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CERTAIN ASPECTS OF THE STABILITY OF SOLUTIONS OF SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS

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INTRODUCTION

In a large class of physical problems one is interested in studying the behavior of solutions of systems of nonlinear differential equations in some vicinity of known solutions, e.g. the steady state of some physicial system. Such systems are conveniently represented in matrix and vector form. If, as is usually the case, the number of equations is large, it is frequently impractical (in fact in many cases impossible) to find exact solutions. One is therefore often content to predict the behavior of solutions without actually finding them.

A situation of interest is the nonlinear system

$$x' = Ax + f(t,x) \qquad (' = d/dt, t \geq 0) \tag{0.1}$$

where A is a real constant matrix with n rows and n columns and x,f are real vectors with n components. It is seen that the linear system

$$y' = Ay \qquad (' = d/dt) \tag{0.2}$$

represents the linearized case of (0.1). It can often be proved that under suitable conditions solutions of the nonlinear system (0.1) have for large t the same behavior as those of the linear system (0.2). Some of the classical results in this direction will be stated. It is the purpose of this study to establish certain generalizations of these classical results.

CHAPTER I

DEFINITIONS AND PRELIMINARY MATERIAL

The material in this chapter is well known and is presented merely for the sake of completeness. It may be found, for example, in Coddington and Levinson, Chapter III [1]*.

1.1. The norm, $|x|$, of a complex vector x with components x_i, $(i = 1, 2, \ldots, n)$, is defined to be

$$|x| = \sum_{i=1}^{n} |x_i|$$

and the Euclidean norm, $||x||$, is defined to be

$$||x|| = \left\{ \sum_{i=1}^{n} |x_i|^2 \right\}^{1/2}$$

where $|x_i|$ represents the absolute value of x_i.

These norms satisfy the following inequalities:

$$||x|| \leq n^{1/2} \; ||x||$$

$$n^{-1/2} \; |x| \leq ||x|| \leq |x|$$

*Numbers in brackets describe the references listed in the bibliography.

1.2. Let A and B be n by n matrices with complex elements (a_{ij}) and (b_{ij}) respectively. The norm, $|A|$, is defined to be

$$|A| = \sum_{i,j=1}^{n} |a_{ij}|$$

And the Euclidean norm is defined to be

$$||A|| = \left\{ \sum_{i,j=1}^{n} |a_{ij}|^2 \right\}^{1/2}$$

where $|a_{ij}|$ represents the absolute value of a_{ij}.

These norms satisfy the following inequalities:

(a) $|A + B| \leq |A| + |B|$, $||A + B|| \leq ||A|| + ||B||$

(b) $|AB| \leq |A| \cdot |B|$, $||AB|| \leq ||A|| \cdot ||B||$

(c) $|Ax| \leq |A| \cdot |x|$, $||Ax|| \leq ||A|| \cdot ||x||$

(d) $||A|| \leq |A|$

where x is an n dimensional vector.

1.3. Let g(t) be a vector function with components $g_i(t)$, $(i = 1, 2, \ldots, n)$. The vector g(t) is said to be a continuous (differentiable or integrable) function of t for t in some real interval I, if and only if each component $g_i(t)$ is a continuous (differentiable or integrable) function of t for t in I.

1.4. If $g(t)$ is a differentiable vector function of t for t in some real interval I such that $||g(t)|| \neq 0$, then

$$\Big| \, ||g(t)||\,' \, \Big| \leq ||g'(t)|| \qquad (' = d/dt)$$

where $g'(t)$ is the vector whose jth component is $g'_j(t)$ and t is in I.

1.5. If $g(t)$ and $|g(t)|$ are integrable functions of t for t in some real interval I: (a,b), then

$$\left| \int_a^b g(t)dt \right| \leq \int_a^b |g(t)|dt$$

where by $\int_a^b g(t)dt$ is meant the vector whose jth component is

$$\int_a^b g_j(t)dt$$

1.6. If $\{A_m\}$ is a sequence of matrices, then this sequence is said to be convergent in norm if given any $\epsilon > 0$ there exists a positive integer N_ϵ such that

$$p,q > N_\epsilon \quad \text{implies that} \quad |A_q - A_p| < \epsilon$$

The sequence $\{A_m\}$ is said to have the limit A if given any $\epsilon > 0$ there exists a positive integer N_ϵ such that

$$m > N_\epsilon \quad \text{implies that} \quad |A_m - A| < \epsilon$$

1.7. The exponential of a matrix A - exp(A) or e^A - is defined by the infinite series

$$\exp(A) = E + \sum_{i=1}^{\infty} \frac{A^i}{i!}$$

where E is the nth order identity matrix. The sequence of partial sums of this series is convergent in the sense of 1.6 for all A.

1.8. Matrices A and B of 1.2 are said to be similar if there exists a nonsingular n by n matrix P with complex elements such that $B = PAP^{-1}$.

Let the characteristic roots of the matrix A described in 1.2 be λ_i, $(i = 1, 2, \ldots, m)$. By a well known theorem from algebra (see [1], p.63) it is possible to find a nonsingular n by n matrix P with complex elements such that $P^{-1}AP = J$, where J is the Jordan (or classical) canonical form of A. The n by n matrix J is given by

$$J = \begin{bmatrix} J_0 & & & & 0 \\ & J_1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & \cdot & J_s \end{bmatrix}$$

where

$$J_0 = \begin{bmatrix} \lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & \cdot & \lambda_q \end{bmatrix} ,$$

and

$$J_k = \begin{bmatrix} \lambda_{q+k} & 1 & & 0 \\ \cdot & & \cdot & \\ & \cdot & & \cdot \\ 0 & & \cdot & 1 \\ & & & \lambda_{q+k} \end{bmatrix} ,$$

$k = 1, 2, \ldots, s$, is a r_k by r_k matrix and λ_{q+k} a characteristic root of multiplicity r_k, and where $q + s = m$ for some natural number m which is less than or equal to n, and all of the characteristic roots of multiplicity one are among the λ_i, $(i = 1, 2, \ldots, q)$. In particular if all of the characteristic roots are distinct, then A is similar to a diagonal matrix.

1.10. It follows by using the canonical form for J in 1.8 and the definition of the exponential of a matrix A in 1.7, that the matrix $\exp(tJ)$, where t is a real variable, is given by the n by n matrix

$$e^{tJ} = \begin{bmatrix} e^{tJ_0} & & & 0 \\ & e^{tJ_1} & & \\ & & \ddots & \\ 0 & & & e^{tJ_s} \end{bmatrix},$$

where

$$e^{tJ_0} = \begin{bmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_2} \end{bmatrix}$$

and

$$e^{tJ_k} = \begin{bmatrix} e^{t\lambda_{q+k}} & te^{t\lambda_{q+k}} & \cdots & \frac{t^{r_k-1}}{(r_k-1)} e^{t\lambda_{q+k}} \\ & \ddots & \ddots & \vdots \\ & & \ddots & te^{t\lambda_{q+k}} \\ 0 & & & e^{t\lambda_{q+k}} \end{bmatrix}.$$

$K = 1, 2, \ldots, s$.

1.11. Let z be a complex number. Denote the real part of z by Re(z).

1.12. Consider the system of differential equations

$$x' = F(t,x) \qquad (' = d/dt, t \geq 0) \tag{1.1}$$

where x,F are real vectors with n components. A solution $x = \phi(t)$ of the system (1.1) is said to be stable if given any $\epsilon > 0$ there exists a $\delta > 0$ and a number $T \geq 0$ such that for any solution $x = \psi(t)$ of (1.1) it is true that

$$|\psi(T) - \phi(T)| < \delta \text{ implies that } |\psi(t) - \phi(t)| < \epsilon$$

for all $t \geq T$. If the above definition is not satisfied, then the solution $x = \phi(t)$ of (1.1) is said to be unstable. The solution $x = \phi(t)$ of (1.1) is said to be asymptotically stable if in addition to being stable it is true that

$$|\psi(t) - \phi(t)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty$$

It should be noted that the number T is either zero or greater than zero. The former case is referred to as stability over the interval $0 \leq t < +\infty$ and the latter has stability over $T \leq t < +\infty$.

1.13. Some classical results concerning the stability of the identically zero solution of systems of nonlinear differential equations of certain forms will now be given.

Consider the following special case of the system (1.1)

$$x' = Ax + f(t,x) \qquad (' = d/dt, t \geq 0) \tag{1.2}$$

where A is a real n by n matrix, x,f are real vectors with n components, and $f(t,0) = 0$. Thus, the vector $\phi \equiv 0$ is a solution of (1.2) by inspection.

<u>Theorem 1.1.</u> (Perron) Let f be continuous for small $|x|$ and

$t \geq 0$, and let

$$\lim_{x \to 0} \frac{|f(t,x)|}{|x|} = 0$$

uniformly in $t, t \geq 0$. If all the characteristic roots of A in (1.2) have negative real parts, then the identically zero solution of (1.2) is asymptotically stable. The proof of this theorem may be found in [1], p. 314.

It is interesting to observe that every solution of the linear system

$$y' = Ay \tag{1.3}$$

obtained from (1.2) by dropping the nonlinear terms $f(t,x)$, has the form

$$\psi(t) = \left\{ \exp(tA) \right\} k$$

where k is a constant vector. Lemma 3.1 of Chapter III enables one to conclude that if all the characteristic roots of the matrix A have negative real parts, then every solution of the system (1.3) approaches zero in the sense of the norm 1.1, as t becomes positively infinite. Thus the zero solution of (1.3) is asymptotically stable.

In view of these considerations one can make an equivalent statement of Theorem 1.1 more enlightening for the purpose of applications:

If

$$\lim_{x \to 0} \frac{|f(t,x)|}{|x|} = 0$$

uniformly in $t, t \geq 0$, then asymptotic stability of the zero solution of the linear system (1.3) implies asymptotic stability of the zero solution of the nonlinear system (1.2).

To apply the theorem one merely needs to check whether the nonlinear terms satisfy the condition given, and to compute the characteristic roots of the matrix A. The former is a simple task and the latter can be done advantageously by employing high speed computers.

The results of Theorem 1.1 are the best possible in the sense that if at least one characteristic root of the matrix A has its real part positive, then the zero solution of (1.2) cannot be stable.

Theorem 1.2. Let at least one characteristic root of the matrix A in (1.2) have its real part positive. Assume that for some positive constant k and $|x|$ sufficiently small

$$|f(t,x)| \leq k|x| \qquad (t \geq 0) \tag{1.4}$$

and assume further that given any $\epsilon > 0$ there exists a $\delta > 0$ and $T \geq 0$ such that

$$|f(t,x)| \leq \epsilon|x| \quad \text{for all } |x| \leq \delta \text{ and } t \geq T \tag{1.5}$$

Then the zero solution of (1.2) is unstable. For proof of this theorem see [1], p. 317.

The special case of the system (1.2) which will be considered in this study is the system

$$x' = Ax + B(t)x + g(t,x) \qquad (' = d/dt, t \geq 0) \tag{1.6}$$

where the matrix A is a real constant matrix with n rows and n columns, B(t) is a real n by n matrix, and x,g are real vector functions with n components such that g(t,0) = 0. Furthermore,

$$\lim_{x \to 0} \frac{|B(t)x + g(t,x)|}{|x|} \neq 0$$

for all t, thus Theorem 1.1 cannot be applied. The well known result which follows is a generalization of Theorem 1.1 and may be found in [1], p. 316.

Theorem 1.3. Let the matrix A in (1.6) have all of its characteristic roots with negative real parts. Let g satisfy the same conditions as f does in Theorem 1.1. Moreover, let $B(t) \to 0$, as $t \to +\infty$ and let B(t) be continuous in t for $t \geq 0$. Then the zero solution of (1.6) is asymptotically stable.

The proof of Theorem 1.3 is a special case of the proof of Theorem 4.1 in Chapter IV.

The chief aim of this study is to generalize Theorem 1.3 and to show that the generalization obtained is the best possible in the same sense as Theorem 1.1 is the best possible. This is done in Chapter IV. It is not surprising that the classical results stated above turn out to be special cases of the theorems in Chapter IV.

It should be pointed out that the case when the matrix A has at least one characteristic root with zero real part would require special investigation. While this situation is of importance in many applications, it will not be considered in this study.

CHAPTER II

EXAMPLES AND MOTIVATION

The examples in this chapter are of a rather simple nature. Their purpose is to illustrate the essential ideas involved. In the cases in which nonlinear terms are not considered, it is assumed that the nonlinear terms would not affect the stability or instability of solutions, e.g. if the nonlinear terms satisfied the hypotheses on f in Theorem 1.1 , then stability would not be affected.

Example 1. Consider the scalar equation

$$x' = \left(-1 + f(t)\right) x \qquad (' = d/dt, t \geq 0) \tag{2.1}$$

where

$$f(t) = \begin{cases} t; & 0 \leq t < T \\ T; & t \geq T \end{cases}$$

Observe that in this case $B(t)$ in Theorem 1.3 is given by T for $t \geq T$. Thus, $B(t) \to T$, as $t \to +\infty$. Hence, Theorem 1.3 does not apply for $T \geq 0$.

It is clear that all solutions of (2.1) are of the form

$$\phi(t) = c_1 \exp\left[(-1 + T)t + T^2/2\right] = e_1 \exp(-1 + T)t \tag{2.2}$$

where c is an arbitrary constant. It is easy to see that the zero solution of (2.1) is asymptotically stable for $0 \leq T < 1$ and is unstable for $T > 1$. If $T = 1$, the zero solution of (2.1) is stable, but not asymptotically. Notice that if $T > 1$, the characteristic root of $A + B(t)$ is positive for all $t > 0$.

Example 2. Consider the system

$$x' = -\left(1 - f(t)\right) x_1 + x_1^2$$
$$x'_2 = -\left(m - h(t)\right) x_2 - kx_1 \qquad (' = d/dt, t \geq 0) \qquad (2.3)$$

where k and m are positive constants and

$$f(t) = \begin{cases} t; & 0 \leq t < T \\ T; & t \geq T \end{cases}$$

and

$$h(t) = \begin{cases} t; & 0 \leq t < T \\ T; & t \geq T \end{cases}$$

It is clear that all solutions of the system (2.3) are of the form

$$\phi_1(t) = c_1 \exp\left[-t(1 - T)\right]$$
$$\phi_2(t) = c_2 \exp\left[-t(m - T)\right] + c_3 \exp\left[-t(1 - T)\right] \qquad (2.4)$$

for arbitrary constants c_1, c_2, c_3 and for $t \geq T$. Thus, the zero solution of the system (2.3) is asymptotically stable for $T < \min(l,m)$, unstable for $T > \min(l,m)$, and stable (but not asymptotically) for $T = \min(l,m)$.

Observe that Theorem 1.3 does not apply since the matrix which corresponds to $B(t)$ does not approach zero as $t \to +\infty$.

Example 3. The example which follows may be found in Bellman, p. 87 [2].

Consider the system

$$\begin{aligned} x'_1 &= -ax_1 \\ x'_2 &= (\sin(\log t) + \cos(\log t) - 2a)x_2 + x_1^2 \end{aligned} \qquad (' = d/dt, t > 0) \tag{2.5}$$

where a is a constant. The linearized case of (2.5) is the linear system

$$\begin{aligned} y'_1 &= -ay_1 \\ y'_2 &= (\sin(\log t) + \cos(\log t) - 2a)y_2 \end{aligned} \qquad (' = d/dt, t > 0) \tag{2.6}$$

It is clear that the system (2.5) is a special case of (1.6) and that solutions of (2.5) are of the form

$$\begin{aligned} \phi_1(t) &= \phi_1(0)\exp(-at) \\ \phi_2(t) &= \Big(\exp(t\sin(\log t) - 2at)\Big)\Big\{\phi_2(0) + \phi_1^2(0)\int_0^t \exp(-u\sin\log u)\,du\Big\} \end{aligned} \tag{2.7}$$

and solutions of (2.6) are of the form

$$\begin{aligned} \psi_1(t) &= \psi_1(0)\exp(-at) \\ \psi_2(t) &= \psi_2(0)\exp(t\sin(\log t) - 2at) \end{aligned} \tag{2.8}$$

It is clear that if $2a > 1$, then

$$|\psi_j(t)| \to 0, \text{ as } t \to +\infty, \quad (j = 1, 2)$$

It is to be shown that if

$$1 + \exp(-\pi/2) > 2a > 1 \quad , \tag{2.9}$$

then

$$|\phi_j(t)| \to 0, \text{ as } t \to +\infty \quad (j = 1,2)$$

only if

$$\phi_1(0) = 0 \quad .$$

Hence, requiring that $|\phi_j(0)|$ be sufficiently small is not sufficient for stability of the identically zero solution of (2.5) even though the solution of the linear system (2.6) tends to zero in norm as $t \to +\infty$ for all a satisfying (2.9)

Consider the integral

$$\int_0^t \exp\left[-s\sin(\log s)\right] ds$$

Since the integrand is positive for t positive, it follows that

$$\int_0^t \exp\left[- s \sin(\log s)\right] ds > \int_{te^{-\pi}}^{te^{-2\pi/3}} \exp\left[-s \sin(\log s)\right] ds$$

Now for

$$t = \exp\left[\pi(2k + 1/2)\right], \quad (k = 1, 2, \ldots)$$

it is true that

$$t \exp\left(-2\pi/3\right) = \exp(2k\pi - \pi/6)$$

and

$$t \exp(- \pi) = \exp(2k\pi - \pi/2)$$

for the specified values of t. Thus, for

$$\exp\left(2k\pi - \pi/2\right) \leq s \leq \exp(2k\pi - \pi/6)$$

it follows that

$$\exp(s/2) \leq \exp\left[- s \sin(\log s)\right] \leq \exp(s)$$

Moreover,

$$\exp\left(t\left[\exp(- \pi/2)\right]\right) < \exp(t/2)$$

So that

$$\int_{te^{-\pi}}^{te^{-2\pi/3}} \exp\left[-s\ \sin(\log\ s)\right]\ ds\ >\ \exp\left(t\left[\exp(-\ \pi/2)\right]\right)\int_{te^{-\pi}}^{te^{-2\pi/3}} ds$$

or

$$\int_0^t \exp\left[-s\ \sin(\log\ s)\right]\ ds\ >\ t(e^{-2\pi/3} - e^{-\pi})\ \exp(te^{-\pi/2})$$

for $t = \exp\left[\pi\ (2k + 1/2)\right]$, $(k = 1, 2, \ldots)$. Hence for the above values of t it is true that

$$\exp(t\ \sin(\log\ t) - 2at)\int_0^t \exp\left[-s\ \sin(\log\ s)\right]\ ds$$

$$> c\ t\ \exp\left\{\left[1 + \exp(-\ \pi/2)\ - 2a\right] t\right\}, \tag{2.10}$$

since

$$t\ \sin(\log\ t) - 2at = t - 2at$$

for $t = \exp\left[\pi(2k + 1/2)\right]$, $(k = 1, 2, \ldots)$, and where

$$c = \exp(-2\pi/3) - \exp(-\pi)$$

Using (2.10) with (2.7) it follows that if

$$1 + \exp(-\pi/2) > 2a > 1$$

then

$$|\phi_j(t)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty, \quad (j = 1, 2)$$

only if $\phi_1(0) = 0$. The above follows since (5.6) implies that $|\phi_j(t)| \longrightarrow +\infty$, as $t \longrightarrow +\infty$ for

$$1 + \exp(-\pi/2) > 2a > 1$$

and for a so restricted the solutions of the linear system (2.6) tend to zero in norm as $t \longrightarrow +\infty$.

The above examples suggest the conjectures which follow.

Conjecture 1. In Theorem 1.3 the condition $B(t) \longrightarrow 0$, as $t \longrightarrow +\infty$ may be made less restrictive. It suffices to require that $B(t)$ be such that

$$|B(t)| \leq M \qquad (t \geq 0)$$

where M is a sufficiently small positive constant, or that for some constant $T \geq 0$ it is true that

$$|B(t)| \leq M \qquad (t \geq T)$$

where M is a sufficiently small positive constant.

This conjecture is suggested by all of the above examples. This conjecture will be established in Chapter IV.

Conjecture 2. One might expect that if at least one characteristic root of the matrix A in (1.6) has its real part positive, then the zero solution of (1.6) is unstable.

This conjecture will be established in Chapter IV.

Conjecture 3. It is sufficient to require that the linear system which represents the linearized case of (1.6) have solutions which tend to zero as t becomes positively infinite in order that the zero solution of (1.6) be asymptotically stable.

This conjecture is not true. Example 3. is a counterexample of this conjecture.

Conjecture 4. Let the matrix A of (1.6) satisfy the hypotheses of Theorem 1.3. Suppose that B(t) fails to satisfy the requirement that $B(t) \rightarrow 0$, as $t \rightarrow +\infty$, but that the matrix A + B(t) is such that at least one of its characteristic roots becomes positive for some $t = T \geq 0$ and remains positive for all $t \geq T$, then the zero solution of (1.6) is unstable.

On the basis of Example 1. this conjecture seems reasonable; however, at the present time its proof is not complete.

CHAPTER III

FOUR LEMMAS

The following result is well known; however, its proof has not been found in the literature.

Lemma 3.1. Let A be any constant n by n matrix with complex elements. Suppose that there exists a constant $\mu > 0$ such that for every characteristic root λ_i of the matrix A,

$$\mathrm{Re}(\lambda_i) \leq -\mu < 0 \qquad (i = 1, 2, \ldots, n)$$

(Note that the λ_i need not be distinct.) Let σ be a constant such that $0 < \sigma < \mu$. Then there exists a positive constant K such that

$$|\exp(tA)| \leq K \exp(-\sigma t) \qquad (t \geq 0) \tag{3.1}$$

where K depends only on μ, σ, and A.

Proof of Lemma 3.1. It follows from paragraph 1.8, Chapter I that there exists a nonsingular, constant, n by n matrix P such that $P^{-1}AP = J$ or $A = PJP^{-1}$, where J is the Jordan canonical form of the matrix A. If paragraph 1.7 is applied, then

$$\exp(tA) = \exp\left[t(P\,J\,P^{-1})\right] = P\left[\exp(tJ)\right]P^{-1}$$

Use of paragraph 1.2 in this equation yields

$$|\exp(tA)| \leq |P|\cdot|P^{-1}|\cdot|\exp(tJ)| \tag{3.2}$$

Thus, if one applies paragraphs 1.1, 1.7, and 1.8, there results

$$|\exp(tJ)| = \sum_{j=1}^{q} |e^{t\lambda_j}| + |e^{t\lambda_{q+1}}| \sum_{j=0}^{r_1-1} \frac{(r_1 - j)}{j!} t^j + \ldots$$

$$+ |e^{t\lambda_{q+s}}| \sum_{j=0}^{r_s-1} \frac{(r_s - j)}{j!} t^j \qquad (3.2a)$$

Since by hypothesis $Re(\lambda_i) \leq -\mu < 0$, $(i = 1,2,\ldots,q,\ldots,q+s = m)$, then

$$|\exp(t\lambda_i)| \leq \exp(-\mu t) \quad (i = 1,2,\ldots,q,\ldots,q + s = m)$$

where $m \leq n$.

This result used with (3.2a) yields

$$|\exp(tJ)| \leq \exp(-\mu t)\Bigg\{ q + \sum_{\nu=0}^{r_1-1} \frac{(r_1-\nu)}{\nu!} t^\nu + \ldots +$$

$$+ \sum_{\nu=0}^{r_s-1} \frac{(r_s-\nu)}{\nu!} t^\nu \Bigg\} \qquad (3.3)$$

Let

$$r = \max(r_i-1), \ (i = 1, 2, \ldots, s) \quad ;$$

then for suitable constants α_i, $i = 1, 2, \ldots, r + 1$, the inequality

(3.3) yields

$$|\exp(tJ)| \leq \exp(-\mu t) \sum_{i=0}^{r} \alpha_{i+1} t^{r-i} \qquad (3.4)$$

Now there exists a positive constant M_j, $0 \leq j \leq r$ such that

$$t^{r-j} \exp(-\mu t) \leq M_j \exp(-\sigma t) \qquad (t \geq 0) \qquad (3.5)$$

where $0 < \sigma < \mu$ or, equivalently such that

$$t^{r-j} \exp\left[(\sigma - \mu)t\right] \leq M_j \qquad (t \geq 0)$$

To show this let $\sigma - \mu = -\delta$. Since $0 < \sigma < \mu$, $-\delta < 0$ the function

$$t^{r-j} \exp(-\delta t)$$

is clearly smooth for $t \geq 0$. Using elementary calculus one finds that for $t = (r-j)/\delta$ the function assumes its maximum given by

$$t^{r-j} \exp(-\delta t) \Big|_{t=(r-j)/\delta} = \left[(r-j)/\delta\right]^{r-j} \exp\left[-(r-j)\right]$$

The above is clearly valid for $j = 0, 1, \ldots, r$. Therefore, take

$$M_j = \left[(r-j)/\delta\right]^{(r-j)} \exp\left[-(r-j)\right] \qquad (0 \leq j \leq r)$$

Thus,

$$t^{r-j}\exp(-\delta t) \leq M_j \qquad (0 \leq j \leq r, t \geq 0)$$

or

$$t^{r-j}\exp(-\mu t) \leq M_j \exp(-\sigma t) \qquad (0 \leq j \leq r, t \geq 0)$$

which establishes (3.5)

Let

$$M = \max(M_j), \ (j = 0, 1, \ldots, r).$$

Thus,

$$t^{r-j}\exp(-\mu t) \leq M \exp(-\sigma t) \qquad (0 \leq j \leq r, t \geq 0) \qquad (3.6)$$

From (3.4) and (3.6) it follows that

$$|\exp(tJ)| \leq M \exp(-\sigma t) \sum_{i=1}^{r+1} \alpha_i$$

Applying this to (3.3) one obtains

$$|\exp(tA)| \leq |P| \cdot |P^{-1}| \, M \sum_{i=1}^{r+1} \alpha_i \cdot \exp(-\sigma t) \quad (t \geq 0)$$

Let

$$K = |P| \cdot |P^{-1}| \ M$$

Hence,

$$|\exp(tA)| \leq K \exp(-\sigma t) \qquad (t \geq 0)$$

where K depends only on μ , σ , and A.

The next result will be needed in the proof of Lemma 3.3.

Lemma 3.2. Let g,B of the system (1.6) be continuous for all small $|x|$ and $t \geq 0$. Let B satisfy the inequality

$$|B(t)| \leq M \qquad (t \geq T)$$

for some $T \geq 0$, where M is a sufficiently small positive constant. For some positive constant k let g satisfy

$$|g(t,x)| \leq k|x| \qquad (t \geq 0)$$

and assume that given any $\epsilon > 0$ there exists a $\delta > 0$ such that for some $T_1 \geq 0$ it is true that

$$|g(t,x)| \leq \epsilon |x| \qquad (|x| \leq \delta, t \geq T_1)$$

Moreover, let $\phi = \phi(t)$ be a solution of the system (1.6). Then for as long as ϕ exists it is true that

$$|\phi(t)| \leq n^{1/2} |\phi(0)| \exp\left[(|A| + M_2 + kn^{1/2})\, t\right] \qquad (3.7)$$

and

$$|\phi(0)| \leq n^{1/2}|\phi(t)| \exp\left[(|A| + M_2 + kn^{1/2})\, t\right] \qquad (3.8)$$

where M_2 is to be specified.

Proof of Lemma 3.2. Since $\phi = \phi(t)$ is a solution of the system (1.6), then

$$\phi'(t) = A\phi(t) + B(t)\phi(t) + g\left[t,\phi(t)\right] \qquad (' = d/dt, t \geq 0)$$

for as long as ϕ exists.

It follows from 1.2 that

$$\begin{aligned} ||\phi'(t)|| &\leq ||A||\cdot||\phi(t)|| + ||B(t)||\cdot||\phi(t)|| + ||g\left[t,\phi(t)\right]|| \\ &\leq ||A||\cdot||\phi(t)|| + |B(t)|\cdot||\phi(t)|| + |g\left[t,\phi(t)\right]| \end{aligned} \qquad (3.9)$$

Since $B(t)$ is continuous for $t \geq 0$, there exists a positive constant M_1 such that

$$|B(t)| \leq M_1 \qquad (0 \leq t \leq T)$$

Hence,

$$|B(t)| \leq M_2 \qquad (t \geq 0)$$

where

$$M_2 = \max(M, M_1)$$

If the hypotheses on g,B and the last result are applied, then, for as long as ϕ exists, (3.9) becomes

$$||\phi'(t)|| \leq ||A||\cdot||\phi(t)|| + M_2||\phi(t)|| + kn^{1/2}||\phi(t)||$$

$$\leq \left\{||A|| + M_2 + kn^{1/2}\right\}||\phi(t)|| \qquad (t \geq 0) \quad .$$

It follows from paragraph 1.4 that the last inequality may be written as

$$||\phi(t)||' \leq \left\{||A|| + M_2 + kn^{1/2}\right\}||\phi(t)|| \quad (' = d/dt) \quad (3.10)$$

This implies that

$$\left\{||\phi(t)|| \exp(-\int_0^t (||A|| + M_2 + kn^{1/2})ds\right\}' \leq 0 \quad (' = d/dt)$$

Thus, integration yields

$$||\phi(s)|| \exp(-(||A|| + M_2 + kn^{1/2})\ s)\ \Big|_0^t \leq 0$$

or

$$||\phi(t)|| \leq ||\phi(0)|| \exp\left[(||A|| + M_2 + kn^{1/2})\ t\right]$$

Now use of 1.2. yields

$$|\phi(t)| \leq |\phi(0)|n^{1/2} \exp\left[(||A|| + M_2 + kn^{1/2})\ t\right] \qquad (t \geq 0)$$

for as long as ϕ exists. Therefore, (3.7) is established by an elementary argument on continuation of solutions.

In order to verify the inequality (3.8) one observes from (3.10) that

$$||\phi(t)||' \geq -\left\{||\phi(t)|| \; (||A|| + M_2 + kn^{1/2})\right\}$$

for as long as ϕ exists. From this result one obtains

$$\left\{||\phi(t)|| \; \exp(\int_0^t (||A|| + M_2 + kn^{1/2}) \, ds)\right\}' \geq 0 \quad .$$

Thus, integration yields

$$||\phi(s)|| \; \exp\left[(||A|| + M_2 + kn^{1/2}) \; s\right] \Big|_0^t \geq 0 \quad ,$$

or

$$||\phi(0)|| \leq ||\phi(t)|| \; \exp\left[(||A|| + M_2 + kn^{1/2}) \; t\right]$$

Now use of paragraph 1.2 yields

$$|\phi(0)| \leq n^{1/2} \; |\phi(t)| \; \exp\left[(||A|| + M_2 + kn^{1/2}) \; t\right]$$

for as long as ϕ exists. This is precisely the inequality (3.8). This completes the proof of Lemma 3.2.

Lemma 3.3. Let g,B in the system (1.6) satisfy the hypotheses of Lemma 3.2. Then the solution $\phi \equiv 0$ of (1.6) is stable over the interval

$T \le t < +\infty$, if and only if it is stable over the interval $0 \le t < +\infty$.

Proof of Lemma 3.3. Suppose that the solution $\tilde{\phi} \equiv 0$ of (1.6) is stable over the interval $T \le t < +\infty$. By the definition given in paragraph 1.12 this implies that given any $\epsilon_2 > 0$ there exists a $\delta_2 > 0$ such that for any solution $\phi(t)$ of (1.6) which satisfies $|\phi(T)| < \delta_2$, it will be true that $|\phi(t)| < \epsilon_2$ for all $t \ge T$. Equation (3.7) implies that if $|\phi(0)|$ is sufficiently small, then $|\phi(t)|$ is small for $t \ge 0$. Specifically, one can find a $\delta^* > 0$ such that

$$|\phi(0)| < \delta^* \text{ implies that } |\phi(t)| < \min(\delta_2, \epsilon_2) \quad (0 \le t \le T)$$

But, by hypothesis one has

$$|\phi(T)| < \delta_2 \text{ implies that } |\phi(t)| < \epsilon_2 \qquad (t \ge T)$$

Hence,

$$|\phi(0)| < \delta^* \text{ implies that } |\phi(t)| < \min(\delta_2, \epsilon_2) \le \epsilon_2 \quad (t > 0)$$

That is, the solution $\phi \equiv 0$ of (1.6) is stable over the interval $0 \le t < +\infty$.

(b) Suppose that the solution $\tilde{\phi} \equiv 0$ of (1.6) is stable over the interval $0 \le t < +\infty$. By the definition given in 1.12 this implies that given any $\epsilon_1 > 0$ there exists a $\delta_1 > 0$ such that for any solution $\phi(t)$ of (1.6) which satisfies $|\phi(0)| < \delta_1$, it will be true that $|\phi(t)| < \epsilon_1$ for all $t \ge 0$. Equation (3.8) implies that if $|\phi(T)|$ is sufficiently small, then $|\phi(0)|$ is small. Specifically, we can find a $\delta^* > 0$ such that

$$|\phi(T)| < \delta^* \text{ implies that } |\phi(0)| < \delta_1$$

But, by hypothesis

$$|\phi(0)| < \delta_1 \text{ implies that } |\phi(t)| < \epsilon_1 \qquad (t \geq 0)$$

Hence,

$$|\phi(T)| < \delta^* \text{ implies that } |\phi(t)| < \epsilon_1 \qquad (t \geq T)$$

That is, the solution $\tilde{\phi} \equiv 0$ of (1.6) is stable over the interval $T \leq t < +\infty$. This completes the proof of Lemma 3.3.

Remark. In the sense of the above lemma stability of a solution of a system over the interval $T \leq t < +\infty$ is said to be equivalent to stability over the interval $0 \leq t < +\infty$.

Lemma 3.4. If A is an n by n constant matrix with complex elements, then the matrix A is similar to a matrix B which is given by

$$B = \begin{bmatrix} \lambda_1 & & & & & & & & & & & \\ & \lambda_2 & & & & & & & & & & 0 \\ & & \ddots & & & & & & & & & \\ & & & \lambda_q & & & & & & & & \\ & & & & \lambda_{q+1} & \gamma & & & & & & \\ & & & & & \lambda_{q+1} & \gamma & & & & & \\ & & & & & & \ddots & \ddots \; \gamma & & & & \\ & & & & & & & \lambda_{q+1} & \ddots & & & \\ & & & & & & & & \ddots & \ddots & & \\ & & & & & & & & & \lambda_{q+s} & \gamma & \\ & 0 & & & & & & & & & \lambda_{q+s} & \gamma \ddots \gamma \\ & & & & & & & & & & & \ddots \; \lambda_{q+s} \end{bmatrix}$$

where γ is a nonzero constant, and the λ_i, $(i = 1, 2, \ldots, q+s)$ are the characteristic roots of the matrix A.

Proof of Lemma 3.4. It must be shown that there exists a nonsingular constant matrix P such that $N = P^{-1} A P$, without loss of generality A can be taken to be in the Jordan form of paragraph 1.8. Let the matrix P be such that $P = (p_{ij})$, $(i, j = 1, 2, \ldots, n)$, where

$$p_{ij} = \gamma^{i-1} \delta_{ij} \qquad (i, j = 1, 2, \ldots, n)$$

and

$$\delta_{ij} = \begin{cases} 1; i = j \\ 0; i \neq j \end{cases}$$

Thus, the matrices P and P^{-1} are given by

$$P = \begin{bmatrix} 1 & & & 0 \\ & \gamma & & \\ & & \gamma^2 & \\ & & & \ddots \\ 0 & & & & \gamma^{n-1} \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 1 & & & 0 \\ & \gamma^{-1} & & \\ & & \gamma^{-2} & \\ & & & \ddots \\ 0 & & & & \gamma^{1-n} \end{bmatrix}$$

and multiplication yields

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & & & & & & & & & \\ & \lambda_2 & & & & & & & & & & \\ & & \ddots & & & & & & & & \mathcal{O} & \\ & & & \lambda_q & & & & & & & & \\ & & & & \lambda_{q+1} & \gamma & & & & & & \\ & & & & & \lambda_{q+1} & \ddots & & & & & \\ & & & & & & \ddots & \gamma & & & & \\ & & & & & & & \lambda_{q+1} & \ddots & & & \\ & & & & & & & & \ddots & \ddots & & \\ & & & & & & & & & \lambda_{q+s} & \gamma & \\ & & \mathcal{O} & & & & & & & & \lambda_{q+s} & \ddots \\ & & & & & & & & & & \ddots & \gamma \\ & & & & & & & & & & & \lambda_{q+s} \end{bmatrix}$$

which is precisely the matrix B. This completes the proof of Lemma 3.4.

It should be pointed out that since $\gamma \neq 0$ is arbitrary, then by proper choice of the matrix P, γ may be made as small as any assigned positive number.

CHAPTER IV

GENERALIZATIONS

Consider the system

$$x' = Ax + B(t)x + g(t,x) \qquad (' = d/dt, t \geq 0) \tag{4.1}$$

where A is a real constant matrix with n rows and n columns, B(t) is a real matrix with n rows and n columns, and x,g are real vectors with n components such that $g(t,0) = 0$.

Remark. Since $g(t,0) = 0$, then the function $\tilde{\phi} \equiv 0$ is a solution of the system (4.1).

<u>Theorem 4.1</u>. Let g,B be continuous for all small $|x|$ and $t \geq 0$. Let g,B be such that

$$|B(t)| \leq M \qquad (t \geq 0)$$

where M is a sufficiently small positive constant and

$$\lim_{|x| \to 0} \frac{|g(t,x)|}{|x|} = 0$$

uniformly in t, $t \geq 0$. Moreover, let the characteristic roots of the matrix A all have negative real parts. Then the identically zero solution of (4.1) is asymptotically stable.

<u>Proof of Theorem 4.1</u>. Consider the solution $\phi = \phi(t)$ of (4.1) with $|\phi(0)|$ small. This solution can be continued for increasing t for as long as $|\phi(t)|$ is small. For as long as the solution $\phi(t)$ exists

it follows that

$$\phi(t) = \left[\exp(tA)\right]\phi(0) + \int_0^t (\exp(t-s)A)\left\{B(s)\phi(s) + g\left[s,\phi(s)\right]\right\} ds \tag{4.2}$$

Since the characteristic roots of the matrix A all have negative real parts, then by Lemma 3.1 there exists a positive constant K and a positive constant σ such that

$$|\exp(tA)| \leq K \exp(-\sigma t) \qquad (t > 0) \tag{4.3}$$

Since

$$\lim_{|x| \to 0} \frac{|g(t,x)|}{|x|} = 0$$

uniformly in t, $t \geq 0$, then given any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|g(t,x)| \leq \epsilon |x| \qquad (|x| \leq \delta, t \geq 0) \tag{4.3a}$$

If one takes the norm in (4.2) and applies (4.3), (4.3a), and the hypotheses on B(t), then

$$|\phi(t)| \leq K|\phi(0)| \exp(-\sigma t) + (KM + \epsilon)\int_0^t \left\{\exp\left[-\sigma(t-s)\right] |\phi(s)|\right\} ds$$

so long as $|\phi(t)| < \delta$. Or

$$|\phi(t)| \exp(\sigma t) \leq K |\phi(0)| + (KM + \epsilon) \int_0^t \left[\exp(\sigma s)\right] |\phi(s)| ds \tag{4.4}$$

so long as $|\phi(t)| < \delta$.

Define

$$R(t) = \int_0^t \left[\exp(\sigma s)\right] |\phi(s)| ds$$

It follows that

$$R'(t) = \left[\exp(\sigma t)\right] |\phi(t)|$$

and that $R(0) = 0$. Now (4.4) may be written as

$$R'(t) - (KM + \epsilon) R(t) \leq K|\phi(0)| \quad (4.5)$$

From the last equation it follows that

$$\frac{d}{dt}\left[R(t) \exp[-(KM + \epsilon) t]\right] \leq K|\phi(0)| \exp[-(KM + \epsilon) t]$$

Integrating from 0 to t there results

$$R(t) \exp[-(KM + \epsilon) t] \leq \frac{K|\phi(0)|}{KM + \epsilon} (1 - \exp[-(KM + \epsilon) t])$$

or

$$R(t) \leq \frac{K|\phi(0)|}{KM + \epsilon}\left\{(\exp[(KM + \epsilon) t]) - 1\right\} \quad (4.6)$$

Substituting into (4.4) there results

$$|\phi(t)| \exp(\sigma t) \leq K|\phi(0)| + K|\phi(0)|\left\{(\exp[KM + \epsilon]t) - 1\right\}$$

or

$$|\phi(t)| \leq K|\phi(0)| \exp[-(\sigma - [KM + \epsilon]) t] \tag{4.7}$$

for $|\phi(t)| < \delta$.

If ϵ and M are such that

$$\sigma > KM + \epsilon$$

then

$$|\phi(t)| \leq K|\phi(0)|$$

so long as $|\phi(t)| < \delta$.

Thus, if

$$|\phi(0)| < \delta/K ,$$

then

$$|\phi(t)| < \delta$$

so that (4.7) is valid for all $t \geq 0$. That is

$$|\phi(t)| < \delta \exp\left[-t(\sigma - (KM + \epsilon))\right] \qquad (t \geq 0)$$

Hence, it is true that if $\sigma - (KM + \epsilon) > 0$, then given any $\epsilon > 0$

$$|\phi(0)| < \epsilon/K \text{ implies that } |\phi(t)| < \epsilon \qquad (t \geq 0)$$

That is, the identically zero solution of (4.1) is stable over $0 \leq t < \infty$. Moreover, by (4.7) $|\phi(t)| \to 0$, as $t \to \infty$, i.e. the stability is asymptotic.

Since,

$$|\exp[(t-T_2)A]| = |[\exp(tA)][\exp(-T_2A)]| \leq |\exp(tA)|\cdot|\exp(-T_2A)| \tag{4.9}$$

then

$$|\exp[(t-T_2)A]| \leq |\exp(tA)|K^*$$

where

$$K^* = |\exp(-T_2A)| \tag{4.10}$$

Taking the norm in (4.8) and applying (4.9) and (4.10) and assumptions (i) and (ii), one obtains that for any $\epsilon > 0$ there exists a $\delta < 0$ such that

$$|\phi(t)| \leq K K^*|\phi(T_2)| \exp(-\sigma t) + (KM+\epsilon)\int_{T_2}^{t}\exp\big(-\sigma(t-s)\big)|\phi(s)|ds$$

for $t \geq T_2$ so long as $|\phi(t)| \leq \delta$. Or

$$|\phi(t)|\exp(\sigma t) \leq K K^*|\phi(T_2)| + (KM+\epsilon)\int_{T_2}^{t}\exp(\sigma s)|\phi(s)|ds \tag{4.11}$$

for $t \geq T_2$ so long as $|\phi(t)| \leq \delta$.

Define

$$R(t) = \int_{T_2}^{t} \exp(\sigma s)|\phi(s)|ds$$

This implies that

$$R'(t) = \exp(\sigma t)|\phi(t)|$$

and $R(T_2) = 0$. Now (4.11) can be written as

$$R'(t) - (KM + \epsilon)R(t) \leq K K^{*}|\phi(T_2)| \tag{4.12}$$

By using the integrating factor

$$\exp(-\int_{T_2}^{t}(KM + \epsilon)ds) = \exp[-(KM + \epsilon)(t - T_2)]$$

there results in the same way as was used to obtain (4.6)

$$R(t) \leq K K^{*}|\phi(T_2)| \frac{1}{(KM + \epsilon)} \left\{ \exp((KM + \epsilon)(t - T_2)) - 1 \right\}$$

Use of this in (4.12) yields

$$|\phi(t)|\exp(\sigma t) \leq K K^{*}|\phi(T_2)| \exp((KM + \epsilon)(t - T_2))$$

or

$$|\phi(t)| \leq K K^{*}|\phi(T_2)| \exp\left[-\left(\sigma - (KM + \epsilon)\right) t\right] \exp(-T_2(KM + \epsilon))$$

Since, T_2, K, M, and ϵ are all non-negative, then

$$|\phi(t)| \leq (K)\ (K^*)\ |\phi(T_2)|\ \exp(-(\sigma - [KM + \epsilon\ ])t) \tag{4.13}$$

If, M and ϵ are such that

$$\sigma > KM + \epsilon$$

then from (4.13) it follows that

$$|\phi(t)| \leq (K)\ (K^*)\ |\phi(T_2)|$$

so long as $|\phi(t)| \leq \delta$. Hence, if

$$|\phi(T_2)| < \delta/(K)\ (K^*) \quad ,$$

then (4.13) is valid for all $t \geq T_2$; i.e. for M and ϵ such that $\sigma > KM + \epsilon$ it is true that $|\phi(t)| \to 0$, as $t \to \infty$. Moreover, for given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|\phi(T_2)| < \delta \text{ implies that } |\phi(t)| < \epsilon \qquad (t \geq T_2)$$

That is, the identically zero solution of (4.1) is stable over the interval $T_2 \leq t < +\infty$; and moreover the stability is asymptotic. By Lemma 3.3 stability over $T_2 \leq t < +\infty$ is equivalent to stability over $0 \leq t < \infty$ under the assumptions (i) and (ii). Hence, the identically zero solution of (4.1) is stable over the interval $0 \leq t < \infty$; moreover, the stability is asymptotic.

A more general statement of Theorem 4.1 makes the hypotheses on B and g less restrictive. Specifically, it suffices to assume that;

(i) For some $T \geq 0$ it is true that

$$|B(t)| \leq M \qquad (t \geq T)$$

where M is a sufficiently small positive constant;

(ii) For some positive constant k it is true that

$$|g(t,x)| \leq k|x| \qquad (t \geq 0)$$

for all small $|x|$, and that given any $\epsilon > 0$ there exists a $\delta > 0$ and a $T_1 \geq 0$ such that

$$|g(t,s)| \leq \epsilon |x| \qquad (|x| \leq \delta, \ t \geq T_1)$$

In fact the following result holds.

Theorem 4.2. Let g,B be real, continuous for small $|x|$ and $t > 0$. Let g,B satisfy the assumptions (i) and (ii) above. Moreover, let the characteristic roots of A in (4.1) all have negative real parts. Then the identically zero solution of (4.1) is asympotically stable.

Proof of Theorem 4.2. Consider the solution $\phi = \phi(t)$ of (4.1) with $|\phi(T_2)|$ small, where $T_2 = \max(T, T_1)$ and T, T_1 are defined in (i) and (ii) above. This solution can be continued for increasing t for as long as $|\phi(t)|$ is small. For as long as $\phi(t)$ exists it follows that

$$\phi(t) = \exp\left[(t - T_2)A\right] \cdot \phi(T_2) + \int_{T_2}^{t} \exp\left[(t - s)A\right]\left\{B(s)\phi(s) + g\left[s, \phi(s)\right]\right\}ds \tag{4.8}$$

The results of Theorems 4.1 and 4.2 are the best possible in the sense that if the matrix A in the system (4.1) has at least one of its characteristic roots with positive real part, and the other assumptions of the theorems hold, then it is impossible for the identically zero solution of (4.1) to be stable.

Theorem 4.3. Let g,B satisfy the assumptions (i) and (ii) of Theorem 4.2 and be continuous for small $|x|$ and $t \geq 0$. Let at least one of the characteristic roots of A in (4.1) have its real part positive. Then the identically zero solution of (4.1) is not stable.

Proof of Theorem 4.3. In the system (4.1) make the change of variable

$$x = Py$$

where P is a nonsingular constant matrix with n rows and n columns and with complex elements. There results a system of the form

$$y' = Cy + D(t)y + h(t,y) \qquad (t \geq 0) \qquad (4.14)$$

where

$$\begin{aligned} C &= P^{-1}A\,P \\ D(t) &= P^{-1}B(t)\,P \\ h(t,y) &= P^{-1}\,g(t,Py) \end{aligned} \qquad (4.15)$$

By proper choice of the matrix P the matrix C may be put in the form

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \qquad (4.16)$$

where C_1 and C_2 are in the classical Jordan canonical form and have the following properties: C_1 is a matrix with k rows and k columns with all its characteristic roots having positive real parts, and C_2 is a matrix with (n - k) rows and (n - k) columns with all its characteristic roots having nonpositive real parts. Since the matrix C is in Jordan canonical form, then by Lemma 4, Chapter III the characteristic roots of C_1 and C_2 (i.e. of the matrix C) are in the main diagonal and the elements of C_1 and C_2 which are off the main diagonal and not zero may be taken to be γ, where γ can be made as small as any assigned positive constant by proper choice of the matrix P.

While y corresponding to real x may be complex (since the matrix P need not be real), Py will be real. Hence, the function

$$h(t,y) = P^{-1} g(t,Py)$$

is well defined.

Let the vector function $\psi = \psi(t)$ with conponents ψ_j, $(j = 1,2,\ldots,n)$ be a solution of the system (4.1). Define

$$R^2 = \sum_{j=1}^{k} |\psi_j|^2 \quad \text{and} \quad \rho^2 = \sum_{j=k+1}^{n} |\psi_j|^2 \qquad (4.17)$$

Where k is the order of the matrix C_1. Let the constant $\sigma > 0$ be such that the real parts of characteristic roots of C_1 all exceed σ. Choose $\epsilon < \sigma/10$ and choose $\eta > 0$ and $T \geq 0$ so that

$$|h(t,y)| \leq \epsilon ||y|| \qquad (||y|| \leq \eta, \ t \geq T) \tag{4.18}$$

The above is possible since

$$|h(t,y)| = |P^{-1} g(t,Py)|$$

and therefore

$$|h(t,y)| \leq |P^{-1}| \cdot |g(t,Py)|$$

and g satisfies assumption (ii) Theorem 4.2.

Suppose that the solution $\phi = 0$ is stable over $T \leq t < \infty$. This implies that for η and T chosen as above there exists a $\delta > 0$ such that if $\psi(t)$ is any solution of the system (4.1) with components ψ_j, $(j = 1, 2, \ldots, n)$, then it is true that

$$\rho(T) + R(T) < \delta \text{ implies that } \rho(t) + R(t) < \eta \quad (t \geq T)$$

Choose such a solution ψ with

$$R(T) = 2\rho(T) > 0 \tag{4.19}$$

It will first be shown that if σ, γ, and ϵ are taken as above, then

$$2RR' = \sum_{i=1}^{k} (\psi'_i \bar{\psi}_i + \psi_i \bar{\psi}'_i) \tag{4.20}$$

and

$$2RR' \geq 2\sigma R^2 - 2\gamma R^2 - 2KM^* R^2 - 2\epsilon(\rho + R)R \qquad (4.21)$$

Define h_j, $(j = 1, 2, \ldots, n)$ to be the components of $h(t,y)$, c_{ij}, $(i, j = 1, 2, \ldots, n)$ to be the elements of the matrix C, and $d_{ij}(t)$, $(i,j = 1, 2, \ldots, n)$ to be the elements of the matrix D(t). Let $\bar{\psi}_i$ denote the complex conjugate of ψ_i. M^* is to be defined.

Since ψ is a solution of the system (4.1), then it follows that

$$2RR' = \left\{ \sum_{j=1}^{k} \psi_j \bar{\psi}_j \right\}' = \sum_{j=1}^{k} (\psi'_j \bar{\psi}_j + \psi_j \bar{\psi}'_j) \quad (' = d/dt)$$

by using differentiation and the fact that

$$|\psi_j|^2 = \psi_j \bar{\psi}_j \qquad (j = 1, 2, \ldots, k)$$

Thus (4.20) is established.

Since ψ is a solution of (4.1), then

$$\psi' = C\psi + D(t)\psi + h(t,y)$$

or

$$\psi'_i = \sum_{j=1}^{k} (c_{ij} \psi_j + d_{ij}(t) \psi_j) + h_i \quad (i = 1,2,\ldots,k) \quad (4.22)$$

It follows from (4.20) and (4.22) that

$$2RR' = \sum_{i=1}^{k} \left\{ \left[\sum_{j=1}^{k} [c_{ij}\Phi_j + d_{ij}(t)\Phi_j] + h_i \right] \bar{\Phi}_i + \Phi_i \left[\sum_{j=1}^{k} [\bar{c}_{ij}\bar{\Phi}_j + \bar{d}_{ij}(t)\bar{\Phi}_j] + \bar{h}_i \right] \right\}$$

$$= \sum_{i=1}^{k} \left\{ \sum_{j=1}^{k} \left[[c_{ij}\bar{\Phi}_i\Phi_j + \bar{c}_{ij}\Phi_i\bar{\Phi}_j + d_{ij}(t)\bar{\Phi}_i\Phi_j + \bar{d}_{ij}(t)\Phi_i\bar{\Phi}_j] \right] + h_i\bar{\Phi}_i + \bar{h}_i\Phi_i \right\}$$

But if F is any complex valued vector function, then

$$F + \bar{F} = 2\mathrm{Re}(F) \qquad (4.22')$$

Hence,

$$2RR' = \sum_{i=1}^{k} \left\{ \sum_{j=1}^{k} [2\mathrm{Re}(c_{ij}\bar{\Phi}_i\Phi_j) + 2\mathrm{Re}(d_{ij}(t)\bar{\Phi}_i\Phi_j)] + 2\mathrm{Re}h_i\bar{\Phi}_i \right\} \qquad (4.23)$$

Recall that the sum of the real parts of complex valued functions is equal to the real part of the sum, and conversely. Thus, writing out the first part of the sum in (4.23) we obtain

$$2RR' = 2\mathrm{Re}\Big\{ c_{11}\bar{\psi}_1\psi_1 + c_{12}\bar{\psi}_1\psi_2 + \dots + c_{1k}\bar{\psi}_1\psi_k$$

$$+ c_{21}\bar{\psi}_2\psi_1 + c_{22}\bar{\psi}_2\psi_2 + \dots + c_{2k}\bar{\psi}_2\psi_k + \dots$$

$$+ c_{k1}\bar{\psi}_k\psi_1 + c_{k2}\bar{\psi}_k\psi_2 + \dots + c_{kk}\bar{\psi}_k\psi_k$$

$$+ d_{11}(t)\bar{\psi}_1\psi_1 + d_{12}(t)\bar{\psi}_1\psi_2 + \dots + d_{1k}(t)\bar{\psi}_1\psi_k$$

$$+ d_{21}(t)\bar{\psi}_2\psi_1 + d_{22}(t)\bar{\psi}_2\psi_2 + \dots + d_{2k}(t)\bar{\psi}_2\psi_k + \dots$$

$$+ d_{k1}(t)\bar{\psi}_k\psi_1 + d_{k2}(t)\bar{\psi}_k\psi_2 + \dots + d_{kk}(t)\bar{\psi}_k\psi_k$$

$$+ \sum_{i=1}^{k} h_i\bar{\psi}_i \Big\}$$

Using the fact that

$$c_{ij} = \begin{cases} \gamma; & i = j-1 \\ 0; & i \neq j,\ i \neq j-1 \\ \lambda_i; & i = j \end{cases} \qquad (i,j = 1, 2, \dots, k)$$

where λ_j, $(j = 1, 2, \dots, k)$, are the characteristic roots of the matrix C_1, and where $\mathrm{Re}(\lambda_j) > \sigma$, $(j = 1, 2, \dots, k)$, in (4.23) there results

$$2RR' \geq 2\sigma\sum_{i=1}^{k} \bar{\psi}_i\psi_i + 2\mathrm{Re}\left\{\gamma(\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_3 + \dots + \bar{\psi}_{k-1}\psi_k)\right\}$$

$$+ \sum_{i=1}^{k}\left\{\sum_{j=1}^{k} [2\mathrm{Re}(d_{ij}(t)\bar{\psi}_i\psi_j)] + 2\mathrm{Re}(h_i\bar{\psi}_i)\right\}$$

Note that for any functions F, G,

$$2\mathrm{Re}(F\bar{G}) \geq -(|F|^2 + |G|^2) \tag{4.24}$$

Applying (4.24) there results

$$2\mathrm{Re}\left\{\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_3 + \dots + \bar{\psi}_{k-1}\psi_k\right\}$$

$$\geq -\left\{|\psi_1|^2 + |\psi_2|^2 + |\psi_2|^2 + |\psi_3|^2 + \dots + |\psi_{k-1}|^2 + |\psi_k|^2\right\}$$

$$\geq -2 \sum_{i=1}^{k} |\psi_i|^2$$

Thus,

$$2RR' \geq 2\sigma \sum_{i=1}^{k} |\psi_i|^2 - 2\gamma \sum_{i=1}^{k} |\psi_i|^2$$

$$+ \sum_{i=1}^{k}\left\{\sum_{j=1}^{k} [2\mathrm{Re}\left(d_{ij}(t)\bar{\psi}_i\psi_j\right)] + 2\mathrm{Re}(h_i\bar{\psi}_i)\right\} \tag{4.24'}$$

Now, since by hypothesis there exists a constant $M > 0$ such that

$$|B(t)| \leq M \qquad (t \geq T)$$

where M is sufficiently small, then for $t \geq T$

$$|D(t)| = |P^{-1} B(t) P| \leq |P^{-1}|\cdot|P|\cdot|B(t)| \leq M^*$$

where M^* is a positive constant depending on M. Moreover, M^* is small if M is small.

$$|D(t)| = \sum_{i,j=1}^{n} |d_{ij}(t)| \leq M^* \qquad (t \geq T)$$

implies that

$$|d_{ij}(t)| \leq M^* \qquad (i,j = 1, 2, \ldots, n) \qquad (4.24'')$$

Since $-|d_{ij}(t)| \leq \mathrm{Re}(d_{ij}(t))$, $(i,j = 1, 2, \ldots, n)$, then

$$2\mathrm{Re} \sum_{i=1}^{k} \left\{ \sum_{j=1}^{k} [d_{ij}(t)\bar{\psi}_i\psi_j] \right\}$$

$$= 2 \sum_{i=1}^{k} \left\{ \sum_{j=1}^{k} [\mathrm{Re}(d_{ij}(t))][\mathrm{Re}\bar{\psi}_i\psi_j] \right. \qquad (4.25)$$

$$\geq -2 \sum_{i=1}^{k} \left\{ \sum_{j=1}^{k} |d_{ij}(t)| \, \mathrm{Re}\bar{\psi}_i\psi_j \right\}$$

$$\geq -2M^* \sum_{i=1}^{k} \left\{ \sum_{j=1}^{k} \mathrm{Re}\bar{\psi}_i\psi_j \right\}$$

But,

$$-2M^* \sum_{i=1}^{k} \left\{ \sum_{j=1}^{k} \mathrm{Re}\bar{\psi}_i\psi_j \right\} = -2M^*\mathrm{Re} \left\{ \sum_{i=1}^{k} \psi_i\bar{\psi}_i + 2 \sum_{i=1}^{k} \bar{\psi}_{i+1}\psi_i \right.$$

$$\left. + 2 \sum_{i=1}^{k-1} \psi_{i+1}\bar{\psi}_i + \ldots + 2\psi_k\bar{\psi}_1 \right\}$$

$$(4.26)$$

Using the fact that

$$2\mathrm{Re}\,\psi_\ell\bar{\psi}_m \le |\psi_\ell|^2 + |\psi_m|^2 \qquad (\ell, m = 1, 2, \ldots, k) \tag{4.27}$$

in (4.26) there results

$$-2M^* \sum_{i=1}^{k}\left\{\sum_{j=1}^{k} \mathrm{Re}\,\bar{\psi}_i\psi_j\right. \ge -2M^*\left\{\sum_{i=1}^{k}|\psi_i|^2 + \sum_{i=1}^{k-1}[|\psi_{i+1}|^2 + |\psi_i|^2]\right.$$

$$\left. + \sum_{i=1}^{k-2}[|\psi_{i+2}|^2 + |\psi_i|^2] + \ldots + |\psi_k|^2 + |\psi_1|^2\right\}$$

Thus, (4.25) becomes

$$2\mathrm{Re} \sum_{i=1}^{k}\left\{\sum_{j=1}^{k}[d_{ij}(t)\bar{\psi}_i\psi_j]\right\} \ge -2M^* kR^2 \tag{4.28}$$

Using (4.28) in (4.24') one obtains

$$2RR' \ge 2\sigma R^2 - 2\gamma R^2 - 2kM^*R^2 + 2\sum_{i=1}^{k}\mathrm{Re}(h_i\bar{\psi}_i) \tag{4.29}$$

where (4.17) was applied. Now

$$2\sum_{i=1}^{k}\mathrm{Re}(h_i\bar{\psi}_i) = 2\mathrm{Re}\sum_{i=1}^{k}(h_i\bar{\psi}_i) \ge -2\left|\sum_{i=1}^{k}h_i\bar{\psi}_i\right|$$

and by the Schwarz inequality

$$-\left|\sum_{i=1}^{k} h_i\bar{\psi}_i\right| \geq -\left\{\sum_{i=1}^{k} |h_i|^2\right\}^{1/2} \left\{\sum_{i=1}^{k} |\psi_i|^2\right\}^{1/2}$$

so that

$$2\sum_{i=1}^{k} \mathrm{Re}(h_i\bar{\psi}_i) \geq -2\left\{\sum_{i=1}^{k} |h_i|^2\right\}^{1/2} \left\{\sum_{i=1}^{k} |\psi_i|^2\right\}^{1/2}$$

But, using (4.17) and 1.4

$$\left\{\sum_{i=1}^{k} |h_i|^2\right\}^{1/2} \leq ||h|| \leq |h|$$

and

$$\left\{\sum_{i=1}^{k} |\psi_i|^2\right\}^{1/2} = R \quad .$$

Moreover, by (4.18) given any $\epsilon > 0$ one can choose $\eta > 0$ and $T \geq 0$, such that

$$|h(t,y)| \leq \epsilon\,||y|| \quad \text{for } ||y|| \leq \eta \quad \text{and} \quad t \geq T.$$

Hence,

$$2\sum_{i=1}^{k} \mathrm{Re}(h_i\bar{\psi}_i) \geq -2R\,\epsilon\,||\psi||$$

where

$$||\psi|| = \left\{\sum_{i=1}^{n} |\psi_i|^2\right\}^{1/2} = (R^2 + \rho^2)^{1/2} .$$

Since $\rho, R > 0$ and $(\rho + R)^2 \geq \rho^2 + R^2$,

$$(R^2 + \rho^2)^{1/2} \leq R + \rho .$$

Thus, (4.29) may be written

$$2RR' \geq 2\sigma R^2 - 2\gamma R^2 - 2kM^* R^2 - 2\epsilon R(\rho + R)$$

as was to be shown.

From (4.21) it follows that

$$R' \geq \sigma R - \gamma R - kM^* - \epsilon(\rho + R)$$

By proper choice of the matrix P the number γ can be made less than $\sigma/20$. Since by proper choice of M it is possible to make:

$$M^* < \sigma/20k$$

(where k is the order of the matrix C_1), (4.21) yields

$$R' \geq 1/2(\sigma R) - \epsilon\rho . \qquad (4.30)$$

Next it will be shown that

$$\rho' \leq \gamma\rho + M^*(n - k)\rho + \epsilon(\rho + R) \qquad (4.31)$$

To this end observe that

$$2\rho\rho' = \left\{\sum_{i=k+1}^{n} \psi_i\bar{\psi}_i\right\}' = \sum_{i=k+1}^{n} (\psi_i'\bar{\psi}_i + \psi_i\bar{\psi}_i') \qquad (4.32)$$

and since ψ is a solution of (4.1) with components ψ_j, $(j=1,2,\ldots,n)$, then

$$\psi_i' = \sum_{j=k+1}^{n} (c_{ij}\psi_j + d_{ij}(t)\psi_j) + h_i \ , \quad (i=k+1, k+2,\ldots,n)$$

Using the last part in (4.32) there results

$$2\rho\rho' = \sum_{i=k+1}^{n} \left\{ 2\mathrm{Re} \sum_{j=k+1}^{n} c_{ij}\bar{\psi}_i\psi_j + 2\mathrm{Re} \sum_{j=k+1}^{n} d_{ij}(t)\bar{\psi}_i\psi_j + 2\mathrm{Re}(h_i\bar{\psi}_i) \right\} \qquad (4.33)$$

in a manner similar to that used to obtain (4.23). Since

$$c_{ij} = \begin{cases} \gamma \, ; & i = j - 1 \\ 0 \, ; & i \neq j, \ i \neq j-1 \\ \lambda_i ; & i = j \end{cases} \qquad (i,j = k+1, k+2,\ldots,n)$$

where λ_j, $(j = k+1, k+2, \ldots, n)$ are the characteristic roots of the matrix c_2 which are such that

$$\mathrm{Re}(\lambda_j) \leq 0 \qquad (j = k+1, k+2, \ldots, n)$$

then

$$\sum_{i=k+1}^{n}\left\{2\mathrm{Re}\sum_{j=k+1}^{n} c_{ij}\bar{\psi}_i\psi_j\right\} = 2\sum_{i=k+1}^{n}(\mathrm{Re}\lambda_i)(\mathrm{Re}\bar{\psi}_i\psi_i)$$

$$+ \; 2\mathrm{Re}\left\{\gamma(\bar{\psi}_{k+1}\psi_{k+2} + \bar{\psi}_{k+2}\psi_{k+3} + \dots + \bar{\psi}_{n-1}\psi_n)\right\}$$

$$\leq 2\mathrm{Re}\left\{\gamma(\bar{\psi}_{k+1}\psi_{k+2} + \bar{\psi}_{k+2}\psi_{k+3} + \dots + \bar{\psi}_{n-1}\psi_n)\right\}$$

where the last follows from the fact that

$$\sum_{i=k+1}^{n}(\mathrm{Re}\lambda_i)(\mathrm{Re}\bar{\psi}_i\psi_i) = \sum_{i=k+1}^{n}\mathrm{Re}(\lambda_i)|\psi_i|^2 \leq 0$$

$$(i = k+1,\ k+2,\ \dots,\ n)$$

Now

$$2\mathrm{Re}(\bar{\psi}_i\psi_j) \leq |\psi_i|^2 + |\psi_j|^2 \qquad (i,j = k+1,\ k+2,\ \dots,\ n)$$

so that

$$\sum_{i=k+1}^{n}\left\{2\mathrm{Re}\sum_{j=k+1}^{n} c_{ij}\bar{\psi}_i\psi_j\right\} \leq \tag{4.34}$$

$$\leq \gamma\left\{|\psi_{k+1}|^2 + |\psi_{k+2}|^2 + |\psi_{k+2}|^2 + |\psi_{k+3}|^2 + \dots + |\psi_{n-1}|^2 + |\psi_n|^2\right\}$$

$$\leq \gamma\left\{2\sum_{i=k+1}^{n}|\psi_i|^2\right\} = 2\gamma\rho^2$$

Moreover,

$$\sum_{i=k+1}^{n}\left\{2\mathrm{Re}\sum_{j=k+1}^{n} d_{ij}(t)\bar{\psi}_i\psi_i\right\}=$$

$$= 2\sum_{i=k+1}^{n}\left\{\sum_{j=k+1}^{n}[\mathrm{Re}\,d_{ij}(t)][\mathrm{Re}\,\bar{\psi}_i\psi_j]\right\}$$

$$\leq 2M^{*}\sum_{i=k+1}^{n}\left\{\sum_{j=k+1}^{n}\mathrm{Re}\,\bar{\psi}_i\psi_j\right\}$$

since $\mathrm{Re}(d_{ij}(t))\leq|d_{ij}(t)| \leq M^{*}$. But

$$2M^{*}\sum_{i=k+1}^{n}\left\{\sum_{j=k+1}^{n}\mathrm{Re}\,\bar{\psi}_i\psi_j\right\}=$$

$$= 2M^{*}\mathrm{Re}\left\{\sum_{i=k+1}^{n}\psi_i\bar{\psi}_i + 2\sum_{i=k+1}^{n-1}\bar{\psi}_i\psi_{i+1} + 2\sum_{i=k+1}^{n-2}\bar{\psi}_i\psi_{i+2}\right.$$

$$\left. + \ldots + 2[\bar{\psi}_{k+1}\psi_{n-1} + \bar{\psi}_{k+2}\psi_n] + \bar{\psi}_{k+1}\psi_n\right\}$$

Recall that

$$2\mathrm{Re}(\bar{\psi}_i\psi_j) \leq |\psi_i|^2 + |\psi_j|^2$$

Hence,

$$2M^* \sum_{i=k+1}^{n} \left\{ \sum_{j=k+1}^{n} \mathrm{Re}\,\bar{\psi}_i \psi_j \right\} \le 2M^* \left\{ \rho^2 + \sum_{i=k+1}^{n-1} \left[|\psi_i|^2 + |\psi_{i+1}|^2 \right] \right.$$

$$+ \sum_{i=k+1}^{n-2} \left[|\psi_i|^2 + |\psi_{i+2}|^2 \right] \tag{4.35}$$

$$\left. + \dots + [\,|\psi_{k+1}|^2 + |\psi_{n-1}|^2 + |\psi_{k+2}|^2 + |\psi_n|^2 \quad + |\psi_{k+1}|^2 + |\psi_n|^2 \right\}$$

$$\le 2(n-k)\, M^* \rho^2$$

Furthermore, by an argument similar to the one used below (4.29) one obtains

$$2\mathrm{Re} \sum_{i=k+1}^{n} h_i \bar{\psi}_i = 2\mathrm{Re} \sum_{i=k+1}^{n} h_i \bar{\psi}_i \le 2 \left| \sum_{i=k+1}^{n} h_i \bar{\psi}_i \right| ,$$

and by the Schwarz inequality

$$2\mathrm{Re} \sum_{i=k+1}^{n} h_i \bar{\psi}_i \le 2 \rho \epsilon (\rho + R) \tag{4.36}$$

Using (4.34), (4.35), and (4.26) in (4.33) there results

$$2\rho\rho' \le 2\gamma\rho^2 + 2M^*(n-k)\rho^2 + 2\epsilon(\rho + R)\rho$$

or

$$\rho' \leq \gamma\rho + M^{*}(n-k)\rho + \epsilon(\rho + R)$$

which is the inequality (4.31).

Since by proper choice of the matrix P, γ may be made less than $\sigma/20$ and since M^{*} may be made less than $\sigma/(n - k)20$. then

$$\rho' \leq \frac{\sigma}{10}\rho + \epsilon(R + \rho) \qquad (4.37)$$

Hence, if

$$M^{*} = \min\left(\sigma/(n - k)20, \sigma/k20\right),$$

then (4.30) and (4.37) are both valid for $t \geq T$.
Since $\epsilon < \sigma/10$, it follows from (4.30) and (4.37) that

$$(R - \rho)' \geq 2/5(R - \rho)\sigma \qquad (' = d/dt)$$

And by integration that

$$R(t) - \rho(t) \geq (R(T) - \rho(T)) \exp(2/5(t - T)\sigma) \text{ for all } t \geq T.$$

But, by (4.19) $R(T) = 2\rho(T) > 0$, hence

$$R(t) > \rho(T) \exp(2/5(t - T)\sigma) \text{ for all } t \geq T.$$

But this contradicts the assumption that the solution $\tilde{\phi} \equiv 0$ of (4.1) is stable, i.e. that

$$R(T) + \rho(T) < \delta \text{ implies that } R(t) + \rho(t) < \eta \quad (t \geq T)$$

Therefore, the solution $\tilde{\phi} \equiv 0$ of (4.1) is not stable. This completes the proof of Theorem 4.3.

It should be pointed out that if in a particular example the sufficient conditions on g,B are not satisfied, then Theorems 4.1, 4.2, and 4.3 do not apply but this does not mean specifically that the identically zero solution of (4.1) is not stable, since the conditions on g,B are sufficient conditions.

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