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THE INHOMOGENEOUS HEAT EQUATION

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CHAPTER I

INTRODUCTION

This thesis is concerned with the one dimensional, non-homogeneous heat equation

$$W_{xx}(x,t) - W_t(x,t) = - f(x,t) \qquad (1)$$

for $-\infty < x < \infty$, $t > t_0$. This equation has been considered from the mathematical point of view by Levi [1], Gevrey [2], Hadamard [3], Dressel [4], and others. Physically (1) represents the temperature, $w(x,t)$, at a point x on a doubly infinite rod at time t due to a prescribed initial temperature distribution $w(x, t_0) = g(x)$ and a given forcing function $f(x,t)$. If $f(x,t) \equiv 0$, (1) becomes the one-dimensional, homogeneous heat equation. The homogeneous heat equation approximates the physical situation of heating a rod to a given temperature, $g(x)$, at $t = t_0$ and then removing the heat source. In either case it is necessary to have some knowledge of the condition of the rod at time $t = t_0$ in order to solve equation (1) for the unique temperature, $w(x, t)$. In the physical problem uniqueness is usually assumed; however, in the mathematical analog the uniqueness problem cannot be ignored.

In particular, this discussion is concerned with the solution of (1) over the infinite rod satisfying an initial condition of the form

$$\lim_{x,t \to x_o, 0} w(x,t) = g(x_o) \tag{2}$$

at each point of continuity of $g(x)$. If $|g(x)|$ is bounded and $g(x)$ piece-wise continuous, and if (3) $f(x,t)$ is continuously differentiable in x with bounded derivative, for example, it has been shown by Gevrey [5] that the function

$$w(x,t) = \int_{-\infty}^{\infty} K(x-z,t)\ g(z)\ dz + \int_{0}^{t}\int_{-\infty}^{\infty} K(x-z,t-y)\ f(z,y)\ dz\ dy \tag{4}$$

where

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{x^2}{4t}\right]$$

is indeed the solution of (1) and (2). For the case of a finite rod, $o \leq x \leq L$, it is easily shown that

$$w(x,t) = \int_{o}^{t}\int_{o}^{L} K(x-z,t-y)\ f(z,y)\ dz\ dy \tag{5}$$

is a formal solution of (1) for $0 < x < L$, $t > 0$. In this case Gevrey [6] has shown that if $f(x,t)$ is Hölder continuous in x and t with exponent γ, $0 < \gamma \leq 1$, in some neighborhood of every point (x,t), $0 < x < L$, $t > 0$; then (5) is an actual solution of (1). In fact, it is apparently known that if (6) $|g(x)| < P \exp[Qx^2]$ and $g(x)$ is piece-wise continuous, and if (7) $f(x,t)$ is Hölder continuous in x and t with exponent γ and $\max_{o \leq t \leq T} |f(x,t)| < P \exp[Qx^2]$; then (4) will be the solution of (1) and (2). However, in a thorough search of the literature no paper was found in which this result was explicitly stated and proved.

The object of this work is to prove in a systematic fashion that (7) is sufficient for

$$u(x, t) = \int_0^t \int_{-\infty}^{\infty} K(x - z, t - y) f(z, y)\, dz dy \tag{8}$$

to be a solution of (1) satisfying an initial condition

$$\lim_{x, t \to x_0, 0} u(x, t) = o . \tag{9}$$

this together with the fact [7] that (6) is a sufficient condition for

$$v(x, t) = \int_{-\infty}^{\infty} K(x - z, t)\, g(z)\, dz$$

to be a solution of the homogeneous heat equation satisfying the initial condition

$$\lim_{x, t \to x_0, 0} v(x, t) = g(x_0)$$

at points of continuity of $g(x)$ will show that $w(x,t) = v(x,t) + u(x,t)$ is a solution of (1) and (2). An application of a theorem of Tikhonov [8] will establish uniqueness.

The method used will be to reduce the original problem to a simpler one in the following way. Define $u(x, t; \lambda)$ as

$$u(x, t; \lambda) = \int_0^{t - \lambda} \int_{-\infty}^{\infty} K(x - z, t - y)\, f(z, y)\, dz\, dy, \quad 0 < \lambda < t, \tag{10}$$

and consider the equation

$$u_t(x, t; \lambda) = u_{xx}(x, t; \lambda) + \int_{-\infty}^{\infty} K(x - z, \lambda)\, f(z, t - \lambda)\, dz . \tag{11}$$

If (12) $f(x, t)$ is continuous and $\max_{o \le t \le T} |f(x, t)| < P \exp[Qx^2]$,

(10) will be a solution of (11) for $x, t \in R$ where

$$R = \left\{x, t: -\infty < x < \infty, o < t < \frac{1}{4Q}\right\} .$$

Also by condition (12) $\lim_{\lambda \to o} \int_{-\infty}^{\infty} K(x - z, \lambda) f(z, t - \lambda) dz = f(x,t)$.

Therefore if it can be shown that

$$\lim_{\lambda \to o} u_t(x, t;\lambda) = u_t(x, t)$$

$$\lim_{\lambda \to o} u_{xx}(x, t;\lambda) = u_{xx}(x, t) \tag{13}$$

where $u(x, t)$ is defined as in (8); then $u(x, t)$ will be a solution of (1) for $x, t \in R$. Showing that $u(x, t)$ satisfies (9) will establish $u(x, t)$ as a solution of (1) and (9) as desired. In order to prove the preceding results essential use is made of the definition of derivative, mean value theorems, and estimation of integrals.

CHAPTER II

THE INHOMOGENEOUS HEAT EQUATION

Consider the following initial value problem

$$w_{xx}(x, t) - w_t(x, t) = - f(x, t) \tag{14}$$

$$\lim_{x,t \to x_o,0} w(x, t) = g(x_o) \tag{15}$$

where x_o is a point of continuity of $g(x)$. It is well known [9] that if $|g(x)| \leq P \exp [Qx^2]$ for some constants P and Q then

$$v(x,t) = \int_{-\infty}^{\infty} K(x - z, t)\, g(z)\, dz, \quad K(x,t) = \frac{\exp\left[\frac{-x^2}{4t}\right]}{\sqrt{4\pi t}} \tag{16}$$

is a solution of the homogeneous heat equation for

$$(x,t) \varepsilon \left\{x,t : -\infty < x < \infty,\ o < t < \frac{1}{4Q}\right\} \quad \text{and}$$

satisfies the initial condition $\lim_{x,t \to x_o,0} v(x,t) = g(x_o)$ at continuity points of $g(x)$. Therefore if a solution $u(x,t)$ of (14) satisfying the initial condition $\lim_{x,t \to x_o,0} u(x,t) = 0$ were known in the region $R = \left\{x,t: -\infty < x < \infty,\ o < t < \frac{1}{4Q}\right\}$ and if $v(x,t)$ is the function defined in (16), then since equation (14) is linear

$$w(x,t) = v(x, t) + u(x, t) \tag{17}$$

would be a solution of (14) and (15) for (x,t) in R .

Suppose that such a solution, $w(x,t)$, of (14) and (15) has been found. It is of interest to know under what conditions this solution of (14) and (15) will be unique. This question may be reduced to a simpler one as follows: Suppose there are two solutions of (14) and (15) in R; call them $w_1(x,t)$ and $w_2(x,t)$. Define

$$W(x,t) = w_1(x,t) - w_2(x,t) \tag{18}$$

then $W(x,t)$ is a solution of the homogeneous heat equation and satisfies the initial condition $\lim_{x,t \to x_0,0} W(x,t) = 0$. If it can be shown that $W(x,t)$ must be identically zero in R, then the uniqueness of (17) will be established in the appropriate class of solutions. The following general uniqueness theorem due to Tikhonov [10] provides sufficient conditions for uniqueness.

TIKHONOV'S THEOREM: Let the following hypothesis be satisfied:

(H_1) $W(x,t)$ belong to the class C^2 for (x,t) in R,

(H_2) $W_{xx}(x,t) - W_t(x,t) = 0$,

(H_3) $\lim_{x,t \to x_0,0} W(x,t) = 0$ for all $-\infty < x_0 < \infty$,

(H_4) $\max_{0 < t \leq T} |W(x,t)| = O(\exp[ax^2])$ as $|x| \to \infty$ for some real a and some fixed but arbitrary T; then $W(x,t) \equiv 0$.

Notice that if $w_1(x,t)$ satisfies (H_4) so does $W(x,t)$. Therefore $w_1(x,t) \equiv w_2(x,t)$ and uniqueness of solutions of (14) and (15) has been established in the class of solutions which satisfy (H_4), which are in C^2, and which are such that $W(x,t)$ as defined in (18) satisfies (H_3).

Thus if a solution $u(x,t)$ of (14) can be determined so that $\lim_{x,t \to x_0,0} u(x,t) = o$ and such that (17) satisfies the appropriate conditions, Tikhonov's theorem can be applied and will guarantee a unique solution of (14) and (15).

To obtain a formal representation of a candidate for $u(x,t)$, assume without loss of generality that $u(x,o) = g(x_0) \equiv o$ and consider the following formal manipulations. The Fourier transform of $u(x,t)$ is

$$U(r,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\,[-\,irz]\, u(z,t)\, dz$$

and the inverse transform is

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\,[ixr]\, U(r,t)\, dr\ . \qquad (19)$$

Formal differentiation under the integral sign yields

$$u_{xx}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\,r^2 \exp\,[ixr]\, U(r,t)\, dr$$

$$u_t(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\,[ixr]\, U_t(r,t)\, dr\ .$$

Also the Fourier transform of $-\,f(x,t)$ is

$$F(r,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\exp\,[-\,irz]\, f(z,t)\, dz \qquad (20)$$

and the inverse transform is

$$- f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\,[ixr]\, F(r,t)\, dr \, .$$

Formally taking the transform of $u_{xx}(x,t) - u_t(x,t) = - f(x,t)$ yields the following differential equation in t with parameter r .

$$- r^2\, U(r,t) - U_t(r,t) = F(r,t) \qquad (21)$$

Formally multiplying (21) by $\exp\,[\int_0^t r^2\, dt]$ gives

$$\frac{\partial\, U(r,t)\, \exp\,[r^2 t]}{\partial t} = - \exp\,[r^2 t]\, F(r,t)$$

which has solution

$$U(r,t) = \int_0^t \exp\,[- r^2(t - y)]\, F(r,y)\, dy \qquad (22)$$

since $g(x_o) \equiv 0$. Substituting (22) in (19) gives

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\,[ixr] \left\{ \int_0^t \exp\,[- r^2(t - y)]\, F(r,y) dy \right\} dr,$$

and interchanging the y and r integration results in

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} \exp\,[ixr - r^2(t - y)]\, F(r,y)\, dr\, dy \, . \qquad (23)$$

Substituting (20) in (23) and interchanging z and r integration gives

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_0^t \int_{-\infty}^{\infty} f(z,y) \left\{ \int_{-\infty}^{\infty} \exp\,[ixr - r^2(t-y) - irz]\, dr \right\} dzdy \, . \qquad (24)$$

Now consider separately

$$I(x-z,\ t-y) = \int_{-\infty}^{\infty} \exp\,[- r^2(t-y) + ir(x-z)]\, dr \, . \qquad (25)$$

Completing the square in the exponent of (25) yields

$$I(x - z, t - y) = \exp\left[\frac{-(x-z)^2}{4(t-y)}\right] \int_{-\infty}^{\infty} \exp\left[-(t-y)\left[r - \frac{i(x-z)}{2(t-y)}\right]^2\right] dr. \quad (26)$$

If $s = r(t-y)^{\frac{1}{2}}$, equation (26) becomes

$$I(x-z, t-y) = \frac{1}{(t-y)^{\frac{1}{2}}} \exp\left[\frac{-(x-z)^2}{4(t-y)}\right] \int_{-\infty}^{\infty} \exp\left[-\left[s - \frac{i(x-z)}{2(t-y)^{\frac{1}{2}}}\right]^2\right] ds. \quad (27)$$

In $\int_{-\infty}^{\infty} \exp\left[-\left[s - \frac{i(x-z)}{2(t-y)^{\frac{1}{2}}}\right]^2\right] ds$ let $a = s$ and $b_o = \frac{-x+z}{2(t-y)^{\frac{1}{2}}}$

and thus consider $\int_{-\infty}^{\infty} \exp\left[-(a + ib_o)^2\right] da$ assuming without loss of generality that $b_o > 0$. Now consider the integral

$\int_c \exp[-(a + ib)^2]\, d(a + ib)$ where c is the boundary of the rectangle $-A \leq a \leq A$, $0 \leq b \leq b_o$ with $A > 0$ and arbitrary. Since $\exp[-(a + ib)^2]$ is analytic in any such rectangle, by Cauchy's Integral Theorem [11] $\int_c \exp[-(a + ib)^2]\, d(a + ib) = 0$.

Also $\int_c \exp[-(a+ib)^2]\, d(a+ib) = i\int_0^{b_o} \exp[-(A+ib)^2]db + \int_A^{-A} \exp[-(a+ib_o)^2]da$

$$+\, i\int_{b_o}^{0} \exp[-(-A + ib)^2]\, db + \int_{-A}^{A} \exp[-a^2]\, da.$$

$\int_c^{b_o} \exp[-(A + ib)^2]\, db$ and $\int_{b_o}^{o} \exp[-(-A + ib)^2]\, db$ tend to zero as A tends to infinity. Therefore as A tends to plus infinity

$\int_A^{-A} \exp[-(a + ib_o)^2]\, da + \int_{-A}^{A} \exp[-a^2]\, da$ tends to zero which implies

$$\int_{-\infty}^{\infty} \exp[-(a + ib_o)^2]\, da = \int_{-\infty}^{\infty} \exp[-a^2]\, da = \sqrt{\pi}\ .$$

Therefore (27) becomes

$$I(x-z,t-y) = \frac{\sqrt{\pi}}{(t-y)^{\frac{1}{2}}} \exp\left[\frac{-(x-z)^2}{4(t-y)}\right] . \tag{28}$$

Substituting (28) in (24) gives

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_0^t \int_{-\infty}^{\infty} \frac{1}{(t-y)^{\frac{1}{2}}} \exp\left[\frac{-(x-z)^2}{4(t-y)}\right] f(z,y)\, dz\, dy . \tag{29}$$

If $K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left[\frac{-x^2}{4t}\right]$, (29) becomes

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-z,t-y)\, f(z,y)\, dz\, dy \tag{30}$$

which formally satisfies (14) and $\lim_{x,t \to x_o,0} u(x,t) = 0$.

From the preceding formal derivation one is led to consider (30) as a particular solution of (14). As usual in a problem of this type the justification of the formal procedure used in the derivation of the solution is too cumbersome and restrictive to be practicable. Therefore it will be verified directly that (30) is an actual solution of (14) under suitable conditions on $f(x,t)$ and that $\lim_{x,t \to x_o,0} u(x,t) = 0$.

As a tcol in showing (30) a solution of (14) a simpler problem will be solved. Define

$$u(x,t;\ \lambda) = \int_0^{t-\lambda} \int_{-\infty}^{\infty} K(x-z,t-y)\ f(z,y)\ dz\ dy,\ 0 < \lambda < t$$

and consider the following formal calculations

$$\frac{\partial\, u(x,t;\ \lambda)}{\partial\, t} = \int_0^{t-\lambda} \int_{-\infty}^{\infty} \frac{\partial\, K(x-z,t-y)}{\partial\, t}\ f(z,y)\ dz\ dy$$

$$+ \int_{-\infty}^{\infty} K(x-z,\lambda)\ f(z,t-\lambda)\ dz$$

$$\frac{\partial^2 u(x,t;\lambda)}{\partial x^2} = \int_0^{t-\lambda} \int_{-\infty}^{\infty} \frac{\partial^2 K(x-z,t-y)}{\partial x^2}\ f(z,y)\ dz\ dy \quad .$$

Notice that

$$\int_0^{t-\lambda} \int_{-\infty}^{\infty} \frac{\partial^2 K(x-z, t-y)}{\partial x^2} f(z,y)\, dz\, dy =$$

$$\int_0^{t-\lambda} \int_{-\infty}^{\infty} \frac{\partial K(x-z, t-y)}{\partial t} f(z,y)\, dz\, dy .$$

Therefore $u(x,t;\lambda)$ formally satisfies

$$\frac{\partial u(x,t;\lambda)}{\partial t} = \frac{\partial^2 u(x,t;\lambda)}{\partial x^2} + \int_{-\infty}^{\infty} K(x-z,\lambda)\, f(z,t-\lambda)\, dz. \tag{31}$$

The following lemma justifies these calculations.

<u>LEMMA</u>: Let the following hypotheses be satisfied:

(H_1) $\max_{0 \le t \le T} |f(x,t)| < P \exp [Qx^2]$, $P > 0$, $Q \geq 0$,

(H_2) $f(x,t)$ continuous for $x, t \in \{x,t: -\infty < x < \infty , t > 0\}$,

(H_3) $0 < \lambda < t$,

then $u(x,t;\lambda)$ satisfies (31) and

$$\lim_{\lambda \to 0} \int_{-\infty}^{\infty} K(x-z,\lambda)\, f(z,t-\lambda)\, dz = f(x,t)$$

at points of continuity of $f(x,t)$.

Therefore if it can be shown that

$$\lim_{\lambda \to o} u_t(x,t: \lambda) = u_t(x,t) \quad \text{and}$$

$$\lim_{\lambda \to o} u_{xx}(x,t: \lambda) = u_{xx}(x,t) ,$$

taking the limit in (31) as λ tends to zero will yield

$$u_t(x,t) = u_{xx}(x,t) + f(x,t) ,$$

and (30) will indeed be a solution of (14) .

It is not at all difficult to show that (30) satisfies (14) if $f(x,t)$ is sufficiently well-behaved, for example $f(x,t)$ continuously differentiable with $\left|\frac{\partial f}{\partial x}\right|$ bounded [12] . However the object here is to obtain weaker sufficient conditions on $f(x,t)$ which will imply that (30) is a solution of (14). It has been shown that $f(x,t)$ bounded and continuous is not sufficient [13] . In view of the remarks in the preceding paragraph proof of the following theorem will establish the desired sufficient conditions.

<u>THEOREM</u> . Let the hypotheses

(H_1) $f(x,t) : \left\{x,t: -\infty < x < \infty,\ t \geq 0\right\} \to$ extended real numbers;

(H_2) $\max\limits_{o \leq t \leq \frac{1}{4Q}} |f(x,t)| \leq P \exp[Qx^2]$, $P > 0,\ Q \geq 0$;

(H_3) $f(x,t)$ satisfies a Hölder condition with exponent $0 < \gamma \leq 1$ in x and t, i.e. there exists a number N such that

$|f(x,t) - f(z,y)| < N\,[|x-z|^{\gamma} + |t-y|^{\gamma}]$;

(H_4) $u(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-z,t-y)\, f(z,y)\, dz\, dy$, $x,t \in \{x,t: -\infty<x<\infty, t>0\}$;

(H_5) $u(x,t;\lambda) = \int_0^{t-\lambda} \int_{-\infty}^{\infty} K(x-z,t-y)\, f(z,y)\, dz\, dy$, $x,t \in \{x,t: -\infty<x<\infty, t>0\}$;

be satisfied. Then

$$\lim_{\lambda \to 0} u_t(x,t;\ \lambda) = u_t(x,t)$$

$$\lim_{\lambda \to 0} u_{xx}(x,t;\ \lambda) = u_{xx}(x,t)$$

for $x,t \in R$.

REMARK: In (H_2) if $Q \leq 0$ then $|f(x,t)|$ is bounded on the infinite strip and the conclusion holds for $x,t \in \{x,t: -\infty < x < \infty, t>0\}$.

If (H_3) is changed so that $f(x,t)$ satisfies a Hölder condition with exponent $0 < \gamma \leq 1$ in x and is bounded and integrable with respect to t, the following result can be proved.

<u>COROLLARY</u>: If (H_1), (H_2), (H_4), and (H_5) of the theorem are satisfied and if (H_3) is modified so that $|f(x,t) - f(z,t)| < h(t)|x-z|^{\gamma}$, $0< \gamma\leq 1$,

where $h(t)$ is bounded and integrable for $t > o$; then the conclusion of the theorem still holds.

It remains to be shown that $\lim_{x,t \to x_o, 0} u(x,t) = 0$ under the hypotheses of the theorem. By applying (H_2) and making the usual estimate it can be shown that

$$|u(x,t)| \leq \int_o^t \frac{P \exp[Qx^2/(1-4(t-y)Q)]}{\sqrt{1-4(t-y)Q}} \, dy$$

and further that

$$|u(x,t)| \leq \frac{P \exp[Qx^2/[1-4tQ)]}{\sqrt{1-4tQ}} \; t \; .$$

Thus given an $\varepsilon > o$ there exists a $\delta > o$ such that $|u(x,t)| < \varepsilon$ for $|x-x_o| < \delta$, $|t| < \delta$ which establishes the desired limit.

Therefore for $u(x,t)$ defined in (30) and $v(x,t)$ defined in (16),

$$w(x,t) = u(x,t) + v(x,t)$$

is a solution of (14) and (15) for $x, t \in R$ under the hypotheses of the theorem and the original problem is therefore solved.

CHAPTER III

PROOF OF LEMMA

The purpose of this chapter is to prove the following lemma.

LEMMA : Let the following hypotheses be satisfied:

(H_1) $\max_{0 \leq t \leq T} |f(x,t)| < P \exp [Qx^2]$, $P > 0$, $Q \geq 0$;

(H_2) $f(x,t)$ continuous for $x, t \in \left\{x,t: -\infty < x < \infty,\ t > 0\right\}$;

(H_3) $0 < \lambda < t$;

then $u(x,t; \lambda)$ satisfies

$$\frac{\partial u(x,t;\lambda)}{\partial t} = \frac{\partial^2 u(x,t;\lambda)}{\partial x^2} + \int_{-\infty}^{\infty} K(x-z,\lambda)\, f(z,t-\lambda)\, dz \tag{31}$$

and

$$\lim_{\lambda \to 0} \int_{-\infty}^{\infty} K(x-z,\lambda)\, f(z,t-\lambda)\, dz = f(x,t)$$

for $x, t \in R$.

PROOF : In order to prove the Lemma, the formal calculations

$$\frac{\partial u(x,t;\lambda)}{\partial t} = \int_0^{t-\lambda} \int_{-\infty}^{\infty} \frac{\partial K(x-z,t-y)}{\partial t} f(z,y)\, dz\, dy + \int_{-\infty}^{\infty} K(x-z,\lambda) f(z,t-\lambda) dz$$

and

$$\frac{\partial^2 u(x,t;\lambda)}{\partial x^2} = \int_0^{t-\lambda} \int_{-\infty}^{\infty} \frac{\partial^2 K(x-z,t-y)}{\partial x^2} f(z,y)\, dz\, dy$$

must be verified. If it can be shown that the resultant integral after differentiation under the integral sign converge uniformly, the differentiation will be justified. Therefore consider the following integral

$$I_1(x,t) = \frac{1}{\sqrt{4\pi}} \int_0^{t-\lambda} \int_{-\infty}^{\infty} \left\{ \frac{-1}{2(t-y)^{\frac{3}{2}}} + \frac{(x-z)^2}{4(t-y)^{\frac{5}{2}}} \right\} \exp\left[-\frac{(x-z)^2}{4(t-y)}\right] f(z,y)dzdy \quad (32)$$

and notice that

$$I_1(x,t) = \int_0^{t-\lambda} \int_{-\infty}^{\infty} K_{xx}(x-z,t-y) f(z,y)dzdy = \int_0^{t-\lambda} \int_{-\infty}^{\infty} K_t(x-z,t-y) f(z,y)dzdy \;.$$

Making the change of variable

$$z - x = 2(t - y)^{\frac{1}{2}} \; s$$

in (32) results in

$$I_1(x,t) = \frac{1}{\sqrt{4\pi}} \int_0^{t-\lambda} (t-y)^{-1} \int_{-\infty}^{\infty} (2s^2 - 1)\exp[-s^2] f(x+2(t-y)^{\frac{1}{2}}s,y)\, ds\, dy \,.$$

Let

$$I_2(x,t) = \int_{-\infty}^{\infty} (2s^2 - 1) \exp\,[-s^2]\; f(x + 2(t-y)^{\frac{1}{2}}s,y)\, ds$$

and thus

$$|I_2(x,t)| \leq \int_{-\infty}^{\infty} |2s^2-1| \exp[-s^2] \, |f(x + 2(t-y)^{\frac{1}{2}}s, y)| \; ds \, . \qquad (33)$$

For any $x, t \in R$ there is a rectangle

$R_1 = \left\{x,t: |x| \leq A, \ 0 < \delta \leq t \leq \frac{1}{4Q} - b\right\}$ where $0 < b < \frac{1}{4Q}$ such that $x,t \in R_1$. Therefore making the substitution $a = 2(t-y)^{\frac{1}{2}}$ in (33) yields

$$|I_2(x,t)| \leq \int_{-\infty}^{\infty} |2s^2 - 1| \exp[-s^2] \, |f(x+as,y)| \, ds$$

where $2\lambda^{\frac{1}{2}} \leq a \leq 2(\frac{1}{4Q} - b)^{\frac{1}{2}}$ for $x, t \in R_1$. By (H_1)

$|f(x + as,y)| \leq P \exp[Q(x + as)^2]$ and therefore

$$|I_2(x,t)| \leq \int_{-\infty}^{\infty} |2s^2 - 1| \exp[-s^2] \, P \exp[Q(x + as)^2] \, ds \, .$$

Completing the square in exponent of the exponential terms gives

$$|I_2(x,t)| \leq P \int_{-\infty}^{\infty} |2s^2 - 1| \exp\left[-(1-Qa^2) \left[s - \frac{aQx}{1-Qa^2}\right]^2\right] \exp\left[\frac{Qx^2}{1-Qa^2}\right] \; ds \, .$$

Making the substitution $w = (1 - Qa^2)^{\frac{1}{2}} \left[s - \frac{a\,Qx}{1-Qa^2}\right]$ implies

$$|I_2(x,t)| \le P \exp\left[\frac{Qx^2}{1-Qa^2}\right] \int_{-\infty}^{\infty} \left| \frac{2w^2}{1-Qa^2} + \frac{4aQxw}{(1-Qa^2)^{\frac{3}{2}}} + \frac{2a^2Q^2x^2}{(1-Qa^2)^2} - 1 \right| \frac{\exp[-w^2]}{(1-Qa^2)^{\frac{1}{2}}}\, dw .$$

Therefore

$$|I_2(x,t)| \le P \exp\left[\frac{Qx^2}{1-Qa^2}\right] (I_{21} + I_{22} + I_{23})$$

where

$$I_{21} = \int_{-\infty}^{\infty} \frac{2w^2 \exp[-w^2]}{(1-Qa^2)^{\frac{3}{2}}}\, dw = \frac{K_1}{(1-Qa^2)^{\frac{3}{2}}}$$

$$I_{22} = \int_{-\infty}^{\infty} \frac{4aQ|x|}{(1-Qa^2)^2} |w| \exp[-w^2]\, dw = \frac{4aQ|x|}{(1-Qa^2)^2}$$

$$I_{23} = \left| \frac{2a^2Q^2x^2}{(1-Qa^2)^{\frac{5}{2}}} - \frac{1}{(1-Qa^2)^{\frac{1}{2}}} \right| \int_{-\infty}^{\infty} \exp[-w^2]\, dw = \sqrt{\pi} \left| \frac{2a^2Q^2x^2 - (1-Qa^2)^2}{(1-Qa^2)^{\frac{5}{2}}} \right| .$$

Since $x, t \in R_1$

$$I_{21} \le \frac{K_1}{(4Qb)^{\frac{3}{2}}}, \quad I_{22} \le \frac{2(1-4Qb)^{\frac{1}{2}} A}{(4Q)^{\frac{3}{2}} b^2} ,$$

$$I_{23} \leq \frac{2\sqrt{\pi}\ (1-4Qb\ )\ A^2 + (1-4Q\lambda)^2\ \sqrt{\pi}}{(4Qb\ )^{\frac{5}{2}}} \quad .$$

Thus

$$|I_2(x,t)| \leq M(A,\ \varepsilon,\ \lambda,\ Q,\ P\ )$$

for $(x,t) \in R$, where M is a constant depending, as indicated, on A, ε, λ, Q, and P. Since

$$|I_1(x,t)| \leq \frac{1}{\sqrt{4\pi}} \int_0^{t-\lambda} |t-y|^{-1} |I_2(x,t)|\ dy$$

it follows that

$$|I_1(x,t)| \leq \frac{M}{\sqrt{4\pi}} \int_0^{t-\lambda} |t-y|^{-1} dy = \frac{M}{\sqrt{4\pi}} [-\ln\ \lambda + \ln\ t]$$

and thus

$$|I_1(x,t)| \leq \frac{M}{\sqrt{4\pi}} |\ln\ \lambda| + |\ln(\frac{1}{4Q} - b\ )\ | \quad .$$

Thus by the Weierstrass M-test [14] the integral $I_1(x,t)$ converges uniformly and absolutely for $(x,t) \in R$, and therefore

$u_{xx}(x,t;\lambda) = I_1(x,t)$. Similar estimates show that $\int_{-\infty}^{\infty} K(x-z,\lambda) f(z,t-\lambda)\, dz$ converges uniformly for $x,t \in R$, and therefore $u_t(x,t;\lambda) = I_1(x,t) + \int_{-\infty}^{\infty} K(x-z,\lambda)\, f(z,t-\lambda)\, dz$. This completes the proof of the first part of the lemma.

To prove the second part of the lemma it must be shown that given an $\varepsilon > 0$ there exists a $\delta > 0$ such that for $|\lambda| < \delta$

$$\left|\int_{-\infty}^{\infty} K(x-z,\lambda)\, f(z,t-\lambda)\, dz - f(x,t)\right| < \varepsilon \quad .$$

Let

$$I(x,t) = \left|\int_{-\infty}^{\infty} K(x-z,\lambda)\, f(z,t-\lambda) dz - f(x,t)\right|$$

and notice that

$$f(x,t) = \int_{-\infty}^{\infty} K(x-z,\lambda)\, f(x,t)\, dz \ .$$

Thus

$$I(x,t) = \left|\int_{-\infty}^{\infty} K(x-z,\lambda) \left\{ f(z,t-\lambda) - f(x,t) \right\} dz\right|$$

and it follows that

$$I(x,t) \leq \int_{-\infty}^{\infty} K(x-z,\lambda)\ |f(z,t-\lambda) - f(x,t)|dz = I_3(x,t) + I_4(x,t) + I_5(x,t)$$

where

$$I_3(x,t) = \int_{-\infty}^{-N} K(x-z,\lambda)|f(z,t-\lambda) - f(x,t)|\ dz$$

$$I_4(x,t) = \int_{-N}^{N} K(x-z,\lambda)|f(z,t-\lambda) - f(x,t)|\ dz$$

$$I_5(x,t) = \int_{N}^{\infty} K(x-z,\lambda)|f(z,t-\lambda) - f(x,t)|\ dz\ .$$

Consider $I_5(x,t)$ in detail and notice that similar calculations apply to $I_3(x,t)$. Making the change of variable $z-x = 2\lambda^{\frac{1}{2}}\ s$ in $I_5(x,t)$ yields

$$I_5(x,t) = \int_{N^1}^{\infty} \frac{\exp[-s^2]}{\sqrt{\pi}}\ |f(x+2\lambda^{\frac{1}{2}}s,t-\lambda) - f(x,t)|\ ds$$

where $N^1 = \frac{N-x}{2\lambda^{\frac{1}{2}}}$. Applying the growth condition (H_2) of the lemma to $f(x,t)$ gives

$$I_5(x,t) \leq \int_{N^1}^{\infty} \frac{\exp[-s^2]}{\sqrt{\pi}} \left\{ P \exp[Q(x + 2\lambda^{\frac{1}{2}}s)^2] + P \exp[Qx^2] \right\} ds$$

which equals $I_{51}(x,t) + I_{52}(x,t)$ where

$$I_{51}(x,t) = \int_{N^1}^{\infty} \frac{P}{\sqrt{\pi}} \exp[-s^2] \exp[Q(x + 2\lambda^{\frac{1}{2}}s)^2] \; ds$$

$$I_{52}(x,t) = \int_{N^1}^{\infty} \frac{P}{\sqrt{\pi}} \exp[-s^2] \exp[Qx^2] \; ds \quad .$$

Detailed consideration of $I_{52}(x,t)$ results in the following inequality:

$$I_{52}(x,t) \leq \frac{P}{\sqrt{\pi}} \exp[QA^2] \int_{N^1}^{\infty} \exp[-s^2] \; ds \; .$$

Since the integral exists, the Cauchy criterion for convergence implies that for N^1 sufficiently large $I_{52}(x,t)$ can be made less than any given $\epsilon_1 > 0$. Notice, by definition of N^1, N^1 is made large by taking N sufficiently large, say $N > N_{52}$. Now consider $I_{51}(x,t)$. Completing the square in the exponent of the integrand yields

$$I_{51}(x,t) = \frac{P}{\sqrt{\pi}} B(x,Q,\lambda) \int_{N^1}^{\infty} \exp\left[-(1-4Q\lambda)\left[s - \frac{2Q\lambda^{\frac{1}{2}}x}{1-4Q\lambda}\right]^2\right] ds$$

where

$$B(x,Q,\lambda) = \exp\left[\frac{(2Q\lambda^{\frac{1}{2}}x\ )^2}{(1-4Q\lambda)^2} + Qx^2\right] .$$

Let $w = (1-4Q\lambda)^{\frac{1}{2}}\left[s - \frac{2Q\lambda^{\frac{1}{2}}x}{1-4Q\lambda}\right]$ and maximize the result. $I_{51}(x,t)$

will then satisfy the following inequality:

$$I_{51}(x,t) \leq \frac{P}{\sqrt{\pi}} C(A,Q,b\ ) \int_{N^{11}}^{\infty} \exp[-w^2]\, dw$$

where

$$C(A,Q,b\ ) = \frac{1}{\sqrt{4Qb}} \exp\left[\frac{[A(1-4Qb\ )^{\frac{1}{2}}]^2}{[4a^{\frac{1}{2}}\ b\ ]^{\frac{1}{2}}} + QA^2\right]$$

and

$$N^{11} = (1-4Q\lambda)^{\frac{1}{2}}\left[\frac{N-x}{2\lambda^{\frac{1}{2}}} - \frac{2Q\lambda^{\frac{1}{2}}x}{1-4Q\lambda}\right] .$$

Therefore given an $\varepsilon_2 > 0$ $I_{51}(x,t)$ can be made less than ε_2 for

N^{11} sufficiently large. The quantity N^{11} is made large by choosing N sufficiently large, say $N > N_{51}$. As indicated previously similar calculations show that $I_3(x,t)$ can be made less than $\varepsilon_3 + \varepsilon_4$ for any given $\varepsilon_3 > 0$, $\varepsilon_4 > 0$ for N sufficiently large; $N > \max(N_{31} + N_{32})$. Now consider $I_4(x,t)$ for $N > \max(N_{31}, N_{32}, N_{51}, N_{52}) = N_o$ and make the change of variable $z-x = 2\lambda^{\frac{1}{2}}s$. The integral $I_4(x,t)$ satisfies the equality

$$I_4(x,t) = \int_{-N}^{N} \frac{\exp[-s^2]}{\sqrt{\pi}} \mid f(x + 2\lambda^{\frac{1}{2}}s, t-\lambda) - f(x,t) \mid ds .$$

By continuity of $f(x,t)$ at (x,t) it is known that given an $\varepsilon_o > 0$ there exists a $\delta_o > 0$ such that for $|x-x_o| < \delta_o$ and $|t-t_o| < \delta$; $|f(x_o,t_o) - f(x,t)| < \varepsilon_o$. In $I_4(x,t)$ $x_o = x + 2\lambda^{\frac{1}{2}}s$ and $t_o < t-\lambda$. Notice that $|x + 2\lambda^{\frac{1}{2}}s - x| \leq 2\lambda^{\frac{1}{2}}N$ and $|t-(t-\lambda)| = \lambda$. Thus for $\lambda < \min\left(\frac{\delta_o^2}{4N^2}, \delta_o\right)$ $|f(x+2\lambda^{\frac{1}{2}}s, t-\lambda) - f(x,t)| < \varepsilon_o$. Choose ε_o such that $I_2(x,t) < \varepsilon_5$ for any given $\varepsilon_5 > o$. Also choose $\max(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5) < \frac{\varepsilon}{5}$. Let $N = \lambda^{-\frac{1}{4}}$ and choose $\lambda < \min\left(\frac{\delta_o^4}{16}, \delta_o, \frac{1}{N_o^4}\right) = \delta$. Therefore for $\lambda < \delta$ it has been shown that $I(x,t) < \varepsilon$ and the proof of the lemma is complete.

CHAPTER IV

PROOF OF THEOREM

The purpose of this chapter is to prove the following theorem.

THEOREM 1 Let the hypotheses

(H_1) $f(x,t)$: $\left\{x,t: -\infty < x < \infty\, ,\ t \geq 0\right\} \rightarrow$ extended real numbers;

(H_2) $\max\limits_{0 \leq t \leq \frac{1}{4Q}} |f(x,t)| \leq P \exp [Qx^2],\ P > 0,\ Q \geq 0$;

(H_3) $f(x,t)$ satisfies a Hölder condition with exponent

$0 < \gamma \leq 1$ in x and t, i.e. $|f(x,t) - f(z,y)| < N[|x-z|^\gamma + |t-y|^\gamma]$;

(H_4) $u(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x-z,t-y)\, f(z,y)\, dzdy,\ x,\ t \in \left\{x,t: -\infty < x < \infty, t>0\right\}$

(H_5) $u(x,t;\ \lambda) = \int_0^{t-\lambda} \int_{-\infty}^{\infty} K(x-z,t-y) f(z,y) dzdy,\ x,t \in \left\{x,t: -\infty < x < \infty, t>0\right\}$

be satisfied then

$$\lim_{\lambda \to 0} u_t(x,t;\ \lambda) = u_t(x,t) \tag{34}$$

$$\lim_{\lambda \to 0} u_{xx}(x,t;\ \lambda) = u_{xx}(x,t) \tag{35}$$

for $x,\ t \in R$.

PROOF: To prove (34) define

$$U(x,t;\ \lambda) = u(x,t) - u(x,t;\ \lambda) = \int_{t-\lambda}^{t} \int_{-\infty}^{\infty} K(x-z,t-y)\ f(z,y)dzdy$$

Consider $U(x,t+h;\ \lambda) - U(x,t;\ \lambda)$ and notice that

$$\lim_{h \to o} \frac{U(x,t+h;\ \lambda) - U(x,t;\ \lambda)}{h} = u_t(x,t) - u_t(x,t;\ \lambda).$$

Therefore if $I(\lambda)$ is defined by

$$I(\lambda) = \frac{U(x,t+h;\ \lambda) - U(x,t;\ \lambda)}{h}$$

and it can be shown that $\lim_{\lambda \to o} [\lim_{h \to o} I(\lambda)] = o$, then $\lim_{\lambda \to o} u_t(x,t;\lambda) = u_t(x,t)$.

Notice that

$$hI(\lambda) = \int_{t+h-\lambda}^{t+h} \int_{-\infty}^{\infty} K(x-z,t+h-y)\ f(z,y)\ dzdy -$$

$$\int_{t-\lambda}^{t} \int_{-\infty}^{\infty} K(x-z,t-y)\ f(z,y)\ dz\ dy$$

and assume $o < h < \lambda < t$. Write $hI(\lambda)$ as

$hI(\lambda) = [I_1(\lambda) + I_2(\lambda) + I_3(\lambda)]$ where

$$I_1(\lambda) = -\int_{t-\lambda}^{t+h-\lambda} \int_{-\infty}^{\infty} K(x-z,t-y)\ f(z,y)\ dzdy$$

$$I_2(\lambda) = \int_{t+h-\lambda}^{t} \int_{-\infty}^{\infty} [K(x-z,t+h-y) - K(x-z,t-y)]f(z,y)dzdy$$

$$I_3(\lambda) = \int_{t}^{t+h} \int_{-\infty}^{\infty} K(x-z,t+h-y)\ f(z,y)dz\ dy\ .$$

In $I_1(\lambda)$ make the substitution $y = r-\lambda$ and thus

$$I_1(\lambda) = -\int_{t}^{t+h}\int_{-\infty}^{\infty} K(x-z,t+\lambda-r)\ f(z,r-\lambda)\ dz\ dr\ .$$

Then letting $r=y$ and combining $I_1(\lambda)$ and $I_3(\lambda)$ one has

$$I_4(\lambda) = \int_{t}^{t+h} \int_{-\infty}^{\infty} [K(x-z,t+h-y)f(z,y) - K(x-z,t+\lambda-y)\ f(z,y-\lambda)]dz\ dy$$

and thus $hI(\lambda) = I_2(\lambda) + I_4(\lambda)$.

Now consider $I_2(\lambda)$ in detail. Define $J_1(\lambda)$ and $J_2(\lambda)$ such that

$$J_1(\lambda) = f(x,t) \int_{t+h-\lambda}^{t} \int_{-\infty}^{\infty}[K(x-z,t+h-y) - K(x-z,t-y)]\ dz\ dy$$

$$J_2(\lambda) = \int_{t+h-\lambda}^{t} \int_{-\infty}^{\infty} [K(x-z,t+h-y)-K(x-z,t-y)][f(z,y)-f(x,t)]dz\ dy$$

and thus $I_2(\lambda) = J_1(\lambda) + J_2(\lambda)$. Since

$$\int_{-\infty}^{\infty} K(x-z,t+h-y)\ dz = \int_{-\infty}^{\infty} K(x-z,t-y)\ dz = 1\ \ ,\ \ J_1(\lambda) \equiv 0\ .$$

By applying (H_4) of the theorem to $|J_2(\lambda)|$ it is seen that

$|J_2(\lambda)| \leq J_{21}(\lambda) + J_{22}(\lambda)$ where

$$J_{21}(\lambda) = \int_{t+h-\lambda}^{t} \int_{-\infty}^{\infty} N|K(x-z,t+h-y) - K(x-z,t-y)| \; |t-y|^{\gamma} \, dzdy$$

$$J_{22}(\lambda) = \int_{t+h-\lambda}^{t} \int_{-\infty}^{\infty} N|K(x-z,t+h-y) - K(x-z,t-y)| \; |x-z|^{\gamma} \, dz\, dy .$$

First considering $J_{21}(\lambda)$ in detail notice that

$$|K(x-z,t+h-y) - K(x-z,t-y)| = \left| \int_0^h \frac{\partial K(x-z,t+r-y)}{\partial r} \, dr \right|$$

and that

$$\frac{\partial K(x-r,t+r-z)}{\partial r} = \frac{1}{\sqrt{4\pi}} \left[\frac{-1}{2(t+r-y)^{\frac{3}{2}}} + \frac{(x-z)^2}{4(t+r-y)^{\frac{5}{2}}} \right] \exp \left[\frac{-(x-z)^2}{4(t+r-y)} \right] .$$

Define

$$J_{211}(\lambda) = \int_{t+h-\lambda}^{t} \int_{-\infty}^{\infty} N \int_0^h \frac{(t-y)^{\gamma}}{2(t+r-y)^{\frac{3}{2}}} \exp \left[\frac{-(x-z)^2}{4(t+r-y)} \right] drdzdy$$

$$J_{212}(\lambda) = \int_{t+h-\lambda}^{t} \int_{-\infty}^{\infty} N \int_0^h \frac{(x-z)^2}{2(t+r-y)^{\frac{5}{2}}} \exp \left[\frac{-(x-z)^2}{4(t+r-y)} \right] dr (t-y)^{\gamma} \, dzdy$$

so that $\sqrt{4\pi}\ J_{21}(\lambda) \leq J_{211}(\lambda) + J_{212}(\lambda)$. In $J_{211}(\lambda)$ consider the

integral

$$L_{rz}(o) = \int_{-\infty}^{\infty} \int_{o}^{h} \frac{(t-y)^{\gamma}}{2(t+r-y)^{\frac{3}{2}}} \exp\left[\frac{-(x-z)^2}{4(t+r-y)}\right] dr\, dz .$$

To justify interchanging the order of integration in this integral, i.e. to show $L_{rz}(o) = L_{zr}(o)$, consider

$$L_{r,z}(c) = \int_{-\infty}^{\infty} \int_{c}^{h} \frac{(t-y)^{\gamma}}{2(t+r-y)^{\frac{3}{2}}} \exp\left[\frac{-(x-z)^2}{4(t+r-y)}\right] dr\, dz$$

where $o < c < h$. Clearly the integrand of the integral $L_{rz}(c)$ is continuous on the rectangular strip $(-\infty,\infty) \times [c,h]$. Consider

$$\int_{-\infty}^{\infty} \frac{(t-y)^{\gamma}}{2(t+r-y)^{\frac{3}{2}}} \exp\left[\frac{-(x-z)^2}{4(t+r-y)}\right] dz$$

and make the change of variable $(z-x) = 2(t+r-y)^{\frac{1}{2}} s$.

The integral becomes

$$\int_{-\infty}^{\infty} \frac{(t-y)^{\gamma}}{2(t+r-y)} \exp\left[-s^2\right] ds \leq \frac{(t-y)^{\gamma}\sqrt{\pi}}{2(t+c-y)}$$

and thus the integral converges uniformly on $[c,h]$ and for all $t + h-\lambda \leq y \leq t$. Therefore by a theorem from analysis [15]

$$L_{rz}(c) = L_{zr}(c) .$$

Now consider $\lim_{c \to o} L_{zr}(c)$ and notice $\lim_{c \to o} L_{zr}(c) = \lim_{c \to o} L_{rz}(c)$. To investigate this limit consider the difference $|L_{zr}(c) - L_{zr}(o)|$ or equivalently

$$\left| \int_{o}^{c} \int_{-\infty}^{\infty} \frac{(t-y)^{\gamma}}{2(t+r-y)^{\frac{3}{2}}} \exp\left[\frac{-(x-z)^2}{4(t+r-y)} \right] dzdr \right| .$$

By making the change of variable $(z-x) = 2(t+r-y)^{\frac{1}{2}} s$ and by noticing that $(t+r-y) \geq (t-y)$ for $t+h-\lambda \leq y \leq t$ it is clear that

$$\left| \int_{o}^{c} \int_{-\infty}^{\infty} \frac{(t-y)^{\gamma}}{2(t+r-y)^{\frac{3}{2}}} \exp\left[\frac{-(x-z)^2}{4(t+r-y)} \right] dzdr \right| \leq \frac{\sqrt{\pi}}{\gamma} \left[(t+c-y)^{\gamma} - (t-y)^{\gamma} \right]$$

which clearly tends to zero as c tends to zero and thus $\lim_{c \to o} L_{zr}(c) = L_{zr}(o)$. This means $\lim_{c \to o} L_{rz}(c) = L_{zr}(o)$, but $\lim_{c \to o} L_{rz}(c) \stackrel{F}{=} L_{rz}(o)$. Thus by uniqueness of limit, $L_{rz}(o) = L_{zr}(o)$ and the interchanging of the order of integration is justified. Therefore interchange the order of z and r integration in $J_{211}(\lambda)$ and consider the change of variable $(z-x) = 2(t+r-y)^{\frac{1}{2}} s$. These calculations yield

$$J_{211}(\lambda) = \int_{t+h-\lambda}^{t} \int_{o}^{h} \int_{-\infty}^{\infty} \frac{N(t-y)^{\gamma}}{(t+r-y)} \exp\left[-s^2\right] ds\, dr\, dy$$

$$J_{211}(\lambda) = \int_{t+h-\lambda}^{t} \int_{o}^{h} \frac{N\sqrt{\pi}\ (t-y)^{\gamma}}{(t+r-y)}\ drdy$$

$$J_{211}(\lambda) \leq \int_{t+h-\lambda}^{t} \int_{o}^{h} N\sqrt{\pi}\ \frac{(t+r-y)^{\gamma}}{(t+r-y)}\ drdy$$

and since $t+r-y > o$ for $t+h-\lambda \leq y \leq t$

$$J_{211}(\lambda) \leq \int_{t+h-\lambda}^{t} \int_{o}^{h} N\sqrt{\pi}\ (t+r-y)^{\gamma -1}\ drdy$$

$$J_{211}(\lambda) \leq \frac{N\sqrt{\pi}}{\gamma} \int_{t+h-\lambda}^{t} [(t+h-y)^{\gamma} - (t-y)^{\gamma}]\ dy$$

$$J_{211}(\lambda) \leq \frac{N\sqrt{\pi}}{\gamma(\gamma+1)} [-h^{\gamma+1} + \lambda^{\gamma+1} - (\lambda-h)^{\gamma+1}] \ .$$

Now by definition of derivative

$$\lim_{h\to o} \frac{J_{211}(\lambda)}{h} = \frac{\sqrt{\pi}\ N}{\gamma(\gamma+1)} \lim_{h\to o} [-h^{\gamma} + \frac{(\lambda-h)^{\gamma+1} - \lambda^{\gamma+1}}{-h}] = \frac{N\sqrt{\pi}}{\gamma}\ \lambda^{\gamma} \ .$$

which tends to zero as λ tends to zero. Thus since $J_{211}(\lambda) \geq 0$,

$$\lim_{\lambda\to o} \left[\lim_{h\to o} \frac{J_{211}(\lambda)}{h} \right] = 0.$$

Applying the same techniques to $J_{212}(\lambda)$ yields

$$J_{212}(\lambda) = \int_{t+h-\lambda}^{t} \int_{o}^{h} \int_{-\infty}^{\infty} N \frac{4\, s^2}{(t+r-y)} |t-y|^{\gamma} \exp[-s^2]\, dsdrdy.$$

Clearly letting $\int_{-\infty}^{\infty} 4s^2 \exp[-s^2]\, ds = C_1$ and noting again that

$t+r-y > t-y > o$ implies that

$$J_{212}(\lambda) \leq C_1 \, N \int_{t+h-\lambda}^{t} \int_{o}^{h} (t+r-y)^{\gamma - 1}\, drdy$$

and as in the case of $J_{211}(\lambda)$

$$\lim_{\lambda \to o} \left[\lim_{h \to o} \frac{J_{212}(\lambda)}{h} \right] = o$$

Now consider $J_{22}(\lambda)$ in much the same way as $J_{21}(\lambda)$. First use the fundamental theorem of calculus to change $J_{22}(\lambda)$ to a triple iterated integral. A calculation similar to that used in connection with $J_{211}(\lambda)$ will justify interchanging the order of the r and z integration. Finally if the change of variable $z-x = 2(t+r-y)^{\frac{1}{2}} s$ is made in $J_{22}(\lambda)$, then

$$\sqrt{4\pi}\; J_{22}(\lambda) \leq \int_{t+h-\lambda}^{t} \int_{o}^{h} \int_{-\infty}^{\infty} N \left[\left| \frac{s^{\gamma}\, 2^{\gamma}}{(t+r-y)^{1-\frac{\gamma}{2}}} \right| + \left| \frac{s^{2+\gamma}\, 2^{\gamma}}{(t+r-y)^{1-\frac{\gamma}{2}}} \right| \right] \exp[-s^2]\, dsdrdy.$$

Define

$$J_{221}(\lambda) = \int_{t+h-\lambda}^{t} \int_{o}^{h} \int_{-\infty}^{\infty} N \left| \frac{2^{\gamma} s^{\gamma}}{(t+r-y)^{1-\frac{\gamma}{2}}} \right| \exp[-s^2]\, dsdrdy$$

$$J_{222}(\lambda) = \int_{t+h-\lambda}^{t} \int_{o}^{h} \int_{-\infty}^{\infty} N \left| \frac{2^{\gamma} s^{2+\gamma}}{(t+r-y)^{1-\frac{\gamma}{2}}} \right| \exp[-s^2]\, dsdrdy$$

so that $\sqrt{4\pi}\ J_{22}(\lambda) \leq J_{221}(\lambda) + J_{222}(\lambda)$.

Carrying out the s integration in $J_{221}(\lambda)$ results in

$$J_{221}(\lambda) = C_2 \int_{t+h-\lambda}^{t} \int_{o}^{h} (t+r-y)^{\frac{\gamma}{2}-1}\, drdy$$

where

$$C_2 = 2^{\gamma} N \int_{-\infty}^{\infty} |s^{\gamma}| \exp[-s^2]\, ds\ .$$

Thus

$$J_{221}(\lambda) = \frac{4C_2}{\gamma(\gamma+2)} \left[-h^{\frac{\gamma}{2}+1} + \lambda^{\frac{\gamma}{2}+1} - (\lambda-h)^{\frac{\gamma}{2}+1} \right]$$

which implies that

$$\lim_{\lambda\to o} \left[\lim_{h\to o} \frac{J_{221}(\lambda)}{h} \right] = 0\ .$$

Applying similar techniques to $J_{222}(\lambda)$ one sees that if C_3 is defined by

$$C_3 = 2^{\gamma} N \int_{-\infty}^{\infty} \left| s^{2+\gamma} \right| \exp[-s^2]\, ds$$

then

$$J_{222}(\lambda) = C_3 \int_{t+h-\lambda}^{t} \int_{c}^{h} (t+r-y)^{\frac{\gamma}{2}-1}\, dr dy \ .$$

This implies that

$$\lim_{\lambda\to o} \left[\lim_{h\to o} \frac{J_{222}(\lambda)}{h} \right] = 0 \ .$$

Thus

$$\sqrt{4\pi}\ | I_2(\lambda) | \leq J_{211}(\lambda) + J_{212}(\lambda) + J_{221}(\lambda) + J_{222}(\lambda)$$

which implies that

$$\lim_{\lambda\to o} \left[\lim_{h\to o} \frac{I_2(\lambda)}{h} \right] = 0 \ .$$

Next consider $I_4(\lambda)$ in detail. Define $J_3(\lambda)$ and $J_4(\lambda)$ as follows:

$$J_3(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} [K(x-z, t+h-y) - K(x-z, t+\lambda-y)] \, f(x,t) \, dzdy$$

$$J_4(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} K(x-z, t+h-y) \, [f(z,y)-f(x,t)] + K(x-z, t+\lambda-y)[f(x,t)-f(z,y-\lambda)]dzdy.$$

Then $I_4(\lambda) = J_3(\lambda) + J_4(\lambda)$. Notice, however, as in the case of $J_1(\lambda)$, $J_3(\lambda) \equiv 0$. Now $J_4(\lambda) = J_{41}(\lambda) + J_{42}(\lambda)$ and $|J_{41}(\lambda)| \leq J_{411}(\lambda) + J_{412}(\lambda)$; $|J_{42}(\lambda)| \leq J_{421}(\lambda) + J_{422}(\lambda)$ for

$$J_{41}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} K(x-z, t+h-y)[f(z,y) - f(x,t)] \, dzdy$$

$$J_{42}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} K(x-z, t+\lambda-y)[f(z,y-\lambda)-f(x,t)]dzdy$$

$$J_{411}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} N|K(x-z, t+h-y)| \, |t-y|^{\gamma} \, dzdy$$

$$J_{412}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} N|K(x-z, t+h-y)| \; |x-z|^{\gamma} \, dzdy$$

$$J_{421}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} N|K(x-z, t+\lambda-y)| \; |t+\lambda-y|^{\gamma} \, dzdy$$

$$J_{422}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} N|K(x-z, t+\lambda-y)| \; |x-z|^{\gamma} \, dzdy$$

Consider $J_{411}(\lambda)$ in detail,

$$J_{411}(\lambda) = \int_{t}^{t+h} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} \frac{N|t-y|^{\gamma}}{|t+h-y|^{\frac{1}{2}}} \exp\left[\frac{-(x-z)^2}{4(t+h-y)}\right] dzdy,$$

and make the change of variable $z-x = 2(t+h-y)^{\frac{1}{2}} s$. Thus $J_{411}(\lambda)$ becomes

$$J_{411}(\lambda) = \int_{t}^{t+h} \int_{-\infty}^{\infty} \frac{2N}{\sqrt{4\pi}} |t-y|^{\gamma} \exp[-s^2]\, dsdy \;,$$

$$J_{411}(\lambda) = \int_{t}^{t+h} N|t-y|^{\gamma}\, dy \;,$$

$$J_{411}(\lambda) = \frac{N}{\gamma+1} h^{\gamma+1} \;.$$

Thus

$$\lim_{\lambda\to o} \left[\lim_{h\to o} \frac{J_{411}(\lambda)}{h} \right] = 0 \;.$$

Similarly

$$J_{412}(\lambda) = \int_{t}^{t+h} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi}} \frac{N|x-z|^{\gamma}}{|t+h-y|^{\frac{1}{2}}} \exp\left[\frac{-(x-z)^2}{4(t+h-y)}\right] dzdy \;.$$

Making the change of variable $z-x = 2(t+h-y)^{\frac{1}{2}} s$ yields

$$J_{412}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} \frac{N2^{\gamma+1}}{\sqrt{4\pi}} (t+h-y)^{\frac{\gamma}{2}} s^{\gamma} \exp[-s^2] \, ds dy$$

and letting

$$C_4 = \int_{-\infty}^{\infty} \frac{N2^{\gamma+1}}{\sqrt{4\pi}} s^{\gamma} \exp[-s^2] \, ds$$

results in

$$J_{412}(\lambda) = \frac{2C_4}{\gamma+2} h^{\frac{\gamma}{2}+1}$$

after the y integration is carried out. Thus as in the case of $J_{411}(\lambda)$

$$\lim_{\lambda \to 0} \left[\lim_{h \to 0} \frac{J_{412}(\lambda)}{h} \right] = 0 .$$

Next consider $J_{421}(\lambda)$

$$J_{421}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} \frac{N \, |t+\lambda-y|^{\gamma}}{|t+\lambda-y|^{\frac{1}{2}}} \exp \left[\frac{-(x-z)^2}{4(t+\lambda-y)} \right] dz dy .$$

Making the canonical change of variable in $J_{421}(\lambda)$ yields

$$J_{421}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} \frac{2N}{\sqrt{4\pi}} |t+\lambda-y|^{\gamma} \exp[-s^2] \, ds \, dy$$

$$J_{421}(\lambda) = \int_t^{t+h} N \, (t+\lambda-y)^{\gamma} \, dy$$

$$J_{421}(\lambda) = -\frac{N}{\gamma+1} [(\lambda-h)^{\gamma+1} - \lambda^{\gamma+1}]$$

which implies

$$\lim_{\lambda\to o} \left[\lim_{h\to o} \frac{J_{421}(\lambda)}{h} \right] = 0 .$$

Similarly

$$J_{422}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} \frac{N}{\sqrt{4\pi}} \frac{|x-z|^{\gamma}}{|t+\lambda-y|^{\frac{1}{2}}} \exp\left[\frac{-(x-z)^2}{4(t+\lambda-y)}\right] dzdy .$$

Making the change of variable $z-x = 2(t+\lambda-y)^{\frac{1}{2}} s$ yields

$$J_{422}(\lambda) = \int_t^{t+h} \int_{-\infty}^{\infty} \frac{N2^{\gamma+1}}{\sqrt{4\pi}} |s|^{\gamma} \exp[-s^2] \, (t+\lambda-y)^{\frac{\gamma}{2}} \, dsdy$$

and letting

$$C_5 = \frac{N2^{\gamma+1}}{\sqrt{4\pi}} \int_{-\infty}^{\infty} |s|^{\gamma} \exp[-s^2] \, ds$$

results in

$$J_{422}(\lambda) = \frac{2C_5}{\gamma+2} \left[-(\lambda-h)^{\frac{\gamma}{2}+1} + \lambda^{\frac{\gamma}{2}+1} \right]$$

which implies

$$\lim_{\lambda\to o} \left[\lim_{h\to o} \frac{J_{422}(\lambda)}{h} \right] = 0 .$$

Therefore since $|I_4(\lambda)| \leq J_{411}(\lambda) + J_{412}(\lambda) + J_{421}(\lambda) + J_{422}(\lambda)$,

$$\lim_{\lambda\to o} \left[\lim_{h\to o} \frac{I_4(\lambda)}{h} \right] = 0 .$$

Thus since $hI(\lambda) = I_1(\lambda) + I_4(\lambda)$ it has been shown that

$$\lim_{\lambda\to o} \left[\lim_{h\to o} I(\lambda) \right] = 0$$

which, as previously remarked, proves that

$$\lim_{\lambda\to o} u_t(x,t;\ \lambda) = u_t(x,t) .$$

To prove (35) it must first be shown that

$$u_x(x,t) = \int_0^t \int_{-\infty}^{\infty} K_x(x-z,t-y)f(z,y)\, dzdy.$$

To prove this define

$$J(x,t) = \int_0^t \int_{-\infty}^{\infty} K_x(x-z,t-y)f(z,y)\, dzdy$$

and notice that

$$|J(x,t)| \le \int_0^t \int_{-\infty}^{\infty} \frac{|x-z|\,|f(z,y)|}{4\sqrt{\pi}(t-y)^{\frac{3}{2}}} \exp\left[\frac{-(x-z)^2}{4(t-y)}\right] dzdy.$$

Letting $z-x = 2(t-y)^{\frac{1}{2}} s$, imposing (H_2) of the theorem, and setting $a = 2(t-y)^{\frac{1}{2}}$ results in the inequality

$$|J(x,t)| \le \int_0^t \int_{-\infty}^{\infty} \frac{P|s|}{\sqrt{\pi}(t-y)^{\frac{1}{2}}} \exp[-s^2] \exp[Q(x+as)^2]\, dsdy \quad .$$

Completing the square in the integrand yields

$$|J(x,t)| \le \frac{P}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} \frac{|s|}{\sqrt{t-y}} \exp\left[\frac{Qx^2}{1-Qa^2}\right] \exp\left[-(1-Qa^2)\left[s-\frac{aQx}{1-Qa^2}\right]^2\right] dsdy \ .$$

If $r = (1-Qa^2)^{\frac{1}{2}}\left[s-\frac{aQx}{1-Qa^2}\right]$ and the r integration is carried out,

then $|J(x,t)|$ satisfies

$$|J(x,t)| \leq \frac{P}{\sqrt{\pi}} \int_0^t \exp\left[\frac{Qx^2}{1-Qa^2}\right]\left(\frac{1}{1-Qa^2} + \frac{\sqrt{\pi}aQ|x|}{(1-Qa^2)^{\frac{3}{2}}}\right)(t-y)^{-\frac{1}{2}}\, dy \, .$$

For every $x, t \in \left\{x,t\colon |x| < A,\ 0 < t \leq \frac{1}{4Q} - b\right\}$ where A is arbitrary and Q is the constant of the growth condition (H_2)

$$|J(x,t)| \leq \frac{P}{\sqrt{\pi}} \exp\left[\frac{A^2}{4b}\right]\left(\frac{1}{4Qb} + \frac{(1-4Qb)^{\frac{1}{2}}A\sqrt{\pi}}{8Qb^{\frac{3}{2}}}\right)\int_0^t (t-y)^{-\frac{1}{2}}\, dy \, .$$

If the y integration is carried out and the result maximized, then $|J(x,t)|$ satisfies

$$|J(x,t)| \leq \frac{P}{\sqrt{\pi}} \exp\left[\frac{A^2}{4b}\right]\left(\frac{1}{4Qb} + \frac{(1-4Qb)^{\frac{1}{2}}A\sqrt{\pi}}{8Qb^{\frac{3}{2}}}\right)\frac{(1-4Qb)^{\frac{1}{2}}}{Q^{\frac{1}{2}}}$$

which shows that $|J(x,t)| \leq D(P, Q, A, b) < +\infty$. Thus by the Weierstrass M-test the integral $J(x,t)$ converges uniformly and absolutely for all $x, t \in \left\{x,t\colon |x| < A,\ 0 < t \leq \frac{1}{4Q} - b\right\}$ and therefore

$$u_x(x,t) = \int_0^t \int_{-\infty}^{\infty} K_x(x-z,t-y)\ f(z,y)\ dz\ dy\ .$$

Now define

$$V(x,t;\lambda) = u_x(x,t) - u_x(x,t;\ \lambda)$$

and consider $V(x+h,t;\ \lambda) - V(x,t;\ \lambda)$. Notice that

$$\lim_{h\to o} \frac{V(x+h,t;\lambda)}{h} - \frac{V(x,t;\lambda)}{h} = u_{xx}(x,t) - u_{xx}(x,t;\lambda).$$

Therefore define

$$J(\lambda) = \frac{V(x+h,t;\lambda)-V(x,t;\lambda)}{h}\ ,$$

and if it can be shown that

$$\lim_{\lambda\to o}\left[\lim_{h\to o}\ J(\lambda)\right] = 0$$

then $u_{xx}(x,t) = \lim_{\lambda\to o} u_{xx}(x,t;\lambda)$. Writing $J(\lambda)$ out gives

$$J(\lambda) = \int_{t-\lambda}^{t} \int_{-\infty}^{\infty} h^{-1}[K_x(x+h-z,t-y)-K_x(x-z,t-y)]\ f(z,y)dzdy\ .$$

Define $J_1(\lambda)$ and $J_2(\lambda)$ as follows:

$$J_1(\lambda) = \int_{t-\lambda}^{t} \int_{-\infty}^{\infty} h^{-1}[K_x(x+h-z,t-y)-K_x(x-z,t-y)]\ [f(z,y)-f(x,t)]\ dzdy$$

$$J_2(\lambda) = f(x,t) \int_{t-\lambda}^{t} \int_{-\infty}^{\infty} h^{-1}[K_x(x+h-z,t-y)-K_x(x-z,t-y)]\ dzdy\ .$$

Thus $J(\lambda) = J_1(\lambda) + J_2(\lambda)$. Notice that

$$\int_{-\infty}^{\infty} K_x(x+h-z,t-y)dz = \int_{-\infty}^{\infty} K_x(x-z,t-y)dz = 0$$

and thus $J_2(\lambda) \equiv 0$. To study $J_1(\lambda)$ in detail define

$$I_{11}(\lambda) = \int_{t-\lambda}^{t} \int_{-\infty}^{x} h^{-1}[K_x(x+h-z,t-y)-K_x(x-z,t-y)]\ [f(z,y)-f(x,t)]dzdy$$

$$I_{12}(\lambda) = \int_{t-\lambda}^{t} \int_{x}^{x+2h} h^{-1}\ [K_x(x+h-z,t-y)-K_x(x-z,t-y)]\ [f(z,y)-f(x,t)]dzdy$$

$$I_{13}(\lambda) = \int_{t-\lambda}^{t} \int_{x+2h}^{\infty} h^{-1}[K_x(x+h-z,t-y)-K_x(x-z,t-y)]\ [f(z,y)-f(x,t)]dzdy$$

and thus $J_1(\lambda) = I_{11}(\lambda) + I_{12}(\lambda) + I_{13}(\lambda)$. Consider $I_{11}(\lambda)$ in detail,

$$I_{11}(\lambda)=\int_{t-\lambda}^{t} \int_{-\infty}^{x} \frac{h^{-1}}{\sqrt{4\pi}} \left\{ -\frac{(x+h-z)}{2(t-y)^{\frac{3}{2}}} \exp\left[\frac{-(x+h-z)^2}{4(t-y)}\right] + \frac{(x-z)}{2(t-y)^{\frac{3}{2}}} \exp\left[\frac{-(x-z)^2}{4(t-y)}\right] \right\} [f(z,y)-f(x,t)]dzdy$$

If the mean value theorem is applied to the integrand of the integral $I_{11}(\lambda)$, then

$$I_{11}(\lambda) = \int_{t-\lambda}^{t}\int_{-\infty}^{x} \frac{\partial}{\partial x}\left[\frac{-(x+\Theta h-z)}{2(t-y)}\, K(x+\Theta h-z,t-y)\right] [f(z,y)-f(x,t)]\, dzdy$$

for some $0 < \Theta < 1$. Thus

$$|I_{11}(\lambda)| \le \int_{t-\lambda}^{t}\int_{-\infty}^{x} \left|\frac{-K(x+\Theta h-z,t-y)-(x+\Theta h-z)K_x(x-\Theta h-z,t-y)}{2(t-y)}\right| |f(z,y)-f(x,t)|\,dzdy.$$

Applying the Hölder condition (H_3) of the theorem to the integrand yields the inequality

$$|I_{11}(\lambda)| \le \int_{t-\lambda}^{t}\int_{-\infty}^{x} \frac{N}{4\sqrt{\pi}}\left|\frac{-1}{(t-y)^{\frac{3}{2}}} + \frac{(x+\Theta h-z)^2}{2(t-y)^{\frac{5}{2}}}\right| \exp\left[\frac{-(x+\Theta h-z)^2}{4(t-y)}\right] [|x-z|^{\gamma}+|t-y|^{\gamma}]\,dzdy.$$

Notice that for $-\infty < z < x$, $|x-z| \le |x+\Theta h-z|$ and thus

$$|I_{11}(\lambda)| \le \int_{t-\lambda}^{t}\int_{-\infty}^{x} \frac{N}{4\sqrt{\pi}}\left[\frac{(x+\Theta h-z)^{\gamma}}{(t-y)^{\frac{3}{2}}} + \frac{(x+\Theta h-z)^{\gamma+2}}{2(t-y)^{\frac{5}{2}}} + \frac{1}{(t-y)^{\frac{3}{2}-\gamma}} + \frac{(x+\Theta h-z)^2}{2(t-y)^{\frac{5}{2}-\gamma}}\right] \exp\left[\frac{-(x+\Theta h-z)^2}{4(t-y)}\right] dzdy.$$

Making the change of variable $(z-\Theta h-x) = 2(t-y)^{\frac{1}{2}}\, s$ results in

$$|I_{11}(\lambda)| \le \int_{t-\lambda}^{t}\int_{-\infty}^{\frac{-\Theta h}{2(t-y)}} \frac{N}{4\sqrt{\pi}}\left[\frac{2^{\gamma+1}|s|^{\gamma}}{(t-y)^{1-\frac{\gamma}{2}}} + \frac{2^{\gamma+2}|s|^{\gamma+2}}{(t-y)^{1-\frac{\gamma}{2}}} + \frac{2}{(t-y)^{1-\gamma}} + \frac{4s^2}{(t-y)^{1-\gamma}}\right] \exp[-s^2]\,dsdy$$

and thus

$$|I_{11}(\lambda)| \leq \int_{t-\lambda}^{t} \int_{-\infty}^{\infty} \frac{N}{4\sqrt{\pi}} \left[\frac{|s|^{\gamma} 2^{\gamma+1}}{(t-y)^{1-\frac{\gamma}{2}}} + \frac{2^{\gamma+2}|s|^{\gamma+2}}{(t-y)^{1-\frac{\gamma}{2}}} + \frac{2}{(t-y)^{1-\gamma}} + \frac{4|s|^2}{(t-y)^{1-\gamma}} \right] \exp[-s^2]\, ds\, dy.$$

If A_1 and A_2 are defined as

$$A_1 = \int_{-\infty}^{\infty} \frac{N}{4\sqrt{\pi}} \left[|s|^{\gamma} 2^{\gamma+1} + |s|^{\gamma+2} 2^{\gamma+2} \right] \exp[-s^2]\, ds$$

$$A_2 = \int_{-\infty}^{\infty} \frac{N}{4\sqrt{\pi}} \left[2 + 4|s|^2 \right] \exp[-s^2]\, ds \,, \tag{36}$$

then $|I_{11}(\lambda)|$ satisfies the inequality

$$|I_{11}(\lambda)| \leq A_1 \int_{t-\lambda}^{t} (t-y)^{\frac{\gamma}{2}-1} dy + A_2 \int_{t-\lambda}^{t} (t-y)^{\gamma-1}\, dy \,.$$

If the y integration is carried out, $|I_{11}(\lambda)|$ satisfies

$$|I_{11}(\lambda) \leq \frac{2A_1}{\gamma} \lambda^{\frac{\gamma}{2}} + \frac{A_2}{\gamma} \lambda^{\gamma}$$

and thus

$$\lim_{\lambda \to 0} \left[\lim_{h \to 0} I_{11}(\lambda) \right] = 0 \,.$$

Now consider $I_{13}(\lambda)$ in detail;

$$I_{13}(\lambda)=\int_{t-\lambda}^{t}\int_{x+2h}^{\infty}\frac{h^{-1}}{\sqrt{\pi}}\left[\frac{-(x+h-z)}{2(t-y)^{\frac{3}{2}}}\exp\left[\frac{-(x+h-z)^2}{4(t-y)}\right]+\frac{(x-z)}{2(t-y)^{\frac{3}{2}}}\exp\left[\frac{-(x-z)^2}{4(t-y)}\right]\right][f(z,y)-f(x,t)]dzdy.$$

Applying the mean value theorem and imposing (H_3) of the theorem on $I_{13}(\lambda)$ as in the case of $I_{11}(\lambda)$ results in $|I_{13}(\lambda)|$ satisfying the inequality

$$|I_{13}(\lambda)|\leq\int_{t-\lambda}^{t}\int_{x+2h}^{\infty}\frac{N}{4\sqrt{\pi}}\left[\frac{|x+\Theta h-z|^{\gamma}}{(t-y)^{\frac{3}{2}}}+\frac{|x+\Theta h-z|^{\gamma+2}}{2(t-y)^{\frac{5}{2}}}+\frac{1}{(t-y)^{\frac{3}{2}-\gamma}}+\frac{|x+\Theta h-z|^2}{2(t-y)^{\frac{5}{2}-\gamma}}\right]\exp\left[\frac{-(x+\Theta h-z)^2}{4(t-y)}\right]dzdy$$

since for $x + 2h \leq z < \infty$, $|x-z| \leq |x+\Theta h-z|$. If the change of variable $z-\Theta h-x = 2(t-y)^{\frac{1}{2}}s$ is made, then $|I_{13}(\lambda)|$ satisfies

$$|I_{13}(\lambda)| \leq A_1\int_{t-\lambda}^{t}(t-y)^{\frac{\gamma}{2}-1}dy + A_2\int_{t-\lambda}^{t}(t-y)^{\gamma-1}dy$$

which implies that

$$|I_{13}(\lambda)| \leq \frac{2A_1}{\gamma}\lambda^{\frac{\gamma}{2}} + \frac{A_2}{\gamma}\lambda^{\gamma}$$

where A_1 and A_2 are defined in expression (36). Thus

$$\lim_{\lambda\to 0}\left[\lim_{h\to 0} I_{13}(\lambda)\right] = 0 \ .$$

Now consider $I_{12}(\lambda)$ in detail;

$$|I_{12}(\lambda)| \leq \int_{t-\lambda}^{t}\int_{x}^{x+2h} Nh^{-1}\left|\frac{-(x+h-z)}{2(t-y)^{\frac{3}{2}}}\exp\left[\frac{-(x+h-z)^2}{4(t-y)}\right]+\frac{(x-z)}{2(t-y)^{\frac{3}{2}}}\exp\left[\frac{-(x-z)^2}{4(t-y)}\right]\right|(|x-z|^{\gamma}+|t-y|^{\gamma})dzdy$$

after imposing (H_3) of the theorem. Define

$$I_{121}(\lambda)=\int_{t-\lambda}^{t}\int_{x}^{x+2h} Nh^{-1}\left[\frac{|x+h-z|}{2(t-y)^{\frac{3}{2}}}\exp\left[\frac{-(x+h-z)^2}{4(t-y)}\right]+\frac{|x-z|}{2(t-y)^{\frac{3}{2}}}\exp\left[\frac{-(x-z)^2}{4(t-y)}\right]\right]|t-y|^{\gamma}dzdy$$

$$I_{122}(\lambda)=\int_{t-\lambda}^{t}\int_{x}^{x+2h} Nh^{-1}\left[\frac{|x+h-z|}{2(t-y)^{\frac{3}{2}}}\exp\left[\frac{-(x+h-z)^2}{4(t-y)}\right]+\frac{|x-z|}{2(t-y)^{\frac{3}{2}}}\exp\left[\frac{-(x-z)^2}{4(t-y)}\right]\right]|x-z|^{\gamma}dzdy$$

and thus $|I_{12}(\lambda)| \leq I_{121}(\lambda) + I_{122}(\lambda)$. It can be shown that for $0 < \ell < 1$, $p > 0$, $\alpha > 0$ [16] there exists a constant $K > 0$ such that

$$\alpha^{p}\exp[-\alpha] < K\exp[-\ell\alpha] \tag{37}$$

and thus, applying this result to $I_{121}(\lambda)$ with $p = \frac{1}{2}$,

$$I_{121}(\lambda) \leq \int_{t-\lambda}^{t}\int_{x}^{x+2h}\frac{KNh^{-1}}{(t-y)^{1-\gamma}}\left[\exp\left[\frac{-\ell(x+h-z)^2}{4(t-y)}\right]+\exp\left[\frac{-\ell(x-z)^2}{4(t-y)}\right]\right]dzdy$$

and therefore

$$I_{121}(\lambda) \le \int_{t-\lambda}^{t} \int_{x}^{x+2h} \frac{4KNh^{-1}}{(t-y)^{1-\gamma}} dzdy = \int_{t-\lambda}^{t} 8NK(t-y)^{\gamma-1} \; dy \; .$$

Carrying out the y integration results in $I_{121}(\lambda)$ satisfying

$$I_{121}(\lambda) \le \frac{8NK\lambda^{\gamma}}{\gamma}$$

and therefore

$$\lim_{\lambda \to o} \left[\lim_{h \to o} I_{121}(\lambda) \right] = 0 \; .$$

If the inequality (37) is applied to $I_{122}(\lambda)$, then

$$I_{122}(\lambda) \le \int_{t-\lambda}^{t} \int_{x}^{x+2h} \frac{h^{-1}NK|x-z|^{\gamma}}{(t-y)} \left[\exp\left[\frac{-\ell(x+h-z)^2}{4(t-y)}\right] + \exp\left[\frac{-\ell(x-z)^2}{4(t-y)}\right] \right] dzdy.$$

Consider

$$\int_{x}^{x+2h} |x-z|^{\gamma} \left[\exp\left[\frac{-\ell(x+h-z)^2}{4(t-y)}\right] + \exp\left[\frac{-\ell(x-z)^2}{4(t-y)}\right] \right] dz$$

and define

$$H_1(h) = \int_x^{x+h} |x-z|^{\gamma} \exp\left[\frac{-\ell(x+h-z)^2}{4(t-y)}\right] dz$$

$$H_2(h) = \int_{x+h}^{x+2h} |x-z|^{\gamma} \exp\left[\frac{-\ell(x+h-z)^2}{4(t-y)}\right] dz ,$$

$$H_3(h) = \int_x^{x+h} |x-z|^{\gamma} \exp\left[\frac{-\ell(x-z)^2}{4(t-y)}\right] dz ,$$

$$H_4(h) = \int_{x+h}^{x+2h} |x-z|^{\gamma} \exp\left[\frac{-\ell(x-z)^2}{4(t-y)}\right] dz .$$

In $H_3(h)$ make the change of variable $z = r-h$ and then change r to z. Thus $H_3(h)$ becomes

$$H_3(h) = \int_{x+h}^{x+2h} |x+h- z|^{\gamma} \exp\left[\frac{-\ell(x+h-z)^2}{4(t-y)}\right] dz,$$

and thus $H_3(h) \leq H_2(h)$. In $H_4(h)$ make the change of variable $z = r + h$ and then change r to z. Then $H_4(h)$ becomes

$$H_4(h) = \int_x^{x+h} |x-h-z|^{\gamma} \exp\left[\frac{-\ell(x-h-z)^2}{4(t-y)}\right] dz .$$

It is easy to see that for $x \leq z \leq x + h$ $|x-z|^{\gamma} \leq |x-h-z|^{\gamma}$, so that if $H_1^*(h)$ is defined as

$$H_1^*(h) = \int_x^{x+h} |x-h-z|^\gamma \exp\left[\frac{-\ell(x+h-z)^2}{4(t-y)}\right] dz$$

$H_1(h) \leq H_1^*(h)$ and since in this range of $z (x-h-z)^2 \geq (x+h-z)^2$, $H_4(h) \leq H_1^*(h)$. Define

$$H_2^*(h) = \int_{x+h}^{x+2h} |x-h-z|^\gamma \exp\left[\frac{-\ell(x+h-z)^2}{4(t-y)}\right] dz$$

and notice that $|x-z|^\gamma \leq |x-h-z|^\gamma$ for $x+h \leq z \leq x + 2h$. Thus $H_2(h) \leq H_2^*(h)$ and therefore

$$H_1(h) + H_2(h) + H_3(h) + H_4(h) \leq 2(H_1^*(h) + H_2^*(h)),$$

Thus

$$I_{122}(\lambda) \leq \int_{t-\lambda}^{t} \int_x^{x+2h} \frac{2h^{-1} NK|x-h-z|^\gamma}{(t-y)} \exp\left[\frac{-\ell(x+h-z)^2}{4(t-y)}\right] dzdy.$$

Since

$$|x-h-z|^\gamma = \frac{|x+h-z|^{1-\gamma} |x-h-z|^\gamma}{|x+h-z|^{1-\gamma}} \leq \frac{h^{1-\gamma}(3h)^\gamma}{|x+h-z|^{1-\gamma}} < \frac{3^\gamma h}{|x+h-z|^{1-\gamma}},$$

then

$$I_{122}(\lambda) \leq \int_{t-\lambda}^{t} \int_x^{x+2h} \frac{2 3 NK|x+h-z|^{\gamma-1}}{(t-y)} \exp\left[\frac{-\ell(x+h-z)^2}{4(t-y)}\right] dzdy .$$

If the change of variable $z-x-h = 2(t-y)^{\frac{1}{2}} s$ is made in the integral, then

$$I_{122}(\lambda) \leq \int_{t-\lambda}^{t} \int_{\frac{-h}{2(t-y)^{\frac{1}{2}}}}^{\frac{h}{2(t-y)^{\frac{1}{2}}}} 2N_1 K (t-y)^{\frac{\gamma-1}{2}} |s|^{\gamma-1} \exp[-\ell s^2] \, ds \, dy$$

$$I_{122}(\lambda) \leq \int_{t-\lambda}^{t} \int_{-\infty}^{\infty} 2N_1 K (t-y)^{\frac{\gamma}{2}-1} |s|^{\gamma-1} \exp[-\ell s^2] \, ds \, dy$$

$$I_{122}(\lambda) \leq \int_{t-\lambda}^{t} C_3 (t-y)^{\frac{\gamma}{2}-1} \, dy = \frac{2C_3}{\gamma} \lambda^{\frac{\gamma}{2}}$$

where

$$C_3 = \int_{-\infty}^{\infty} 2^{\gamma+1} N_1 K \, |s|^{\gamma-1} \exp[-\ell s^2] \, ds$$

and

$$N_1 = 2^2 \cdot 3^{\gamma} \; .$$

Thus

$$\lim_{\lambda \to 0} \left[\lim_{h \to 0} I_{122}(\lambda) \right] = 0$$

which implies that

$$\lim_{\lambda \to 0} \left[\lim_{h \to 0} I_{12}(\lambda) \right] = 0 \; .$$

Since $J_1(\lambda) = I_{11}(\lambda) + I_{12}(\lambda) + I_{13}(\lambda)$,

$$\lim_{\lambda \to 0} \left[\lim_{h \to 0} J_1(\lambda) \right] = 0$$

and thus

$$\lim_{\lambda \to 0} \left[\lim_{h \to 0} J(\lambda) \right] = 0 .$$

Therefore as stated previously

$$\lim_{\lambda \to 0} u_{xx}(x,t;\lambda) = u_{xx}(x,t)$$

and the theorem is proved.

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