ERGODIC THEORY WITH APPLICATIONS TO SYSTEMS OF DIFFÈRENTIAL EQUATIONS

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SUMMARY

This paper is an exposition of the application of the techniques and results of ergodic theory to systems of ordinary differential equations. In particular, let

$$\frac{dx}{dt} = F(x)$$

where $x = (x_1, x_2, \ldots, x_n)$ and $F = (F_1, F_2, \ldots, F_n)$. Let F be such that there is a unique function $x = x(t)$ which satisfies the above equation on some real t interval I and for which

$$x(t_o) = x_o, \quad t_o \varepsilon I .$$

This solution defines a set of transformations $\{T_t\}$; viz., if u is any point of a (sufficiently small) neighborhood U of x_o, then

$$T_t : u \rightarrow x(t_o + t)$$

whenever $(t_o + t) \varepsilon I$. The properties of this set of transformations are studied for the case when I is the real line, $-\infty < t < +\infty$.

It is shown that if $\operatorname{div} F = 0$, then each T_t, $-\infty < t < +\infty$, is a measurable and measure-preserving transformation. Further, if $\operatorname{div} F = 0$ and if D is an invariant set, $T_t : D \rightarrow D$, then almost every point of D is Poisson stable. This result is important in the study of conservative physical systems.

The ergodic theorem of G. D. Birkhoff is developed with an explanation of the meaning and importance of the conditions under which

time means can be replaced by phase means. An example is given for the case of ergodic motions on the surface of a torus. This example arises in the study of physical systems described by periodic position coordinates and a Hamiltonian which is independent of the potential energy.

A theorem presented recently by A. N. Kolmogorov is stated. This theorem demonstrates the direction of current research in ergodic theory.

CHAPTER I

GENERAL THEORY OF DYNAMICAL SYSTEMS

Let G denote a conservative dynamical system having s degrees of freedom and having constraints which are independent of time. Let the state of G at any time t be described by a set of $2s$ Hamiltonian coordinates $(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s)$, or more briefly, (q,p). The motions of the system G are then determined by the $2s$ canonical equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2, \ldots, s \tag{1.1}$$

where $H = H(q,p)$ is the Hamiltonian function for G. Note that the given system is autonomous; that is, H is assumed not to depend explicitly on t.

Let Φ denote the 2s-dimensional Euclidean space of coordinates (q,p); that is, Φ is the phase space for the system G. The initial value problem associated with (1.1) is: given any point (q_o, p_o) in Φ and a real number t_o, find a set of functions

$$(q(t), p(t)) = (q_1(t), q_2(t), \ldots, q_s(t), p_1(t), \ldots, p_s(t))$$

defined on a real interval I such that the set satisfies both (1.1) and the initial condition on I

$$(q(t_o), p(t_o)) = (q_o, p_o) \tag{1.2}$$

This Hamiltonian problem is a special case of finding a solution of the system of differential equations

$$\frac{dx}{dt} = F(t,x) \tag{1.3}$$

which satisfies the initial condition

$$x(t_o) = x_o \tag{1.4}$$

where $x = (x_1, x_2, \ldots, x_n)$, $F = (F_1, F_2, \ldots, F_n)$, x_o is a point of n-dimensional Euclidean space E_n, and t_o is a real number. To display the dependence of a solution on the initial condition, $x(t) = x(t;t_o,x_o)$ shall denote that solution of (1.3) which satisfies (1.4).

Define the <u>norm</u> of a vector $x = (x_1, x_2, \ldots, x_n)$ as

$$\| x \| = \sum_{i=1}^{n} |x_i|$$

A function F is said to satisfy a <u>Lipschitz condition</u> with respect to x in a domain D, a nonempty open connected set, of the (t,x) plane if there is a positive constant k such that

$$\| F(t,x') - F(t,x'') \| \leq k \| x' - x'' \|$$

for every (t,x') and (t,x'') in D.

The following three theorems are of fundamental importance. Theorem 1 given conditions for the local existence of a solution to the above initial value problem. Theorem 2 concerns the uniqueness of this solution and its continuity and differentiability as a function of the initial condition. Theorem 3 concerns the continuation of solutions.

Theorem 1. Let F be continuous on a domain D in the (t,x) space, and let (t_o,x_o) be any point in D. Then there exists a function $x = x(t;t_o,x_o)$ which satisfies equations (1.3) and (1.4) on some interval (t_1, t_2) containing t_o.

Theorem 2. Let F be continuous in t and $\mathbf{x}$ and satisfy a Lipschitz condition with respect to $\mathbf{x}$ in a domain $\mathbf{D}$ and let (t_o,x_o) be a point in D.

i. If x_1 and x_2 are any two solutions of (1.3) and (1.4) on (t_1,t_2), $t_1 < t_o < t_2$, such that $x_1(t_o) = x_2(t_o) = x_o$, then $x_1 = x_2$.

ii. Let $x = x(t;t_o,x_o)$ be a solution of (1.3) on a closed interval $[t_1,t_2]$. Then there exists a $d > 0$ such that for every $u = (u_1,u_2, \ldots, u_n)$ satisfying $\| u - x_o \| < d$, there is a unique solution φ of (1.3) on $[t_1,t_2]$ with $\varphi(t_o;t_o,u) = u$. Moreover, φ is a continuous function of t and u for $t_1 < t < t_2$, and $\| u - x_o \| < d$.

iii. Assume the hypothesis of ii. and let φ be the described solution. If $\partial F_i / \partial x_j$ for $i,j = 1,2, \ldots, n$ is continuous on D, then $\partial\varphi / \partial u_j$, $i,j = 1,2, \ldots, n$ is continuous for $t_1 < t < t_2$ and $\| u - x_o \| < d$.

Theorem 3. Let F be continuous in a domain D of the (t,x) plane, and suppose F is bounded on D. If $\mathbf{x}$ is a solution of (1.3) on an interval (t_1,t_2), then the limits $x(t_1 + 0)$ and $x(t_2 - 0)$ exist. If $x(t_1 + 0)$ [or $x(t_2 - 0)$] is in D, then the solution x may be continued to the left of t_1 [or to the right of t_2].

The proofs of these well-known theorems can be found in Chapter 1 of Coddington and Levinson [1].

Note that under the conditions of Theorem 3 every solution can be continued without bound as $t \to +\infty$ (or $t \to -\infty$) or else, for some finite value $t = T$, reaches the boundary of the domain D.

CHAPTER II

TRANSFORMATIONS ASSOCIATED WITH A SYSTEM OF DIFFERENTIAL EQUATIONS

Consider the autonomous system of differential equations

$$\frac{dx}{dt} = F(x) \tag{2.1}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $F = (F_1, \ldots, F_n)$. Let F satisfy a Lipschitz condition with respect to x in a domain D of E_n. By Theroems 1 and 2, for x in D there exists a unique function $x = x(t, t_o, x_o)$ which satisfies (2.1) on some interval (t_1, t_2) and for which

$$x(t_o, t_o, x_o) = x_o \ .$$

where $t_1 < t_o < t_2$. Let $[t_3, t_4]$ be a nondegenerate closed subinterval of (t_1, t_2)

$$t_1 < t_3 \leq t_o \leq t_4 < t_2 .$$

By Theorem 2, part ii, there is a neighborhood U of x_o such that for every $u \varepsilon U$, the solution $x = x(t, t_o, u)$ exists on $[t_3, t_4]$. The solution $x = x(t, t_o, u)$ defines a set of mappings $\{T_t\}$ as follows. If t is such that $t_o + t$ is in $[t_3, t_4]$, for each $u \varepsilon U$,

$$T_t : (t_o u) \rightarrow (t_o + t, \ x(t_o + t, t_o, u))$$

or more briefly,

$$T_t(u) = x(t_o + t, t_o, u) .$$

Figure 1 shows, for the scalar case $n = 1$, the interval of existence (t_1, t_2), the subinterval $[t_3, t_4]$, the solution which passes through (t_o, x_o) and how this solution determines the point $(t_o+t, T_t(x_o))$. The shaded region on the t_o line represents the neighborhood U of x_o. Through each point of this neighborhood passes a solution which determines a corresponding point on the $t_o + t$ line. Thus T_t maps U onto the region T_tU which is represented by the shading on the line $t_o + t$.

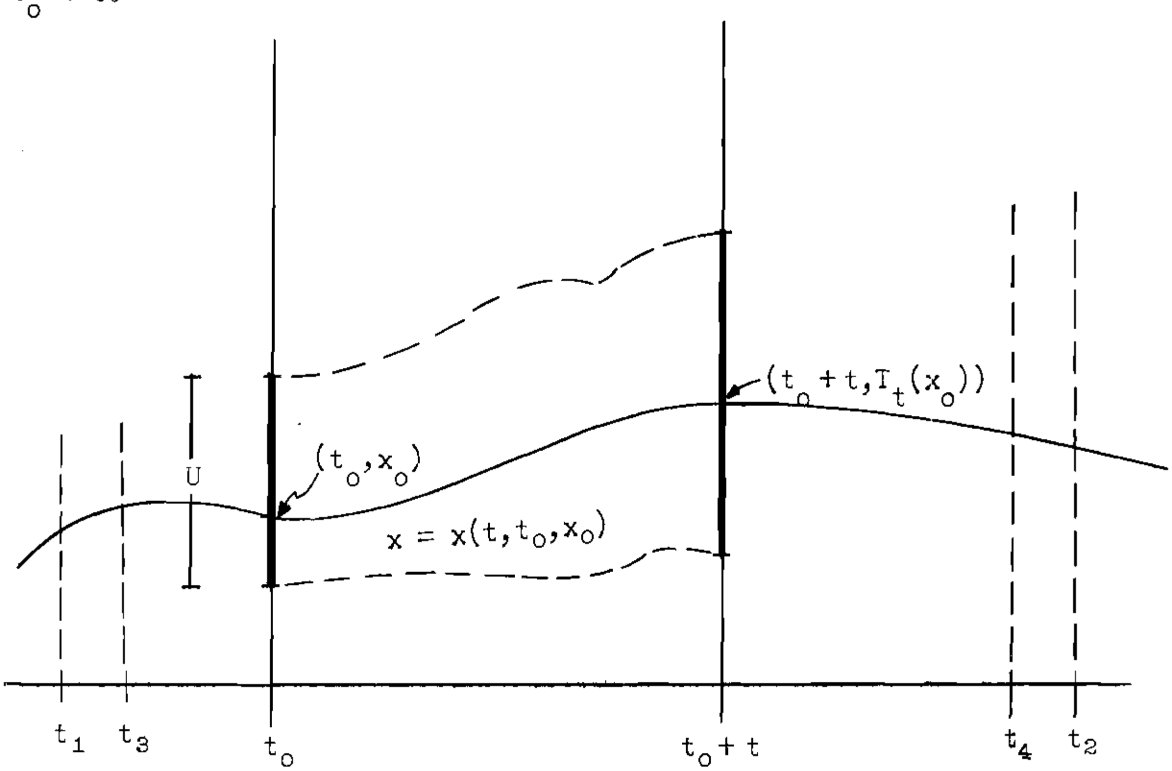

Figure 1. Transformation Associated with a Solution Curve

Theorem 4. Each T_t is a topological mapping; that is, for each t, $-\infty < t < +\infty$, T_t has an inverse T_t^{-1}. Both T_t and T_t^{-1} are continuous.

Proof. By assumption, the solution $x = x(t, t_o, u)$ is unique. Hence $T_t u_1 = T_t u_2$ if and only if $u_1 = u_2$. Thus T_t is one-to-one, so the inverse transformation T_t^{-1} exists. By Theorem 2, part ii, x is a continuous function of u, so T_t is a continuous function of u since $T_t u = x(t_o + t, t_o, u)$. Let

$$\bar{u} = x(t_o + t, t_o, u) = x(t_o + t, t_o + t, \bar{u}).$$

Then the transformation T_t^{-1} is given by the equation

$$u = T_t^{-1}(\bar{u}) = x(t_o, t_o + t, \bar{u})$$

and the continuity of x implies that T_t^{-1} is continuous.

Example 1. Consider the differential equation

$$\frac{dx}{dt} = 1 + x^2 \tag{2.2}$$

with the initial condition

$$x(0) = 0. \tag{2.3}$$

Since $F(x) = 1 + x^2$ is continuous, Theorem 1 guarantees there is a solution of (2.2) which satisfies (2.3) on some interval containing $t = 0$. Since $\frac{dF(x)}{dx} = 2x$ is also continuous, F satisfies a Lipschitz condition on any closed and bounded x-interval.

It is easy to see that the desired solution of (2.2) is

$$x(t,0,0) = \tan t .$$

Note here that Theorem 1 assures only the local existence of a solution through the point $(t_o,x_o) = (0,0)$. In the present example, the interval of existence is

$$-\frac{\pi}{2} < t_1 < t < t_2 < \frac{\pi}{2} .$$

The solution cannot be extended to any t interval containing $\pm\frac{\pi}{2}$ since $\tan t \to \pm\infty$ as $t \to \pm\frac{\pi}{2}$ and the solution will not stay in any domain D as required in Theorem 3.

Consider now any closed subinterval

$$-\frac{\pi}{2} < t_3 \leq t_4 < \frac{\pi}{2} .$$

Theorem 2, part ii, asserts the existence of a $d > 0$ such that the solution

$$x(t,0,u) = \tan (t + \mathrm{Tan}^{-1} u) \tag{2.4}$$

of (2.2) exists on $[t_3,t_4]$ for all u with $\| u - x_o \| = |u| < d$. If the solution (2.4) is to exist on $[t_3,t_4]$, it is necessary that

$$t_4 + \mathrm{Tan}^{-1} u < \frac{\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < t_3 + \mathrm{Tan}^{-1} u .$$

Hence, permissible values of u are given by the conditions

$$u < \tan (\frac{\pi}{2} - t_4) \quad \text{and} \quad u > \tan (-\frac{\pi}{2} - t_3) = -\tan (\frac{\pi}{2} + t_3) .$$

Thus it is sufficient to take

$$d = \min\left[\tan\left(\frac{\pi}{2} - t_4\right),\ \tan\left(\frac{\pi}{2} + t_3\right)\right] .$$

This example thus shows the extent of the interval on which the solutions are to exist may limit the size of the set U.

A general property satisfied by the transformations associated with an autonomous system of differential equations is the so-called group property $T_s T_t = T_{s+t}$. Just how the composition $T_s T_t$ is effected must be understood. Suppose

$$T_t : (0,u) \rightarrow (t,\ x(t,0,u))$$

so that

$$T_t(u) = x(t,0,u) .$$

The composite mapping is formed by applying first T_t and T_s; that is,

$$T_s : (t,\ x(t,0,u)) \rightarrow (s+t,\ x(s+t,\ t,\ x(t,0,u)))$$

so that

$$T_s T_t(u) = x(s+t,\ t,\ x(t,0,u)) .$$

If the function $x(t,0,u)$ is determined uniquely by the initial condition $x(0) = u$, then

$$T_s T_t(u) = x(s+t,\ t,\ x(t,0,u)) = x(s+t,\ 0,\ u)$$
$$= T_{s+t}(u) . \qquad (2.4)$$

To exemplify the property (2.4) consider again Example 1 where

$$T_t(u) = \tan(t + \mathrm{Tan}^{-1} u) .$$

Let

$$s = s, \quad t = \frac{\pi}{4}, \quad \text{and} \quad u = 0 .$$

Then

$$T_{s+t}(u) = T_{s+\frac{\pi}{4}}(0) = \tan(s + \tfrac{\pi}{4})$$

and

$$T_s T_t(u) = T_s T_{\frac{\pi}{4}}(0) .$$

The transformation $T_{\frac{\pi}{4}}$ takes the point $(0,0)$ into the point $(\frac{\pi}{4},1)$ so $T_{\frac{\pi}{4}}(0) = \tan(\frac{\pi}{4}) = 1$. In forming the composite function $T_s T_{\frac{\pi}{4}}$, T_s operates on $(\frac{\pi}{4}, 1)$

$$T_s : (\tfrac{\pi}{4},1) \to (s+\tfrac{\pi}{4},\ x(s+\tfrac{\pi}{4}, \tfrac{\pi}{4}, 1)) = (s+\tfrac{\pi}{4},\ \tan(s+\tfrac{\pi}{4}))$$

so that

$$T_s T_{\frac{\pi}{4}}(0) = \tan(s + \tfrac{\pi}{4}),$$

the desired result. Note that for $s = \frac{\pi}{4}$, $T_s T_{\frac{\pi}{4}}(0)$ is not defined. The example thus shows that if the solution $x(t,0,u)$ exists only on a finite interval $[t_3, t_4]$, it may occur that $T_s T_t$ does not belong to $\{T_t\}$ even though T_s and T_t separately belong to $\{T_t\}$. To avoid such a situation it is sufficient to assume that the solution can be extended to exist for all t, $-\infty < t < +\infty$.

In summation, it has been shown that the system of differential equations

$$\frac{dx}{dt} = F(x)$$

defines a set of transformations $\{T_t\}$. In case the solution $x = x(t,t_o,u)$ is unique and exists for all t, $-\infty < t < +\infty$, the set $\{T_t\}$ is said to form a dynamical group.

Definition 1. A dynamical group is a set of transformations $\{T_t\}$ with the properties

i. The domain of the group is a set U in E_n so that for each t, $-\infty < t < +\infty$, $T_t : U \to E_n$.

ii. The set $\{T_t\}$ has T_o as the identity element; that is, for each $u \varepsilon U$, $T_o(u) = u$.

iii. $T_t(u)$ is a continuous function of t and u for all t and for $u \varepsilon U$.

iv. $T_s T_t(u) = T_{s+t}(u)$ for all $u \varepsilon U$.

CHAPTER III

INTEGRAL INVARIANTS AND LIOUVILLE'S THEOREM

In the autonomous system of differential equations

$$\frac{dx}{dt} = F(x) \tag{3.1}$$

let F and its first partial derivatives $\frac{\partial F_i}{\partial x_j}$, $i, j, = 1, 2, \ldots, n$, be continuous functions in a domain D of E_n. Let the derivatives $\frac{\partial F_i}{\partial x_j}$, $i, j = 1, 2, \ldots, n$, also be bounded in D. By the mean value theorem, F then satisfies a Lipschitz condition with respect to $\mathbf{x}$ in D, so the solution $x = x(t,0,u)$ is uniquely determined by the initial condition

$$x(0,0,u) = u .$$

Suppose further that the solution $x = x(t,0,u)$ exists for all t, $-\infty < t < +\infty$, so that a dynamical group $\{T_t\}$ is defined. Let $T_t(D) = D_t$ for each t, $-\infty < t < +\infty$.

<u>Definition 2.</u> An <u>integral invariant</u> [for (3.1)] is a quantity that can be expressed in the form

$$\int_D M(x)\,dx = \int\int \underset{D}{\cdots} \int M(x_1, x_2, \ldots, x_n)\,dx_1\,dx_2 \cdots dx_n$$

if this quantity possesses the property

$$\int_D M(x)\,dx = \int_{D_t} M(x)\,dx$$

for all t, $-\infty < t < +\infty$.

Example 2. Let the system (3.1) with $x = (x_1,x_2,x_3)$ and $F = (F_1,F_2,F_3)$ determine the velocity of the steady state motion of a fluid in E_3. If $r(x)$ denotes the density of the fluid at the point x, then $\int_D r(x)\,dx$ is the mass of the fluid filling the domain D.

In t units of time, D flows into D_t. On physical grounds the mass of the fluid remains constant in this transformation so that

$$\int_D r(x)\,dx = \int_{D_t} r(x)\,dx .$$

Thus the equations representing the steady state flow of a fluid possess an integral invariant. Note that for an incompressible fluid $r(x)$ is constant so that the volume is an integral invariant.

At this point it is natural to ask if there are analytical conditions which insure the existence of an integral invariant. More particularly, when will the volume be an integral invariant. Liouville's theorem answers this question.

Theorem 5. (Liouville's Theorem). If

$$\operatorname{div} F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = 0$$

then the n-dimensional volume is an integral invariant for (3.1).

<u>Proof</u>. The volume $\int_D dx$ is an integral invariant if and only if

$$\frac{d}{dt} \int_{D_t} dx = 0 . \tag{3.2}$$

Assuming the set of transformations $\{T_t\}$ is defined by the solution $x = x(t,0,u)$ of (3.1), an application of the formula for change of variables in a multiple integral yields

$$\int_{D_t} dx = \int_D J(t,u)\ du$$

where $J(t,u)$, the Jacobian of the transformation, is the determinant of the square matrix $\left(\frac{\partial x_i}{\partial u_j}\right)$, $i,j = 1,2, \ldots, n$. Theorem 2, part iii, insures the existence and the continuity of $J(t,u)$. Thus the requirement (3.2) becomes

$$\frac{d}{dt} \int_D J(t,u)\ du = \int_D \frac{d}{dt} J(t,u)\ du = 0 .$$

Thus a sufficient condition for (3.2) is

$$\frac{d}{dt} J(t,u) = 0 .$$

Since

$$J(t,u) = \det \left(\frac{\partial x_i}{\partial u_j}\right)$$

$$\frac{d}{dt} J(t,u) = \sum_{j=1}^{n} D_j$$

where

$$D_j = \det \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{du_n} \\ \vdots & & \vdots \\ \frac{d}{dt}\frac{\partial x_j}{\partial u_1} & \cdots & \frac{d}{dt}\frac{\partial x_j}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_n} \end{pmatrix}, \quad j = 1,2, \ldots, n .$$

Than since

$$\frac{d}{dt}\frac{\partial x_j}{\partial u_k} = \frac{\partial}{\partial u_k}\frac{dx_j}{dt} = \frac{\partial}{\partial u_k} F_j(x) = \sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i}\frac{\partial x_i}{\partial u_k}$$

each D_j is the sum of n determinants

$$D_j = \det \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \vdots & & \vdots \\ \sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i}\frac{\partial x_i}{\partial u_1} & \cdots & \sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i}\frac{\partial x_i}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_n} \end{pmatrix}$$

$$= \sum_{i=1}^{n} \frac{\partial F_j}{\partial x_i} \det \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial x_i}{\partial u_1} & \cdots & \frac{\partial x_i}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_n} \end{pmatrix} .$$

Each determinant in the last sum is zero except when $i = j$. Thus

$$\frac{d}{dt} J(t,u) = \sum_{j=1}^{n} D_j = \sum_{j=1}^{n} \left(\frac{\partial F_j}{\partial x_j}\right) J(t,u) .$$

So a sufficient condition for (3.2) is

$$\operatorname{div} F = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j} = 0 .$$

Furthermore, $\operatorname{div} F = 0$ is also a necessary condition that the volume $\int_{D'_t} dx$ of the image D'_t of every (arbitrarily small) subset D' of D be invariant. For suppose $\operatorname{div} F \neq 0$. Recall that

$$\frac{d}{dt} \int_{D'_t} dx = \frac{d}{dt} \int_{D'} J(t,u)\, du$$

$$= \int_{D'} \frac{d}{dt} J(t,u)\, du = \int_{D'} J(t,u) \operatorname{div} F\, du .$$

Since $J(t,u)$ and $\operatorname{div} F$ are continuous and since $J(t,u) \neq 0$, if $\operatorname{div} F \neq 0$ it is possible to find a neighborhood on which the integrand is strictly positive (or negative). Let this neighborhood be D'.

Theorem 5 is particularly important when $n = 2s$ and the system (3.1) represents a Hamiltonian system (1.1). In this case,

$$(x_1, x_2, \ldots, x_n) = (q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s)$$

and

$$(F_1, \ldots, F_n) = \left(\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \ldots, \frac{\partial H}{\partial p_s}, -\frac{\partial H}{\partial q_1}, \ldots, -\frac{\partial H}{\partial q_s} \right)$$

so that

$$\text{div } F = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j} = \sum_{j=1}^{s} \left[\frac{\partial}{\partial q_j}\left(\frac{\partial H}{\partial p_j}\right) + \frac{\partial}{\partial p_j}\left(-\frac{\partial H}{\partial q_j}\right)\right] = 0 .$$

Therefore n-dimensional volume is an integral invariant for any mechanical system whose motions are described by the canonical equations (1.1).

CHAPTER IV

LEBESGUE MEASURE AND DYNAMICAL SYSTEMS

The previous chapters have shown how the study of dynamical systems often reduces to the study of the properties of a set of transformations defined by the equations of motion for the system. This method of study has proven to be particularly effective when the number of dimensions of the solution space is very large. In such cases it is impossible in practice to specify completely the initial point (t_o, x_o). However the described transformations $\{T_t\}$ operate on <u>sets</u> of initial conditions, and hence it is possible to formulate theorems of the following type: if a set S of initial conditions has property P, then for each t, $-\infty < t < +\infty$, the set $T_t S$ will have property Q. The most important questions are the asymptotic ones: what will happen to $T_t S$ as $t \to \infty$? The most useful results are given as holding for "almost all" points u of S; that is, for all points of S with the possible exception of a set of measure zero. The measure chosen should be appropriate to the problem under consideration. In the case of dynamical systems in E_n, Lebesgue measure generally is used.

A brief exposition of Lebesgue measure in E_n follows. The purpose of the development is to introduce the main definitions and theorems. For proofs of the theorems see, for example, Kolmogorov and Fomin [2].

Definition 3. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ where $a_i < b_i$, $i = 1, 2, \ldots, n$, are finite real numbers. The set of points $x = (x_1, x_2, \ldots, x_n)$ in E_n for which $a_i < x_i < b_i$, $i = 1, 2, \ldots, n$, is called an open, bounded, n-dimensional rectangle. Similarly, a rectangle defined by $a_i \leq x_i \leq b_i$, $i = 1, 2, \ldots, n$, is called a closed rectangle. If in the defining relationship both symbols $<$ and $\leq$ occur, the rectangle is called half-open.

Definition 4. The measure m of a rectangle R (whether closed, open, or half-open) is a nonnegative, real-valued function defined by

1) If φ represents the empty set, then $m(\varphi)$ is taken as zero.

2) The measure of a rectangle R is

$$m(R) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n) .$$

3) The measure m is finitely additive; that is, if $R = \bigcup_{i=1}^{n} R_i$ where the R_i, $i = 1, 2, \ldots, n$, are pairwise-disjoint rectangles, $R_i \cap R_j = \varphi$ for $i \neq j$, then

$$m(R) = \sum_{k=1}^{n} m(R_k) .$$

The concept of measure is next extended to a larger class of sets, the elementary sets.

Definition 5. A set S in E_n is called an elementary set if S can be represented as the union of a finite number of pairwise-disjoint rectangles

$$S = \bigcup_{i=1}^{n} R_i \; .$$

Definition 6. Let $S = \bigcup_{i=1}^{n} R_i$ be an elementary set. The measure of S, $m'(S)$, is the number

$$m'(S) = \sum_{i=1}^{n} m(R_i) \; .$$

It is easy to show that $m'(S)$ is independent of the particular makeup of S by rectangles. It is also easy to show that m' is finitely additive. Furthermore, m' has the following important property.

Theorem 6. Let A be an elementary set and let $\{R_k\}$ be a countable collection of rectangles such that $A \subset \bigcup_k R_k$. Then

$$m'(A) \leq \sum_k m(A_k) \; .$$

Next the concept of measure is extended to a class of sets which can be approximated (in a sense to be made precise) by elementary sets.

Let A be a bounded set in E_n. Then it is possible to find sets of rectangles $\{R_j\}$ which cover A; that is, for each set $\{R_j\}$, $A \subset \bigcup_j R_j$.

Definition 7. The outer measure μ^* of A is the real number

$$\mu^*(A) = \inf \{ \sum_j m(R_j) : A \subset \bigcup_j R_j \}$$

where the infinum is taken over all countable collections of rectangles $\{R_j\}$ which cover A.

If the set A is bounded, it is possible to find a single rectangle R which covers A.

Definition 8. The inner measure μ_* of the bounded set A is the real number

$$\mu_*(A) = mR - \mu^*(R - A) .$$

It can be shown that $\mu_*(A)$ is independent of the particular choice of R.

It is clear that for any bounded set A,

$$\mu_*(A) \leq \mu^*(A) .$$

Definition 9. A bounded set A in E_n is saif to be Lebesgue measurable if

$$\mu_*(A) = \mu^*(A) .$$

The common value of the inner and outer measures is called the Lebesgue measure of A and is denoted μA. Whenever the term measurable is used without qualification, Lebesgue measurable is to be understood.

An unbounded set S in E_n is said to be Lebesgue measurable if the intersections of S with all bounded rectangles R are measurable. The measure of S is the real number (perhaps infinite)

$$\mu S = \sup_R \{\mu(S \cap R)\} .$$

Theorem 7. Let A be a set in E_n. A is measurable if and only if, for every $\varepsilon > 0$, there is an elementary set B such that

$$\mu^*(A \cup B - A \cap B) < \varepsilon .$$

Thus a set in E_n is said to be Lebesgue measurable if it can be approximated by rectangles.

Some fundamental properties of measurable sets and the measure μ are stated in the following theorem.

Theorem 8. Let $\{A_n\}$ be a countable or finite collection of measurable sets.

1) The union $\bigcup_n A_n$ and the intersection $\bigcap_n A_n$ are measurable sets.

2) If $A_i \subset A_j$ for any i and j, then $A_j - A_i$ is measurable and

$$\mu(A_j - A_i) = \mu A_j - \mu A_i .$$

3) Lebesgue measure is countably subadditive; that is,

$$\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$$

with equality holding if the sets are pairwise disjoint.

Unfortunately, the class of Lebesgue measurable sets is too inclusive for direct application to the theory of dynamical systems, for there are examples which show that the image of a Lebesgue measurable set under a topological mapping is not necessarily measurable. Thus it is necessary to introduce a class of sets "between" the elementary sets and Lebesgue measurable sets. These sets are called the Borel sets.

Definition 10. Let A be any collection of subsets of E_n with the following properties:

1) $E_n \varepsilon A$.

2) If $A_1 \varepsilon A$ then $E_n - A_1 \varepsilon A$.

3) If $A_1, A_2, \ldots, A_n, \ldots$ is a finite or countable collection of subsets of A, then $\bigcup_i A_i \varepsilon A$.

4) If R is an elementary set, then $R \varepsilon A$.

The smallest such collection of sets is called the collection of Borel sets of E_n. Clearly the set of Borel sets is the intersection of all collections of sets which satisfy the conditions of Definition 10. The following theorem lists some relevant properties of Borel sets.

Theorem 9.

1) If T is a topological mapping and B is a Borel set, then $T(B)$ is also a Borel set.

2) Every open set and every closed set in E_n is a Borel set.

3) Every Borel set is Lebesgue measurable.

4) If X is any Lebesgue measurable subset of E_n, then there are Borel sets A and B such that

$$A \subset X \subset B$$

and

$$\mu A = \mu X = \mu B .$$

Finally, it is necessary to introduce the concept of measurable functions and the Lebesgue integral.

Definition 11. A function f from E_n into the real number system is said to be a measurable function if, for every real number $c, -\infty < c < +\infty$, the set

$$\{x = (x_1, x_2, \ldots, x_n) : f(x) > c\}$$

is measurable.

Let A be a Lebesgue measurable subset of E_n having finite Lebesgue measure. Let

$$A = L_1 \cup L_2 \cup L_3 \cup \cdots \cup L_n$$

where $L_1, L_2, \ldots, L_n$ is a finite collection of pairwise disjoint Lebesgue measurable sets. Let $f : A \to E_n$ be a function defined and bounded on A with

$$m_j = \inf \{f(x) : x \in L_j\}$$

and

$$M_j = \sup \{f(x) : x \in L_j\}, \quad j = 1, 2, \ldots, n$$

Definition 12. The sums

$$\underline{L} = \sum_{j=1}^{n} m_j \, \mu L_j \quad \text{and} \quad \overline{L} = \sum_{j=1}^{n} M_j \, \mu L_j$$

are called respectively the lower and the upper Darboux sum of f on A with respect to the partition $\{L_j\}$, $j = 1, 2, \ldots, n$.

The supremum of all lower Darboux sums over all partitions of A into a finite number of pairwise disjoint measurable sets is called the lower (Lebesgue) integral of f on A. Similarly, the infimum of all upper Darboux sums over all such partitions is called the upper (Lebesgue) integral of f on A. If the upper and the lower integrals of f on A are equal, then f is said to be integrable on A, and the common value of the upper and lower integrals is called the Lebesgue integral. This integral is denoted by

$$\int_A f(x)\ d\mu .$$

The following results are of fundamental importance.

Theorem 10.

1) If a function g has a Riemann integral on a set A, then the Lebesgue integral of g on A exists and equals the value of the Riemann integral.

2) If a set A has finite Lebesgue measure and f and g are two functions having Lebesgue integrals on A, then for every constant k, the function $kf + g$ has a Lebesgue integral on A with

$$\int_A (k\ f(x) + g(x))\ d\mu = k \int_A f(x)\ d\mu + \int_A g(x)\ d\mu .$$

3) If A and B are two disjoint sets having finite Lebesgue measure, and if a function f has a Lebesgue integral on $A \cup B$, then f has a Lebesgue integral on A and on B with

$$\int_{A \cup B} f(x)\ d\mu = \int_A f(x)\ d\mu + \int_B f(x)\ d\mu .$$

To return to the application of Lebesgue measure to dynamical systems, two terms require definition. Let T be a transformation $T : E_n \to E_n$.

Definition 12. The transformation T is said to be a measurable transformation if the image of every measurable set M in the domain of T is a measurable set $T(M)$ in the range of T.

Definition 13. Let M be a measurable set in the domain of a measurable transformation T, and let $\mu(M)$ be the Lebesgue measure of M. The transformation T is said to be a measure-preserving transformation if

$$\mu(M) = \mu(TM) .$$

Now let the differential equation

$$\frac{dx}{dt} = F(x)$$

define a dynamical group $\{T_t\}$.

Theorem 11. Let $\operatorname{div} F = 0$. Then for each t, $-\infty < t < +\infty$, the transformation T_t is measurable and measure-preserving.

Proof. Assume the theorem is true for Riemann-measurable sets (see Theorem 5). Let B be a bounded Borel set and let

$$T_t(B) = B_t .$$

By Theorem 9, the image B_t is also a Borel set and hence measurable. Thus the measure of B_t is equal to the outer measure of B_t :

$$\mu(B_t) = \mu^*(B_t) = \inf \{ \sum_i \mu R_i \}$$

where the infimum is taken over all finite or countable collections of rectangles which cover B_t. Since the theorem is assumed true for Riemann measurable sets, for each rectangle R_i, $i = 1,2, \ldots,$

$$\mu R_i = \mu(T_{-t}(R_i)$$

and hence

$$\sum_i \mu R_i = \sum \mu(T_{-t}R_i) \ .$$

Since $B_t \subset \bigcup_i R_i$

$$T_{-t}(B_t) = B \subset T_{-t}(\bigcup_i R_i) \ .$$

These remarks imply that

$$\mu(B_t) = \mu^*(B_t) \geq \mu^*(B) = \mu(B) \ .$$

Let $S_t = T_t(S)$ be a rectangle covering B_t. Since $S_t - B_t$ is a Borel set, the argument above applied to $S_t - B_t$ implies that

$$\mu(S_t - B_t) \geq \mu(T_{-t}(S_t - B_t)) \ .$$

Since, for each t, T_t is a topological mapping it is one-to-one so that

$$T_t(S_t - B_t) = S - B$$

and hence

$$\mu(S_t - B_t) \geq \mu(S - B) \ .$$

Now assume that

$$\mu(B_t) > \mu(B) \ .$$

Then

$$\mu(S_t) = \mu(S_t - B_t) + \mu(B_t) > \mu(S - B) + \mu(B) = \mu(S)$$

so that

$$\mu(S_t) > \mu(S)$$

which contradicts the assumption that the theorem is true for Riemann measurable sets. The proof for unbounded Borel sets if clear.

Now let $L_t = T_t(L)$ be any Lebesgue measurable subset of E_n. By Theorem 9, there are two Borel sets A_t and B_t such that

$$A_t \subset L_t \subset B_t$$

with

$$\mu A_t = \mu L_t = \mu B_t .$$

Then

$$T_{-t} A_t = A \subset T_{-t} L_t = L \subset T_{-t} B_t = B$$

so that

$$\mu A = \mu_* A \leq \mu_* L \leq \mu^* L \leq \mu^* B = \mu B .$$

Since A and B are Borel sets

$$\mu A = \mu B$$

so that

$$\mu_* L = \mu^* L$$

so that L is Lebesgue measurable and

$$\mu L_t = \mu L .$$

CHAPTER V

RECURRENCE

In the study of dynamical systems, an important topic is stability. There are many definitions of stability, each being appropriate for some class of dynamical systems. For example, in the case of nonconservative systems, Lyapunov stability is used most often; the method usually is to determine separate asymptotically stable motions. For conservative systems, however, asymptotically stable motion is impossible. One of the more successful definitions of stability for conservative systems is "stability in the sense of Poisson."

Definition 14. Let u be any point in the domain U of a dynamical group $\{T_t\}$, and let V be any neighborhood of u, $V \subset U$. The point u is said to be positively stable in the sense of Poisson if, for every number $N > 0$, there is a $t \geq N$ such that $T_t(u)$ is in V. Similarly the point u is said to be negatively stable in the sense of Poisson if there is a $t \leq -N$ such that $T_t(u)$ is in V. If a point is both positively and negatively stable, it is said to be stable (in the sense of Poisson).

Thus a point u is (Poisson) stable if, for arbitrarily large and for arbitrarily small values of t, the point returns to every neighborhood of its initial position.

It is easy to see that any periodic motion is stable, for let $\{T_t\}$ be a dynamical group. Suppose there is a number $p > 0$ such that

$$T_{t+p}(u) = T_t(u), \quad -\infty < t < +\infty$$

for each u in the domain of the group. Then

$$T_o(u) = T_{kp}(u)$$

for each $k = 0, \pm 1, \pm 2, \dots,$ so for $t = kp$, the point $T_{kp}(u) = u$ and hence is in any neighborhood V of u .

The definition of Poisson stability given in Definition 14 can be reduced to an equivalent and more operative statement.

Theorem 12. The point u of U is positively Poisson stable if, for every neighbothood V of u, there is a $t \geq 1$ such that $T_t(u)$ is in V.

Proof. Assume the hypotheses of Theorem 12, and assume that the point u is not positively Poisson stable. Then there is a neighborhood V_1 of u and an $N > 1$ such that $T_t(u) \cap V_1 = \varphi$ for $t \geq N$. Consider the set $\{T_t(u)\}$ for $1 \leq t \leq N$. If there is a τ, $1 \leq \tau \leq N$ for which $T_\tau(u) = u$, then the motion is periodic and hence Poisson stable. So assume $T_t(u) \neq u$ for $1 \leq t \leq N$. The set $\{T_t(u)\}$ for $1 \leq t \leq N$ is a compact set in E_n, since T is continuous, and hence is a closed set and hence is a (finite) positive distance from u. Thus there is a neighborhood V_2 of u, $V_2 \subset V_1$ such that $T_t(u) \cap V_2 = \varphi$ for

$1 \leq t < +\infty$. This contradicts the hypotheses of Theorem 12.

Similarly, it can be shown that u is negatively Poisson stable if, for every neighborhood V of u, there is a $t \leq -1$ such that $T_t(u)$ is in V.

An analytical criterion for Poisson stability is given by the recurrence theorems of Poincare-Caratheodory. These theorems apply to the class of invariant sets for the group $\{T_t\}$.

Definition 15. A measurable set D in E_n is said to be an invariant set for the dynamical group $\{T_t\}$ if, for each t, $-\infty < t < +\infty$,

$$T_t(D) = D .$$

Now let the dynamical group be defined by a system of differential equations

$$\frac{dx}{dt} = F(x) .$$

Let the domain D of the group be a measurable set having finite (Lesbesgue) measure. Let D be invariant with respect to the group $\{T_t\}$; that is, $T_t(D) = D$ for each t, $-\infty < t < +\infty$.

Theorem 13. (Recurrence of sets) Let M be any measurable subset of the invariant set D, and let $\mu(M) > 0$. If $\operatorname{div} F = 0$, then there are positive and negative values of t, $|t| \geq 1$, such that

$$\mu(M \cap T_t M) > 0 .$$

Proof. The proof will demonstrate there are integer values of t, ($t = 0, \pm 1, \pm 2, \ldots$) for which the conclusion is valid. For convenience, let $T_n(M) = M_n$ for each integer n. Consider first $n = 0,1,2, \ldots, k$, and suppose the sets M_n, $n = 0,1,2, \ldots, k$, are pairwise disjoint (or at least the intersection of any pair is a set of measure zero). By the finite additivity of Lebesgue measure,

$$\mu\left(\bigcup_{n=o}^{k} M_n\right) = \sum_{n=o}^{k} \mu M_n .$$

Since $\operatorname{div} F = 0$, Theorem 11 implies that T_n, for each $n = 0,1, \ldots$, k, is a measure-preserving transformation. It was assumed that $\mu D < +\infty$, so

$$+\infty > \mu D > \mu\left(\bigcup_{n=o}^{k} M_n\right) = \sum_{n=o}^{k} \mu M_n = k\mu M_o$$

for every k. This is clearly impossible, so there must exist at least two sets M_i and M_j such that

$$\mu(M_i \cap M_j) > 0 . \tag{5.1}$$

For definiteness, let $0 \leq i < j \leq k$ and consider

$$T_{-i}(M_i \cap M_j) = M_o \cap M_{j-i} .$$

Since T_{-i} is a measure-preserving transformation,

$$\mu(M_o \cap M_{j-i}) = T_{-i}(\mu(M_i \cap M_j)) > 0 .$$

But $M_o = M$, so for $t = j - i \geq 1$,

$$\mu(M \cap M_t) > 0 .$$

Applying T_{-j} to (5.1) gives

$$\mu(M \cap M_{i-j}) > 0$$

and $i - j \leq -1$, so the theorem is proved.

Corollary. Theorem 13 implies an even stronger recurrence property, namely that the values of t such that $\mu(M \cap M_t) > 0$ can be chosen arbitrarily large (in absolute value). For, let $N > 0$ be given and let n be an integer $n > N$. Apply the method of proof of Theorem 13 to the sets

$$M_o, M_n, M_{2n}, \ldots$$

and obtain that

$$0 < T_{-ni}(\mu[M_{ni} \cap M_{nj}]) = \mu(M_o \cap M_{nj-ni})$$

where $nj - ni \geq n > N$. A similar argument applies for $t < -N$.

Theorem 14. (Recurrence of points) If $\operatorname{div} F = 0$, then almost every point u of D is stable in the sense of Poisson.

Proof. Let $M \subset D$ be a measurable set having positive measure, $\mu(M) > 0$. As before, let $M_n = T_n(M)$ for $n = 0, \pm 1, \pm 2, \ldots$. Consider, for $n = 0,1,2, \ldots,$ the set of points R_n of M_n which are not recurrent; that is,

$$R_n = M_n - \bigcup_{k=n+1}^{\infty} (M_n \cap M_k), \qquad n = 0,1,2, \ldots .$$

To see that $\mu R_o = 0$, suppose that $\mu R_o > 0$. Since $T_1 M_n = M_{n+1}$, then $T_1(M_o \cap M_n) = M_1 \cap M_{n+1}$, and $T_1 R_o = R_1$. In a similar fashion, $T_1 R_1 = R_2$, $T_1 R_2 = R_3$, ... etc., so $T_n R_o = R_n$. By hypothesis, div $F = 0$, so that each T_t, $-\infty < t < +\infty$, is measure-preserving. Thus

$$\mu R_o = \mu R_1 = \cdots = \mu R_n > 0 .$$

By construction of R_o,

$$R_o \cap M_n = \varphi \qquad \text{for} \quad n = 1,2, \ldots$$

and since $M_n \supset R_n$,

$$R_o \cap R_n = \varphi \qquad \text{for} \quad n = 1,2, \ldots .$$

In a similar fashion,

$$R_m \cap R_n = \varphi \qquad \text{for} \quad n = m+1, m+2, \ldots .$$

Thus $\{R_n\}$ for $n = 0,1,2, \ldots$, is a collection of pairwise disjoint measurable sets, so that

$$\mu(\bigcup_{n=o}^{k} R_k) = \sum_{n=o}^{k} \mu R_n = k\mu R_o .$$

This number can be made arbitrarily large in contradiction to the initial assumption that the measure of the invariant set D is finite. Thus $\mu R_o = 0$.

Let C be a countable basis for D. To be definite, let

$$C = \{C_i\} \qquad i = 1,2, \ldots$$

where each C_i is an n-dimensional sphere contained in D having rational radius and center at a point with all coordinates rational. For each C_i, $i = 1,2, \ldots$, construct the set of nonrecurrent points

$$R_o(C_i) = C_i - \bigcup_{k=1}^{\infty} [C_i \cap T_k(C_i)] .$$

Since each C_i is a measurable subset of D, the previous argument for M implies that

$$\mu R_o(C_i) = 0 \qquad i = 1,2, \ldots .$$

Consider the set $\bigcup_i R_o(C_i)$. By the countable additivity of Lebesgue measure,

$$\mu(\bigcup_i R_o(C_i)) = 0 .$$

It will be shown that all points in $D - \bigcup_i R_o(C_i)$ are negatively stable in the sense of Poisson.

Let u be in $D - \bigcup_i R_o(C_i)$ and let C_j be any sphere in C containing u. By definition of $\bigcup_i R_o(C_i)$, there is an integer $n \geq 1$ such that u is in $T_n(C_j)$. Thus $T_{-n}(u)$ is in C_j. Now if V is any neighborhood of u, for some k there is a set C_k of C such that $C_k \subset V$. By the preceding argument for the sphere C_j, there is an integer n, $-n \leq -1$ for which $T_{-n}(u)$ is in C_k. Thus $T_{-n}(u)$ is in the neighborhood V and u is negatively Poisson stable.

In a similar manner, starting with the sets M_o, M_{-1}, M_{-2}, ... , and defining the set

$$R_{-o} = M_o - \bigcup_{k=+1}^{+\infty} M_o \cap M_{-k}$$

construct the set

$$R_{-o}(C_i) = C_i - \bigcup_{k=1}^{\infty} C_i \cap T_{-k}(C_i)$$

for each C_i, $i = 1,2, \ldots$. Then

$$\mu(\bigcup_i R_{-o}(C_i)) = 0$$

and every point in the set $D - \bigcup_i R_{-o}(C_i)$ is positively Poisson stable. Thus every point in

$$D - [(\bigcup_i R_o(C_i)) \cup (\bigcup_i R_{-o}(C_i))]$$

is Poisson stable, and

$$\mu[(\bigcup_i R_o(C_i)) \cup (\bigcup_i R_{-o}(C_i))] = 0 .$$

CHAPTER VI

BIRKHOFF'S ERGODIC THEOREM

In 1931, G. D. Birkhoff proved a theorem of great importance for the general theory of dynamical systems. For the statement of the theorem, let the system of differential equations

$$\frac{dx}{dt} = F(x)$$

define a dynamical group $\{T_t\}$. Let $\operatorname{div} F = 0$ so that by Theorem 11, each transformation T_t is measure-preserving with respect to n-dimensional Lebesgue measure. Let the domain D of the group $\{T_t\}$ be an invariant measurable subset of E_n having finite Lebesgue measure, $\mu(D) < +\infty$. Let f be any function defined and absolutely summable on D; that is,

$$\int_D |f(x)| \, d\mu < +\infty .$$

<u>Theorem 15</u>. The limit

$$\lim_{C \to \pm\infty} \frac{1}{C} \int_0^C f(T_t(u)) \, dt$$

exists for almost all points u of the invariant set D.

The proof of this theorem will not be given here. For a very clear presentation see Khinchin [3]. Another excellent demonstration of the proof is given by Nemytskii and Stepanov [4].

An application of Theorem 15 is the case where V is any measurable subset of D and f is the characteristic function of V

$$f(u) = \begin{cases} 1 & \text{if } u \in V \\ 0 & \text{if } u \in D - V \end{cases} .$$

Theorem 15 then asserts that almost every point $u \in V$ spends a definite average amount of time in V under the action of T_t, $-\infty < t < +\infty$.

Definition 16. Let f be the function described in Theorem 15. The function $\bar{f}$ defined on D by the equation

$$\bar{f}(u) = \lim_{C \to \infty} \frac{1}{C} \int_0^C f(T_t(u))\, dt \qquad (6.1)$$

shall be called the time average of f along the solution curve or "trajectory" through the point u.

It is indeed proper to call the expression (6.1) the time average of f, for (6.1) does not depend on what point of the solution curve through the point u is chosen as the initial point. This will be proven.

Theorem 16. Let $u \in D$, and let $\bar{f}(u)$ exist. Then for all t, $-\infty < t < +\infty$,

$$\bar{f}(u) = \bar{f}(T_t u) .$$

Proof. To be definite, let $t > 0$. By hypothesis,

$$\lim_{C \to \infty} \frac{1}{C + t} \int_0^{C+t} f(T_\tau u)\, d\tau = \bar{f}(u) \qquad (6.2)$$

exists. Note that

$$\frac{1}{C}\int_0^{C+t} f(T_\tau u)\, d\tau - \frac{1}{C+t}\int_0^{C+t} f(T_\tau u)\, d\tau \qquad (6.3)$$

$$= \frac{t}{C}\,\frac{1}{C+t}\int_0^{C+t} f(T_\tau u)\, d\tau \;.$$

Taking the limit of the right hand side

$$\lim_{C\to\infty} \frac{t}{C}\,\frac{1}{C+t}\int_0^{C+t} f(T_\tau u)\, d\tau \;=\; \left(\lim_{C\to\infty}\frac{t}{C}\right)\bar{f}(u) \;=\; 0$$

so from (6.2) and (6.3)

$$\lim_{C\to\infty} \frac{1}{C}\int_0^{C+t} f(T_\tau u)\, d\tau \;=\; \bar{f}(u)\;. \qquad (6.4)$$

By definition

$$\bar{f}(T_t u) \;=\; \lim_{C\to\infty}\int_0^{C} f(T_s(T_t u))\, ds\;. \qquad (6.5)$$

By hypothesis, the set of transformations $\{T_t\}$ is a dynamical group, so that

$$T_s(T_t u) \;=\; T_{s+t}\, u\;.$$

Hence

$$\int_0^{C} f(T_s(T_t u))\, ds \;=\; \int_0^{C} f(T_{s+t}\, u)\, ds\;.$$

In the last integral, make the change of variable $v = s + t$:

$$\int_0^{C} f(T_{s+t}\, u)\, ds \;=\; \int_t^{C+t} f(T_v\, u)\, dv \qquad (6.6)$$

$$= \int_0^{C+t} f(T_v\, u)\, dv - \int_0^{t} f(T_v\, u)\, dv\;.$$

From Equations (6.5) and (6.6)

$$\overline{f}(T_t\, u) = \lim_{C\to\infty} \frac{1}{C} \int_0^{C+t} f(T_\tau\, u)\, d\tau) - \lim_{C\to\infty} \frac{1}{C} \int_0^{t} f(T_\tau u)\, d\tau. \tag{6.7}$$

Since

$$\lim_{C\to\infty} \frac{1}{C} \int_0^{t} f(T_\tau u)\, d\tau = 0$$

Equations (6.4) and (6.7) imply that

$$\overline{f}(u) = \overline{f}(T_t\, u) .$$

The most important case of Theorem 15 occurs "when the set of transformations $\{T_t\}$ do a good job of mixing up the points of the space" (Halmos). Stated more formally, let $\{T_t\}$ be a measure-preserving, dynamical group of transformations defined on a measurable and invariant set D of E_n with $\mu(D) < +\infty$.

Definition 17. The set D is said to be decomposable if there exists two disjoint measurable sets A and B, each having positive measure, such that

$$D = A \cup B$$

and

$$T_t A = A \text{ and } T_t B = B, \quad -\infty < t < \infty .$$

Otherwise, the set D is said to be indecomposable. In case D is indecomposable, the group $\{T_t\}$ is said to be metrically transitive or ergodic.

Theorem 17. Let the conditions of Theorem 15 apply. If the set D is indecomposable, then

$$\bar{f}(u) = \lim_{C \to \infty} \frac{1}{C} \int_0^C f(T_t u) \, dt$$

is a constant for almost every $u \in D$.

Proof. Assume $\bar{f}(u)$ is not a constant almost everywhere on D. Let M be the supremum of $\bar{f}$ on D with the possible exception of a set of measure zero; that is,

$$\mu \{u : \bar{f}(u) > M\} = 0$$

and for every $\varepsilon > 0$,

$$\mu \{u : \bar{f}(u) > M - \varepsilon\} > 0 .$$

Similarly, let m be the infimum of $\bar{f}$ on D with the possible exception of a set of measure zero. Since $\bar{f}$ is not constant, there is a real number r such that

$$m < r < M.$$

Let

$$D_1 = \{u : \bar{f}(u) < r\}$$

and let

$$D_2 = \{u : \bar{f}(u) \geq r\} = D - D_1 .$$

By construction,

$$\mu D_1 > 0 \quad \text{and} \quad \mu D_2 > 0 .$$

By Theorem 16, the sets D_1 and D_2 are invariant, for if $u \varepsilon D_1$

$$\bar{f}(u) = \bar{f}(T_t\, u) < a$$

so that $T_t u \varepsilon D_1$. A similar statement holds if $u \varepsilon D_2$. Thus D is decomposable in contradiction to the hypotheses. Hence $\bar{f}$ must be constant almost everywhere on D.

There is an expression for computing the value of the constant $\bar{f}(u)$. It will be shown that $\bar{f}(u)$ is equal to a quantity called the phase average of the function f.

Definition 18. Let f be the function described in Theorem 15. The number $\hat{f}$ defined on the set D by the equation

$$\hat{f} = \frac{1}{\mu D} \int_D f(u)\, d\mu$$

shall be called the phase average of f over the set D.

Theorem 18. Let the hypotheses of Theorem 17 be satisfied. Then

$$\bar{f}(u) = \frac{1}{\mu D} \int f(u)\, d\mu = \hat{f}$$

almost everywhere on D.

For the proof of Theorem 18, it is convenient first to prove a lemma. For convenience of notation, let

$$f_C(u) = \frac{1}{C} \int_0^C f(T_t\, u)\, dt .$$

Lemma. The collection of functions $f_C(u)$, $0 < C < +\infty$, is summable in D uniformly with respect to C; that is, for every $\varepsilon > 0$, there is a $\delta > 0$ such that for every subset X of D with $\mu X < \delta$, the inequality

$$\int_X |f_C(u)| \, d\mu < \varepsilon$$

holds. The choice of $\delta > 0$ is independent of C.

Proof.

$$\int_X |f_C(u)| \, d\mu = \int_X |\frac{1}{C} \int_o^C f(T_t u) \, dt| \, d\mu$$

$$\leq \int_X [\frac{1}{C} \int_o^C |f(T_t u)| \, dt] \, d\mu .$$

By Fubini's theorem (Kolmogorov and Fomin [5]),

$$\int_X |f_C(u)| \, d\mu \leq \frac{1}{C} \int_o^C [\int_X |f(T_t u)| \, d\mu] \, dt$$

$$= \frac{1}{C} \int_o^C [\int_{T_t(X)} |f(u)| \, d\mu] \, dt .$$

From the hypotheses of Theorem 15, f is absolutely summable on D and hence on $X \subset D$. Thus for $\varepsilon > 0$ given, there is a $\delta > 0$ such that

$$\int_X |f(u)| \, d\mu < \varepsilon$$

whenever $\mu X < \delta$. Also by hypotheses, each transformation T_t is measure-preserving so that

$$\mu(T_t X) < \delta$$

which implies that

$$\int_X |f_C(u)|\ d\mu \ < \ \frac{1}{C} \int_o^C \varepsilon\ dt \ = \ \varepsilon$$

for every C, $0 < C < +\infty$.

Proof of Theorem 18. Recall that

$$\hat{f} \ = \ \frac{1}{\mu D} \int_D f(u)\ d\mu \ .$$

It will be shown for almost all $u \varepsilon D$ that

$$\bar{f}(u) \ = \ \hat{f}$$

when D is indecomposable.

By Theorem 17, $\bar{f}(u)$ is constant almost everywhere on D, say

$$\bar{f}(u) \ = \ a$$

almost everywhere on D. Then

$$a \ = \ \frac{1}{\mu D} \int_D a\ d\mu$$

$$= \ \frac{1}{\mu D} \int_D [a - f_C(u)]\ d\mu + \frac{1}{\mu D} \int_D f_C(u)\ d\mu \ . \qquad (6.8)$$

Inserting the definition of f_C and applying Fubini's theorem,

$$\frac{1}{\mu D} \int_D f_C(u)\ d\mu \ = \ \frac{1}{\mu D} \int_D [\frac{1}{C} \int_o^C f(T_t\ u)\ dt]\ d\mu$$

$$= \ \frac{1}{C\mu D} \int_o^C [\int_D f(T_t\ u)\ d\mu]\ dt$$

$$= \ \frac{1}{C\mu D} \int_o^C [\int_{T_t D} f(u)\ d\mu]\ dt \ .$$

From the hypotheses of Theorem 15, D is an invariant set, so the last equation implies that

$$\frac{1}{\mu D}\int_D f_C(u)\ d\mu = \frac{1}{C\mu D}\int_o^C [\int_D f(u)\ d\mu]\ dt$$

$$= \frac{1}{\mu D}\int_D f(u)\ d\mu = \hat{f}\ .$$

Therefore, from Equation (6.8),

$$a = \frac{1}{\mu D}\int_D [a - f_C(u)]\ d\mu + \hat{f}$$

so that

$$a - \hat{f} = \frac{1}{\mu D}\int_D [a - f_C(u)]\ d\mu\ . \tag{6.9}$$

It will be shown that $a - \hat{f}(u) = 0$. Let $\varepsilon > 0$ be given. Let $D_1(C)$ be the set of points $u \varepsilon D$ such that

$$|a - f_C(u)| < \varepsilon$$

and let

$$D_2(C) = D - D_1(C)\ .$$

Then

$$|\int_D [a - f_C(u)]\ d\mu| \leq \int_{D_1} |a - f_C(u)|\ d\mu + \int_{D_2} |a - f_C(u)|\ d\mu \tag{6.10}$$

$$\leq \varepsilon\mu D_1 + |a|\mu D_2 + \int_{D_2} |f_C(u)|\ d\mu$$

$$\leq \varepsilon\mu D + |a|\mu D_2 + \int_{D_2} |f_C(u)|\ d\mu\ .$$

By Theorem 15, as $C \to \infty$, $f_C(u) \to 0$ almost everywhere on D, so

$$\mu D_2(C) \to 0 \quad \text{as} \quad C \to \infty .$$

Thus for sufficiently large C, μD_2 can be made arbitrarily small. For the given $\varepsilon > 0$, let C be chosen so large that

$$\mu D_2 < \min(\varepsilon, \delta)$$

where δ in the last expression is sufficient to guarantee that

$$\int_{D_2} |f_C(u)| \, d\mu < \varepsilon$$

in accordance with the preceding lemma. From the inequality (6.10),

$$0 \leq \int_D |a - f_C(u)| \, d\mu \leq \varepsilon \mu D + |a| \, \varepsilon + \varepsilon$$

so that from (6.9),

$$\hat{f} = a = \overline{f}(u)$$

almost everywhere on D.

Finally, indecomposability is a necessary condition that

$$\overline{f}(u) = \hat{f}$$

almost everywhere on D. to show this, let D be decomposable:

$$D = A \cup B$$

in accordance with Definition. Consider the summable function f defined by

$$f(u) = \begin{cases} 0 & \text{if } u \,\varepsilon\, A \\ 1 & \text{if } u \,\varepsilon\, B \end{cases} .$$

Then $\overline{f}(u) = 1$ or 0 while

$$\hat{f} = \frac{1}{\mu D} \int_D f(u)\, d\mu$$

$$= \frac{1}{\mu D} \left[\int_A f(u)\, d\mu + \int_B f(u)\, d\mu \right]$$

$$= \frac{1}{\mu D} \left[\int_B d\mu \right]$$

$$= \frac{\mu B}{\mu D} \neq \overline{f}(u) .$$

CHAPTER VII

THE ERGODIC PROBLEM

In the physical sciences, a common occurrence is the comparison of experimental data with the results predicted by theory. In case the physical system under consideration has a large number of degrees of freedom, a nontrivial problem arises: Physical quantities, in general, are functions of all the coordinates of the system. Thus, in order to compare the predicted and the measured values of a physical quantity at any time t, it is often necessary to know all the dynamical coordinates of the system for time t. This however is usually not possible. Consider, for example, the physical system of a box containing a gas. A complete determination of the dynamical coordinates would mean knowing the position and momentum of every molecule of the gas.

Furthermore, physical quantities are usually not measured instantaneously. During the time interval while the measurement is being made, the system may undergo changes which, in turn, change the quantity being measured. Thus, to be strictly proper, experimental values should be compared with theoretical time averages of the quantity being measured. This however leads to another problem: The average value of a given function over a time interval may vary considerably with the length of the interval. Here the results of the previous chapters find application. It was shown, under certain conditions, that the time average of any summable function is a constant. Thus approximately the same average value will be obtained for every large time interval, and here "large" means with respect to the physical system under consideration.

This however does not solve the first problem; that is, it is not known which solution curve is being traversed, and hence it is not possible to compute the time average for the system. Suppose, for example, that the system has five degrees of freedom and the motions of the system are described by a set of $2s$ differential equations (see Chapter I). To determine the appropriate solution curve, it would be necessary to know $2s - 1$ integrals for the set of differential equations. In general, only one integral, the total energy of the system, is known, and hence it is known only that the solution curve lies on a $2s - 1$ dimensional surface of constant energy. Suppose however that this surface is indecomposable. The results of the previous chapters imply that the time averages of any summable function are the same on almost every solution curve, and moreover this common time average is equal to the phase average of the function over the indecomposable surface of constant energy.

Thus the study of dynamical systems having many degrees of freedom can be reduced to two problems:

1) The ergodic problem - the justification of the replacement of time averages by phase averages.

2) The calculation of phase averages.

There are two historical precursors of Birkhoff's approach to the ergodic problem. First, Boltzmann and Maxwell hypothesized that each solution curve for the motions of a dynamical system will eventually pass through every point on the surface determined by the given energy of the system. This hypothesis indeed justified the replacement of time averages by phase averages. However Poincare [6] showed that the Boltzmann-Maxwell hypothesis is impossible. (The solution curve, representing the unique solution of a system of differential equations, can have no multiple points. This is incompatible with the requirement that the curve must cover the $2s - 1$ dimensional surface of constant energy). Poincare moreover indicated the only practical modification of the Boltzmann-Maxwell hypothesis, namely, the quasi-ergodic hypothesis of P. and T. Ehrenfest [7]: Let the surface of constant energy E be a bounded subset of E_n. Then almost every solution curve passes arbitrarily close to every point of E.

Unfortunately, no one has been able to show the quasi-ergodic hypothesis justifies replacing time averages by phase averages.

CHAPTER VIII

THE STANDARD EXAMPLE OF AN ERGODIC SYSTEM

Consider a system having two degrees of freedom q_1 and q_2. Let q_1 and q_2 be circular coordinates of period 1; that is, for any integers k_1 and k_2, the points

$$(q_1,q_2) \quad \text{and} \quad (q_1 + k_1,\ q_2 + k_2)$$

represent the same state of the system. Let p_1 and p_2 be momentum coordinates conjugate to q_1 and to q_2. This means that $p_i = \frac{d}{dt} q_i$, $i = 1,2$, but in the Hamiltonian formulation, q_1, q_2, p_1, and p_2 are all considered as independent variables.

The motions of the system are determined by the system of differential equations

$$\frac{dq_i}{dt} = + \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{\partial t} = - \frac{\partial H}{\partial q_i}, \qquad i = 1,2 \tag{8.1}$$

where $H = H(q_1,q_2,p_1,p_2)$ is the Hamiltonian function.

Definition 19. A function f of q_1,q_2,p_1,p_2 is called an integral for the system (8.1) if

1) f does not depend explicitly on t.

2) f is not identically a constant.

3) The system (8.1) can be used to show that $\frac{df}{dt} = 0$; that is, for values of q_1,q_2,p_1,p_2 satisfying (8.1), $f(q_1,q_2,p_1,p_2)$ is constant.

The system of equations (8.1) would be solved if four independent integrals could be found. Usually only those integrals determined by fixing the energy of the system are known. It is here that ergodic theory finds application; that is, in describing the properties of solutions of a system of equations which cannot be solved completely.

It is always necessary to fix the value of every single-valued integral if a system is to be ergodic. To show this, let the energy of the system under consideration be fixed so the motions of the system take place on the surface of constant energy Ω. Let f be a single-valued integral independent of the total energy; that is, f is not identically constant on each Ω. Since f is differentiable, it is continuous and hence cannot remain constant almost everywhere on Ω. Thus there is a number k,

$$\inf \{f(u) : u \in \Omega\} < k < \sup \{f(u) : u \in \Omega\} .$$

Since f is an integral, the sets

$$\Omega_1 = \{u \in \Omega : f(u) > k\}$$

and

$$\Omega_2 = \Omega - \Omega_1$$

are two disjoint invariant sets of positive measure such that

$$\Omega = \Omega_1 \cup \Omega_2$$

and the surface Ω is decomposable.

To return to the example, consider the Hamiltonian function

$$H(q_1,q_2,p_1,p_2) = \frac{1}{2}(p_1^2 + p_2^2)$$

so that Equations (8.1) become

$$\frac{dq_1}{dt} = p_1, \quad \frac{dq_2}{dt} = p_2, \quad \frac{dp_1}{dt} = 0, \quad \frac{dp_2}{dt} = 0 . \qquad (8.2)$$

Three integrals for the system (8.2) are the functions

$$p_1, \quad p_2, \quad \text{and} \quad q_1p_2 - q_2p_1 .$$

The integrals p_1 and p_2 are single-valued and hence must be fixed, say

$$p_1 = C_1 \quad \text{and} \quad p_2 = C_2 .$$

The third integral is not single-valued. To see this, let k_1 and k_2 be integers, and for the point (q_1,q_2,p_1,p_2) let

$$x = q_1p_2 - q_2p_1 .$$

Then $(q_1 + k_1, q_2 + k_2, p_1, p_2)$ represents the same point as (q_1, q_2, p_1,p_2) in the phase space of the system, but the value of the third integral at this point is

$$x + k_1p_2 - k_2p_1$$

which, in general, is different from x .

When p_1 and p_2 have been fixed, the motions of the system are described by the two circular coordinates q_1 and q_2. Thus the motions occur on the surface of a torus.

Theorem 19. Let the motions of a dynamical system be described by the set of equations (8.2). The two-dimensional torus T_2 determined in the phase space of the system by fixing the integrals

$$p_1 = C_1 \quad \text{and} \quad p_2 = C_2$$

is indecomposable if $C_1 \neq 0$ and C_2/C_1 is an irrational number.

Proof. Let $p_1 = C_1$ and $p_2 = C_2$ so the system (8.2) becomes

$$\frac{dq_1}{dt} = C_1, \quad \frac{dq_2}{dt} = C_2$$

or

$$q_1(t) = q_1(0) + C_1 t, \quad q_2(t) = q_2(0) + C_2 t .$$

If $C_1 \neq 0$,

$$q_2(t) = \frac{C_2}{C_1} q_1(t) + q_2(0) - \frac{C_2}{C_1} q_1(0) . \tag{8.3}$$

Let M be any measurable subset of T_2. From the remarks concerning Hamiltonian systems made following Theorem 5 in Chapter III, the two-dimensional volume (area)

$$\iint_M dq_1 \, dq_2$$

is an integral invariant for transformations of M defined by (8.3).

In Figure 2, the torus T_2 has been flattened into the unit square. In this planar representation, the points

$$(q_1,q_2) \quad \text{and} \quad (q_1 - [q_1],\ q_2 - [q_2])$$

are considered identical. Here $[x]$ denotes the greatest integer less than or equal to x, so $x - [x]$ is the fractional part of x.

Let the torus T_2 be decomposable, so there are two disjoint, invariant, measurable sets A and B having positive measure with

$$T_2 = A \cup B .$$

Let S be the set of points

$$S = \{q_2 : (0,q_2) \in A\} .$$

In Figure 2, A is the shaded region in the unit square. The set S is the intersection of A with the line $q_1 = 0$. The dotted line through the point (q_1,q_2) in A has slope C_2/C_1. As time varies, the transformations defined by (8.3) move the point (q_1,q_2) along the dotted line. For some value of t, the point $(0,b)$ is in A. Thus the set S is nonempty.

By assumption,

$$\iint_A dq_1\, dq_2 = \int_0^1 [\int_S dq_2]\, dq_1 > 0$$

so that the linear measure m_1 of S is positive

$$m_1(S) = \int_S dq_2 > 0 .$$

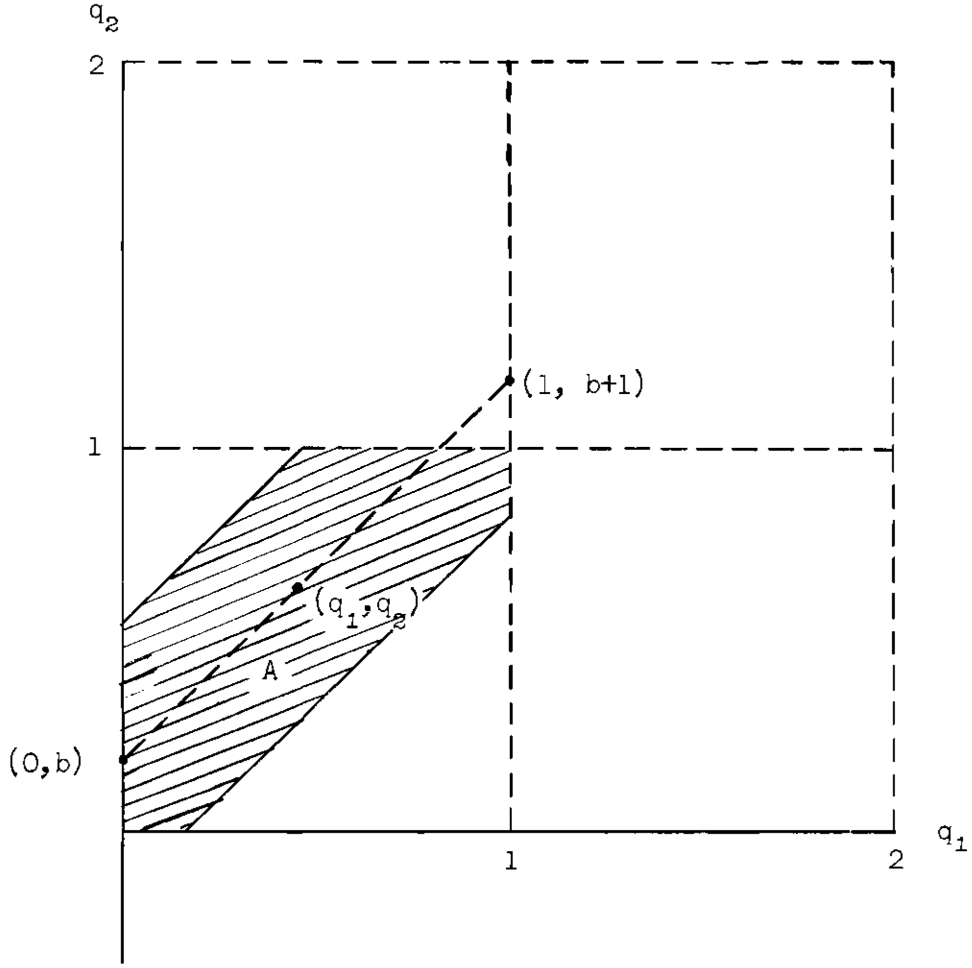

Figure 2. Planar Representation of T_2

There is a point u_o of S such that, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\frac{1}{2\delta} m_1[S \cap (u_o - \delta, u_o + \delta)] > 1 - \varepsilon .$$

If the ratio C_2 / C_1 is irrational, it can be shown that the set of points

$$\{u_n\} = \{u_o + n \frac{C_2}{C_1}\}, \qquad n = 0,1,2, \ldots$$

is everywhere dense in the set $\{(0,q_2) : 0 \leq q_2 < 1\}$ (see, for example, Nemytakii and Stepanov [9]). Then there is an integer N such that, for every point u of $\{(0,q_2) : 0 \leq q_2 < 1\}$, there is a u_i, $0 \leq i \leq N$ such that $|u - u_i| < \delta$. In other words, the intervals $(u_i - \delta, u_i + \delta)$, $i = 1,2, \ldots, N$ cover the set $\{(0,q_2) : 0 \leq q_2 < 1\}$. The linear measure m_1 is invariant so that

$$m_1[S \cap (u_i - \delta, u_i + \delta)] = m_1[S \cap (u_0 - \delta, u_0 + \delta)], \qquad i = 1,2, \ldots, N$$

and hence

$$\frac{1}{2\delta} m_1[S \cap (u_i - \delta, u_i + \delta)] > 1 - \varepsilon, \qquad i = 1,2, \ldots, N .$$

It follows that

$$m_1 S > 1 - \varepsilon$$

or, since the number $\varepsilon > 0$ is arbitrary,

$$m_1 S = \int_S dq_2 = 1 .$$

Thus the measure of A is given by

$$\iint_A dq_1 \, dq_2 = \int_0^1 [\int_S dq_2] \, dq_1 = \int_0^1 dq_1 = 1 .$$

It was assumed that $T_2 = A \cup B$ where A and B are disjoint sets having positive measure. Since

$$1 = \iint_{T_2} dq_1 \, dq_2 = \iint_{A \cup B} dq_1 \, dq_2 = \iint_A dq_1 \, dq_2 + \iint_B dq_1 \, dq_2$$

$$= 1 + \iint_B dq_1 \, dq_2$$

it follows that

$$\iint_B dq_1\, dq_2 \;=\; 0$$

in contradiction to the assumption that B has positive measure. Therefore the torus T_2 is indecomposable.

If the ratio C_2/C_1 is rational, the torus T_2 given by

$$p_1 \;=\; C_1, \qquad p_2 \;=\; C_2$$

is decomposable. In fact, the motions

$$q_1(t) \;=\; q_1(0) + C_1 t, \qquad q_2(t) \;=\; q_2(0) + C_2 t$$

are conditionally periodic (the word "conditionally" refers to the condition that C_2/C_1 be rational). To see this, let q_1 and q_2 have period $p > 0$ so that

$$q_1(t + p) \;=\; q_1(t) + C_1 t + C_1 p \;=\; q_1(t) + C_1 p$$

and similarly

$$q_2(t + p) \;=\; q_2(t) + C_2 p$$

It was assumed that the points

$$(q_1, q_2) \qquad \text{and} \qquad (q_1 + k,\; q_2 + \ell)$$

are identical for every pair of integers k and ℓ. Thus the motions on T_2 are periodic if

$$C_1 p = k \quad \text{and} \quad C_2 p = \ell$$

for some pair of integers k and ℓ. Thus

$$p = \frac{C_1}{k} = \frac{C_2}{\ell}$$

so that

$$\frac{C_2}{C_1} = \frac{\ell}{k} = \text{a rational number.}$$

Consider the pair of sets

$$M_1 = \{(0,q_2) : 0 \leq q_2 < \tfrac{1}{2}\}$$

$$M_2 = \{(0,q_2) : \tfrac{1}{2} \leq q_2 < 1\} .$$

The sets $\{T_t(M_1)\}$ and $\{T_t(M_2)\}$, $-\infty < t < +\infty$, are two disjoint invariant sets of positive measure which form a decomposition of the two dimensional torus

$$p_1 = C_1, \quad p_2 = C_2, \quad C_2/C_1 \text{ rational .}$$

CHAPTER IX

A THEOREM OF A. N. KOLMOGOROV

In 1957, A. N. Kolmogorov presented to the International Congress of Mathematicians a paper entitled, "General Theory of Dynamical Systems and Classical Mechanics." The most important part of the paper was a theorem concerning the conservation of conditionally periodic solutions.

For a statement of the theorem, consider a system of canonical equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1,2, \ldots, S \tag{9.1}$$

where the variables $q = (q_1, \ldots, q_s)$ are periodic with period 1. Let the Hamiltonian function H be a real analytic function of q,p and a small parameter θ, and assume H has the form

$$H(q,p,\theta) = W(p) + \theta\, S(q,p)$$

where W and S are real analytic functions of their arguments.

In case $\theta = 0$,

$$H(q,p,0) = W(p)$$

and (9.1) becomes

$$\frac{dq_i}{dt} = \frac{\partial W}{\partial p_i}, \quad \frac{dp_i}{dt} = 0, \qquad i = 1,2, \ldots, S \tag{9.2}$$

The phase space for the system (9.1) is then reduced to the invariant S-dimensional tori

$$p_i = C_i, \quad i = 1,2, \ldots, S$$

on which the conditionally periodic motions

$$q_i(t) = q_i(0) + t \frac{\partial}{\partial p_i} W(C_1, C_2, \ldots, C_S), \quad i = 1,2, \ldots, S$$

take place.

Theorem 20. Let B be any bounded region of the plane of points $p = (p_1, p_2, \ldots, p_S)$, and let $p = (C_1, C_2, \ldots, C_S) \varepsilon B$ determine a region G in the 2s-dimensional phase space for the system (9.1). For $p = (C_1, C_2, \ldots, C_S)$, let

$$1. \quad \det \left(\frac{\partial^2 W}{\partial p_i \, \partial p_j} (C_1, \ldots, C_S) \right) \neq 0, \quad i,j = 1,2, \ldots, S$$

and

$$2. \quad \left| \sum_{j=1}^{S} n_j \frac{\partial W}{\partial p_j} (C_1, \ldots, C_S) \right| \geq C \left(\sum_{j=1} |n_j| \right)^{-k}$$

for all integers n_j, $j = 1,2, \ldots, S$, and some $C > 0$ and $k > 0$. Then, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|\theta| < \delta$ implies that the region G, with the possible exception of a set of measure smaller than ε, consists of invariant S-dimensional tori. Moreover, on each of these tori, for appropriate coordinates y_i, $i = 1,2,\ldots,S$, the equations of motion (9.1) take the form

$$\frac{dy_i}{dt} = \frac{\partial W}{\partial p_i} (C_1, C_2, \ldots, C_S), \quad p_i = C_i, \quad i = 1,2, \ldots, S. \qquad (9.3)$$

The outline of a proof for Theorem 20 was given (in Russian) in 1954 by Kolmogorov [8]. However, a convergence argument in this outline is not convincing. Some results of Hamilton-Jacobi theory seem to imply that Theorem 20 is valid, but no complete proof has been published.

The importance of the theorem is easy to see. For $\theta = 0$, conditionally periodic motions take place on the invariant tori given by

$$p_i = C_i, \quad i = 1,2, \ldots, S .$$

Then for small perturbations of the Hamiltonian, all solutions (excepting possibly a set of small measure) still lie on invariant tori. Moreover, the solutions are conditionally periodic with the periodicity being described by the same constants

$$\frac{\partial W}{\partial p_i} (C_1, C_2, \ldots, C_S)$$

as the solutions for $\theta = 0$.

This implies that if a system is not ergodic for $\theta = 0$, it remains not ergodic when θ is sufficiently small. Kolmogorov's theorem then implies that there are large classes of nonergodic physical systems.