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STABILITY OF A SYSTEM OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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CHAPTER I

INTRODUCTION

There are several well-known sufficient conditions for the asymptotic stability, in the sense of Poincarè-Liapounov, of solutions of systems of ordinary differential equations. The simplest result is the following one, due to Perron [1]: let x, g be real vectors with n components, let A be a real $n \times n$ matrix, and suppose t is real. If all of the characteristic roots of A have negative real parts, if g(t, x) is continuous for small $||x||$, if $t \geq 0$, and if

$$g(t, x) = o(||x||)$$

as $||x|| \to 0$, uniformly in t for $t \geq 0$, then the identically zero solution of the nonlinear system

$$x' = Ax + g(t, x)$$

where

$$x' = \frac{dx}{dt} ,$$

is asymptotically stable.

The situation is more complicated in the case of nonlinear functional differential equations. Bellman [2] and others have extended the above theory for nonlinear ordinary differential equations to single nonlinear differential-difference equations, and recently Nohel [3] has established criteria for the stability and instability of solutions of the equation

$$u' = a \int_0^{\Theta}(\Theta - h)\, u(t - h)\, dh + \int_0^{\Theta}(\Theta - h)\, g\left[t - h,\, u(t - h)\right] dh, \qquad (1)$$

$t > \Theta$, where u and g are real functions, a is a real constant, and Θ is a given positive constant. The present study is an extension of the stability theorem for equation (1) to a system of equations of this type of the form (3) below.

The problem here considered is not without physical motivation. The equation

$$u' = - \frac{c}{\Theta} \int_0^{\Theta} (\Theta - h)\, [e^{u(t - h)} - 1]\, dh, \qquad (2)$$

where c is a given positive constant, and u represents the logarithm of the power in a one-region "circulating fuel" nuclear reactor having "transit time" Θ , is clearly a special case of the system (2) below with $n = 1$, $g(t, u)$ a power series in u beginning with the second-degree term, and t not entering explicitly. A stability theory for the identically zero solution of (2) follows easily from Theorem 3. The derivation of this equation is given by Ergen [4]. Physically, the stability criterion for the identically zero solution of (2) describes conditions under which the equilibrium state of the reactor is stable.

Preliminaries. -- Consider the system

$$y_j' = \sum_{i=1}^{j} a_{ji} \int_0^{\Theta_j} (\Theta_j - h)\, y_i(t - h)\, dh$$

$$+ \int_0^{\Theta_j} (\Theta_j - h)\, g_j[t - h,\, y_1(t - h), \ldots, y_j(t - h)]\, dh, \qquad (3)$$

$t > \theta_j$, with initial conditions

$$y_j(t) = q_j(t), \quad 0 \leq t \leq \theta_j, \; j = 1, \dots, n.$$

It will suffice to assume that each g_j is a real function which is continuous in (t, u) for $t \geq 0$, and that

$$g_j(t, u) = o(||u||)$$

as $||u|| \to 0$, uniformly in t, for $t \geq 0$.

Definition 1.-- A solution $\phi(t)$ of (3) is a real continuous n-vector defined on $0 \leq t \leq \theta + \alpha$, where

$$\theta = \max_j \theta_j,$$

$\alpha > 0$, and the components ϕ_j and ϕ'_j are continuous on $\theta_j \leq t \leq \theta + \alpha$, and satisfy the system (3) on this interval. Note that the zero vector, $y_j \equiv 0$, $j = 1, \dots, n$, is a solution of (3).

It is necessary to remark that the existence of each component of the solution over the entire interval above follows only as a consequence of Theorem 3.

Definition 2.-- The norm $||x||$ of a vector x with components $x_1, \dots, x_k$ is defined by

$$||x|| = \sum_{i=1}^{k} |x_i|.$$

To simplify the notation in the proof of the stability theorem, the following definition is also made:

Definition 3.-- Let x be an n-vector with components $x_1,\dots,x_n$. Then

$$||x||_j = \sum_{i=1}^{j} |x_i|.$$

Definition 4.-- The solution $y_j \equiv 0$, $j = 1,\dots,n$ of system (3) is said to be stable if, given an $\varepsilon > 0$, there exists a $\delta > 0$, such that if $\phi(t)$ is any solution of (3) for which

$$\max_{0 \leq t \leq \theta} ||\phi(t)|| < \delta ,$$

then

$$||\phi(t)|| < \varepsilon$$

for $t > \theta$.

Definition 5.-- The solution $y_j \equiv 0$, $j = 1,\dots,n$ of system (3) is said to be asymptotically stable if it is stable and if

$$\lim_{t \to +\infty} ||\phi(t)|| = 0,$$

where $\phi(t)$ is as above.

Definition 6.-- The solution $y_j \equiv 0$, $j = 1,\dots,n$ of system (3) is said to be unstable if it is not stable.

Procedure.-- The following procedure is employed: first, a linear problem is formulated by replacing the nonlinear terms in the original system (3) by known functions of t, and a solution of the linear problem is derived (in the form of a complex contour integral) by formal calculations with the Laplace transformation. This formal solution is then shown (Theorem 1) to be an actual solution. Next, a real representation theorem (Theorem 2), analgous to the variation-of-constants formula for ordinary differential equations, is obtained. By using this representation theorem, conditions are obtained for the stability, asymptotic stability, or instability of the identically zero solution of the associated linear system. Finally, the solution of the nonlinear problem is shown to have a representation in the form of a nonlinear system of integral equations, from which is derived a sufficient condition for the asymptotic stability of the identically zero solution of the nonlinear system (3) (Theorem 3).

CHAPTER II

SOLUTION OF THE LINEAR PROBLEM

Derivation of the formal solution. -- Consider the system

$$y_j' = \sum_{i=1}^{j} a_{ji} \int_0^{\Theta_j} (\Theta_j - h)\, y_i(t-h)\, dh$$
$$+ \int_0^{\Theta_j} (\Theta_j - h)\, g_j[t-h, y_1(t-h), \ldots y_j(t-h)]\, dh, \tag{4}$$

$t > \Theta_j$, with initial conditions

$$y_j(t) = q_j(t), \; 0 \leq t \leq \Theta_j, \tag{5}$$

where q_j is a real continuous function, a_{ji} and Θ_j are real constants with $\Theta_j > 0$, i, j = 1,..., n. First, one considers the associated linear system:

$$y_j' = \sum_{i=1}^{j} a_{ji} \int_0^{\Theta_j} (\Theta_j - h)\, y_i(t-h)\, dh + W_j(t), \tag{6}$$

$t > \Theta_j$, where

$$y_j'(\Theta_j) = y_j'(\Theta_j^+) \,,$$

which satisfies (5) on $0 \leq t \leq \Theta_j$, j = 1,..., n. To ensure that $W_j(t)$ has a Laplace integral, assume that $W_j(t)$ is continuous for $t < \Theta_j$, and that there exist constants c_{j1}, c_{j2}, such that

$$|W_j(t)| \leq c_{j1}\, e^{c_{j2} t} \,, \quad t > \Theta_j. \tag{7}$$

Now define

$$L_j(y) = \int_{\Theta_j}^{\infty} e^{-st}\, y(t)\, dt \,,$$

whenever the integral exists. All formal aspects which are not justified here will be established in the next section. Thus, proceeding formally, the j'th equation of the system (6) is now considered. Multiply both sides of (6) by e^{-st} and integrate on t from Θ_j to infinity. This yields, after an integration by parts on the left-hand side,

$$-y_j(\Theta_j)\, e^{-\Theta_j s} + sL_j(y_j) = \sum_{i=1}^{j} a_{ji}\, I_{ji}(s) + L_j(W_j), \qquad (8)$$

where

$$I_{ji}(s) = \int_{\Theta_j}^{\infty} e^{-st} \int_0^{\Theta_j} (\Theta_j - h)\, y_i(t - h)\, dh\, dt.$$

To evaluate $I_{ji}(s)$, let

$$t - h = r$$

in the inside integral, giving

$$I_{ji}(s) = \int_{\Theta_j}^{\infty} e^{-st} \left[\int_{t-\Theta_j}^{t} (\Theta_j - t - r)\, y_i(r)\, dr\right] dt\ .$$

Interchange of the order of integration yields

$$I_{ji}(s) = \int_0^{\Theta_j} y_i(r) \int_{\Theta_j}^{r+\Theta_j} (\Theta_j - t + r)\, e^{-st} dt\, dr$$

$$+ \int_{\Theta_j}^{\infty} y_i(r) \int_{r}^{r+\Theta_j} (\Theta_j - t + r)\, e^{-st}\, dt\, dr\ .$$

It is now assumed that the system (6), with initial conditions (5) can be solved recursively for the y_i's, i = 1, 2, ..., (j - 1), as known functions of t, which are continuous and of exponential order. (Observe that this is immediate in the case j = 1.) Using also the initial condition (5) and evaluating the inside integrals, one obtains

$$I_{jj}(s) = \frac{e^{-\theta_j s}}{s} \int_0^{\theta_j} r q_j(r) dr - \frac{e^{-\theta_j s}}{s^2} \int_0^{\theta_j} (1 - e^{-sr})\, q_j(r)\, dr + \left[\frac{\theta_j}{s} - \frac{1 - e^{-\theta_j s}}{s^2} \right] L_j(y_j) \tag{9}$$

and, for $i < j$,

$$I_{ji}(s) = \frac{e^{-\theta_j s}}{s} \int_0^{\theta_j} r\, y_i(r)\, dr - \frac{e^{-\theta_j s}}{s^2} \int_0^{\theta_j} (1 - e^{-sr})\, y_i(r)\, dr + \left[\frac{\theta_j}{s} - \frac{1 - e^{-\theta_j s}}{s^2} \right] L_j(y_i) .$$

To simplify equation (9), let

$$A_j = \int_0^{\theta_j} r\, q_j(r)\, dr \tag{10}$$

and

$$B_j = \int_0^{\theta_j} q_j(r)\, dr ; \tag{11}$$

since q_j is given, A_j and B_j are known constants, $j = 1,\ldots, n$. Further, since all y_i's are known functions, $i = 1,\ldots, (j - 1)$, $I_{ji}(s)$ is a known function for $i = 1,\ldots, (j - 1)$. If one uses (9), (10), and (11), and recalls that

$$y_j(\theta_j) = q_j(\theta_j),$$

one can write (8) in the form

$$\left\{ s - a_{jj} \left[\frac{\Theta_j}{s} - \frac{1 - e^{-\Theta_j s}}{s^2} \right] \right\} L_j(y_j) = q_j(\Theta_j)\, e^{-\Theta_j s} + a_{jj} A_j \frac{e^{-\Theta_j s}}{s} \tag{12}$$

$$- a_{jj} B_j \frac{e^{-\Theta_j s}}{s^2} + a_{jj} \frac{e^{-\Theta_j s}}{s^2} \int_0^{\Theta_j} e^{-sr} q_j(r)\, dr + \sum_{i=1}^{j-1} a_{ji} I_{ji}(s) + L_j(W_j) \, .$$

But observe that

$$\frac{A_j}{s} + \frac{B_j}{s^2} + \frac{1}{s^2} \int_0^{\Theta_j} e^{-sr} q_j(r)\, dr = \int_0^{\Theta_j} \int_0^{r} (r - p)\, e^{-ps} dp\; q_j(r)\, dr. \tag{13}$$

Using (13) in (12), one obtains

$$s - \frac{a_{jj}\Theta_j}{s} + \frac{a_{jj}}{s^2} (1 - e^{-\Theta_j s})\, L_j(y_j) = q_j(\Theta_j)\, e^{-\Theta_j s}$$

$$+ a_{jj} e^{-\Theta_j s} \int_0^{\Theta_j} \int_0^{r} (r - p)\, e^{-ps}\, dp\; q_j(r)\, dr + \sum_{i=1}^{j-1} a_{ji} I_{ji}(s) \tag{14}$$

$$+ L_j(W_j) \equiv R_j(s) \, .$$

Note that, as a result of this equation, the Laplace integral of the unknown function y_j is now given in terms of quantities which are completely known. To simplify the notation further, observe that the coefficient of $L_j(y_j)$ in (14) has a removable singularity at $s = 0$, and define

$$G_j(s) = \begin{cases} s - \dfrac{a_j \Theta_j}{s} + \dfrac{a_j}{s^2} (1 - e^{-\Theta_j s}), & s \neq 0, \\ - \dfrac{a_j \Theta_j}{2}, & s = 0 \, . \end{cases}$$

Further, let $R_j(s)$ represent the right-hand side of (14). Thus, after one applies the inversion theorem for the Laplace transformation, [5], one obtains formally the solution

$$y_j(t) = \frac{1}{2\pi i} \int_{\beta_j - i\infty}^{\beta_j + i\infty} \frac{e^{ts}}{G_j(s)} R_j(s)\, ds, \quad \beta_j > \alpha_j , \tag{15}$$

where the path of integration is along the line $\mathrm{Re}\, s = \beta_j$, and the line $\mathrm{Re}\, s = \alpha_j$ is located to the right of all the zeroes of G_j . The existence of such a line is proved in the next section.

<u>Formal solution is actual solution</u> . --

<u>Theorem 1</u>. Let $q_j(t)$ be continuous on $0 \leq t \leq \theta_j$, and let $W_j(t)$ satisfy (7), $j = 1,\ldots, n$. Then the components of the solution of system (6) with initial conditions (5) are given by (15), provided that β_j is sufficiently large, $j = 1,\ldots, n$.

<u>Proof</u>: It suffices to prove that each component $y_j(t)$ of the solution of (6) satisfies an inequality of the form

$$|y_j(t)| \leq c_{j3}\, e^{c_{j4} t} , \quad t > \theta_j , \tag{16}$$

for suitably chosen constants c_{j3} and c_{j4} , since this implies that the Laplace integral of each y_j converges absolutely, and thus represents an analytic function of s for $\mathrm{Re}\, s > c_{j4}$, and consequently also $G_j(s) \neq 0$ for $\mathrm{Re}\, s > c_{j4}$. Further, this implies the existence of the line $\mathrm{Re}\, s = \beta_j$ such that all the zeroes of $G_j(s)$ are located to the left of this line. By the method of successive approximations it can be shown that each y_j is continuous and of bounded variation for $t \geq \Theta_j$, and thus that the application of the inversion theorem is justified. Moreover, the absolute convergence of the Laplace integral justifies all interchanges of order of integration which were made in the formal derivation of the preceding section. The proof is by induction.

The inequality (16) for the case $j = 1$ has been established by a theorem of Nohel [6]. Since the proof in that case is essentially similar to the induction step of the present theorem, it will be omitted here. Assume, then, that (16) holds for $j = 1, \ldots, (k - 1)$. Now observe that the solution of the **k'th** equation of (6) can be found from the integral equation

$$y_k(t) = q_k(\Theta_k) + a_{kk} \int_{\Theta_k}^{t} (\Theta_k - h)\, y_k(\tau - h)\, dh\, d\tau + \int_{\Theta_k}^{t} W_k(\tau)\, d\tau \qquad (17)$$

$$+ \sum_{i=1}^{k} a_{ki} \int_{\Theta_k}^{t} \int_{0}^{\Theta_k} (\Theta_k - h)\; y_i\,(\tau - h)\, dh\, d\tau\,.$$

Since $q_k(t)$ is continuous on $0 \leq t \leq \Theta_k$, it is uniformly bounded there, and thus $y_k(t)$ satisfies (16) on this interval. Suppose that the inequality (16) holds on $0 \leq t \leq T$, $T > \Theta_k$, and T is arbitrary. Without loss of generality, it can be assumed that c_{k2} and c_{k4} are both positive. If one uses (7) and both induction assumptions in (17), one obtains for the interval $T \leq t \leq T + \theta_k$,

$$|y_k(t)| \leq |q_k(\theta_k)| + |a_{kk}| \int_{\theta_k}^{t} \int_{0}^{\theta_k} (\theta_k - h)\, c_{k3}\, e^{c_{k4}(\tau - h)}\, dh\, d\tau$$

$$+ \int_{\theta_k}^{t} c_{k1}\, e^{c_{k2}\tau}\, d\tau + \sum_{i=1}^{k-1} |a_{ki}| \int_{\theta_k}^{t} \int_{0}^{\theta_k} (\theta_k - h)\, c_{i3} e^{c_{i4}(\tau - h)}\, dh\, d\tau .$$

After one performs the integrations, and makes crude estimates,

$$|y_k(t)| \leq |q_k(\theta_k)| + |a_{kk}| \left|\frac{c_{k3}}{c_{k4}}\right| \left[\theta_k + \frac{e^{-c_{k4}\theta_k}}{c_{k4}}\right] \left(e^{c_{k4}t} + \frac{c_{k1}}{c_{k2}} e^{c_{k2}t}\right)$$

$$+ \sum_{i=1}^{k-1} |a_{ki}| \left|\frac{c_{i3}}{c_{i4}}\right| \left[\theta_k + \frac{e^{-c_{i4}\theta_k}}{c_{i4}}\right] e^{c_{i4}t} .$$

If one chooses $c_{k4} > c_{i4}$ for $i < k$, and $c_{k4} > c_{k2}$,

$$|y_k(t)| \leq |q_k(\theta_k)| + |a_{kk}| \left|\frac{c_{k3}}{c_{k4}}\right| \left[\theta_k + \frac{e^{-c_{k4}\theta_k}}{c_{k4}} + \frac{c_{k1}}{c_{k2}}\right] e^{c_{k4}t}$$

$$+ \sum_{i=1}^{k-1} |a_{ki}| \left|\frac{c_{i3}}{c_{i4}}\right| \left[\theta_k + \frac{e^{-c_{i4}\theta_k}}{c_{i4}}\right] e^{c_{k4}t} .$$

If c_{k2}, c_{k4}, and c_{i4} are chosen sufficiently large, the desired inequality will thus hold for all $t \geq \theta_k$.

Real representation theorem. -- For the purpose of establishing criteria for stability of the identically zero solution of (4), the solution of the system (6), (5) in the form of the contour integral (15) is unsatisfactory. Rather, an analogue of the variation-of-constants formula in real form is needed, and this can be obtained when one applies the convolution theorem below to the form of the solution (15), after showing that $\frac{1}{G_j(s)}$ has an inverse transform, $j = 1,\ldots, n$; it is clear that each term of $R_j(s)$ has an inverse transform.

It will be shown that $\frac{1}{G_j(s)}$ satisfies the hypotheses of the following theorem [7]: let $f(s)$ be analytic in the half-plane $\operatorname{Re} s > \alpha \geq 0$ and have the representation

$$f(s) = \frac{c}{s} + \frac{u(s)}{s^{1+\delta}}, \quad \delta > 0,$$

where $u(s)$ is bounded. Then $f(s)$ is the Laplace transform of the function

$$F(t) = \text{pr.v.}\ \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{ts} f(s)\, ds, \quad x > \alpha,$$

where pr.v. denotes the Cauchy principal value of the integral. Observe that, since $G_j(s)$ is analytic for all s, as can be seen readily, $\frac{1}{G_j(s)}$ is also analytic for $s > \alpha_j$, where $\operatorname{Re} s = \alpha_j$ is the line with the property that all zeroes of $G_j(s)$ lie to the left of this line. Observe also that

$$\frac{1}{G_j(s)} = \frac{1}{s} + \frac{s - G_j(s)}{sG_j(s)}, \tag{18}$$

and it will be shown in the next lemma that

$$\frac{s - G_j(s)}{sG_j(s)} = O\left(\frac{1}{|s|^3}\right).$$

Thus $\frac{1}{G_j(s)}$ is the Laplace transform of the function

$$K_j(t) = \lim_{x \to +\infty} \frac{1}{2\pi i} \int_{\beta_j - ix}^{\beta_j + ix} \frac{e^{ts}}{G_j(s)}\, ds, \quad \beta_j > \alpha_j , \tag{19}$$

where the existence of the line $\operatorname{Re} s = \alpha_j$ has been established by Theorem 1.

Lemma 1. $K_j(t)$, defined by (19), has the following properties:

i) $K_j(t) = 0$ for $t < 0$

ii) $K_j'(t) = a_{jj} \int_0^{\theta_j} j(\theta_j - h)\, K_j(t - h)\, dh, \; t > \theta_j$

iii) $K_j(0^+) = 1$.

Proof: i) If $t < 0$, the integral defining $K_j(t)$ is zero, since no new singularities are encountered in shifting the contour to $+\infty$.

ii) Observe that this states that $K_j(t)$ is a solution of the homogeneous equation

$$y_j'(t) = a_{jj} \int_0^{\theta_j} j(\theta_j - h)\, y_j(t - h)\, dh, \quad t > \theta_j ,$$

for each $j = 1, \ldots, n$. To prove this, one would like to differentiate the integral (19) which defines K_j . But since this cannot be done directly, we observe first that (18) can be written

$$\frac{1}{G_j(s)} = \frac{1}{s} + \frac{s - G_j(s)}{sG_j(s)} = \frac{1}{s} + Q_j(s) ,$$

where

$$Q_j(s) = \frac{\frac{a_{jj}\theta_j}{s} - \frac{a_{jj}}{s^2}\left(1 - e^{-\theta_j s}\right)}{sG_j(s)} = O\left(\frac{1}{|s|^3}\right) \text{ as } |s| \to +\infty , \tag{20}$$

if $\operatorname{Re} s > \alpha_j$. Thus, for $t > 0$, we obtain

$$K_j(t) = \lim_{x \to +\infty} \frac{1}{2\pi i} \int_{B_j - ix}^{B_j + ix} \frac{e^{ts}}{s} + e^{ts} Q_j(s)\, ds,$$

or

$$K_j(t) = 1 + \lim_{x \to +\infty} \int_{B_j - ix}^{B_j + ix} e^{ts} \left[\frac{\frac{a_{jj}\Theta_j}{s} - \frac{a_{jj}}{s^2}(1 - e^{-\Theta_j s})}{sG_j(s)} \right] ds. \quad (21)$$

Now from (20) it is clear that (21) can be differentiated to give for $t > \Theta_j$ [8]

$$K_j'(t) = \lim_{x \to +\infty} \frac{1}{2\pi i} \int_{B_j - ix}^{B_j + ix} e^{ts} \left[\frac{\frac{a_{jj}\Theta_j}{s} - \frac{a_{jj}}{s^2}(1 - e^{-\Theta_j s})}{G_j(s)} \right] ds. \quad (22)$$

To show that the integral in (22) is the right-hand side of ii), observe that

$$\frac{\Theta_j}{s} - \frac{1 - e^{-\Theta_j s}}{s^2} = \int_0^{\Theta_j} (\Theta_j - h)\, e^{-sh}\, dh.$$

After this expression is substituted into (21) and the order of integration is interchanged,

$$K_j'(t) = a_{jj} \int_0^{\Theta_j} (\Theta_j - h) \left[\lim_{x \to +\infty} \frac{1}{2\pi i} \int_{\beta_j - ix}^{\beta_j + ix} \frac{e^{s(t-h)}}{G_j(s)}\, ds \right] dh, \; t > \Theta_j ,$$

which is just the result stated in ii), using the definition of $K_j(t)$ given by (19).

iii) The proof follows from (21); we can write

$$K_j(0^+) = \lim_{t \to 0^+} \lim_{x \to \infty} \frac{1}{2\pi i} \int_{\beta_j - ix}^{\beta_j + ix} \frac{e^{ts}}{s} + e^{ts} Q_j(s)\, ds, \ \beta_j > \alpha_j .$$

or by uniform convergence in t, for small t, the limits and the integration can be interchanged, and thus

$$K_j(0^+) = 1 + \lim_{x \to \infty} \frac{1}{2\pi i} \int_{\beta_j - ix}^{\beta_j + ix} \lim_{t \to 0^+} e^{ts} Q_j(s)\ ds ,$$

yielding

$$K_j(0^+) = 1 .$$

<u>Theorem 2</u>. The solution of the functional system (6) satisfying the initial conditions (5) is given by

$$y_j(t) = q_j(\theta_j)\, K_j(t - \theta_j) + a_{jj} \int_0^{\theta_j} \int_0^r (r - p)\, K_j(t - \theta_j - p)\, dp\, q_j(r)\, dr$$

$$+ \int_{\theta_j}^t K_j(t - \tau)\, W_j(\tau)\, d\tau + \sum_{i=1}^{j-1} a_{ji} \int_{\theta_j}^t K_j(t - \tau) \int_0^{\theta_j} (\theta_j - h)\, y_i(\tau - h)\, dh\, d\tau,$$

$j = 1, \ldots, n$, where $K_j(t)$ is defined by (19).

<u>Proof</u>: In <u>Theorem 1</u> it was shown that the formal solution (15) of (6) for $t > \theta_j$ is an actual solution. Now (15) may be written in the form

$$y_j(t) = q_j(\theta_j) \lim_{x \to +\infty} \int_{\beta_j - ix}^{\beta_j + ix} \frac{e^{-(t - \theta_j)s}}{G_j(s)}\ ds$$

$$+ a_{jj} \lim_{x \to +\infty} \frac{1}{2\pi i} \int_{\beta_j - ix}^{\beta_j + ix} \frac{e^{s(t - \theta_j)}}{G_j(s)} \int_0^{\theta_j} \int_0^r (r - p) e^{-ps}\, dp\, q_j(r)\, dr\, ds$$

$$+ \lim_{x \to +\infty} \frac{1}{2\pi i} \int_{\beta_j - ix}^{\beta_j + ix} [\frac{e^{ts}}{G_j(s)} \int_{\theta_j}^{\infty} e^{-sr}\, W_j(r)\, dr]\, ds$$

$$+ \lim_{x \to +\infty} \frac{1}{2\pi i} \sum_{i=1}^{j-1} a_{ji} \int_{\beta_j - ix}^{\beta_j + ix} \left\{ \frac{e^{ts}}{G_j(s)} \int_{\theta_j}^{\infty} [e^{-sr} \int_0^{\theta_j} (\theta_j - h) y_i(r-h)\, dh]\, dr \right\} ds,$$

$\beta_j > \alpha_j$, where the definition of R_j from (14) is used. The definition of $K_j(t)$ and the convolution theorem [9] applied to this equation give the desired result. Note that

$$\lim_{t \to \theta_j^+} y_j(t) = \lim_{t \to \theta_j^+} K_j(t - \theta_j) \; q_j(\theta_j) = q_j(\theta_j) \quad .$$

CHAPTER III

BEHAVIOR OF $K_j(t)$ FOR LARGE t

Before the nonlinear system can be considered, the behavior of solutions of the linear system as t tends to infinity must be examined more specifically. The following results, obtained by Nohel [10], [11] are important for this purpose.

Lemma 1.-- The constants a_{jj} in (6) play the following role:

i) if $a_{jj} > 0$, $G_j(s)$ has at least one zero with a positive real part.

ii) if $a_{jj} < 0$, let $a_{jj} = \frac{-(w_j)2}{\theta_j}$. Then, if $w_j \neq \frac{2n\pi}{\theta_j}$, n = 1, 2, . . . , all zeroes of $G_j(s)$ lie strictly in the left s-half-plane.

iii) if $a_{jj} = \frac{-w_j^2}{\theta_j}$, $w_j = \frac{2n\pi}{\theta_j}$, $n = \pm 1, \pm 2, \ldots$, $G_j(s)$ has exactly one pair of zeroes on the imaginary axis for each n, while all remaining zeroes lie in the left s-half-plane.

iv) if $a_{jj} = 0$ for some $j \leq n$, the function y_j can be found immediately by integration of the j'th equation, since each y_i, for $i < j$, is a known function of t. Note that in this case $G_j(s) = s$, and the only zero of this function is at $s = 0$.

Proof: i) For each $j \geq 1$, $G_j(s)$ is continuous for all s, and $G_j(0) < 0$.

Further,

$$\lim_{s \to +\infty} G_j(s) = +\infty .$$

Thus $G_j(s)$ must have at least one real positive zero as a consequence of the intermediate-value theorem.

ii) Let $a_{jj} = \frac{-w_j^2}{\theta_j}$, and let $s = i\eta$ for some real number η.

Then

$$G_j(i\eta) = \begin{cases} \frac{w_j^2}{\theta_j}\left[\frac{1-\cos\theta_j\eta}{\eta^2}\right] + i\left[\frac{\eta^3 - {}_j^2 + {}_j^2\sin\theta_j\eta}{\eta^2}\right] & \text{if } \eta \neq 0 \\ \frac{w_j^2\,\theta_j}{2} & \text{if } \eta = 0 . \end{cases} \tag{23}$$

Observe that the real part of $G_j(i\eta)$ will be zero if, and only if, $\theta_j\eta = 2\,n\pi$, i.e., $\eta = \frac{2n\pi}{\theta_j}$, $n = \pm 1, \ldots$. Since $G_j(s)$ is entire in s, the argument principle can be used. A contour in the form of a large semicircle C is taken in the right half-plane, and the change in $(\arg G_j(s))$ in going around the boundary of C is computed.

Let s trace out the contour ABCDA in the s-plane shown in Fig. 1, and compute $\Delta \arg G_j(s)$ on ABCDA. First, on AB, let $s = Re^{i\phi}$, assuming R large. From (15),

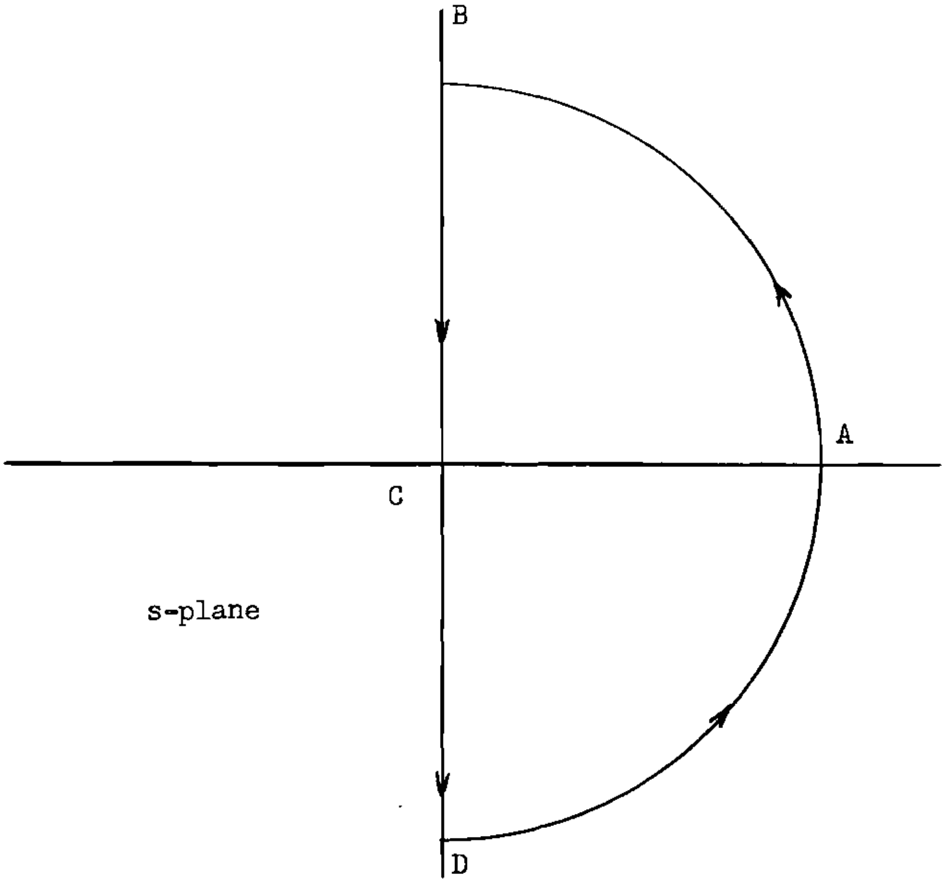

Figure 1. Contour for Integral in Lemma 1.

$$\Delta \arg G_j(s) = \Delta \arg R^3 e^{3i\phi} + \arg (1 + O(\frac{1}{R^2})) - \Delta \arg (R^2 e^{2i\phi}) ,$$

or

$$\Delta \arg G_j(s) = \Delta \arg R e^{i\phi} + \Delta \arg(1 + O(\frac{1}{R^2})) .$$

Thus, on AB, $\Delta \arg G_j(s) = \frac{\pi}{2}$, noting that, for large real s, $G_j(s)$ is real and positive, and $\arg G_j(A)$ is zero. On BC, let $s = i\eta$. Then, from (23), since $w_j \neq \frac{2n\pi}{\Theta_j}$, $G_j(i\eta) \neq 0$, and $\arg G_j(i\eta)$ is a continuous function of n with $\arg G_j(i - 0)= 0$. Thus, as s traces out the line segment BC, $\arg G_j(s)$ decreases from $\frac{\pi}{2}$ to zero. The segment CD is handled in the same way as the segment BC, and; in this case, $\arg G_j(s)$ decreases from zero to $-\frac{\pi}{2}$, while the arc DA is handled in the same way as the arc AB, and $\arg G_j(s)$ increases from $-\frac{\pi}{2}$ to zero. When these results are combined, one obtains

$$\arg G_j(s) = \arg G_j(s)\ [\widehat{AB}] + \arg G_j(s)\ [\overline{BC}] + \arg G_j(s)\ [\overline{CD}] + \arg G_j(s)\ [\widehat{DA}] ,$$

or,

$$\arg G_j(s) = \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} + \frac{\pi}{2} ,$$

which yields

$$\Delta \arg G_j(s) = 0$$

on ABCDA, provided that $w_j \neq \frac{2n\pi}{\theta_j}$, $n = \pm 1, \pm 2, \ldots$, and thus, by the argument principle, all zeroes of $G_j(s)$ lie in the left s-half-plane.

iii) If $w_j = \frac{2n\pi}{\theta_j}$, $n = \pm 1, \pm 2, \ldots$, we find from (23) that $\eta = \frac{2n\pi}{\theta_j}$, $n = \pm 1, \pm 2, \ldots$, is a zero of the function $G_j(i\eta)$. If a contour similar to Fig. 1, but indented at the zeroes is taken, it is seen that there are no zeroes of G_j in the right half-plane.

This lemma yields a result better than <u>Theorem 1</u> about the existence of the line Re $s = \alpha_j$ which has the property that all zeroes of $G_j(s)$ lie to the left of this line in the case $a_{jj} < 0$.

<u>Lemma 2</u>.-- If $a_{jj} < 0$, $a_{jj} = \frac{-w_j^2}{\theta_j}$, $w_j \neq \frac{2n\pi}{\theta_j}$, $n = \pm 1, \pm 2, \ldots$, there exists a constant $\lambda_j > 0$, such that

$$K_j(t) = O(e^{-\frac{\lambda_j}{2} t}), \quad t > \theta_j, \quad j = 1, 2, \ldots, n.$$

<u>Proof</u>: If $w_j \neq \frac{2n\pi}{\theta_j}$, and if $a_{jj} = \frac{-w_j^2}{\theta_j}$, then, by Lemma 1, all zeroes of $G_j(s)$ lie in the left half-plane; hence, there exists a constant $\lambda_j > 0$, such that all zeroes of $G_j(s)$ lie in the half-plane

$$\text{Re } s < -\lambda_j < 0 .$$

Recalling the definition of $K_j(t)$ in (19), the path Re $s = \beta_j$ may taken as

$$\text{Re } s = \frac{-\lambda_j}{2}, \quad \lambda_j > 0 .$$

Thus

$$K_j(t) = \frac{1}{2\pi i} \lim_{x \to +\infty} \int_{-\frac{\lambda_j}{2} - ix}^{-\frac{\lambda_j}{2} + ix} \frac{e^{ts}}{G_j(s)} \, ds, \quad t > \Theta_j .$$

As before,

$$\frac{1}{G_j(s)} = \frac{1}{s} + Q_j(s) ,$$

where

$$Q_j(s) = O\left(\frac{1}{|s|^3}\right)$$

as $|s| \to \infty$, for $\text{Re } s > - \lambda_j$; therefore

$$K_j(t) = \frac{1}{2\pi i} \lim_{x \to +\infty} \int_{-\frac{\lambda_j}{2} - ix}^{-\frac{\lambda_j}{2} + ix} \frac{e^{ts}}{s} \, ds + \lim_{x \to +\infty} \int_{-\frac{\lambda_j}{2}}^{-\frac{\lambda_j}{2} + ix} e^{ts} Q_j(s) \, ds,$$

or, equivalently,

$$K_j(t) = \frac{1}{2\pi i} \lim_{x \to +\infty} \int_{-\frac{\lambda_j}{2} - ix}^{-\frac{\lambda_j}{2} + ix} e^{ts} \, Q_j(s) \, ds .$$

Now let

$$s = \frac{-\lambda_j}{2} + iv .$$

Then

$$K_j(t) = \frac{e^{-\frac{\lambda_j}{2}t}}{2\pi} \lim_{x \to +\infty} \int_{-x}^{x} e^{itv} \, Q_j\left(-\frac{\lambda_j}{2} + iv\right) dv, \quad t > \Theta_j .$$

Since

$$\int_{-\infty}^{\infty} |Q_j(-\frac{\lambda_j}{2} + iv)| \, dv$$

is convergent, the result follows. It is clear that if $a_{jj} = -\frac{w_j^2}{\theta_j}$, $w_j = \frac{2n\pi}{\theta_j}$, $n = \pm 1, \pm 2, \ldots$, $K_j(t)$ is merely uniformly bounded, and if $a_{jj} > 0$, $K_j(t)$ becomes unbounded exponentially.

CHAPTER IV

STABILITY

Before proceeding to the stability theorem for the nonlinear system, note that Lemma 2, combined with the representation theorem, implies that if $a_{jj} = -\frac{w_j^2}{\theta_j}$, $w_j \neq \frac{2k\pi}{\theta_j}$, $k = \pm 1, \pm 2, \ldots$, the identically zero solution of the linear system associated with (4),

$$y_j' = \sum_{i=1}^{j} a_{ji} \int_0^{\theta_j} (\theta_j - h)\, y_i(t-h)\, dh \,, \tag{24}$$

is asymptotically stable; if $w_j = \frac{2k\pi}{\theta_j}$, it is merely stable, and if $a_{jj} > 0$, it is unstable. However, as is well known for ordinary differential equations, it does not follow without proof that the same results hold for the corresponding nonlinear problem.

The following result [12] will be employed:

Lemma 3.-- Let F, G, and H be continuous real-valued functions for $t \geq a$, with $H(t) > 0$, and suppose that, for $t \geq a$,

$$F(t) \leq G(t) + \int_a^t H(s)\, F(s)\, ds \,. \tag{25}$$

Then, for $t \geq a$,

$$F(t) \leq G(t) + \int_a^t H(s)\, G(s) \exp\left(\int_s^t H(u)\, du\right) ds \,.$$

Proof: Let

$$R(t) = \int_a^t H(s)\, F(s)\, ds \,. \tag{26}$$

Then

$$R'(t) = H(t)\, F(t) \quad ,$$

and, from (30),

$$R'(t) - H(t)\, R(t) \le H(t)\, G(t) \; .$$

Multiplying by $\exp(- \int_a^t H(u)\, du$,

$$\frac{d}{dt}\left(R(t) e^{- \int_a^t H(u)\, du}\right) \le H(t)\, G(t) e^{- \int_a^t H(u)\, du} \; .$$

Integrating from a to t,

$$R(t) e^{- \int_a^t H(u)\, du} \le \int_a^t H(s)\, G(s)\, e^{- \int_a^t H(u)\, du}\, ds \; ,$$

or

$$R(t) \le \int_a^t H(s)\, G(s)\, e^{- \int_s^t H(u)\, du}\, ds \; . \qquad (27)$$

Using (27) and (26) in (25) ,

$$F(t) \le G(t) + \int_a^t H(s)\, G(s) \exp(\int_s^t H(u)\, du)\, ds \; .$$

The principal result can now be stated.

<u>Theorem 3</u>.-- In the nonlinear functional system (4), let $a_{jj} < 0$,

$a_{jj} = \frac{-w_j^2}{\theta_j}$, $w_j \neq \frac{2k\pi}{\theta_j}$, $k = \pm 1, \pm 2, \ldots$, $j = 1, 2, \ldots, n$.

Let g_j be continuous in (t, u) for $t \geq 0$ and $||u||_j$ small. Moreover, let

$$g_j(t, u) = o(||u||_j) \quad ,$$

as $||u||$ tends to zero, uniformly in t, for $t \geq 0$, and assume each of the initial conditions q_j to be continuous for $0 \le t \le \theta_j$, $j = 1, \ldots, n$.

Then the identically zero solution of (4) is asymptotically stable.

Proof: The solution components $\phi_j(t)$ of (4), with

$$\delta = \max_{\substack{0 \le t \le \Theta_j \\ \text{all } j}} |q_j(t)| \tag{28}$$

can be continued for increasing $t > \Theta_j$, so long as $||\phi||_j$ remains small. As long as $\phi_j(t)$ exists, the representation theorem for the linear system suggests that the components $\phi_j(t)$ satisfy the integral equation

$$\phi_j(t) = x_{j0}(t) + \int_{\Theta_j}^{t} K_j(t - \tau) \int_0^{\Theta_j} (\Theta_j - h)\, g_j[\tau - h, \phi_1(\tau - h), \ldots, \phi_j(\tau - h)]dh\, d\tau$$

$$+ \int_{\Theta_j}^{t} K_j(t - \tau) \sum_{i=1}^{j-1} a_{ji} \int_0^{\Theta_j} (\Theta_j - h)\, \phi_i(\tau - h)\, dh\, d\tau ,$$

where

$$x_{j0}(t) = q_j(\Theta_j)K_j(t - \Theta_j) - \frac{w_j^2}{\Theta_j} \int_0^{\Theta_j} \int_0^{r} [(r - p)K_j(t - \Theta_j - p]dp\, q_j(r)\, dr \tag{30}$$

and, by Theorem 2,

$$x_{j0}(t) + \int_{\Theta_j}^{t} K_j(t - \tau) \sum_{i=1}^{j-1} a_{ji} \int_0^{\Theta_j} (\Theta_j - h)\, \phi_i(\tau - h)\, dh\, d\tau$$

is that solution of (24) satisfying

$$y_j(t) = q_j(t),\ 0 \le t \le \Theta_j ,$$

where a_{jj} in Theorem 2 has been replaced by $\left(\frac{-w_j^2}{\theta_j}\right)$. This can be verified by direct substitution.

First, it is shown that there exist constants $\delta_{j1} > 0$ and $\lambda_j > 0$, such that

$$|x_{j0}(t)| \leq \delta_{j1} e^{-\frac{\lambda_j}{2}(t - \theta_j)} \quad t > \theta_j, \; j = 1, \ldots, n, \tag{31}$$

where δ_{j1} is a constant which will be small provided only that δ_j in (28) is small. To prove this, recall that, by Lemma 2, there exist constants $\lambda_j > 0$ and $c_{j1} > 0$, such that

$$K_j(t) \leq c_{j1} e^{-\frac{\lambda_j}{2} t}, \; t > 0, \; j = 1, 2, \ldots, n. \tag{32}$$

If (28) and (32) are used in (30), (30) becomes

$$|x_{j0}(t)| \leq \delta \, c_{j1} e^{-\frac{\lambda_j}{2}(t - \theta_j)} + \frac{w_j^2}{\theta_j} c_{j1} \delta_j e^{-\frac{\lambda_j}{2}(t - \theta_j)} \int_0^{\theta_j} \int_0^{r} (r - p) \, e^{\frac{\lambda_j}{2} p} \, dp \, dr.$$

Now define

$$c_{j2} = \int_0^{\theta_j} \int_0^{r} (r - p) \, e^{\frac{\lambda_j}{2} p} \, dp \, dr,$$

and

$$\delta_{j1} = \delta \, c_{j1} \left(1 + \frac{w_j^2}{\theta_j} c_{j2}\right). \tag{33}$$

This proves (31), and (33) shows that δ_{j1} can be made small by taking δ small.

If one uses (31) and (32) in the integral equation (29), one obtains for $t > \theta_j$,

$$|\phi_j(t)| \leq \delta_{j1}\, e^{-\frac{\lambda_j}{2}(t-\Theta_j)}$$

$$+ c_{j1} \int_{\Theta_j}^{t} e^{-\frac{\lambda_j}{2}(t-\tau)} \int_0^{\Theta_j} j(\Theta_j - h)\left|g_j\left[\tau-h,\ \phi_1(\tau-h),\ldots\phi_{jj}(\tau-h)\right]\right| dh\, d\tau$$

$$+ c_{j1} \int_{\Theta_j}^{t} e^{-\frac{\lambda_j}{2}(t-\tau)} \left|\sum_{i=1}^{j-1} a_{ji} \int_0^{\Theta_j} j(\Theta_j - h)\, \phi_i(\tau-h)\, dh\right| d\tau .$$

If one employes the order condition on g_j and Definition 3, one obtains for $t > \Theta_j$, and $||\phi||_j$ sufficiently small,

$$|\phi_j(t)| \leq \delta_{j1}\, e^{-\frac{\lambda_j}{2}(t-\Theta_j)} + \varepsilon c_{j1} \int_{\Theta_j}^{t} e^{-\frac{\lambda_j}{2}(t-\tau)} \int_0^{\Theta_j} j(\Theta_j - h) \sum_{i=1}^{j-1} |\phi_i(\tau-h)|\, dh\, d\tau$$

$$+ \varepsilon c_{j1} \int_{\Theta_j}^{t} e^{-\frac{\lambda_j}{2}(t-\tau)} \int_0^{\Theta_j} j(\Theta_j - h)\, |\phi_j(\tau-h)|\, dh\, d\tau$$

$$+ c_{j1} \int_{\Theta_j}^{t} e^{-\frac{\lambda_j}{2}(t-\tau)} \int_0^{\Theta_j} j(\Theta_j - h) \sum_{i=1}^{j-1} |a_{ji}|\, |\phi_i(\tau-h)|\, dh\, d\tau .$$

After the terms involving sums are combined,

$$|\phi_j(t)| \leq \delta_{j1} e^{-\frac{\lambda_j}{2}(t-\Theta_j)} + \varepsilon c_{j1} \int_{\Theta_j}^{t} e^{-\frac{\lambda_j}{2}(t-\tau)} \int_0^{\Theta_j} j(\Theta_j - h)\, |\phi_j(\tau-h)|\, dh\, d\tau \tag{34}$$

$$+ c_{j1} \sum_{i=1}^{j-1} \left\{|a_{ji}| + \varepsilon\right\} \int_{\Theta_j}^{t} e^{-\frac{\lambda_j}{2}(t-\tau)} \int_0^{\Theta_j} j(\Theta_j - h)\, |\phi_i(\tau-h)\, dh\, d\tau .$$

If both sides of (34) are multiplied by $e^{\frac{\lambda_j}{2} t}$, and U_j is defined by

$$U_j(t) = e^{\frac{\lambda_j}{2}}\, |\phi_j(t)| ,$$

one obtains, after interchanging the order of integration in the first double integral,

$$U_j(t) \leq \delta_{j1} e^{\lambda_j \frac{\Theta_j}{2}} + \varepsilon c_{j1} \int_0^{\Theta_j} (\Theta_j - h) \int_{\Theta_j}^{t} e^{\frac{\lambda_j h}{2}} U_j(\tau - h)\, d\tau\, dh$$

$$+ c_{j1} \sum_{i=1}^{j-1} [|a_{ji}| + \varepsilon] \int_{\Theta_j}^{t} e^{\frac{\lambda_j}{2}\tau} \int_0^{\Theta_j} (\Theta_j - h) |\phi_i(\tau - h)|\, dh\, d\tau .$$

if $||\phi||_j$ is sufficiently small. Let $(\tau - h)$ be a new variable in the first double integral, and call it τ again. Then

$$U_j(t) \leq \delta_{j1} e^{\lambda_j \frac{\Theta_j}{2}} + \varepsilon c_{j1} \int_0^{\Theta_j} (\Theta_j - h)\, e^{\frac{\lambda_j}{2} h} \int_{\Theta_j - h}^{\Theta_j} U_j(\tau)\, d\tau\, dh \tag{35}$$

$$+ \varepsilon c_{j1} \int_0^{\Theta_j} (\Theta_j - h)\, e^{\frac{\lambda_j}{2} h} \int_{\Theta_j}^{t-h} U_j(\tau)\, d\tau\, dh$$

$$+ c_{j1} \sum_{i=1}^{j-1} [|a_{ji}| + \varepsilon] \int_{\Theta_j}^{t} e^{\frac{\lambda_j}{2}\tau} \int_0^{\Theta_j} (\Theta_j - h) |\phi_i(\tau - h)|\, dh\, d\tau$$

for $t > \Theta_j$, provided that $||\phi||_j$ is sufficiently small.

Define

$$F_j(t) = \sum_{i=1}^{j-1} [|a_{ji}| + \varepsilon] \int_{\Theta_j}^{t} e^{\frac{\lambda_j}{2}\tau} \int_0^{\Theta_j} (\Theta_j - h) |\phi_i(\tau - h)|\, dh\, d\tau \tag{36}$$

$j = 2, \ldots n$, and $F_1(t) \equiv 0$.

The following lemma will be employed:

<u>Lemma 4</u>. -- Under the hypotheses of <u>Theorem 3</u>, there exist positive constants γ_j, σ_j, such that

$$F_j(t) \leq \gamma_j e^{\left(\frac{\lambda_j}{2} - \sigma_j\right) t},$$

where $\gamma_j \to 0$ as $\delta \to 0$, $j = 1, \ldots, n$, and where λ_j is defined as in

(31). Further, without loss of generality, it can be assumed that

$$\sigma_j < \min_{i = 1, \ldots, n} \frac{\lambda_i}{4} \quad . \tag{37}$$

Proof will be postponed.

When Lemma 4 is used in (35), one obtains

$$U_j(t) \leq \delta_{j1} e^{\frac{\lambda_j \Theta_j}{2}} + \varepsilon c_{j1} \int_0^{\Theta_j} (\Theta_j - h) e^{\frac{\lambda_j}{2}h} \int_{\Theta_j - h}^{\Theta_j} U_j(\tau)\, d\tau\, dh \tag{38}$$

$$+ \varepsilon c_{j1} \int_0^{\Theta_j} (\Theta_j - h) e^{\frac{\lambda_j}{2} h} \int_{\Theta_j}^{t - h} U_j(\tau)\, d\tau\, dh + c_{j1}\, {}_j e^{(\frac{\lambda_j}{2} - \sigma_j)t} \quad ,$$

if $t > \Theta_j$, and $||\phi||_j$ is sufficiently small. Note that $U_j(t)$ is known on $0 \leq t \leq \Theta_j$, and can be expressed in terms of $q_j(t)$ there. Thus the first integral in (38) is a constant. Let

$$\delta_{j2} = \int_0^{\Theta_j} (\Theta_j - h) e^{\frac{\lambda_j}{2} h} \int_{\Theta_j - h}^{\Theta_j} U_j(\tau)\, d\tau\, dh \ , \tag{39}$$

it is evident that δ_{j2} will be small if δ_j is sufficiently small. Using (39) and the fact that $U_j(t) \geq 0$ in (38), we obtain, for $t > \Theta_j$, and $||\phi||_j$ sufficiently small,

$$U_j(t) \leq \delta_{j1} e^{\frac{\lambda_j \Theta_j}{2}} + \varepsilon c_{j1} \delta_{j2} + \varepsilon c_{j1} \int_0^{\Theta_j} (\Theta_j - h) e^{\frac{\lambda_j}{2} h} \int_{\Theta_j}^{t} U_j(\tau)\, d\tau\, dh$$

$$+ c_{j1}\, \gamma_j\, e^{(\frac{\lambda_j}{2} - \sigma_j)\, t} \quad .$$

Letting

$$c_{j3} = \int_0^{\theta_j} (\theta_j - h) e^{\frac{\lambda_j}{2} h}\, dh,$$

a positive constant, (39) simplifies to

$$U_j(t) \leq \delta_{j1} e^{\frac{\lambda_j}{2} \theta_j} + \varepsilon\, c_{j1} \delta_{j2} + c_{j1}\, \gamma_j e^{(\frac{\lambda_j}{2} - \sigma_j)t} + \varepsilon c_{j1} c_{j3} \int_{\theta_j}^{t} U_j(\tau)\, d\tau, \quad (40)$$

if $t > \theta_j$, and $||\phi||_j$ is sufficiently small. Applying Lemma 3 to (40) yields, for $||\phi||_j$ sufficiently small,

$$U_j(t) \leq \delta_{j1}\, e^{\frac{\lambda_j}{2} \theta_j} + \varepsilon\, c_{j1}\, \delta_{j2} + c_{j1}\, \gamma_j e^{(\frac{\lambda_j}{2} - \sigma_j)\, t}$$

$$+ \int_{\theta_j}^{t} \varepsilon\, c_{j1}\, c_{j3} \left[\delta_{j1}\, e^{\frac{\lambda_j}{2} \theta_j} + \varepsilon c_{j1}\, \delta_{j2} + c_{j1}\, \gamma_j e^{(\frac{\lambda_j}{2} - \sigma_j)s} \right] e^{\varepsilon c_{j1} c_{j3} (t-s)}\, ds.$$

Recall that, from (37),

$$0 < \sigma_j < \frac{\lambda_j}{4};$$

thus, if one chooses

$$\varepsilon < \frac{\lambda_j}{8\, c_{j1} c_{j3}},$$

which can be done by choosing $||\phi||_j$ small, one obtains, after integrating,

$$U_j(t) \leq \delta_{j1} e^{\frac{\lambda_j}{2} \theta_j} + \varepsilon\, c_{j1} \delta_{j2} + c_{j1}\, \gamma_j e^{(\frac{\lambda_j}{2} - \sigma_j)t} + \delta_{j1} e^{\frac{\lambda_j}{2} \theta_j} e^{\varepsilon c_{j1} c_{j3} (t - \theta_j)}$$

$$+ \varepsilon c_{j1} \delta_{j2} e^{\varepsilon c_{j1} c_{j3} (t - \theta_j)} - \delta_{j1} e^{\frac{\lambda_j}{2} \theta_j} - \varepsilon c_{j1} \delta_{j2}$$

$$+ \frac{c_{j1}^2\, {}_j\, \varepsilon c_{j3}}{\frac{\lambda_j}{2} - \sigma_j - \varepsilon c_{j1} c_{j3}} \left[e^{(\frac{\lambda_j}{2} - \sigma_j)t} - e^{\varepsilon c_{j1} c_{j3} t} e^{(\frac{\lambda_j}{2} - \sigma_j - \varepsilon c_{j1} c_{j3})\theta_j} \right].$$

After simplifying, and making crude estimates, this reduces to

$$U_j(t) \leq \left\{ \frac{(\frac{\lambda_j}{2} - \sigma_j)\, c_{ji}\, \gamma_j}{\frac{\lambda_j}{2} - \sigma_j - \varepsilon c_{j1} c_{j3}} + \delta_{j1} e^{\frac{\lambda_j}{2}\theta_j} + \varepsilon c_{j1} c_{j2} \right\} e^{(\frac{\lambda_j}{2} - \sigma_j)\, t}. \quad (41)$$

Noting that

$$\frac{\frac{\lambda_j}{2} - \sigma_j}{\frac{\lambda_j}{2} - \sigma_j - \varepsilon c_{j1} c_{j3}} < 2 ,$$

when ε and σ_j are chosen as above, (41) becomes

$$U_j(t) \leq \left[2\, c_{j1}\, \gamma_j + \delta_{j1} e^{-\theta_j} + \varepsilon c_{j1} c_{j2} \right] e^{(\frac{\lambda_j}{2} - \sigma_j)\, t},$$

if $t > \theta_j$, and $||\phi||_j$ is small. If now the definition of $U_j(t)$ is employed,

$$|\phi_j(t)| \leq |2c_{j1}\, \gamma_j + \delta_{j1} e^{\frac{\lambda_j}{2}\theta_j} + \varepsilon c_{j1} c_{j2}|\, e^{-\sigma_j t}, \quad (42)$$

if $t > \theta_j$, and $||\phi||_j$ is sufficiently small. If one lets

$$u_j = 2c_{j1}\, \gamma_j + \delta_{j1} e^{\frac{\lambda_j}{2}\theta_j} + \varepsilon c_{j1} c_{j2},$$

a positive number which goes to zero with δ, (42) reduces to

$$|\phi_j(t)| \leq u_j e^{-\sigma_j t} ,$$

if $t > \theta_1$, and $||\phi||_1$ is sufficiently small. Using Definition 3,

$$||\phi(t)||_1 = |\phi_1(t)| ,$$

it is clear that if δ is chosen sufficiently small, (43) will hold for all $t > \Theta_1$; moreover, since

$$|\phi_1(t)| < \delta ,$$

if $0 \leq t \leq \Theta_1$, $\phi_1(t)$ is uniformly bounded for all $t > 0$ by a constant which can be chosen arbitrarily small, provided only that δ is chosen sufficiently small. Next, we assume that the components $\phi_1, \phi_2, \ldots, \phi_{k-1}$ satisfy (43); as above, this will imply that there is a uniform bound for each component $\phi_1, \ldots, \phi_{k-1}$ for all $t > 0$, provided only that δ is sufficiently small. Thus, $||\phi||_k$ will be small so long as $|\phi_k(t)|$ and δ are both small enough. Thus $|\phi_k(t)|$ will satisfy (43) for all $t > \Theta_k$, and thus (43) holds for all $j = 1, \ldots, n$, provided δ is chosen sufficiently small. Moreover, (43) implies that $||\phi(t)||_j \to 0$ as $t \to +\infty$, $j = 1, \ldots, n$. Thus, by Definition 5, the identically zero solution of (4) is asymptotically stable, and hence is also stable. This completes the proof of the theorem, except for the lemma, which remains to be proved.

Lemma 4.-- Under the hypotheses of Theorem 3, there exist positive constants γ_j, σ_j, such that

$$F_j(t) \leq \gamma_j e^{(\frac{\lambda_j}{2} - \sigma_j) t},$$

$t > \Theta_j$, where $\gamma_j \to 0$ as $\delta \to 0$, $j = 1, \ldots, n$, and where λ_j is defined as in (31).

<u>Proof</u>: Recall the definition of $F_j(t)$:

$$F_j(t) = \sum_{i=1}^{j-1} \left\{ |a_{ji}| + \varepsilon \right\} \int_{\Theta_j}^{t} e^{\frac{\lambda_j}{2}\tau} \int_0^{\Theta_j} (\Theta_j - h) |\phi_i(\tau - h)| dh \, d\tau, \quad j=2,3,\ldots, n; \quad (36)$$

$F_1(t) = 0.$

The lemma is trivially true in the case $j = 1$. Using induction, suppose the conclusion of the lemma to hold for $j = 1,\ldots,(k-1)$, for some $k \geq 2$. In particular, by this induction hypothesis, the estimates on each $|\phi_j|$ given by (43) will hold for $t > \Theta_j$, and for $t < \Theta_j$,

$$|\phi_j(t)| < \delta,$$

$j = 1,\ldots,(k-1)$. Define Δ_j by

$$\Delta_j = \max(u_j, \delta e^{\sigma_j \Theta_j}) . \quad (44)$$

Then

$$|\phi_j(t)| \leq \Delta_j e^{-\sigma_j t}. \quad (45)$$

for all $t \geq 0$. If one takes absolute values in (36), and uses (45), (36) becomes, for $j = k$,

$$|F_k(t)| \leq |c_{k1}| \sum_{i=1}^{k-1} \left[|a_{ki}| + \varepsilon \right] \int_{\Theta_k}^{t} e^{\frac{\lambda_k \tau}{2}} \int_0^{\Theta_k} (\Theta_k - h) \Delta_i e^{-\sigma_i(\tau - h)} d\tau \, dh,$$

$t > \Theta_k$. Let

$$\sigma_k = \min\left(\min_{i<k} \sigma_2 , \frac{\lambda_k}{4}\right)$$

Then, making crude estimates,

$$|F_k(t)| \le |c_{k1}| \sum_{i=1}^{k-1} \left[|a_{ki}| + \varepsilon \right] \int_{\theta_k}^{t} e^{\frac{\lambda_k}{2}\tau} \int_0^{\theta_k} (\theta_k - h) \sum_{j=1}^{k-1} \Delta_j e^{-\sigma_k(\tau-h)} dh\, d\tau,$$

$t > \theta_k$. Integration yields

$$|F_k(t)| \le |c_{k1}| \sum_{j=1}^{k-1} \Delta_j \left[\sum_{i=1}^{k-1} |a_{ki}| + \varepsilon(k-1) \right] \theta_k \int_{\theta_k}^{t} e^{\frac{\lambda_k}{2}\tau} \left[\frac{e^{\sigma_k \theta_k} - 1}{\sigma_k} \right] e^{\sigma_k \tau} d\tau. \quad (46)$$

To simplify the notation, let

$$\rho_k = |c_{k1}| \left(\sum_{j=1}^{k-1} \Delta_j \right) \left[\sum_{i=1}^{k-1} |a_{ki}| + \varepsilon(k-1) \right] \left\{ \frac{\theta_k}{\sigma_k} \left(e^{\sigma_k \theta_k} - 1 \right) \right\} ;$$

note that $\rho_k \to 0$ as $\delta \to 0$, by (44). If one substitutes this into (46), inequality becomes

$$F_k(t) \le \rho_k \int_{\theta_k}^{t} e^{\left(\frac{\lambda_k}{2} - \sigma_k\right)\tau} d\tau .$$

Another integration yields

$$F_k(t) \le \frac{\rho_k}{\frac{\lambda_k}{2} - \sigma_k} e^{\left(\frac{\lambda_k}{2} - \sigma_k\right) t} .$$

If one lets

$$\gamma_k = \frac{\rho_k}{\frac{\lambda_k}{2} - \sigma_k} ,$$

the inequality becomes

$$F_k(t) \le \gamma_k e^{\left(\frac{\lambda_k}{2} - \sigma_k\right) t} ,$$

$t > \theta_k$, $k = 1, \ldots, n$, and the proof is complete.

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