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DISSIPATIVE SETS AND SEMIGROUPS IN HILBERT SPACES

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To Cheryl

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CHAPTER I

INTRODUCTION

The purpose of this work is to present, in some detail, the development of two frequently-cited but seldom-proved results in the theory of semigroups. The first is a proof by G. J. Minty that in a Hilbert space every dissipative set can be extended to a hyper-dissipative set; this is done in Chapter II. The second is a proof by T. Kato of a theorem of Y. Kōmura that a strongly continuous, non-expansive semigroup on a closed, convex subset of a Hilbert space has a generator which is defined on a dense subset of the domain of the semigroup; this is done in Chapter III.

Chapter II is concerned with the equivalence (in Hilbert spaces) of maximal dissipative sets and hyper-dissipative sets. Definitions and examples of the various kinds of dissipative sets are presented in Section 1. Section 2 introduces a technique used by G. J. Minty in [1] which relates dissipative sets and non-expansive sets, and Section 3 contains the major theorems on maximal dissipative and hyper-dissipative sets. Section 4 contains propositions on dissipative sets which are used in Chapter III.

Chapter III is a survey of the use and occurrence of dissipative sets in semigroup theory. The chapter begins with two examples which illustrate certain properties of semigroups and the "approximation" semigroups which occur in the proof of the non-linear Hille-Yosida theorem. Section 2 contains a detailed presentation of a proof by T. Kato [2] of a theorem of Y. Kōmura on the generator of a semigroup. Section 3 states (without proof) recent results on generation of semigroups in Hilbert spaces.

Chapter IV is a brief introduction to the use of semigroup theory in the formulation and solution of the abstract Cauchy problem. An illustrative example of J. R. Dorroh and S. Oharu is presented in detail.

Functions will often be identified with their graphs. If X is a linear space, each of A and B is a subset of X×X, and h is a scalar, then the following notational conventions will be used:

(i) $D(A) = \{x \text{ in } X \mid (x,y) \text{ is in } A \text{ for some } y \text{ in } X\}$

(ii) $R(A) = \{y \text{ in } X \mid (x,y) \text{ is in } A \text{ for some } x \text{ in } X\}$

(iii) $Ax = \{y \text{ in } X \mid (x,y) \text{ is in } A\}$ for each x in D(A)

(iv) $A^{-1} = \{(x,y) \text{ in } X\times X \mid (y,x) \text{ is in } A\}$

(v) $hA = \{(x,hy) \text{ in } X\times X \mid (x,y) \text{ is in } A\}$, and

(vi) $A + B = \{(x,y+z) \mid (x,y) \text{ is in } A \text{ and } (x,z) \text{ is in } B\}$

If reference is made to some Hilbert space without indication as to whether X is a real Hilbert space or a complex Hilbert space, it is to be assumed that X may be either real or complex. All proofs are written for the complex case, but all propositions, lemmas, and theorems are true for either case.

CHAPTER II

DISSIPATIVE AND NON-EXPANSIVE SETS

This chapter includes the basic definitions and properties of dissipative and non-expansive sets in Hilbert spaces. Section I gives the definitions of various types of dissipative sets, and ends with examples of dissipative subsets in $E^1 \times E^1$. In Section II the duality between dissipative sets and non-expansive sets is explored using the functions E_λ. The functions E_λ are a generalization of the function Φ in [1]. The major result of Section III is the theorem that every dissipative set is contained in a hyper-dissipative extension. Section IV contains definitions and propositions used in Chapter III.

Section II.1. Dissipative Sets

This section gives the definitions of various types of dissipative sets in Hilbert spaces. The definitions given here are those used by M. G. Crandall and A. Pazy in [3]. An alternate definition of dissipative sets (which can be generalized to Banach spaces) appears in Proposition II.1.3. Examples II.1.1 and II.1.2 consider dissipative sets in $E^1 \times E^1$. The various quantities introduced in these examples are developed for general Hilbert spaces in Sections II.3 and II.4.

Remark II.1.1

Let X denote a Hilbert space over the field of complex numbers. Then $[\ ,\ ]_X$ will denote the inner product in X, and $|\ |_X$ will denote the norm in X determined by the inner product. If each of x and y is a member of X, then (x,y) will denote a member of $X^2 = X \times X$.

Definition II.1.1

Let X be a Hilbert space. A subset A of X×X is called dissipative if $\mathrm{Re}[v-y,u-x]_X \leq 0$ whenever each of (x,y) and (u,v) is a member of A.

Definition II.1.2

Let X be a Hilbert space. A subset A of X×X is called maximal dissipative if A is dissipative and A is equal to any dissipative subset of X×X which contains A. An immediate consequence of this definition is the following proposition.

Proposition II.1.1

If A is a maximal dissipative subset of X×X, and (x,y) is a member of X×X with the property that $\mathrm{Re}[(y-v,x-u)]_X \leq 0$ whenever (u,v) is contained in A, then (x,y) is contained in A.

Definition II.1.3

Let X be a Hilbert space. A subset A of X×X is called hyper-dissipative if A is dissipative and the set to which z belongs only in case there is a member (x,y) of A such that z = y-x is all of X.

Proposition II.1.2

If X is a Hilbert space and A is a hyper-dissipative subset of X×X, then A is maximal dissipative.

Proof. Let A be a hyper-dissipative subset of X×X. Suppose that B is a dissipative subset of X×X and B contains A. If (x,y) is a member of B, then y-x is a member of X, and there is a member (u,v) of A such that $v-u = y-x$. Thus $v-y = u-x$. Since B is dissipative it follows that

$$\begin{aligned} 0 &\le [v-y,v-y]_X \\ &= \mathrm{Re}[v-y,v-y]_X \\ &= \mathrm{Re}[v-y,u-x]_X \\ &\le 0. \end{aligned}$$

Therefore $0 = v-y = u-x$ and (x,y) is (u,v). Hence A contains B, and A is maximal dissipative.

Theorem II.1.1

If X is a Hilbert space and A is a dissipative subset of $X \times X$, then there is a maximal dissipative subset B of $X \times X$ such that B contains A.

Proof. Zorn's lemma states that if S is a non-empty partially ordered set, and if every linearly ordered subset of S has an upper bound, then S has a maximal element. Let A be a dissipative set and let S be the collection of dissipative subsets of $X \times X$ which contain A. Then A is in S. Let S be partially ordered by set-theoretic inclusion. Let $\{A_\lambda | \lambda \in \Lambda\}$ be a linearly ordered chain in S indexed by Λ. Let $B = \bigcup_{\lambda \in \Lambda} A_\lambda$. Suppose that (x,y) and (u,v) are in B. Then there are indices α and β in Λ such that (x,y) is in A_α and (u,v) is in A_β. Since $\{A_\lambda | \lambda \in \Lambda\}$ is linearly ordered, either $A_\alpha \subset A_\beta$ or $A_\beta \subset A_\alpha$. Assume (without loss of generality) that $A_\alpha \subset A_\beta$. Then each of (x,y) and (u,v) is in A_β, hence $\mathrm{Re}[y-v,x-u]_X \le 0$. It follows that B is dissipative.

Let λ be a member of Λ, then $A \subset A_\lambda \subset B$, so B is in S, therefore B is an upper bound for $\{A_\lambda | \lambda\varepsilon\Lambda\}$. The hypothesis of Zorn's Lemma is satisfied by S, therefore there is a member B of S which is maximal under the partial ordering of inclusion. Such a set B is a maximal dissipative extension of A.

Proposition II.1.3

Let X be a Hilbert space and let A be a subset of X×X. These are equivalent:

(i) A is dissipative.

(ii) $|(x-\lambda y)-(u-\lambda v)|_X \geq |x-u|_X$ whenever λ is a positive number and each of (x,y) and (u,v) is in A.

Proof. Suppose that A is dissipative and each of (x,y) and (u,v) is in A. Let $\lambda>0$ be given. Then

$$|x-\lambda y-u+\lambda v|_X^2 = [x-\lambda y-u+\lambda v, x-\lambda y-u+\lambda v]_X$$

$$= |x-u|_X^2 + \lambda^2|y-v|_X^2 - 2\lambda \mathrm{Re}[v-y,u-x]_X$$

$$\geq |x-u|_X^2 - 2\lambda \mathrm{Re}[v-y,u-x]_X$$

$$\geq |x-u|_X^2, \text{ and (ii) holds.}$$

The last inequality follows from the requirement that A be dissipative.

Now suppose that condition (ii) holds. Let $\lambda>0$ and (x,y) and (u,v) in A be given. Then from the work above,

$$2\text{Re}[v-y,u-x]_X = (|x-u|_X^2 + \lambda^2|y-v|_X^2 - |x-y-u+\lambda v|_X^2)/\lambda$$

$$\leq \lambda|y-v|_X^2$$

for each positive number λ. Thus

$$2\text{Re}[v-y,u-x]_X \leq 0 = \lim_{\lambda\to 0^+} \lambda|v-y|_X^2.$$

Hence A is dissipative.

Example II.1.1

Let E^1 be the Euclidean space of real numbers. The inner product in E^1 is ordinary multiplication and the norm is the absolute value function. Let A be a non-increasing (not necessarily continuous) function from E^1 to E^1. If each of (x,y) and (u,v) is in A, then $y \leq v$ if $x \geq u$ and $v \leq y$ if $u \geq x$; hence $(v-y)(u-x) \leq 0$ and A is dissipative. If A is not continuous, then there is a point z in E^1 such that $\lim_{x\to z^-} Ax > \lim_{x\to z^+} Ax$. If w is a point in E^1 such that $\lim_{x\to z^-} Ax > w > \lim_{x\to z^+} Ax$, then $(w-y)(z-x) \leq 0$, and A can be extended to a larger dissipative set. Let $\hat{A} = \{(z,w) \in E^1 \times E^1 | \lim_{x\to z^-} Ax \geq w \geq \lim_{x\to z^-} Ax\}$. $\hat{A}$ is a maximal dissipative set in $E^1 \times E^1$ (the proof is left to the reader) but in general $\hat{A}$ is not a function, however, if A^{-1} is a function, then $\hat{A}^{-1}$ is a function. A portion of a particular set A and its extension $\hat{A}$ are shown in Figure 1 and Figure 2.

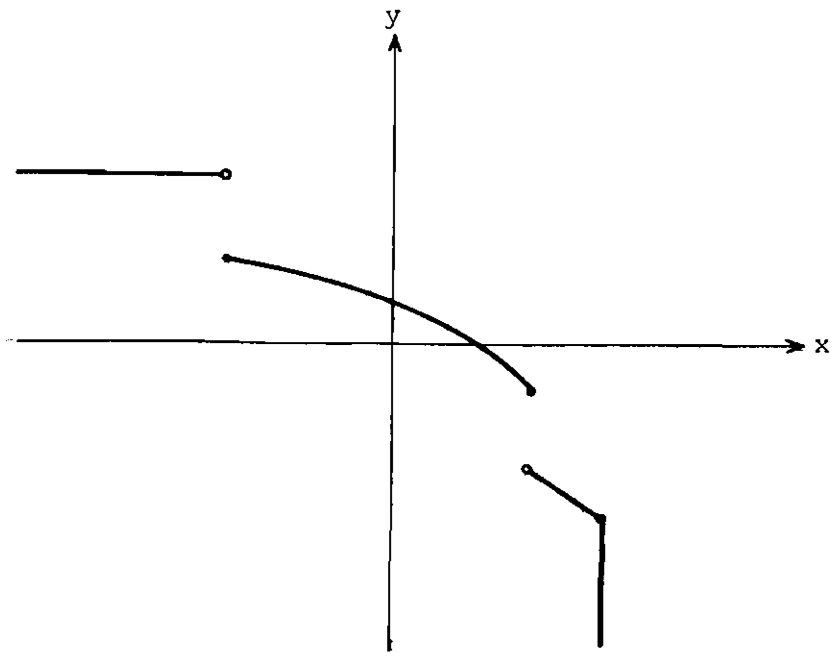

Figure 1. A Dissipative Set A in $E^1 \times E^1$

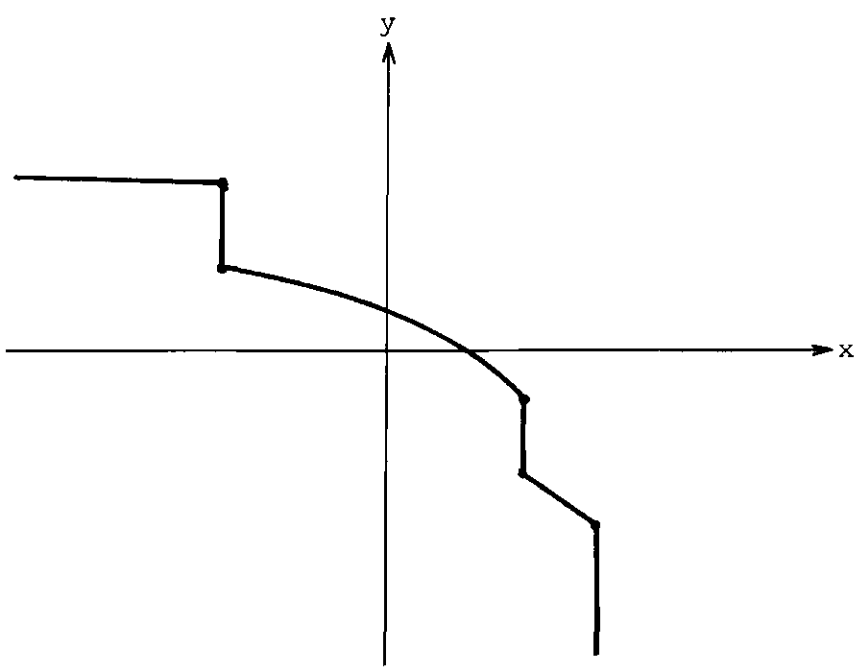

Figure 2. A Maximal Dissipative Extension $\hat{A}$ of the Dissipative Set A in Figure 1

Example II.1.2

Another dissipative set in $E^1 \times E^1$ is $A = \{(x,y) \text{ in } E^2 | y=1 \text{ if } x<0, y=0 \text{ if } x\geq 0\}$. The graph of A is shown in Figure 3. A maximal dissipative extension of A is $B = \{(x,y) \text{ in } E^2 | x=0, 0\leq y\leq 1\} \cup A$. The graph of B is shown in Figure 4. Although B is not a function, it can be "approximated" by Lipschitz continuous, dissipative functions in the following manner:

For each positive number λ, let

$$B_\lambda = \{(x-\lambda y, y) | (x,y) \text{ is in } B\}.$$

In the notation of Chapter I, $B_\lambda = \left((I-\lambda B)^{-1}-I\right)/\lambda$. The graphs of $I-\lambda B$ and B_λ are shown (for $\lambda=1$) in Figures 5 and 6, respectively. Recall that B is maximal dissipative. It is obvious from Figure 5 that $R(I-\lambda B)$ (at least for $\lambda=1$) is all of E^1. That this is true for an arbitrary maximal dissipative set in a Hilbert space X is a major result of Section II.3. Now we show that B_λ is dissipative and Lipschitz continuous. Suppose each of (u_1,v_1) and (u_2,v_2) is in B_λ. Then there exist (x_1,y_1) and (x_2,y_2) in B such that $x_1-\lambda y_1 = u_1$, $y_1 = v_1$, $x_2-\lambda y_2 = u_2$, $y_2 = v_2$. Therefore

$$\begin{aligned} \mathrm{Re}[v_1-v_2,u_1-u_2]_{E^1} &= \mathrm{Re}[y_1-y_2,x_1-\lambda y_1-x_2+\lambda y_2]_{E^1} \\ &= \mathrm{Re}[y_1-y_2,x_1-x_2]_{E^1} - \lambda|y_1-y_2|^2_{E^1} \\ &\leq 0. \end{aligned}$$

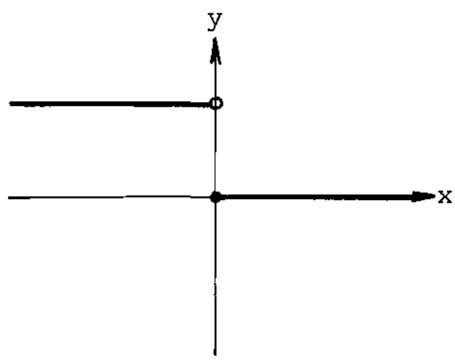

Figure 3. The Dissipative Set A of Example II.1.1

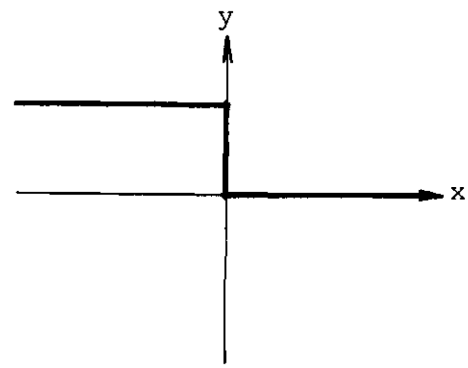

Figure 4. The Maximal Dissipative Extension B of A

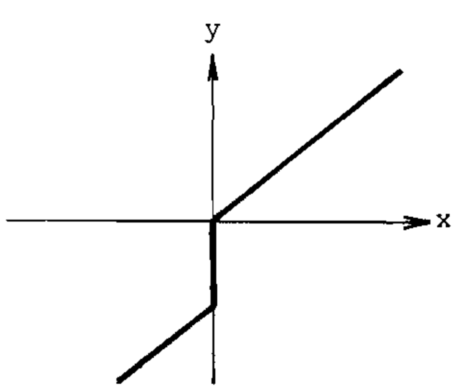

Figure 5. The Graph of I-λB for λ=1

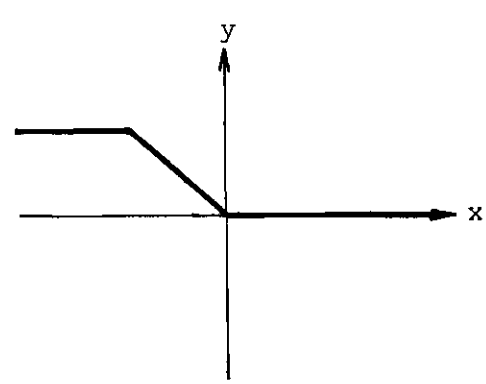

Figure 6. The Graph of Bλ for λ=1

Moreover $|u_1-u_2|_{E^1} = |x_1-\lambda y_1-x_2+\lambda y_2|_{E^1}$. We note here that $[y-v,x-u]_{E^1} = [x-u,y-v]_{E^1}$, whenever each of (x,y) and (u,v) is in $E^1\times E^1$, so a subset C of $E^1\times E^1$ is dissipative if and only if C^{-1} is dissipative. Therefore B^{-1} is dissipative. By Proposition II.1.3,

$$|x_1-\lambda y_1-x_2+\lambda y_2|_{E'} = \lambda\left|\frac{1}{\lambda}x_1-y_1-\frac{1}{\lambda}x_2+y_2\right|_{E'}$$

$$\geq \lambda|y_1-y_2|_{E'}$$

Hence B_λ is Lipschitz continuous with Lipschitz constant λ^{-1}. This example is continued in Example III.1.2.

Section II.2. Non-Expansive Sets

Definition II.2.1

Let X be a Hilbert space. A subset M of $X\times X$ is non-expansive if $|y-v|_X \leq |x-u|_X$ whenever each of (x,y) and (u,v) is a member of M.

Definition II.2.2

Let X be a Hilbert space. A subset L of $X\times X$ is Lipschitz (with Lipschitz constant $\lambda>0$) if $|y-v|_X \leq \lambda|x-u|_X$ whenever each of (x,y) and (u,v) is a member of L.

Definition II.2.3

Let X be a Hilbert space. For each positive number λ, define a function E_λ from $X\times X$ into $X\times X$ as follows:

$$E_\lambda(x,y) = (x-\lambda y,-x-\lambda y) \text{ for each } (x,y) \text{ in } X\times X.$$

Proposition II.2.1

If X, λ and the function E_λ are as in Definition II.2.3, then E_λ has all of X×X as its range and has an inverse given by

$$E_\lambda^{-1}(x,y) = \big((x-y)/2,(-x-y)/2\lambda\big) \text{ for each } (x,y) \text{ in } X\times X.$$

Proof. For each (u,v) in X×X, we solve the equations

$$x-\lambda y = u \quad \text{and} \quad -x-\lambda y = v$$

for x and y. Thus

$$x = (u-v)/2 \quad \text{and} \quad y = (-u-v)/2\lambda.$$

Lemma II.2.1

Let X, λ, and E_λ be as in Definition II.2.3. If A and M are two subsets of X×X, and if M is the image of A under the function E_λ, then these are equivalent:

(i) A is dissipative.

(ii) M is non-expansive.

Proof. Suppose that A is dissipative. Then $A = E_\lambda^{-1}M$. Let (x,y) and (u,v) be two members of M. Then each of $\big((x-y)/2,(-x-y)/2\lambda)\big)$ and $\big((u-v)/2,(-u-v)/2\lambda\big)$ is a member of A. Since A is dissipative it follows that

$$0 \geq \mathrm{Re}[\{(-x-y)/2\lambda\}-\{(-u-v)/2\lambda\},\{(x-y)/2\}-\{(u-v)/2\}]_X$$

$$= \frac{-1}{4\lambda}\mathrm{Re}[(x+y-u-v),(x-y-u+v)]_X,$$

and

$$0 \leq \mathrm{Re}[(x-u)+(y-v),(x-u)-(y-v)]_X$$

$$= [x-u,x-u]_X + \mathrm{Re}([y-v,x-u]_X - [x-u,y-v]_X) - [y-v,y-v]_X$$

$$= |x-u|_X^2 + \mathrm{Re}(2\mathrm{Im}[y-v,x-u]_X) - |y-v|_X^2$$

$$= |x-u|_X^2 - |y-v|_X^2.$$

Thus $|x-u|_X \geq |y-v|_X$. It follows that M is non-expansive.

Suppose that M is non-expansive. Let (x,y) and (u,v) be two members of A. Then each of $(x-\lambda y,-x-\lambda y)$ and $(u-\lambda v,-u-\lambda v)$ is a member of M. Since M is non-expansive, it follows that

$$0 \geq |(-x-\lambda y)-(-u-\lambda v)|_X^2 - |(x-\lambda y)-(u-\lambda v)|_X^2$$

$$= [(-x-\lambda y+u+\lambda v),(-x-\lambda y+u+\lambda v)]_X - [(x-\lambda y-u+\lambda v),(x-\lambda y-u+\lambda v)]_X$$

$$= \mathrm{Re}\{[(-x-\lambda y+u+\lambda v),(-x-\lambda y+u+\lambda v)]_X - [(-x-\lambda y+u+\lambda v),(x-\lambda y-u+\lambda v)]_X$$

$$+ [(x-\lambda y-u+\lambda v),(-x-\lambda y+u+\lambda v)]_X$$

$$- [(x-\lambda y-u+\lambda v),(x-\lambda y-u+\lambda v)]_X\}$$

$$= \mathrm{Re}[(-x-\lambda y+u+\lambda v) + (x-\lambda y-u+\lambda v),(-x-\lambda y+u+\lambda v) - (x-\lambda y-u+\lambda v)]_X$$

$$= \mathrm{Re}[2\lambda(v-y),2(u-x)]_X$$

$$= 4\lambda \mathrm{Re}[(v-y),(u-x)]_X.$$

Thus A is dissipative.

Definition II.2.4

Let X be a Hilbert space. A subset M of X×X is maximal non-expansive if M is non-expansive and M is equal to any non-expansive subset N of X×X which contains M.

Theorem II.2.1

Let X, λ and E_λ be as in Definition II.2.3. If A and M are two subsets of X×X, and if M is the image of A under the function E_λ, then these are equivalent:

(i) A is maximal dissipative.

(ii) M is maximal non-expansive.

Proof. Suppose that A is maximal dissipative. By Lemma II.2.1, M is non-expansive. Suppose that N is a non-expansive subset of X×X which contains M. By Proposition II.2.1 and Lemma II.2.1, $E_\lambda^{-1}N$ is a dissipative subset of X×X and $E_\lambda^{-1}N \supset A$. Since A is maximal, $E_\lambda^{-1}N = A$, and $N = E_\lambda A = M$. Therefore M is maximal non-expansive. The proof of the converse is similar.

Section II.3. Hyper-Dissipative Sets

The main result of this section is Theorem II.3.3 and its Corollary II.3.4 which state that if X is a Hilbert space and A is a maximal dissipative subset of X×X, then A is hyper-dissipative (the converse of Proposition II.1.2); moreover, if λ is a positive number, then λA is also hyper-dissipative. The technique is basically that of Crandall and Pazy [3] and Minty [1]. The section begins with a theorem of Kirzbraum which states that if m closed balls in Euclidean space with non-empty intersection are displaced in a manner which does not move the diameters further apart, then the closed balls retain their non-empty intersection. The proof follows the techniques of Schoenburg [4]. A theorem of Minty [1] generalized Kirzbraum's theorem to Hilbert spaces. It is thus shown that a maximal non-expansive set in X×X must have all of X as its domain. The main theorem follows immediately using the function E_λ introduced in Section II.2. The major theorem is sharpened so as to include Theorem 2.1 of [3].

Remark II.3.1

To simplify the notation, the norm in the Euclidean space E^n is written $| \ |$ rather than $| \ |_{E^n}$ in Theorem II.3.1 and Corollary II.3.1.

Theorem II.3.1 [4, Theorem 2, p. 620]

Suppose that m is a positive integer, each of $\{x_p\}_{p=1}^m$ and $\{y_p\}_{p=1}^m$ is a sequence of points in Euclidean space E^n such that

$$|y_i - y_j| \le |x_i - x_j|$$

whenever each of i and j is a positive integer not greater than m, and p is a point in E^n; then there is a point q in E^n such that

$$|y_i - q| \le |x_i - p|$$

whenever i is a positive integer not greater than m.

Proof. If $p=x_j$ for some integer j, $1 \le j \le m$, then $q=y_j$ satisfies the inequalities in the conclusion of the theorem. In what follows, assume that $p \ne x_j$ for $1 \le j \le m$. For each integer j, $1 \le j \le m$, define a function f_j from E^n to the set of non-negative real numbers by

$$f_j(x) = \frac{|x - y_j|}{|p - x_j|}$$

for each x in E^n, and let the function f be defined by

$$f(x) = \max\{f_i(x) \mid 1 \le i \le m\}$$

for each x in E^n. Each f_i is continuous, therefore f is continuous. Note that $\lim_{|x| \to \infty} f_j(x) = +\infty$ for $1 \le j \le m$, so $\lim_{|x| \to \infty} f(x) = +\infty$, and f attains its minimum value in some bounded subset of E^n. Suppose $\lambda = \inf\{f(x) \mid x \in E^n\}$ and $f(z) = \lambda$. If $\lambda = 0$, then $f_j(z) = 0$ for $1 \le j \le m$ and $z = y_j$, $1 \le j \le m$. If $z = y_j$, $1 \le j \le m$, the inequalities in the conclusion of the theorem are satisfied by $q = z$. In what follows, assume $\lambda > 0$.

Let N be the set of positive integers not greater than m to which j belongs only in case $f_j(z) = \lambda$. Let M be the set of positive integers not greater than m which are not members of N. If j is in M, then $f_j(z) < \lambda$. Let K be the convex hull of $\{y_k | k \in N\}$, i.e. K is the set to which w belongs only in case there is a sequence of non-negative numbers $\{a_i\}_{i \in N}$ such that $\sum_{i \in N} a_i = 1$ and $\sum_{i \in N} a_i y_i = w$. Suppose, for the moment, that z is not in K. Since K is bounded and closed (and hence compact), there is a point w in K such that $|z-w| \leq |z-u|$ whenever u is in K. For each j in N, let c_j be the cosine of the angle between z-w and y_j-w. If $c_j > 0$ for some j in N, then by the law of cosines the function

$$g(\varepsilon) = \left|z-\left(w+\varepsilon(y_j-w)\right)\right|^2 = |z-w|^2 - 2\varepsilon|z-w|\,|y_j-w|c_j + \varepsilon^2|y_j-w|$$

is strictly decreasing for small positive values of ε, and for some $\varepsilon_o > 0$ $(1 \geq \varepsilon_o)$,

$$g(\varepsilon_o) = \left|z-\left(w+\varepsilon_o(y_j-w)\right)\right|^2 < |z-w|^2,$$

but since $(1-\varepsilon_o)w + \varepsilon_o y_j$ is in K, we must have

$$\left|z-\left(w+\varepsilon_o(y_j-w)\right)\right| \geq |z-w|.$$

Hence $c_j \leq 0$ for each j in N. For each positive number ε less than 1, let $z_\varepsilon = z + \varepsilon(w-z)$. Then for each j in N,

$$
\begin{aligned}
|z_\varepsilon - y_j|^2 &= |z_\varepsilon - w + w - y_j|^2 \\
&= |z_\varepsilon - w|^2 - 2c_j |z_\varepsilon - w||y_j - w| + |y_j - w|^2 \\
&= |z + \varepsilon(w-z) - w|^2 - 2c_j |z+\varepsilon(w-z)-w||y_j - w| + |y_j - w|^2 \\
&= (1-\varepsilon)^2 |z-w|^2 - 2c_j(1-\varepsilon)|z-w||y_j-w| + |y_j-w|^2 \\
&= |z-w|^2 - 2c_j|z-w||y_j-w| + |y_j-w|^2 \\
&\qquad + \varepsilon^2|z-w|^2 + 2\varepsilon c_j|z-w||y_j-w| - 2\varepsilon|z-w|^2 \\
&= |z-y_j|^2 + \varepsilon^2|z-w|^2 + \varepsilon(2c_j|z-w||y_j-w| - 2|z-w|^2)
\end{aligned}
$$

Since $c_j \leq 0$, there is a positive number δ_j so that if $0<\varepsilon\leq\delta_j$, then $|z_\varepsilon - y_j| < |z-y_j|$ and $f_j(z_\varepsilon) < f_j(z) = \lambda$. Because $f_i(x)$ is continuous for each i in M and $f_i(z) < \lambda$ for each i in M, it follows that for each i in M, there is a positive number δ_i so that if $0<\varepsilon\leq\delta_i$, $f_i(z_\varepsilon) < \lambda$. Now let $\delta = \min\{\delta_j \mid 1\leq j\leq m\}$. Then for $0<\varepsilon\leq\delta$, $f_j(z_\varepsilon) < \lambda$ for each integer j, $1\leq j\leq m$, and $f(z) = \lambda$ is not the absolute minimum of f on E^n. This contradiction implies that z must be in K.

It must now be shown that $\lambda \leq 1$. Suppose for the moment that $\lambda>1$. Then for each j in N, $f_j(z) = \lambda>1$, so that

$$\frac{|z-y_j|}{|p-x_j|} > 1$$

and

$$|z-y_j|^2 > |p-x_j|^2$$

for each j in N. Also, for each i and j in N,

$$\begin{aligned} |y_i-z+z-y_j|^2 &= |y_i-y_j|^2 \\ &\leq |x_i-x_j|^2 \\ &= |x_i-p+p-x_j|^2 \end{aligned}$$

Thus

$$\begin{aligned} &|z-y_i|^2 - 2\mathrm{Re}[z-y_i,z-y_j] + |z-y_j|^2 \\ &\leq |p-x_i|^2 - 2\mathrm{Re}[p-x_i,p-x_j] + |p-x_j|^2 \\ &\lneq |z-y_i|^2 - 2\mathrm{Re}[p-x_i,p-x_j] + |z-y_j|^2. \end{aligned}$$

Simplification of the preceding inequality yields

$$\mathrm{Re}[p-x_i,p-x_j] < \mathrm{Re}[z-y_i,z-y_j]$$

whenever each of i and j is in N. Since z is in K, there are non-negative numbers c_i, for each i in N, such that

$$\sum_{i \in N} c_i = 1$$

and

$$\sum_{i \in N} c_i y_i = z.$$

Then

$$0 = z - \sum_{i \in N} c_i y_i = \sum_{i \in N} c_i (z - y_i).$$

It follows that

$$\begin{aligned} 0 &= \mathrm{Re}[\sum_{i \in N} c_i (z-y_i), \sum_{j \in N} c_j (z-y_j)] \\ &= \sum_{i \in N} \sum_{j \in N} c_i c_j \mathrm{Re}[z-y_i, z-y_j] \\ &\lneq \sum_{i \in N} \sum_{j \in N} c_i c_j \mathrm{Re}[p-x_i, p-x_j] \\ &= \mathrm{Re}[\sum_{i \in N} c_i (p-x_i), \sum_{j \in N} c_j (p-x_j)] \\ &\le 0. \end{aligned}$$

The assumption that $\lambda > 1$ leads to a contradiction, therefore λ is not greater than 1. Choose q=z. Then $f(q) = f(z) = \lambda \le 1$, and $f_i(q) \le 1$

for $1 \le i \le m$. Thus $|q-y_i| \le |p-x_i|$ for $1 \le i \le m$.

Corollary II.3.1 [4, Theorem 1, p.620 and 1, Theorem 1, p. 341]

Let E^n be n-dimensional Euclidean space and let m be a positive integer. For each positive integer i not greater than m, let each of x_i and y_i be given points in E^n, let a_i be given non-negative numbers, and let

$$R_i = \{z \epsilon E^n \mid |z-x_i| \le a_i\}$$

and

$$S_i = \{z \epsilon E^n \mid |z-y_i| \le a_i\}.$$

If $|y_i-y_j| \le |x_i-x_j|$ whenever each of i and j is a positive integer not greater than m and if $\bigcap_{i=1}^{m} R_i$ is not empty, then $\bigcap_{i=1}^{m} S_i$ is not empty.

Proof. Suppose that $|y_i-y_j| \le |x_i-x_j|$ for $1 \le i \le m$ and $1 \le j \le m$, and that there is a point p in $\bigcap_{i=1}^{m} R_i$. Then $|p-x_i| \le a_i$ for $1 \le i \le m$. By Theorem II.3.1, there is a point q in E^n such that

$$|q-y_i| \le |p-x_i|$$

for each i, $1 \le i \le m$. Thus

$$|q-y_i| \le a_i$$

for $1 \le i \le m$, and $\bigcap_{i=1}^{m} S_i$ is not empty.

Theorem II.3.2 [1, Theorem 1, p.341]

Let X be a Hilbert space and let Λ be an index set. Suppose that for each λ in Λ there are points x_λ and y_λ are points in X and a_λ is a non-negative number. For each λ in Λ, let

$$R_\lambda = \{z\varepsilon X \mid |z-x_\lambda| \le a_\lambda\}$$

and

$$S_\lambda = \{z\varepsilon X \mid |z-y_\lambda| \le a_\lambda\}.$$

Suppose also that $|x_\alpha - x_\beta| \ge |y_\alpha - y_\beta|$ whenever each of α and β is in Λ. Then, if there is a point p which is in R_λ for each λ in Λ, there is a point q which is in S_λ for each λ in Λ.

Proof. Let δ be a member of Λ. Then $\bigcap_{\lambda\varepsilon\Lambda} S_\lambda = \bigcap_{\lambda\varepsilon\Lambda} (S_\lambda \cap S_\delta)$. Suppose that $\bigcap_{\lambda\varepsilon\Lambda} S_\lambda$ is empty. S_δ is weakly compact and $S_\lambda \cap S_\delta$ is weakly closed for each λ in Λ. Since $\bigcap_{\lambda\varepsilon\Lambda} S_\lambda = \bigcap_{\lambda\varepsilon\Lambda} (S_\lambda \cap S_\delta)$ is empty, it follows that $\{(S_\lambda \cap S_\delta)^c \mid \lambda\varepsilon\Lambda\}$* is a weakly open cover of S_δ. There is a finite subset L of Λ such that $\{(S_\lambda \cap S_\delta)^c \mid \lambda\varepsilon L\}$ is a finite subcover of S_δ. We assume, without loss of generality, that δ is in L. Hence $\bigcap_{\lambda\varepsilon L} S_\lambda = S_\delta \cap \left(\bigcap_{\lambda\varepsilon L} (S_\lambda \cap S_\delta)\right)$ is empty. Let Y be the finite dimensional subspace of X generated by $\{x_\lambda \mid \lambda\varepsilon L\}$, $\{y_\lambda \mid \lambda\varepsilon L\}$ and $\{p\}$. There is an isometry J between Y and a Euclidean space E^n. For each λ in L, let $\bar{R}_\lambda = Y \cap R_\lambda$, and $\bar{S}_\lambda = Y \cap S_\lambda$. Then for each λ in L, $J\bar{R}_\lambda$ is a sphere in E^n with center

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*If S is a subset of X, S^c denotes the set-theoretic complement of S in X.

Jx_λ and radius a_λ, $J\bar{S}_\lambda$ is a sphere in E^n with center Jy_λ and radius a_λ, and for λ and δ in L,

$$\begin{aligned} |Jy - Jy_\delta| &= |y_\lambda - y_\delta| \\ &\le |x_\lambda - x_\delta| \\ &= |Jx_\lambda - Jx_\delta| \end{aligned}$$

Jp is in $\bigcap_{\lambda \varepsilon L} J\bar{R}_\lambda$, so by Corollary II.3.1, there is a point q in E^n such that q is in $\bigcap_{\lambda \varepsilon L} J\bar{S}_\lambda$. Since J preserves distances, $J^{-1}q$ is in $\bigcap_{\lambda \varepsilon L} \bar{S}_\lambda$, hence $\bigcap_{\lambda \varepsilon L} \bar{S}_\lambda$, hence $\bigcap_{\lambda \varepsilon L} S_\lambda$ is not empty. This contradicts the assumption that $\bigcap_{\lambda \varepsilon \Lambda} S_\lambda$ was empty, hence $\bigcap_{\lambda \varepsilon \Lambda} S_\lambda$ is not empty and the theorem is proved.

Corollary II.3.2

Let X be a Hilbert space, and let M be a non-expansive subset of X×X. Then there is a non-expansive subset N of X×X with the properties that N contains M and D(N) is all of X.

Proof. Let N be a maximal non-expansive subset of X×X which contains M. Such a set exists by Lemma II.2.1, Proposition II.1.2, and Theorem II.2.1. Suppose that D(N) is not all of X. Let p be a point of X which is not in D(N). For each x in D(N) let

$$R_x = \{z \text{ in } X \mid |z-x|_X \le |p-x|_X\},$$

and let

$$S_x = \{z \text{ in } X \,|\, |z-Nx|_X \le |p-x|_X\}.$$

Note that for x and y in D(N),

$$|x-y|_X \ge |Nx-Ny|_X.$$

Also, p is in $\bigcap_{x \in D(N)} R_x$, therefore the closed balls R_x and S_x satisfy all hypotheses of Theorem II.3.2; therefore, there is a point q in X which is in S_x for each x in D(N). It follows that

$$|q-Nx|_X \le |p-x|_X.$$

Hence N can be extended by including the pair (p,q), but N is maximal, hence p is in D(N). Thus, by contradiction, D(N) must be all of X.

Corollary II.3.3

Let X be a Hilbert space, and let M be a non-expansive subset of X×X. These are equivalent:

(a) M is maximal non-expansive.

(b) D(M) is all of X.

Proposition II.3.1

Let X be a Hilbert space, let λ be a positive number, let E_λ be the function in Definition II.2.3, and let A and M be subsets of X×X such that $M = E_\lambda A$. Then $D(M) = R(I-\lambda A)$.

Theorem II.3.3

Let X be Hilbert space and let A be a dissipative subset of X×X. These are equivalent:

(a) A is maximal dissipative.

(b) A is hyper-dissipative.

(c) For some positive number λ, $R(I-\lambda A) = X$.

(d) For every positive number λ, $R(I-\lambda A) = X$.

Proof. It is clear that (d) implies (c) and that (d) implies (b). By Proposition II.1.2, (b) implies (a), hence (d) also implies (a).

Suppose that A is maximal dissipative. Let λ be an arbitrary positive number. Let E_λ be the function in Definition II.2.3, and let $M = E_\lambda A$. By Theorem II.2.1, M is maximal non-expansive. By Corollary II.3.3, $D(M) = X$. By Proposition II.3.1, $R(I-\lambda A) = D(M) = X$. Thus (a) implies each of (b), (c) and (d). It now remains to show that (c) implies (a).

Suppose that for some positive number λ, $R(I-\lambda A) = X$. Let E_λ be the function in Definition II.2.3 and let $M = E_\lambda A$. By Proposition II.3.1, $D(M) = R(I-\lambda A) = X$. By Corollary II.3.3, M is maximal non-expansive. By Theorem II.2.1, A is maximal dissipative. Thus (c) implies (a) and the proof is complete.

Remark II.3.2

There is yet another useful result (primarily due to F. Browder) on the extension of dissipative operators which we now state without proof.

Theorem II.3.4 [5, Proposition 1, p. 93; 3, Lemma A2, p.417]

Let X be a Hilbert space, let D be a closed, convex subset of X, and let A be a dissipative subset of $X \times X$ such that D(A) is contained in D. Then there is a maximal dissipative (hence hyper-dissipative) subset B of X×X such that B contains A and D(B) is contained in D.

Corollary II.3.4

Let X be a Hilbert space and let A be a dissipative subset of X×X. Then A is contained in a hyper-dissipative subset B of X×X. Moreover, if D(A) is contained in some closed, convex subset of X, then there is a hyper-dissipative extension B with D(B) contained in D.

Indication of Proof. The existence of a maximal dissipative extension B of A is guaranteed by Theorem II.1.1 or Theorem II.3.4. The maximal dissipative extension is hyper-dissipative by Theorem II.3.3.

Section II.4. Special Results

Lemma II.4.1

If X is a Hilbert space and A is a subset of X×X, then A is dissipative if and only if A^{-1} is dissipative, and A is maximal dissipative if and only if A^{-1} is maximal dissipative.

Proof. If (x,y) and (u,v) are members of A (respectively, A^{-1}) then (y,x) and (v,u) are members of A^{-1} (respectively, A), and since $\mathrm{Re}[y-v,x-u]_X = \mathrm{Re}[v-y,u-x]$, it follows that if A (respectively, A^{-1}) is dissipative, then so is A^{-1} (respectively, A). If A is maximal dissipative, $B \subset X\times X$ is dissipative, and $A^{-1} \subset B$, then $A \subset B^{-1}$, and B^{-1} is dissipative; hence $B^{-1} = A$, $A^{-1} = B$, and A^{-1} is maximal dissipative.

Lemma II.4.2 [3, Lemma 2.2, p.386]

If X is a Hilbert space and A is a maximal dissipative subset of X×X, then for each x in D(A), Ax is convex and closed.

Proof. Let x be given in D(A). Suppose that each of $\{y_p\}_{p=0}^{\infty}$ is a member of Ax and $\lim_{p\to\infty} y_p = z$, z in X. Then

$$\mathrm{Re}[x-u,y_p-v]_X \le 0$$

for each (u,v) in A. Thus by the continuity of the inner product of X,

$$\mathrm{Re}[x-u,z-v]_X = \lim_{p\to\infty} [x-u,y_p-v]_X \le 0.$$

Then by Proposition II.1.1, (x,z) is in A and z is in Ax. Thus Ax is closed. For x in X, suppose that each of y and z is in Ax. Suppose that λ is a number such that $0\le\lambda\le1$. Then for each (u,v) in A,

$$\begin{aligned}\mathrm{Re}[\lambda y+(1-\lambda)z-v,x-u]_X &= \lambda\mathrm{Re}[y-v,x-u]_X + (1-\lambda)\mathrm{Re}[z-v,x-u]_X \\ &\le 0\end{aligned}$$

Thus by Proposition II.1.1, $(x,\lambda y+(1-\lambda)z)$ is in A, and $\lambda y + (1-\lambda)z$ is in Ax. It follows that Ax is convex.

CHAPTER III

SEMIGROUPS IN HILBERT SPACES

Chapter III concerns the connection between dissipative sets and non-expansive semigroups. Section 1 gives the definition of a semigroup on a subset of a Hilbert space and certain properties which semigroups might have. Three examples are given. Section 2 concerns differentiability properties of semigroups and generators of semigroups and generation of semigroups by various types of operators.

Section III.1. Definitions and Examples

Definition III.1.1

Let D be a subset of a Hilbert space H. A semigroup on D is a function T from the number interval $[0,\infty)$ into the space of functions from D into D such that $T(t)T(s)u = T(s+t)u$, whenever each of s and t is in $[0,\infty)$ and u is in D, and $T(0)u = u$ whenever u is in D.

Definition III.1.2

A semigroup T on a subset D of a Hilbert space X is non-expansive if $|T(t)u-T(t)v|_X \leq |u-v|_X$ whenever t is in $[0,\infty)$ and each of u and v is in D.

Definition III.1.3

A semigroup T on a subset D of a Hilbert space X is strongly continuous if $\lim_{s\to t} T(s)u = T(t)u$ whenever t is in $[0,\infty)$ (the limit from the right is understood for t=0) and u is in D.

Example III.1.1

Consider the initial value problem

$$\left.\begin{aligned} \frac{dy}{dt} &= -y^{1/3}, \\ y(0) &= p, \\ y:[0,\infty)\to E^1 &\text{ continuously.} \end{aligned}\right\} \qquad (1)$$

The solutions* are

$$y(t) = \begin{cases} \operatorname{sgn}(p)\left(p^{2/3}-(2t/3)\right)^{3/2} & \text{for } 0\le t\le (3/2)p^{2/3} \\ 0 & \text{for } t\ge (3/2)p^{2/3}. \end{cases}$$

The graphs of the solutions are shown in Figure 7. Define a semigroup T on E^1 as follows:

For each non-negative number t, let T(t) be that function from E^1 to E^1 such that

$$T(t)p = \begin{cases} \operatorname{sgn}(p)\left(p^{2/3}-(2t/3)\right)^{3/2} & \text{for } 0\le t\le (3/2)p^{2/3}, \\ 0 & \text{for } (3/2)p^{2/3}\le t \end{cases}$$

---

*The function sgn is the signum function: sgn x = 1 if x>0, sgn 0 = 0, and sgn x = -1 if x<0.

If p is in E^1, then $T(0)p = p$. If each of s and t is a non-negative number, then for every p in E^1,

$$T(t)T(s)p = \begin{cases} \operatorname{sgn}(p)T(t)\left(p^{2/3}-(2s/3)\right)^{3/2} & \text{for } 0 \le s \le (3/2)p^{2/3}, \\ 0 & \text{for } (3/2)p^{2/3} \le s. \end{cases}$$

$$= \begin{cases} \operatorname{sgn}(p)(p^{2/3}-(2s/3)-(2t/3) \;{}^{3/2} & \text{for } 0 \le s+t \le (3/2)p^{2/3}, \\ 0 & \text{for } (3/2)p^{2/3} \le s+t. \end{cases}$$

$$= T(s+t)p.$$

Thus T is a semigroup on E^1. It will not be shown here, but T is a strongly continuous, non-expansive semigroup.

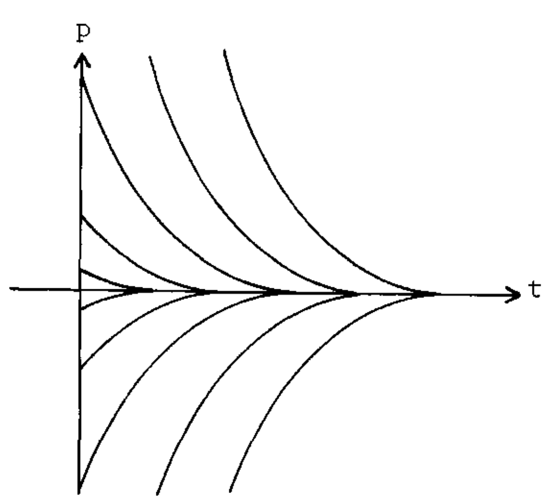

Figure 7. Solution of the Initial Value Problem (1)

Example III.1.2 (This example is a continuation of Example II.1.2.)

Let A and B be the dissipative sets in Example II.1.2. Recall that A is the function

$$Ax = \begin{cases} 1 & \text{for } x<0, \\ 0 & \text{for } x\geq 0. \end{cases}$$

B is the set to which (x,y) belongs only in case (x,y) is in A or x = 0 and $0<y\leq 1$. For each positive number h, B_h is the set to which (u-hv,v) belongs only in case (u,v) is in B. It was shown in Example II.1.2 that B_h is a dissipative Lipschitz continuous function with Lipschitz constant 1/h. The graphs of B and B_h (for h=1) are shown in Figures 4 and 6, respectively. It is now necessary to calculate B_h explicitly.

$$B_h x = \begin{cases} 1 & \text{for } x\leq -h, \\ -(1/h)x & \text{for } -h\leq x\leq 0, \\ 0 & \text{for } x\geq 0. \end{cases}$$

We now wish to consider the initial value problem (for each number $h \gtrneqq 0$)

$$\left.\begin{aligned} u_h'(t) = B_h u_h(t) \quad \text{for } t\geq 0, \\ u_h(0) = p \text{ in } E^1, \\ u_h \text{ maps } [0,\infty) \text{ into } E^1 \text{ continuously.} \end{aligned}\right\} \qquad (2)$$

Thus

$$u_h' = \begin{cases} 1 & \text{for } u_h \le -h, \\ -(1/h)u_h & \text{for } -h \le u_h \le 0, \\ 0 & \text{for } u_h \ge 0. \end{cases}$$

Suppose that $p \ge 0$, then $u_h(t) = p$ for $t \ge 0$. Suppose that $-h \le p \le 0$, then $u_h(t) = pe^{-(1/h)t}$ for $t \ge 0$. Suppose that $p \le -h$, then $u_h(t) = p+t$ for $p + t \le -h$, i.e. for $0 \le t \le -h-p$. At $t = -h-p$, $u_h(-h-p) = -h$. Thus for $t \ge -h-p$, $u_h(t) = -he^{-(1/h)(t+p+h)} = -he^{-(t+p+h)/h}$. Now let the function T_h from $[0,\infty)$ to the space of functions from E^1 to E^1 be defined by

$$T_h(t)x = \begin{cases} x & \text{for } x \ge 0, \\ xe^{-(1/h)t} & \text{for } -h \le x \le 0, \\ x+t & \text{for } x \le -h \quad \text{and} \quad 0 \le t \le -h-x, \\ -he^{-(1/h)(t+x+h)} & \text{for } x \le -h \quad \text{and} \quad t \ge -h-x. \end{cases}$$

It will not be verified here, but for each positive number h, T_h is a non-expansive, strongly continuous semigroup. The importance of this example is that the semigroup T defined by

$$T(t)x = \lim_{h \to 0^+} T_h(t)x \quad \text{(for t and x fixed)}$$

generates the solutions of

$$\left.\begin{array}{c} u'(t) = Au(t) \quad \text{a.e. in } t \text{ on } [0,\infty) \\ u(0) = p \text{ in } E^1, \\ u \text{ maps } [0,\infty) \text{ into } E^1 \text{ absolutely continuously.} \end{array}\right\} \qquad (3)$$

To show this, we note that

$$T(t)x = \begin{cases} x & \text{for } x\geq 0 \\ 0 & \text{for } 0\leq x\leq 0 \\ x+t & \text{for } x\leq 0 \text{ and } 0\leq t\leq -x \\ 0 & \text{for } x\leq 0 \text{ and } t\geq -x. \end{cases}$$

Thus $u(t) = p$ if $p\geq 0$, $u(t) = p+t$ if $p\leq 0$ and $0\leq t\leq -p$, $u(t) = 0$ if $p\leq 0$ and $t\geq -p$, where $u(t) = T(t)p$.

Section III.2. Generators of Semigroups

Definition III.2.1

Let T be a semigroup on a subset D of a Hilbert space X. For each positive number h, let $A^h\cdot$ be the function from D to X defined by

$$A^h x = (1/h)\big(T(h)x-x\big) \quad \text{for each } x \text{ in } D.$$

Proposition III.2.1

Let T be a semigroup on a subset D of a Hilbert space X. For each positive number h, let A^h be the function in Definition III.2.1.

If T is non-expansive, then A^h is dissipative for every positive number h.

Proof. Suppose that T is non-expansive. For each positive number h, let A^h be the function described above. Suppose that x and y are in D. Then

$$\begin{aligned} \mathrm{Re}[A^h x - A^h y, x-y]_X &= \mathrm{Re}[\{T(h)x-x-T(h)y+y\}/h, x-y]_X \\ &= (1/h)\mathrm{Re}[T(h)x-T(h)y, x-y]_X - (1/h)|x-y|_X^2 \\ &\le (1/h)|T(h)x-T(h)y|_X \cdot |x-y|_X - (1/h)|x-y|_X^2 \\ &= (1/h)|x-y|_X\left(|T(h)x-T(h)y|_X - |x-y|_X\right) \\ &\le 0. \end{aligned}$$

Lemma III.2.1 [2, Lemma 1, p.92]

Let T be a non-expansive semigroup on a closed, convex subset D of a Hilbert space X. For every positive number h, let A^h be the function in Definition III.1.1. Then $R(I-kA^h)$ contains D for every positive number k, and $(I-kA^h)^{-1}$ exists and is jointly continuous in h and k at each point in D.

Proof. By Proposition III.2.1, A^h is dissipative, and by Proposition II.4. $I-kA^h$ is reversible whenever each of h and k is a positive number. Let x be a point in D and let each of h and k be positive numbers. Define a function G from D into X by

$$G(y) = \left(hx/(h+k)\right) + \left(kT(h)y/(h+k)\right),$$

for each y in D. Each of x and T(h)y is a point in D, and D is convex, so G(y) is in D. Let y and z be two points in D. Then

$$|G(y)-G(x)|_X = \left(k/(h+k)\right)|T(h)y-T(h)z|_X$$

$$\leq \left(k/(h+k)\right)|y-z|_X$$

because T is non-expansive. The number $\left(k/(h+k)\right)$ is less than one, hence G is a contraction of a closed, convex subset of a Hilbert space into itself. By the Banach fixed point theorem [5, p.157] G has a fixed point y in D. Thus

$$y = \left(hx/(h+k)\right) + \left(kT(h)y/(h+k)\right).$$

It now follows that

$$hx/(h+k) = y - \left(kT(h)y/(h+k)\right),$$

$$hx = (h+k)y - \left(kT(h)y\right),$$

$$x = y - (k/h)\left(T(h)y-y\right),$$

$$x = (I-kA^h)y.$$

Hence $D \subset R(I-kA^h)$.

Definition III.2.2

Let T, D and X be as in Lemma III.2.1. For each positive number h, let A^h be the function in Definition III.2.1. For each pair of positive numbers h and k, let J(h,k) be the function from D into D defined by $J(h,k)x = (I-kA^h)^{-1}x$ for each x in D. Lemma III.2.1 guarantees the existence of such a function.

Proposition III.2.2

Let T, D, X, h, k, J(h,k) and A^h be as in Definition III.2.2. Let x be a point in D. Then each of the following holds:

(i) $T(h)J(h,k)x - x = (1+h/k)\left(J(h,k)x-x\right)$,

(ii) $-A^h J(h,k)x = (1/k)\left(x-J(h,k)x\right)$, and

(iii) $|J(h,k)x-x|_X \leq (k/h)|x-T(h)x|_X$.

Proof. For simplicity, let $y(h,k) = J(h,k)x = (I-kA^h)^{-1}x$. Then

$$y(h,k) - k\left(\frac{T(h)-I}{h}\right)y(h,k) = (I-kA^h)y(h,k) = x.$$

So that

$$-(k/h)T(h)y(h,k) = x - y(h,k) - (k/h)y(h,k).$$

Multiplication of the last equation by -(h/k) followed by subtraction of x from both sides yields condition (i):

$$T(h)y(h,k) - x = -(h/k)x - x + (h/k)y(h,k) + y(h,k)$$

$$= \{1+(h/k)\}\{y(h,k)-x\}.$$

On the other hand

$$(I-kA^h)y(h,k) = (I-kA^h)(I-kA^h)^{-1}x = x$$

can be rearranged to give

$$-kA^h y(h,k) = x - y(h,k),$$

which, after division by k, is (ii):

$$-A^h y(h,k) = (1/k)\{x-y(h,k)\}.$$

A consequence of condition (i) is that

$$|T(h)y(h,k)-T(h)x+T(h)x-x|_X = |T(h)y(h,k)-x|_X$$

$$= \{1+(h/k)\}|y(h,k)-x|_X.$$

The triangle inequality yields

$$|T(h)y(h,k)-T(h)x|_X + |T(h)x-x|_X \geq \{1+(h/k)\}|y(h,k)-x|_X.$$

Because T is non-expansive, it follows that

$$|y(h,k)-x|_X + |T(h)x-x|_X \geq \big(1+(h/k)\big)|y(h,k)-x|_X.$$

Rearrangement in the last equation gives (iii):

$$|T(h)x-x|_X \geq (h/k)|y(h,k)-x|_X.$$

This completes the proof.

Lemma III.2.2

Let T be a strongly continuous, non-expansive semigroup on a closed, convex subset D of a Hilbert space X. Let J(h,k) be the function in Definition III.2.1 for each pair of positive numbers (h,k). Let x be a given point in D and let ε and k be given positive numbers. Let δ be a positive number such that $|T(h)x-x| \leq \varepsilon$ whenever $0<h\leq\delta$. If n is a positive integer, h is a positive number, and $0<nh\leq\delta$, then

$$|J(h,k)x-J(nh,k)x|_X \leq 2\varepsilon|J(h,k)x-x|.$$

Proof. Since k is fixed, the notation can be simplified by denoting J(h,k)x by y(h) for each positive number h. There will be no loss of generality (and considerably less notation) if we assume that x=0. In what follows, suppose that x=0. Suppose that n is a positive integer, h is a positive number, and $nh\leq\delta$. If m is a positive integer not greater than n, the condition that T be non-expansive implies that

$$|y(h)-T\big((m-1)h\big)y(nh)|_X \geq |T(h)y(h)-T(mh)y(nh)|_X.$$

From Proposition III.2.2, part (i), it follows (with x=0 and k fixed) that

$$|T(h)y(h)-T(mh)y(nh)|_X = \big|\big(1+(h/k)\big)y(h)-T(mh)y(nh)\big|_X.$$

From the preceding equation and the previous inequality, it is clear that

$$|y(h)-T\big((m-1)h\big)y(nh)|_X^2 \geq \big|\big(1-(h/k)\big)y(h)-T(mh)y(nh)\big|_X^2.$$

The right side of this inequality can be expanded (using the inner product) to yield

$$\begin{aligned}
&|y(h)-T\big((m-1)h\big)y(nh)|_X^2 \\
&\quad \geq [y(h)-T(mh)y(nh)+(h/k)y(h),\ y(h)-T(mh)y(n,h)+(h/k)y(h)]_X \\
&\quad = |y(h)-T(mh)y(nh)|_X^2 + 2\mathrm{Re}(h/k)[y(h),y(h)-T(mh)y(nh)]_X + \\
&\qquad\qquad + (h^2/k^2)|y(h)|_X^2 \\
&\quad \geq |y(h)-T(mh)y(nh)|_X^2 + 2\mathrm{Re}(h/k)[y(h),y(h)-T(mh)y(nh)]_X.
\end{aligned}$$

Thus, if m is a positive integer not greater than n,

$$|y(h)-T\big((m-1)h\big)y(nh)|_X^2 - |y(h)-T(mh)y(nh)|_X^2$$

$$\geq 2\mathrm{Re}(h/k)[y(h),y(h)-T(mh)y(nh)]_X.$$

Also, notice that by Proposition III.2.2, part (i),

$$|y(h)-T(nh)y(nh)|_X^2 = |y(h)-\big(1+(nh/k)\big)y(nh)|_X^2.$$

Expanding the right side of this equation with the inner product yields

$$|y(h)-T(nh)y(nh)|_X^2 = |y(h)-y(nh)|_X^2 + 2(nh/k)\mathrm{Re}[y(nh),y(nh)-y(h)]$$

$$+ (n^2h^2/k^2)|y(nh)|_X^2$$

$$\geq |y(h)-y(nh)|_X^2 + 2(nh/k)\mathrm{Re}[y(nh),y(nh)-y(h)]_X,$$

and

$$|y(h)-T(nh)y(nh)|_X^2 - |y(h)-y(nh)|_X^2 \geq 2(nh/k)\mathrm{Re}[y(nh),y(nh)-y(h)]_X$$

Observe that

$$0 = \sum_{m=1}^{n}\left\{|y(h)-T\big((m-1)h\big)y(nh)|_X^2 - |y(h)-T(mh)y(nh)|_X^2\right\}$$

$$+ |y(h)-T(nh)y(nh)|_X^2 - |y(h)-y(nh)|_X^2$$

$$\geq \sum_{m=1}^{n} 2\mathrm{Re}(h/k)[y(h),y(h)-T(mh)y(nh)]_X$$

$$+\ 2(nh/k)\mathrm{Re}[y(nh),y(nh)-y(h)]_X.$$

$$= 2(nh/k)|y(h)|_X^2 - 2(h/k)\sum_{m=1}^{n}\mathrm{Re}[y(h),T(mh)y(nh)]_X$$

$$+\ 2(nh/k)|y(nh)|_X^2 - 2(nh/k)\mathrm{Re}[y(nh),y(h)]_X.$$

Hence

$$2(nh/k)\left(|y(h)|_X^2+|y(nh)|_X^2\right) \leq 2(h/k)\sum_{m=1}^{n}\mathrm{Re}[y(h),T(mh)y(nh)]_X$$

$$+\ 2(nh/k)\mathrm{Re}[y(h),y(nh)]_X.$$

Division by 2(nh/k) yields

$$|y(h)|_X^2 + |y(nh)|_X^2 \leq (1/n)\sum_{m=1}^{n}\mathrm{Re}[y(h),T(mh)y(nh)]_X$$

$$+\ \mathrm{Re}[y(h),y(nh)]_X.$$

By the non-expansive property of T,

$$|T(mh)y(nh)-T(mh)0|_X \geq |y(nh)|_X.$$

Since $mh\leq\delta$ for $1\leq m\leq n$, we have

$$|T(mh)0|_X = |T(mh)0-0|_X \le \varepsilon.$$

The last two inequalities may be combined with the triangle inequality to give

$$|T(mh)y(nh)|_X \le |T(mh)y(nh)-T(mh)0|_X + |T(mh)0|_X$$

$$\le |y(nh)|_X + \varepsilon.$$

We may now expand our bound for $|y(h)|_X^2 + |y(nh)|_X^2$ using the Cauchy-Schwartz inequality, i.e.

$$|y(h)|_X^2 + |y(nh)|_X^2 \le (1/n) \sum_{m=n}^{n} \left(|y(h)|_X \left(|y(nh)|_X + \varepsilon \right) \right) + \mathrm{Re}[y(h),y(nh)]_X.$$

$$= |y(h)|_X |y(nh)|_X + \varepsilon |y(h)|_X + \mathrm{Re}[y(h),y(nh)]_X$$

Now it is useful to recall that

$$2|y(h)|_X |y(nh)|_X \le |y(h)|_X^2 + |y(nh)|_X^2.$$

Thus

$$|y(h)|_X^2 + |y(nh)|_X^2 \le \frac{1}{2} |y(h)|_X^2 + \frac{1}{2} |y(nh)|_X^2 + \varepsilon |y(h)|_X + \mathrm{Re}[y(h),y(nh)]_X,$$

and

$$\frac{1}{2}\,|y(h)|_X^2 - \mathrm{Re}[y(h),y(nh)] + \frac{1}{2}\,|y(nh)|_X^2 \le \varepsilon|y(h)|_X$$

$$\frac{1}{2}\,|y(h)-y(nh)|_X^2 \le \varepsilon|y(h)|_X.$$

Multiplication of the last inequality by two gives the desired result.

Lemma III.2.3

Let T, D, X, x, k, ε and δ be as in the hypothesis of Lemma III.2.2. Let J(h,k) be the function in Definition III.2.2 for every pair (h,k) of positive numbers. Then

$$|J(h,k)x-x|_X \le 2\varepsilon(1+4k/\delta)$$

for each h in (0,δ].

Proof. As in the proof of Lemma III.2.2, assume that x=0, and write y(h) for y(h,k). If s is in [δ/2,δ], then by Proposition III.2.2, part (iii),

$$\begin{aligned}|y(s)-0|_X &= |J(s,k)0-0|_X \\ &\le (k/s)|T(s)0-0|_X \\ &\le (2k/\delta)|T(s)0|_X \\ &\le 2h\varepsilon/\delta.\end{aligned}$$

If h is a number in $(0,\delta]$, then there is an integer n such that $s=nh$ is in $[\delta/2,\delta]$. Then by Lemma III.2.2,

$$|y(h)-y(nh)|_X^2 \le 2\varepsilon|y(h)|_X$$

$$\le 2\varepsilon|y(h)-y(nh)|_X + 2\varepsilon|y(nh)|_X.$$

But

$$|y(nh)|_X \le 2k\varepsilon/\delta$$

so

$$|y(h)-y(nh)|_X^2 \le 2\varepsilon|y(h)-y(nh)|_X + 4k\varepsilon^2/\delta,$$

$$|y(h)-y(nh)|_X^2 - 2\varepsilon|y(h)-y(nh)|_X + \varepsilon^2 \le \varepsilon^2(1+4k/\delta),$$

and

$$|y(h)-y(nh)|_X - \varepsilon \le r\sqrt{(1 + 4k/\delta)}$$

Thus

$$|y(h)-y(nh)|_X \le \varepsilon\left(1+(1+4k/\delta)^{1/2}\right),$$

We can now estimate the size of $|y(h)|$ by

$$|y(h)|_X \le |y(h)-y(nh)|_X + |y(nh)|_X$$

$$\le \varepsilon\left(1+(1+4k/\delta)^{1/2}+2k/\delta\right)$$

$$\le \varepsilon(2+6k/\delta)$$

and the lemma is proved.

Lemma III.2.4

Let T, D, X, x, k, ε and δ be as in the hypotheses of Lemma III.2.2. Let J(h,k) be the function in Definition III.2.2, and for each positive number h, let y(h) = J(h,k)x. Then $\lim_{h\to 0^+} y(h)$ exists and is a point in D.

Proof. By Lemma III.2.3, y(h) is bounded for h in $(0,\delta]$. Suppose that $|y(h)-x|_X \in M$ for h in $(0,\delta]$. Let s and t be two numbers in $(0,\delta]$ such that s/t is rational. Then there are integers m and n and a positive number h in $(0,\delta]$ such that s=mh and t=nh. By Lemma III.2.2,

$$|y(h)-y(s)|_X \le \left(2\varepsilon|y(h)-x|_X\right)^{1/2} \le (2\varepsilon M)^{1/2}$$

and

$$|y(h)-y(t)|_X \le \left(2\varepsilon|y(h)-x|_X\right)^{1/2} \le (2\varepsilon M)^{1/2}.$$

Thus

$$|y(s)-y(t)|_X \le 2(2\varepsilon M)^{1/2}.$$

It follows that $\{y_p\}_{p=1}^{\infty}$ given by $y_p = y(2^{-p})$ is a Cauchy sequence, hence $\lim_{p\to\infty} y_p$ exists. By Lemma III.2.1, y(h) is continuous for h in $(0,\delta]$, thus $\lim_{h\to 0^+} y(h) = \lim_{p\to\infty} y_p$. Since each member of y_p is in D and D is closed, it follows that $\lim_{h\to 0^+} y(h)$ is in D.

<u>Lemma III.2.5</u>

Let T, D, X and x be as in Lemma III.2.2. Let J(h,k) be the function in Definition III.2.2. For each positive number k, let $z(k) = \lim_{h\to 0^+} J(h,k)x$. Then $\lim_{k\to 0^+} z(k) = x$.

Proof. Let $\varepsilon>0$ be given. By Lemma III.2.3, there is a number $\delta>0$ such that if $0<h\le\delta$, then $|J(h,k)x-x|_X \le 2\varepsilon(1+4k/\delta)$ for every positive number k. It follows from the continuity of the norm that $|z(k)-x|_X \le 2\varepsilon(1+4k/\delta)$ for every positive number k. Thus $|z(k)-x|_X \le 4\varepsilon$ whenever $0<k\le\delta/4$. It follows that $\lim_{k\to 0} z(k) = x$.

<u>Lemma III.2.6</u>

Let T, D, X and x be as in Lemma III.2.2. Let A^h be the function in Definition III.2.1. Let z(k) be as in Lemma III.2.5. If each of h and k is a positive number, then

$$|A^h z(k)|_X \le (1/k)|z(k)-x|_X.$$

Proof. As before, let $y(h,k) = J(h,k)x$. If each of h and k is a positive number and n is a positive integer, then

$$h|A^h y(h/n,k)|_X = |\{T(h)-I\}y(h/n,k)|_X$$

$$\leq \sum_{j=1}^{n} |\{T(jh/n)-T((j-1)h/n)\}y(h/n,k)|_X$$

$$\leq n|(T(h/n)-I)y(h/n,k)|_X.$$

The last inequality follows from the condition that T be non-expansive. Recall that

$$A^{h/n}y(h/n,k) = (n/h)(T(h/n)y(h/n,k)-y(h/n,k)),$$

therefore

$$h|A^{h/n}y(h/n,k)| = n|\{T(h/n)-I\}y(h/n,k)|.$$

It now follows that

$$|A^h y(h/n,k)|_X \leq |A^{h/n}y(h/n,k)|_X.$$

Because T is non-expansive, A^h is continuous and

$$\lim_{n\to\infty}|A^h y(h/n,k)|_X = |A^h z(k)|_X.$$

By Proposition II.2.2, part (ii),

$$A^{h/n}y(h/n,k) = A^{h/n}J(h/n,k)x$$

$$= (1/k)\left(J(h/n,k)x-x\right)$$

$$= (1/k)\left(y(h/n,k)-x\right).$$

Thus

$$|A^{h/n}y(h/n,k)|_X = (1/k)|y(h/n,k)-x|_X.$$

and

$$\lim_{n\to\infty}|A^{h/n}y(h/n,k)|_X = (1/k)|z(k)-x|_X.$$

It now follows that

$$|A^h z(k)|_X = \lim_{n\to\infty}|A^h y(h/n,k)|_X$$

$$\geq \lim_{n\to\infty}|A^{h/n}y(h/n,k)|_X$$

$$= (1/k)|z(k)-x|_X,$$

and the proof is complete.

<u>Theorem III.1.1</u> [2]

Let T be a strongly continuous, non-expansive semigroup on a closed, convex subset D of a Hilbert space X. Then there is a subset C of D which is dense in D with the property that $\lim_{h\to 0^+} (1/h)\big(T(h)x-x\big)$ exists for each x in C.

Proof. Let each of s and k be positive numbers. Then for each positive number h,

$$|T(s+h)z(k)-T(s)z(k)|_X \le |T(h)z(k)-z(k)|$$

$$= h|A^h z(k)|_X.$$

By Lemma III.2.6,

$$h|A^h z(k)|_X \le h(1/k)|z(k)-x|_X.$$

Thus if s and t are positive numbers, and $|s-t| < \delta$,

$$|T(t)z(k)-T(s)z(k)|_X \le |t-s|(1/k)|z(k)-x|_X.$$

It follows that T(t)z(k) is Lipschitz in t on $(0,\infty)$; hence

$$\frac{d}{dt}\big(T(t)z(k)\big) = \lim_{h\to 0}(1/h)\big(T(t+h)z(k)-T(t)z(k)\big)$$

exists almost everywhere in t on $(0,\infty)$. Let C be the subset of D to which u belongs only in case there exists

$$\lim_{h\to 0^+}(1/h)\big(T(h)u-u\big).$$

Then $T(t)z(k)$ is in C for each $t>0$ and each $k>0$. Recall that $z(k) = \lim_{t\to 0^+} T(t)z(k)$, so $z(k)$ is in the closure of C for each positive number k. By Lemma III.2.5, $\lim_{k\to 0^+} z(k) = x$, therefore x is in the closure of C. Since x was an arbitrary point in D, the closure of C is all of D.

Definition III.2.3

Let X be a Hilbert space and let A be a maximal dissipative subset of $X\times X$. By Proposition II.4.2, Ax is closed and convex for each x in $D(A)$, hence Ax has a unique element of minimum norm. Let A^0 be the subset of $X\times X$ such that $D(A^0) = D(A)$, and A^0x is the element of Ax of minimum norm for each x in $D(A)$. The function A^0 is the canonical restriction of A.

Remark III.2.1

Theorem III.2.1 has been proved following the techniques of T. Kato [2]. We now state (without proof) a stronger theorem which forms half of the nonlinear Hille-Yosida theorem.

Theorem III.2.2 [7, Theorem 3, p.385]

Let T be a strongly continuous, non-expansive semigroup on a subset D of a Hilbert space X. Suppose that T is maximal in the sense that T is not a restriction of a larger semigroup. Then D is closed and convex. Furthermore the function A defined by

$$Ax = \lim_{h\to 0^+} (1/h)\left(T(h)x-x\right)$$

for x in some dense subset of D is a maximal member of the class of canonical restrictions of maximal dissipative subsets of $X\times X$.

Definition III.2.4

Let T be a semigroup on a subset D of a Hilbert space X. Let C be the subset of D to which x belongs only in case $\lim_{h \to 0^+} (1/h)\big(T(h)x - x\big)$ exists. Let A be the function from C to X defined by

$$Ax = \lim_{h \to 0^+} (1/h)\big(T(h)x - x\big).$$

Then A is called the generator of T.

Remark III.2.2

We now state (without proof) a weak form of the second half of the nonlinear Hille-Yosida theorem. A strong form of this theorem appears in [7,p.401].

Theorem III.3.1 [3, Theorem I, p.394]

Let X be a Hilbert space and let A be a maximal dissipative (hence hyper-dissipative) subset of X×X. Let A^0 be the canonical restriction of A. Then there is a unique semigroup T on D(A) with A^0 as the generator of T.

CHAPTER IV

THE ABSTRACT CAUCHY PROBLEM

The basic application of semigroup theory is the study of differential equations, particularly nonlinear partial differential equations. This chapter defines the abstract Cauchy problem and gives an example of a nonlinear partial differential equation which has a semigroup solution generated by a dissipative set.

Example IV.1.1 [8, p.530; 9]

Let X be the Hilbert space $L^2[0,1]$ of real-valued, measurable, square-summable functions on [0,1]. Let D be the subset of X to which u belongs only in case u has a representative x in X such that $x(0)=0$, $|x(s)-x(t)| \le |s-t|$ whenever s and t are in [0,1], and x is non-decreasing. D is a closed convex subset of X. Let A be a subset of $D \times X$ defined as follows:

For each x in D, Ax is the set of representatives of $-x\,x'$. Note that if x is in D, then x is absolutely continuous, hence x' exists almost everywhere in [0,1], and $0 \le x'(s) \le 1$ for almost all s in [0,1]. Also, $1 \ge x(1) = \int_0^1 x'(s)ds \ge \int_0^1 \left(x'(s)\right)^2 ds$, so x' is in X. We now wish to show that A is dissipative. To obtain this result, let x and y be in D and let u and v be representatives of Ax and Ay, respectively. Then

$$[u-v,x-y]_X = \int_0^1 \left(-x(s)x'(s)+y(s)y'(s)\right)\left(x(s)-y(s)\right)ds$$

$$= -\frac{1}{2}\int_0^1 \left(x^2(s)-y^2(s)\right)'\left(x(s)-y(s)\right)ds$$

$$= -\frac{1}{2}\int_0^1 \left(x'(s)+y'(s)\right)\left(x(s)-y(s)\right)^2 ds$$

$$-\frac{1}{2}\int_0^1 \left(x'(s)-y'(s)\right)\left(x^2(s)-y^2(s)\right)ds.$$

Also,

$$\lfloor u-v,x-y\rfloor_X = \int_0^1 \left(-x(s)x'(s)+y(s)y'(s)\right)\left(x(s)-y(s)\right)ds$$

$$= -\frac{1}{2}\int_0^1 \left(x^2(s)-y^2(s)\right)'\left(x(s)-y(s)\right)ds$$

$$= -\frac{1}{2}\int_0^1 \left[\left(x^2(s)-y^2(s)\right)\left(x(s)-y(s)\right)\right]'ds$$

$$+\frac{1}{2}\int_0^1 \left(x^2(s)-y^2(s)\right)\left(x(s)-y(s)\right)'ds.$$

Adding these two equations gives

$$2[u-v,x-y]_X = -\frac{1}{2}\int_0^1 \left(x'(s)+y'(s)\right)\left(x(s)-y(s)\right)^2 ds$$

$$-\frac{1}{2}\int_0^1 \left[\left(x^2(s)-y^2(s)\right)\left(x(s)-y(s)\right)\right]'ds$$

$$= -\frac{1}{2}\int_0^1 \left(x'(s)+y'(s)\right)\left(x(s)-y(s)\right)^2 ds$$

$$-\frac{1}{2}\left[\left(x(s)-y(s)\right)^2\left(x(s)+y(s)\right)\right]_0^1.$$

There are sufficient conditions on the members of D to insure that each of

$$\int_0^1 \left(x'(s)+y'(s)\right)\left(x(s)-y(s)\right)^2 ds$$

and

$$\left(x(1)-y(1)\right)^2\left(x(1)+y(1)\right)$$

is non-negative, therefore A is dissipative.

Statement of the Problem

Let X be a Hilbert space, let D be a subset of X, and let A be a subset of $D \times X$ with $D(A) = D$. We wish to find a function y with domain $[0,\infty) \times D$ and range in D such that

(a) for each x in D, $y(t,x)$ is strongly absolutely continuous on finite intervals of $[0,\infty)$.

(b) for each x in D, $y(0,x) = x$, and

(c) for each x in D, $(d/dt)y(t,x)$ is in $Ay(t,x)$ for almost all t in $[0,\infty)$.

If p is a point in D and A is a function, the abstract Cauchy problem takes on the appearance of an initial value problem, i.e. the related Cauchy problem is

(a) $y(t)$ is a function from $[0,\infty)$ into D which is strongly abs. cont. on finite intervals of $[0,\infty)$,

(b) $(d/dt)y \in Ay$ a.e. in $[0,\infty)$,

(c) $y(0) = p$.

The following example will illustrate this connection.

Example IV.2.1

Let A, D, and X be as in Example IV.1.1. We wish to find a real-valued function y(,) defined on $[0,\infty) \times [0,1]$ which satisfies the initial-value problem

$$\left.\begin{aligned} &\frac{\partial y(t,s)}{\partial t} + y(t,s)\,\frac{\partial y(t,s)}{\partial s} = 0, \\ &y(0,s) = x(s) \quad \text{for each } s \text{ in } [0,t] \\ &\text{for some } x \text{ in } D. \end{aligned}\right\} \qquad (4)$$

The initial value problem can be written as the abstract Cauchy problem

$$\left.\begin{aligned} &\frac{du(t)}{dt} = Au(t) \\ &u(0) = x \text{ in } D. \end{aligned}\right\} \qquad (5)$$

The proof of the existence of solutions to the abstract Cauchy problem (5) is rather complicated and requires results stronger than those presented in Chapters II and III.

Condition (R). A dissipative subset B of X×X satisfies condition (R) if $R(I-\lambda B) \supset D(B)$ for each positive number λ. This definition is given by S. Oharu in [8, p.527]. J. R. Dorroh [9, Example 4.10, p.453] has shown that the operator A of this example satisfies condition (R).

Demiclosed Sets. A subset B of X×X is demi-closed if, whenever each of $\{x_p\}_{p=1}^{\infty}$ and $\{y_p\}_{p=1}^{\infty}$ is a sequence of points in X with strong limit x and weak limit y, respectively, and (x_p, y_p) is a member of B for each positive integer p, it follows that (x,y) is a member of B. It has been shown by S. Oharu [8, p.530] that the operator A of this example is demi-closed.

A Theorem of Oharu. S. Oharu [8, Theorem 6.1, p. 544] has shown that if A is a single-valued, demi-closed, dissipative operator satisfying condition (R), then A is the infinitesimal generator of a unique non-expansive semigroup T on D(A) such that for each x in D(A), T(t)x is weakly continuously differentiable on [0,∞) and

$$T(t)x = x + \int_0^1 AT(\tau)x \, d\tau \qquad \text{for } t \geq 0 \; ;$$

this theorem guarantees the existence of solutions for (5) and hence for (4).

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