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PERIODIC SOLUTIONS OF A
NONLINEAR SECOND ORDER DIFFERENTIAL EQUATION

A THESIS

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NONLINEAR SECOND ORDER DIFFERENTIAL EQUATION

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CHAPTER I

INTRODUCTION

Consider the differential equation

\[ \ddot{x} + x = p f(t, x, \dot{x}, p) \quad \left( \frac{\text{d}}{\text{d}t} \right)^2 \]

(1.1)

together with the initial conditions

\[ x(0) = a + p(p), \quad p(0) = 0 \quad (1.2) \]

\[ \dot{x}(0) = b + q(p), \quad q(0) = 0 \]

and the generating equation obtained by letting \( p = 0 \)

\[ \ddot{x} + x = 0 \quad (1.3) \]

together with the initial conditions

\[ x(0) = a \quad (1.4) \]

\[ \dot{x}(0) = b \]

where \( f \) is a given real function which is periodic in \( t \) with period \( 2\pi \). It is desired to construct a solution of (1.1) satisfying (1.2) which is periodic in \( t \) with period \( 2\pi \) and which for \( p = 0 \) reduces to the solution

\[ x = a \cos t + b \sin t \]

of (1.3) satisfying (1.4). Hereafter this will be referred to as
the "basic problem" for (1.1)-(1.2).

In this study the problem is treated by first introducing a phase shift in (1.1)-(1.2) in order that the second initial condition in (1.2) may be taken as zero. This is done by defining

\[ h(\rho) = \arctan \frac{b + a(\rho)}{a + p(\rho)} = \frac{\pi}{2} - \arctan \frac{a + p(\rho)}{b + q(\rho)} \]

where \( p \) and \( q \) are taken to be analytic functions of \( \rho \) at \( \rho = 0 \) and

\[ p(\rho) = o(|\rho|^r), r \geq 1 \]

and

\[ q(\rho) = o(|\rho|^s), s \geq 1. \]

Without loss of generality it may be assumed that either \( a \) or \( b \) is different from zero. For if \( a = b = 0 \) and \( r \geq s \), \( p \) may be put in the form

\[ p = a_1 \rho^{r-s} + \overline{p}(\rho) \]

where \( \overline{p}(0) = \overline{q}(0) = 0 \) and \( b_1 \neq 0 \). Similarly for \( r < s \) and \( q/p \).

Hence, \( h(\rho) \) will be analytic at \( \rho = 0 \). Define \( g(\rho) = -h(\rho) \). Then (1.1)-(1.2) take the form

\[ \ddot{x} + x = p^p(t + g(\rho), x, \dot{x}, \rho) \]  \hspace{1cm} (1.5)

\[ x(0) = A_0 + \lambda(\rho) \]  \hspace{1cm} (1.6)

\[ \dot{x}(0) = 0 \]
and (1.3)-(1.4) become

\[ \ddot{x} + x = 0 \quad (1.7) \]
\[ x(0) = A_0 \quad (1.8) \]
\[ \dot{x}(0) = 0 \]

The basic problem will be to construct a solution, \( x(t, p) \) of (1.5) satisfying (1.6) which is periodic in \( t \) with period 2\( \pi \) and which for \( p = 0 \) reduces to the solution

\[ x(t, 0) = A_0 \cos t \]

of (1.7) satisfying (1.8).

The quantity \( A_0 \) and the functions \( g \) and \( \lambda \) are to be determined as functions of \( p \) such that the basic problem will have a solution.

It will be seen that there can be a solution to the basic problem only for certain pairs \( (A_0, g_0) \), called admissible, where \( g_0 = g(0) \).

An admissible pair \( (A_0, g_0) \) then determines a pair \( (\lambda'(0), g'(0)) \) and this in turn determines the pair \( (\lambda''(0), g''(0)) \), and so on, where the primes denote differentiation with respect to \( p \).

The basic problem connected with the corresponding autonomous system

\[ \ddot{x} + x = p(x, \dot{x}, p) \quad (1.9) \]
\[ x(0) = a + p(p), \quad p(0) = 0 \quad (1.10) \]
\[ \dot{x}(0) = b + q(p), \quad q(0) = 0 \]
is considered by Malkin [1], Stoker [2], Andronow and Chaikin [3], Coddington and Levinson[4] and others. Here a simple change of the independent variable always enables one to take the second initial condition in (1,10) to be zero without changing the differential equation (1.9). Once a solution is found, the inverse change of variable yields a solution of (1.9)-(1.10). Proskuryakov [5] solves the basic problem for (1.9)-(1.10) with the second initial condition zero and with $f$ analytic in its three arguments. In this work a similar method of solution is used to solve the basic problem for the system (1.5)-(1.6).

After independent work was begun on this study, Proskuryakov [6] published the solution of the basic problem for (1.1)-(1.2). He constructs a solution under the assumption that $f$ is periodic in $t$ with period $2\pi$, continuous in all its arguments, and analytic in $(x,\dot{x},p)$ for each fixed $t$. The method of solution requires that $f$ be expanded in powers of $p$. By treating the differential equation (1.1)-(1.2) as it stands (i.e., without first reducing the second initial condition to zero) he was able to avoid all differentiation of $f$ with respect to $t$. However, in the differential equation (1.5)-(1.6) treated in this study the phase shift in the first argument of $f$ makes it necessary to differentiate $f$ with respect to $t$ and hence a hypothesis stronger than continuity in $t$ is essential. In fact $f$ is assumed to be analytic in all its arguments. The method of solution also requires $g$ and $\lambda$ to be expanded in powers of $p$ and hence these functions are assumed to be analytic functions of $p$. 
The thesis is divided into three parts. Chapter Two is devoted to an exposition of basic concepts and theorems from the theory of differential equations which are needed to accomplish the main task. Most of these are known in some sense but are included for the sake of completeness. Chapter Three contains an exposition of the main theoretical problem described above, together with an example. The appendix contains some theorems from the theory of functions of several complex variables pertinent to the development of chapters two and three.
CHAPTER II

AUXILIARY CONCEPTS AND THEOREMS

In this chapter various basic concepts and theorems are presented to support the work done in Chapter Three. The existence and uniqueness theorem (Theorem 1) and the theorem on periodic solutions (Theorem 3) are basic in the theory of differential equations. The definition of multiple roots and the theorem on interchange of order of differentiation (Theorem 2) are more specialized to handle this particular problem.

Definition of Multiple Roots for a System of Equations.— For the equation \( h(x) = 0 \), the definition of multiple roots is quite simple. Suppose \( h \) is sufficiently smooth so that there is no question about differentiability. The equation is said to have an \( n \)-fold root at \( x = x_0 \) if \( h(x_0) = h'(x_0) = \ldots = h^{(n-1)}(x_0) = 0 \), but \( h^{(n)}(x_0) \neq 0 \).

In the case of a system

\[
\begin{align*}
f(x, y) &= 0 \\
g(x, y) &= 0
\end{align*}
\]

one considers the two curves in the \((x, y)\) plane determined by \( f(x, y) = 0 \) and \( g(x, y) = 0 \). Let \((x_0, y_0)\) be a point at which \( f(x_0, y_0) = g(x_0, y_0) = 0 \). In a neighborhood of \((x_0, y_0)\) \( f(x, y) = 0 \) defines \( y \) as a function of \( x \), \( y = y_1(x) \), and \( g(x, y) = 0 \) defines \( y \) as a function of \( x \), \( y = y_2(x) \). The system \( S \) is said to have an \( n \)-fold
root at \((x_0, y_0)\) if

\[ D^k y_1(x_0) = D^k y_2(x_0) \text{ for } k = 1, 2, \ldots, n - 1 \]

and

\[ D^n y_1(x_0) \neq D^n y_2(x_0), \text{ where} \]

\[ D^k \text{ is the operator } \frac{d^k}{dx^k}. \]

Note that the definition for the system \(S\) reduces to the simpler case above when \(f(x, y) = y - h(x)\) and \(g(x, y) = y\).

To illustrate the concept of multiple roots the case of double roots will be considered. Suppose \(S\) has a double root at \((x_0, y_0)\) and further suppose that \(f_y\) and \(g_y\) are different from zero. By the definition of double root it must be true that

\[ \frac{dy}{dx} = \frac{f_x}{f_y} = \frac{g_x}{g_y} \tag{2.1} \]

at \((x_0, y_0)\) and

\[ -(f_y)^{-2} [f_y (f_{xx} + f_{xy} \frac{dy}{dx}) - f_x (f_{xy} + f_{yy} \frac{dy}{dx})] \tag{2.2} \]

\[ \neq -(g_y)^{-2} [g_y (g_{xx} + g_{xy} \frac{dy}{dx}) - g_x (g_{xy} + g_{yy} \frac{dy}{dx})]. \]

Note that (2.1) is equivalent to saying that the Jacobian

\[ \frac{\partial (f, g)}{\partial (x_0, y_0)} = 0. \]
Using (2.1) in (2.2) one sees that for the case of double roots,

\[
\begin{align*}
\frac{f_y^2}{y} f_{xx} - 2 f_x g_y f_{xy} + g_x^2 f_{yy} &= - (2.3) \\
\frac{g_y^2}{y} g_{xx} - 2 g_x g_y g_{xy} + f_x^2 g_{yy} &\neq 0.
\end{align*}
\]

The following general existence and uniqueness theorem for complex systems is needed:

**Theorem 1.**—Let

1. \( \mathcal{I}_\rho \) be the region \( |\rho - \rho_0| < c, \, c > 0 \), where \( \rho \) is the vector \( \rho = (\rho_1, \ldots, \rho_n), \, \rho_i, \, i = 1, \ldots, n \) complex;

2. \( \mathcal{I}_g \) be the interval \( |g - g_0| < c', \, c' > 0 \), \( g \) real;

3. \( \mathcal{D} \) be a domain of \((t, w)\) space, \( t \) real and \( w \) the vector \( w = (w_1, \ldots, w_n), \, w_i \) complex \( i = 1, \ldots, n \);

4. \( \mathcal{D}_\rho \) be the set of points \((t, w, \rho)\) such that \((t, w) \in \mathcal{D}\) and \(\rho \in \mathcal{I}_\rho\) and;

5. \( \mathcal{D}_g \) be the set of points \((t + g, w, \rho)\) such that \((t, w, \rho) \in \mathcal{D}_\rho\), \(g \in \mathcal{I}_g\).

Further, let \( f(s, w, \rho) \) be continuous in \((s, w, \rho)\) on \( \mathcal{D}_\rho \) and for each fixed \( s \) let \( f \) be analytic in \((w, \rho)\), where \( f = (f_1, \ldots, f_n) \).

For \( \rho = \rho_0 \) let \( p(t, g_0, \rho_0) \) be a solution of

\[ w' = f(t + g_0, w, \rho_0) \]

on some interval \( I: \, a \leq t \leq b \) [i.e. \((t, p(t, g_0, \rho_0)) \in \mathcal{D} \) for \( t \in I \)]
satisfying \( p(t) = w_0 \) where \( r \in I \).

Then there exists a \( \delta > 0 \) such that for any \((w, p, g) \in U_{g, p}\),

\[
U_{g, p} : |w - w_0| + |g - g_0| + |p - p_0| < \delta,
\]

there exists a unique solution

\[
h = h(t, g, w_0, p) \quad \text{of}
\]

\[
w' = f(t + g, w_0, p)
\]

for \( t \in I \) with \( h(t, g, w_0, p) = \bar{w} \). Moreover, \( h \) is continuous in

\((t, g, w_0, p)\) and for each fixed \( t \) and \( g \), \( h \) is analytic in \((w, p)\).

**Proof.**— Choose a \( \delta_1 \) such that the closed \((t + g, w, p)\) region

\[
R_{g, p} = \{ (t + g, w, p) : a < t < b, |w - w_0| + |p - p_0| + |g - g_0| < \delta_1 \}
\]

is in \( D_{g, p} \). Let

\[
\bar{R} = \{ (t, g, w, p) : a < t < b, |w - w_0| + |p - p_0| + |g - g_0| < \delta_1 \}
\]

Define the successive approximations \( h_1 \) by

\[
h_0(t, \tau, g, w_0, p) = p(t, g_0, p_0) + \bar{w} - w_0 \quad (2.1)
\]

\[
h_n + 1(t, \tau, g, w_0, p) = \bar{w} +
\]

\[
\int_\tau^t f(s + g_n h_n(s, \tau, g, w_0, p), p_0) \, ds
\]

Since \( p \) is a solution of a differential equation on \([a, b] \),

\( p \) is a continuous function of \( t \) on \([a, b] \) and from (2.1) it is seen

that \( h_0 \) is continuous in \((t, g, w, p)\) for \( t \) on \([a, b] \) and any choice.
of the other variables. In particular then, \( h_0 \) is continuous in \((t, g, \bar{w}, p)\) on \( \mathbb{R}^k \). Similarly for each fixed \( t \in [a, b] \), \( h_0 \) is analytic in \((\bar{w}, p)\) and hence this is true at least on \( \mathbb{R}^* \).

As an induction hypothesis assume that \( h_n \) is continuous for \((t, g, \bar{w}, p) \in \mathbb{R}^k \) and that for each fixed \( t \) and \( g \), \( h_n \) is analytic in \((\bar{w}, p)\). By Theorem 3 of the appendix \( h_{n+1} \) is a continuous function of \((t, g, \bar{w}, p)\) in \( \mathbb{R}^k \), and by Theorem 5 of the appendix for each fixed \( t \) and \( g \), \( h_{n+1} \) is analytic in \((\bar{w}, p)\). Hence all the iterants possess the desired property. The next step is to show the uniform convergence of the series

\[
\sum_{n=1}^{\infty} [h_n - h_{n-1}] \text{ on } \mathbb{R}^k.
\]

From (2.1),

\[
|h_0(t, \tau, g, \bar{w}, p) - p(t, g_0, p_0)| = |\bar{w} - w_0| \quad (2.6)
\]

and

\[
h_1(t, \tau, g, \bar{w}, p) - h_0(t, \tau, g, \bar{w}, p) =
\]

\[
\int_{\tau}^{t} f(s + g, h_0(s, \tau, g, \bar{w}, p), p) \, ds - [p - w_0].
\]

But \( p - w_0 = \int_{\tau}^{t} p'(s) \, ds = \int_{\tau}^{t} f(s + g_0, p(s), \rho_0) \, ds \).

Thus,

\[
|h_1(t, \tau, g, \bar{w}, p) - h_0(t, \tau, g, \bar{w}, p)| = (2.7)
\]

\[
|\int_{\tau}^{t} f(s + g, h_0(s, p), p) - f(s + g_0, p(s), \rho_0) \, ds|.
\]
Since \( f \) is continuous on the compact set \( \mathbb{R}_p \), it is uniformly continuous so that given \( \varepsilon > 0 \), there exists a \( \delta_{\varepsilon} \) such that

\[
|f(s + g, h_0(s), p) - f(s + g_0, p(s), p)| < \varepsilon \quad (2.8)
\]
for \( a \leq s \leq b \) and

\[
|w - w_0| + |g - g_0| + |p - p_0| < \min(\delta_1, \delta_{\varepsilon}). \quad (2.9)
\]

Using (2.8) in (2.7), one obtains

\[
|h_1(t, \tau, g, \bar{w}, p) - h_0(t, \tau, g, \bar{w}, p)| < \varepsilon |t - \tau|
\]

provided (2.9) holds. Now since \( f \) is analytic in \( w \), \( f_w \) is continuous on the compact set \( \mathbb{R}_p \) and hence is bounded; \( |f_w| \leq K \). Also since \( \mathbb{R}_p \) is convex, the mean value theorem may be applied to show that \( f \) satisfies a Lipschitz condition in \( w \) on \( \mathbb{R}_p \) uniformly with respect to its other arguments. Note that since \( f \) and \( w \) are vectors, \( f_w \) is a matrix \( A = (a_{ij}) \), and the norm is defined by the relation

\[
|A| = \sum_{i,j=1}^{n} |a_{ij}|
\]

As an induction hypothesis assume that for some \( n \)

\[
|h_n - h_{n-1}| \leq \varepsilon K \frac{|t - \tau|^{n-1}}{n!}
\]

where \( K \) is the Lipschitz constant for \( f \).

Then

\[
|h_n - h_{n-1}| \leq \frac{t}{\tau} \int_{\tau}^{t} |f(s + g, h_n, p) - f(s + g, h_{n-1}, p)| \, ds
\]

\[
\leq \frac{t}{\tau} K |h_n - h_{n-1}| \, ds \leq \frac{t}{\tau} K^n \frac{|s - \tau|^{n-1}}{n!} \, ds
\]
Thus, for all $n$

$$|h_n - h_{n-1}| \leq eK^{n-1}\frac{|t - \tau|^{n}}{n!}.$$

Thus, for all $n$

$$|h_n - p| \leq |h_n - h_{n-1}| + |h_{n-1} - h_{n-2}| + \ldots + |h_1 - h_0| + |h_0 - p| \leq \frac{e}{K}\left(\frac{[K(b - a)]^n}{n!} + \frac{[K(b - a)]^{n-1}}{(n-1)!} + \ldots + K(b - a)\right) + 8\epsilon \leq \frac{e}{K}(e^{K(b - a)} - 1) + 8\epsilon.$$

Now let $\epsilon$ be given such that $\epsilon(e^{K(b - a)} - 1) < \frac{8\epsilon}{2}$. Then choose $\frac{8\epsilon}{2}$.

$8\epsilon < \frac{8\epsilon}{2}$. Finally let the $8$ in the statement of the theorem be equal to $\frac{8\epsilon}{2}$. Hence $|h_n - p| < \frac{8\epsilon}{2}$ for all $n$ and the point $(t, h_n(t, \tau, g, w, p))$ remains in the region $\left\{a \leq t \leq b, |w - p| < \frac{8\epsilon}{2}\right\}$ for all $(\tau, w, p) \in D$.

The series

$$h_0 + \sum_{n=1}^{\infty} [h_n - h_{n-1}] = \lim_{n \to \infty} h_n$$

is majorized by the series for $e^{K(b-a)}$ and hence converges.
uniformly to a limit function
\[ \lim_{n \to \infty} h_n = h(t, \tau, g, \overline{w}, \rho). \]

Since the convergence is uniform, \( h \) is continuous in \((t, g, \overline{w}, \rho)\) on \( R^* \) and by Theorem 6 of the Appendix, \( h \) is analytic in \((\overline{w}, \rho)\) for each fixed \( t \) and \( g \).

To see that \( h \) is a solution let \( f_n = f(s + g, h_n, \rho) \). Then for any \( r \) and for \( n \) sufficiently large,
\[
|f_n - f_{n} + r - |f(s + g, h_n, \rho) - f(s + g, h_n + r, \rho)|
\leq K |h_n - h_n + r| < \epsilon
\]
for all \((t, g, \overline{w}, \rho) \in R^* \) since \( h_n \to h \) uniformly in \( R^* \). Thus,
\[
\lim_{n \to \infty} \int_{\tau}^{t} f_n ds = \int_{\tau}^{t} \lim_{n \to \infty} f_n ds = \int_{\tau}^{t} f(s + g, \lim_{n \to \infty} h_n, \rho) ds = \int_{\tau}^{t} f(s + g, h, \rho) ds.
\]

Thus, in (2.5) the limit may be taken on both sides to obtain
\[
h(t, \tau, g, \overline{w}, \rho) = \overline{w} + \int_{\tau}^{t} f(s + g, h(s), \rho) ds
\]
which is equivalent to
\[
h' = f(t + g, h, \rho).
\]

Uniqueness of \( h \) is obtained as in Coddington and Levinson [7].

**Corollary.**— In the previous theorem if \( f \) is assumed to be analytic in all its arguments then the solution
\[
h = h(t, g, \overline{w}, \rho)
\]
will be analytic in all its arguments.

**Proof.**— This follows from the uniform convergence of the successive approximations and Theorem 6 of the Appendix.
Theorem 2. Consider the differential equation

\[ x + x = \rho f(t + g(\rho), x, x, \rho) \]  
\[ x(t = 0) = A + \lambda \]  
\[ \dot{x}(t = 0) = 0 \]

where \( x = x(t, A, g, \lambda, \rho) \) and \( f(s, x, x, \rho) \) is analytic in \((s, x, x, \rho)\).

Suppose that the solution \( x \) can be written as

\[ x(t, A, g, \lambda, \rho) = (A + \lambda) \cos t + \sum_{n=1}^{\infty} \left\{ C_n(t, A, g) + \frac{\delta C_n}{\delta \lambda} \lambda + \frac{1}{2} \frac{\delta^2 C_n}{\delta \lambda^2} \lambda^2 + \cdots \right\} \rho^n \]

with its first derivative given by

\[ \dot{x}(t, A, g, \lambda, \rho) = -(A + \lambda) \sin t + \sum_{n=1}^{\infty} \left\{ \dot{C}_n(t, A, g) + \frac{\delta \dot{C}_n}{\delta \lambda} \lambda + \frac{1}{2} \frac{\delta^2 \dot{C}_n}{\delta \lambda^2} \lambda^2 + \cdots \right\} \rho^n \]

Then in (2.11) and (2.12), differentiation with respect to \( \lambda \) can be replaced by differentiation with respect to \( A \) at \( \rho = \lambda = 0 \), i.e.,

\[ \frac{\delta^m + n}{\delta \lambda^m \delta \rho^n} x(t, A, g, \lambda, 0) = \frac{\delta^m + n}{\delta A^m \delta \rho^n} x(t, A, g, 0, 0) \]  
\[ m, n \geq 0 \] and similarly for \( \dot{x} \).
Proof. Let

\[ p\ell(t + g(p), x(t, A_o, g_o, \lambda, p), x(t, A_o, g_o, \lambda, p), P) = p\ell(t, A_o, g_o, \lambda, p). \]

Now expand \( p\ell \) about \( p = \lambda = 0 \)

\[ p\ell = \sum_{m, n = 0}^{\infty} \frac{1}{m! n!} \frac{\delta^{m + n} \ell(t, A_o, g_o, o)}{\partial \lambda^m \partial \rho^n} \chi^m \rho^n + 1 \quad (2.14) \]

Substitute (2.14) and (2.11) into (2.10) and equate coefficients of \( \lambda^{m \rho + 1} \) to get

\[ \frac{\delta^{m + n + 1}}{\partial \lambda \partial \rho} x(t, A_o, g, o, o) + \frac{\delta^{m + n + 1}}{\partial \lambda \partial \rho} \chi(t, A_o, g, o, o) = (n + 1) \frac{\delta^{m + n} \ell(t, A_o, g, o, o)}{\partial \lambda \partial \rho} \quad (2.15) \]

Using the fact that \( x(o, A_o, g, \lambda, p) = A_o + \lambda \) and \( x(o, A_o, g, \lambda, p) = 0 \)

the following initial conditions for (2.15) must be satisfied,

\[ \frac{\delta^{m + n + 1}}{\partial \lambda \partial \rho} x(o, A_o, g, o, o) = 0 \quad (2.16) \]

By variation of parameters, the solution of (2.15) and (2.16) is

\[ \frac{\delta^{m + n + 1}}{\partial \lambda \partial \rho} x(t, A_o, g, o, o) = 0 \quad (2.17) \]
\[ (n + 1) \int_0^t \frac{\partial^{m + n} f(s, A_0, g_s, o, o)}{\partial \lambda^m \partial \rho^n} \sin (t - s) \, ds. \]

Now to start an induction argument, notice from (2.11) and (2.12) that (2.13) is true for \( n = 0 \) and all \( m \). As an induction step assume that (2.13) is true for \( n = 1, 2, \ldots, k \) and all \( m \). Then

\[
\frac{\partial^{m + k} f(t, A_0, g_s, o, o)}{\partial \lambda^m \partial \rho^k} = \quad \text{(2.18)}
\]

Upon substituting (2.18) under the integral sign in (2.17) one gets

\[
\frac{\partial^{m + k + 1} x(t, A_0, g_s, o, o)}{\partial \lambda^m \partial \rho^{k + 1}} = \frac{\partial^{m + k + 1} x(t, A_0, g_s, o, o)}{\partial A_0^m \partial \rho^{k + 1}}
\]

Since the procedure was independent of \( m \), it has been shown that (2.13) now holds for \( n = k + 1 \) and all \( n \). This completes the induction.

The following criterion for periodic solutions of nonlinear systems is needed.
Theorem 3.-- Consider the following system of differential equations and initial conditions:

\[ x' = f(t, x, \rho) \]
\[ x(0) = x_0 \]  \hspace{1cm} (2.19)

where \( x \) and \( f \) are real vectors and \( \rho \) is a parameter. Suppose that \( f \) is periodic in \( t \) with period \( 2\pi \) and that \( f \) and \( f_x \) are continuous in \((t, x, \rho)\) for \(-\infty < t < \infty\), \(|x| \leq a\), \(|\rho| \leq \rho_0\) where \( a \) and \( \rho_0 \) are constants. Let \( x(t, x_0, \rho) \), or simply \( x(t) \), be a solution of (2.19) defined on \(-\infty < t < \infty\). Then a necessary and sufficient condition that \( x(t) \) be periodic with period \( 2\pi \) is that

\[ x(0) = x(2\pi). \hspace{1cm} (2.20) \]

Proof.-- First assume that \( x(t) \) is \( 2\pi \) periodic, i.e., \( x(t + 2\pi) = x(t) \) for all \( t \). Then in particular for \( t = 0 \), (2.20) holds.

Now assume that (2.20) holds. Define \( \bar{x}(t) = x(t + 2\pi) \). The solution of (2.19) satisfies

\[ t \]
\[ x(t) = x_0 + \int_{\tau}^{t} f(s, x(s), \rho) \, ds \]

and therefore

\[ t + 2\pi \]
\[ \bar{x}(t) = x_0 + \int_{\tau}^{t} f(s, x(s), \rho) \, ds. \]

The new function \( \bar{x}(t) \) satisfies the differential equation in (2.19) since

\[ \bar{x}'(t) = f(t + 2\pi, x(t + 2\pi), \rho) \]
\[ = f(t, \bar{x}(t), \rho) \]
and furthermore

\[ \overline{x}(0) = x_0 + \int_0^{2\pi} f(s, x(s), p) \, ds. \]

However, (2.20) implies that

\[ \int_0^{2\pi} f(s, x(s), p) \, ds = 0 \]

so that

\[ \overline{x}(0) = x_0. \]

Thus \( \overline{x}(t) \) and \( x(t) \) satisfy the same differential equation and initial conditions and by uniqueness \( \overline{x}(t) = x(t) \), or

\[ x(t + 2\pi) = x(t). \]

The last relation states that \( x(t) \) is a 2\( \pi \) periodic solution.
CHAPTER III
CONSTRUCTION OF PERIODIC SOLUTIONS

In this chapter a procedure for the construction of periodic solutions is presented and the results are summarized near the end of the chapter in the form of two theorems. An example is worked at the end to illustrate the construction procedure.

Consider the differential equation

$$\dddot{x} + x = pF(t + g(p), x, \dot{x}, \rho)$$

(3.1)

with initial conditions

$$x(0) = A_0 + \lambda(p)$$

(3.2)

$$\dot{x}(0) = 0$$

Suppose the following hypotheses are satisfied:

$$(H_1) \quad F \text{ is periodic in } t \text{ with period } 2\pi$$

$$(H_2) \quad F(s, x, \dot{x}, \rho) \text{ is analytic in } (s, x, \dot{x}, \rho)$$

$$(H_3) \quad g(\rho) \text{ and } \lambda(\rho) \text{ are analytic functions of } \rho$$

in some neighborhood of \(\rho = 0\), \(g(0) = g_0\),

and \(\lambda(0) = 0\).

The generating equation \((\rho = 0)\) is

$$\dddot{x} + x = 0$$

(3.3)

with initial conditions
\[ x(0) = A_0 \]  
\[ x(0) = 0 \]

which has the solution

\[ x(t,A_0) = A_0 \cos t. \]  

(3.5)

It is required to find a periodic solution of (3.1) which for \( p = 0 \) reduces to the solution (3.5) of (3.3). It is known by Theorem 1 of Chapter I that the solution \( x(t,A_0,\lambda,\rho) \) of (3.1) exists for \( 0 \leq t \leq 2\pi \) and is analytic in \( \lambda \) and \( \rho \) in some neighborhood of \( \lambda = \rho = 0 \) and thus may be expanded in powers of \( \lambda \) and \( \rho \).

\[ x(t,A_0,\rho,\lambda) = x_0 + B_1 \lambda + C_1 \rho + D_1 \lambda \rho + B_2 \lambda^2 + \ldots, \]  

(3.6)

where the coefficients are appropriate derivatives of \( x \) evaluated at \( \lambda = \rho = 0 \) and are functions of \( t, A_0, \rho, \lambda \) and various other derivatives of \( \rho \). Since at \( \lambda = \rho = 0 \), \( x = A_0 \cos t \), it must be the case that \( x_0 = A_0 \cos t \) in expression (3.6).

Expand \( F \) in terms of \( \rho \) and \( \lambda \) and substitute (3.6) into (3.1). Notice that since every term on the right of (3.1) has at least one factor of \( \rho \), the following set of relations must hold:

\[ B_n(t) + B_n(t) = 0, \quad n = 1, 2, \ldots \]

Since at \( t = 0 \), (3.6) must reduce to \( A_0 + \lambda \), the following set of initial conditions must hold:

\[ B_1(o) = 1 \quad B_1(o) = 0 \]
\[ B_n(o) = 0 \quad B_n(o) = 0, \quad n = 2, 3, \ldots \]
Hence $B_n(t) = \cos t$ and $B_n(t) = 0$ for $n = 2, 3, \ldots$. Now apply Theorem 7 of the Appendix to rewrite the solution (3.6) as

$$x(t, A_o, g, \lambda, \rho) = (A_o + \lambda) \cos t +$$

$$\sum_{n=1}^{\infty} \left\{ c_n(t, A_o, g) + \frac{\partial c_n}{\partial \lambda} \lambda + \frac{1}{2} \frac{\partial^2 c_n}{\partial \lambda^2} \lambda^2 + \ldots \right\} \rho^n,$$

where all the coefficients are evaluated at $\lambda = \rho = 0$ and

$$c_n = \frac{1}{n!}, \quad \frac{\partial x}{\partial \rho^n}.$$

Applying Theorem 2 of Chapter II, one readily sees that the solution may finally be written as

$$x(t, A_o, g, \lambda, \rho) = (A_o + \lambda) \cos t +$$

$$\sum_{n=1}^{\infty} \left\{ c_n(t, A_o, g) + \frac{\partial c_n}{\partial A_o} \lambda + \frac{1}{2} \frac{\partial^2 c_n}{\partial A_o^2} \lambda^2 + \ldots \right\} \rho^n.$$

Note that

$$x(t, A_o, g, \lambda, \rho) = -(A_o + \lambda) \sin t +$$

$$\sum_{n=1}^{\infty} \left\{ c_n^* + \frac{\partial c_n^*}{\partial A_o} \lambda + \frac{1}{2} \frac{\partial^2 c_n^*}{\partial A_o^2} \lambda^2 + \ldots \right\} \rho^n.$$

From (3.7) one sees that it is necessary only to calculate the $c_n$, $n = 1, 2, \ldots$; the remaining terms may be obtained by differentiation.
with respect to $A_0$.

Now expand $F$ in terms of $p$.

$$p^n = \sum_{n=0}^{\infty} \left( \frac{d^n F}{dp^n} \right)_o \frac{p^{n+1}}{n!}$$  \hspace{1cm} (3.8)

where the subscript $(o)$ indicates that the term in parentheses is to be evaluated at $p = \lambda = o$, i.e.,

$$(F)_o = F(t + g_o, A_o \cos t_o, -A_o \sin t_o).$$

Note that

$$\left( \frac{dF}{dp} \right)_o = \left( \frac{\partial F}{\partial t} \right)_o \cdot (o) + \left( \frac{\partial F}{\partial x} \right)_o \cdot c_1 + \left( \frac{\partial F}{\partial x} \right)_o \cdot c_2 + \left( \frac{\partial F}{\partial p} \right)_o \cdot.$$

The coefficient of $p^n$ in (3.8) is

$$H_n(t) = \frac{1}{(n-1)!} \left( \frac{d^n - 1}{dp^n - 1} \right)_o .$$  \hspace{1cm} (3.9)

Substitute (3.8) and (3.7) into (3.1) and equate to zero the coefficients of $p^n$. The result is

$$\left[ c_{\lambda} + \frac{\partial c_{\lambda}}{\partial A_0} \lambda + \ldots \right] + \left[ c_{\lambda} + \frac{\partial c_{\lambda}}{\partial A_0} \lambda + \ldots \right] = H_n(t),$$

but since this must hold for all $\lambda$ in a neighborhood of $\lambda = o$, it holds also for $\lambda = o$ with the result,

$$c_{\lambda}(t) + c_{\lambda}(t) = H_n(t).$$  \hspace{1cm} (3.10)
Since at \( t = 0 \), \( x(0, A_0, \varepsilon, \lambda, p) = A_0 + \lambda \) one sees from (3.7) that

\[
    c_n(0) = c'_n(0) = 0, \quad n = 1, 2, \ldots
\]

(3.11)

It should be realized that \( c_n \) is actually a function of \( t, A_0 \) and \( g(0) \), \( i = 1, \ldots, n-1 \). However for the present it is not necessary to show this dependence explicitly. It will be shown later just how the parameters \( \frac{d^i g(o)}{dp^i} \) can be handled.

The solution of (3.10) and (3.11) is, for \( n = 1, 2, \ldots \),

\[
    c_n(t) = \int_0^t H_n(s) \sin (t - s) \, ds
\]

(3.12)

\[
    c'_n(t) = \int_0^t H_n(s) \cos (t - s) \, ds.
\]

A few of the \( H_n \) will now be calculated.

\[
    H_1(t) = F(t + g_0, A_0 \cos t, -A_0 \sin t, o)
\]

(3.13)

\[
    H_2(t) = (F_t \cdot g_0) + (F_x \cdot c_1(t)) + (F_{\varepsilon} \cdot c_1(t)) + (F_{\lambda} \cdot c_1(t))
\]

(3.14)

\[
    H_3(t) = \frac{1}{2} (F_{tt} \cdot g_0^2) + (F_{tx} \cdot c_1 g'(0)) + (F_{t\varepsilon} \cdot c_1 g'(0)) + (F_{t\lambda} \cdot c_1 g'(0))
\]

\[
    + \frac{1}{2} (F_{xx} \cdot c_1^2) + (F_{x\varepsilon} \cdot c_1 c_1) + (F_{x\lambda} \cdot c_1) \cdot c_1 + (F_{p\varepsilon} \cdot c_1) \cdot c_1 + (F_{p\lambda} \cdot c_1) \cdot c_1.
\]
The next step is to expand \( \lambda \) and \( g \) in powers of \( \rho \) and to determine the coefficients in their expansions.

\[
\lambda(\rho) = \sum_{n=1}^{\infty} A_n \rho^n \quad (3.16)
\]

\[
g(\rho) = g_0 + \sum_{n=1}^{\infty} G_n \rho^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{d^n g}{d\rho^n} \right)_0 \rho^n \quad (3.17)
\]

Since periodic solutions of (3.1) are desired one may now impose some periodicity conditions on \( x \) and \( \dot{x} \). By Theorem 3 of Chapter II, it is enough to require that \( x(2\pi) = x(0) \) and \( \dot{x}(2\pi) = \dot{x}(0) \). Expressing the fact that \( x(2\pi) = x(0) \), one obtains

\[
\sum_{n=1}^{\infty} \left\{ c_n (2\pi) + \frac{d^n}{d\lambda_0} \lambda + \frac{1}{2} \frac{d^2 c_n}{d\lambda_0^2} \lambda^2 + \ldots \right\} \rho^n = 0 \quad (3.18)
\]

Substitute (3.16) into (3.18) and equate to zero the coefficients of like powers of \( \rho \) to obtain the following:

\[
c_1 (2\pi) = 0
\]
\[ c_2(2\pi) + A_1 c_{1A_o} = 0 \]

\[ c_3(2\pi) + A_2 c_{1A_o} + \frac{1}{2} A_1^2 c_{1A_o A_o} + A_1 c_{2A_o} = 0 \] (3.19)

\[ c_4(2\pi) + A_3 c_{1A_o} + A_1 A_2 c_{1A_0 A_0} + \frac{1}{6} A_1^3 c_{1A_o A_o A_o} + A_1 c_{3A_o} + A_2 c_{2A_o} + \]

\[ + \frac{1}{2} A_1^2 c_{2A_o A_o} = 0 \]

\[ \vdots \]

where \( c_{1A_o A_o} = \frac{\delta^2 c_{1}}{\delta A_o^2} \) etc.

Now expressing the fact that \( x(2\pi) = x(0) \), one obtains in the same way

\[ \sum_{n=1}^{\infty} \left\{ c_n(2\pi) + \frac{\delta c_n}{\delta A_o} \lambda + \frac{1}{2} \frac{\delta^2 c_n}{\delta A_o^2} \lambda^2 + \ldots \right\} \rho^n = 0 \] (3.20)

and

\[ c_1(2\pi) = 0 \]

\[ c_2(2\pi) + A_1 c_{1A_o} = 0 \]

\[ c_3(2\pi) + A_0 c_{1A_o} + \frac{1}{2} A_1^2 c_{1A_o A_o} + A_1 c_{2A_o} = 0 \] (3.21)

\[ c_4(2\pi) + A_3 c_{1A_o} + A_1 A_2 c_{1A_0 A_0} + \frac{1}{6} A_1^3 c_{1A_o A_o A_o} + A_1 c_{3A_o} + A_2 c_{2A_o} + \]

\[ + \frac{1}{2} A_1^2 c_{2A_o A_o} = 0 \]

\[ \vdots \]
\[ + A_1 c_{2A_0} + A_2 c_{2A_0} + \frac{1}{2} A_1^2 c_{2A_0} A_0 = 0 \]

\[ \vdots \]

where \( c_{1A_0 A_0} = \frac{\partial^2 c}{\partial A_0^2} \), etc.

Equations (3.19) and (3.21) thus form a set of necessary and sufficient conditions for the existence of a periodic solution of (3.1).

Some relations will now be derived showing explicitly the dependence of \( c_n \) on \( \frac{d^i g(o)}{d\rho^i} \), \( i = 1, 2, \ldots, n-1 \). Let \( D \) be an operator which takes the total derivative with respect to \( \rho \) of a function holding \( g \) fixed, i.e.,

\[ D F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \rho} \frac{\partial x}{\partial \rho} + \frac{\partial F}{\partial \rho} \frac{\partial x}{\partial \rho} \frac{\partial \rho}{\partial \rho} \]

where

\[ \left( \frac{\partial^n x}{\partial \rho^n} \right)_0 = n! E_n, \quad \left( \frac{\partial^n x}{\partial \rho^n} \right)_0 = n! E_n, \]

\[ E_n = \int_0^t H^*_n(s) \sin(t-s) \, ds, \]

\[ E_n = \int_0^t H^*_n(s) \cos(t-s) \, ds, \]

and

\[ H^*_n = \frac{1}{(n-1)!} \left( D^F g \right)_0 \]

\[ + A_1 c_{2A_0} + A_2 c_{2A_0} + \frac{1}{2} A_1^2 c_{2A_0} A_0 = 0 \]
Note that the $E_\text{n}$ are functions of $t$, $A_0$, and $g_0$. The $c_\text{n}$ and $E_\text{n}$ now satisfy the following relationships:

\[ c_1(t) = E_\text{l}(t) \]
\[ c_2(t) = g'(o) E_{1g_0} + E_\text{2}(t) \]
\[ c_3 = \frac{1}{2} [g'(o)]^2 E_{1g_0} + g'(o) E_{2g_0} + \]
\[ + \frac{1}{2} g''(o) E_{1g_0} + E_3 \]
\[ c_4 = \frac{1}{6} [g'(o)]^3 E_{1g_0} + \frac{1}{2} [g'(o)]^2 E_{2g_0} + \]
\[ + \frac{1}{2} g''(o) E_{2g_0} + g'(o) E_{3g_0} + \]
\[ + \frac{1}{2} g'(o) g''(o) E_{1g_0} + \frac{1}{2} g''(o) E_{2g_0} + E_4 \]
\[ \vdots \]

and similarly for the $c_\text{n}$. Use of the relations (3.22) to rewrite conditions (3.19) and (3.21) yields the following:

\[ E_\text{l}(2\pi, g_0, A_0) = 0 \]  
(3.23)

\[ E_\text{l}(2\pi, g_0, A_0) = 0 \]

\[ E_2(2\pi, g_0, A_0) + G_1 E_{1g_0} + A_1 E_{1A_0} = 0 \]  
(3.24)
\[ E_2(2\pi, g_o, A_o) + \frac{\dot{G}_E}{1\, lg_o} + \frac{A_1\dot{E}}{1\, lg_o} = 0, \]

\[ E_3 + \frac{G_2\, E_2}{1\, lg_o} + \frac{1}{2} \frac{G^2}{1\, lg_o g_o} + \frac{G\, E_2}{1\, lg_o} = 0, \]

(3.25)

\[ + A_2\, E_2 A_o + \frac{1}{2} A_1^2 \frac{E_2}{1\, lg_o A_o} + A_1\, E_2 A_o + A_2\, G_1 E_2 A_o g_o = 0, \]

\[ E_3 + \frac{G_2\, E_3}{1\, lg_o} + \frac{1}{2} \frac{G^2}{1\, lg_o g_o} + \frac{G\, E_3}{1\, lg_o} = 0, \]

(3.26)

\[ + A_2\, E_3 A_o + \frac{1}{2} A_1^2 \frac{E_3}{1\, lg_o A_o} + A_1\, E_3 A_o + A_2\, G_1 E_3 A_o g_o = 0, \]

\[ E_4 + \frac{A_2\, E_4}{1\, lg_o} + \frac{G_3\, E_4}{1\, lg_o} + \frac{1}{6} \frac{G^3}{1\, lg_o g_o} = 0, \]

\[ + \frac{1}{2} \frac{G^2}{1\, lg_o g_o} + \frac{G\, E_4}{1\, lg_o} + \frac{G\, E_4}{2\, lg_o} + \frac{G\, E_4}{1\, lg_o} + \frac{G\, E_4}{1\, lg_o} = 0, \]

\[ + A_2\, E_2 A_o + \frac{1}{2} A_1^2 \frac{E_2}{1\, lg_o A_o} + \frac{1}{2} A^2 \frac{E_2}{1\, lg_o A_o} + \]

\[ + \frac{1}{2} \frac{A_1 G_2}{1\, lg_o A_o} g_o + \frac{A_1 G_2}{1\, lg_o A_o} + \frac{A_1\, E_2 A_o}{1\, lg_o} = 0, \]

\[ E_4 + \frac{A_2\, E_4}{1\, lg_o} + \frac{G_3\, E_4}{1\, lg_o} + \frac{1}{6} \frac{G^3}{1\, lg_o g_o} = 0, \]
\[ + \frac{1}{2} A_1^4 E_2 A_0^2 g_o + G_1 E_2 g_o + G_1 E_3 g_o + G_1 G_2 E_1 g_o g_o + \]
\[ + A_1 A_2 E_1 A_o A_o + \frac{1}{6} A_1^3 E_1 A_o A_o + (A_2 G_1 + A_1 G_2) E_1 g_o A_o + \]
\[ + A_2 E_1 A_o + \frac{1}{2} A_1^2 G_1 E_1 A_o A_o + \frac{1}{2} A_1^2 E_2 A_0 A_o + \]
\[ + \frac{1}{2} A_1 G_1^2 E_1 A_o g_o g_o + A_1 G_1 E_2 A_0 g_o + A_1 E_3 A_o = 0 \]

Equations (3.23) form a system for the determination of \( g_o \) and \( A_o \).

If (3.23) has simple roots (see definition in Chapter II) then the Jacobian

\[ J = \frac{a(E_1, E_1)}{a(A_o, g_o)} \neq 0 \]

and the system (3.24) may be solved for a unique pair \( A_1, G_1 \).

Substitute these values of \( A_1 \) and \( G_1 \) into (3.25) and note that there results a linear system for the determination of \( A_2 \) and \( G_2 \).

The determinant of coefficients is again \( J \) and (3.25) may be solved for a unique pair \( A_3 \) and \( G_3 \). Now looking at formulas (3.9) and (3.15) and also noting that

\[ \frac{a c_1(t)}{ag_o} = \int_0^t D_1 F \sin (t - s) \, ds \]
where \( D_F(t,x,x',p) = \frac{\partial F}{\partial x} \), one sees that \( c^{(2n)} \) will contain terms at 

\[ \text{in } G_{n-2} G_{n-3} \ldots, G_2, G_1 \text{ and a term } \frac{ac}{\partial g_0} G_{n-1} \] 

which, due to \((3.22)\) is equal to \( \frac{\partial E}{\partial g_0} G_{n-1} \). Moreover, it is seen that \( c^{(2n)} \) will contain no other terms in \( G_{n-1} \). Now consider \((3.16)\) and \((3.18)\) and notice that upon collecting the coefficients of \( p^n \) one obtains

\[
c^{(2n)} + \frac{ac}{\partial A_o} A_{n-1} + \text{other terms in } A_{n-2}, \ldots, A_2, A_1
\]

and furthermore \( \frac{ac}{\partial A_o} A_{n-1} \) is the only term in \( A_{n-1} \). Again due to \((3.22)\) \( \frac{ac}{\partial A_o} = \frac{\partial E}{\partial A_o} \). The same comments apply to \( c^{(2n)} \) and the series \((3.20)\).

Hence, just as \((3.24)\) and \((3.25)\) give recursive linear systems (with determinant \( J \)) for the determination of \( A_1, G_1 \) and \( A_2, G_2 \), the system obtained by equating to zero the coefficients of \( p^n \) in \((3.18)\) and \((3.20)\) gives a linear system (with determinant \( J \)) for the determination of \( A_{n-1} \) and \( G_{n-1} \). Thus the coefficients \( A_n \) and \( G_n \) may be uniquely determined in a recursive manner.

It should be noted that the Jacobian \( J \) above is the same as the crucial Jacobian in the work of Coddington and Levinson [8] for
the case of a second order equation, and that throughout their work this Jacobian does not vanish.

If the system (3.23) has double roots then the Jacobian $J$ vanishes for these roots and the procedure just described fails. The vanishing of the Jacobian $J$ gives rise to some supplementary conditions which must be satisfied.

It is well known that the equation

$$
\begin{bmatrix}
E_{1A_0} & E_{1g_0} \\
\cdot & \cdot \\
E_{1A_0} & E_{1g_0}
\end{bmatrix}
\begin{bmatrix}
A_1 \\
G_1
\end{bmatrix}
= 
\begin{bmatrix}
E_2 \\
E_2
\end{bmatrix},
$$

which is just equation (3.24) in matrix form, will have a solution if and only if the rank of

$$
\begin{bmatrix}
E_{1A_0} & E_{1g_0} \\
\cdot & \cdot \\
E_{1A_0} & E_{1g_0}
\end{bmatrix}
(3.27)
$$

is equal to the rank of

$$
\begin{bmatrix}
E_{1A_0} & E_{1g_0} & E_2 \\
\cdot & \cdot & \cdot \\
E_{1A_0} & E_{1g_0} & E_2
\end{bmatrix}
(3.28)
$$

But, since the determinant of (3.27) is just the zero Jacobian $J$, the matrix (3.28) must have rank < 2. Thus, every 2 X 2 sub-matrix of (3.28) must have determinant zero.
Hence,

$$\begin{align*}
\frac{E_{1A_0}}{E_{1A_0}} &= \frac{E_{1g_0}}{E_{1g_0}} = \frac{E_2}{E_2} \\
\end{align*}$$

(3.29)

must be satisfied.

A procedure will now be described for solving (3.25) for $A_1$. Since $J = 0$, we may eliminate $A_2$, $G_2$ and $G_1$ from the relations (3.25) in the following manner. Multiply the first equation in (3.25) by $(E_{1g_0})^2$ and the second by $(E_{1g_0})^2$. Then from (3.24) substitute for $G_1$ and finally multiply the first equation by $E_{1g_0}$, the second by $E_{1g_0}$, and subtract the second from the first. The result is the following quadratic expression for the determination of $A_1$:

$$P_0 A_1^2 + P_1 A_1 + P_2 = 0$$

(3.30)

where

$$P_0 = E_{1g_0} \left[ E_{1g_0} E_{1A_0} A_{1}^2 - 2E_{1g_0} E_{1A_0} E_{1g_0} A_{0} + E_{1A_0} E_{1g_0} g_{0} \right]$$

$$P_1 = E_{1g_0} \left[ E_{2A_0} E_{1g_0} - E_{2g_0} E_{1g_0} E_{1A_0} \right]$$

$$- E_2 E_{1A_0} E_{1g_0} E_{1g_0} E_{1A_0}$$
If it happens that both $E_1^{\dot{g}_0}$ and $E_2^{\dot{g}_0}$ are zero, then an analogous procedure using $E_1^{\dot{A}_0}$ and $E_2^{\dot{A}_0}$ yields a quadratic in $B_1$.

Having determined $A_1$ from (3.30), one may obtain $G_1$ from one of the relations (3.24).

Since it was assumed that $(g_0, A_0)$ is a double root of (3.23), equation (2.3) holds and thus $P_0 \neq o$. Hence, (3.30) determines two values for $A_1$. In the case that the roots of (3.30) are complex conjugate, no real solution exists. Even in the case of real roots they might be distinct giving rise to two separate determinations of $\lambda$ and $g$ as functions of $p$ and hence two separate solutions. Note that this does not contradict the uniqueness theorem since there is one and only one solution for each set of initial conditions,

$$x(0) = A_0 + \lambda, \quad x(0) = 0.$$ 

There is however no guarantee of finding a unique solution to the
basic problem as stated in the introduction, i.e., there may be several periodic solutions of (3.1) which for \( p = 0 \) reduce to the solution (3.5) of the generating equation (3.3). Moreover, since (3.23) may have several distinct real roots, there may be several available solutions of the generating system.

Having obtained \( A_1 \) and \( G_1 \), one may calculate the quantities \( A_2 \) and \( G_2 \) in the following manner. Multiply the first equation of (3.26) by \( E_2 \) and the second by \( E_2 \) and subtract the second from the first. This eliminates \( A_3 \) and \( G_3 \) and leaves one equation in the two unknowns, \( A_2 \) and \( G_2 \). Another equation in \( A_2 \) and \( G_2 \) is obtained by adding the two equations in (3.25). The system is now linear in \( A_2 \), \( G_2 \) with determinant

\[
\Delta = (E_2 + E_2) \Delta_1
\]

where

\[
\Delta_1 =
\begin{align*}
&\begin{bmatrix}
E_{2A_0} & E_{1g_0} & -E_{1g_0} E_{2A_0} - E_{1A_0} E_{2g_0} + E_{1A_0} E_{2g_0} + \\
E_{1A_0} G_1 & E_{1A_0} & -E_{1A_0} E_{1A_0} G_1 - E_{1A_0} E_{1A_0} G_1 + E_{1A_0} E_{1A_0} G_1 \\
E_{1A_0} G_1 & E_{1A_0} & -E_{1A_0} E_{1A_0} G_1 - E_{1A_0} E_{1A_0} G_1 + E_{1A_0} E_{1A_0} G_1 \\
\end{bmatrix}
\end{align*}
\]
Jacobian with three zero elements. — Suppose here, without loss of
generality, that the Jacobian

\[ J = \frac{\partial (E_1, E_2)}{\partial (A_0, g_0)} \]

vanishes at \((A_0, g_0)\) with \(E_{1A_0} = 0\) and \(E_{1g_0} = E_{2g_0} = 0\). The
first equation of (3.24) may be solved for \(A_1\) with the supplementary
condition that \(E_2 = 0\). With this value of \(A_1\), the second equation
of (3.25) may be solved for \(G_1\). The first equation of (3.25) may
then be solved for \(A_2\) and the second equation of (3.26) for \(G_2\) and
so on. If, however, all the coefficients of \(G_1\) in the second
equation of (3.25) are zero another supplementary condition,
\(E_2(2\pi, g_0, A_0) = 0\), arises and the second equation of (3.26) must be used
to determine \(A_1\).

Jacobian with four zero elements. — Suppose that the Jacobian \(J\)
vanes at \((A_0, g_0)\) with four zero elements. Then from (3.24)
it is seen that the two supplementary conditions

\[ E_2(2\pi, g_0, A_0) = 0 \]

\[ E_2(2\pi, g_0, A_0) = 0 \]

must be satisfied. Equations (3.25) now give a system of two
non-linear equations in the two unknowns \(A_1\) and \(G_1\). As another
supplementary condition, assume that these nonlinear equations can be solved for a pair \((A^*, G^*)\). Substituting these values of \(A^*\) and \(G^*\) into equations (3.26) one obtains a linear system for the determination of \(A_2\) and \(G_2\) and so on. If, however, all the coefficients of \(A_1\) and \(G_1\) in equations (3.25) are zero then the supplementary conditions

\[
E_3(2\pi, g_0, A_0) = 0 \quad E_3(2\pi, g_0, A_0) = 0
\]

must be satisfied and \(A_1, G_1\) may be determined from equations (3.26).

Jacobian with a row of zeros.-- Suppose the Jacobian \(J\) vanishes at \((A_0, g_0)\) with \(E_{1A_0} = E_{1g_0} = 0\) but \(E_{2A_0} \neq 0\) and \(E_{2g_0} \neq 0\). This gives rise to the supplementary condition

\[
E_2(2\pi, g_0, A_0) = 0
\]

which must be satisfied. The second equation in (3.24) and the first in (3.25) now provide a non-linear system for the determination of \(A_1\) and \(G_1\). Substituting these values of \(A_1\) and \(G_1\) into the second equation of (3.25) and the first equation of (3.26), one obtains a linear system for \(A_2\) and \(G_2\) and so on. If, however, in the first equation of (3.25) the coefficients of \(A_1\) and \(G_1\) are all zero then the supplementary condition

\[
E_3(2\pi, g_0, A_0) = 0
\]

must be satisfied and \(A_1\) and \(G_1\) may be determined from the first equation of (3.26) and the second equation of (3.24).
Construction of solution.-- Having determined the coefficients $A_n$ and $G_n$ one may now construct the solution of (3.1). Collecting coefficients of like powers of $p$ in (3.7) one obtains

$$x(t,p) = x_0(t) + x_1(t) p + x_2(t) p^2 + \ldots$$

where

$$x_0(t) = A_0 \cos t$$
$$x_1(t) = A_1 \cos t + E_1(t)$$
$$x_2(t) = A_2 \cos t + E_2(t) + G_1 E_0(t) + A_1 E_1 A_0(t)$$
$$x_3(t) = A_3 \cos t + E_3(t) + G_2 E_0(t) + \frac{1}{2} G_1 E_1 E_0(t) + A_2 E_2 A_0(t) + A_1 E_1 A_2 A_0(t) + \frac{1}{2} G_1 E_1 E_1 A_0(t) + \ldots$$

The principal results of this investigation may be summarized in the following two theorems.

**Theorem 1.**-- Consider the differential equation (3.1) with initial conditions (3.2) and suppose that $H_1$, $H_2$ and $H_3$ stated after formula (3.2) are satisfied. Suppose that the Jacobian $J$ evaluated at $(A_0, G_0)$ is different from zero. Then it is possible to determine the coefficients $A_n$ and $G_n$ recursively and to construct a unique solution of (3.1) satisfying (3.2) which for $p = 0$ reduces to the solution of
(3.3) which satisfies (3.4).

Remark.— The solution is unique in the sense that given a root \( (A_o, G_o) \) of (3.23) there is one and only one solution of (3.1) which for \( \rho = 0 \) reduces to (3.5). This corresponds exactly to the case discussed in Coddington and Levinson [9] where it is assumed that the Jacobian does not vanish.

Theorem 2.— Consider the differential equation (3.1) with the initial conditions (3.2) and suppose that \( H_1, H_2, \) and \( H_3 \) stated after formula (3.2) are satisfied. Suppose that the Jacobian \( J \) evaluated at \( (A_o, G_o) \) vanishes in one of the following ways:

1. \( (A_o, G_o) \) is a double root of (3.23);

2. \( J \) has a row of zeros, the other row consists of non-zero elements;

3. \( J \) has three zero elements, the other element is non-zero;

4. \( J \) has four zero elements.

Also assume that all of the supplementary conditions imposed by the vanishing of \( J \) are satisfied. Then it is possible to determine recursively the coefficients \( A_n \) and \( G_n \) and to construct a solution (not necessarily unique) of (3.1) which satisfies (3.2) and which for \( \rho = 0 \) reduces to the solution of (3.3) which satisfies (3.4).

Remark.— The vanishing of \( J \) with two zeros in one column or by simple cancellation of its terms is treated in Case 1.
Example.

Consider the following equation:

\[ \ddot{x} + x = \rho(ax + dx^2 + bx^3 + c \cos t) \]

\[ x(0) = A_0 + p(\rho) \]

\[ \dot{x}(0) = \bar{B}_0 + q(\rho) \]

where \( \rho \) is a small parameter. By introducing the phase shift \( g \), one may assume that \( x(0) = 0 \). Thus the equation under consideration is

\[ \ddot{x} + x = \rho(ax + dx^2 + bx^3 + c \cos [t + g(\rho)]) \]

\[ x(0) = A_0 + \lambda(\rho) \quad (3.31) \]

\[ x(0) = 0 \ . \]

Suppose

\[ a = 4 \quad b = -4 \quad (3.32) \]

\[ c = -16 \quad d = \frac{\sqrt{3}}{10} \ . \]

After carrying out the necessary integrations, one sees that

\[ E_1(2\pi) = n\epsilon \sin g_0 \quad (3.33) \]

\[ \dot{E}_1(2\pi) = n(aA_0 + \frac{3bA_0^3}{4} + c \cos g_0) . \]

Now substitute (3.32) into (3.33) and equate both expressions of (3.33) to zero so that relations (3.23) may be satisfied. The result is
\[-\frac{16}{9} \pi \sin g_0 = 0\]
and
\[\pi (\mu A_0 - 3A^3 - \frac{16}{9} \cos g_0) = 0\]
with roots
\[g_0 = 0 \quad (3.34)\]
\[A_0 = 2/3, 2/3, -4/3.\]

Now suppose that it is desired to construct a solution of (3.31) which for \(\rho = 0\) reduces to
\[x(t, 0) = 2/3 \cos t,\]
corresponding to the double root 2/3.

The four elements in the Jacobian \(J\) are
\[E_{1g_0} = \pi c \cos g_0 \quad (3.35)\]
\[E_{1A_0} = 0\]
\[E_{2g_0} = -\pi c \sin g_0\]
\[E_{2A_0} = \pi (a + \frac{2}{3} b A^2_0) \cdot\]

For the values (3.32), \(g_0 = 0\), and \(A_0 = 2/3\) the first expression in (3.35) is different from zero and the last two expressions vanish. Hence this is the case discussed in chapter two in which the Jacobian \(J\) vanishes with three zero elements and the new condition \(E_2(2\pi) = 0\) must be satisfied. After a little calculation, one finds
\[ E_2(2\pi) = \pi a^3 \left\{ \frac{5}{6} a d^2 - \left( \frac{a + 3 b A^2}{4} \right) d + \left( \frac{ab}{32} + \frac{3 b^2 A^2}{64} \right) \right\} \]  

(3.36)

which indeed vanishes for the values (3.32), \( g = 0 \), and \( A = 2/3 \).

The first equation of (3.24) reduces to

\[ E_2(2\pi) + g E_1 g_0 = 0 , \]

and upon calculating \( E_2 \) it is seen that \( E_2(2\pi) \neq 0 \). Hence \( g_1 = 0 \) and the second equation of (3.25) reduces to

\[ E_3(2\pi) + \frac{1}{2} A_1^2 E_{1A} A_0 \dot{A}_0 + A_1 E_{2A} = 0 \]  

(3.37)

from which \( A_1 \) may be determined. For the example under consideration,

\[ E_3(2\pi) = \pi a^3 (K_1 A_0^4 + K_2 A_0^3 + K_3 A_0^2 + K_4 A_0 + K_5) \]

where

\[ K_1 = \frac{219}{(64)^2} b^3 + \frac{19}{24} b d^2 - \frac{113}{256} d b^2 \]

\[ K_2 = \frac{763}{384} b d^2 - \frac{5}{18} d^3 \]

\[ K_3 = \frac{1}{16} b d^2 - \frac{5}{9} d^3 + \frac{165}{6164} a b^2 - \frac{255}{576} a b d \]

\[ K_4 = \frac{201}{1144} a d^2 \]
\[ K_5 = \frac{3}{142} a^2 b - \frac{1}{9} a^2 d, \]

\[ E_{1 \omega} \omega = \frac{9}{2} \pi b \omega, \]

and

\[ E_{2 \omega} = n \omega^3 \left( \frac{5}{6} d^2 - \frac{3}{2} b \omega A + \frac{3}{32} b^2 A \right). \]

It is easy to show that (3.37) will have real roots if the values (3.32), \( \omega_0 = 0 \), and \( A_0 = 2/3 \) are used.

The calculated value for \( E_1 \) is:

\[ E_1(t) = \frac{1}{8} b \omega^3 \cos t \sin^2 t + \]

\[ \frac{dA^3}{\omega} \left( 2 - \cos t - \cos^2 t \right). \]

It is now possible to write down the first two terms in the expanded solution.

\[ x(t, \rho) = \omega_0 \cos t + \]

\[ + \rho [ A_1 \cos t + \frac{1}{8} b \omega^3 \cos t \sin^2 t + \]

\[ + \frac{dA^3}{\omega} \left( 2 - \cos t - \cos^2 t \right)] + \ldots \]

Higher order terms may be found recursively.

The first equation in (3.25) reduces to

\[ E_3(2n) + C_2 E_{1 \omega} = 0 \]
and upon carrying out the necessary calculations one sees that

\[ E_3(2\pi) = 0. \]  

Hence \( G_2 = 0 \) and the expansion of \( g \) in powers of \( \rho \)
contains no powers of \( \rho \) less than three.
APPENDIX

In this appendix some theorems from the theory of functions of several complex variables are presented. Most of the theorems are well known for the case of functions of one complex variable and are proved in standard textbooks such as Ahlfors[10] and Titchmarsh [11]. For the case of functions of several complex variables those proofs which could not be found in the literature are presented here for the sake of completeness. The principle reference is Bochner and Martin [12].

Definition 1.-- By the norm \(|w|\) of a complex n component vector \(w\) is meant the following:

\[
|w| = \sum_{i=1}^{n} |w_i|, \quad |w_i| = [(Re w_i)^2 + (Im w_i)^2]^{1/2}
\]

where \(Re w_i\) and \(Im w_i\) denote respectively the real and imaginary parts of \(w_i\).

Definition 2.-- Let \(F = (F_1, \ldots, F_n)\) be a vector function defined on a region \(D\) of the n complex dimensional \(w\) space where \(w = (w_1, \ldots, w_n)\) with \(w_i\) complex. The function \(F\) is said to be analytic at a point \(w_0 = (w_{10}, w_{20}, \ldots, w_{no})\) if each \(F_j\) can be represented by an absolutely convergent power series
\[ F_j(w_1, \ldots, w_n) = \sum_{m_1=0}^{\infty} A_{m_1 m_2 \cdots m_n} (w_1 - w_1^0)^{m_1} (w_2 - w_2^0)^{m_2} \cdots (w_n - w_n^0)^{m_n} \]

in some neighborhood \(|w - w_0| < \rho, \rho > 0\). A function is said to be analytic in a domain \(D\) if it is analytic at each point of \(D\). By a theorem in Bochner and Martin [13],

\[ A_{m_1 m_2 \cdots m_n} = \frac{1}{m_1! m_2! \cdots m_n!} \frac{m_1 + m_2 + \cdots + m_n}{m_1 \omega_1 \omega_2 \cdots \omega_n} \]

An equivalent definition of analyticity is supplied by the following theorem which is quoted without proof from Bochner and Martin [13].

Theorem 1. — If a function \(F(w_1, \ldots, w_n)\), all \(w_i\) complex, is continuous in a domain \(D\), and if in the neighborhood of every point it is analytic in each variable, then \(F(w)\) is analytic in \(D\).

Note. — "\(F\) analytic in \(D\)" implies analytic in the sense of definition 2.

In all of the theorems which follow \(F\) is taken to be a scalar function. The same theorem may be stated for vector functions since each component of \(F\) may be treated separately.

Theorem 2. — Let \(F\) be a scalar function of \(p\) complex vector arguments,

\[ F = F(w_1, \ldots, w_p), w_i = (w_{i1}, \ldots, w_{in}) \; i = 1, \ldots, p \]. Suppose \(F\) is
defined and analytic on a region $D$ of $(w_1, \ldots, w_p)$ space of $np$ complex dimensions. Let $G_{jk}$, $j = 1, \ldots, p$ be a region in the $w_{jk}$ plane. Let $k = 1, \ldots, n$

the boundary of $G_{jk}$ be a closed curve $c_{jk}$ which is piecewise differentiable. Define

$$c = c_{11} \times c_{12} \times \cdots \times c_{np}$$

and

$$G = G_{11} \times G_{12} \times \cdots \times G_{np}$$

where the symbol $\times$ denotes topological product. Suppose $\bigcup_{G} \subset D$

Then for $(w_1, \ldots, w_p) \in G$, Cauchy's integral formula takes the

following form:

$$F(w_1, \ldots, w_p) = \sum_{1}^{np} \int_{c_1}^{c_2} \cdots \int_{c_p}^{c_2} \frac{F(w_1, w_2, \ldots, w_p) d^*(\overline{w_1, w_2, \ldots, w_p})}{(w_1 - w_1)(w_2 - w_2) \cdots (w_p - w_p)}$$

$$\int_{c_j}^{c_j} \int_{c_{j1}}^{c_{j2}} \cdots \int_{c_{jn}}^{c_{jn}}$$

$$\overline{w}_j \cdot w_j = (\overline{w}_{j1} - w_{j1})(\overline{w}_{j2} - w_{j2}) \cdots (\overline{w}_{jn} - w_{jn})$$

$j = 1, \ldots, p$
and
\[
d^{*} (\overline{w}_1, \ldots, \overline{w}_n) = \prod_{j=1}^{p} \prod_{k=1}^{n} \overline{\text{d}w}_{jk}
\]
where
\[
j = 1, \ldots, p
\]
\[
k = 1, \ldots, n
\]
the symbolic product may be arranged in any order, i.e., the integration may be carried out in any order.

**Proof.**— Since \( F \) is analytic in \( D \), it is analytic in each variable separately for all combinations of the other variables. By repeated application of Cauchy's integral formula for functions of one complex variable, one may write
\[
F(w_1, \ldots, w_p) = \left( \frac{1}{2\pi i} \right)^{np} \int_{c_11} \frac{\overline{\text{d}w}_{11}}{\overline{w}_1 - w_1} \int_{c_{12}} \frac{\overline{\text{d}w}_{12}}{\overline{w}_{12} - w_{12}} \ldots
\]
\[
\ldots \int_{c_{np}} \frac{F(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_p)}{\overline{w}_p - w_p} \overline{\text{d}w}_{np}
\]
Now since \( F \) is continuous on the manifold \( c \), it is uniformly continuous and bounded on \( c \) and thus the integration may be carried out in any order.

**Theorem 3.**— Let \( D \) be a region of \((\zeta, w, z)\) space and let \( D_s \) be a region of \( \zeta \) space where \( w = (w_1, \ldots, w_n) \) and \( z = (z_1, \ldots, z_n) \) and \( w_1, z_1, \zeta \) are complex. Let \( c \) be a rectifiable path lying in \( D_s \) with \( \tau_1 \) as one endpoint and \( \tau \) as a variable point on \( c \). Define
and suppose $F(t, w, z)$ is continuous at each point of $D_{\xi s}$ where $F$ is a scalar function. Then

$$G(\tau, \xi, w, z) = \int_{\tau_0}^{\tau} F(s + \xi, w, z) \, ds$$

is continuous at each $(\tau, \xi, w, z) \in X \cdot D_s$.

**Proof.**— Let $(\tau_0, \xi_0, w_0, z_0)$ be an arbitrary point of $c X D$ and define the set

$$V = \left\{ (s + \xi_0, w_0, z_0) : s \in c \right\}.$$  

Now $V$ is a compact subset of the open set $D_{\xi s}$ and hence there exists a compact subset $K$ of $D_{\xi s}$ with $V$ on its interior $V \subset K \subset D_{\xi s}$. Let $d$ be the distance from $V$ to the boundary of $K$. Now for $(\tau, \xi, w, z) \in c X D,$

$$|G(\tau_0, \xi_0, w_0, z_0) - G(\tau, \xi, w, z)| \leq$$

$$\int_c |F(s + \xi_0, w_0, z_0) - F(s + \xi, w, z)| \, |ds| +$$

$$\int_{\tau_0}^{\tau} |F(s + \xi_0, w_0, z_0)| \, |ds|,$$

where the latter integral is taken along $c$.

Continuity of $F$ on $K$ implies that $F$ is bounded on $K$ by $M$ and that $F$ is uniformly continuous on $K$. Hence given $\varepsilon > 0$, there exists
\( \varepsilon > 0 \) such that

\[
|\zeta - \zeta_0| + |w - w_0| + |z - z_0| < \min (\varepsilon, d)
\]

implies

\[
|F(s + \zeta_0, w_0, z_0) - F(s + \zeta, w, z)| < \frac{\varepsilon}{2L}
\]

for all \( s \) on \( c \) where \( L \) is the length of \( c \).

Also, since the path \( c \) is regular, there exists a \( \varepsilon_1 > 0 \)

such that

\[
\int_{\tau_0}^{\tau} |ds| < \frac{\varepsilon}{2M}
\]

whenever \( |\tau - \tau_0| < \varepsilon_1 \). Hence, for

\[
|\tau - \tau_0| + |\zeta - \zeta_0| + |w - w_0| + |z - z_0| < \min (\varepsilon, d, \varepsilon_1),
\]

\[
\left| G(\tau_0, \zeta_0, w_0, z_0) - G(\tau, \zeta, w, z) \right| < \\
\int_{\tau_0}^{\tau} \frac{\varepsilon}{2L} |ds| + \int_{\tau}^\tau M |ds| < \varepsilon.
\]

Since \((\tau_0, \zeta_0, w_0, z_0)\) is an arbitrary point of \( c \times D \), \( G \) is continuous on \( c \times D \).

**Corollary.** Theorem 3 is true when \( D \) and \( D_s \) are compact sets.

This follows from the estimate (2) and the uniform continuity and boundedness of \( F \) on \( D_{\zeta^s} \).
Theorem 1.-- Let $F(z_1, \ldots, z_n)$ be a function of $n$ complex variables. Let $s_i$ be a region in $z_i$ plane, $i = 1, \ldots, n$, and suppose that $s_i$ has a rectifiable boundary curve $c_i$. Let $s = s_1 \times s_2 \times \ldots \times s_n$ and $c = c_1 \times c_2 \times \ldots \times c_n$ where $\times$ denotes topological product. Suppose that $F$ is continuous on the manifold $c$. Then for all $m_i > 1$, $i = 1, \ldots, n$, ($m_i$ integers)

$$G_{m_1 \cdots m_n}(z_1, \ldots, z_n) = \int_{c_1} \int_{c_2} \cdots \int_{c_n} \frac{F(\overline{z}_1, \ldots, \overline{z}_n)}{\left(\overline{z}_1 - z_1\right)^{m_1} \cdots \left(\overline{z}_n - z_n\right)^{m_n}} \, \overline{dz}_1 \cdots \overline{dz}_n$$

is analytic in $s$ and for $i = 1, 2, \ldots, n$, and

$$\frac{\partial G_{m_1 \cdots m_n}}{\partial z_i} = m_i G_{m_1 \cdots m_i - 1 \cdots m_n} \cdot (m_i + 1)^{m_i + 1} \cdots m_n$$

Proof.-- In view of Theorem 1 of the Appendix, it is necessary only to show that $G_{m_1 \cdots m_n}$ is continuous in $(z_1, \ldots, z_n)$ and it is analytic in each variable separately for all combinations of the other variables. A proof will be given for the case $n = 2$ with $z_1 = z$, $z_2 = w$, $m_1 = r$, and $m_2 = s$. Define

$$g_{rs}(z, w; \overline{z}, \overline{w}) = \frac{F(z, w)}{(\overline{z} - z)^r (\overline{w} - w)^s}$$

for $\overline{z}$ on $c_1$, $\overline{w}$ on $c_2$, $z \in s_1$, and $w \in s_2$. The function $g_{rs}$ satisfies the hypotheses of Theorem 3 of the Appendix and the Corollary
following that theorem. Thus,

\[ g_{rs}(z, w; \bar{z}) = \int_{c_2} g_{rs}(z, w; \bar{z}, \bar{w}) \, d\bar{w} \]

is continuous in all its variables. Applying the theorem again to \( g \), one obtains

\[ G_{rs}(z, w) = \int_{c_1} g_{rs}(z, w; \bar{z}) \, d\bar{z} \]

as a continuous function of \((z, w)\).

Now let \( z \) be arbitrary but fixed. Define the function

\[ h_r(z, w) = \int_{c_1} F(z, w) \, d\bar{w} \frac{1}{(\bar{z} - z)^s} \]

For fixed \( z \) and for \( \bar{w} \) on \( c_2 \), the integrand satisfies the hypotheses of the corollary to Theorem 3 of the appendix and thus \( h_r(z, \bar{w}) \) is continuous in \( \bar{w} \) on \( c_2 \). Now \( h_r \) satisfied the hypothesis of Lemma 3 in Ahlfors [15] and thus

\[ G_{rs}(z, w) = \int_{c_2} \frac{h_r(z, \bar{w}) \, d\bar{w}}{(\bar{w} - w)^s} \]

for fixed \( z \), is an analytic function of \( w \) for \( w \) in \( s_2 \), and

\[ \frac{\partial G_{rs}}{\partial w} = sG_r(s + 1) \cdot \]

A similar argument applies for \( z \) by letting \( w \) be fixed. This proves the theorem.

**Theorem 5.**—Let \( F(s, w_1, \ldots, w_p) \) be a scalar complex function of
(s, w₁, ..., wᵢ) for (w₁, ..., wᵢ) ∈ D and s on a regular contour c in s space where \( wᵢ = (w_{i1}, ..., w_{in}) \), \( w_{ij} \) complex, and D is a region of \( (w₁, ..., wₙ) \) space of np complex dimensions. Further let F be analytic in D for each s on c and continuous in \( (s, w₁, ..., wᵢ) \). Then

\[
G(w₁, ..., wᵢ) = \int_\Gamma F(s, w₁, ..., wᵢ) \, ds
\]

is an analytic function of \( (w₁, ..., wᵢ) \) in D.

Proof.— By Theorem 3 of the appendix G is continuous in all its arguments. Since \( F \) is analytic for fixed s, it is analytic in \( w_{i1} \) for fixed s and all other \( w_{ij} \) fixed. By a theorem for one complex variable (Titchmarsh [16]), one obtains that G is analytic in \( w_{i1} \) for any combination of the other \( w_{ij} \). A similar argument applies to all the other \( w_{ij} \) taken one at a time. Hence G is continuous in all its variables and analytic in each variable separately for all combinations of the other variables. Now apply Theorem 1 of the appendix to obtain the conclusion.

Theorem 6.— Suppose that each member of the sequence

\[
\left\{ F_i(z₁, ..., zₚ) \right\}
\]

is analytic in a region D, where the \( F_i \) are scalar complex functions and \( z_j = (z_{j1}, ..., z_{jn}) \), each \( z_{jk} \) complex. Further suppose that

\[
\sum_{i=1}^{∞} F_i = F(z₁, ..., zₚ)
\]
converges uniformly on every compact subset of \( D \). Then the function \( F \) is analytic in \( D \). Moreover,

\[
\sum_{i=1}^{\infty} \frac{\partial F_i}{\partial z_{jk}} = \frac{\partial F}{\partial z_{jk}}
\]

converges uniformly on every compact subset of \( D \).

**Proof.**-- Actually each \( F_i \) is a function of the \( np \) complex variable \( z_{jk}; j = 1, \ldots, p; k = 1, \ldots, n \). Since each \( F_i \) is analytic in all \( z_{jk} \) it is continuous in all \( z_{jk} \). By uniform convergence the limit function \( F \) is continuous in all \( z_{jk} \). Consider \( F_i \) as a function of \( z_{11} \) with all other \( z_{jk} \) fixed. Now one may apply the theorem for the case of one complex variable (Ahlfors [17]) to obtain that \( F \) is analytic in \( z_{11} \) for all combinations of the other \( z_{jk} \) and that

\[
\frac{\partial F}{\partial z_{11}} = \sum_{i=1}^{\infty} \frac{\partial F_i}{\partial z_{11}}
\]

converges uniformly on every compact subset of \( D \). A similar argument applies to the other \( z_{jk} \) taken one at a time. Analyticity of \( F \) can now be inferred from Theorem 1 of the appendix.

**Theorem 7.**-- Let \( F(z_1, z_2) \) be a scalar function of two scalar complex variables. Suppose that \( F \) is analytic at \( P_0 = (z_{10}, z_{20}) \) and its double series expansion converges in \( R \), a neighborhood of \( P_0 \). Then at each point of \( R \), \( F \) has the following alternative representation:
\[ F(z_1, z_2) = \sum_{n=0}^{\infty} \left\{ a_{on} + a_{1n}(z_1 - z_{10}) + a_{2n}(z_1 - z_{10})^2 + \ldots \right\} (z_2 - z_{20})^n \]  \hfill (3)

where

\[ a_{on}(z_1) = \frac{1}{n!} \frac{\partial^n F}{\partial z_2^n} (z_1, z_{20}) \]

and

\[ a_{kn} = \frac{1}{k!} \frac{\partial^k a_{on}}{\partial z_1^k} (z_{10}) \]

**Proof.** Since \( F \) is analytic at \( P_0 \),

\[ F(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} (z_1 - z_{10})^m (z_2 - z_{20})^n \] \hfill (4)

where

\[ a_{mn} = \frac{1}{m! n!} \frac{\partial^m + \partial^n F(z_{10}, z_{20})}{\partial z_1^m \partial z_2^n} \]

and the series \((4)\) converges absolutely in \( R \). Let \( P: (\bar{z}_1, \bar{z}_2) \)

be an arbitrary point of \( R \) and let

\[ h_{mn} = a_{mn} (\bar{z}_1 - z_{10})^m (\bar{z}_2 - z_{20})^n. \]

Then

\[ \sum_{m,n=0}^{\infty} h_{mn} \]

Converges absolutely and by a theorem in Apostol [18],
\[ F(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h(m,n) = \]

\[ \sum_{n=0}^{\infty} (z_2 - z_{20})^n \left\{ \sum_{m=0}^{\infty} a_{mn} (z_1 - z_{10})^m \right\} \]

which is just (3) evaluated at \((z_1, z_2) = (z_{11}, z_{21})\). Since \(P\) was an arbitrary point of \(R\), the theorem follows.
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8. Ibid., p. 362.


13. Ibid., p. 31.


