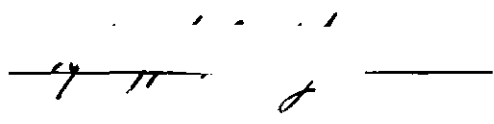


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A handwritten signature, possibly "J. H. ...", is written across two horizontal lines. The signature is in dark ink and appears to be a cursive or semi-cursive style.

7/25/68

THE BEHAVIOR OF SOLUTIONS OF STIELTJES INTEGRAL EQUATIONS

A THESIS

Presented to

The Faculty of the Division of Graduate

Studies and Research

by

David Lowell Lovelady

In Partial Fulfillment
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February, 1971

THE BEHAVIOR OF SOLUTIONS OF STIELTJES INTEGRAL EQUATIONS

Approved:

[Handwritten initials]

Chairman

[Handwritten signature]

Date approved by Chairman:

April 5, 1971

To my friend,

With a little help from whom

I get by.

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CHAPTER I

INTRODUCTION

In 1954, H. S. Wall [31] investigated equations of the form

$$h(t) = p + \int_t^0 (dF)h \quad (\text{LE})$$

where F is a continuous square-matrix-valued function on $[0, \infty)$ which is of bounded variation on each bounded interval of $[0, \infty)$, p is a member of the matrix ring which contains the values of F , and the integral is the ordinary Riemann-Stieltjes integral. J. S. MacNerney [15,16] took this development further by allowing the functions to have their values in a complete normed ring with identity. A nonlinear analogy was presented by J. W. Neuberger [23] who studied

$$h(t) = p + \int_t^0 dF[h] \quad (\text{NLE})$$

where each value of F is a nonlinear function from a Banach space into itself, and continuity conditions are imposed on F so as to make solutions continuous. Discontinuities were introduced in the linear equation (LE) by MacNerney [17] who simply required F to be of bounded variation on each bounded interval, and used the Cauchy right integral. In [19] this work was extended to allow for discontinuities in the solutions of the nonlinear equation (NLE). The equation (NLE), taken in the

context of [19], shall be referred to as the nonlinear Cauchy-Stieltjes integral equation, or, for brevity, the C-S equation.

The C-S equation was extended to a more general nonlinear integral operation by J. V. Herod in [7]. Also, Herod [8,10] studied the problem of coalescence of solutions of the C-S equation, and R. H. Martin, Jr. [20,21] obtained numerical bounds for solutions of the C-S equation.

In this work we shall continue the analysis of the nonlinear Cauchy-Stieltjes integral equation. In Chapter III, algebraic operations will be described with respect to which the integrator class and the solution class become isomorphic algebraic semigroups. Under fairly lenient hypotheses, the algebraic operations will show how to solve perturbed equations in terms of the solutions of unperturbed equations. There will be introduced in Chapter IV a product integral with the aid of which we shall derive a variation-of-parameters formula for nonlinear systems and shall solve a coalescence problem. We shall, in Chapter V, extend the work of Martin on bounds. Bounds will be obtained for solutions of equations subject to integrator perturbations and to forcing function perturbations.

To achieve a measure of self-containment, we have included, in Chapter II, many known results due to MacNerney, Herod, and Martin, but we have not included the proofs. In few, if any, cases will these proofs be essential to the present discussion, and each cited result is fully referenced. In the last section of Chapter II there is a discussion designed to point up the relation between our subject and more widely studied problems of differential equations with interface conditions.

CHAPTER II

THE GENERAL THEORY

SECTION II.1: Definitions
and the Fundamental Correspondence

Let S be the set of all nonnegative real numbers, and let Y be a Banach space with norm N_1 (in none of what follows will it matter whether Y is a Banach space over the real field or a Banach space over the complex field). Let H be the set to which A belongs only in case A is a function from Y to Y , $A[0] = 0$, and there is a number b such that $N_1[A[p] - A[q]] \leq bN_1[p-q]$ whenever (p,q) is in $Y \times Y$. If A is in H , let $N_2[A]$ be the least number b such that $N_1[A[p] - A[q]] \leq bN_1[p-q]$ whenever (p,q) is in $Y \times Y$, and let $N_3[A]$ be the least number b such that $N_1[A[p]] \leq bN_1[p]$ whenever p is in Y . Note that if A is in H and is linear, then $N_2[A] = N_3[A]$. If each of A and B is in H , then AB will denote that member C of H having the property that if p is in Y then $C[p] = A[B[p]]$.

If h is a function from S to a metric space, we define $h(0-) = h(0)$, $h(0+) = \lim_{t \rightarrow 0} h(t)$, and if $t > 0$, then $h(t-)$ and $h(t+)$ will denote the usual left- and right-hand limits, respectively. Let QC be the set to which f belongs only in case f is a function from S to Y , and f is quasicontinuous, i.e., each of $f(t-)$ and $f(t+)$ exists whenever t is in S . Let BV be the set to which f belongs only in case f is a function from S to Y and f has bounded variation on each bounded interval of S . Note that BV is a subset of QC .

If (a,b) is in $S \times S$, t will be called a chain from a to b only in case t (also denoted $(t_k)_{k=0}^n$) is a monotone sequence into S and $t_0 = a$ and $t_n = b$. If each of s and t is a chain from a to b , s will be called a refinement of t only in case t is a subsequence of s . If h is a function from $S \times S$ to H , and (a,b) is in $S \times S$, and p is in Y , then by $\prod_a^b h[p]$ and $\sum_a^b h[p]$ we mean the limits, in the sense of successive refinements of chains t , of members of Y of the forms $\prod_{k=1}^n h(t_{k-1}, t_k)[p]$ (where $\prod_{k=1}^n h(t_{k-1}, t_k) = h(t_0, t_1)h(t_1, t_2) \cdots h(t_{n-1}, t_n)$) and $\sum_{k=1}^n h(t_{k-1}, t_k)[p]$, respectively. If h is a real-valued function on $S \times S$, and (a,b) is in $S \times S$, we define $\prod_a^b h$ and $\sum_a^b h$ analogously. If (x,y,z) is in $S \times S \times S$, we will refer to y as being between x and z only in case $|x-y| + |y-z| = |x-z|$.

Let OA^+ be the set to which α belongs only in case α is a function from $S \times S$ to S and $\alpha(x,y) + \alpha(y,z) = \alpha(x,z)$ whenever (x,y,z) is in $S \times S \times S$ and y is between x and z . Let OM^+ be the set to which μ belongs only in case μ is a function from $S \times S$ to $[1, \infty)$ and $\mu(x,y)\mu(y,z) = \mu(x,z)$ whenever (x,y,z) is in $S \times S \times S$ and y is between x and z . Let OA be the set to which V belongs only in case V is a function from $S \times S$ to H such that

$$(OA1) \quad V(x,y) + V(y,z) = V(x,z) \text{ whenever } (x,y,z) \text{ is } S \times S \times S$$

and y is between x and z , and

$$(OA2) \quad \text{there is } \alpha \text{ in } OA^+ \text{ such that } N_2[V(a,b)] \leq \alpha(a,b)$$

whenever (a,b) is in $S \times S$.

If α and V are related as in (OA2), α will be said to dominate V . Let OM be the set to which W belongs only in case W is a function from $S \times S$ to H such that

(OM1) $W(x,y)W(y,z) = W(x,z)$ whenever (x,y,z) is in $S \times S \times S$ and y is between x and z , and

(OM2) there is μ in OM^+ such that $N_2[W(a,b) - I] \leq \mu(a,b) - 1$ whenever (a,b) is in $S \times S$, where I in H is given by $I[p] = p$.

If μ and W are related as in (OM2), μ will be said to dominate W .

The following three theorems are due to MacNerney, and the latter two delineate what MacNerney calls the fundamental correspondences between OA^+ and OM^+ and between OA and OM .

THEOREM II.1.1 ([17, Lemmas 2.1 and 2.2], [19, Theorem 1.1])

Let α be in OA^+ , μ be in OM^+ , V be in OA , W be in OM , (a,b) be in $S \times S$, and p be in Y . Then each of ${}_a\Pi^b[1+\alpha]$, ${}_a\sum^b[\mu-1]$, ${}_a\Pi^b[I+V][p]$, and ${}_a\sum^b[W-I][p]$ exists.

THEOREM II.1.2 ([17, Theorem 2.2])

There is a bijection E^+ from OA^+ onto OM^+ such that if α is in OA^+ and μ is in OM^+ , then (i), (ii), and (iii) are equivalent.

- (i) $\mu = E^+[\alpha]$.
- (ii) $\mu(a,b) = {}_a\Pi^b[1+\alpha]$ whenever (a,b) is in $S \times S$.
- (iii) $\alpha(a,b) = {}_a\sum^b[\mu-1]$ whenever (a,b) is in $S \times S$.

THEOREM II.1.3 ([19, Theorem 1.1])

There is a bijection E from OA onto OM such that if V is in OA and W is in OM , then (i), (ii), (iii), and (iv) are equivalent.

- (i) $W = E[V]$.

(ii) $W(a,b)|p] = \int_a^b [I+V][p]$ whenever (a,b,p) is in $S \times S \times Y$.

(iii) $V(a,b)[p] = \int_a^b [W-I][p]$ whenever (a,b,p) is in $S \times S \times Y$.

(iv) There is a member (α, μ) of E^+ such that $N_3[W(a,b)-I-V(a,b)] \leq \mu(a,b) - 1 - \alpha(a,b)$ whenever (a,b) is in $S \times S$.

SECTION II.2: Integral Equations

If h is a function from $S \times S$ to Y , and (a,b) is in $S \times S$, we define $\int_a^b h$ in the obvious way. If V is in OA , g is in QC , and (a,b) is in $S \times S$, then we define $(R) \int_a^b V[g]$ to be $\int_a^b h$, where h is given by $h(s,t) = V(s,t)[g(t)]$. This integration process yields what is called the right Cauchy-Stieltjes integral. The following theorem is due to MacNerney.

THEOREM II.2.1 ([19, Theorem 2])

Let V be in OA with $W = E[V]$, let (a,p) be in $S \times Y$, and let h be in QC . Then (i) and (ii) are equivalent.

(i) $h(t) = p + (R) \int_t^a V[h]$ whenever t is in S .

(ii) $h(t) = W(t,a)[p]$ whenever t is in S .

Furthermore, if (ii) holds, then h is in BV .

THEOREM II.2.2

Let V be in OA , let a be in S , and let f be in QC . Then there is exactly one member h of QC such that

$$h(t) = f(t) + (R) \int_t^a V[h]$$

whenever t is in S . Furthermore, h is in BV only in case f is in BV .

REMARK II.2.1: Although Theorem II.2.2 appears not to have been stated in this form by MacNerney, it follows immediately from an iterative scheme analogous to that cited in [18].

SECTION II.3: Coalescence of Solutions

If V is in OA , p is in Y , (a,b) is in $S \times S$, and $a \neq b$, then Theorem II.2.1 does not make it clear whether there exists a member h of QC such that $h(b) = p$ and such that

$$h(t) = h(a) + (R) \int_t^a V[h]$$

whenever t is in S . When does there exist such an h , when is h unique, and when does $h(a)$ depend, in a Lipschitz-continuous fashion, on p ? Since $h(b) = W(b,a)[h(a)]$ (where $W = E[V]$), it is clear that the following theorem of Herod deals with these questions.

THEOREM II.3.1 ([10], see also [8])

If (V,W) is in E , then (i) and (ii) are equivalent.

(i) Whenever a is in S , each of $I+V(a,a+)$, $I+V(a,a-)$, $I+V(a+,a)$, and $I+V(a-,a)$ has inverse in H .

(ii) Whenever (a,b) is in $S \times S$, $W(a,b)$ has inverse in H .

In fact, Herod has shown more. Let OAI be that subset of OA to which V belongs only in case V satisfies (i) of Theorem II.3.1.

THEOREM II.3.2 ([10], see also [8])

There is a bijection G from OAI onto OAI such that if V is in OAI , then each of (i), (ii), (iii), and (iv) is true.

$$(i) \quad G[G[V]] = V.$$

(ii) $G[V](a,b) = -V(b,a)$ whenever (a,b) is in $S \times S$ only in case $\int_a^b N_3[V[I-V]-V] = 0$ whenever (a,b) is in $S \times S$.

(iii) $E[G[V]](a,b)E[V](b,a) = E[V](b,a)E[G[V]](a,b) = I$ whenever (a,b) is in $S \times S$.

(iv) $G[V](a,b)[p] = - \int_b^a V[I+V]^{-1}[p]$ whenever (a,b,p) is in $S \times S \times Y$.

SECTION II.4: Bounds for Solutions

Since Y is a Banach space, the set of all real numbers can be thought of as a subset of H . Let OAR and OMR denote the subsets of OA and OM, respectively, consisting of real-valued functions. Martin has proved the following two theorems.

THEOREM II.4.1 ([21, Lemma 3.3, Theorem 3.1], see also [20])

Let V be in OA. If (a,b) is in $S \times S$, then $\int_a^b (N_3[I+V] - 1)$ exists, and if γ is given on $S \times S$ by $\gamma(a,b) = \int_a^b (N_3[I+V] - 1)$, then γ is in OAR.

THEOREM II.4.2 ([21, Theorem 3.1], see also [20])

Let V be in OA, and let γ in OAR be given by $\gamma(a,b) = \int_a^b (N_3[I+V] - 1)$. Let $W = E[V]$, and let $\lambda = E[\gamma]$. Then $N_3[W(a,b)] \leq \lambda(a,b)$ whenever (a,b) is in $S \times S$, and λ is the least member of OMR for which this is so, i.e., $\lambda(a,b) = \int_a^b N_3[W]$ whenever (a,b) is in $S \times S$.

SECTION II.5: Differential
Equations with Interface Conditions

We shall describe a basic interface problem and show how it can be solved with the theory of nonlinear Cauchy-Stieltjes integral equations. Let A be an N_2 -continuous function from S to H , and let K be a countable subset of S such that K does not contain 0 . Let each of B and C be a function from K to H such that if M is a bounded subset of K , then each of $\sum_{t \in M} N_2[B(t) - I]$ and $\sum_{t \in M} N_2[C(t) - I]$ is finite. Let q be in Y . The objective of this interface problem is to find a member h of QC such that if t is in K , then $h(t) = B(t)[h(t-)]$ and $h(t+) = C(t)[h(t)]$, and such that if N is an open connected set in S which does not intersect K then h is continuously differentiable on N and $h'(t) = A(t)[h(t)]$ whenever t is in N , and such that $h(0) = q$.

Let U_1 and U_2 be members of OA such that if $0 \leq a \leq b$ and p is in Y then $U_1(b,a)[p] = \sum_{a < t \leq b} [B(t)-I][p]$ and $U_2(b,a)[p] = \sum_{a \leq t < b} [C(t)-I][p]$. Now if t is in K then $U_1(t,t-) = B(t)$ and $U_2(t+,t) = C(t)$. Let V be a member of OA such that if $0 \leq a \leq b$ and p is in Y , then $V(b,a)[p] = \int_a^b A(s)[p]ds + U_1(b,a)[p] + U_2(b,a)[p]$. Let $W = E[V]$, and let h be given by $h(t) = W(t,0)[q]$. Now h fulfills the requirements of our interface problem.

The question of uniqueness of solutions merits some discussion here. The integral equation we constructed has, of course, a unique solution, but the lack of uniqueness of interface solutions arises since it need not be the case that every solution to the interface problem also solves our integral equation. In particular, suppose (a,b) is an open interval in the closure of K , and D is an N_2 -continuous function

from S to H such that $D(t) = 0$ whenever t is outside (a,b) . Then $E[Z](\cdot, 0)[p]$ also solves our interface problem, where Z in OA is given by

$$Z(d,c)[p] = \int_c^d A(s)[p]ds + \int_c^d D(s)[p]ds \\ + U_1(d,c)[p] + U_2(d,c)[p].$$

Several authors have studied differential interface systems, and the connection with Cauchy-Stieltjes integral equations has long been recognized. In particular, we refer the reader to the works of M. Fréchet [5], T. H. Hildebrandt [11], W. H. Ingram [12], G. B. Price [24], W. T. Reid [25], W. C. Sangren [26], H. Schärf [27], and F. W. Stallard [28,29]. For a more complete treatment of differential interface systems, we refer to F. V. Atkinson [1, Chapter 11, Section 8] and the references cited there.

CHAPTER III

ALGEBRAIC STRUCTURE

SECTION III.1: The θ OperationLEMMA III.1.1

If each of α and β is in OA^+ , and (a,b) is in $S \times S$, then $\sum_a^b \alpha[1+\beta]$ exists and is the greatest lower bound of the set to which r belongs only in case there is a chain $(t_k)_{k=0}^n$ from a to b such that $r = \sum_{k=1}^n \alpha(t_{k-1}, t_k)[1+\beta(t_{k-1}, t_k)]$.

PROOF: Let (a,b,c) be in $S \times S \times S$ with b between a and c . Now $\alpha(a,c) \geq \alpha(a,b)$ and $\alpha(a,c) \geq \alpha(b,c)$, so

$$\begin{aligned} \alpha(a,c)\beta(a,c) &= \alpha(a,c)\beta(a,b) + \alpha(a,c)\beta(b,c) \\ &\geq \alpha(a,b)\beta(a,b) + \alpha(b,c)\beta(b,c), \end{aligned}$$

and

$$\alpha(a,c)[1+\beta(a,c)] \geq \alpha(a,b)[1+\beta(a,b)] + \alpha(b,c)[1+\beta(b,c)].$$

It is now clear that if (a,b) is in $S \times S$, each of s and t is a chain from a to b , and s refines t , then

$$\alpha(a,b) \leq \sum_s \alpha[1+\beta] \leq \sum_t \alpha[1+\beta].$$

This completes the proof.

THEOREM III.1.1

If each of V_1 and V_2 is on OA, and (a,b,p) is in $S \times S \times Y$, then $\int_a^b V_1[I+V_2][p]$ exists. If, for $i=1$ or $i=2$, α_i dominates V_i , then

$$\begin{aligned} N_3[V_1(a,b)[I+V_2(a,b)] - \int_a^b V_1[-+V_2]] \\ \leq \alpha_1(a,b)[1+\alpha_2(a,b)] - \int_a^b \alpha_1[1+\alpha_2], \end{aligned}$$

whenever (a,b) is in $S \times S$. Furthermore, if U is given by $U(a,b)[p] = \int_a^b V_1[I+V_2][p]$, then U is in OA.

PROOF: Let (a,b,c,p) be in $S \times S \times S \times Y$, with b between a and c . Now

$$\begin{aligned} N_1[V_1(a,c)[I+V_2(a,c)][p] - V_1(a,b)[I+V_2(a,b)][p] \\ - V_1(b,c)[I+V_2(b,c)][p] \\ = N_1[V_1(a,b)[I+V_2(a,c)][p] - V_1(a,b)[I+V_2(a,b)][p] \\ + V_1(b,c)[I+V_2(a,c)][p] - V_1(b,c)[I+V_2(b,c)][p] \\ \leq N_1[p][\alpha_1(a,b)\alpha_2(b,c) + \alpha_1(b,c)\alpha_2(a,b)] \\ = N_1[p][\alpha_1(a,c)[1+\alpha_2(a,c)] - \alpha_1(a,b)[1+\alpha_2(a,b)] \\ - \alpha_1(b,c)[1+\alpha_2(b,c)]]. \end{aligned}$$

Consequently, if (a,b,p) is in $S \times S \times Y$, if each of s and t is a chain from a to b , and if s refines t , then

$$\begin{aligned}
& N_1[\sum_t V_1[I+V_2][p] - \sum_s V_1[I+V_2][p]] \\
& \leq N_1[p][\sum_t \alpha_1[1+\alpha_2] - \sum_s \alpha_1[1+\alpha_2]].
\end{aligned}$$

It is now clear that $\sum_a^b V_1[I+V_2][p]$ exists and that the inequality of the conclusion holds. Let (a,b,p,q) be in $S \times S \times Y \times Y$. Now

$$N_1[\sum_a^b V_1[I+V_2][p] - \sum_a^b V_1[I+V_2][q]] \leq N_1[p-q] \sum_a^b \alpha_1[1+\alpha_2],$$

and the proof is complete.

DEFINITION III.1.1: If each of V_1 and V_2 is in OA, then $V_1 \oplus V_2$ will be that member U of OA given by

$$U(a,b)[p] = V_2(a,b)[p] + \sum_a^b V_1[I+V_2][p].$$

If V is in OA, then V^* will be that member of OA given by $V^*(a,b) = V(b,a)$.

Our next two theorems will be concerned with the \oplus operation. Of particular importance here will be the discovery of necessary and sufficient conditions for \oplus to reduce to ordinary addition. Also of interest is the fact that \oplus turns out to be associative.

THEOREM III.1.2

If each of V_1 , V_2 , and V_3 is in OA, then $V_1 \oplus (V_2 \oplus V_3) = (V_1 \oplus V_2) \oplus V_3$, and consequently (OA, \oplus) is a semigroup. (OAI, \oplus) is a subgroup of (OA, \oplus) ,

each subgroup of (OA, θ) is contained in OAI, and if V is in OAI, then

$$V \theta G[V]^* = G[V]^* \theta V = 0.$$

PROOF: If α is in OA^+ , and (a, b) is in $S \times S$, let $(L) \int_a^b \alpha(a,) \alpha$ denote $\int_a^b h$, where h is given by $h(s, t) = \alpha(a, s) \alpha(s, t)$. It is known [17, Lemma 4.2] that if α is in OA^+ and (a, b) is in $S \times S$, then $(L) \int_a^b \alpha(a,) \alpha$ exists and is the least upper bound of the set to which r belongs only in case there is a chain $(t_k)_{k=0}^n$ from a to b such that $r = \sum_{k=1}^n \alpha(a, t_{k-1}) \alpha(t_{k-1}, t_k)$, and the analogous statement holds for $(R) \int_a^b \alpha(, b)$. Let each of V_1 , V_2 , and V_3 be in OA , and choose α in OA^+ such that α dominates each of V_1 , V_2 , and V_3 . Let (a, b, p) be in $S \times S \times Y$, and let (x, y) be in $S \times S$ such that x is between a and b and y is between x and b . Let $(s_k)_{k=0}^m$ be a chain from x to y , and let j be an integer in $[1, m]$. Now

$$\begin{aligned} & N_1[[I+V_3(x,y)+\sum_{k=1}^m V_2(s_{k-1},s_k)][I+V_3(s_{k-1},s_k)]] [p] \\ & \quad - [I+V_2(s_{j-1},s_j)][I+V_3(x,y)] [p] \\ & = N_1[\sum_{k=1}^m V_2(s_{k-1},s_k)[I+V_3(s_{k-1},s_k)]] [p] \\ & \quad - V_2(s_{j-1},s_j)[I+V_3(x,y)] [p] \\ & = N_1[\sum_{\substack{k=1 \\ k \neq j}}^m V_2(s_{k-1},s_k)[I+V_3(s_{k-1},s_k)]] [p] \\ & \quad + V_2(s_{j-1},s_j)[I+V_3(s_{j-1},s_j)] [p] \\ & \quad - V_2(s_{j-1},s_j)[I+V_3(x,y)] [p] \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\substack{k=1 \\ k \neq j}}^m \alpha(s_{k-1}, s_k) [1 + \alpha(s_{k-1}, s_k)] N_1[p] \\
&\quad + \alpha(s_{j-1}, s_j) [\alpha(x, s_{j-1}) + \alpha(s_j, y)] N_1[p] \\
&\leq N_1[p] [\alpha(x, s_{j-1}) [1 + \alpha(x, s_{j-1})] + \alpha(s_j, y) [1 + \alpha(s_j, y)]] \\
&\quad + \alpha(x, y) [\alpha(x, s_{j-1}) + \alpha(s_j, y)] \\
&\leq [1 + 2\alpha(x, y)] N_1[p] [\alpha(x, s_{j-1}) + \alpha(s_j, y)].
\end{aligned}$$

The penultimate inequality in the above computation is a direct application of Lemma III.1.1. Now

$$\begin{aligned}
&N_1[V_2(x, y) [I + V_3(x, y)] [p] + v_1(x, y) [I + V_3(x, y) \\
&\quad + \sum_{k=1}^m v_2(s_{k-1}, s_k) [I + V_3(s_{k-1}, s_k)]] [p] - v_2(x, y) [I + V_3(x, y)] [p] \\
&\quad - [\sum_{k=1}^m v_1(s_{k-1}, s_k) [I + V_2(s_{k-1}, s_k)]] [I + V_3(x, y)] [p]] \\
&\leq \sum_{i=1}^m \alpha(s_{i-1}, s_i) N_1[[I + V_3(x, y) + \sum_{k=1}^m v_2(s_{k-1}, s_k) [I + V_3(s_{k-1}, s_k)]] [p] \\
&\quad - [I + V_2(s_{i-1}, s_i)] [I + V_3(x, y)] [p]] \\
&\leq [1 + 2\alpha(x, y)] N_1[p] \sum_{i=1}^m \alpha(s_{i-1}, s_i) [\alpha(x, s_{i-1}) + \alpha(s_i, y)].
\end{aligned}$$

Now let each of $(s_k)_{k=0}^m$ and $(t_k)_{k=0}^n$ be a chain from a to b , and suppose $(s_k)_{k=0}^m$ refines $(t_k)_{k=0}^n$. Let J be a nondecreasing integer-valued sequence on the domain of t such that $J(0) = 0$, $J(n) = m$, and $s_{J(k)} = t_k$ whenever k is in the domain of t . It now follows that

$$\begin{aligned}
& N_1 \left[\sum_{k=1}^n [V_2(t_{k-1}, t_k) [I+V_3(t_{k-1}, t_k)] [p] + V_1(t_{k-1}, t_k) [I+V_3(t_{k-1}, t_k) \right. \\
& \quad + \sum_{i=J(k-1)+1}^{J(k)} V_2(s_{i-1}, s_i) [I+V_3(s_{i-1}, s_i)]] [p] \\
& \quad - V_2(t_{k-1}, t_k) [I+V_3(t_{k-1}, t_k)] [p] \\
& \quad \left. - \sum_{i=J(k-1)+1}^{J(k)} V_1(s_{i-1}, s_i) [I+V_2(s_{i-1}, s_i)] [I+V_3(t_{k-1}, t_k)] [p]] \right] \\
& \leq \sum_{k=1}^n [1+2\alpha(t_{k-1}, t_k)] N_1 [p] \\
& \quad \cdot \sum_{i=J(k-1)+1}^{J(k)} \alpha(s_{i-1}, s_i) [\alpha(t_{k-1}, s_{i-1}) + \alpha(s_i, t_k)] \\
& \leq N_1 [p] [1+2\alpha(a, b)] \sum_{k=1}^n \sum_{i=J(k-1)+1}^{J(k)} [\alpha(t_{k-1}, s_{i-1}) \alpha(s_{i-1}, s_i) \\
& \quad \quad \quad + \alpha(s_{i-1}, s_i) \alpha(s_i, t_k)] \\
& = N_1 [p] [1+2\alpha(a, b)] [(L) \sum_S \alpha(a,) \alpha + (R) \sum_S \alpha \alpha(, b) \\
& \quad \quad \quad - (L) \sum_t \alpha(a,) \alpha - (R) \sum_t \alpha \alpha(, b)].
\end{aligned}$$

But since this is true for every refinement s of t , it follows that

$$\begin{aligned}
& N_1 \left[\sum_t [V_2 [I+V_3] + V_1 [I+V_2 \oplus V_3] - (V_1 \oplus V_2) [I+V_3]] [p] \right] \\
& \leq N_1 [p] [1+2\alpha(a, b)] [(L) \int_a^b \alpha(a,) \alpha + (R) \int_a^b \alpha \alpha(, b) \\
& \quad \quad \quad - (L) \sum_t \alpha(a,) \alpha + (R) \sum_t \alpha \alpha(, b)].
\end{aligned}$$

The associativity is now clear.

Now note that if A is in H , and $I+A$ has inverse in H , then

$$-A[I+A]^{-1} + A[I-A[I+A]^{-1}] = -A[I+A]^{-1} + A[[I+A]-A][I+A]^{-1} = 0.$$

Let V in OAI , let α dominate V , and let (a,b,p) be in $S \times S \times Y$. Let s be a chain from a to b such that if t is a refinement of s , then

$[I+V(t_{k-1},t_k)]^{-1}$ exists whenever k is a positive member of the domain of t (Herod [10] has shown that such s exists). Let $(t_k)_{k=0}^n$ be a refinement of s . Now

$$\begin{aligned} & N_1 \left[\sum_{k=1}^n [-V(t_{k-1},t_k)][I+V(t_{k-1},t_k)]^{-1}[p] \right. \\ & \quad \left. + V(t_{k-1},t_k) \left[I - \sum_{k-1}^t V[I+V]^{-1} \right][p] \right] \\ &= N_1 \left[\sum_{k=1}^n [V(t_{k-1},t_k) \left[I - \sum_{k-1}^t V[I+V]^{-1} \right][p] - V(t_{k-1},t_k) \right. \\ & \quad \left. [I - V(t_{k-1},t_k)][I+V(t_{k-1},t_k)]^{-1}][p] \right] \\ &\leq \sum_{k=1}^n \alpha(t_{k-1},t_k) N_1 \left[\sum_{k-1}^t V[I+V]^{-1}[p] \right. \\ & \quad \left. - V(t_{k-1},t_k)[I+V(t_{k-1},t_k)]^{-1}[p] \right]. \end{aligned}$$

Thus it is clear (see [10, Theorem 1]) that $V \oplus G[V]^* = 0$. Similarly, $G[V]^* \oplus V = 0$, so (OAI, \oplus) is a group.

Now let U and V be in OA , and let $U \oplus V = V \oplus U = 0$. Let t be in S . Now $[U \oplus V](t, t+) = 0$, so

$$U(t, t+)[I+V(t, t+)] + V(t, t+) = 0,$$

$$U(t,t+)[I+V(t,t+)] + [I+V(t,t+)] = I,$$

and

$$[I+U(t,t+)][I+V(t,t+)] = I.$$

Also, since $[V\oplus U](t,t+) = 0$, we have $[I+V(t,t+)][I+U(t,t+)] = I$.

Similar computations for $(t,t-)$, $(t+,t)$, and $(t-,t)$ show that each of U and V is in OAI , and the proof of Theorem III.2.1 is complete.

LEMMA III.1.2

Let each of α_1 and α_2 be in OA^+ , and let β be a continuous member of OA^+ . Suppose that $\beta(a,b) \leq \int_a^b \alpha_1 \alpha_2$ whenever (a,b) is in $S \times S$. Then $\beta(a,b) = 0$ whenever (a,b) is in $S \times S$.

PROOF: Let (a,b) be in $S \times S$ with $a \leq b$. If $\alpha_2(a,b) = 0$, then $\beta(a,b) = 0$. Suppose $\alpha_2(a,b) > 0$. Let $\epsilon > 0$. Find a chain t from a to b such that $\alpha_1(t_{k-1}^+, t_k^-) < \epsilon / \alpha_2(a,b)$ whenever k is a positive member of the domain of t . Let n be the largest member of the domain of t . Now

$$\begin{aligned} \beta(a,b) &= \sum_{k=1}^n \beta(t_{k-1}, t_k) = \sum_{k=1}^n \beta(t_{k-1}^+, t_k^-) \\ &\leq \sum_{k=1}^n \alpha_1(t_{k-1}^+, t_k^-) \alpha_2(t_{k-1}^+, t_k^-) < \epsilon. \end{aligned}$$

An analogous argument holds if $a > b$, and the proof is complete.

THEOREM III.1.3

Let each of V_1 and V_2 be in OA . Then (i) and (ii) are equivalent and (iii) and (iv) are equivalent.

$$(i) \quad V_1 \oplus V_2 = V_1 + V_2.$$

(ii) $V_1[I+V_2] - V_1 = 0$ at all "pairs" of the forms $(t,t+)$, $(t,t-)$, $(t+,t)$, and $(t-,t)$ for t in S .

$$(iii) \quad V_1 \oplus V_2 = V_2 \oplus V_1.$$

(iv) $V_1 - V_2 = V_1[I+V_2] - V_2[I+V_1]$ at all "pairs" of the forms $(t,t+)$, $(t,t-)$, $(t+,t)$, and $(t-,t)$ for t in S .

PROOF: We shall indicate the first equivalence and leave the second to the reader. From the definition of \oplus , it is clear that (i) implies (ii). Now suppose (ii). For $i=1$ or $i=2$, let α_i dominate V_i . Let β in OA^+ be given by $\beta(a,b) = \int_a^b W_3[V_1[I+V_2] - V_1]$. Now, by (ii), β is continuous, and clearly $\beta(a,b) \leq \int_a^b \alpha_1 \alpha_2$ whenever (a,b) is in $S \times S$. Thus $\beta = 0$, (i) follows, and the proof is complete.

SECTION III.2: The \otimes Operation and the Exponential Identity

THEOREM III.2.1

Let each of V_1 and V_2 be in OA , with $W_1 = E[V_1]$ and $W_2 = E[V_2]$. Let (a,b,p) be in $S \times S \times Y$. Then each of

$$\int_a^b [I+V_1][I+V_2][p] \quad \text{and} \quad \int_a^b W_1 W_2 [p]$$

exists, and they are equal. Furthermore, if M is given by

$$M(a,b)[p] = \int_a^b W_1 W_2 [p],$$

then M is in OM .

PROOF: Let $U = V_1 \oplus V_2$. Let α be a member of OA^+ such that α dominates each of U , V_1 , and V_2 , and let $\mu = E^+[\alpha]$. Let (a,b,p) be in $S \times S \times Y$, and let t be a chain from a to b . Now

$$\begin{aligned}
N_1[\Pi_t[I+U][p] - \Pi_t[I+V_1][I+V_2][p]] &= N_1[\Pi_{k=1}^n[I+U(t_{k-1}, t_k)][p] \\
&\quad - \Pi_{k=1}^n[I+V_1(t_{k-1}, t_k)][I+V_2(t_{k-1}, t_k)][p]] \\
&\leq \sum_{k=1}^n N_1[\Pi_{j=1}^k[I+U(t_{j-1}, t_j)]\Pi_{j=k+1}^n[I+V_1(t_{j-1}, t_j)][I+V_2(t_{j-1}, t_j)][p] \\
&\quad - \Pi_{j=1}^{k-1}[I+U(t_{j-1}, t_j)]\Pi_{j=k}^n[I+V_1(t_{j-1}, t_j)] \\
&\hspace{20em} [I+V_2(t_{j-1}, t_j)][p]] \\
&\leq \sum_{k=1}^n \Pi_{j=1}^{k-1}[1+\alpha(t_{j-1}, t_j)]N_3[[I+U(t_{k-1}, t_k)] \\
&\hspace{10em} - [I+V_1(t_{k-1}, t_k)][I+V_2(t_{k-1}, t_k)]] \\
&\hspace{15em} \Pi_{j=k+1}^n[1+\alpha(t_{j-1}, t_j)]^2 N_1[p] \\
&\leq N_1[p]\mu(a,b)^2 \sum_{k=1}^n N_3[V_1(t_{k-1}, t_k)[I+V_2(t_{k-1}, t_k)] - \sum_{t_{k-1}}^{t_k} V_1[I+V_2]] \\
&\leq N_1[p]\mu(a,b)^2 [\sum_t \alpha[1+\alpha] - \sum_a^b \alpha[1+\alpha]].
\end{aligned}$$

It is now clear that ${}_a\Pi^b[I+V_1][I+V_2][p]$ exists and equals ${}_a\Pi^b[I+U][p]$ whenever (a,b,p) is in $S \times S \times Y$.

Since

$$\begin{aligned}
 & N_1 \left[\prod_{k=1}^n W_1(t_{k-1}, t_k) W_2(t_{k-1}, t_k) \right] [p] \\
 & - \prod_{k=1}^n [I+V_1(t_{k-1}, t_k)] [I+V_2(t_{k-1}, t_k)] [p] \\
 & \leq N_1 [p] \left[\prod_{k=1}^n \mu(t_{k-1}, t_k)^2 - \prod_{k=1}^n [1+\alpha(t_{k-1}, t_k)]^2 \right]
 \end{aligned}$$

(see [19, Lemma 1.2]), it is clear that

$${}_a \Pi^b [I+V_1] [I+V_2] [p] = {}_a \Pi^b W_1 W_2 [p]$$

whenever (a, b, p) is in $S \times S \times Y$. Since these products describe $E[U]$, it follows that M is in OM , and the proof is complete.

DEFINITION III.2.1: If each of W_1 and W_2 is in OM , then $W_1 \otimes W_2$ is that member M of OM given by

$$M(a, b) [p] = {}_a \Pi^b W_1 W_2 [p].$$

There emerges from the proof of Theorem III.2.1 a fact which we now record.

THEOREM III.2.2

If each of V_1 and V_2 is in OA , then

$$E[V_1 \oplus V_2] = E[V_1] \otimes E[V_2].$$

REMARK III.2.1: It is now clear that (OM, \otimes) is a semigroup and is semigroup-isomorphic to (OA, \otimes) , with E serving as an isomorphism. In Theorem 6 of [13], this author showed that if V_1 and V_2 are in OA , with $W_1 = E[V_1]$ and $W_2 = E[V_2]$, and $\sum_a^b N_3[V_1[I+V_2]-V_1] = 0$ whenever (a,b) is in $S \times S$, then $\sum_a^b W_1 W_2[p]$ exists whenever (a,b,p) is in $S \times S \times Y$, and, if M is given by $M(a,b)[p] = \sum_a^b W_1 W_2[p]$, then $M = E[V_1+V_2]$. Note that Theorem III.2.2, together with the first equivalence of Theorem III.1.3, includes this result.

THEOREM III.2.3

Let V_1 be in OA , and let V_2 be in OAI . Let U in OA be given by

$$U(a,b)[p] = \sum_a^b V_1[I+V_2]^{-1}[p].$$

Then

$$E[V_1+V_2] = E[U] \otimes E[V_2].$$

INDICATION OF PROOF: A line of argument similar to that used by Herod in [10] can be used to show that the sums indicated in the definition of U actually exist and to show that U is in OA . Let (a,b,p) be in $S \times S \times Y$. Now

$$\begin{aligned} [E[U] \otimes E[V_2]](a,b)[p] &= \sum_a^b E[U]E[V_2][p] \\ &= \sum_a^b [I+U][I+V_2][p] \end{aligned}$$

$$\begin{aligned}
&= {}_a\Pi^b[I+V_1[I+V_2]^{-1}][I+V_2][p] \\
&= {}_a\Pi^b[I+V_1+V_2][p] \\
&= E[V_1+V_2](a,b)[p].
\end{aligned}$$

This completes the proof.

REMARK III.2.2: Note that by using Theorems III.3.3, III.2.2, and III.2.3, we can compute, under two different sets of hypotheses, $E[V_1+V_2]$ in terms of the \otimes operation.

REMARK III.2.3: The notion of continuously multiplying solutions for generators in order to construct the solution for a sum of generators has been used by Trotter [30] and Chernoff [2], [3] for the case of autonomous linear differential equations with discontinuous linear operators, by Helton [6] for the case of linear Stieltjes integral equations, and by Mermin [22] for the case of autonomous nonlinear differential equations with accretive operators.

CHAPTER IV

EVOLUTION SYSTEMS WITH QUASICONTINUOUS TRAJECTORIES

SECTION IV.1: A Product Integral

It should be noted that the definition of product integral requires in no way that we restrict our attention to operators mapping zero onto zero. If q is in Y , then K_q , the q -constant function, will be that function from Y to Y such that if p is in Y then $K_q[p] = q$. For the remainder of Section IV.1, W will be a member of OM , and μ will be a member of OM^+ dominating W .

LEMMA IV.1.1

Suppose that (a,b) is in $S \times S$, $(t_k)_{k=0}^n$ is a chain from a to b , f is in QC , and p is in Y . Then if k is an integer in $[1,n]$ it follows that

$$\begin{aligned} \prod_{j=k}^n [W(t_{j-1}, t_j) - K_{[f(t_j) - f(t_{j-1})]}][p] &= p - f(b) + f(t_{k-1}) \\ &+ \sum_{j=k}^n [W(t_{j-1}, t_j) - I][\prod_{i=j+1}^n [W(t_{i-1}, t_i) - K_{[f(t_i) - f(t_{i-1})]}][p]]. \end{aligned}$$

PROOF: With the suppositions of the lemma, let $d_n = p$, and if k is an integer in $[0, n-1]$, let

$$d_k = W(t_k, t_{k+1})[d_{k+1}] + f(t_k) - f(t_{k+1}).$$

Now

$$d_k = \prod_{j=k+1}^n [W(t_{j-1}, t_j) - K_{[f(t_j) - f(t_{j-1})]}] [p]$$

whenever k is an integer in $[0, n-1]$. Note that

$$\begin{aligned} d_{n-1} &= W(t_{n-1}, t_n) [p] + f(t_{n-1}) - f(t_n) \\ &= p - f(b) + f(t_{n-1}) + [W(t_{n-1}, t_n) - I] [d_n]. \end{aligned}$$

Suppose that k is an integer in $[1, n-1]$, and

$$d_k = p - f(b) + f(t_k) + \sum_{j=k+1}^n [W(t_{j-1}, t_j) - I] [d_j].$$

Now

$$\begin{aligned} d_{k-1} &= f(t_{k-1}) - f(t_k) + W(t_{k-1}, t_k) [d_k] \\ &= d_k + f(t_{k-1}) - f(t_k) + [W(t_{k-1}, t_k) - I] [d_k] \\ &= p - f(b) + f(t_{k-1}) + \sum_{j=k}^n [W(t_{j-1}, t_j) - I] [d_j]. \end{aligned}$$

Thus this last equation holds whenever k is an integer in $[1, n]$, and the proof is complete.

LEMMA IV.1.2

Let (a,b) be in $S \times S$, let t be a chain from a to b , and let p be in Y . Let each of f and g be in QC , and let β on S be given by $\beta(s) = N_1[f(b)-g(b)] + N_1[f(s)-g(s)-f(b)+g(b)]$. Then

$$N_1[\Pi_t[W-K_{df}][p] - \Pi_t[W-K_{dg}][p]] \leq \beta(a) + (R)\int_t d[\mu(a,)]\beta.$$

PROOF: Let n be the largest member of the domain of t . Let $\eta_n = \beta(b)$, and if k is an integer in $[0, n-1]$ let $\eta_k = \mu(t_k, t_{k+1})\eta_{k+1} + \beta(t_k) - \beta(t_{k+1})$. Now

$$\eta_{k-1} = \beta(t_{k-1}) + \sum_{j=k}^n [\mu(t_{j-1}, t_j) - 1]\eta_j$$

whenever k is an integer in $[1, n]$. Since $N_1[f(t_n)-g(t_n)] = \beta(t_n)$, Lemma IV.1.1 and an easy induction argument tell us that

$$\begin{aligned} N_1[\Pi_t[W-K_{df}][p] - \Pi_t[W-K_{dg}][p]] &\leq \eta_0 \\ &= \mu(a,b)\beta(b) - (L)\int_t \mu(a,)d\beta \\ &= \beta(a) + (R)\int_t d[\mu(a,)]\beta. \end{aligned}$$

This completes the proof.

LEMMA IV.1.3

Let (a,b) be in $S \times S$, and let p be in Y . Let $(f_n)_{n=1}^{\infty}$ be a sequence into QC , and let g in QC be such that

$$\lim_{n \rightarrow \infty} [\sup\{N_1[f_n(t) - g(t)]: t \text{ is between } a \text{ and } b\}] = 0.$$

Suppose that ${}_a\Pi^b[W-K_{df_n}][p]$ exists whenever n is a positive integer. Then each of

$$\lim_{n \rightarrow \infty} ({}_a\Pi^b[W-K_{df_n}][p]) \quad \text{and} \quad {}_a\Pi^b[W-K_{dg}][p]$$

exists, and they are equal.

PROOF: The existence of $\lim_{n \rightarrow \infty} ({}_a\Pi^b[W-K_{df_n}][p])$ is clear from Lemma IV.1.2. Let q be this limit. Let ϵ be a positive number. Find a positive integer n_0 such that if m and n are integers and $n > n_0$ and $m > n_0$, then

$$N_1[q - {}_a\Pi^b[W-K_{df_n}][p]] < \epsilon/4$$

and

$$N_1[{}_t\Pi^b[W-K_{df_n}][p] - {}_t\Pi^b[W-K_{df_m}][p]] < \epsilon/4$$

for any chain t from a to b (Lemma IV.1.2 tells us we can do this).

Let n be a positive integer, $n > n_0$. Find a chain s from a to b such that if t refines s then

$$N_1[\pi_a^b[W-K_{df_n}][p] - \pi_t[W-K_{df_n}][p]] < \epsilon/4.$$

Let t be a refinement of s , and let m be an integer, $m > n_0$, such that

$$N_1[\pi_t[W-K_{df_m}][p] - \pi_t[W-K_{dg}][p]] < \epsilon/4.$$

Now

$$\begin{aligned} N_1[q - \pi_t[W-K_{dg}][p]] &\leq N_1[q - \pi_a^b[W-K_{df_n}][p]] \\ &+ N_1[\pi_a^b[W-K_{df_n}][p] - \pi_t[W-K_{df_n}][p]] \\ &+ N_1[\pi_t[W-K_{df_n}][p] - \pi_t[W-K_{df_m}][p]] \\ &+ N_1[\pi_t[W-K_{df_m}][p] - \pi_t[W-K_{dg}][p]] \\ &< \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon. \end{aligned}$$

Thus

$$N_1[q - \pi_t[W-K_{dg}][p]] < \epsilon$$

whenever t refines s , so

$$q = \pi_a^b[W-K_{dg}][p],$$

and the proof is complete.

DEFINITION IV.1.1: A member f of QC shall be called a step function only in case there is a nondecreasing unbounded sequence $(t_k)_{k=0}^{\infty}$ into S such that $t_0 = 0$ and two sequences $(p_k)_{k=0}^{\infty}$ and $(q_k)_{k=1}^{\infty}$ into Y such that if k is a positive integer then $f(t_{k-1}) = p_{k-1}$ and $f(s) = q_k$ whenever s is in S and $t_{k-1} < s < t_k$.

LEMMA IV.1.4

Let (a,b) be in $S \times S$, let p be in Y , and let f be a step function in QC. Then

$${}_a \Pi^b [W-K_{df}][p]$$

exists.

INDICATION OF PROOF: The lemma is clear from the following two observations:

(i) If (c,d) is in $S \times S$, q is in Y , and $f(t) = q$ whenever t is between c and d , then ${}_c \Pi^d [W-K_{df}][p] = W(c,d)[p]$.

(ii) If c is in S , then

$${}_{c+} \Pi^{c-} [W-K_{df}][p] = W(c+,c)[W(c,c-)[p] - f(c-) + f(c)] - f(c) + f(c+).$$

Since it is well known that each member of QC can be written as the locally uniform limit of step functions, Lemmas IV.1.3 and IV.1.4 now make the following theorem clear.

THEOREM IV.1.1

Let (a,b) be in $S \times S$, let p be in Y , and let f be in QC . Then

$${}_a \Pi^b [W-K_{df}][p]$$

exists.

SECTION IV.2: Solving Forced Equations

Throughout Section IV.2, V will be a member of OA , $W = E[V]$, α will be a member of OA^+ dominating V , and $\mu = E^+[\alpha]$. Note that this implies that μ dominates W .

LEMMA IV.2.1

Let a be in S , and let each of f , g , and h be a quasicontinuous function from S to S . Suppose that

$$h(t) = f(t) + (R) \int_t^a \alpha h$$

and

$$g(t) \leq f(t) + (R) \int_t^a \alpha g$$

whenever t is in S . Then

$$g(t) \leq h(t)$$

whenever t is in S .

PROOF: Let P be a sequence such that $P_0 = g$ and

$$P_n(t) = f(t) + (R) \int_t^a \alpha^n P_{n-1}$$

whenever n is a positive integer and t is in S . Now $g(t) \leq P_n(t)$ whenever t is in S and n is a positive integer. But according to MacNerney [18], $\lim_{n \rightarrow \infty} P_n(t) = h(t)$ whenever t is in S , so the proof is complete.

REMARK IV.2.1: The iterative technique used in the proof of Lemma IV.2.1 is similar to that developed by Herod in [9].

LEMMA IV.2.2

Let a be in S . Let $(f_n)_{n=0}^{\infty}$ and $(h_n)_{n=0}^{\infty}$ be sequences into QC such that each of (i) and (ii) is true.

(i) $h_n(t) = f_n(t) + (R) \int_t^a V[h_n]$ whenever t is in S and n is a nonnegative integer.

(ii) $\lim_{n \rightarrow \infty} f_n(t) = f_0(t)$ whenever t is in S , the convergence being uniform on bounded subsets of S .

Then $\lim_{n \rightarrow \infty} h_n(t) = h_0(t)$ whenever t is in S , the convergence being uniform on bounded subsets of S .

PROOF: Let b and ϵ be positive numbers, $b > a$, and let c be a number such that $c > \mu(b, 0)$. Let n_0 be a positive integer such that $N_1[f_n(t) - f_m(t)] < \epsilon/c$ whenever t is in $[0, b]$ and m and n are integers such that $n > n_0$ and $m > n_0$. Let m and n be integers such that $n > n_0$ and $m > n_0$, and let P from S to S be such that $P(t) = \epsilon/c$ if $0 \leq t \leq b$ and $P(t) =$

$N_1[f_n(t) - f_m(t)]$ if $t > b$. Let Q be that quasicontinuous function from S to S such that

$$Q(t) = P(t) + (R) \int_t^a \alpha Q$$

whenever t is in S . Now

$$N_1[h_n(t) - h_m(t)] \leq P(t) + (R) \int_t^a \alpha N_1[h_n - h_m]$$

whenever t is in S , so according to Lemma IV.1.1,

$$N_1[h_n(t) - h_m(t)] \leq Q(t)$$

whenever t is in S . But $Q(t) = \mu(t, a)(\epsilon/c) < \epsilon$ whenever $0 \leq t \leq b$, so it is now clear that there is a member U of QC such that $\lim_{n \rightarrow \infty} h_n(t) = U(t)$ whenever t is in S , the convergence being uniform on bounded subsets of S . Now $\lim_{n \rightarrow \infty} (R) \int_t^a V[h_n] = (R) \int_t^a V[U]$ whenever t is in S , so $U(t) = f_0(t) + (R) \int_t^a V[U]$ whenever t is in S . Now Theorem II.2.2 tells us $U = h_0$, and the proof is complete.

LEMMA IV.2.3

Let f be a step function in QC , and let (a, p) be in $S \times Y$. Let h be in QC . Then (i) and (ii) are equivalent.

- (i) $h(t) = p - f(a) + f(t) + (R) \int_t^a V[h]$ whenever t is in S .
- (ii) $h(t) = {}_t \Pi^a [W-K_{df}][p]$ whenever t is in S .

REMARK IV.2.2: Lemma IV.2.2 follows from Theorem II.2.1 and a straightforward computation.

THEOREM IV.2.1

Let f be in QC . There is a function M on $S \times S$, each value of which is a function from Y to Y , such that each of (i), (ii), (iii), and (iv) is true.

$$(i) \quad M(a,b)[p] = {}_a\Pi^b[W-K_{df}][p] \text{ whenever } (a,b,p) \text{ is in } S \times S \times Y.$$

(ii) $M(a,b)[M(b,c)[p]] = M(a,c)[p]$ whenever (a,b,c,p) is in $S \times S \times S \times Y$ and b is between a and c .

(iii) $M(b,a)[p] = p - f(a) + f(b) + (R) \int_b^a V[M(\cdot, a)[p]]$ whenever (a,b,p) is in $S \times S \times Y$.

$$(iv) \quad N_1[[M(a,b)-I][p] - [M(a,b)-I][q]] \\ \leq [\mu(a,b) - 1]N_1[p-q]$$

whenever (a,b,p,q) is in $S \times S \times Y \times Y$.

PROOF: Define M according to (i). Now (ii) follows immediately, and (iii) follows from Lemmas IV.2.2 and IV.2.3. Thus it remains only to show (iv). Let (a,p,q) be in $S \times Y \times Y$, and let $h = M(\cdot, a)[p]$ and $g = M(\cdot, a)[q]$. Now

$$h(t) = p + f(t) - f(a) + (R) \int_t^a V[h]$$

and

$$g(t) = q + f(t) - f(a) + (R) \int_t^a V[g]$$

whenever t is in S . Thus, if t is in S ,

$$N_1[h(t)-g(t)] \leq N_1[p-q] + (R) \int_t^a \alpha_1[h-g].$$

Since

$$\mu(t,a) = 1 + (R) \int_t^a \alpha_1(,a)$$

whenever t is in S , this says

$$N_1[h(t)-g(t)] \leq N_1[p-q]\mu(t,a)$$

whenever t is in S . Now, if t is in S ,

$$[h(t)-p] - [g(t)-q] = (R) \int_t^a V[h] - (R) \int_t^a V[g],$$

so

$$\begin{aligned} N_1[[h(t)-p] - [g(t)-q]] &\leq (R) \int_t^a \alpha_1 N_1[h-g] \\ &\leq N_1[p-q] (R) \int_t^a \alpha_1(,a) \\ &= N_1[p-q][\mu(t,a) - 1], \end{aligned}$$

and the proof is complete.

COROLLARY IV.2.1

Suppose that each value of V is linear and f is in QC. Let a be in S , and let h be in QC. Then (i) and (ii) are equivalent.

- (i) $h(t) = f(t) + (R) \int_t^a V[h]$ whenever t is in S .
- (ii) $h(t) = W(t,a)[f(a)] - (L) \int_t^a W(t,) [df]$ whenever t is in S .

REMARK IV.2.3: This corollary can be viewed as a companion result to Theorem 5.2 of [17]. An easy integration-by-parts shows that the formula in (ii) can also be written

$$h(t) = f(t) + (R) \int_t^a (d[W(t,)])[f].$$

PROOF: Note that if each value of V is linear, then each value of W is linear. To prove the corollary it suffices to show that if f is in QC and (a,b,p) is in $S \times S \times Y$ then

$$\int_a^b [W-K_{df}][p] = W(a,b)[p] - (L) \int_a^b W(a,) [df].$$

Let (a,b,p) be in $S \times S \times Y$, and let $(t_k)_{k=0}^n$ be a chain from a to b . Let $(d_k)_{k=0}^n$ be as in the proof of Lemma IV.1.1. Suppose that k is an integer in $[1, n-1]$, and

$$d_k = W(t_k, b)[p] - \int_{j=k+1}^n W(t_k, t_{j-1}) [f(t_j) - f(t_{j-1})].$$

Now

$$d_{k-1} = W(t_{k-1}, t_k) [d_k] - [f(t_k) - f(t_{k-1})]$$

$$\begin{aligned}
&= W(t_{k-1}, b)[p] - \sum_{j=k+1}^n W(t_{k-1}, t_{j-1})[f(t_j) - f(t_{j-1})] \\
&\quad - W(t_{k-1}, t_{k-1})[f(t_k) - f(t_{k-1})] \\
&= W(t_{k-1}, b)[p] - \sum_{j=k}^n W(t_{k-1}, t_{j-1})[f(t_j) - f(t_{j-1})].
\end{aligned}$$

Hence this last equation holds whenever k is an integer in $[1, n]$, and the corollary follows.

SECTION IV.3: A Coalescence Problem

In Section IV.3, V will be a member of OAI, and $W = E[V]$. We will take λ to be a member of OMR such that $N_2[W(a, b)^{-1}] \leq \lambda(a, b)$ whenever (a, b) is in $S \times S$. (This is clearly possible by Theorem II.3.2).

THEOREM IV.3.1

Let f be in QC, and let (a, b, p, q) be in $S \times S \times Y \times Y$. Then

$$N_1[p - q] \leq \lambda(a, b) N_1[a \prod_a^b [W - K_{df}][p] - a \prod_a^b [W - K_{df}][q]].$$

PROOF: Let $(t_k)_{k=0}^n$ be a chain from a to b . If k is an integer in $[0, n]$, let

$$d_k = \prod_{j=k+1}^n [W(t_{j-1}, t_j) - K_{[f(t_j) - f(t_{j-1})]}][p],$$

and

$$e_k = \prod_{j=k+1}^n [W(t_{j-1}, t_j) - K_{[f(t_j) - f(t_{j-1})]}][q].$$

Now, if k is an integer in $[0, n-1]$,

$$d_k = W(t_k, t_{k+1})[d_{k+1}] + f(t_k) - f(t_{k+1})$$

and

$$e_k = W(t_k, t_{k+1})[e_{k+1}] + f(t_k) - f(t_{k+1}),$$

so

$$d_{k+1} = W(t_k, t_{k+1})^{-1}[d_k + f(t_{k+1}) - f(t_k)]$$

and

$$e_{k+1} = W(t_k, t_{k+1})^{-1}[e_k + f(t_{k+1}) - f(t_k)].$$

Thus

$$N_1[d_{k+1} - e_{k+1}] \leq \lambda(t_k, t_{k+1})N_1[d_k - e_k]$$

whenever k is an integer in $[0, n-1]$. Consequently,

$$\begin{aligned} N_1[p-q] &= N_1[d_n - e_n] \\ &\leq \lambda(a, b)N_1[d_0 - e_0] \\ &= \lambda(a, b)N_1[\Pi_t[W-K_{df}][p] - \Pi_t[W-K_{df}][q]]. \end{aligned}$$

Since this last inequality holds for each chain t from a to b , the proof is complete.

COROLLARY IV.3.1

Let (a,p,q) be in $S \times Y \times Y$, and let each of f , g , and h be in QC.

Suppose that whenever t is in S , then

$$h(t) = p + f(t) + (R) \int_t^a V[h]$$

and

$$g(t) = q + f(t) + (R) \int_t^a V[g].$$

Then, if there exists t in S such that $h(t) = g(t)$, it follows that

$p = q$.

CHAPTER V

BOUNDS FOR SOLUTIONS OF PERTURBED EQUATIONS

SECTION V.1: Integrator Perturbations

Our results for integrator perturbations will largely follow from the following lemma. The first inequality in the conclusion of the lemma was established in Lemma 5 of [13].

LEMMA V.1.1

Let $(A_k)_{k=1}^n$ and $(B_k)_{k=1}^n$ be sequences into H . Let $(a_k)_{k=1}^n$, $(b_k)_{k=1}^n$, and $(c_k)_{k=1}^n$ be real-valued sequences such that if k is an integer in $[1, n]$, then $N_2[A_k] \leq a_k$, $N_3[B_k] \leq b_k$, and $N_3[B_k - I] \leq c_k$. Then neither of

$$N_3[\prod_{k=1}^n A_k B_k - \prod_{k=1}^n A_k] \quad \text{and} \quad N_3[\prod_{k=1}^n B_k A_k - \prod_{k=1}^n A_k]$$

exceeds $(\prod_{k=1}^n a_k) \sum_{k=1}^n [c_k (\prod_{j=k+1}^n b_j)]$.

INDICATION OF PROOF: We shall indicate how to prove

$$N_3[\prod_{k=1}^n B_k A_k - \prod_{k=1}^n A_k] \leq (\prod_{k=1}^n a_k) \sum_{k=1}^n [c_k (\prod_{j=k+1}^n b_j)].$$

Note that

$$N_3[\prod_{k=1}^n B_k A_k - \prod_{k=1}^n A_k]$$

$$\begin{aligned}
&= N_3[B_1 A_1 \prod_{k=2}^n B_k A_k - A_1 \prod_{k=2}^n B_k A_k + A_1 \prod_{k=2}^n B_k A_k - A_1 \prod_{k=2}^n A_k] \\
&\leq N_3[B_1 - I] N_2[A_1] \prod_{k=2}^n N_3[A_k] N_3[B_k] + N_2[A_1] N_3[\prod_{k=2}^n B_k A_k - \prod_{k=2}^n A_k] \\
&\leq (\prod_{k=1}^n a_k) c_1 (\prod_{k=2}^n b_k) + a_1 N_3[\prod_{k=2}^n B_k A_k - \prod_{k=2}^n A_k].
\end{aligned}$$

The remainder of the proof now follows from an obvious induction argument.

We now state without proof our theorem on integrator perturbations. The proof will be obvious from Lemma V.1.1.

THEOREM V.1.1

Let each of W_1 and W_2 be in OM, let U be in OA, and suppose $W_2 = E[U]$. Suppose λ_1 and λ_2 in OMR are given by $\lambda_1(a,b) = a \prod N_2[W_1]$ and $\lambda_2(a,b) = a \prod^b N_3[W_2]$. Let β be a member of OA^+ such that $N_3[U(a,b)] \leq \beta(a,b)$ whenever (a,b) is in $S \times S$. Let (a,b) be in $S \times S$. Then neither of $N_3[W_1 \otimes W_2(a,b) - W_1(a,b)]$ and $N_3[W_2 \otimes W_1(a,b) - W_1(a,b)]$ exceeds $\lambda_1(a,b) \cdot (R) \int_a^b \beta \lambda_2(\cdot, b)$.

REMARK V.1.1: In Theorem 11 of [13], this author showed that if V_1, V_2, W_1, W_2 , and M are as in Remark III.2.1, then $N_3[M(a,b) - W_1(a,b)] \leq \lambda_1(a,b) \cdot (R) \int_a^b \beta \lambda_2(\cdot, b)$ whenever (a,b) is in $S \times S$. Note that Theorem V.1.1, together with the first equivalence of Theorem III.1.3, includes this result. It is also worthy of note that there is an obvious result to be obtained by the conjunction of Theorems III.2.3 and V.1.1.

SECTION V.2: Forcing Function Perturbations

THEOREM V.2.1

Let (V,W) be in E . Let λ_1 and λ_2 in OMR be given by $\lambda_1(a,b) = {}_a\Pi^b N_2[W]$ and $\lambda_2(a,b) = {}_a\Pi^b N_3[W]$. Let each of f_1 and f_2 be in BV. Let a be in S . Let h_1 and h_2 be in BV, and suppose

$$h_i(t) = f_i(t) + (R) \int_t^a V[h_i]$$

whenever t is in S and $i=1$ or $i=2$. Then, if t is in S ,

$$N_1[h_1(t)] \leq \lambda_2(t,a)N_1[f_1(a)] + (L) \int_t^a \lambda_2(t,)N_1[df_1]$$

and

$$\begin{aligned} N_1[h_1(t) - h_2(t)] &\leq \lambda_1(t,a)N_1[f_1(a) - f_2(a)] \\ &\quad + (L) \int_t^a \lambda_1(t,)N_1[d(f_1-f_2)]. \end{aligned}$$

REMARK V.2.1: Our proof of Theorem V.2.1 will involve an application of Theorem IV.2.1. The first inequality of the conclusion was known prior to the discovery of Theorem IV.2.1 (see [14, Theorem A]). It is clear from Theorem II.4.2 that, if γ in OAR is given by $\gamma(a,b) = {}_a\int^b (N_3[I+V] - 1)$, then the function g , given by

$$g(t) = \lambda_2(t,a)N_1[f_1(a)] + (L) \int_t^a \lambda_2(t,)N_1[df_1],$$

satisfies

$$g(t) = N_1[f_1(a)] + \int_t^a N_1[df_1] + (R) \int_t^a \gamma g$$

whenever t is in S . This author conjectures that analogous results hold for λ_1 , but these analogous results have not yet been shown.

INDICATION OF PROOF OF THEOREM V.2.1: We shall indicate the second inequality of the conclusion, and it will then be evident how to demonstrate the first. Let t be in S , and let $(s_k)_{k=0}^n$ be a chain from t to a . Now

$$\begin{aligned} & N_1 \left[\prod_{k=1}^n [W(s_{k-1}, s_k) - K_{[f_1(s_k) - f_1(s_{k-1})]}] \right] [f_1(a)] \\ & - \prod_{k=1}^n [W(s_{k-1}, s_k) - K_{[f_2(s_k) - f_2(s_{k-1})]}] [f_2(a)] \\ & \leq \lambda_1(s_0, s_0) N_1 [[f_1(s_1) - f_2(s_1)] - [f_1(s_0) - f_2(s_0)]] \\ & + \lambda_1(s_0, s_1) N_1 \left[\prod_{k=2}^n [W(s_{k-1}, s_k) - K_{[f_1(s_k) - f_1(s_{k-1})]}] \right] [f_1(a)] \\ & - \prod_{k=2}^n [W(s_{k-1}, s_k) - K_{[f_2(s_k) - f_2(s_{k-1})]}] [f_2(a)]. \end{aligned}$$

An induction argument similar to those in the proofs of Lemma IV.1.1 and Corollary IV.2.1 now makes it clear that

$$\begin{aligned}
& N_1 \left[\prod_{k=1}^n [W(s_{k-1}, s_k) - K_{[f_1(s_k) - f_1(s_{k-1})]}] [f_1(a)] \right. \\
& \quad \left. - \prod_{k=1}^n [W(s_{k-1}, s_k) - K_{[f_2(s_k) - f_2(s_{k-1})]}] [f_2(a)] \right] \\
& \leq \lambda_1(t, a) N_1 [f_1(a) - f_2(a)] + \sum_{k=1}^n \lambda_1(t, s_{k-1}) N_1 [[f_1(s_k) - f_2(s_k)] \\
& \quad - [f_1(s_{k-1}) - f_2(s_{k-1})]],
\end{aligned}$$

and, from this, the theorem follows.

BIBLIOGRAPHY

- [1] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Mathematics in Science and Engineering, Vol. 8, Academic Press, New York, 1964.
- [2] P. R. Chernoff, Note on product formulas for operator semigroups, *Journal of Functional Analysis*, 2 (1968), 238-242.
- [3] _____, Semigroup product formulas and addition of unbounded operators, *Bulletin of the American Mathematical Society*, 76 (1970), 395-398.
- [4] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, D. C. Heath and Co., Boston, 1965.
- [5] M. Fréchet, Sur l'intégration d'un système canonique d'équations différentielles linéaires à coefficients discontinus, *Proc. Benares Math. Soc.*, 1 (1939), 1-34.
- [6] B. W. Helton, Integral equations and product integrals, *Pacific Journal of Mathematics*, 16 (1966), 297-322.
- [7] J. V. Herod, Solving integral equations by iteration, *Duke Journal of Mathematics*, 34 (1967), 519-534.
- [8] _____, Multiplicative inverses of solutions for Volterra-Stieltjes integral equations, *Proceedings of the American Mathematical Society*, 22 (1969), 650-656.
- [9] _____, A Gronwall inequality for linear Stieltjes integrals, *Proceedings of the American Mathematical Society*, 23 (1969), 34-36.
- [10] _____, Coalescence of solutions for nonlinear Stieltjes equations, *Journal für die Reine und Angewandte Mathematik* (to appear).
- [11] T. H. Hildebrandt, On systems of linear differentio-Stieltjes-integral equations, *Illinois Journal of Mathematics*, 3 (1959), 352-373.
- [12] W. H. Ingram, The j -differential and its integral, *Proceedings of the Edinburgh Mathematical Society* (2), 12 (1960-61), 85-93; 13 (1962), 85-86.
- [13] D. L. Lovelady, Perturbations of solutions of Stieltjes integral equations, *Transactions of the American Mathematical Society*, 155 (1971), 158-178.

- [14] _____, A variation-of-parameters inequality, *Proceedings of the American Mathematical Society*, 26 (1970), 598-602.
- [15] J. S. MacNerney, Stieltjes integrals in linear spaces, *Annals of Mathematics*, 61 (1955), 354-367.
- [16] _____, Continuous products in linear spaces, *Journal of the Elisha Mitchell Scientific Society*, 71 (1955), 185-200.
- [17] _____, Integral equations and semigroups, *Illinois Journal of Mathematics*, 7 (1963), 148-173.
- [18] _____, Note on successive approximations, *Rendiconti Circolo Matematico di Palermo* (2), 12 (1963), 87-90.
- [19] _____, A nonlinear integral operation, *Illinois Journal of Mathematics*, 8 (1964), 621-638.
- [20] R. H. Martin, Jr., A bound for solutions to Volterra-Stieltjes integral equations, *Proceedings of the American Mathematical Society*, 23 (1969), 506-512.
- [21] _____, Bounds for solutions to a class of nonlinear integral equations, *Transactions of the American Mathematical Society* (to appear).
- [22] J. L. Mermin, Accretive operators and nonlinear semigroups, Ph.D. Thesis, University of California, Berkeley (1968).
- [23] J. W. Neuberger, Continuous products and nonlinear integral equations, *Pacific Journal of Mathematics*, 8 (1958), 529-549.
- [24] G. B. Price, Cauchy-Stieltjes and Riemann-Stieltjes integrals, *Bulletin of the American Mathematical Society*, 49 (1943), 625-630.
- [25] W. T. Reid, Generalized linear differential systems, *Journal of Mathematics and Mechanics*, 8 (1959), 705-726.
- [26] W. C. Sangren, Differential equations with discontinuous coefficients, *Oak Ridge Nat. Lab. Publ. No. 1566 (Physics)*.
- [27] H. Schärf, Über links- und rechtseitige Stieltjes-integrale und deren Anwendungen, *Portugaliae Mathematica*, 4 (1944), 73-118.
- [28] F. W. Stallard, Differential systems with interface conditions, *Oak Ridge Nat. Lab. Publ. No. 1876 (Physics)*.
- [29] _____, Functions of bounded variation as solutions of differential systems, *Proceedings of the American Mathematical Society*, 13 (1962), 366-373.

- [30] H. F. Trotter, On the product of semigroups of operators, *Proceedings of the American Mathematical Society*, 10 (1959), 545-551.
- [31] H. S. Wall, Concerning harmonic matrices, *Archiv Mathematik*, 5 (1954), 160-167.

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