$. Hence $Z^*_G(e)$ satisfies (4) and hence contradicts the maximality of $Z'_G(e)$.

Thus, we may assume $Z_G(e)$ exists.

Suppose $n' \geq 7$.

If $f \in E(Z_G(e))$ or if $V(f) \cap V(Z_G(e)) = \emptyset$, then by Lemma (3.3.1) we find a path P' in $G' - x_r - y_s$ from $N(x_r)$ (say x') to $N(y_s)$ (say y') such that $E(P') \subseteq E(G)$, if $f \notin E(Z_G(e))$ then $f \in P'$, and $|E(P')| \geq \alpha(n'+2)^\gamma + 2$. $C := P' \cup Z_G(e) \cup \{x'x_r, y'y_s\}$ is the desired cycle for the Theorem.

Thus we may assume $f \notin E(Z_G(e))$ but $V(f) \cap V(Z_G(e)) \neq \emptyset$. Note that $f \neq x_r y_s$, by definition of $Z_G(e)$ (in particular property (1)). Thus without loss of generality, $f = x_r x$ where $x \in G' - x_r - y_s$. By property (4) of the definition of $Z_G(e)$, $\{x, x_r, y_s\}$ do not form a 3-cut in G'. Thus by Lemma (3.3.2), we find a path P' in $G' - x_r - y_s$ from x to $N(y_s)$, $E(P') \subseteq E(G)$, $|E(P')| \geq \alpha(n'+2)^\gamma + 2$. $C := P' \cup Z_G(e) \cup \{xx_r, y'y_s\}$ is the desired cycle for the Theorem.

Thus we may assume $n' \leq 6$.

Hence $|E(Z_G(e))| \geq 2$. If $x_r y_s \in E(G)$, then by Lemma (2.1.1), we find a Hamilton cycle C' in G' such that $e, x_r y_s \in C'$. $C := (C' - x_r y_s) \cup Z_G(e)$ is the desired cycle for the Theorem. Thus we may assume $x_r y_s \notin E(G)$. By Lemma (2.1.2), we find a path P' in G' from x_r to y_s such that $f \in P'$, $|E(P')| \geq n' - 2$. $C := P' \cup Z_G(e)$ is the desired cycle for the Theorem. □

4.2 Future work

We have proven Theorem (1.2.2) and hence improved the bound for the circumference of 3-connected claw-free graphs.

However, we believe we can improve the bound even further using the methods of this thesis more extensively. Specifically we believe that if G is a 3-connected claw-free graph on n vertices, then we can find a cycle of length at least $\alpha n^{\gamma} + 5$ where $\alpha \geq 1/7$ and $\gamma = \log_6 4 \sim 0.77$. Such a result (or even a slightly weaker one) would improve the bound for the circumference of 3-connected cubic graphs.

We conclude this thesis with intuition about how we can adapt our methods to further improve our bound. In short, we would want to use Tutte decomposition more extensively. In the proof of the main theorem, we define the path $Z_G(e)$ (with ends a_1, a_2), $G' = G - (Z_G(e) - \{a_1, a_2\})$, and then use two lemmas to find a path P' in G' from a_1 to a_2 with the desired properties. In those two lemmas, we find P' by taking the Tutte decomposition of $G' - a_1$ and then constructing P' through the 3-blocks of that decomposition by exhaustive case analysis. We could, instead, perform a "double decomposition". We could consider $G - a_1 - a_2$. In full generality, this might be a very complicated proposition. However, in our original case analysis, when a_2 was not an internal vertex of a 3-connected 3-block of the decomposition of $G' - a_1$, then it is easier to make the path P' go through more of the 3-blocks of the decomposition and hence satisfy the length requirement for a larger value of γ. Thus for these "easier" cases, we perform a single decomposition as before. We only perform a double decomposition for the cases where a single decomposition is sufficient. However, in this second decomposition, we can now look to decompose the 3-connected 3-block in the first decomposition which contains a_2. This doubly decomposed structure will have more sections than the singly decomposed structure. In particular we will have multiple sections where there was previously just one. Thus in our case analysis, where we may have previously neglected the contribution of the entire 3-block containing a_2, we may now neglect only some of the sections from its decomposition, but not all of them. Another way to see this, is there are simply more sections and so we may have the flexibility to neglect more sections than before. Note that we do employ this technique in some of the proofs throughout the thesis. However, we can apply this concept more extensively to improve our bound.

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