

**SOME PROBLEMS IN THE THEORY OF OPEN
DYNAMICAL SYSTEMS AND DETERMINISTIC WALKS
IN RANDOM ENVIRONMENTS.**

A Dissertation
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Mathematics

Georgia Institute of Technology
December 2008

**SOME PROBLEMS IN THE THEORY OF OPEN
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IN RANDOM ENVIRONMENTS.**

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*To my mom, dad, sister, and grandma, who love and believe in me;
to the love of my life Olya, who is always there for me;
to my uncle Dr. Yan Umanskiy, who inspired me to become a
mathematician.*

ACKNOWLEDGEMENTS

I want to thank my dissertation advisor Dr. Leonid Bunimovich for suggesting the problems. Thank you for always standing by me and for all your help. Without your guidance it would be impossible to complete this work. I am also grateful to Dr. Valentin Afraimovich, Dr. Mark Demers, Dr. Mark Pollicott, and Dr. Tamas Tél for numerous helpful discussions.

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SUMMARY

The first part of this work deals with open dynamical systems. A natural question of how the survival probability depends upon a position of a hole was seemingly never addressed in the theory of open dynamical systems. We found that this dependency could be very essential. The main results are related to the holes with equal sizes (measure) in the phase space of strongly chaotic maps. Take in each hole a periodic point of minimal period. Then the faster escape occurs through the hole where this minimal period assumes its maximal value. The results are valid for all finite times (starting with the minimal period), which is unusual in dynamical systems theory where typically statements are asymptotic when time tends to infinity. It seems obvious that the bigger the hole is the bigger is the escape through that hole. Our results demonstrate that generally it is not true, and that specific features of the dynamics may play a role comparable to the size of the hole.

In the second part we consider some classes of cellular automata called Deterministic Walks in Random Environments on \mathbb{Z}^1 . At first we deal with the system with constant rigidity and Markovian distribution of scatterers on \mathbb{Z}^1 . It is shown that these systems have essentially the same properties as DWRE on \mathbb{Z}^1 with constant rigidity and independently distributed scatterers. Lastly, we consider a system with non-constant rigidity (so called process of aging) and independent distribution of scatterers. Asymptotic laws for the dynamics of perturbations propagating in such environments with aging are obtained.

CHAPTER I

WHERE TO PLACE A HOLE TO ACHIEVE A MAXIMAL ESCAPE RATE.

1.1 Introduction.

The theory of open dynamical systems is (naturally) much less developed than of the closed ones. Basically, so far the problems studied were on the existence of conditionally invariant measures, their properties, and the existence of the escape rates [5], [11], [25], [26], [27], [28], [32], [47], [53], [57].

Here we address a natural question which, to the best of our knowledge, has not been studied so far. Obviously, if one enlarges a hole, then the escape rate of the orbits will increase as well (or, at least, it cannot decrease). Consider, however, two holes of the same size (measure), placed at the different positions in the phase space of the dynamical system under study. Would the escape rates through these holes be equal?

We demonstrate that the answer to this question could be both "yes" and "no". In case when there exists a group of measure preserving translations of the phase space which commute with the dynamics the answer is "yes". It is quite natural and intuitive answer which is justified in Section 1.4. However the dynamics of these systems is quite regular.

Much less trivial is the question of what other factors, besides the size of the hole, can influence the escape rate. In particular, what can generate different escape rates through two holes of the same size?

Consider a system with strongly chaotic dynamics. For many classes of such systems it is known that there exists infinitely many periodic orbits of infinitely

many periods and that the periodic orbits are everywhere dense in the phase space. Therefore in each hole there are infinitely many periodic points.

Our approach is based on the idea that the faster escape occurs through a hole whose preimages overlap less than the ones of another hole. This idea leads to the following procedure (algorithm): 1) find in each hole a periodic point with minimal period; 2) compare these periods. We claim that the escape will be faster through a hole where this minimal period is bigger. This claim is justified for various classes of dynamical systems with strongly chaotic behavior and Markov holes in Section 1.5.

We also computed the local escape rate and demonstrated that for all non-periodic points this value is the same, while at the periodic points the escape "slows down" and assumes smaller values at the periodic points with smaller period.

Thus we demonstrate that the dynamical factors could be as important for the escape as the size of the hole. In fact it is possible that the escape rate through a larger hole could be less than the escape rate through a smaller hole.

An important and a new feature of our results is that they hold for all finite times starting with some moment of time, in comparison to the usual setup in the theory of dynamical systems where one deals with the asymptotic properties at infinite time.

For more general classes of dynamical systems not only the distribution of the periodic points, but other characteristics of dynamics, e.g. distortion, may also contribute to the process of escape. This will be considered in a future publication.

The structure of this chapter is the following one. Section 1.2 contains necessary definitions and some auxiliary results. In Section 1.3 we formally pose the questions. Section 1.4 deals with the case where escape rate does not depend on the position of the hole. Section 1.5 presents the main results of this chapter. Section 1.6 deals with some generalization and, finally, Section 1.8 contains concluding remarks. The results of this chapter were submitted for publication [37].

1.2 Definitions and some technical results.

Consider a measure-preserving map

$$\hat{T} : \hat{M} \rightarrow \hat{M},$$

where \hat{M} is a Borel probability space with the measure λ . Let \mathcal{B} be the Borel σ -algebra on \hat{M} with respect to λ . The *measure-preserving transformation* is a transformation that (obviously) preserves the measure, i.e. $\lambda(\hat{T}^{-1}A) = \lambda(A)$ for all measurable (with respect to \mathcal{B}) sets A . The *measure-preserving discrete time dynamical system* is the probability space with a measure-preserving transformation on it, i.e. it is a system

$$\left(\hat{M}, \mathcal{B}, \lambda, \hat{T}\right).$$

Denote by \hat{T}^n the n th iterate of \hat{T} , i.e.

$$\hat{T}^n = \underbrace{\hat{T} \circ \dots \circ \hat{T}}_n.$$

The (forward) *orbit* of a point $x \in \hat{M}$ is a union of all images of x under \hat{T} , i.e. the orbit of x is the set $\{\hat{T}^n x\}_{n=0}^{\infty}$.

1.2.1 Recurrences.

Here we define some notions related to the recurrence properties of the dynamical system.

Definition 1.2.1.1. *The Poincaré recurrence time of a subset $A \in \mathcal{B}$ of a positive measure is a positive integer $\tau(A) \leq +\infty$ given by*

$$\tau_{\hat{T}}(A) = \inf_{n \geq 1} \{n : \lambda(\hat{T}^n(A) \cap A) > 0\}. \quad (1.2.1)$$

If there is no ambiguity about which map we are considering, then we drop the subscript and use $\tau(A)$ instead. If the Poincaré recurrence time is finite, then it is the smallest integer n such that the n th iterate of A under \hat{T} intersects A nontrivially (in this case nontrivially means that intersection has a non-zero measure), see Figure 1.

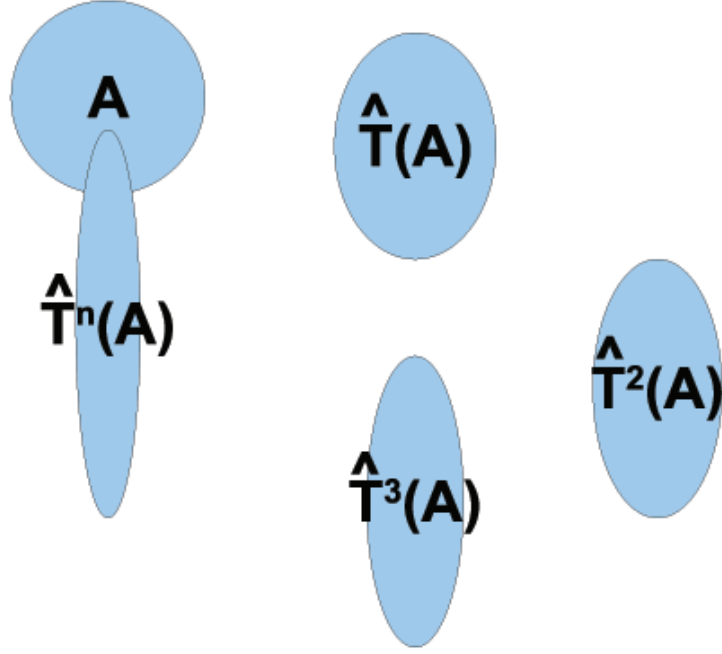


Figure 1: Poincaré recurrence time of a set A .

According to the Poincaré Recurrence Theorem (see, for example Theorem 1.4 in [58]) for the spaces of finite measure the Poincaré recurrence time of any measurable set of positive measure is finite.

Next, we list a few properties of Poincaré recurrence time which will be used later. These statements follow easily from the definition.

Proposition 1.2.1.2. *Let A and B be two measurable sets. Then*

a) *if $A \subset B$ then $\tau(A) \geq \tau(B)$;*

b) $\tau(A) = \tau(\hat{T}^{-1}(A))$,

where $\hat{T}^{-1}(A)$ is a complete preimage of A .

For $n \geq 0$ and $A \in \mathcal{B}$, define the following (measurable) sets [32],

$$\Omega_n(A) = \left\{ x \in \hat{M} : \exists j \in \mathbb{N}, 0 \leq j \leq n, \hat{T}^j x \in A \right\} = \cup_{i=0}^n \hat{T}^{-i}(A),$$

$$\Theta_n(A) = \left\{ x \in \hat{M} : \hat{T}^n x \in A, \hat{T}^j x \notin A, j = 0, \dots, n-1 \right\},$$

where $\hat{T}^{-i}(A)$ is a complete preimage of A under \hat{T}^i . The set $\Omega_n(A)$ consists of all points which orbits enter A after no more than n iterates. The set $\Theta_n(A)$ consists of all points which orbits enter A at first time exactly on n th iterate. Note that $\Omega_0(A) = A$ and $A \subset \Omega_n(A)$, $\forall n \in \mathbb{N}$. It is easy to see that these sets have the following properties.

Proposition 1.2.1.3. *Let A be a measurable set. Then*

- a) $\Omega_i(A)$ is a nondecreasing sequence of sets;
- b) $\Theta_i(A) \cap \Theta_j(A) = \emptyset$ if $i \neq j$, $i, j > 0$;
- c) $\Omega_n(A) = \cup_{i=0}^n \Theta_i(A)$.

1.2.2 Open dynamical systems.

Let A be a measurable set and let $M = \hat{M} \setminus A$. We define an open dynamical system (system with a "hole" A) by a map

$$T : M \rightarrow \hat{M},$$

where $T := \hat{T}|_M$ is a restriction of \hat{T} to M . We keep track of the orbits while they stay outside the "hole" A , and after they enter a hole we no longer consider these orbits (they just "disappear"). So we can talk about iterates of T instead of \hat{T} as long as orbit stays outside A . Note that we use a hat over a letter to denote an object in a closed system and letters without a hat for corresponding objects in the open system.

1.2.3 Escape rate.

In this chapter we want to study the rate at which different orbits enter the hole. The number

$$\lambda \left(\hat{M} \setminus \Omega_n(A) \right) = 1 - \lambda(\Omega_n(A)),$$

sometimes called a *survival probability*, is the measure of the set that does not escape into the hole during the first n iterations of the map. The escape rate, on the other hand, represents the average rate at which orbits enter the hole.

Definition 1.2.3.1. *The (exponential) escape rate into the hole A is a nonnegative number $\rho(A)$ given by*

$$\rho(A) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \lambda \left(\hat{M} \setminus \Omega_n(A) \right), \quad (1.2.2)$$

if this limit exists.

Note that the larger the escape rate is, the faster the "mass" escapes from the system into the hole A .

We will only consider systems in which almost every orbit eventually enters the hole, i.e. systems which satisfy the following condition

$$\sum_{i=0}^{\infty} \lambda(\Theta_i(A)) = 1. \quad (\text{H1})$$

Any ergodic system would be an example of such a system. In that case property H1 holds for any measurable hole of positive measure. On the other hand if we consider a system with a globally attracting set A , then the property H1 holds only for that set A and any set which contains A .

The next proposition lists a few simple but useful properties of the escape rate.

Proposition 1.2.3.2. *Let A and B be two measurable sets. Assume that $\rho(A)$ and $\rho(B)$ exist. Then,*

- a) *if $A \subset B$ then $\rho(A) \leq \rho(B)$;*
- b) *$\rho(A) = \rho(\hat{T}^{-1}(A))$;*
- c) *for any finite positive integer m , $\rho(A) = \rho \left(\bigcup_{i=0}^m \hat{T}^{-i} A \right)$;*
- d) *if $B \subset \hat{T}^{-k} A$ for some $k > 0$ then $\rho(A \cup B) = \rho(A)$;*

e) if there exists m such that $\hat{M} = \cup_{i=0}^m \hat{T}^{-i} A$ then $\rho(A) = +\infty$.

The first part of the proposition says that the size of the hole is one of the factors that determines the escape rate. As we will see later, it is not necessarily the only one or even the dominant one. Moreover, c) and d) state that, in principle, we can have a system in which holes of different size have the same escape rate.

Instead of looking at the measure of the set that does not enter a hole during the first n iterations, sometimes it is more convenient to consider the set which enters the hole for the first time on exactly n th iteration (but not earlier). The following lemma illustrates how we can accomplish that.

Lemma 1.2.3.3. *Suppose that condition H1 holds and the escape rate, $\rho(A)$, exists.*

Then

$$\rho(A) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \lambda(\Theta_n(A)).$$

Proof. Let $a_n = \lambda(\Theta_n(A))$ and assume that $-\lim_{n \rightarrow \infty} \frac{1}{n} \ln a_n = \alpha$. Then $\forall \epsilon \in (0, \alpha)$ $\exists N \in \mathbb{N}$ such that $\forall n \geq N$ one has that

$$-\epsilon - \alpha \leq \frac{1}{n} \ln a_n \leq \epsilon - \alpha$$

or, equivalently,

$$e^{-n(\alpha+\epsilon)} \leq a_n \leq e^{-n(\alpha-\epsilon)}.$$

Next, observe that if $\rho(A)$ exists, then it is given by

$$\begin{aligned} \rho(A) &= - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(1 - \sum_{i=0}^n \lambda(\Theta_i(A)) \right) \\ &= - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\sum_{i=n+1}^{\infty} \lambda(\Theta_i(A)) \right) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\sum_{i=n+1}^{\infty} a_i \right). \end{aligned}$$

For $n \geq N$ we have

$$\sum_{i=n+1}^{\infty} e^{-i(\alpha+\epsilon)} \leq \sum_{i=n+1}^{\infty} a_i \leq \sum_{i=n+1}^{\infty} e^{-i(\alpha-\epsilon)}$$

or, equivalently,

$$\frac{e^{-(n+1)(\alpha+\epsilon)}}{1 - e^{-(\alpha+\epsilon)}} \leq \sum_{i=n+1}^{\infty} a_i \leq \frac{e^{-(n+1)(\alpha-\epsilon)}}{1 - e^{-(\alpha-\epsilon)}}.$$

Taking the logarithm of both sides, dividing by n , and letting n tend to infinity we complete the proof. \square

Recall now the notion of metric conjugacy which will play an important role in what follows. Note that for Lebesgue probability spaces metric conjugacy is equivalent to two maps being isomorphic (see, for example, Theorem 2.5 and 2.6 in [58]). We use the following definition.

Definition 1.2.3.4. *Let T_i be a measure-preserving transformation of the Lebesgue probability space $(X_i, \mathcal{B}_i, \lambda_i)$, $i = 1, 2$, where \mathcal{B}_i is a Borel σ -algebra on X_i and λ_i is a probability measure. We say that T_1 and T_2 are metrically conjugate if there exist $M_i \in \mathcal{B}_i$ with $\lambda_i(M_i) = 1$ and $T_i(M_i) \subset M_i$ and there is an invertible measure-preserving transformation (called metric conjugacy) $F : M_2 \rightarrow M_1$ such that*

$$F \circ T_2(x) = T_1 \circ F(x), \quad \forall x \in M_2.$$

The following result states that if two systems are metrically conjugate, then the escape rates into the corresponding holes and Poincaré return times of these holes are the same for both systems.

Lemma 1.2.3.5. *Let T_1 and T_2 be two metrically conjugate measure-preserving transformations on the Borel probability spaces $(X_1, \mathcal{B}_1, \lambda_1)$ and $(X_2, \mathcal{B}_2, \lambda_2)$, correspondingly, with a conjugacy map $F : (\mathcal{B}_2, \lambda_2) \rightarrow (\mathcal{B}_1, \lambda_1)$. Suppose also that T_2 satisfy condition H1. Then $\forall A \in \mathcal{B}_2$ we have*

$$a) \quad \rho_{T_2}(A) = \rho_{T_1}(F(A)),$$

$$b) \quad \tau_{T_2}(A) = \tau_{T_1}(F(A)).$$

Proof. a) Let $A \in \mathcal{B}_2$ and, as above, define two sets

$$\begin{aligned}\Theta_n^2(A) &= \{x \in X_2 : T_2^n x \in A, T_2^j x \notin A, j = 0, \dots, n-1\}; \\ \Theta_n^1(F(A)) &= \{y \in X_1 : T_1^n y \in F(A), T_1^j y \notin F(A), j = 0, \dots, n-1\}.\end{aligned}$$

Then the escape rates for two systems are given by

$$\begin{aligned}\rho_{T_1}(A) &= -\lim_{n \rightarrow \infty} \frac{1}{n} \ln \lambda_1(\Theta_n^1(F(A))); \\ \rho_{T_2}(A) &= -\lim_{n \rightarrow \infty} \frac{1}{n} \ln \lambda_2(\Theta_n^2(A)).\end{aligned}\tag{1.2.3}$$

Claim 1.2.3.6. $\forall n \geq 1$,

$$F(\Theta_n^2(A)) = \Theta_n^1(F(A)).$$

Proof.

$$\begin{aligned}\Theta_n^1(F(A)) &= \{y \in X_1 : T_1^n y \in F(A), T_1^j y \notin F(A), j = 0, \dots, n-1\} \\ &= \{y \in X_1 : F^{-1}T_1^n y \in A, F^{-1}T_1^j y \notin A, j = 0, \dots, n-1\} \\ &= \{y \in X_1 : T_2^n F^{-1}y \in A, T_2^j F^{-1}y \notin A, j = 0, \dots, n-1\} \\ &= \{F(x) \in X_1 : T_2^n x \in A, T_2^j x \notin A, j = 0, \dots, n-1\} \\ &= F(\{x \in X_2 : T_2^n x \in A, T_2^j x \notin A, j = 0, \dots, n-1\}) = F(\Theta_n^2(A)).\end{aligned}$$

□

By the previous claim and definition of the conjugacy map we have

$$\lambda_2(\Theta_n^2(A)) = \lambda_1(F(\Theta_n^2(A))) = \lambda_1(\Theta_n^1(F(A))).$$

Hence one has that $\rho_{T_2}(A) = \rho_{T_1}(F(A))$, $\forall A \in \mathcal{B}_2$.

b) Suppose that $\tau_{T_2}(A) = N < \infty$. That means that

$$\lambda(T_2^N(A) \cap A) > 0$$

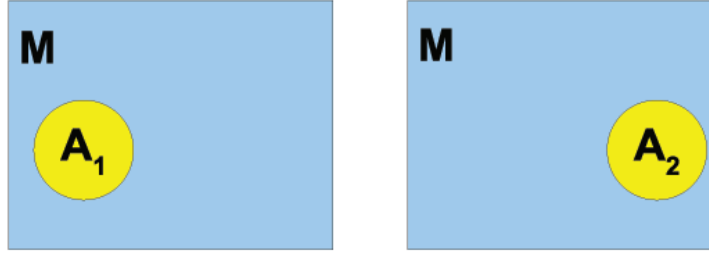


Figure 2: Two holes of the same size in different positions.

and

$$\lambda(T_2^k(A) \cap A) = 0, \quad k < N.$$

But

$$\lambda(T_1^N(F(A)) \cap F(A)) = \lambda(F(T_2^N(A)) \cap F(A)) > 0$$

and similarly

$$\lambda(T_1^k(F(A)) \cap F(A)) = 0, \quad k < N.$$

Hence, $\tau_{T_1}(F(A)) = N$.

□

1.3 Questions.

The main questions of this chapter are the following. What factors (dynamical, geometric, etc) can determine the escape rate? Given two holes of the same size in the different position in the phase space, which system will have a larger escape rate (see Figure 2)? For which systems is the position of the hole irrelevant? Given a large hole and a smaller hole, can we have the escape rate larger for the system with a smaller hole (see Figure 3 and Figure 4)? What are the other factors (besides the size and the position) that can play a role in escape rate?

In the following sections we will answer these questions with the help of examples.

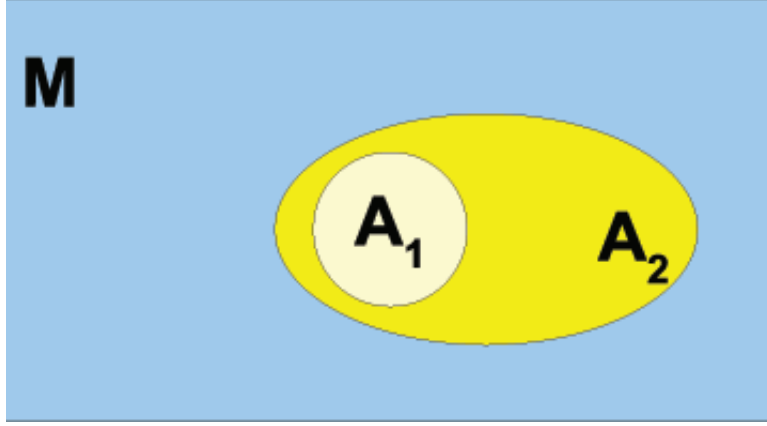


Figure 3: Two nested holes.

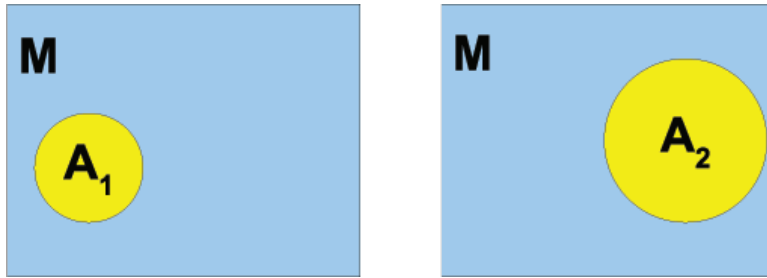


Figure 4: Two holes of different sizes in different positions.

1.4 Escape rate for the ergodic group rotations.

Suppose that the phase space \hat{M} is a compact connected metric group. Let $S_a : \hat{M} \rightarrow \hat{M}$ be a group rotation defined as

$$S_a(x) = ax,$$

for some $a \in \hat{M}$. Then there is the unique any rotation invariant probability measure, λ , called Haar measure (see, e.g. [44]).

The following simple statement claims that if a group rotation S commutes with the dynamics, i.e.

$$\hat{T}^{-1}S = S\hat{T}^{-1}, \tag{H2}$$

then the escape rate is invariant when we rotate the hole by S .

Lemma 1.4.0.7. *Suppose that $\hat{T}^{-1}S(A) = S\hat{T}^{-1}(A)$ for some $S \in \mathbb{G}$ and $A \in \mathbb{B}$. Also, assume that $\rho_{\hat{T}}(A)$ exists. Then $\rho_{\hat{T}}(A) = \rho_{\hat{T}} \circ S(A)$.*

Proof. Let $A \in \mathbb{B}$ as in the statement of the theorem. Then,

$$\begin{aligned}\Theta_n(S(A)) &= \left\{ x \in \hat{M} : \hat{T}^n x \in S(A), \hat{T}^j x \notin S(A), j = 0, \dots, n-1 \right\} \\ &= \left\{ x \in \hat{M} : \hat{T}^n S^{-1}x \in A, \hat{T}^j S^{-1}x \notin A, j = 0, \dots, n-1 \right\} \\ &= \left\{ Sy \in \hat{M} : \hat{T}^n y \in A, \hat{T}^j y \notin A, j = 0, \dots, n-1 \right\} = S(\Theta_n(A)).\end{aligned}$$

But $\lambda(S(\Theta_n(A))) = \lambda(S\Theta_n(A))$. Therefore, by Lemma 1.2.3.3 the result follows. \square

Assume now that $\hat{T} = \hat{T}_a$ is an ergodic rotation of \hat{M} given by

$$\hat{T}_a(x) = ax,$$

for some $a \in \hat{M}$. For the rotations we use the following property as a definition of ergodicity.

Proposition 1.4.0.8 (see, for example, Theorem 1.9 in [58]). *Let \hat{M} be a compact group and \hat{T}_a a rotation of \hat{M} . Then \hat{T}_a is ergodic iff $\{a^n\}_{n=-\infty}^{+\infty}$ is dense in \hat{M} .*

The simplest example of this class of dynamical systems is the irrational rotations of the circle. It is known (see, for example, Theorem 1.9 in [58]) that if there is an ergodic rotation of \hat{M} then \hat{M} must be Abelian. In that case conditions H1 and H2 are satisfied so Lemma 1.4.0.7 is applicable. Hence, the escape rate for any hole is independent of the position of that hole. Moreover, one can compute the escape rate for any hole that contains an open set. It turns out that it is infinite because in the case of ergodic rotation all orbits escape within finite amount of time.

Theorem 1.4.0.9. *For any ergodic rotation $\hat{T}_a = ax$ of the compact connected metric group \hat{M} and any hole A that contains an open ball the escape rate is infinite.*

Proof. Let $dist(\cdot, \cdot)$ be a metric on \hat{M} . Suppose that $V(x, \varepsilon)$ is an open ball of radius ε centered at x that is contained in the hole A . Since \hat{M} is a compact metric space we can find a finite covering by open balls of radius $\frac{1}{4}\varepsilon$. Let $\{V_i(x_i, \frac{1}{4}\varepsilon)\}_{i=1}^m$ be that covering.

It follows from ergodicity that if \hat{T}_a is ergodic then the set $\{a^i\}_{i=1}^{+\infty}$ is dense in \hat{M} . Therefore we can find n and $\{n_j\}_{j=1}^m$ such that

$$\begin{aligned} \text{dist}(a^n, x) &< \frac{1}{4}\varepsilon; \\ \text{dist}(a^{n_j}, x_j) &< \frac{1}{4}\varepsilon, \quad \forall j = 1 \dots m; \\ n &> \max_{j=1 \dots m} \{n_j\}. \end{aligned}$$

Then we have that

$$\begin{aligned} \text{dist}(\hat{T}_a^{n-n_j} x_j, x) &\leq \text{dist}(\hat{T}_a^{n-n_j} x_j, a^n) + \text{dist}(a^n, x) \\ &= \text{dist}(\hat{T}_a^{n-n_j} x_j, \hat{T}_a^{n-n_j} a^{n_j}) + \text{dist}(a^n, x) \\ &= \text{dist}(x_j, a^{n_j}) + \text{dist}(a^n, x) < \frac{1}{2}\varepsilon. \end{aligned}$$

Therefore, $\forall j = 1 \dots m$, $\hat{T}_a^{n-n_j} V_j \subset V$. Thus every ball V_j will be mapped into the hole in finite number of steps. Since every set can be covered by these balls we get that any set is mapped into the hole in finite number of iterations. This finishes the proof. \square

1.5 Escape rate for the expanding maps of the interval.

In this section we look at the examples of the dynamical systems in which the position of the hole plays an important role in determining the escape rate. We consider some classes of the uniformly expanding maps of the interval that have a finite Markov partition. These systems are the examples of so called chaotic dynamical systems.

Consider first the map of a unit interval to itself, $\hat{T} : [0, 1] \rightarrow [0, 1]$, given by

$$\hat{T}x = \kappa x \pmod{1},$$

where κ is an integer larger than one. This map preserves the Lebesgue measure on $[0, 1]$. Without any loss of generality (to be justified bellow) one can assume that $\kappa = 2$.

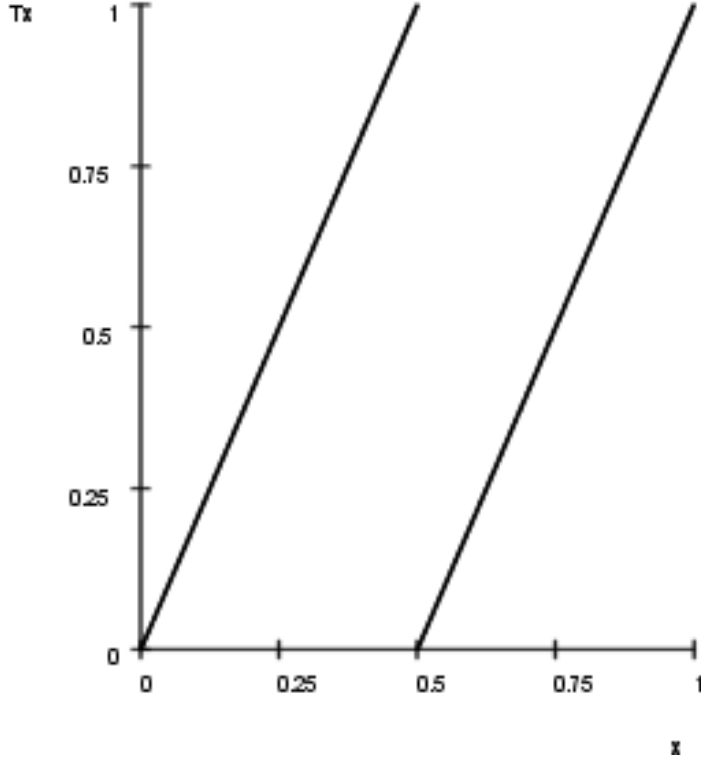


Figure 5: Doubling Map of the unit interval.

We will use a notion of the *Markov partition* without properly defining it (for a definition see, for example, Definition 18.7.1 in [44]). For our purposes Markov partition is a partition of the unit interval by closed subinterval with disjoint interiors having some "nice" properties with respect to the dynamics (intersection properties). We will mention these properties as we need them.

Fix $N \in \mathbb{N}$ and let $\mathcal{I}_N = \{I_{i,N}\}_{i=1}^{2^N}$ be the partition consisting of the pre-images of the elements of Markov partition $\{[0, 0.5], [0.5, 1]\}$ of $[0, 1]$ given by

$$I_{i,N} = \left[\frac{i-1}{2^N}, \frac{i}{2^N} \right], \quad i = 1 \dots 2^N.$$

Define a sequence of partitions $\mathcal{I}_N^k = \{I_{j,N+k}\}_{j=1}^{2^{N+k}}$ as k^{th} preimage of the partition \mathcal{I}_N , i.e. for each j , $j = 1 \dots 2^{N+k}$, there is i , $i = 1 \dots 2^N$, such that $\hat{T}^k I_{j,N+k} = I_{i,N}$. Observe that \mathcal{I}_N^k are Markov partitions themselves.

Consider now an open dynamical system defined by the map \hat{T} and the hole $I_{i,N}$,

$$T_{i,N} : [0, 1] \setminus I_{i,N} \rightarrow [0, 1]$$

as in section 1.2.2. For each i , $i = 1 \dots 2^N$, we have different open dynamical system with a corresponding hole $I_{i,N}$ (we refer to this hole as to a *Markov hole* because $I_{i,N}$ is an element of Markov partition). Define the Poincaré recurrence time of the hole, $\tau(I_{i,N})$, escape rate into the hole, $\rho(I_{i,N})$, and the set

$$\Omega_n(I_{i,N}) = \left\{ x \in [0, 1] : \exists j, 0 \leq j \leq n, \quad \hat{T}^j x \in I_{i,N} \right\}, \quad n \geq 0.$$

In Section 1.5.3 we will show that escape rate is well defined and then, in Section 1.5.5, that it depends not only on the size of the hole, but also on its position. More precisely, we prove the following statement.

Theorem (Main Theorem). *Let $I_{i,N}$ and $I_{j,N}$ be two Markov holes for the doubling map. Suppose that $\tau(I_{j,N}) > \tau(I_{i,N})$. Then,*

$$\rho(I_{j,N}) > \rho(I_{i,N}).$$

Moreover, for all $n \geq \tau(I_{i,N})$,

$$1 - \lambda(\Omega_n(I_{j,N})) < 1 - \lambda(\Omega_n(I_{i,N})).$$

In the section 1.5.6 we show that asymptotically as we decrease the size of the hole, escape rate is proportional to the size of the hole.

Theorem (Local Escape). *Let $x \in [0, 1]$ and $\{A_n(x)\}_{n=1}^\infty$ is a sequence of nested decreasing intervals with $x = \bigcap_{n=1}^\infty A_n(x)$ for all n . The following statements hold:*

a) *if x is a periodic point of period m then*

$$\lim_{n \rightarrow \infty} \frac{\rho(A_n(x))}{\lambda(A_n(x))} = 1 - \frac{1}{2^m};$$

b) if x is a non-periodic point and $x \neq s2^{-k}$, $s, k \in \mathbb{Z}^+$, then

$$\lim_{n \rightarrow \infty} \frac{\rho(A_n(x))}{\lambda(A_n(x))} = 1.$$

Moreover, for a sequence of shrinking Markov holes (Section 1.5.7) the corresponding sequence of escape rates is a monotone one.

Theorem (Monotonicity).

$$\max_{1 \leq i \leq 2^{N+1}} \rho(I_{i,N+1}) = \min_{1 \leq j \leq 2^N} \rho(I_{j,N}).$$

First, we state some known results about doubling map and then reformulate the problem in terms of the symbolic dynamics.

1.5.1 Preliminary results.

A point $x \in \hat{M}$ is a *periodic point* of \hat{T} of period $n > 0$ if $\hat{T}^n x = x$. If n is the smallest such integer then it is called the *minimal period* of x . Let $p_{i,N}(n)$ be a number of points of period (not necessary the minimal one) n in the hole $I_{i,N}$. Due to the fact that \mathcal{I}_N is a Markov partition for the doubling map, for each j , $j = 1 \dots 2^{N+k}$, $\hat{T}_{i,N}^k(I_{j,N+k}) = I_{s,N}$ for some s , $s = 1 \dots 2^N$ (see, for example Section 1.7 in [44]). Let $f_{i,N}^k$ be a number of elements of partition \mathcal{I}_N^k that do not enter the hole $I_{i,N}$ in the first k iterations, i.e.

$$f_{i,N}^k = \# \left\{ j : I_{j,N+k} \in \mathcal{I}_N^k; \lambda \left(\hat{T}^s(I_{j,N+k}) \cap I_{i,N} \right) = 0; s = 0, \dots, k \right\}.$$

The next two results show how to compute the Poincaré recurrence time and the escape rate for any Markov hole.

Proposition 1.5.1.1. *The Poincaré recurrence time of the hole $I_{i,N}$ is equal to the period of a periodic point contained in $I_{i,N}$ having the smallest period, i.e.*

$$\tau(I_{i,N}) = \min_{n \geq 1} \{n : p_{i,N}(n) > 0\}.$$

Proof. Suppose that a point $x \neq 0$ is a periodic point of the smallest period in the hole $I_{i,N}$. Let that period be equal to $p > 1$ (the case $p = 1$ is considered separately). All periodic points have the form $\frac{l}{2^k-1}$, $k \in \mathbb{N}$, $l \in \mathbb{N}$, $0 \leq l \leq 2^k - 1$. The endpoints of the elements of a Markov partition have the form $\frac{n}{2^s}$, $k \in \mathbb{N}$, $n \in \mathbb{N}$, $0 \leq n \leq 2^k - 1$. Therefore all periodic points except for zero and one are in the interior of the elements of the partition. Thus

$$\lambda\left(\hat{T}^p(I_{i,N}) \cap I_{i,N}\right) > 0,$$

and, therefore, $\tau(I_{i,N}) \leq p$.

To obtain the opposite inequality we need to use the Markov property of the partition. Suppose that $\lambda\left(\hat{T}^k(I_{i,N}) \cap I_{i,N}\right) > 0$ for some $k \leq p$. Since partition is Markov we have $I_{i,N} \subset \hat{T}^k(I_{i,N})$. Thus by the standard application of Intermediate Value Theorem to \hat{T}^k we conclude that $I_{i,N}$ contains a periodic point of period k . Therefore, $\tau(I_{i,N}) \geq p$. This finishes the proof for the case $x \neq 0$.

Now consider the case of a fixed point, $x = 0$ (the case of $x = 1$ can be treated similarly). We only need to consider one hole, $I_{0,N}$. Clearly,

$$\lambda\left(\hat{T}(I_{0,N}) \cap I_{0,N}\right) = \lambda(I_{0,N}) > 0,$$

and, therefore, $\tau(I_{0,N}) = 1$. □

Proposition 1.5.1.2.

$$\rho(I_{i,N}) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{f_{i,N}^n}{2^n},$$

if the limit exists.

Proof. Consider a Markov partition \mathcal{I}_N of a unit interval. There are $f_{i,N}^n$ elements of this partition that do not enter the hole $I_{i,N}$ in the first n iterations of T . The measure of each element is $\frac{1}{2^{n+N}}$. Thus

$$1 - \lambda(\Omega_n(I_{i,N})) = \frac{f_{i,N}^n}{2^{n+N}}.$$

Therefore,

$$\rho(I_{i,N}) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln (1 - \lambda(\Omega_n(I_{i,N}))) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{f_{i,N}^n}{2^n}.$$

□

We now describe the distribution of periodic points among different holes. But at first we look at the distribution of the periodic points in the whole interval. The following Proposition is a well known result(see, e.g. Proposition 1.7.2 in [44]).

Proposition 1.5.1.3. *The number, $p(k)$, of periodic points of period k (not necessary minimal) of the doubling map of the unit interval is equal to $2^k - 1$ and the distance between two neighboring periodic points of the same period is equal to $\frac{1}{2^k - 1}$.*

In other words, periodic points of the same period are distributed uniformly in the unit interval. Therefore, short intervals have few periodic points of small periods. In particular the following statements hold.

Corollary 1.5.1.4. *An interval of the size $\frac{1}{2^N}$, $N \geq 1$, contains at most one periodic point of a period $k \leq N$.*

Corollary 1.5.1.5. *Suppose x is a non-periodic point. Then for any positive integer n there exists $\delta(x, n) > 0$ such that δ -neighborhood of x does not contain any periodic points of periods less or equal n .*

Proof. Proposition 1.5.1.3 claims that doubling map has finitely many periodic points of periods less or equal n . Hence $\delta(x, n) = \min_{y \in Per_k, k \leq n} |x - y|$, where Per_k is the set of all periodic points of period k , is well defined. Moreover, the interval $(x - \delta(x, n), x + \delta(x, n))$ does not contain any periodic points of period less then or equal to n . Recall that all numbers are considered $\pmod 1$ and we identify zero and one. □

1.5.2 Symbolic dynamical system.

We can view all real numbers between 0 and 1 as binary numbers represented by one-sided infinite sequences of zeros and ones. In that case, the result of applying a doubling map to a number in the unit interval is a number whose binary representation is obtained from the original one by erasing the first symbol and leaving the rest unchanged. This allows us to introduce the symbolic dynamics for the map under study.

Let $\Omega(\kappa)$ be a finite alphabet (set of symbols) of size κ . A word w is a sequence of symbols from $\Omega(\kappa)$ of a finite or infinite length, $w = \{w_i\}_{i=1}^j, w_i \in \Omega(\kappa)$. Let $|w|$ denote the length of a word w . The set $\Lambda_{\Omega(\kappa)}^+$ consists of all one-sided infinite words, i.e. $\Lambda_{\Omega(\kappa)}^+ = \{\{w_i\}_{i=1}^\infty : w_i \in \Omega(\kappa)\}$.

For a word $w = \{w_i\}_{i=1}^k$ of a finite length $|w| = k$ and a positive integer n define a *cylinder set* in $\Lambda_{\Omega(\kappa)}^+$,

$$C_w(n) = \left\{ W \in \Lambda_{\Omega(\kappa)}^+ : W_{n+i-1} = w_i, i = 1 \dots |w| \right\} \subset \Lambda_{\Omega(\kappa)}^+.$$

Consider now a Bernoulli measure $\hat{\lambda}$ on the collection of cylinder sets and extend it to the σ -algebra generated by this collection (see, for example, Section 1.9 in [44]). In particular, the measure of the cylinder $C_w(1)$ is then given by (see, for example, Section 9.4 in [46])

$$\hat{\lambda}(C_w(1)) = \kappa^{-|w|}.$$

The shift map of $\Lambda_{\Omega(\kappa)}^+$ into itself is defined as $(\sigma(w))_i = w_{i+1}$, i.e. σ drops the first symbol and shifts the whole sequence to the left. The shift map preserves the Bernoulli measure. Then the triplet $(\Lambda_{\Omega(\kappa)}^+, \sigma, \hat{\lambda})$ together with the σ -algebra generated by the cylinder sets define a measurable dynamical system.

The doubling map is metrically equivalent to this one-sided shift on the space of one-sided infinite binary ($\kappa = 2$) sequences (see, for example [46] for details). A Markov hole (see Section 1.5) of the size 2^{-N} corresponds to a cylinder defined by a

word w of the size N and located at the first position, $C_w(1)$. For example, a hole of the size 2^{-3} in the third position, $I_{3,4} = [\frac{3}{2^3}, \frac{4}{2^3}]$, corresponds to a cylinder generated by the word $w = [010]$, $C_{[010]}(1)$. The periodic points for the doubling map correspond to infinite periodic words in the symbolic space. For example, a periodic point $x = \frac{2}{3}$ of period 2 of a doubling map corresponds to the infinite word $w = [1010\dots]$.

1.5.3 Escape rate.

Suppose that Markov hole $I_{i,N}$ corresponds to the cylinder $C_w(1)$, $|w| = N$. Then the set of points that do not enter the hole during the first n iterations of the doubling map corresponds to the set of points in $\Lambda_{\Omega(2)}^+$ that do not enter the $C_w(1)$ after applying a shift map n times, i.e. infinite words that do not contain the word w in the first $n + N$ positions. Let $c_w(n + N)$ be the number of such infinite words,

$$c_w(k) = \text{card} \left\{ v \in \Lambda_{\Omega(2)}^+ : \sigma^i(v) \notin C_w(1), i = 1 \dots k - |w| + 1 \right\}.$$

Since the doubling map are metrically conjugate, by Lemma 1.2.3.5 the escape rate into the hole $I_{i,N}$ equals to the escape rate for the shift map into a corresponding cylinder set $C_w(1)$:

$$\rho(I_{i,N}) = \rho(C_w(1)) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{c_w(n + N)}{2^{n+N}},$$

if the limit exists. The next lemma shows that this limit exists indeed.

Lemma 1.5.3.1. *The escape rate $\rho(C_w(1))$ is well defined and depends only on w .*

Moreover,

$$\rho(C_w(1)) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{c_w(n + N)}{2^n} = - \ln \frac{\theta_w}{2}, \quad (1.5.1)$$

where $\theta_w < 2$ is a constant depending on w .

Proof. It is known [40] that there exists a positive integer n_0 such that for all $n \geq n_0$

$$c_1 \theta_w^n \leq c_w(n) \leq c_2 \theta_w^n, \quad (1.5.2)$$

for some constants c_1 , c_2 , and $\theta_w < 2$ that depend only on w . Therefore,

$$\begin{aligned} \frac{n+N}{n} \ln \frac{\theta_w}{2} + \frac{1}{n} \ln c_1 &\leq \frac{1}{n} \ln \frac{c_w(n+N)}{2^n} \\ &\leq \frac{n+N}{n} \ln \frac{\theta_w}{2} + \frac{1}{n} \ln c_2. \end{aligned}$$

By letting n go to infinity we obtain that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{c_w(n+N)}{2^n} = \ln \frac{\theta_w}{2}.$$

□

Our next goal is to determine how does $c_w(n)$ depend on w .

1.5.4 Combinatorics on words.

As we have seen above, in order to compute escape rate we need to count the number of binary words of a fixed length that do not contain a certain subword, $c_w(k)$. We use some results from the theory of combinatorics on words to obtain this number.

In [40] Guibas and Odlyzko studied (introduced, according to Guibas and Odlyzko, by J. Conway) a function from the set of finite words to itself called an autocorrelation function, $corr(w)$. Let w be a binary word of the size k . Then the value $corr(w) = [b_1 \dots b_k]$ is determined in the following way. Place a copy of the word w under the original and shift it to the right by ℓ digits. If the overlapping parts match then $b_{\ell+1} = 1$. Otherwise we set $b_{\ell+1} = 0$. In particular, we always have $b_1 = 1$ (since the word matches itself).

Consider the following example. Suppose that $w = [10100101]$, then we have

l		b_{l+1}
0	1 0 1 0 0 1 0 1	1
1	1 0 1 0 0 1 0	0
2	1 0 1 0 0 1	0
3	1 0 1 0 0	0
4	1 0 1 0	0
5	1 0 1	1
6	1 0	0
7	1	1

We can now see that $corr(w) = [10000101]$.

Note that we can also view $corr(w)$ as a binary number and slightly abusing notations we denote both, a binary word and a binary number,

$$corr(w) = b_1 \cdot 2^{k-1} + b_2 \cdot 2^{k-2} + \dots + b_k.$$

In the example above we have

$$corr([10100101]) = 1 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0.$$

Similarly, we can define a correlation polynomial,

$$f_w(z) = b_1 \cdot z^{k-1} + b_2 \cdot z^{k-2} + \dots + b_k, \quad z \in \mathbb{C}, \quad (1.5.3)$$

so that $f_w(2) = corr(w)$.

Autocorrelation function, among other things, describes periodicities in the word. Consider a cylinder set $C_w(1)$ generated by the word w and suppose that $C_w(1)$ contains a periodic point $v = \{v_i\}_{i=1}^{+\infty} \in C_w(1)$ of period ℓ , $\ell < |w|$. Since v is a periodic point we have $v_i = v_{i+\ell}$, $\forall i \in \mathbb{N}$, that is

$$v = \underbrace{[v_1 \dots v_\ell \dots v_1 \dots v_\ell \dots]}_{|w|}.$$

Therefore, if $\text{corr}(w) = [b_1 \dots b_k]$ then $b_{\ell+1} = 1$. (The autocorrelation function was used in [45] for computing of the dynamical Zeta function of subshifts of finite type.)

In order to compute the number of words of the size n avoiding a given word w of a length k , $c_w(n)$, consider a generating function for $c_w(n)$,

$$F_w(z) = \sum_{n=0}^{\infty} c_w(n) z^{-n}.$$

It was shown in [40] that $F_w(z)$ is a rational function and the following asymptotic estimates on $c_w(n)$ were obtained.

Lemma 1.5.4.1. *Suppose that w and u are two words of the same length and $\text{corr}(w) > \text{corr}(u)$. Then,*

$$\lim_{n \rightarrow \infty} \frac{\ln c_w(n)}{n} > \lim_{n \rightarrow \infty} \frac{\ln c_u(n)}{n}.$$

The following non-asymptotic result relating the number of sequences that do not contain certain word to the autocorrelation function of that word was proved in [35].

Lemma 1.5.4.2. *Suppose that w and u are two words of the same length and $\text{corr}(w) > \text{corr}(u)$. Then there exists \tilde{n}_0 , such that for all $n \geq \tilde{n}_0$ one has that*

$$c_w(n) > c_u(n). \tag{1.5.4}$$

Then in [24] and [49] this result was improved by finding an explicit value of \tilde{n}_0 and expanding the result to the words of different lengths and to the systems with the alphabet of any finite size. Specifically, consider binary words of equal length, w and u . Suppose that $\text{corr}(w) = [b_1 \dots b_N]$ and $\text{corr}(u) = [a_1 \dots a_N]$. Then,

$$\tilde{n}_0 = N + \min\{i : b_i \neq a_i\} - 1. \tag{1.5.5}$$

1.5.5 Main result.

Lemma 1.5.4.2 leads to the following relationship between the correlation function on one hand and the escape rate into and survival probability of the cylinder generated by the corresponding word on the other hand.

Lemma 1.5.5.1. *Suppose that w and u are two words of the same length. Let $C_w(1)$ and $C_u(1)$ be two cylinder sets generated by these two words. Then*

$$\text{corr}(w) > \text{corr}(u) \Rightarrow \rho(C_w(1)) < \rho(C_u(1)).$$

Proof. By Lemma 1.5.3.1

$$\begin{aligned} \rho(C_u(1)) &= - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{c_u(n+N)}{2^n}, \\ \rho(C_w(1)) &= - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{c_w(n+N)}{2^n}, \end{aligned}$$

Using the results of Lemma 1.5.4.1 we complete the proof. \square

As before, the number $1 - \lambda(\Omega_n(C_w(1)))$ is called a survival probability of the set $C_w(1)$.

Lemma 1.5.5.2. *Suppose that w and u are two words of the same length with $\text{corr}(w) > \text{corr}(u)$, Let $\text{corr}(w) = [b_1 \dots b_N]$ and $\text{corr}(u) = [a_1 \dots a_N]$. Let $C_w(1)$ and $C_u(1)$ be two cylinder sets generated by these two words. Then for all $n \geq \min\{i : b_i \neq a_i\} - 1$,*

$$1 - \lambda(\Omega_n(C_w(1))) > 1 - \lambda(\Omega_n(C_u(1))).$$

Proof. In view of 1.5.4 there exists $\tilde{n}_0 > 0$ such that $c_w(n+N) > c_u(n+N)$ for all $n+N \geq \tilde{n}_0$. Therefore,

$$\frac{c_w(n+N)}{2^{n+N}} > \frac{c_u(n+N)}{2^{n+N}}, \quad n+N \geq \tilde{n}_0.$$

But,

$$1 - \lambda(\Omega_n(C_w(1))) = \frac{c_w(n+N)}{2^{n+N}}, \quad 1 - \lambda(\Omega_n(C_u(1))) = \frac{c_u(n+N)}{2^{n+N}}.$$

It follows from Equation 1.5.5 that we must have $n+N \geq N + \min\{i : b_i \neq a_i\} - 1$.

Therefore,

$$n \geq \min\{i : b_i \neq a_i\} - 1.$$

\square

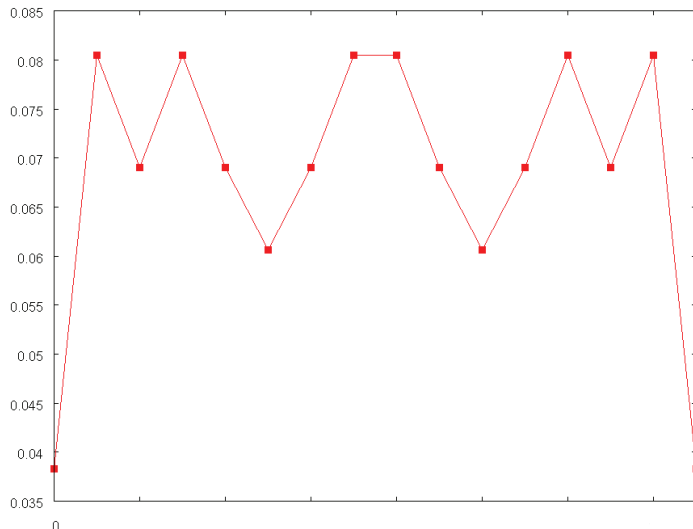


Figure 6: Escape rate vs. position of the Markov hole of size 2^{-N} , $N = 4$.

Finally, we turn to the proof of the main theorems of this section, which state that the escape rate is larger for the hole that has a larger Poincaré recurrence time (asymptotic result). Moreover, the survival probability is smaller for the hole that has a larger Poincaré recurrence time (non-asymptotic result).

Theorem 1.5.5.3. *Let $I_{i,N}$ and $I_{j,N}$ be two Markov holes for the doubling map of a unit interval. Suppose that $\tau(I_{j,N}) > \tau(I_{i,N})$. Then,*

$$\rho(I_{j,N}) > \rho(I_{i,N}).$$

Moreover, for all $n \geq \tau(I_{i,N})$,

$$1 - \lambda(\Omega_n(I_{j,N})) < 1 - \lambda(\Omega_n(I_{i,N})).$$

Proof. Let w , $|w| = N$, be the word that codes the hole $I_{i,N}$ and let $C_w(1)$ be the cylinder generated by that word. Similarly, let u , $|u| = N$, be the word that codes the hole $I_{j,N}$ and let $C_u(1)$ be the cylinder generated by that word. Consider the autocorrelation functions of w and u , $\text{corr}(w) = [b_1 \dots b_N]$ and $\text{corr}(u) = [a_1 \dots a_N]$. As we have seen before, $b_1 = a_1 = 1$. Also, $b_{\ell+1} = 1$ if and only if $C_w(1)$ contains a periodic point of period ℓ , $1 \leq \ell < N$. The same is true for $a_{\ell+1}$. Hence, the first

non-zero element of $[b_1 \dots b_N]$ after b_1 is $b_{\tau(I_{i,N})+1}$. Thus, if $\tau(I_{j,N}) > \tau(I_{i,N})$ then $\text{corr}(u) < \text{corr}(w)$. But by Lemma 1.5.5.1

$$\rho(I_{j,N}) = \rho(C_{u(1)}) > \rho(C_{w(1)}) = \rho(I_{i,N}).$$

In order to compare surviving probabilities we use Lemma 1.5.5.2. In addition, the argument in the preceding paragraph shows that $\min\{i : b_i \neq a_i\} = \tau(I_{i,N}) + 1$. Since $\lambda(\Omega_n(I_{j,N})) = \lambda(\Omega_n(C_u(1)))$ and $\lambda(\Omega_n(I_{i,N})) = \lambda(\Omega_n(C_w(1)))$, we obtain that

$$1 - \lambda(\Omega_n(I_{j,N})) < 1 - \lambda(\Omega_n(I_{i,N})),$$

for all $n \geq \tau(I_{i,N})$. □

1.5.6 Local escape rate.

Our next task is to investigate what happens to the escape rate as the size of the hole is decreasing. Let $I_N(x)$ be an element of a partition \mathcal{I}_N (see Section 1.5.1) that contains a point x . If there are two elements that contain x (when x is an end point of the element of Markov partition) then pick any.

It is known [3] that for a large class of symbolic systems, which includes the expanding maps of the interval considered here, for almost every point Poincaré recurrence time grows linearly with N as a size of the hole exponentially decreases, $\lim_{N \rightarrow \infty} \frac{\tau(I_N(x))}{N} = 1$.

We want to obtain a similar result for the escape rate. The following theorem answers this question. In particular, it says that the escape rate decreases linearly with respect to the size of the hole.

Theorem 1.5.6.1. *Consider the doubling map of the unit interval. Suppose that $x \in [0, 1]$. Then the following statements are true:*

a) *if x is a periodic point of period m then*

$$\lim_{N \rightarrow \infty} \frac{\rho(I_N(x))}{\lambda(I_N(x))} = 1 - \frac{1}{2^m};$$

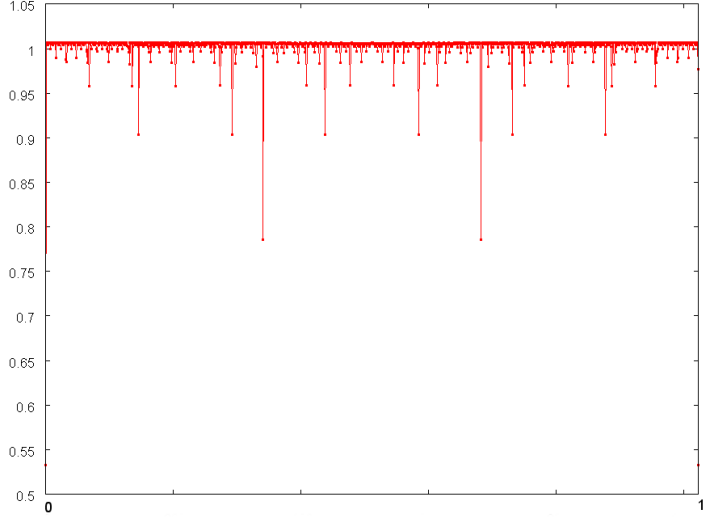


Figure 7: Plot of $\frac{\rho(I_N(x))}{\lambda(I_N(x))}$ vs x , $N = 10$.

b) if x is a non-periodic point then

$$\lim_{N \rightarrow \infty} \frac{\rho(I_N(x))}{\lambda(I_N(x))} = 1.$$

Proof. Using equality 1.5.1 and the fact that $\lambda(I_N(x)) = 2^{-N}$, we obtain,

$$\lim_{N \rightarrow \infty} \frac{\rho(I_N(x))}{\lambda(I_N(x))} = - \lim_{N \rightarrow \infty} 2^N \ln \frac{\theta_w}{2},$$

where θ_w depends on N .

Recall that for any word w of a length N we can define a correlation polynomial 1.5.3 as

$$f_w(z) = \sum_{i=1}^N b_i z^{N-i},$$

where $\text{corr}(w) = [b_1, \dots, b_N]$, b_i depends on N , $b_i = b_i(N)$, and $z \in \mathbb{C}$. Clearly, $b_1 = 1$, thus $\text{deg} f_w(z) = N - 1$.

It was shown in [38] that asymptotically the constant θ_w satisfies

$$\ln \theta_w = \ln 2 - \frac{1}{2f_w(2)} + \mathcal{O}(2^{-2N}).$$

Then,

$$\ln \theta_w = \ln 2 - \frac{1}{2^N + \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i}} + \mathcal{O}(2^{-2N}).$$

Suppose now that x is non-periodic point. Then one has that,

$$\lim_{N \rightarrow \infty} 2^{-N} \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i} = 0.$$

Indeed,

$$2^{-N} \sum_{i=\tau(I_N(x))}^N b_i 2^{N-i} = \sum_{i=\tau(I_N(x))+1}^N b_i(N) 2^{-i} \leq 1.$$

But, $\lim_{N \rightarrow \infty} b_i(N) = 0$ for all $i \geq \tau(I_N(x))$.

Therefore,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\rho(I_N(x))}{\lambda(I_N(x))} &= \lim_{N \rightarrow \infty} \frac{2^N}{2^N + \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i}} \\ &= \lim_{N \rightarrow \infty} \frac{1}{1 + 2^{-N} \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i}} = 1. \end{aligned}$$

Suppose now that x is a periodic point of (minimum) period m . Then,

$$\begin{aligned} \lim_{N \rightarrow \infty} 2^{-N} \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i} \\ = \sum_{k=1}^{\infty} 2^{-km} = \frac{1}{1 - 2^{-m}} - 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\rho(I_N(x))}{\lambda(I_N(x))} &= \lim_{N \rightarrow \infty} \frac{2^N}{2^N + \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i}} \\ &= \frac{1}{1 + \lim_{N \rightarrow \infty} \left(2^{-N} \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i} \right)} \\ &= 1 - \frac{1}{2^m}. \end{aligned}$$

This completes the proof. □

Remark 1.5.6.2. *All non-periodic points form a set of full measure. Thus for almost every point in the interval $[0, 1]$ the escape rate into the hole "centered" at a given point asymptotically, as the size of the hole goes to zero, equals to the measure of the hole.*

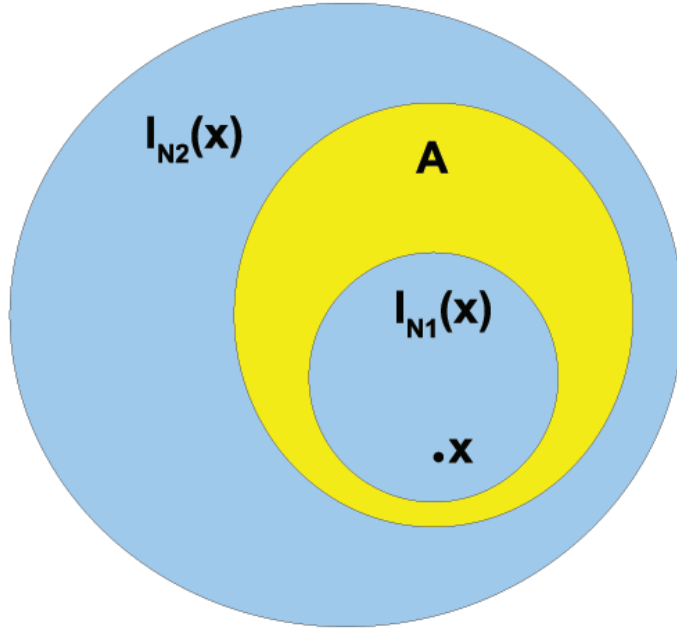


Figure 8: Sandwiching a hole between elements of two different Markov partitions.

Suppose that an interval A does not contain points $x = s2^{-k}$ for all $k \leq n$ for some $n > 1$, $s, k, n \in \mathbb{N}$, i.e. A does not contain an end point of an element of any Markov partition \mathcal{I}_N , $N \leq n$. Then one can find two elements of the Markov partitions $I_{N_1}(x)$ and $I_{N_2}(x)$ so that

$$I_{N_1}(x) \subseteq A \subseteq I_{N_2}(x).$$

Thus, the following result holds for the arbitrary decreasing nested sequence of intervals.

Corollary 1.5.6.3. *Consider the doubling map of a unit interval. Let $x \in [0, 1]$ and suppose $\{A_n(x)\}_{n=1}^{\infty}$ is a sequence of nested decreasing intervals with $x = \bigcap_{n=1}^{\infty} A_n$ for all n . The following statements are true:*

a) *if x is a periodic point of period m , then*

$$\lim_{n \rightarrow \infty} \frac{\rho(A_n(x))}{\lambda(A_n(x))} = 1 - \frac{1}{2^m};$$

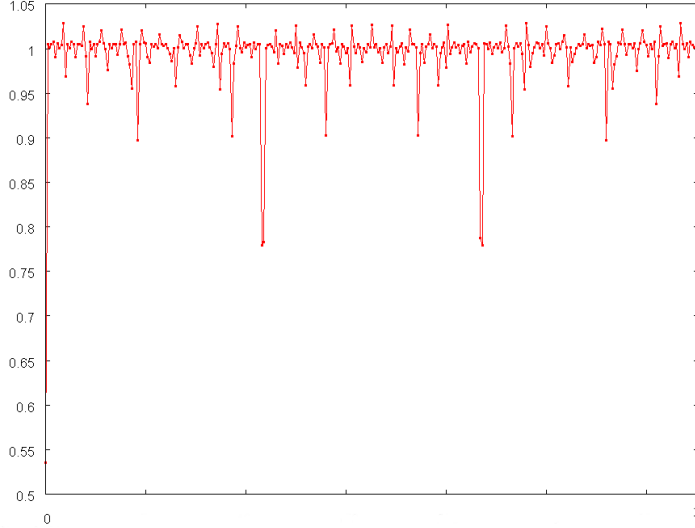


Figure 9: Plot of $\frac{\rho(A(x))}{\lambda(A(x))}$ vs. x , $\lambda(A(x)) = \frac{1}{327}$.

b) if x is a non-periodic point and $x \neq s2^{-k}$, $s, k \in \mathbb{Z}^+$, then

$$\lim_{n \rightarrow \infty} \frac{\rho(A_n(x))}{\lambda(A_n(x))} = 1.$$

Remark 1.5.6.4. Observe that in Fig. 9 there are many local maxima of $\frac{\rho(A(x))}{\lambda(A(x))}$ which are essentially larger than one. These happen at the places where the intervals $A(x)$ have maximum Poincaré return time among all such intervals $A(x)$ with length of $A(x)$ being fixed. Indeed, consider a set of holes that are Markov, say \mathcal{I}_N . The ratio $\frac{\rho(I_{i,N})}{\lambda(I_{i,N})}$ attains its maximum (see Corollary 1.5.7.1 below) in the holes that have the largest Poincaré return time (in this case N). Now, suppose that $\lambda(I_{i,N+1}) \leq \lambda(A(x)) \leq \lambda(I_{i,N})$, $A(x) \subset I_{i,N}$, and $\tau(A(x)) = \tau(I_{i,N})$. Then $\rho(A(x)) \approx \rho(I_{i,N})$, but

$$\frac{\rho(A(x))}{\lambda(A(x))} \geq \frac{\rho(I_{i,N})}{\lambda(I_{i,N})}.$$

Thus, by shrinking a Markov hole while keeping the same Poincaré return time we can increase the ratio of the escape rate to the size of the hole.

1.5.7 More results for one hole.

In the case of the doubling map with Markov holes we know precisely where to make a hole to achieve maximum (or minimum) escape rate.

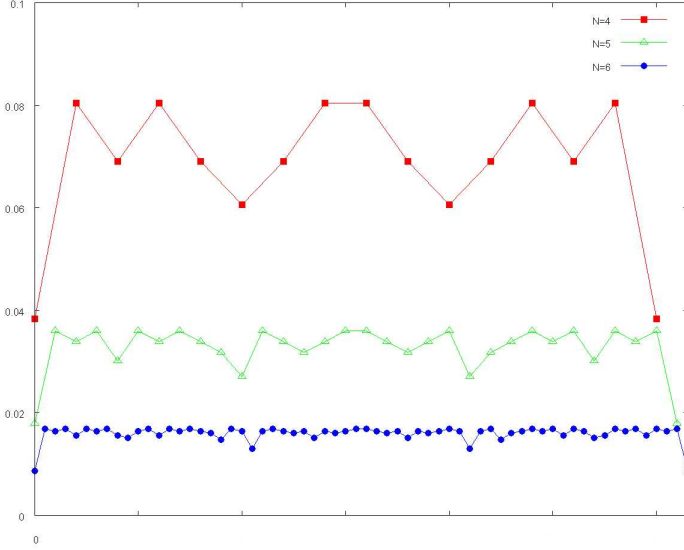


Figure 10: Escape rate vs. position of the Markov hole of size 2^{-N} .

Corollary 1.5.7.1.

$$\min_{1 \leq i \leq 2^N} \rho(I_{i,N}) = \rho(I_{1,N}), \quad \max_{1 \leq i \leq 2^N} \rho(I_{i,N}) = \rho(I_{2,N}),$$

although the holes that give these extremes are not unique.

Proof. Clearly, by Proposition 1.5.1.4, one has that $1 \leq \rho(I_{i,N}) \leq N$. Then, it is easy to check that $\rho(I_{1,N}) = 1$ and $\rho(I_{2,N}) = N$. Thus the result follows from the Main theorem. \square

Note that we obtained minimum escape rate in one more interval that contains a fixed point, namely $I_{2^N,N}$. The maximum is obtained in the intervals that have a minimum period equal to N , as Figure 6 illustrates this.

Next theorem states that the escape rate decreases monotonically as we decrease the size of the hole.

Theorem 1.5.7.2.

$$\max_{1 \leq i \leq 2^{N+1}} \rho(I_{i,N+1}) = \min_{1 \leq i \leq 2^N} \rho(I_{i,N}).$$

Proof. It follows from Corollary 1.5.7.1 that

$$\max_{1 \leq i \leq 2^{N+1}} \rho(I_{i,N+1}) = \rho(I_{2,N+1}), \quad \min_{1 \leq i \leq 2^N} \rho(I_{i,N}) = \rho(I_{1,N}).$$

Let w_1 and w_2 be two binary words that define holes $I_{2,N+1}$ and $I_{1,N}$, respectively, $|w_1| = N + 1$ and $|w_2| = N$.

We now proceed in the following fashion. First, we compute the generating functions $F_{w_1}(z) = \sum_{j=0}^{\infty} c_{w_1}(j)z^{-j}$ and $F_{w_2}(z) = \sum_{j=0}^{\infty} c_{w_2}(j)z^{-j}$ defined in Section 1.5.4. Next, we show that $c_{w_1}(n) \sim c_{w_2}(n)$ for $n \gg 1$. Finally, using Lemma 1.5.3.1 we conclude that $\rho(I_{2,N+1}) = \rho(I_{1,N})$.

The explicit analytic expression for the generating function was found in [40],

$$F_w(z) = \frac{z \cdot f_w(z)}{1 + (z - 2) \cdot f_w(z)},$$

where $f_w(z)$, as before, is a correlation polynomial of w . It is easy to check that

$$w_1 = \underbrace{0 \dots 0}_N 1, \quad w_2 = \underbrace{0 \dots 0}_N$$

and thus,

$$\text{corr}(w_1) = [1 \underbrace{0 \dots 0}_N], \quad \text{corr}(w_2) = [\underbrace{1 \dots 1}_N].$$

Therefore,

$$f(w_1) = z^N, \quad f(w_2) = \sum_{j=0}^{N-1} z^j.$$

After some tedious but straightforward algebra we arrive at

$$F_{w_1}(z) = \frac{1}{1 - (2z^{-1} - z^{-(N+1)})},$$

$$F_{w_2}(z) = \frac{1}{1 - (2z^{-1} - z^{-(N+1)})} - \frac{z^{-N}}{1 - (2z^{-1} - z^{-(N+1)})}.$$

Let $t = z^{-1}$ and expand the above equations into power series. Then we get

$$F_{w_1}(z) = \sum_{j=0}^{\infty} (2t - t^{N+1})^j,$$

$$F_{w_2}(z) = \sum_{j=0}^{\infty} (2t - t^{N+1})^j - t^N \cdot \sum_{j=0}^{\infty} (2t - t^{N+1})^j.$$

Therefore, $\sum_{j=0}^{\infty} (2t - t^{N+1})^j = \sum_{j=0}^{\infty} c_{w_1}(j)t^j$. Then,

$$F_{w_1}(z) = \sum_{j=0}^{\infty} c_{w_1}(j)t^j,$$

$$F_{w_2}(z) = \sum_{j=0}^{N-1} c_{w_1}(j)t^j + \sum_{j=N}^{\infty} (c_{w_1}(j) - c_{w_1}(j-N))t^j.$$

Thus, for $j > N$ we obtain the following relationship:

$$c_{w_2}(j) = c_{w_1}(j) - c_{w_1}(j-N).$$

The equation 1.5.2 implies that for $j \gg 1$,

$$C_1\theta_{w_1}^j \leq c_{w_1}(j) \leq C_2\theta_{w_1}^j,$$

for some constant C_1 and C_2 . Thus,

$$\theta_{w_1}^j (C_1 - C_2\theta_{w_1}^{-N}) \leq c_{w_1}(j) - c_{w_1}(j-N) \leq \theta_{w_1}^j (C_2 - C_1\theta_{w_1}^{-N})$$

Hence, for $j \gg 1$ we have that $c_{w_2}(j)$ asymptotically behaves as $\theta_{w_1}^j$, $c_{w_2}(j) \sim \theta_{w_1}^j$.

This finishes the proof. \square

The next theorem deals with arbitrary (not necessarily the elements of Markov partition), but sufficiently small holes.

Theorem 1.5.7.3. *Suppose x_1 and x_2 are two periodic points with periods m_1 and m_2 , respectively, $m_1 < m_2$. Then there exists a positive number $\varepsilon_0 = \varepsilon_0(m_1, m_2)$ such that for all subintervals A_1 and A_2 of the unit interval with*

$$x_i \in \text{int}(A_i), \quad i = 1, 2, \quad \lambda(A_1) = \lambda(A_2) < \varepsilon_0,$$

one has

$$\rho(A_1) < \rho(A_2).$$

Proof. The result follows directly from Corollary 1.5.6.3. \square

1.5.8 Contributions to the escape rate: the size of the hole vs. dynamics.

The doubling map shows that the escape rate is not determined by the size of the hole alone. The following examples demonstrate the possibility of a larger escape rate into the smaller hole.

Consider two sets $A = [0, \frac{1}{4})$ and $B = [\frac{1}{4}, \frac{1}{2})$. Then $\lambda(A) = \lambda(B) = \frac{1}{4}$. It is easy to check that $\tau(A) = 1$ and $\tau(B) = 2$. Thus it follows from Theorem 1.5.5.3 that $\rho(A) < \rho(B)$.

On the other hand by Proposition 1.2.3.2 it follows that $\rho(\hat{T}^{-1}A \cup A) = \rho(A)$. Thus, for two holes,

$$\hat{T}^{-1}A \cup A = \left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{5}{8}\right] \text{ and } B = \left[\frac{1}{4}, \frac{1}{2}\right],$$

one has

$$\rho\left(\left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{5}{8}\right]\right) < \rho\left(\left[\frac{1}{4}, \frac{1}{2}\right]\right), \quad \lambda\left(\left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{5}{8}\right]\right) > \lambda\left(\left[\frac{1}{4}, \frac{1}{2}\right]\right).$$

The next example shows that even if we have two holes which are connected sets, it is still possible to have a faster escape through a smaller one. Consider two holes $A \cup B$ and C , where $A = [0, \frac{1}{4}]$, $B = [\frac{1}{4}, \frac{5}{16}]$, and $C = [\frac{1}{2}, \frac{3}{4}]$. Note that $B \subset \hat{T}^{-2}A$. Clearly, $\lambda(A \cup B) = \frac{5}{16} > \lambda(C) = \frac{4}{16}$. It is easy to check that $\tau(A) = 1$ and $\tau(C) = 3$, thus $\rho(A) < \rho(C)$. On the other hand, by Proposition 1.2.3.2 it follows that $\rho(A) = \rho(A \cup B)$. Therefore we have that $\rho(A \cup B) < \rho(C)$. Hence,

$$\rho\left(\left[0, \frac{5}{16}\right]\right) < \rho\left(\left[\frac{1}{2}, \frac{3}{4}\right]\right), \quad \lambda\left(\left[0, \frac{5}{16}\right]\right) > \lambda\left(\left[\frac{1}{2}, \frac{3}{4}\right]\right).$$

In general, the following result shows that there are holes of the arbitrarily large size with arbitrarily small escape rate.

Theorem 1.5.8.1. *For any $\varepsilon \in (0, 1)$ and any $r > 0$ there exists a measurable set $A \subset [0, 1]$ such that*

$$\lambda(A) > 1 - \varepsilon, \quad \rho(A) < r.$$

Proof. We can always pick N such that $\rho(I_{i,N}) < r$ for some i . One has that

$$\lambda \left(\bigcup_{j=0}^{\infty} \hat{T}^{-j} I_{i,N} \right) = 1,$$

but

$$\lambda \left(\bigcup_{j=0}^n \hat{T}^{-j} I_{i,N} \right) < 1$$

for any finite n . Thus for any $\varepsilon > 0$ we can find n_0 such that

$$1 > \lambda \left(\bigcup_{j=0}^{n_0} \hat{T}^{-j} I_{i,N} \right) > 1 - \varepsilon.$$

By Proposition 1.2.3.2 one has that $\rho(I_{i,N}) = \rho(\bigcup_{j=0}^{n_0} \hat{T}^{-j} I_{i,N})$. Now, set

$$A = \bigcup_{j=0}^{n_0} \hat{T}^{-j} I_{i,N}.$$

□

1.6 Some generalizations.

Lemma 1.2.3.5 allows us to extend the results of the previous sections to a wider class of maps.

1.6.1 Linear expanding map.

As was mentioned at the beginning of this section all of the proofs go through if we replace the binary alphabet by the alphabet of size κ . Then all of the results in the preceding sections hold for $x \mapsto \kappa x \pmod{1}$, $\kappa \in \mathbb{N}$ and $\kappa > 1$.

1.6.2 Tent map.

The tent map is a map \hat{T} of a unit interval to itself given by

$$\hat{T}(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \leq x \leq 1 \end{cases}.$$

This map preserves a Lebesgue measure on $[0, 1]$. We can use the following correspondence between the tent map and the symbolic dynamics: $s_n = 0$ if $\hat{T}^n x < 0.5$

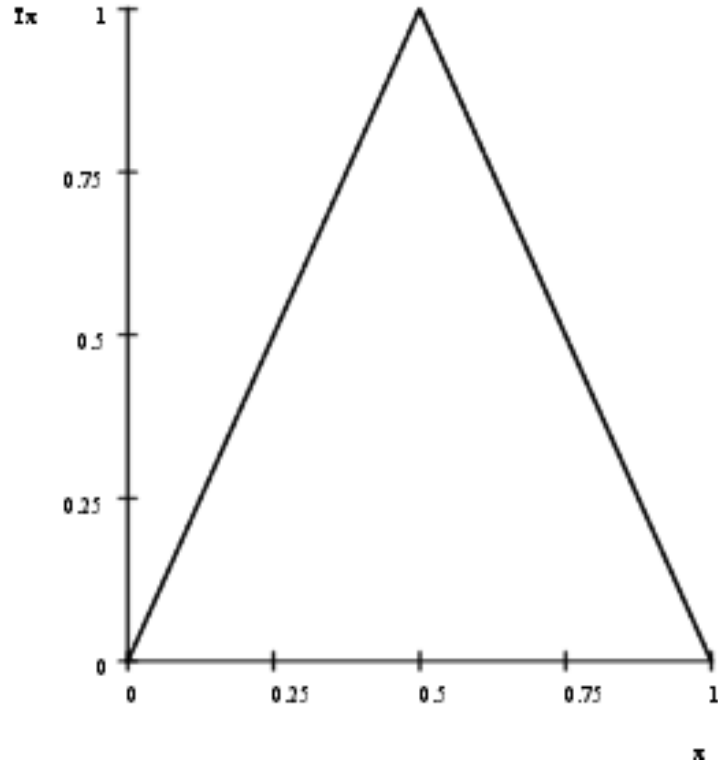


Figure 11: Tent Map.

and $s_n = 1$ otherwise. It can be easily shown that mapping $x \mapsto \{s_n\}$ is a metric conjugacy onto the left shift symbolic space. Then one can repeat all the arguments that we used for the doubling map to obtain similar result. Namely, for $I_{i,N}$ defined as before,

$$I_{i,N} = \left[\frac{i-1}{2^N}, \frac{i}{2^N} \right], \quad i = 1 \dots 2^N,$$

the following statement holds.

Theorem 1.6.2.1. *Consider a tent map. $\forall N \in \mathbb{N}$ and $\forall i, j = 1, \dots, 2^N$ we have that if $\tau(I_{j,N}) > \tau(I_{i,N})$ then $\rho(I_{j,N}) > \rho(I_{i,N})$.*

1.6.3 Logistic map.

Consider a logistic map $\hat{T} : [0, 1] \rightarrow [0, 1]$ given by

$$\hat{T}(x) = 4x(1-x), \quad x \in [0, 1].$$

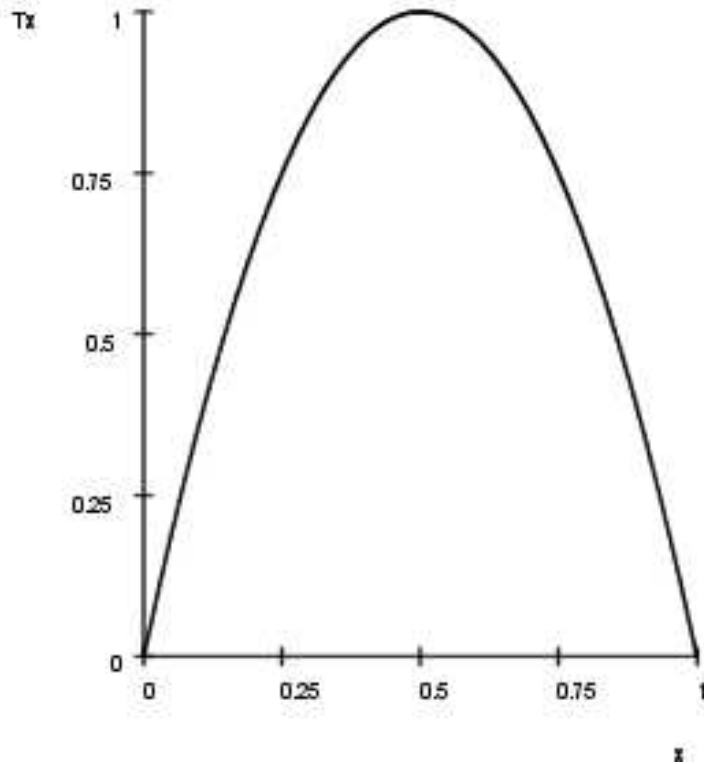


Figure 12: Logistic map.

It preserves a measure, λ , (absolutely continuous with respect to Lebesgue measure) with the density $\frac{1}{\pi\sqrt{x(1-x)}}$.

A logistic map and the tent map are metrically conjugate: it is easy to check that conjugacy is given by the transformation $y = \sin^2 \frac{\pi x}{2}$.

Thus, by Lemma 1.2.3.5, for $I_{i,N}$ (preimages of a finite Markov partition for \hat{T}) defined as

$$I_{i,N} = \left[\frac{2}{\pi} \arcsin \sqrt{\frac{i-1}{2^N}}, \frac{2}{\pi} \arcsin \sqrt{\frac{i}{2^N}} \right], \quad i = 1 \dots 2^N,$$

the following theorem holds.

Theorem 1.6.3.1. *Consider a logistic map. Suppose that $\tau(I_{j,N}) > \tau(I_{i,N})$. Then, $\rho(I_{j,N}) > \rho(I_{i,N})$.*

1.6.4 Baker's map.

A Baker's map is an example of two dimensional invertible hyperbolic chaotic map.

This map, $\hat{T} : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$, is defined in the following way:

$$\hat{T}(x, y) = \left(2x \bmod 1, \frac{1}{2}(y + \lfloor 2x \rfloor) \bmod 1 \right),$$

where $\lfloor x \rfloor$ is an integer part of x . It preserves the Lebesgue measure and is mixing (and, therefore, ergodic).

It was observed numerically in [55] that escape rate in this system depends on the position of the hole. The Markov holes for this map are the rectangles given by

$$I_{i,j,N,M} = \left[\frac{i-1}{2^N}, \frac{i}{2^N} \right] \times \left[\frac{j-1}{2^M}, \frac{j}{2^M} \right], \quad i = 1 \dots 2^N, \quad j = 1 \dots 2^M.$$

It is well known that Baker's map is conjugate to the full binary shift. Thus the following statement holds.

Theorem 1.6.4.1. *Consider the Baker's map \hat{T} . Then $\forall N, M \in \mathbb{N}, \forall i_1, i_2 = 1, \dots, 2^N$, and $\forall j_1, j_2 = 1, \dots, 2^M$ we have that if $\tau(I_{i_1, j_1, N, M}) > \tau(I_{i_2, j_2, N, M})$ then $\rho(I_{i_1, j_1, N, M}) > \rho(I_{i_2, j_2, N, M})$.*

1.7 Future directions.

There are several directions for the future research and there are still many unanswered questions.

We have shown that dynamical factors determine escape rate. What other factors can influence escape rate? In order to investigate that, consider a map of a unit interval to itself given by

$$f(x) = \frac{x}{p}, x \in [0, p]; \quad f(x) = \frac{1-x}{1-p}, x \in (p, 1),$$

where $p \in (0, 1)$. Note that the case when $p = 0.5$ is the one studied extensively in this work. The system has a Markov partition. Numerical evidence suggests that if

we consider Markov holes of the same size, then the escape rate is still determined by the periodic orbit structure.

Question 1.7.0.2. *How do other factors, besides the size of the hole and the structure of periodic orbits in the hole, affect the escape rate? Specifically, how does the local distortion enter into the calculations of the escape rate?*

In this work we obtained asymptotic results as the size of the hole (not necessarily Markov) tends to zero. In order to study escape rate through arbitrary holes of any size, first we need to understand how to compute the escape rate through several Markov holes. Equivalently, we can look at the subshifts of finite type.

Question 1.7.0.3. *What can we say about the escape rate for the subshifts of finite type?*

So how do we compare escape rate for non-Markov holes? In many cases we believe that small-size limits will follow from the results for Markov holes. Our goal is to be able to say something about large holes. Any hole can be approximated arbitrary well by a finite union of Markov holes.

Question 1.7.0.4. *What can we say about the escape rate through non-Markov holes for the strongly chaotic maps?*

In our study of the escape in the symbolic systems we came across a combinatorial function called correlation function on words. In [45] author used this function in connection to the dynamical zeta function.

Question 1.7.0.5. *What are the other application of correlation function in the theory of dynamical systems?*

Question 1.7.0.6. *What can we say about zeta function and entropy of the system using the information about the escape rate?*

Many results in the theory of dynamical systems are asymptotic in nature. A very unusual nature of our results for the doubling map is finite time phenomena.

Question 1.7.0.7. *What are the implications of the finite time phenomena of our results?*

Escape rate can be generalized to a larger class of functions on the underlying dynamical systems.

Question 1.7.0.8. *What is the proper class of functions (which include the escape rate) that have properties similar to what we obtained in this paper?*

We have seen several examples of the systems where escape rate is independent of the position of the hole.

Question 1.7.0.9. *What is the proper class of dynamical systems in which the escape rate does not depend on the position of the hole? Conversely, suppose that the escape rate does not depend on the position of the hole. What kind of information can we learn from that about the dynamical system (topological, probabilistic, algebraic etc properties)?*

Basically, we are trying to learn about the system (its dynamics, properties of the phase space, etc) through a "hole", i.e. by computing certain observable averages (escape rate). This kind of analysis could be very useful in the industrial applications.

1.8 Conclusions.

Apparently there are still other natural and interesting unexplored questions on the dynamics of open dynamical systems. In this chapter we dealt with one such problem, the dependence of the escape rate on the position of a hole. We demonstrated that dynamical characteristics can play as important a role for the escape rate as the size (measure) of the hole does.

Here we just scratch the surface of this new area in the theory of open dynamical systems dealing with the uniformly hyperbolic systems. In general, the effect of distortion (variability of values of the derivative or Jacobian of the dynamical system) can also essentially contribute to the escape through a hole of a given size. This problem will be addressed in the future research.

CHAPTER II

DETERMINISTIC WALKS IN RANDOM ENVIRONMENTS.

2.1 *Introduction.*

Deterministic walks in random environments (DWRE) are discrete in time dynamical systems on any graph \mathcal{G} which describe a motion of an object (in what follows we call it a particle) that jumps between vertices of \mathcal{G} . We assume that graph \mathcal{G} is a lattice, i.e. a non-directed graph, and edges of \mathcal{G} are line segments of the length one. The particle moves between the neighboring vertices of the graph on these straight segments. By the *structure of an orbit* we mean a sequence of such line segments. If the particle occupies a vertex g then a choice of the next vertex \tilde{g} is completely determined by a *scatterer* (local scattering rule) which currently occupies the vertex g and by *edge* along which the particle came to g .

There is an initially prepared deterministic program at each vertex $g \in \mathcal{G}$, which is formed by fixing from the very beginning scatterers that can occupy this vertex and the rule that determines how the scatterers can change (if at all). The dynamics of DWRE is defined as following: when the object comes to a vertex g it gets scattered by the scatterer located at that vertex. The collection of all scatterers in all vertices of \mathcal{G} form an (instant) *environment* in which the particle moves. Two types of environments are possible: fixed environment and evolving environment. In the *fixed environment* every time the particle arrives at a vertex g it encounters one and the same scatterer. On the other hand, in the *evolving environment* the type of the scatterer at the vertex g can change upon visits by the particle to the vertex and/or with time. There are two possibilities: either the particle first gets scattered then

changes the scatterer or changes the scatterer and then gets scattered by the new scatterer. The dynamics is similar in both cases so we use the first option.

One can contrast this dynamics with random walks where a moving object at each vertex performs a random trial ("throws a dice") to choose an edge along which to continue the motion. In DWRE the evolution of the environment is deterministic, hence the choice of the next vertex is deterministic, not random. The randomness comes into play when we set up the model: at the time $t = 0$ the scatterers are assumed to be randomly distributed over vertices of the graph \mathcal{G} .

For these models two major problems are of interest. At first, we study a geometric structure of a "typical" orbit, i.e. dynamical properties of a model. After that we introduce some initial distribution of scatterers and then investigate statistical properties of the ensemble of all orbits.

2.2 The rigidity. Deterministic walks in rigid environments.

There is not much hope to rigorously study DWRE with an arbitrary program allowed at a vertex. Therefore we need to identify some sufficiently general classes of DWRE that allow rigorous analysis. One such class of DWRE has been introduced in [14]. In these models the scatterers can change only due to interaction with the particle and an environment is characterized by a function

$$r : \mathcal{G} \times \mathbb{N} \rightarrow \mathbb{N},$$

called *rigidity*, that describes the manner at which the scatterers that are located on each vertex can change: a type of scatterer at the vertex g changes at the time $t + 1$ if the particle visits site g for the $r(g, t)$ th instance at the time t since the last change. For example, if rigidity is a constant function, $r(g, t) = r$, the scatterers change type after each r th visit of the particle to a given site.

In this chapter we study Walks in Rigid Environments with constant rigidity and with *aging*. In the case of aging the rigidity function is a piecewise constant in time

and independent of the vertex function defined as

$$r(g, t) = r(t) = r_j,$$

$$j \in \mathbb{N}, \quad \forall g \in \mathcal{G}, \quad t \in [\tau_{j-1}, \tau_j) \cap \mathbb{N},$$

for some $\tau_0 = 0 \leq \tau_1 \leq \tau_2 \leq \dots, \tau_j \in \mathbb{N}, r_j \in \mathbb{N}$.

Let $\eta(z, t)$, called an *index of a scatterer* at the site g at the time t , be the number of visits of the particle to a site g , which occurred between the last moment of time when scatterer at g changed type and t . Then the type of a scatterer at a vertex $g \in \mathcal{G}$ changes at the moment of time when the index of a scatterer at the vertex g , $\eta(g, t)$, is equal to the rigidity, $r(t)$, that is at the smallest time t_0 such that $r(t_0) = \eta(g, t_0)$.

It is well known that properties of materials, cells, species, environments, etc, do change with time ([1]-[9], [30]-[56]). This process is universal in natural as well as in artificial systems. It involves any individual subsystems as well as their networks, populations, etc. Walks in Rigid Environments with constant rigidity and aging mimic the changing physical environment and, therefore, have many practical applications.

2.3 Deterministic walks in rigid environments on \mathbb{Z}^1 .

Here we study the models on \mathbb{Z}^1 . For \mathbb{Z}^1 there are only four types of scatterers: the left scatterer ('LS'), the right scatterer ('RS'), the backward scatterer ('BS'), and the forward scatterer ('FS'). Picture 13 shows all four types of scatterers on \mathbb{Z}^1 . The reflection is the only nontrivial symmetry of \mathbb{Z}^1 . Two of the scatterers are invariant under reflection ('BS' and 'FS') and two are not ('LS' and 'RS'). Therefore, it is natural to consider two types of models ([14], [15]), each of them having only two types of scatterers, call them S_1 and S_2 . The first one with $S_1 = \text{'FS'}$ and $S_2 = \text{'BS'}$ we call *NOS-model* (model with non-oriented scatterers). The second with $S_1 = \text{'LS'}$ and $S_2 = \text{'RS'}$ we call *OS-model* (model with oriented scatterers).

Let us first introduce some notation. Denote by $z(t)$ and $v(t)$ the position and the velocity of the particle at the time t , respectively. Let t and t_+ be the times

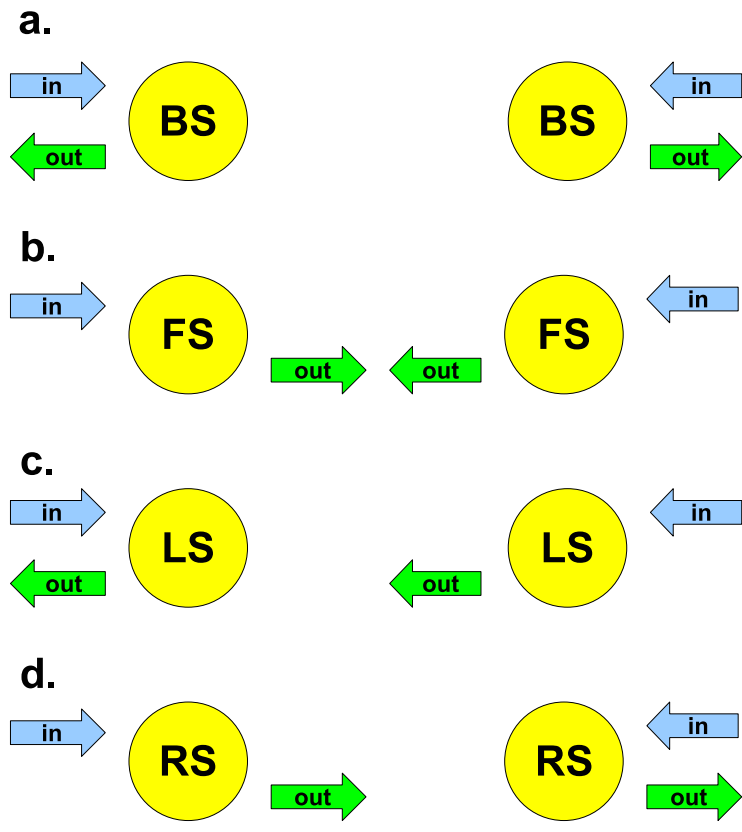


Figure 13: All possible types of scatterers on \mathbb{Z}^1 : a. the backward scatterer, b. the forward scatterer, c. the left scatterer, d. the right scatterer.

just before the interaction with the scatterer and after the interaction. As before let $\eta(z, t)$ be the index of a scatterer at the site z at the time t .

Let Ω be a collection of all initial configurations of scatterers, in our case $\Omega = \{S_1, S_2\}^{\mathbb{Z}}$. Also let $\omega \in \Omega$ be some fixed initial configuration of scatterers and, finally, let $\omega(z, t)$ be the scatterer that is located at the position z and the time t .

Definition 2.3.0.10. *We say that a configuration of scatterers ω has a positive tail of a scatterer S if there exists an integer $n > 0$ such that all sites $z > n$ in the configuration are occupied by the scatterer S . We can similarly define a negative tail.*

Without loss of generality we can always assume that the particle starts moving to the right, i.e. $v(0) = +1$.

2.4 Models with constant rigidity on \mathbb{Z}^1 . Geometric structure of the orbits.

In this section we study qualitative properties of deterministic walks when rigidity is constant, $r(g, t) = r$. The results in this sections were obtained in [14] - [16]. A basic, although simple fact in the theory of DWRE is provided by the following theorem.

Theorem 2.4.0.11 (Fundamental Theorem of the theory of DWRE, [16]). *Consider any two models of DWRE on some graph \mathcal{G} that admit the same set of initial configurations of scatterers. Then the structure of orbits of such models does not depend on the probability distribution of scatterers among the vertices $g \in \mathcal{G}$.*

The theorem allows us to study the geometric structure of all orbits without introducing the probability distribution of scatterers.

2.4.1 OS-model.

At first we consider a OS-model. In this model only two types of scatterers are allowed, $S_1 = \text{'LS'}$ and $S_2 = \text{'RS'}$. The geometric structure of the orbits is described in the following theorem.

Theorem 2.4.1.1 ([15]). *Consider a OS-model with constant rigidity. If the initial configuration does not belong to an exceptional set, the particle will visit any site of the lattice \mathbb{Z}^1 infinitely many times, i.e. the particle will oscillate. The exceptional set of initial configurations contains a set of configurations of scatterers in which there is a positive tail of 'RS' or/and a negative infinite tail of 'LS'. For these initial conditions, a particle will eventually propagate in one direction with velocity of ± 1 .*

Proof. This theorem was proven in [15]. For the sake of completeness and to use the details of the proof in the theorems below, we present an outline of the proof here.

Without loss of generality we can assume that initially there is a LS at $z = 0$, i.e. $\omega(0, 0) = \text{'LS'}$. Let $a_i \leq 0$ be the locations of 'RS' on the negative semi-axis in the initial distribution of scatterers ω and $b_i > 0$ are the locations of 'LS' on the positive semi-axis. Assume at first that there are no positive tail of 'RS' nor negative tail of 'LS'.

Let us break the motion of the particle into several steps. We assume that $r > 1$. The case when $r = 1$ is similar and much simpler.

Step 1. The particle will move to the right until it meets the first left scatterer at the site $z = b_1$. At this moment, $t = b_1$, the particle will turn, i.e. $v((b_t)_+) = -1$.

Step 2. At the time $t = b_1$ we have

$$\begin{aligned}\omega(b_1 - 1, b_1) &= \text{'RS'}, & \eta(b_1 - 1, (b_1)_+) &= 1; \\ \omega(b_1, b_1) &= \text{'LS'}, & \eta(b_1, (b_1)_+) &= 1.\end{aligned}$$

Thus the particle will be confined between site $z = b_1 - 1$ and $z = b_1$ until the scatterer at the site $z = b_1 - 1$ flips from 'RS' to 'LS' at the moment of time $t = b_1 + 2r - 1$. At that time we have

$$\begin{aligned}\omega(b_1 - 1, (b_1 + 2r - 1)_+) &= \text{'LS'}, & \eta(b_1 - 1, (b_1)_+) &= 0; \\ \omega(b_1, b_1 + 2r - 1) &= \text{'RS'}, & \eta(b_1, b_1) &= 0,\end{aligned}$$

and $v(b_1 + 2r - 1) = -1$. At $t = b_1 + 2r$ the particle will be at $z = b_1 - 2$, hence the particle is moving toward the origin while oscillating between neighboring sites on the way.

Step 3. At the moment of time $t = b_1 + (2r - 1)(b_1 - 1) + 1$ the particle returns to the origin with $v((2r - 1)(b_1 - 1) + 1) = -1$.

Step 4. We can repeat this argument for the motion of the particle to the left by interchanging 'RS' and 'LS', $v = -1$ and $v = 1$.

Thus at the time $t = 2r(b_1 - a_1)$ we have

$$v(2r(b_1 - a_1)) = +1, \quad z(2r(b_1 - a_1)) = 0,$$

and the following distribution of scatterers at the sites $z = a_1, \dots, b_1$:

$$\begin{aligned} \omega(z, 2r(b_1 - a_1)) &= \text{'RS'}, & \eta(z, 2r(b_1 - a_1)) &= 0, & z &= 1, \dots, b_1; \\ \omega(z, (2r(b_1 - a_1))_+) &= \text{'LS'}, & \eta(z, 2r(b_1 - a_1)) &= 0, & z &= a_1, \dots, 0. \end{aligned}$$

Therefore after time $t = 2r(b_1 - a_1)$ the particle will be able to travel into the positive direction b_2 steps and a_2 steps into the negative direction. Thus the particle will penetrate further into both directions. Since we do not have positive tail of 'RS' nor negative tail of 'LS' we can continue this procedure forever by induction.

If we have positive tail of 'RS' or negative tail of 'LS' then the particle will eventually propagate in one direction with velocity $v = \pm 1$ and there will be no oscillation. \square

Remark 2.4.1.2 (Description of the typical trajectory). *For the set of initial conditions for which a particle oscillates we can describe the motion of the particle more precisely. Namely, there exists a sequence of times τ_i such that within the interval $[\tau_i, \tau_{i+1}]$ the particle moves inside the interval $B_{i+1} = [a_{i+1}, b_{i+1}]$, $i = 1, 2, \dots$, where a_i are the locations of 'RS' on the negative semi-axis in the initial distribution of scatterers ω and b_i are the locations of 'LS' on the positive semi-axis. The length of this*

time interval is

$$\tau_{i+1} - \tau_i = 2r (b_{i+1} - a_{i+1}).$$

Moreover, the particle will visit each site in B_{i+1} within this time interval exactly $2r$ times.

Proof. This follows directly from the proof of the previous theorem. \square

2.4.2 NOS-model. Even rigidity.

The NOS-model has only two types of scatterers, $S_1 = \text{'BS'}$ and $S_2 = \text{'FS'}$. The geometric structure of the orbits for even values of rigidity is described in the following theorem.

Theorem 2.4.2.1 ([15]-[18]). *Consider a NOS-model with constant even rigidity r . If the initial configuration does not belong to an exceptional set the particle will visit any site of the lattice infinitely many times, i.e. the particle will oscillate. The exceptional set of initial configurations is equal to the set of initial configurations of scatterers in which there is a positive or negative tail of FS. For these initial conditions, a particle will eventually propagate in one direction with velocity of ± 1 .*

Proof. For the sake of completeness and to use in the succeeding sections we will present the proof given in [15]-[18].

First, we need some insight on the motion of the particle in the special initial configuration of the scatterers.

Proposition 2.4.2.2. *Assume that initially all sites are occupied by the back scatterers. Let the particle visit at the first time a site $z_0 > 0$. Then the next site which the particle will be visited at the first time will be $z = -z_0 + 1$.*

Proof. See Main Lemma in [14]. \square

Let T_n be the time it takes a particle to brake the barrier (i.e. to change a back scatterer to the forward scatterer) at the sites $z = n$ and $z = -n + 1$ and return back to the origin $z = 0$.

Proposition 2.4.2.3. *At time T_n there are 'BS' with index zero at each site of $(-n + 1, n)$ and there are 'FS' with index zero at $z = -n + 1$ and $z = n$. Moreover,*

$$T_n = 3T_{n-1} + 4r = 2r(3^n - 3^{n-1} - 1), \quad T_1 = 2r. \quad (2.4.1)$$

Proof. The first part of the formula is clear from the discussion above. Indeed, at the moment of time T_n the particle does three times the same thing as it does to the moment T_{n-1} - first it flips the back scatterers at $z = n - 1$ and $z = -n + 2$, then it hits the back scatterers at $z = n$ and $z = -n + 1$ $\frac{r}{2}$ times and change their index to $\frac{r}{2}$ (to do that we again need to flip back scatterers at $z = n - 1$ and $z = -n + 2$), and, finally, it has to hit the back scatterers at $z = n$ and $z = -n + 1$ $\frac{r}{2}$ more times and flip them.

The second part follows from the fact that $T_1 = 2r$ together with the recurrence relationship. \square

Thus, before the first visit to the site $z = n + 1$, we have the following configuration of the scatterers:

$$\begin{aligned} \omega(i, t_+) &= \text{'FS'}, \quad \eta(i, t_+) = 0, \quad i = -n + 1, \dots, n; \\ \omega(n + 1, t_+) &= \omega(-n, t_+) = \text{'BS'}, \quad \eta(n + 1, t_+) = \eta(-n, t_+) = 0 \end{aligned}$$

So to pass through a back scatterer at $z = n$ the particle has to flip the back scatterers at $z = n$ and $z = -n + 1$, $z = n - 1$ and $z = -n + 2$ etc. after which the particle will be at the origin moving to the right and there are forward scatterers with index zero at $z = -n + 1, \dots, n$. Then the particle has to travel $n + 1$ steps to the right.

Now, let us come back to the general situation. Suppose that the back scatterers situated at the sites $\{n_i^-\}_{i=1}^{+\infty}$ and $\{n_i^+\}_{i=1}^{+\infty}$, where

$$\dots < n_2^- < n_1^- < 0 < n_1^+ < n_2^+ < \dots$$

We break the motion of the particle into several steps. As always we assume that the particle starts moving to the right.

Step 1. Initially the particle starts moving to the right until it hits the first back scatterer at $z = n_1^+$. Then it travels left until the first back scatterer on the negative semi-axis at $z = n_1^-$. At the moment of time $t_0 = (n_1^+ - n_1^-)r$ the particle will be at the site $z = 0$ and moving into the positive direction. Moreover, there are back scatterers with index zero in all sites in the interval (n_1^-, n_1^+) and back scatterers with index $r/2$ at n_1^- and n_1^+ , i.e. we have the following configuration (only affected sites are shown):

$$\begin{aligned} z(([n_1^+ - n_1^-]r)_+) &= 0, & v(([n_1^+ - n_1^-]r)_+) &= +1; \\ \omega(i, ([n_1^+ - n_1^-]r)_+) &= \text{'BS'}, & \eta(i, ([n_1^+ - n_1^-]r)_+) &= 0, \\ & & i &= n_1^- + 1, \dots, n_1^+ - 1; \\ \omega(n_1^+, ([n_1^+ - n_1^-]r)_+) &= \omega(n_1^-, ([n_1^+ - n_1^-]r)_+) = \text{'BS'}, \\ \eta(n_1^+, ([n_1^+ - n_1^-]r)_+) &= \eta(n_1^-, ([n_1^+ - n_1^-]r)_+) = \frac{r}{2}. \end{aligned}$$

Step 2. Now we are back to the situation above, namely, we have back scatterers in the interval $[n_1^-, n_1^+]$. Without loss of generality assume that $n_1^- < -n_1^+ + 1$. Thus, by Proposition 2.4.2.2, before visiting site $z = n_1^+ + 1$ (i.e. passing through a back scatterer at $z = n_1^+$), the particle must visit at least the sites $z = n_1^- - 1, \dots, -n_1^+ + 1$, (i.e. it must pass through all the back scatterers between $z = n_1^-$ and $z = -n_1^+ + 1$). Next two steps will describe that process in more details.

Step 3. After the particle finally flips the back scatterer at $z = n_1^-$ but before it visited the site $z = -n_1^+ - 1$ for the first time at some later time t_1 we have the

following situation:

$$\begin{aligned}
z((t_1)_+) &= 0, & v((t_1)_+) &= +1; \\
\omega(i, (t_1)_+) &= \text{'FS'}, & \eta(i, (t_1)_+) &= 0, & i &= n_1^-, \dots, -n_1^-; \\
\omega(-n_1^- + 1, (t_1)_+) &= \omega(n_1^+, (t_1)_+) &= \text{'BS'}, \\
\eta(-n_1^- + 1, (t_1)_+) &= \eta(n_1^+, (t_1)_+) &= \frac{r}{2}.
\end{aligned}$$

Moreover, if $-n_1^- + 2 < n_1^+ - 1$ then

$$\omega(i, (t_1)_+) = \text{'BS'}, \quad \eta(i, (t_1)_+) = 0, \quad i = -n_1^- + 2, \dots, n_1^+ - 1.$$

Step 4. Next, after r passes of the interval $[n_2^-, -n_1^- + 1]$ we are back to the situation of the Step 1 with the new endpoint equal to n_2^- and a forward scatterer at $z = -n_1^- + 1$.

We can continue this process inductively.

Clearly, if the initial configuration has a tail (negative or/and positive) of forward scatterers then there will be no oscillations and a particle will eventually propagate into one direction with the velocity $v = \pm 1$. \square

Remark 2.4.2.4 (Symmetry). *Let $a(t) \in \{n_i^-\}_{i=1}^{+\infty}$ be the maximum absolute value of the position of a back scatterer on the negative semi-axis that the particle reached by the time t , say $a(t) = n_i^-$. Similarly, let $b(t) \in \{n_i^+\}_{i=1}^{+\infty}$ be the maximum value of the position of a back scatterer on the positive semi-axis that the particle reached by the time t , say $b(t) = n_j^+$. Then*

$$b(t) + 1 \leq |n_{i+1}^-|, \quad |a(t)| \leq n_{j+1}^+ + 1.$$

In other words, the oscillation is "almost" symmetric.

2.4.3 NOS-model. Odd rigidity.

Theorem 2.4.3.1 ([15] and [14]). *Consider a NOS-model with constant odd rigidity r . Then for any initial distribution of scatterers the particle will eventually propagate*

in one direction on the lattice. This direction depends only on the initial velocity of the particle $v(0)$ and the initial configuration of scatterers at the sites $z = \pm 1$.

Proof. Again, for the sake of completeness and to use some details later we will outline the proof presented in [15] and [14].

Remember that we always assume that $v(0) = +1$. Let $a_i \leq 0$ be the locations of 'BS' on the negative semi-axis in the initial distribution of scatterers ω and $b_i > 0$ are the locations of 'BS' on the positive semi-axis. Clearly if there are no back scatterers on one of the semi-axis then the particle will eventually travel in that direction. So we assume that there are infinitely many back scatterers on each semi-axis.

Suppose, at first, that initially there is a 'BS' at the origin, i.e. $a_0 = \omega(0, 0) = \text{'BS'}$. As before, we will break the motion of the particle into several steps.

Step 1. The particle starts moving to the right until it encounters a back scatterer at $z = b_1$. Then the particle starts oscillating between $z = 0$ and $z = b_1$. After $rb_1 - 1$ moments of time the particle will be at location $z = b_1 - 1$ moving to the right. It has traveled the interval $(0, b_1)$ r times. All forward scatterers in the interior of this interval thus flipped. Back scatterers at $z = 0$ and $z = b_1$ were hit only $\frac{r-1}{2}$ times. Hence, at the moment of time $t = rb_1 - 1$ we have the following configuration:

$$\begin{aligned} z(rb_1 - 1) &= b_1 - 1, & v((rb_1 - 1)_+) &= +1; \\ \omega(i, (rb_1 - 1)_+) &= \text{'BS'}, & \eta(i, (rb_1 - 1)_+) &= 0, & i &= 1, \dots, b_1 - 1; \\ \omega(0, (rb_1 - 1)_+) &= \omega(b_1, (rb_1 - 1)_+) &= \text{'BS'}, \\ \eta(0, (rb_1 - 1)_+) &= \eta(b_1, (rb_1 - 1)_+) &= \frac{r-1}{2}. \end{aligned}$$

Step 2. Next, the particle will be trapped between two neighboring back scatters at $z = b_1 - 1$ and $z = b_1$. After r more moments of time the back scatterer at $z = m$ will flip and the one at $z = b_1 - 1$ will have an index of $\frac{r-1}{2}$ at the time $t = r + 1$. At that moment of time we will have the following configuration (only the sites that has

changed are listed):

$$\begin{aligned} z(r(b_1 + 1)) &= b_1 - 1, & v((r(b_1 + 1))_+) &= +1; \\ \omega(b_1 - 1, (r(b_1 + 1))_+) &= \text{'BS'}, & \eta(b_1 - 1, (r(b_1 + 1))_+) &= \frac{r-1}{2} \\ \omega(b_1, (r(b_1 + 1))_+) &= \text{'FS'}, & \eta(b_1, (r(b_1 + 1))_+) &= 0. \end{aligned}$$

We have shifted the initial configuration $b_1 - 1$ units to the left.

Step 3. We can continue with the same argument to construct a sequence of the intervals which shift to the right in which the particle oscillate. Namely, this intervals are given by $B_i = [b_{i-1} - 1, b_i]$, $i \geq 1$, $b_0 = 0$.

Thus we proved the statement of the theorem for the case when there is a back scatterer at the origin.

Now let us assume that initially there is a forward scatterer at the origin, i.e. $w(0, 0) = \text{'FS'}$. We have to consider two possibilities, $b_1 > 1$ and $b_1 = 1$.

In the first case the particle will move to the right until the site $z = b_1$. Then it moves left until $z = a_1$. After the particle traveled the interval $[0, b_1)$ r times and the interval $(a_1, 0)$ $(r - 1)$ times (that happens at the moment of time $t = rb_1 + (r - 1)|a_1| - 1$), the particle will return to the site $z = b_1 - 1$ and it is moving to the right. At that time there is a back scatterers at the sites $z = b_1 - 1$ and $z = b_1$. The index of the site $z = b_1 - 1$ is zero, but the index of the site $z = b_1$ is $\frac{r-1}{2}$. This is the configuration we already looked at in the **Step 2** above. Thus, in this case, the particle will propagate to the right.

Now assume that $b_1 = 1$. At the moment of time $t = (|a_1| + 1)r$ the particle will be at the site $z = a_1 + 1$ moving to the left. There are back scatterers at the sites $z = a_1 + 1$ and $z = a_1$. The index of the site $z = a_1$ is zero, but index of the site $z = a_1 + 1$ is $\frac{r-1}{2}$. Following the same argument above we can see that the particle will propagate to the left. \square

2.5 Models with constant rigidity on \mathbb{Z}^1 and independent distribution of scatterers. Statistical properties.

In this section we review the statistical properties of the models with constant rigidity on \mathbb{Z}^1 and independent distribution of scatterers. Most of these results were obtained in [14]-[16] and will be used in the later sections of this chapter.

In all models that we are considering there are only two possible types of scatterers that can be present on any site of the lattice. As before, let Ω be a collection of all initial configurations of scatterers, $\Omega = \{S_1, S_2\}^{\mathbb{Z}}$. We assume that initially scatterers are distributed independently and identically among the sites. let q be the probability that S_1 is located at any given site of the lattice Then $(1 - q)$ is the probability that S_2 is located at any given site of the lattice. Thus we have a usual Bernoulli measure on the collection of all initial distributions Ω .

2.5.1 OS-model.

We start with OS-model which has two scatterers, $S_1 = \text{'LS'}$ and $S_2 = \text{'RS'}$. Let q be the probability that 'LS' is located at any given site of the lattice. Then the probability that 'RS' is located at any given site of the lattice is $(1 - q)$.

Theorem 2.5.1.1 ([14]-[16]). *Consider the OS-models. There exists $t_0 > 0$ such that for all values of $t > t_0$ and all finite values of rigidity r , $\mathbb{E}z(t) = 0$. Moreover,*

$$\mathbb{E}z^2(t) = \mathcal{O}(t),$$

as $t \rightarrow \infty$.

Next, we refine the results for the OS-models with constant rigidity to be used for analysis of the models with aging, i.e. with piecewise constant rigidity.

Lemma 2.5.1.2. *Consider the OS-models with constant rigidity on \mathbb{Z}^1 . Then, there exist $T_0 = 0 \leq T_1 \leq T_2 \leq \dots$, $T_j = T_j(q, r_j) \in \mathbb{N}$ such that $\forall r_j \in \mathbb{N}$ one can find*

$C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ with the following property

$$\mathbb{E}z_j(t) = 0, \quad C_1 \frac{t}{r_j} \leq \mathbb{E}z_j^2(t) \leq C_2 \frac{t}{r_j}, \quad \forall t \geq T_j, \quad \forall j \in \mathbb{N},$$

where z_j is a position of a particle at the time t in a system with a constant rigidity $r(z, t) = r_j$.

Proof. By Theorem 2.5.1.1 for each fixed $j \in \mathbb{N}$ one there exists T_j such that $\mathbb{E}z_j(t) = 0$ for all values of $t > T_j$. Moreover, there exists $T_j = T_j(q, r_j)$ (see proof of Theorem 2 in [15]) such that for all $t \geq T_j$ one can find two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ (both independent of r_j) such that

$$C_1 \frac{t}{r_j} \leq \mathbb{E}z_j^2(t) \leq C_2 \frac{t}{r_j}, \quad \forall t \geq T_j.$$

We can apply this argument for all values of $j \in \mathbb{N}$ and obtain the sequence $\{T_j(q, r_j)\}_{j=0}^{\infty}$ with the same two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ for all values of j .

Thus the result follows. □

2.5.2 NOS-model.

Now we consider NOS-models with constant rigidity. Let q be the probability that 'FS' is located at any given site of the lattice. Then the probability that 'BS' is located at any given site of the lattice is $(1 - q)$.

Theorem 2.5.2.1 ([14]-[16]). *Consider the NOS-model. If the value of the rigidity r is even, then for almost all initial configurations of scatterers the particle will visit each site of the lattice infinitely many times. Moreover, there exists $t_0 > 0$ such that $\mathbb{E}z(t) = 0$ for all values of $t > t_0$ and all finite values of rigidity r and*

$$\mathbb{E}z^2(t) = \mathcal{O}(\ln t),$$

as $t \rightarrow \infty$.

On the other hand, if the value of the rigidity r is odd then for all initial configurations of scatterers the particle is eventually propagating into one direction. This

direction is determined by the initial velocity, $v(0_+)$, and the types of scatterers that are located initially at $z = \pm 1$. Moreover,

$$\mathbb{E}|z(t)| = \mathcal{O}(t),$$

as $t \rightarrow \infty$.

Next, we look at the family of systems with the constant even rigidity and refine the theorem above.

Lemma 2.5.2.2. *Consider the NOS-models with constant even rigidity. There exist $T_1, T_2, \dots, T_j = T_j(q, r_j) \in \mathbb{N}$ such that for each $r_j \in \mathbb{N}$ one has $\mathbb{E}z_j(t) = 0$ for all $t > T_j$. Moreover, one can find two positive constants C_1 and C_2 depending only on q (and not on r_j) satisfying the relationship*

$$\frac{C_1}{\ln r_j} \ln t \leq \mathbb{E}z_j^2(t) \leq \frac{C_2}{\ln r_j} \ln t, \quad \forall t \geq T_j, \quad \forall j \geq 1,$$

where $z_j(t)$ is a position of a particle at the time t in a system with a constant rigidity $r(z, t) = r_j$.

Proof. By Theorem 2.5.2.1 for each fixed $j \in \mathbb{N}$ there exists T_j such that $\mathbb{E}z_j(t) = 0$ for all values of $t \in T_j$. Moreover, one can select these values T_j (see the proof of Theorem 4 in [15]) in such a way so one can find two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ (both independent of r_j) such that

$$\frac{C_1}{\ln r_j} \ln t \leq \mathbb{E}z_j^2(t) \leq \frac{C_2}{\ln r_j} \ln t, \quad \forall t \geq T_j.$$

We can apply this result for all values of $j \in \mathbb{N}$ and pick the sequence $\{T_j(q, r_j)\}_{j=0}^{\infty}$ and the same two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ for all values of j .

Thus the result follows. \square

The following conditions determine the direction of propagation for the case of constant odd rigidity.

Lemma 2.5.2.3 ([14]-[16]). *Consider the NOS-models and suppose that rigidity r is odd. Then for sufficiently large $T = T(q, r)$ we have the following:*

a) $\mathbb{E}v(t) > 0, \forall t \geq T$ if either

- $v(0_+) = +1$ and $\omega(0, 0) = \text{'BS'}$ or
- $v(0_+) = +1$ and $\omega(0, 0) = \omega(+1, 0) = \text{'FS'}$ or
- $v(0_+) = -1, \omega(0, 0) = \text{'FS'}, \omega(-1, 0) = \text{'BS'}$;

b) $\mathbb{E}v(t) < 0, \forall t \geq T$ if either

- $v(0_+) = +1, \omega(0, 0) = \text{'FS'}, \omega(+1, 0) = \text{'BS'}$ or
- $v(0_+) = -1$ and $\omega(0, 0) = \omega(-1, 0) = \text{'FS'}$ or
- $v(0_+) = -1$ and $\omega(0, 0) = \text{'BS'}$.

We also have another refinement of the Theorem 2.5.2.1.

Lemma 2.5.2.4. *Consider the NOS-models with constant odd rigidity. There exist $T_1, T_2, \dots, T_j = T_j(q, r_j) \in \mathbb{N}$ such that for each $r_j \in \mathbb{N}$, odd, one can find two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ satisfying relationship*

$$C_1 \frac{t}{r_j} \leq \mathbb{E}|z_j(t)| \leq C_2 \frac{t}{r_j}, \quad \forall t \geq T_j, \quad \forall j \geq 1,$$

where z_j is a position of the particle at the time t in a system with a constant rigidity $r(z, t) = r_j$.

Proof. By Theorem 2.5.2.1 for each fixed $j \in \mathbb{N}$ there exists $T_j = T_j(q, r_j)$ such that for all $t \geq T_j$ one can find (see proof of Theorem 1 in [14]) two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ (both independent of r_j) such that

$$C_1 \frac{t}{r_j} \leq \mathbb{E}|z_j(t)| \leq C_2 \frac{t}{r_j}, \quad \forall t \geq T_j.$$

We can apply this result for all values of $j \in \mathbb{N}$ and obtain the sequence $\{T_j(q, r_j)\}_{j=0}^{\infty}$ and the same two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ for all values of j . Thus, the result follows. □

2.5.3 Summary for the models with constant rigidity and independent initial distribution of scatterers.

Here we summarize the statistical properties of the models with constant rigidity on \mathbb{Z}^1 and independent distribution of scatterers. It is appropriate to compare these results to the classical random walk on \mathbb{Z}^1 .

As before, $z(t)$ is the position of the particle at time t . From the preceding sections we have:

- **OS-model:** for almost all initial configuration of scatterers particle will oscillate around the origin and as $t \rightarrow \infty$,

$$\mathbb{E}z^2(t) = \mathcal{O}(t), \quad \mathbb{E}z(t) = 0.$$

- **NOS-model, even rigidity:** for almost all initial configuration of scatterers particle will oscillate around the origin and as $t \rightarrow \infty$,

$$\mathbb{E}z^2(t) = \mathcal{O}(\ln t), \quad \mathbb{E}z(t) = 0.$$

- **NOS-model, odd rigidity:** for all initial configuration of scatterers particle will eventually propagate (on average) into one direction and as $t \rightarrow \infty$,

$$\mathbb{E}|z(t)| = \mathcal{O}(t).$$

2.6 *Models with constant rigidity on \mathbb{Z}^1 and Markovian distribution of scatterers. Statistical properties.*

As it has been mentioned in [16] (see the Theorem 2.4.0.11 in this chapter), qualitative properties of DWRE do not depend upon an exact form of the distribution of scatterers in the (random) environments. Rather, they depend upon a structure of the graph (lattice) and upon a collection of (admissible) scatterers.

We now turn to a quantitative analysis of DWRE under study and introduce a distribution of scatterers on the lattice \mathbb{Z}^1 . We assume that our system allows only two

type of scatterers, S_1 and S_2 (we associate a number 1 with the first one and 2 with the second one to ease the notation), and the initial configuration of scatterers form a double infinite Markov chain with a binary state space and a stochastic transition matrix A given by

$$A = (A_{i,j}) = \begin{pmatrix} 1 - p_1 & p_1 \\ 1 - p_2 & p_2 \end{pmatrix}.$$

The case $p_1 = p_2$ corresponds to the independent distribution of scatterers and was considered above. Assume that $0 < p_1 < 1$ and $0 < p_2 < 1$ to avoid the triviality.

Let $\vec{\pi}$ be the left stochastic (meaning that the norm of $\vec{\pi}$ equals to one) eigenvector of A with eigenvalue 1,

$$\vec{\pi} = (\pi_1, \pi_2) = \left(\frac{p_1}{1 + p_1 - p_2}, \frac{1 - p_2}{1 + p_1 - p_2} \right).$$

Then we can define a Markov probability measure \mathbb{P} on the set of double infinite sequences of scatterers in the following way (see [54]). First we define a measure on the cylinder sets and then extend it to the entire σ -algebra generated by them. The measure of the cylinder $[X_m \dots X_n]$ is given by

$$\mathbb{P}([X_1 \dots X_n]) = \pi_{X_m} A_{X_m, X_{m+1}} \dots A_{X_{n-1}, X_n},$$

where X_i is either one (which represents S_1) or two (which represents S_2). The cylinders were defined in 1.5.2.

Denote by τ_z a moment of time when the particle passes through the site $z \in \mathbb{Z}$ for the first time and $\mathbb{E}\tau_z$ is its expectation.

As was mentioned above, the case when initially scatterers were distributed independently was completely solved in [14] - [16]. The results of this section were published in [21].

2.6.1 OS-model.

In this model we only have two scatterers, $S_1 = \text{'LS'}$ and $S_2 = \text{'RS'}$. Since geometric structure of the orbits is independent of the initial distribution of scatterers (as long as

they admit the same configurations) for almost all configurations particle will oscillate around the origin (see Theorem 2.4.1.1).

Theorem 2.6.1.1. *Consider a OS-model with the Markovian initial distribution of scatterers. For any positive value of rigidity $r > 0$ as $t \rightarrow \infty$, one has that*

$$\mathbb{E}z^2(t) = \mathcal{O}(t).$$

Proof. The proof is similar to the proof of the Theorem 2 in [15] with independent distribution of scatterers replaced with a Markov chain.

Let us fix a configuration of scatterers ω . Assume that for this configuration a particle will oscillate (the set of these configurations has measure one by Theorem 2.4.1.1). Without loss of generality, assume also that the particle starts moving into the positive direction.

Let a_i and b_i be defined as in the Remark 2.4.1.2. We will need to estimate the growth of the region visited by the particle, i.e the growth of interval $B_i = [a_i, b_i]$.

First note that we can view $b_1, b_{i+1} - b_i$, and $a_1, a_{i+1} - a_i, i = 1, 2, \dots$ as random variables. Assume $b_{i+1} - b_i = k > 1, i > 1$. Then there are $k - 1$ right scatterers starting at $b_i + 1$ followed by one left scatterer at the position b_{i+1} . If $k = 1$ then there is one left scatterer at b_{i+1} . Therefore we have

$$\mathbb{P}\{b_{i+1} - b_i = k\} = \begin{cases} \frac{(1-p_2)^2}{1+p_1-p_2} p_2^{k-2}, & k > 1, \\ \frac{p_1}{1+p_1-p_2}, & k = 1. \end{cases} \quad (2.6.1)$$

Similarly, we get

$$\mathbb{P}\{a_i - a_{i+1} = k\} = \begin{cases} \frac{(1-p_2)^2}{1+p_1-p_2} (1-p_1)^{k-2}, & k > 1, \\ \frac{1-p_2}{1+p_1-p_2}, & k = 1. \end{cases} \quad (2.6.2)$$

We always assume that $a_0 = b_0 = 0$.

Proposition 2.6.1.2. *The random variables $b_1, b_{i+1} - b_i$, and $a_1, a_{i+1} - a_i, i = 1, 2, \dots$ are independent random variables on the measure space of all initial configurations of scatterers (Ω, \mathbb{P}) , where \mathbb{P} is a Markov measure defined above.*

Now we can compute expected values of b_1 , $b_{i+1} - b_i$, and a_1 , $a_{i+1} - a_i$, $i = 1, 2, \dots$:

$$\begin{aligned} \mathbb{E}(b_1) &= \mathbb{E}(b_{i+1} - b_i) \\ &= \sum_{k=2}^{+\infty} k \frac{(1-p_2)^2}{1+p_1-p_2} p_2^{k-2} + 1 \cdot \frac{p_1}{1+p_1-p_2} \\ &= \frac{2}{1+p_1-p_2}; \end{aligned} \quad (2.6.3)$$

$$\begin{aligned} \mathbb{E}(a_1) &= \mathbb{E}(a_i - a_{i+1}) \\ &= \sum_{k=2}^{+\infty} k \frac{(1-p_2)^2}{1+p_1-p_2} (1-p_1)^{k-2} + 1 \cdot \frac{1-p_2}{1+p_1-p_2} \\ &= \frac{(p_1+1)(1-p_2)^2}{p_1^2(1+p_1-p_2)} + \frac{1-p_2}{1+p_1-p_2}. \end{aligned} \quad (2.6.4)$$

According to Remark 2.4.1.2 at any moment of time τ the particle will be confined in some segment $B(\tau) = [a(\tau), b(\tau)]$, where $a(\tau)$ and $b(\tau)$ are maximal and minimal coordinates of the site visited by the particle until the moment τ . We want to compute the expected values of $a(\tau)$ and $b(\tau)$. Let $m_+(\tau)$ be a number of 'LS' located between the origin and $b(\tau-1)$ in ω and $m_-(\tau)$ be a number of 'RS' located between the origin and $a(\tau-1)$ (including origin). Then $m_+(\tau) - m_-(\tau) = 1$.

By Remark 2.4.1.2 there is a sequence of times such that the particle visits the endpoints of the intervals $B_i = [a_i, b_i]$, $i = 1, 2, \dots$ for the first time. So at some time τ a particle is in the process of visiting each site of the interval $B_{m_+(\tau)}$ $2r$ times. It might not have had enough time yet to visit the endpoints, but it certainly visited the endpoints of $B_{m_+(\tau)-1}$. Hence for any moment of time τ we can write

$$\tau = 2r \sum_{i=1}^{m_+(\tau)-1} (b_i - a_i) + \gamma(\tau), \quad (2.6.5)$$

where $\gamma(\tau)$ is the length of the interval of time between the moment when the particle returns to the origin with velocity $+1$ after visiting each site of the interval $B_{m_+(\tau)-1}$ $2r$ times and the moment τ . Clearly,

$$\gamma(\tau) \leq 2r(b_{m_+(\tau)} - a_{m_+(\tau)}). \quad (2.6.6)$$

We have that

$$b_{m_+(\tau)} = \sum_{i=0}^{m_+(\tau)-1} (b_{i+1} - b_i).$$

There are $m_+(\tau)$ independent identically distributed random variables (see Proposition 2.6.1.2) in the sum (therefore each term has the same expectation given by 2.6.3).

Hence by *Wald's equality* (see, for example, Theorem 5.5.3 in [29]) we have that

$$\mathbb{E}b_{m_+(\tau)} = \mathbb{E}m_+(\tau) \cdot \mathbb{E}(b_{i+1} - b_i).$$

Similar argument can be applied to $\mathbb{E}a_{m_+(\tau)}$. Thus using 2.6.3 and 2.6.4 we can compute that

$$\mathbb{E}b_{m_+(\tau)} = \frac{2}{1 + p_1 - p_2} \mathbb{E}m_+(\tau), \quad (2.6.7)$$

$$\mathbb{E}a_{m_+(\tau)} = \frac{(p_1 + 1)(1 - p_2)^2 + (1 - p_2)p_1^2}{p_1^2(1 + p_1 - p_2)} \mathbb{E}m_+(\tau). \quad (2.6.8)$$

It follows from 2.6.5 and 2.6.6 that

$$\begin{aligned} \tau &= 2r \sum_{i=1}^{m_+(\tau)-1} (b_i - a_i) + \gamma(\tau) \\ &\leq 2r \sum_{i=1}^{m_+(\tau)-1} (b_i - a_i) + 2r(b_{m_+(\tau)} - a_{m_+(\tau)}) \\ &\leq 2r \sum_{i=1}^{m_+(\tau)} (b_i - a_i) \\ &= 2r \sum_{i=1}^{m_+(\tau)} (m_+(\tau) - i + 1) [(b_i - b_{i-1}) + (a_{i-1} - a_i)] \end{aligned}$$

and

$$\begin{aligned} \tau &\geq 2r \sum_{i=1}^{m_+(\tau)-1} (b_i - a_i) \\ &= 2r \sum_{i=1}^{m_+(\tau)-1} (m_+(\tau) - i) [(b_i - b_{i-1}) + (a_{i-1} - a_i)]. \end{aligned}$$

Finally,

$$\begin{aligned}
& 2r \sum_{i=1}^{m_+(\tau)-1} (m_+(\tau) - i) [(b_i - b_{i-1}) + (a_{i-1} - a_i)] \leq \tau \\
& \leq 2r \sum_{i=1}^{m_+(\tau)} (m_+(\tau) - i + 1) [(b_i - b_{i-1}) + (a_{i-1} - a_i)].
\end{aligned}$$

Using Wald's equality again we obtain,

$$\begin{aligned}
& J(p_1, p_2) \frac{\mathbb{E}m_+(\tau) (\mathbb{E}m_+(\tau) - 1)}{2} \leq \tau \\
& \leq J(p_1, p_2) \frac{\mathbb{E}m_+(\tau) (\mathbb{E}m_+(\tau) + 1)}{2},
\end{aligned}$$

where $J(p_1, p_2)$ is given by

$$\begin{aligned}
J(p_1, p_2) &= \mathbb{E}(b_{i+1} - b_i) + \mathbb{E}(a_i - a_{i+1}) \\
&= \frac{(p_1 + 1)(1 - p_2)^2}{p_1^2(1 + p_1 - p_2)} + \frac{3 - p_2}{1 + p_1 - p_2}.
\end{aligned}$$

Therefore there exist two positive constants C_1 and C_2 such that

$$C_1 t \leq (\mathbb{E}m_+(t))^2 \leq C_2 t$$

for sufficiently large t . Denote by z_{max} and z_{min} the sites with maximal and minimal coordinates respectively visited by the particle before time t . Thus by 2.6.7

$$C'_1 t \leq (\mathbb{E}z_{max})^2 \leq C'_2 t.$$

Similar argument shows that

$$C''_1 t \leq (\mathbb{E}z_{min})^2 \leq C''_2 t$$

for some positive constants C''_1 and C''_2 .

But at any time the particle is confined to some interval B_i and spends equal amount of time at each site inside that interval. Therefore position of the particle is uniformly distributed inside that interval in the time interval $[\tau_i, \tau_{i+1}]$. Thus the asymptotic behavior of $\mathbb{E}z^2$ is the same as for $\mathbb{E}z_{max}^2$ and $\mathbb{E}z_{min}^2$, i.e.

$$At \leq \mathbb{E}z^2 \leq Bt,$$

for all sufficiently large t and some constants $A = A(p_1, p_2)$ and $B = B(p_1, p_2)$.

□

2.6.2 NOS-model. Even Rigidity.

In this model we have two types of scatterers, $S_1 = \text{'FS'}$ and $S_2 = \text{'BS'}$.

Theorem 2.6.2.1. *Consider a NOS-model with the Markovian initial distribution of scatterers. Assume that the rigidity r is even and set $\mathcal{R}(z) = \frac{1}{\mathbb{E}\tau_{z+1} - \mathbb{E}\tau_z}$, the average rate of the increase of the amplitude of the oscillation of the particle from z to $z + 1$. Then as $z \rightarrow \infty$ one has that*

$$\mathcal{R}(z) = \mathcal{O}(\alpha^{-z}), \quad \alpha > 1.$$

Proof. Without loss of generality assume that a particle starts moving into the positive direction. We also assume that initial configuration is such that the particle will oscillate (see Theorem 2.4.2.1).

Consider now the difference $\mathbb{E}\tau_{n+1} - \mathbb{E}\tau_n$, that is the expected time it takes the particle to pass the site $n + 1$ for the first times after it passed the site n after the first time. Let us consider different contributions to this quantity.

We have two possible scatterers at the site $z = n + 1$, 'BS' and 'FS'. In the second case, $\tau_{n+1} - \tau_n = 1$ and probability of this is equal to $\frac{p_1}{1+p_1-p_2}$. Hence, in this case

$$\mathbb{E}\tau_{n+1} - \mathbb{E}\tau_n = \mathcal{O}(1) \tag{2.6.9}$$

The first case, when there is a back scatterer at $z = n+1$, is a bit more complicated.

Assume that the closest from the left to $z = n$ back scatterer is located at b_n , $n \geq b_n \geq 1$. Thus, before passing through $z = n$ the particle had to break that back scatterer at $z = b_n$.

The proof of the Theorem 2.4.2.1 shows that just before the particle passes the

site $z = n$ for the first time we have the following configuration:

$$\begin{aligned} v((\tau_n)_+) &= +1, & z(\tau_n) &= 0; \\ \omega(i, (\tau_n)_+) &= \text{'FS'}, & \eta(i, (\tau_n)_+) &= 0, & i &= -b_n + 1, \dots, n; \\ \omega(n+1, \tau_n) &= \text{'BS'}, & \eta(n+1, \tau_n) &= 0. \end{aligned}$$

The amount of time it takes to pass the next site at $z = n+1$ (which is occupied by 'BS') depends on the initial configuration of scatterers in the interval $[-\infty, -b_n]$.

At first, assume that there are forward scatterers (they must have an index of zero at this time) on each site of the interval $[-n+1, -b_n]$. And assume that the closest to the site $z = -n$ back scatterer is located at $z = -n - k$, $k \geq 0$ (since we do not have negative tail of 'FS' k must be finite). Then $\tau_{n+1} - \tau_n = 2T_n + r(2n+1+k) + 2r$. The probability of this configuration is

$$\begin{aligned} & \frac{p_2(1-p_2)}{1+p_1-p_2} \cdot \frac{1-p_2}{1+p_1-p_2}, & k &= 0; \\ & \frac{p_2(1-p_2)}{1+p_1-p_2} \cdot \frac{(1-p_1)^{k-1}(1-p_2)^2}{1+p_1-p_2}, & k &> 1. \end{aligned}$$

Thus, in this case

$$\mathbb{E}\tau_{n+1} - \mathbb{E}\tau_n = \mathcal{O}(3^n). \quad (2.6.10)$$

Since we might have some back scatterers in $[-n+1, -b_n]$ we must compute the delay time t_D it takes to pass all the back scatterers in that interval and get back to the situation above. Clearly, $\max t_D = \mathcal{O}(T_n)$ and $\min t_D = \mathcal{O}(n)$. There are at most 2^{n-1} configuration of scatterers in the interval $[-n+1, -b_n]$ and each of those configuration has the probability of $\mathcal{O}(p_1^{n-2}, p_2^{n-2})$. Thus

$$\mathbb{E}t_D = \mathcal{O}(a^n) \quad (2.6.11)$$

for some $a > 0$.

Combining the results of equations 2.6.9-2.6.11 we finish the proof. □

2.6.3 NOS-model. Odd Rigidity.

Again, in this model we have only two types of scatterers, $S_1 = \text{'FS'}$ and $S_2 = \text{'BS'}$. We proved that for *all* distributions of scatterers the particle will eventually propagate in one direction.

Lemma 2.6.3.1. *Consider a NOS-model with the Markovian initial distribution of scatterers and assume that the rigidity r is odd. Then the particle will eventually propagate with probability $1 - \frac{p_1^2}{1+p_1-p_2}$ into the direction of its initial velocity.*

Proof. Assume that initially the particle moves to the right. The proof of the Theorem 2.4.3.1 shows that the particle will propagate to the left only if we have a forward scatterer at $z = 0$ and back scatterer at $z = 1$. The probability of this event is $\frac{p_1^2}{1+p_1-p_2}$. The result follows. \square

Since the particle will eventually propagate in one direction, it will visit each site (if at all) only finite number of times. For sites z that belong to a trajectory of the particle, let us define a function $R(z)$, the number of visits by the particle to a site z . Next lemma gives the distribution of this function.

Lemma 2.6.3.2. *Consider a NOS-model with the Markovian initial distribution of scatterers and odd rigidity r . Assume that the particle will eventually propagate to the right. Then $R(z)$ has the following distribution for $z \geq 1$*

$$R(z) = \begin{cases} r, & \text{with probability } \frac{p_1(1-p_1)}{1+p_1-p_2} \\ 2r, & \text{with probability } \frac{(1-p_2)^2+p_1^2}{1+p_1-p_2} \\ 3r, & \text{with probability } \frac{p_2(1-p_2)}{1+p_1-p_2} \end{cases} .$$

Proof. The number of visits to a site $z \geq 1$ depends on the initial configuration of the scatterers at the site z and $z + 1$. There are four cases.

Case 1 [FF]. Initially there are forward scatterers at z and $z + 1$, i.e. $\omega(z, 0) = \omega(z + 1, 0) = \text{'FS'}$. Then, according to the proof of the Theorem 2.4.3.1, the site $z \geq 3$

will be visited only r times until there will be back scatterers at z and $z + 1$ and the particle will never return to z . The probability of this event is

$$\frac{p_1(1 - p_1)}{1 + p_1 - p_2}. \quad (2.6.12)$$

Case 2[BF]. Initially there is a back scatterer at z and forward scatterer at $z + 1$, i.e. $\omega(z, 0) = \text{'BS'}$ and $\omega(z + 1, 0) = \text{'FS'}$. The site z will be visited exactly $2r$ times. Indeed, first it takes r visits to flip the back scatterer and after that we are back to the Case 1. The probability of this event is

$$\frac{(1 - p_2)^2}{1 + p_1 - p_2}. \quad (2.6.13)$$

Case 3[FF]. Initially there is a forward scatterer at z and a back scatterer at $z + 1$ and , i.e. $\omega(z, 0) = \text{'FS'}$ and $\omega(z + 1, 0) = \text{'BS'}$. In this case, first after r passes through the site z it will flip and the index of the back scatterer at $z + 1$ will be $\frac{r-1}{2}$. Then after $\frac{r-1}{2}$ more hits to the site z the back scatterer at $z + 1$ will flip. After $\frac{r-1}{2}$ hits of the site z the back scatterer will reappear at the site $z + 1$ and the particle will never return to z . Thus the site z will be visited $2r$ times. The probability of this event is

$$\frac{p_1^2}{1 + p_1 - p_2}. \quad (2.6.14)$$

Case 4[BB]. Initially there are back scatterers at z and $z + 1$, i.e. $\omega(z, 0) = \omega(z + 1, 0) = \text{'BS'}$. It takes r hits to flip a back scatterer at z . Then we are back to the Case 3. Thus the site z will be visited exactly $3r$ times. Probability of this event is

$$\frac{p_2(1 - p_2)}{1 + p_1 - p_2}. \quad (2.6.15)$$

Combining 2.6.12 - 2.6.15 we finish the proof of the lemma. \square

The same argument can be used in the case when the particle will eventually propagate to the left.

Corollary 2.6.3.3. *Consider a NOS-model with the Markovian initial distribution of scatterers and odd rigidity r . Assume that the particle will eventually propagate to the right. Then the particle will visit any site no more than $3r$ times. An average number of visits by an orbit to any site $z \geq 1$ is given by*

$$\mathbb{E}R(z) = Q(p_1, p_2)r, \quad (2.6.16)$$

where

$$Q(p_1, p_2) = \frac{2 + (p_1 - p_2)(p_1 + p_2 + 1)}{1 + p_1 - p_2}.$$

Theorem 2.6.3.4. *Consider a NOS-model with the Markovian initial distribution of scatterers and odd rigidity r . Assume that the particle will eventually propagate to the right. Then after a sufficient amount of time (the time it takes to pass the first back scatterer on the positive semi-axis) the average velocity of the particle is constant given by*

$$[rQ(p_1, p_2)]^{-1}$$

Proof. After the particle leaves the initial 'trapping' region $[a_1, b_1]$, it will never return to the negative semi-axis. Then the average number of visits by a particle to any site is given by 2.6.16. The result of the theorem then follows. \square

2.6.4 Summary for the models with constant rigidity and Markovian initial distribution of scatterers.

Here we put together the results of this section on model with constant rigidity and Markovian initial distribution of scatterers. The asymptotic results look exactly the same as for the case of independent initial distribution of scatterers. The following are the corollaries from the results in the preceding section:

- **OS-model:** for almost all initial configuration of scatterers particle will oscillate around the origin and as $t \rightarrow \infty$,

$$\mathbb{E}z^2(t) = \mathcal{O}(t), \quad \mathbb{E}z(t) = 0.$$

- **NOS-model, even rigidity:** for almost all initial configuration of scatterers particle will oscillate around the origin and as $t \rightarrow \infty$,

$$\mathbb{E}z^2(t) = \mathcal{O}(\ln t), \quad \mathbb{E}z(t) = 0.$$

- **NOS-model, odd rigidity:** for all initial configuration of scatterers particle will eventually propagate (on average) into one direction and as $t \rightarrow \infty$,

$$\mathbb{E}|z(t)| = \mathcal{O}(t).$$

2.7 Models with aging on \mathbb{Z}^1 and independent distribution of scatterers. Statistical properties.

In this section we study models in environments with aging and independent initial distribution of scatterers on \mathbb{Z}^1 . Each model in this section has two type of scatterers, S_1 and S_2 . Let q be the probability that S_1 is located at any given site of the lattice. Then the probability that S_2 is located at any given site of the lattice is $(1 - q)$. Again, we consider separately OS- and NOS-models. In the models with aging the rigidity function is defined as

$$r(z, t) = r(t) = r_j, \quad j \in \mathbb{Z}^+, \quad \forall z \in \mathbb{Z}, \quad t \in [\tau_{j-1}, \tau_j) \cap \mathbb{N}, \quad (2.7.1)$$

for some $\tau_0 = 0 \leq \tau_1 \leq \tau_2 \leq \dots, \tau_j \in \mathbb{N}$ to be specified below and $r_j \in \mathbb{N}, r_j \leq r_{j+1}$.

Here we extensively use the materials from Section 2.5 that deals with the case of constant rigidity. In what will follow we will describe a typical orbit in the system and will obtain statistical properties of the ensemble of the system that have independent initial distribution of scatterers. The results of this section were published in [22].

2.7.1 OS-model.

Again, in this model we only have two type of scatterers, $S_1 = \text{'LS'}$ and $S_2 = \text{'RS'}$. Assume that initial configuration of scatterers is such that there is no positive tail of right scatterers nor negative tail of left scatterers. The set of such configurations has

a full measure. We show here that the dynamics of this model is qualitatively similar to the case of constant rigidity.

The next theorem provides the results on the statistical properties of the OS-models with aging.

Theorem 2.7.1.1. *Consider the OS-model with the rigidity function defined by 2.7.1. Then, there is $t_0 > 0$ such that*

$$\mathbb{E}z(t) = 0, \quad \forall t > t_0.$$

Moreover, for any positive integer n there exist an increasing sequence of integers $\{T_j\}_{j=1}^n$, $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$, such that for any $\tau_1 \leq \tau_2 \leq \dots \leq \tau_n$ (as defined in 2.7.1), $\tau_j \in \mathbb{N}$, $\tau_j \geq T_{j+1}$, $j = 1, 2, \dots$ the following holds

$$C_1 \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right] \leq \mathbb{E}z^2(\tau_n) \leq C_2 \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right].$$

Proof. Theorem 2.5.1.1 states that the particle will oscillate regardless of the value of the rigidity and that there exists t_0 such that $\mathbb{E}z(t) = 0, \forall t > t_0$.

Select the sequence of integers $\{T_j\}_{j=1}^n$ as in Lemma 2.5.1.2 and let $\tau_j \geq T_{j+1}$, $j = 1, 2, \dots$. We now compute $\mathbb{E}z^2(\tau_n)$ for all n .

At $t = \tau_1$, by Theorem 2.5.1.1, there exist two constants $C_1 = C_1(1) > 0$ and $C_2 = C_2(q) > 0$ such that

$$C_1 \frac{\tau_1}{r_1} \leq \mathbb{E}z^2(\tau_1) \leq C_2 \frac{\tau_1}{r_1}.$$

At the moment of time $t = \tau_1$ we change the rigidity from r_1 to r_2 . When the rigidity of the environment is $r(t) = r_1$ the distance a particle travels in τ_1 units of time equals to the distance the particle travels in $\frac{r_2}{r_1}\tau_1$ units of time but with the rigidity of the environment $r(t) = r_2$. This follows from the relations

$$C_1 \frac{\frac{r_2}{r_1}\tau_1}{r_2} = C_1 \frac{\tau_1}{r_1} \leq \mathbb{E}z_1^2(\tau_1) \leq C_2 \frac{\tau_1}{r_1} = C_2 \frac{\frac{r_2}{r_1}\tau_1}{r_2},$$

and

$$C_1 \frac{\frac{r_2}{r_1}\tau_1}{r_2} \leq \mathbb{E}z_2^2\left(\frac{r_2}{r_1}\tau_1\right) \leq C_2 \frac{\frac{r_2}{r_1}\tau_1}{r_2}.$$

Next, particle continues its movement on the lattice with a new rigidity, r_2 . So the particle first travels τ_1 units of times when the rigidity of environment equals r_1 or, equivalently, for $\frac{r_2}{r_1}\tau_1$ units of time when the rigidity of environment equals r_2 , and $\tau_2 - \tau_1$ units of time when the rigidity of environment equals r_2 .

Hence, at the moment of time $t = \tau_2$ the following relationship holds

$$C_1 \frac{\frac{r_2}{r_1}\tau_1 + (\tau_2 - \tau_1)}{r_2} \leq \mathbb{E}z^2(\tau_2) \leq C_2 \frac{\frac{r_2}{r_1}\tau_1 + (\tau_2 - \tau_1)}{r_2}.$$

Therefore,

$$C_1 \left[\left(\frac{1}{r_1} - \frac{1}{r_2} \right) \tau_1 + \frac{\tau_2}{r_2} \right] \leq \mathbb{E}z^2(\tau_2) \leq C_2 \left[\left(\frac{1}{r_1} - \frac{1}{r_2} \right) \tau_1 + \frac{\tau_2}{r_2} \right].$$

We continue inductively to finish the proof. Suppose that at the time $t = \tau_n$ we have the following bounds

$$C_1 \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right] \leq \mathbb{E}z^2(\tau_n) \leq C_2 \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right].$$

Then by the same argument as above one gets at time $t = \tau_{n+1}$ that

$$\begin{aligned} \frac{C_1}{r_{n+1}} \left(r_{n+1} \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right] + \tau_{n+1} - \tau_n \right) &\leq \\ &\mathbb{E}z^2(\tau_{n+1}) \leq \\ \frac{C_2}{r_{n+1}} \left(r_{n+1} \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right] + \tau_{n+1} - \tau_n \right) &. \end{aligned}$$

or, equivalently,

$$C_1 \left[\sum_{j=1}^n \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_{n+1}}{r_{n+1}} \right] \leq \mathbb{E}z^2(\tau_{n+1}) \leq C_2 \left[\sum_{j=1}^n \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_{n+1}}{r_{n+1}} \right].$$

□

Observe that $\mathbb{E}z^2(t)$ grows linearly in time, similarly to the case of constant rigidity considered in [14]-[16].

2.7.2 NOS-model. Even rigidity.

In this model we have only two types of scatterers, $S_1 = \text{'FS'}$ and $S_2 = \text{'BS'}$. The dynamics of NOS-model with aging is more complicated than the one of the OS-model with aging. We first consider the cases when a parity of rigidity $r(t)$ does not change.

At first, we assume that r_j is even for all $j \in \mathbb{N}$.

Theorem 2.7.2.1 (NOS-model, even rigidity). *Consider the NOS-model. Suppose that r_j is even for all $j \in \mathbb{N}$. Then, there is t_0 such that*

$$\mathbb{E}z(t) = 0, \quad \forall t > t_0.$$

Moreover, for any fixed positive integers n there exist an increasing sequence of integers $\{T_j\}_{j=1}^n$ and $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ such that for any $\tau_1 \leq \tau_2 \leq \dots \leq \tau_n$ (as defined in 2.7.1), $\tau_j \in \mathbb{N}$,

$$\tau_j \geq T_{j+1}, \quad j = 1, 2, \dots \quad (2.7.2)$$

the following holds

$$C_1 \left[\sum_{j=1}^{n-1} \left(\frac{1}{\ln r_j} - \frac{1}{\ln r_{j+1}} \right) \ln \tau_j + \frac{\ln \tau_n}{\ln r_n} \right] \leq \mathbb{E}z^2(\tau_n) \leq C_2 \left[\sum_{j=1}^{n-1} \left(\frac{1}{\ln r_j} - \frac{1}{\ln r_{j+1}} \right) \ln \tau_j + \frac{\ln \tau_n}{\ln r_n} \right].$$

Proof. The geometric structure of the orbits does not change if we keep the parity constant. Thus if all r_j 's are even then almost all orbits will oscillate. Consider the collection of these orbits.

Suppose that

$$\tau_j \geq T_j, \quad \forall j \in \mathbb{N},$$

where integers T_j as in Lemma 2.5.2.2. Then we can repeat the same argument as in Theorem 2.7.1.1 to arrive at the result. \square

Hence, if we allow enough time for a system to evolve (i.e. each $T_j(q, r_j)$ is large enough) then the behavior of the systems with aging is similar to the ones with constant rigidity.

2.7.3 NOS-model. Odd rigidity.

In this model we have only two types of scatterers, $S_1 = \text{'FS'}$ and $S_2 = \text{'BS'}$. We assume that r_j is odd for all $j \in \mathbb{N}$. In this model particle will eventually propagate into one direction and that direction is determined by the scatterers around the origin and the initial velocity (see Theorem 2.4.3.1).

Theorem 2.7.3.1 (NOS-model, odd rigidity). *Suppose that r_j is odd for all $j \in \mathbb{N}$ and $\tau_j - \tau_{j-1} \geq T_j$ for all $j \in \mathbb{N}$. Then there are $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ such that*

$$C_1 \sum_{j=1}^n \frac{\tau_j - \tau_{j-1}}{r_j} \leq \mathbb{E} |z(\tau_n)| \leq C_2 \sum_{j=1}^n \frac{\tau_j - \tau_{j-1}}{r_j}, \quad \forall n \in \mathbb{N}.$$

Proof. During each interval $[\tau_{j-1}, \tau_j)$ of constant rigidity $r(z, t) = r_j$, $t \in [\tau_{j-1}, \tau_j)$, by Lemma 2.5.2.4 we have the following estimate on the average displacement of the particle during that time interval,

$$C_1 \frac{\tau_j - \tau_{j-1}}{r_j} \leq |\mathbb{E} z(\tau_j) - \mathbb{E} z(\tau_{j-1})| \leq C_2 \frac{\tau_j - \tau_{j-1}}{r_j}.$$

Since particle propagate into one direction on each of these intervals the result follows. □

Again, we have a behavior similar to the systems with constant rigidity: if we allow enough time for each interval of constant rigidity then we observe a propagation with $\mathbb{E} |z(\tau_n)|$ is growing linearly in time.

2.7.4 NOS-model. Alternating parity of the rigidity.

Again, we have only two types of scatterers, $S_1 = \text{'FS'}$ and $S_2 = \text{'BS'}$. Assume now that r_{2j} is even and r_{2j+1} is odd for all $j \in \mathbb{N}$. Also, assume that $\tau_1 = 0$, that is

we start with even rigidity (which corresponds to an oscillatory behavior). We will alter now the way rigidity changes, namely, we now assume that at the time τ_j (the moment of time we change a value of rigidity) we reset the index of all sites to zero, i.e. $\eta(z, \tau_{j+}) = 0, \forall z \in \mathbb{Z}, \forall j \in \mathbb{N}$.

Assume that τ_j satisfies the inequality

$$\tau_j - \tau_{j-1} \geq T_j, \quad j \geq 2, \quad \tau_1 = 0, \quad (\text{H1})$$

where T_j defined as in Lemmas 2.5.2.2 or 2.5.2.4 (depending on parity of j). That means that on each interval of time $[\tau_j, \tau_{j+1})$ on which rigidity is constant, we can apply the asymptotic estimates obtained earlier.

Also, assume that

$$\tau_{2j+1} - \tau_{2j} \gg \log(\tau_{2i} - \tau_{2i-1}), \quad \forall i, j \geq 1, \quad \tau_1 = 0. \quad (\text{H2})$$

This relationship says that the time intervals when a particle oscillates are much shorter compare to the time intervals when the particle propagates in one direction. This ensures that two consecutive oscillations do not overlap.

First, we consider the interfaces between intervals of time with even rigidity and with odd rigidity.

Lemma 2.7.4.1. *Suppose that at the time $t = \tau_{2n}$ rigidity switches from the even value $r = 2n$ to the odd value $r = 2n + 1$. Then the probability that $v(\tau_{2n+}) = +1$ is the same as the probability that $v(\tau_{2n+}) = -1$ and equals to $\frac{1}{2}$.*

Proof. During the times when rigidity is even particle undergoes oscillations with a zero mean. Thus the average velocity is zero. The result follows. \square

We now know that when rigidity switches to the an odd value, the particle will have the same probability of moving in either direction. During the time interval when rigidity is an odd number the particle eventually propagate in one direction.

Combining the results from Lemma 2.5.2.3 and Lemma 2.7.4.1 we obtain the following statement.

Lemma 2.7.4.2. *There exists increasing sequence of natural numbers $\{\hat{T}_{2j}\}_{j=1}^{\infty}$ such that if*

$$\tau_{2j} - \tau_{2j-1} > \hat{T}_{2j}, \quad \forall j \geq 1, \quad (\text{H3})$$

then the probability of

$$\mathbb{E}[z(t_{2j+1}) - z(t_{2j})] > 0$$

is the same as probability of

$$\mathbb{E}[z(t_{2j+1}) - z(t_{2j})] < 0$$

and is equal to 0.5.

Hence, if we pick the sequence $\{\tau_j\}$ in such a way so it satisfies conditions H1 - H3, when the value of the rigidity switches from even to odd number with the probability 0.5 the particle will propagate into one or another direction. Thus we have a situation resembling a standard random walk.

The oscillatory behavior during the intervals of time when rigidity is even plays a role of a "dice" (i.e. it randomizes the motion of the particle), but unlike the standard random walk the decision about the direction of the eventual propagation during the stage with odd rigidity is not made instantaneously. Instead, it takes $\tau_{2j} - \tau_{2j-1}$ units of time to make a decision. By Lemma 2.5.2.4 the displacement during the intervals of odd rigidity equals

$$|\mathbb{E}z(\tau_{2n+1}) - \mathbb{E}z(\tau_{2n})| \sim \frac{\tau_{2n+1} - \tau_{2n}}{r_{2n+1}}.$$

2.7.5 Summary for the models with aging and independent initial distribution of scatterers.

In the system with aging the value of the rigidity increases with time. Therefore we have a "slowing down" of the particle - it takes longer for the particle to "discover"

unvisited territory. On the other hand, after each time we change the rigidity (the values of τ_i) we have to wait for some period of time in order to obtain any meaningful statistical results. Therefore we introduce the following notation, similar to big-O notation. We say that $a(t, r) = \tilde{\mathcal{O}}(b(t, r))$ as $t \rightarrow \infty$ and $r \rightarrow \infty$, where r is the value of rigidity and t is time, if for any prescribed sequence r_i , $r_i \rightarrow \infty$, there exist a positive number N , two sequences, τ_i and t_i , $\tau_i \rightarrow \infty$, $t_i \rightarrow \infty$, $t_i \in (\tau_{i-1}, \tau_i)$, and two positive constants C_1 and C_2 , such that

$$C_1 b(t, r_i) \leq a(t, r_i) \leq C_2 b(t, r_i), \quad \forall t \in [t_i, \tau_i], \quad i \geq N.$$

Thus our results for the models with aging and independent initial distribution of scatterers can be written in the following way.

- **OS-model:** for almost all initial configuration of scatterers particle will oscillate around the origin and as $t \rightarrow \infty$ and $r \rightarrow \infty$,

$$\mathbb{E}z^2(t, r) = \tilde{\mathcal{O}}\left(\frac{t}{r}\right), \quad \mathbb{E}z(t) = 0.$$

- **NOS-model, even rigidity:** for almost all initial configuration of scatterers particle will oscillate around the origin and as $t \rightarrow \infty$ and $r \rightarrow \infty$ taking only even values,

$$\mathbb{E}z^2(t, r) = \tilde{\mathcal{O}}\left(\frac{\ln t}{\ln r}\right), \quad \mathbb{E}z(t) = 0.$$

- **NOS-model, odd rigidity:** for all initial configuration of scatterers particle will eventually propagate (on average) into one direction and as $t \rightarrow \infty$ and $r \rightarrow \infty$ taking only odd values,

$$\mathbb{E}|z(t, r)| = \tilde{\mathcal{O}}(t \ln r).$$

- **NOS-model, alternating rigidity:** the model resembles a complicated random walk with propagation alternating with oscillation.

2.8 Future directions.

Our next goal is to obtain statistical properties of DWRE for an arbitrary initial distribution of scatterers on \mathbb{Z}^1 and constant rigidity. The many-particle systems need to be investigated also.

After that we will look for the systems with interesting and rich behavior in higher dimensions (for example, when the graph is a square lattice; the case when the graph is a tree was investigated in [20] and triangular lattices were studied in [17]) and try to obtain some geometric and statistical results for these systems. But first some numerical simulations will be needed.

Lastly, it was mentioned in [22] that DWRE seem to have many applications in science and industries (for example in material science, communication theory, computer science, etc). These connections and applications need to be looked at more closely. I envision many exciting and promising opportunities for the future research that is of interest to the industry and NSF.

2.9 Conclusions.

In the first part of this chapter we demonstrated that deterministic walks in Markov environments have qualitatively the same dynamics as deterministic walks in environments with independently distributed scatterers. Such qualitative stability with the respect to non small but large perturbations of environment is another strong indication that real network may well be more similar to DWRE than to purely random systems. Indeed, living systems often demonstrate an amazing stability in changing environments. Another natural field of application of DWRE is a design of robust industrial systems, e.g. robotic networks.

In the second part we introduced a broad class of models which describe propagation of signals (particles, waves, information, etc) in discrete environments with aging. An environment could be any graph \mathcal{G} (directed or undirected). The process

of aging is described by a time-dependent rigidity of an environment. The exact results were obtained for the case when \mathcal{G} is \mathbb{Z}^1 and the rigidity is a piecewise constant function of time. It is unlikely that sufficiently complete analytical results could be obtained for the general class of graphs (general structures of networks). However, these models are very amendable for numerical studies. Therefore, it seems feasible that deterministic walks in random environments with aging could provide a lot of useful information about the behavior of systems with aging.

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