

RESULTS IN SEMI-INNER-PRODUCT SPACES AND
GENERALIZED COSINE OPERATOR FUNCTIONS

A THESIS

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To my wife without whose
help this would have been impossible

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INTRODUCTION

In an effort to introduce the simplicity and elegance of operator theory on a Hilbert space into the realm of a general Banach space, G. Lumer (15) introduced the notion of a semi-inner-product on the Banach space X as a function $(\cdot, \cdot): X \times X \rightarrow \mathbb{C}(\mathbb{R})$ which is linear in the first argument, strictly positive, and satisfies the Schwarz inequality $|(x, y)|^2 \leq (x, x)(y, y)$. The form (\cdot, \cdot) induces a norm in the natural way by setting $\sqrt{(x, x)} = \|x\|$. Lumer showed that every normed linear space has at least one semi-inner-product which is compatible with the norm in this fashion.

Later, J. R. Giles (8) extended the result to include conjugate homogeneity in the second coordinate.

The principal thrust of the theory of semi-inner-products has been the study of operator theory in Banach spaces. A notable exception to this is a theorem by Giles to the effect that in a smooth, uniformly convex Banach space, a Riesz Representation theorem holds. This is to say that for every $x^* \in X^*$ there exists an $x \in X$ so that for all $y \in X$, $x^*(y) = (y, x)$. Our intent is to turn attention toward the use of semi-inner-products in the geometric theory of Banach spaces. In fact we prove a generalization of Giles' theorem amounting to a characterization of reflexive Banach spaces. In other words, a space is reflexive if and only if every $x^* \in X^*$ can

be represented by some semi-inner-product.

In (9) James introduced the notion of orthogonality in Banach spaces as follows: $x \in X$ is orthogonal to $y \in X$, denoted $x \perp y$, if and only if for each scalar α , $\|x\| \leq \|x + \alpha y\|$. We are able to prove a theorem relating this notion of orthogonality to the natural concept of orthogonality (normality) engendered by the semi-inner-product. Using these two key results we explore some of the geometric implications of semi-inner-product theory.

In addition, we use semi-inner-products to investigate somewhat the structure of scalar operators with a cyclic vector on a certain class of Banach spaces.

In Chapter 2 we turn our attention to families of operators in a Hilbert space satisfying D'Alembert's functional equation $2C(s)C(t) = C(s+t) + C(s-t)$ ($s, t \in \mathbb{R}$). Such families are called cosine operator functions and have been extensively studied under various assumptions of continuity, notably by Sova (24), Sz.-Nagy (17), and Kurepa (12), (13), (14). We define a cosine representation of a *-semigroup and give a characterization of operator families which may be dilated to cosine representations. This theorem is analogous to the celebrated "principal theorem" of Sz.-Nagy (18) which gives similar conditions for semigroup representations. We are able to characterize generalized cosine operator functions as an application of this theorem. A generalized cosine operator function is a family of operators $C(t)$ on a Hilbert

space H which can be realized as $C(t) = P_H \tilde{C}(t)|_H$, where $\tilde{C}(t)$ is a cosine operator function on a Hilbert space \tilde{H} containing H as a subspace.

We are able to give two integral representations of generalized cosine operator functions, the scalar versions of which solve two cosine moment problems related to the cosine-Stieltjes transform.

For the general theory of dilations which is substantially used in this chapter, the reader is referred to (18), (23), (16).

In what follows \mathbb{R} will denote the real numbers, \mathbb{C} the complex numbers, $B(H)$ the bounded linear operators on the Hilbert space H , $L(H)$ the linear operators on H , X^* the dual of the Banach space X , and $N(f)$ the null space of the transformation f . $D(A)$ and $R(A)$ will denote the domain and range respectively of the operator A .

CHAPTER I

SEMI-INNER-PRODUCT SPACES

SECTION 1: Geometry

In the following, all Banach spaces will be considered to be over the field of complex numbers with the understanding that, unless otherwise specified, all of the conclusions also apply to real Banach spaces with the obvious natural modifications.

Definition 1.1.1 Let X be a Banach space. A semi-inner-product on X is a function $(\cdot, \cdot): X \times X \rightarrow \mathbb{C}$ satisfying the following: For all $x, y, z \in X$

1. $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$ For all $\alpha, \beta \in \mathbb{C}$
2. $(x, x) = \|x\|^2 > 0$ For $x \neq 0$
3. $|(x, y)|^2 \leq (x, x)(y, y)$
4. $(x, \beta y) = \bar{\beta}(x, y)$

Semi-inner-products were first considered by G. Lumer (15) as a form (\cdot, \cdot) which satisfied (1)-(3) of the above definition. Later, J. R. Giles (8) showed that without sacrifice of applicability (\cdot, \cdot) can be chosen to satisfy (4) in addition. That is to say that Giles gave a proof that every normed linear space X has a (possibly infinitely many)

semi-inner-product, (\cdot, \cdot) , satisfying all of the axioms of Definition 1.1.1. We will not attempt to give a proof of Giles's result here in as much as it is a consequence of a more general theorem to follow.

Definition 1.1.2 A Banach space X is said to be uniformly convex if and only if for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ so that if $\|x\| = \|y\| = 1$ and $\|x-y\| > \epsilon$, then $\|\frac{x+y}{2}\| < 1-\delta$.

Definition 1.1.3 A Banach space X is said to be strictly convex if and only if $\|x\| + \|y\| = \|x+y\|$, where $x, y \neq 0$, implies $x = \lambda y$ for some $\lambda > 0$.

In (4) Clarkson showed that every separable Banach space can be renormed so as to be strictly convex.

Definition 1.1.4 A Banach space X is reflexive if and only if the $J: X \rightarrow X^{**}$ given by $(Jx)(x^*) = x^*(x)$ is surjective.

Every Hilbert space is reflexive and apparently, as we will later observe, reflexivity in a Banach space in some sense says that the Banach space is approximately a Hilbert space.

It is well known that uniform convexity implies both strict convexity and reflexivity. As a corollary to theorem 1.1.7 we will obtain an alternate route to proving the Milman-Pettis theorem that every uniformly convex Banach space is reflexive.

We will now state without formal proof several well

known theorems.

Theorem 1.1.1 In a uniformly convex Banach space every closed convex set has an element of minimum norm.

As a consequence of this, if M is a closed subspace of the uniformly convex Banach space X and $x \in X - M$, then the closed convex set $x - M$ has a unique point of minimum norm so that we have:

Theorem 1.1.2 Let X be a uniformly convex Banach space, M a closed subspace of X , and $x \in X - M$, then there exists a unique element $x_0 \in M$ so that

$$\|x_0 - x\| = \inf_{y \in M} \|y - x\| = d(x, M)$$

In reflexive Banach spaces, Theorems 1.1.1 and 1.1.2 may be retained with the sacrifice of uniqueness so that we have:

Theorem 1.1.3 Let X be a reflexive Banach space, then the following are true:

- (i) Every closed convex set in X has element of minimum norm.
- (ii) For every subspace M of X and every $x \in X - M$ there exists an element $x_0 \in M$ so that

$$\|x_0 - x\| = \inf_{y \in M} \|y - x\| = d(x, M).$$

Definition 1.1.5 Following James (9), in a normed

vector space X , an element $x \in X$ is said to be orthogonal to an element $y \in X$ if and only if for every $\lambda \in \mathbb{C}$, $\|x + \lambda y\| \geq \|x\|$. This is written as $x \perp y$. If for each $x \in M$ and $y \in N$, $M, N \subseteq X$, we have $x \perp y$, we will write $M \perp N$.

This notion of orthogonality generalizes the familiar notion of orthogonality in inner-product space. However, unlike the inner-product engendered orthogonality, the relation \perp is neither symmetric nor additive, (where additive means $z \perp \{x, y\} \longrightarrow z \perp \alpha x + \beta y$). In fact, Birkhoff (3) has shown that for normed spaces of dimension strictly greater than 2, the symmetry of \perp implies that the normed space is a Hilbert space. James (9) has shown that the additivity of \perp is equivalent to the Gateaux differentiability of the norm at every nonzero vector, where we have the following:

Definition 1.1.6 A functional f defined on a normed linear space X is Gateaux differentiable at $x \in X$ if and only if $\lim_{h \rightarrow 0} (f(x+hy) - f(x))/h$ exists for each $y \in Y$ with $\|y\| = 1$.

Always when referring to the Gateaux differentiability of the norm, we assume this to be at nonzero vectors.

Definition 1.1.7 The relation \perp is said to be right-unique if and only if for no element $x (\neq 0)$ and y there is more than one number a for which $x \perp ax + y$.

We have the following by James (9):

Theorem 1.1.4 Orthogonality is additive in a normed linear space X if and only if it is right unique, or if and only if the norm is Gateaux differentiable.

We now show that in reflexive spaces, closed subspaces

have orthogonal complements.

Theorem 1.1.5 Let X be a reflexive Banach space and $M \subseteq X$ a subspace with $M \neq X$, then there exists a non-zero element $x_0 \in X-M$ such that

$$x_0 \perp M.$$

Proof: According to Theorem 1.1.3, if M is a closed subspace of X and $x \in X-M$, then there exists an element $z_0 \in M$ such that

$$\|z_0 - x\| = \inf_{y \in M} \|y - x\|$$

Let y_0 be an arbitrary element of M and put $x_0 = z_0 - x$, then for all $\lambda \in \mathbb{C}$

$$\|x_0\| = \inf_{y \in M} \|y - x\| \leq \|x_0 - \lambda y_0\|$$

so that $x_0 \perp M$.

Definition 1.1.8 For $x, y \in X$ we say that x is normal to y with respect to, or relative to, the semi-inner-product (\cdot, \cdot) if and only if $(y, x) = 0$. If M and N are subsets of X , we say that M is normal to N if and only if for each $x \in M$ and $y \in N$ we have $(y, x) = 0$.

Definition 1.1.9 A Banach space X is smooth if and only if for each $x \in X$ with $\|x\| = 1$ there is a unique $x^* \in X^*$ such that $x^*(x) = \|x^*\|$. We may note that in all cases the Hahn-Banach theorem insures the existence of at least one

such x^* . Geometrically, the condition of smoothness is that the unit ball of the space possess unique supporting hyperplanes. From this definition the following is clear:

Theorem 1.1.6 (15) A Banach space is smooth if and only if there is a unique semi-inner-product.

We now pass to a theorem which is central to our considerations of semi-inner-products and their relations to orthogonality and Banach space geometry.

Theorem 1.1.7 Let M and N be subspaces of a normed linear space X . A necessary and sufficient condition for $M \perp N$ is that there exist a s.i.p. (\cdot, \cdot) relative to which M is normal to N .

Proof: Suppose that M is normal to N with respect to (\cdot, \cdot) . If $x \in M$ and $y \in N$ we have

$$\|x+y\| \|x\| \geq (x+y, x) = \|x\|^2,$$

from which it follows that $M \perp N$.

Let us now suppose that $M \perp N$, then $M \cap N = \{0\}$. Hence, for each $x \in M$ we may define a linear functional f_x on $s_x = \text{span}\{x, N\} = \{\alpha x + n \mid \alpha \in \mathbb{C}, n \in N\}$ as follows:

$$f_x(\alpha x + n) = \alpha \|x\|^2.$$

Now f_x is clearly linear and in view of

$$|f_x(\alpha x + n)| = |\alpha| \|x\|^2 \leq \|x\| |\alpha| \left\|x + \frac{n}{\alpha}\right\| = \|x\| \|\alpha x + n\|,$$

f_x is bounded with $\|f_x\| \leq \|x\|$. By observing that

$f_x\left(\frac{x}{\|x\|}\right) = \|x\|$ we obtain $\|f_x\| = \|x\|$. Also, f_x satisfies

$f_x(x) = \|x\|^2$ and $f_x(n) = 0 \forall n \in \mathbb{N}$. For $z \notin M$ we may define

$f_z(\alpha z) = \alpha \|z\|^2$ on the span of $\{z\}$. Clearly, for these

$z \in X$, $\|f_z\| = \|z\|$ and $f_z(z) = \|z\|^2$. Thus for each $x \in X$ we

obtain a bounded linear functional f_x which may be extended

to the entire space by the Hahn-Banach theorem without in-

creasing the norm. We therefore consider f_x to be defined

throughout X . Now, let Λ be a well ordering of $X - \{0\}$, and

let x be the initial element of Λ . Define the functional ϕ_x

to be f_x ; and if $z = \lambda x$, define $\phi_z = \bar{\lambda} \phi_x$. Similarly for x' ,

the initial element of Λ not in the span of x , define

$\phi_{x'} = f_{x'}$, and $\phi_{\lambda x'} = \bar{\lambda} \phi_{x'}$. Continuing in this fashion we

may, by transfinite induction, define ϕ_z for each $z \in X$. Since

$z \in X$ has a unique initial generator ω relative to the order Λ

(i.e., ω is the least element of Λ for which $z = \lambda \omega$), the

indexing of the functionals ϕ_z is clearly well-defined. We

may now set $(x, z) = \phi_z(x)$, and we need only verify that (1)-

(4) of Definition 1.1.1 holds, since clearly for $x \in M$, $y \in N$ we

have $(y, x) = \phi_x(y) = 0$. The condition (1) is immediate from

the linearity of ϕ_z . For condition (2) suppose that ω is the

initial generator of $x \in X$, say $x = \lambda \omega$, then $(x, x) = \phi_{\lambda \omega}(\lambda \omega) =$

$|\lambda|^2 f_\omega(\omega) = \|\lambda \omega\|^2 = \|x\|^2 > 0$. Similarly, for condition (3),

if $x = \lambda \omega$ and $y = \mu \nu$ for both ω and ν initial in Λ , then we

have

$$\begin{aligned}
|(x, y)|^2 &= |\phi_y(x)|^2 = |\phi_{\mu\nu}(\lambda\omega)|^2 = |\mu|^2 |\lambda|^2 |\phi_\nu(\omega)|^2 \\
&\leq |\mu|^2 |\lambda|^2 \|\xi_\nu\|^2 \|\omega\|^2 = \|\mu\nu\|^2 \|\lambda\omega\|^2 = (y, y)(x, x).
\end{aligned}$$

Finally for part (4) $(x, \beta y) = (x, (\beta\mu)\nu) = \beta\mu\phi_\nu(x) = \bar{\beta}\phi_{\mu\nu}(x) = \bar{\beta}(x, y)$. This concludes the proof.

We may observe that there exist subspaces M and N , both with dimension larger than one, that satisfy the hypothesis of this theorem. For example if the Banach space has a monotone base $\{x_i\}$ then for every n , $\text{span}\{x_1, \dots, x_n\}$ is orthogonal to its algebraic complement. As a consequence of the previous theorem we obtain:

Corollary 1.1.1 (5) In a normed linear space X the norm is Gateaux differentiable if and only if X is smooth.

Proof: If the norm of X is Gateaux differentiable, then \perp is right-unique. Suppose that there exist two semi-inner-products with $(y, x)_1 \neq (y, x)_2$. It follows then that $a_1 = -\frac{(y, x)_1}{\|x\|^2}$ and $a_2 = -\frac{(y, x)_2}{\|x\|^2}$ are not equal. But $(a_1x + y, x)_1 = -\frac{(y, x)_1}{\|x\|^2} \|x\|^2 + (y, x)_1 = 0$, and similarly $(a_2x + y, x)_2 = 0$. As in the proof of Theorem 1.1.6, this implies that $x \perp a_1x + y$, and $x \perp a_2x + y$, which contradicts the assumption that \perp is right unique. So X must be smooth. Suppose now that X is smooth, so that (\cdot, \cdot) is unique. Suppose that $x \perp ax + y$. By theorem 1.1.7 there exists a semi-inner-product for which

$(ax+y, x) = 0$. This is equivalent to $a\|x\|^2 + (y, x) = 0$ which implies that $a = - (y, x) / \|x\|^2$. In other words, a is unique which by theorem 1.1.4 implies that the norm is Gateaux differentiable. This concludes the proof.

In (2) Berkson showed that the following holds:

Lemma 1.1.1 A semi-inner-product space is strictly convex if and only if whenever $(x, y) = \|x\| \|y\|$, with $x, y \neq 0$, then $y = \lambda x$ for some $\lambda > 0$. We will use this in the following theorem which generalizes the celebrated Riesz Representation theorem for inner-products. The theorem has been proven independently in part by Papini (21).

Theorem 1.1.8 Let X be a Banach space, then a necessary and sufficient condition for X to be reflexive is that for every $f \in X^*$ there exists a semi-inner-product (\cdot, \cdot) and an element $y \in X$ so that $f(x) = (x, y)$ for all $x \in X$. In the event that X is strictly convex, then relative to the semi-inner-product (\cdot, \cdot) , the element y is unique.

Proof: (Necessity) If the null space $N(f) = X$, any semi-inner-product will suffice with $y = 0$. If $N = N(f) \neq X$, since X is reflexive, by theorem 1.1.5 there exists an $x_0 \perp N$. The orthogonality relation is homogeneous, thus if M is the span of x_0 we have $M \perp N$. By theorem 1.1.7 let (\cdot, \cdot) be chosen so that M is normal to N with respect to (\cdot, \cdot) . For $x \in X$, consider the element z given by $z = f(x)x_0 - f(x_0)x$. Clearly, $z \in N$ so

$$0 = (z, x_0) = f(x) \|x_0\|^2 - f(x_0) (x, x_0).$$

Consequently we have

$$f(x) = (x, \frac{\overline{f(x_0)}}{\|x_0\|^2} x_0) = (x, y).$$

(Sufficiency) For sufficiency we need only observe that every functional representable by a semi-inner-product assumes its norm on the unit sphere and hence by James (10) X is reflexive.

(Uniqueness) Suppose there exist vectors y and y' such that $f(x) = (x, y) = (x, y')$ for all $x \in X$, then $(y, y) = (y, y') \leq \|y\| \|y'\|$. Thus $\|y\| \leq \|y'\|$, or $\|y\| = \|y'\|$. Since $\|y\|^2 = (y, y')$, it follows that $\|y\| \|y'\| = (y, y')$ so that $y = y'$ by Lemma 1.1.1.

J. R. Giles (8) defined a continuous semi-inner-product as a semi-inner-product (\cdot, \cdot) which for every $x, y \in X$

$$\operatorname{Re}\{ (y, x + \lambda y) \} \rightarrow \operatorname{Re}\{ (y, x) \}$$

for all real $\lambda \rightarrow 0$.

Using this definition he proved the following:

Theorem 1.1.9 Suppose X is a continuous semi-inner-product space X which is uniformly convex and complete in its norm. Then, corresponding to every functional $f \in X^*$ there exists a unique vector $y \in X$ such that $f(x) = (x, y)$ for all $x \in X$.

Theorem 1.1.8 is clearly a generalization of this re-

sult. If we observe that the uniform convexity is required only to obtain a vector orthogonal to $N(f)$ we are led to the following famous result as a corollary to Theorem 1.1.8.

Theorem 1.1.9 (Milman-Pettis) Every uniformly convex Banach space is reflexive.

Theorem 1.1.10 Let X be a smoothable Banach space. Then X is isomorphic to a Hilbert space if and only if for each subspace $M \subseteq X$ and each $y \in X$ there exists a smooth re-norming of X and an $m \in M$ such that $M \perp y-m$.

Proof: If X is a Hilbert space then clearly such an m exists by virtue of the projection theorem. Conversely if for each subspace M and each $y \in X$, an $m \in M$ exists so that $M \perp y-m$, we can show that every subspace in X is complemented in X . To this end, let $M^\perp = \{x \in X \mid (x, m) = 0 \text{ for all } m \in M\}$. By the continuity of (\cdot, m) , M^\perp is closed. In addition, $M \cap M^\perp = \{0\}$. Since M is smooth, $M \perp y-m$ implies that $y-m \in M^\perp$ (Theorem 1.1.7) from which we may conclude that each y has a unique representation $y = (y-m) + m$, i.e. $X = M \oplus M^\perp$. By the Lindenstrauss-Tzafrini theorem on complemented subspaces; X is a Hilbert-space. We may observe that as a consequence of this, in a smooth, reflexive Banach space X , if \perp is symmetric, then X is a Hilbert space. As was previously mentioned, the following was shown by James (9).

Theorem 1.1.11 In a smooth Banach space the relation \perp is additive.

Proof: Suppose $x \perp y$ and $x \perp z$. We must show that

$x \perp \alpha y + \beta z$ for complex numbers α, β . Let M_x, M_y, M_z be the subspaces generated by x, y , and z respectively. By hypothesis, since \perp is homogeneous, $M_x \perp M_y$ and $M_x \perp M_z$. If $M_y = M_z$ we are finished. If $M_y \neq M_z$, then relative to the semi-inner-product on X , M_x is normal to both M_y and M_z and consequently M_x is normal to $M_y \oplus M_z$. But this implies that $x \perp \alpha y + \beta z$.

Theorem 1.1.12 Suppose that M and N are closed subspaces of a Banach space X . If $M \perp N$ then $M \oplus N$ is closed.

Proof: Suppose $z_n \in M \oplus N$ and $z_n \rightarrow z$. Let $z_n = x_n + y_n$ where $x_n \in M$ and $y_n \in N$ for each $n \in \mathbb{N}$. Let (\cdot, \cdot) be any semi-inner-product with respect to which M is normal to N . Then

$$\|z_n - x_m\| \|x_n - x_m\| \geq |(z_n - z_m, x_n - x_m)| = \|x_n - x_m\|^2.$$

Thus, since $\|z_n - z_m\| \geq \|x_n - x_m\|$, $\{x_n\}$ must be Cauchy with $x_n \rightarrow x$ for some $x \in M$. In addition, $y_n = z_n - x_n$ must be a Cauchy sequence so that $y_n \rightarrow y \in N$. So $z = \lim z_n = \lim x_n + \lim y_n = x + y \in M \oplus N$.

SECTION 2: An Application to Operator Theory

1.2 The original introduction of semi-inner-products (15) was motivated by an attempt to introduce the elegance and simplicity of operator theory, as it appears in the context of a Hilbert space, to a general Banach space setting. This attempt must fall far short of this goal since it is necessary to maintain the obvious distinction between Hilbert spaces and Banach spaces in general. There is, however, a great deal of insight to be gained through the attempt.

In the following we will apply the semi-inner-product structure to operators in a Banach space.

Definition 1.2.1 A scalar operator in a Banach space X is a linear operator A for which there is spectral family $E(\cdot)$ such that

$$A = \int \lambda dE(\lambda).$$

The general theory of such operators can be found in (6). If $\sigma(A) \subseteq (a, b)$ for some $a, b \in \mathbb{R}$, then A is clearly a generalization of a bounded self-adjoint operator in a Hilbert space.

Definition 1.2.2 Let A be a bounded scalar operator and $E(\cdot)$ its spectral family. A will be said to have simple

spectra if and only if there is some vector $g \in X$ such that the linear manifold spanned by vectors of the form $E(\Delta)g$ is dense in X , where Δ is an interval in \mathbb{R} . The vector g will be called a generating vector.

It is well known that the self-adjoint operators in a Hilbert space which have simple spectrum are isometrically equivalent to multiplication by the independent variable on a suitably constructed L_2 space (1). The purpose of this section is to extend this result.

In finite dimensional spaces or in the elementary theory of integral equations, an operator has simple spectra if the multiplicity of each eigenvalue equals one. For general operators the eigenvalues do not exhaust the spectrum. Consequently we have the Definition 1.2.2.

Definition 1.2.3 Let μ be a vector valued measure whose domain is the sigma algebra of Borel sets in \mathbb{R} on \mathcal{C} , and whose range is in the Banach space X . The semi-variation of μ , denoted $\|\mu\|$, is given by

$$\|\mu\|(E) = \sup \left\| \sum \alpha_i \mu(E_i) \right\| ,$$

where the supremum is taken over all finite collections of scalars α_i with $|\alpha_i| \leq 1$ and all finite measurable disjoint partitions of E .

Lemma 1.2.1 A vector valued measure has finite semi-variation. For the proof of this lemma and the following

theorem the reader is referred to (6, sec. IV. 10).

Theorem 1.2.1 If μ is a vector valued measure taking values in the Banach space X , and f a μ -essentially bounded scalar valued function on $\mathbb{R}(\mathcal{C})$, then f is μ -integrable and

$$\left\| \int f(\lambda) d\mu(\lambda) \right\| \leq \{ \mu\text{-ess sup}_{s \in \mathbb{R}(\mathcal{C})} |f(s)| \} \{ \|\mu(E)\| \}.$$

Definition 1.2.4 Let X be a smooth, strictly convex, reflexive Banach space and let A be a bounded operator on X . The uniquely determined operator A^+ defined by $(Ax, y) = (x, A^+y)$ will be called the generalized adjoint of A .

By Section 1 we see that A^+ is well defined though not necessarily linear. For an excellent treatment of generalized adjoints see Koehler (11) and Stampfli (25). Our interest in them is only their use in the text of the next theorem.

Now let A be a bounded scalar operator in X with $\sigma(A) \subseteq (a, b)$ for some $a, b \in \mathbb{R}$. Suppose, in addition that A has simple spectra with g a generating vector. If E is the spectral family for A , let the vector measure μ be defined by

$$\mu(M) = E(M)g \quad \text{for all } M \in \mathcal{B}.$$

Using the vector measure μ we can define a linear operator $U: L_\infty(\mu) \rightarrow X$ by

$$Uf = \int f(\lambda) d\mu(\lambda).$$

By Theorem 1.2.1 U is bounded, and by virtue of

$$E(\Delta)g = \int 1(\Delta\lambda) d\mu(\lambda),$$

the range of U is dense in X .

We are now in a position to state and prove the principal result of this section.

Theorem 1.2.2 Let A be a scalar operator on a smooth, strictly-convex, reflexive Banach space X with $\sigma(A) \subseteq (a, b)$ for some $a, b \in \mathbb{R}$. Suppose that A has simple spectra with generating vector g , and, in addition, that $E(\cdot)$ is the spectral family for A . Then, if $U: L_\infty(\mu) \rightarrow X$ is given by

$$Uf = \int f(\lambda) d\mu(\lambda),$$

for $\mu(\cdot) = E(\cdot)g$, and $Q: L_\infty(\mu) \rightarrow L_\infty(\mu)$ is "multiplication by the independent variable", i.e. $Qf(\lambda) = \lambda f(\lambda)$, the following diagram commutes.

$$\begin{array}{ccc}
 X & \xleftarrow{U} & L_\infty(\mu) \\
 \downarrow A & & \downarrow Q \\
 X & \xleftarrow{U} & L_\infty(\mu)
 \end{array} \tag{1.}$$

If $N = N(U)$ is the null space of U and $M = L_\infty(\mu)/N(Q)$, the corresponding operator in the equivalence classes of M , Q' and U' are well defined and for any $f \in X$ and any $\epsilon > 0$ there exists an $f' \in X$ such that

$$\left\| Af - U'Q(U')^{-1}f' \right\| < \epsilon. \quad (2.)$$

This implies that on a dense subset of X , A is essentially a multiplication.

Proof: We will first show that the diagram (1.) commutes. Let h be an arbitrary vector in X , (\cdot, \cdot) the semi-inner-product for X . We remark that (\cdot, \cdot) is unique by the smoothness of X . We calculate as follows: for any $f \in L_\infty(\mu)$

$$\begin{aligned} (AUf, h) &= \int_{-\infty}^{\infty} \lambda d(E(\lambda)Uf, h) = \int_{-\infty}^{\infty} \lambda d(Uf, E^+(\lambda)h) \\ &= \int_{-\infty}^{\infty} \lambda d \left\{ \int_{-\infty}^{\infty} f(\mu) d(E(\mu)g, E^+(\lambda)h) \right\} = \int_{-\infty}^{\infty} \lambda d \left\{ \int_{-\infty}^{\lambda} f(\mu) d(E(\mu)g, h) \right\} \\ &= \int_{-\infty}^{\infty} \lambda f(\lambda) d(E(\lambda)g, h) = (U(\lambda f(\lambda)), h). \end{aligned}$$

By the arbitrariness of h , it follows that

$$AUf(\lambda) = U(\lambda f(\lambda)) = UQf(\lambda) \quad (3.)$$

so that (1.) commutes.

The operator $U': M \rightarrow X$ given by $U'[f] = Uf$ is clearly well defined and injective. We define $Q'[f] = [Qf]$. To show that Q' is well defined suppose $f(\lambda) - g(\lambda) \in N$, then we require $\lambda(f(\lambda) - g(\lambda)) \in N$. However, $f - g \in N \Rightarrow AU(f - g) = 0$. But, $AU(f - g) = UQ(f - g) = 0$ so $\lambda(f - g) \in N$. It follows that Q' is well defined and that $AU' = U'Q'$, which is to say that for $f' \in R(U')$ $= R(U)$

$$Af' = U'Q(U')^{-1}f'$$

Since by hypothesis $R(U')$ is dense in X , the inequality (2.) holds. This completes the theorem.

CHAPTER II

GENERALIZED COSINE OPERATOR FUNCTIONS

Definition 2.1.1 A semigroup Γ with identity ϵ will be called a $*$ -semigroup provided there is a "star" operation $*: \Gamma \rightarrow \Gamma$ satisfying

- (i) $\epsilon^* = \epsilon$
- (ii) $s^{**} = s$
- (iii) $(st)^* = t^*s^*$

Definition 2.1.2 Let Γ be an abelian $*$ -semigroup with identity ϵ and $L(H)$ the collection of all linear operators on the Hilbert space H . A function $C: \Gamma \rightarrow L(H)$ will be called a cosine representation of Γ provided C satisfies

- (i) $C(\epsilon) = I$
- (ii) $2C(s)C(t) = C(st) + C(s^*t)$ for all $s, t \in \Gamma$.

Definition 2.1.3 Let A be a linear operator on a Hilbert space H , then an operator \tilde{A} on a Hilbert space \tilde{H} containing H as a subspace will be said to be a dilation of A provided

$$A = P_H \tilde{A}|_H$$

where P_H is the orthogonal projection of \tilde{H} onto H . This relationship will be denoted $A = \text{pr} \tilde{A}$.

The study of dilations was initiated in order to generalize the conventional concept of the extension of an operator. It may, in fact, be thought of as an extension which goes beyond the space on which the original operator was defined.

We will first consider an extension problem of the conventional sense.

Definition 2.1.4 Let A be a symmetric operator on a Hilbert space H . Then the dimensions of the orthogonal complements of the subspaces $R(A-iI)$ and $R(A+iI)$, denoted m^- and m^+ respectively, are called the deficiency indices of the operator A . A symmetric operator is self adjoint if and only if $m^+ = m^- = 0$ (1).

The next well known theorem relates the deficiency indices to the possibility of extending a symmetric operator to a self-adjoint operator. We will not give a proof here.

Theorem 2.1.1 (1) A symmetric operator A has a self adjoint extension if and only if $m^- = m^+$.

In the event that $m^- \neq m^+$, then, within the original Hilbert space containing the domain of the symmetric operator A , no self-adjoint extension can be obtained. However, if we allow extensions to extend beyond this original space we have the following theorem which is a slight generalization of one by Naimark (19).

Theorem 2.1.2 Let A_α ($\alpha \in \Lambda$) be a collection of pairwise commuting symmetric operators on a Hilbert space H , then

there exists a Hilbert space \tilde{H} containing H and a pairwise commuting collection of self-adjoint operators \tilde{A}_α ($\alpha \in \Lambda$) on \tilde{H} satisfying

$$A_\alpha = \tilde{A}_\alpha \Big|_H \quad \text{For each } \alpha \in \Lambda$$

Proof: Let $\tilde{H} = H \oplus H$ endowed with inner-product $\langle (f, g), (f', g') \rangle = \langle f, f' \rangle + \langle g, g' \rangle$. Define \hat{A}_α on \tilde{H} by $\hat{A}_\alpha = A_\alpha \oplus \{-A_\alpha\}$, then clearly, $\hat{A}_\alpha \hat{A}_\beta = \hat{A}_\beta \hat{A}_\alpha$ and \hat{A} is symmetric for each $\alpha \in \Lambda$. If we identify $x \rightarrow (x, 0)$, then \hat{A}_α is clearly an extension of A_α for each $\alpha \in \Lambda$. We show that the deficiency indices of \hat{A}_α are identical for each α

$$\begin{aligned} R(\hat{A}_\alpha - iI) &= (\hat{A}_\alpha - iI)D(\hat{A}_\alpha) = \\ &= (\hat{A}_\alpha - iI)D(A_\alpha) \oplus D(-A_\alpha) = \\ &= (A_\alpha - iI)D(A_\alpha) \oplus (-A_\alpha - iI)D(-A_\alpha) \\ &= (A_\alpha - iI)D(A_\alpha) \oplus (A_\alpha + iI)D(A_\alpha) \\ &= R(A_\alpha - iI) \oplus R(A_\alpha + iI). \end{aligned}$$

So, the orthogonal complement of $R(\hat{A}_\alpha - iI)$ has dimension $m^+ + m^-$. Similarly, $R(\hat{A}_\alpha + iI)$ has orthogonal complement of dimension $m^+ + m^-$. We must conclude that each of \hat{A}_α has a self-adjoint extension \tilde{A}_α , and consequently so does A_α . In addition, since $\hat{A}_\alpha \subseteq \tilde{A}_\alpha \subseteq \hat{A}_\alpha^*$, the collection \tilde{A}_α is pairwise commuting.

Theorem 2.1.3 Let H be a Hilbert space and A_i ($i \in I$) a countable collection of pairwise commuting symmetric operators on H . Then, there exists a Hilbert space \tilde{H} containing H , an orthogonal spectral function \tilde{E} on \tilde{H} and an \tilde{E} -measurable function $\phi(s, i)$, so that for each $x \in D(A_i)$

$$A_i x = \int \phi(s, i) dE_s x.$$

Proof: This follows from Theorem 2.1.2 and (18, p 67). In what follows we will need the following:

Theorem 2.1.4 Let E be a real spectral family, $g: \mathbb{R} \rightarrow \mathbb{C}$ a Borel measurable function, and f a Borel measurable function $\mathbb{C} \rightarrow \mathbb{C}$ for which, given $x \in H$

$$\int_{\mathbb{R}} f \circ g \, dE x \quad \text{exists,}$$

then the function $\hat{E}(\Delta) = E(g^{-1}(\Delta))$ is a complex spectral family,

$$\int_{\mathbb{C}} f \, d\hat{E} x \quad \text{exists,}$$

and the two integrals are equal.

Proof: It is immediate that \hat{E} is a complex spectral family. If $f = 1_{\Delta}$ where Δ is a Borel subset of \mathbb{C} the conclusion obviously holds. Thus we are lead to conclude that for all simple functions f ,

$$\int f \circ g \, dE_x = \int f \, d\hat{E}_x.$$

Now, as is usual, if f is non-negative, we approximate f from below by simple functions f_n . Since the dominated convergence theorem holds for vector valued measures, $\int f \, d\hat{E}_x$ exists and

$$\int f \circ g \, dE_x = \lim_{n \rightarrow \infty} \int f_n \circ g \, dE_x = \lim_{n \rightarrow \infty} \int f_n \, d\hat{E}_x = \int f \, d\hat{E}_x.$$

The extension to arbitrary functions is routine.

The desirability of being able to extend a symmetric operator to a self-adjoint operator is evident in that when this is possible we are able to apply the spectral theorem to such an operator. Suppose now that we have a symmetric operator A and a self-adjoint dilation of A . That is to say that if A has domain in the Hilbert space H , then there is a Hilbert space \tilde{H} and a self-adjoint operator \tilde{A} on \tilde{H} such that

$$\text{pr}\tilde{A} = A.$$

Let \tilde{E} be the spectral family associated with \tilde{A} . In other words

$$\tilde{A} = \int \lambda d\tilde{E}_\lambda.$$

In this case, the family $E_\lambda = \text{pr}\tilde{E}_\lambda$ satisfies the following:

- (i) For $\lambda_2 > \lambda_1$, the difference $E_{\lambda_2} - E_{\lambda_1}$ is a

bounded positive operator

$$(ii) \quad E_{\lambda-0} = E_{\lambda}$$

$$(iii) \quad \lim_{\lambda \rightarrow -\infty} E_{\lambda} = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow +\infty} E_{\lambda} = I.$$

In contrast to an orthogonal spectral family, the operators E_{λ} need not be orthogonal projections. As a consequence of this, we do not have $E_{\lambda_1} E_{\lambda_2} = E_{\min(\lambda_1, \lambda_2)}$.

Families of operators satisfying the conditions (i)-(iii) are called generalized spectral families. Such a function generates an operator valued measure in the same fashion that a distribution function generates a Stieltjes measure. In full generality we have the following:

Definition 2.1.5 A generalized spectral function E is a positive operator valued measure defined on a σ -algebra S satisfying

$$(i) \quad E(X) = I$$

$$(ii) \quad E(\phi) = 0$$

$$(iii) \quad \text{For } E_i \in S \text{ with } E_i \cap E_j = \phi$$

$$E\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} E(E_j).$$

In (20) Naimark showed that every generalized spectral family $E(\cdot)$ can be realized as

$$E(\cdot) = \text{pr} \tilde{E}(\cdot)$$

where $\tilde{E}(\cdot)$ is projection valued.

For the symmetric operator A with self adjoint dilation \tilde{A} , with

$$\tilde{A} = \int \lambda d\tilde{E}_\lambda,$$

we have the following:

If $E_\lambda = \text{pr}\tilde{E}_\lambda$, then for $x \in D(A)$ and $y \in H$

$$(i) \quad \langle Ax, y \rangle = \int \lambda d \langle \tilde{E}_\lambda x, y \rangle = \int \lambda d \langle E_\lambda x, y \rangle$$

and

$$(ii) \quad \|A_x\|^2 = \int \lambda^2 d \langle \tilde{E}_\lambda x, x \rangle = \int \lambda^2 d \langle E_\lambda x, x \rangle.$$

E_λ is said to be a generalized spectral family associated with the operator A . In general, E_λ is not unique.

The proof of Naimark's theorem that all generalized spectral families may be dilated to orthogonal spectral functions may be made to rely on the following theorem by Sz.-Nagy which, in the full range of its applicability, is unsurpassed in generality and elegance.

Theorem 2.1.5 (18) Let Γ be a *-semigroup with identity ϵ and suppose that $T: \Gamma \rightarrow B(H)$ (the bounded operators on a Hilbert space H) satisfies the following conditions:

- (a) $T(\epsilon) = I$, $T(s^*) = T(s)^*$
- (b) $T(s)$ considered as a function of s is of positive type. That is to say that for every finite

$\{x_1, x_2, \dots, x_n\} \subseteq H$ and $\{s_1, \dots, s_n\} \subseteq \Gamma$ we have

$$\sum_{i,j=1}^n \langle T(s_i^* s_j) x_j, x_i \rangle \geq 0.$$

(c) For finite $\{x_1, \dots, x_n\} \subseteq H$ and $\{s_1, \dots, s_n\} \subseteq \Gamma$ and $t \in \Gamma$ we have

$$\sum_{i,j=1}^n \langle T(s_i^* t^* t s_j) x_j, x_i \rangle \leq C_\alpha^2 \sum_{i,j=1}^n \langle T(s_i^* s_j) x_j, x_i \rangle.$$

Then there exists a representation $D: \Gamma \rightarrow B(\tilde{H})$ on a dilation space \tilde{H} of H such that $T(s) = \text{pr}D(s)$ for each $s \in \Gamma$.

In actuality, under subsidiary conditions, Sz.-Nagy is able to conclude more. We are not, however, concerned here with the remaining conclusions. We may note in passing that in the case that Γ is a topological group and H is the complex numbers, then this theorem reduces to the Gelfand-Riakov theorem (7) on the representation of positive definite functions on Γ .

Definition 2.1.6 Let E be an abstract set, H a Hilbert space and $K: E \times E \rightarrow B(H)$. Then the kernel K is said to be of positive type if and only if for finite subsets $\{s_1, s_2, \dots, s_n\} \subseteq E$ and $\{x_1, x_2, \dots, x_n\} \subseteq H$ we have

$$\sum_{i=1}^n \sum_{j=1}^n \langle K(s_i, s_j) x_j, x_i \rangle \geq 0.$$

We now come to the principal result of this section.

Theorem 2.1.6 Let Γ be an Abelian $*$ -semigroup with identity ε . Suppose $C:\Gamma \rightarrow B(H)$ is a family of operators satisfying

$$(a) \quad C(\varepsilon) = I, \quad C(s) = C(s^*)$$

(b) the kernel $K(s,t) = \frac{1}{2}\{C(st)+C(s^*t)\}$ is of positive type,

then there exists a dilation space \tilde{H} containing H and a family of operators $\tilde{C}:\Gamma \rightarrow L(\tilde{H})$ satisfying

(a') \tilde{C} is a cosine representation of Γ

(b') $\text{pr}\tilde{C}(t) = C(t)$ for all $t \in \Gamma$.

Proof: Let H_0 be the linear space formed by the collection of all linear combinations of functions of the form $\phi(s) = K(s,t)x$, where $x \in H$. We may define a bilinear form (\cdot, \cdot) on H_0 as follows: if $\phi(s) = \sum_{i=1}^n K(s,t_i)x_i$ and $v(s) = \sum_{j=1}^m K(s,t'_j)x'_j$ are functions in H_0 , we define $(\phi(s), v(s))$ by

$$(\phi, v) = \sum_{i,j=1}^n \langle K(t'_j, t_i)x_i, x'_j \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner-product on H . We observe that

$$\begin{aligned} (\phi(s), v(s)) &= \sum_{i=1}^n \sum_{j=1}^m \langle K(t'_j, t_i)x_i, x'_j \rangle \\ &= \sum_{j=1}^m \langle \phi(t'_j), x'_j \rangle = \sum_{i=1}^n \langle x_i, v(t_i) \rangle, \end{aligned}$$

where the last equation follows from the relation $\langle K(s,t)x,y \rangle = \langle x,K(s,t)y \rangle$, which is a consequence of K being of positive type. In any case, this implies that $(\phi(s),v(s))$ is independent of the representation of either ϕ or v . It follows from this that (\cdot,\cdot) is well defined.

We now show that (\cdot,\cdot) satisfies the axioms of an inner-product. It is clear that (\cdot,\cdot) is both bilinear and symmetric. In addition, by the positivity of K , (\cdot,\cdot) is non-negative. Suppose now that $(\phi(s),\phi(s)) = 0$. Since the Cauchy-Schwarz inequality is valid for (\cdot,\cdot) , we have for any $x \in H$ and any $t \in \Gamma$

$$\begin{aligned} 0 &\leq |\langle \phi(t),x \rangle|^2 = |(\phi(s),K(s,t)x)|^2 \\ &\leq (\phi(s),\phi(s))(K(s,t)x,K(s,t)x) = 0 \end{aligned}$$

from which it follows that (\cdot,\cdot) is positive definite. Thus with (\cdot,\cdot) , H_0 is a pre-Hilbert space. Let \tilde{H} denote the completion of H_0 .

We may embed H into \tilde{H} by identifying $x \rightarrow K(s,\epsilon)x = C(s)x$. Since $(K(s,\epsilon)x,K(s,\epsilon)x) = \langle K(\epsilon,\epsilon)x,x \rangle = \langle x,x \rangle$, the embedding is an isometry. Henceforth, H will denote the image of this identification.

If $P = P_H$ is the orthogonal projection of \tilde{H} onto H ,

$$P K(s,t)x = K(s,\epsilon)K(\epsilon,t)x = K(s,\epsilon)C(t)x.$$

That this is true can be shown by observing that for every

$\phi \in H_0$ and $v \in H$,

$$(Pv - v, \phi) = 0.$$

We now define $\tilde{C}: \Gamma \rightarrow L(H_0)$ by

$$\tilde{C}(s') (K(s, t)x) = \frac{1}{2}(K(s, s't)x + K(s, s'^*t)x).$$

Direct calculation shows that for $\phi, v \in H_0$

$$(\tilde{C}(t)\phi, v) = (\phi, \tilde{C}(t)v)$$

so that, since H_0 is dense in \tilde{H} , if $\phi = 0$, it follows that $\tilde{C}(t)\phi = 0$. That is to say that $\tilde{C}(t)$ is well defined for each $t \in \Gamma$.

We note that

$$\tilde{C}(\varepsilon) (K(s, t)x) = \frac{1}{2}K(s, \varepsilon t)x + \frac{1}{2}K(s, \varepsilon^*t)x = K(s, t)x$$

so that $\tilde{C}(\varepsilon) = I$.

Now for a function $\phi(s) = K(s, t)x$ we obtain

$$2\tilde{C}(v)\tilde{C}(\mu) (K(s, t)x) = \frac{1}{2}(K(s, v\mu t)x + K(s, v^*\mu t)x +$$

$$K(s, v\mu^*t)x + K(s, v^*\mu^*t)x) = (\tilde{C}(v\mu) + \tilde{C}(v^*\mu)) (K(s, t)x)$$

so that \tilde{C} satisfies (i) and (ii) of Definition 2.1.2. We observe that at this point we make essential use of the commutativity of Γ .

In order to complete the proof, we need only show that

$\text{pr}\tilde{C}(t) = C(t)$. But we have

$$\text{pr}\tilde{C}(t)(K(s, \varepsilon)x) = P_H \left(\frac{1}{2}K(s, t)x + \frac{1}{2}K(s, t^*)x \right) =$$

$$K(s, \varepsilon) \left(\frac{1}{2}K(\varepsilon, t)x + \frac{1}{2}K(\varepsilon, t^*)x \right) = K(s, \varepsilon)C(t)x,$$

and this completes the proof.

Definition 2.1.7 Let X be a Banach space and $C: \mathbb{R} \rightarrow B(X)$. C is said to be a cosine operator function provided C satisfies

$$(a) \quad 2C(s)C(t) = C(s+t) + C(s-t)$$

$$(b) \quad C(0) = I.$$

Cosine operator functions have been extensively studied, as was mentioned in the introduction.

Definition 2.1.8 Let $C(t)$ be a family of operators in $B(H)$ on a Hilbert space H . If there is a Hilbert space \tilde{H} containing H and a cosine operator function $\tilde{C}(t)$ in $B(\tilde{H})$ satisfying

$$C(t) = \text{pr}\tilde{C}(t),$$

then $C(t)$ is said to be a generalized cosine operator function.

As a consequence of Theorems 2.1.6-2.1.7, we have the following:

Theorem 2.1.8 Let $C: \mathbb{R} \rightarrow B(H)$ satisfy

$$(a) \quad C(0) = I, \quad C(s) = C(-s)$$

$$(b) \quad K(s, t) = \frac{1}{2}(C(s+t) + C(s-t)) \text{ is of positive type,}$$

then, $C(t)$ is a generalized cosine operator function. In

addition, if the dilated operator \tilde{C} is self adjoint, then conditions (a) and (b) are necessary.

Proof: We need only take $\Gamma = \mathbb{R}$ and "*" as the additive inverse in Theorem 2.1.6. We note that it is possible to have unbounded cosine operator functions so that without further hypotheses we cannot assume that $\tilde{C}(t) \in B(\tilde{H})$. As an example of an unbounded cosine operator function, let P be an unbounded projection on the Hilbert space H and $P_c = I - P$. Direct calculation shows that $C(t) = P_c + (\cos t)P$ is an unbounded cosine operator function. We have the following parallel to condition (c) of Theorem 2.1.5.

Theorem 2.1.9 If the following inequality holds, then $\tilde{C}(v)$ is bounded: for all $\{t_1, \dots, t_n\} \subseteq \Gamma$ and $\{x_1, \dots, x_n\} \subseteq H$,

$$\begin{aligned} & \sum_{i,j=1}^n \langle (K(vt_j, vt_i) + K(v^*t_j, vt_i) + K(vt_j, v^*t_i) \\ & \quad + K(v^*t_j, v^*t_i))x_i, x_j \rangle \\ & \leq M_v \left[\sum_{i,j=1}^n \langle K(t_j, t_i)x_i, x_j \rangle \right] \end{aligned}$$

Proof: Direct calculation yields

$$\|\tilde{C}(v) \sum K(s, t_i)x_i\| = \frac{1}{2} \sum_{i,j=1}^n \langle (K(vt_j, vt_i) + K(vt_j, v^*t_i)$$

$$\begin{aligned}
& + K(v^*t_j, vt_i) + K(v^*t_j, v^*t_i) \rangle_{x_i, x_j} \\
& \leq M_v \left[\sum_{i,j=1}^n \langle K(t_j, t_i) x_i, x_j \rangle \right] = \frac{M_v}{2} \left\| \sum K(s, t_i) x_i \right\|^2
\end{aligned}$$

We will see later that even with the apparent artificiality of the hypothesis of Theorem 2.1.9, it is, in fact, satisfied for the case of principal interest. We will need the following later.

Lemma 1.1.1 Let $\tilde{C}(t)$ and $C(t)$ be as in Theorem 2.1.8, then the following are true:

- (a) If $C(t)$ is weakly continuous, then $\tilde{C}(t)$ is weakly continuous
- (b) If $\omega\text{-}\lim_{t \rightarrow 0} \frac{C(s+t) - C(s)}{t}$ exists for each $s \in \mathbb{R}$, then $\omega\text{-}\lim_{t \rightarrow 0} \frac{\tilde{C}(t) - I}{t}$ exists.

Proof: (a) Consider $\lim_{t \rightarrow t_0} \langle \tilde{C}(t) K(s, u)x, K(s, v)y \rangle$. By

direct calculation we have

$$\begin{aligned}
\langle \tilde{C}(t) K(s, u)x, K(s, v)y \rangle &= \frac{1}{2} \langle K(s, u+t)x + K(s, u-t)x, K(s, v)y \rangle \\
&= \frac{1}{2} \langle \{K(v, u+t) + K(v, u-t)\}x, y \rangle = \frac{1}{4} \langle C(v+u+t)x + C(v-u-t)x \\
&\quad + C(v+u-t)x + C(v-u+t)x, y \rangle.
\end{aligned}$$

So $\lim_{t \rightarrow t_0} \langle \tilde{C}(t) K(s, u)x, K(s, v)y \rangle = \frac{1}{4} \langle C(v+u+t_0)x + C(v-u-t_0)x$

$$+ C(v+u-t_0)x + C(v-u+t_0)x, y \rangle = \langle \tilde{C}(t_0)K(s, u)x, K(s, v)y \rangle,$$

from which (a) follows.

(b) We calculate

$$\begin{aligned} & \langle \frac{\tilde{C}(t)-I}{t} K(s, v)x, K(s, u)y \rangle = \\ & \frac{1}{t} \langle \tilde{C}(t)K(s, v)x, K(s, u)y \rangle - \frac{1}{t} \langle K(s, v)x, K(s, u)y \rangle \\ & = \frac{1}{4t} \langle C(v+u+t)x + C(v-u-t)x + C(v+u-t)x + C(v-u+t)x, y \rangle \\ & \quad - \frac{1}{2t} \langle C(v+u)x + C(v-u)x, y \rangle \\ & = \frac{1}{4} \langle \frac{C(v+u+t)-C(v+u)}{t} x, y \rangle + \frac{1}{4} \langle \frac{C(v+u-t)-C(v+u)}{t} x, y \rangle \\ & \quad + \frac{1}{4} \langle \frac{C(v-u+t)-C(v-u)}{t} x, y \rangle + \frac{1}{4} \langle \frac{C(v-u-t)-C(v-u)}{t} x, y \rangle. \end{aligned}$$

So if $\omega\text{-}\lim_{t \rightarrow 0} \frac{C(s+t)-C(s)}{t}$ exists for each $s \in \mathbb{R}$ then $\omega\text{-}\lim_{t \rightarrow 0} \frac{C(t)-I}{t}$

exists.

In (13), Kurepa showed that every function $C: \mathbb{R} \rightarrow B(H)$ satisfying

- (a) $C(s)$ is a normal transformation
- (b) $2C(s)C(t) = C(s+t) + C(s-t)$
- (c) $C(s)$ is weakly continuous in s ,

can be represented as

$$C(t) = \int \cos t\lambda dE_\lambda$$

where E_λ is a spectral family with compact support and integration is taken over the complex plane. We now apply this result to obtain such a representation for generalized cosine operator.

Theorem 2.1.10 Let $C: \mathbb{R} \rightarrow B(H)$ satisfy the following:

- (1) $C(0) = I$, $C(s) = C(-s)$
- (2) the kernel $K(s,t) = \frac{1}{2}\{C(s+t) + C(s-t)\}$ satisfies
 - (a) K is of positive type

$$(b) \sum_{i,j=1}^n \left[\langle K(s+t_j, s+t_i) x_i, x_j \rangle + \langle K(s+t_j, t_i-s) x_i, x_j \rangle + \langle K(t_j-s, t_i+s) x_i, x_j \rangle + \langle K(t_j-s, t_i-s) x_i, x_j \rangle \right] \leq M_s \left[\sum_{i,j=1}^n \langle K(t_j, t_i) x_i, x_j \rangle \right],$$

then $C(t)$ has a representation given by

$$\langle C(t)x, y \rangle = \int \cos \lambda t d\langle E_\lambda x, y \rangle.$$

$$\|C(t)x\|^2 \leq \int \cos^2 \lambda t d\langle E_\lambda x, x \rangle$$

For all $x, y \in H$, where E_λ is a generalized spectral family with compact support, and the integration is over the complex plane. In addition, the support of $E(\cdot)$ is contained in the two axes of the plane.

Proof: By Theorems 2.1.8 and 2.1.9 the family can be dilated to a cosine operator function $\tilde{C}(t)$ on a Hilbert space \tilde{H} containing H . In addition, since $\tilde{C}(t)$ is symmetric and bounded (by virtue of condition 2b), $\tilde{C}(t)$ is self adjoint. By Kurepa's result we have

$$\tilde{C}(t) = \int \cos \lambda t d\tilde{E}_\lambda$$

where $\tilde{E}(\cdot)$ is a regular orthogonal spectral family defined on the Borel sets of \mathbb{C} . However, since $C(t)$ is self adjoint, $\cos \lambda t$ must be real-valued on the support of E . This implies that the support of E is contained in $\mathbb{R} \cup i\mathbb{R}$. Now we have for $x, y \in H$

$$\begin{aligned} \langle C(t)x, y \rangle &= \langle P_H \tilde{C}(t)x, y \rangle \\ &= \int \cos \lambda t d\langle P_H \tilde{E}_\lambda x, y \rangle = \int \cos \lambda t d\langle E_\lambda x, y \rangle \end{aligned}$$

where $E(\cdot) = P_H \tilde{E}(\cdot)$. In addition, for $x \in H$,

$$\begin{aligned} \|C(t)x\|^2 &= \|P_H \tilde{C}(t)x\|^2 \leq \\ \| \tilde{C}(t)x \|^2 &= \int \cos^2 \lambda t d\langle \tilde{E}_\lambda x, x \rangle \\ &= \int \cos^2 \lambda t d\langle \tilde{E}_\lambda x, P_H x \rangle \end{aligned}$$

$$= \int \cos^2 \lambda t d\langle E_\lambda x, x \rangle.$$

This concludes the theorem.

Definition 2.1.9 Let E be an abstract set and g a function $E \times E \rightarrow \mathbb{C}(\mathbb{R})$ then g is said to be positive definite if for all finite sets $\{A_1, A_2, \dots, A_n\} \subseteq E$ and $\{a_1, a_2, \dots, a_n\} \subseteq \mathbb{C}(\mathbb{R})$ we have

$$\sum_{i,j=1}^n g(t_i, t_j) a_j \bar{a}_i \geq 0.$$

We now will apply Theorem 2.1.10 to obtain a solution to the cosine moment problem.

Theorem 2.1.11 Let $f(t)$ be a real valued function of a complex variable. Then, a necessary and sufficient condition for $f(t)$ to be representable as

$$f(t) = \int_{\mathbb{C}} \cos(\lambda t) d\alpha(\lambda) \quad (1),$$

where α is a regular measure in the complex plane with compact support, is that the following hold:

- (a) $f(t) = f(-t)$ and $f(0) \neq 0$
- (b) The kernel $K(s, t) = \frac{1}{2}\{f(s+t) + f(s-t)\}$ satisfy
 - (i) K is positive definite

$$(ii) \sum_{i,j=1}^n [K(t_j+s, t_i+s) + K(t_j-s, t_i+s)]$$

$$\begin{aligned}
& + K(t_j+s, t_i-s) + K(t_j-s, t_i-s) \Big] a_i \bar{a}_j \\
& \leq M_s \left[\sum_{i,j=1}^n K(t_j, t_i) a_i a_j \right]
\end{aligned}$$

Proof: Let us first show that the conditions are sufficient to guarantee the representation (1). Let the Hilbert space H be the complex numbers with the inner-product given by $\langle a, b \rangle = a \bar{b}$. Without loss of generality, we may assume that $f(0) = 1$, and we may define $C(t)(a) = f(t) \cdot a$ for each $t \in \mathbb{C}$ and each $a \in \mathbb{R}$. Clearly, $C(t)$ so defined satisfies the conditions of Theorem 2.1.10. Thus, if we set $\alpha(\cdot) = \langle E(\cdot)1; 1 \rangle$ we obtain

$$f(t) = \int_{\mathbb{C}} \cos \lambda t d\alpha(t),$$

where as before the support of α is a compact subset of $\mathbb{R} \cup i\mathbb{R}$.

To show necessity, we note that on $\mathbb{R} = \mathbb{H}$, the measure α defines a generalized spectral family if we put $E(\cdot)(a) = \alpha(\cdot) \cdot a$. By Naimark's theorem (20), there is a Hilbert space \tilde{H} containing H and an orthogonal spectral function \tilde{E} on \tilde{H} satisfying

$$\text{pr} \tilde{E} = E. \quad \text{and} \quad E(\Delta) = 0 \iff \tilde{E}(\Delta) = 0.$$

The operator $\tilde{C}(t)$ given by

$$\tilde{C}(t) = \int \cos \lambda t d\tilde{E}_\lambda$$

is therefore a self-adjoint cosine operator function satisfying

$$\text{pr}\tilde{C}(t) = C(t).$$

Consequently by Theorem 2.1.8 the kernel K is of positive type.

Now under the assumption that

$$\begin{aligned} f(t) &= \int \cos \lambda t d\alpha(\lambda), \\ &\sum_{i,j=1}^n [K(t_j+s, t_i+s) + K(t_j-s, t_i+s) \\ &+ K(t_j+s, t_i-s) + K(t_j-s, t_i-s)] a_i a_j = \\ &\frac{1}{2} \sum_{i,j=1}^n \{f(t_i+t_j+2s) + f(t_j-t_i) + f(t_j+t_i) + f(t_j-t_i-2s) \\ &+ f(t_j+t_i) + f(t_j-t_i+2s) + f(t_j+t_i-2s) + f(t_j-t_i)\} a_i a_j \\ &= \frac{1}{2} \sum_{i,j=1}^n [\{ \cos(\lambda(t_i+t_j+2s)) + \cos(\lambda(t_i+t_j-2s)) \} + \\ &\quad \cos(\lambda(t_j-t_i+2s)) + \cos(\lambda(t_j-t_i-2s)) + \\ &\quad 2 \cos(\lambda(t_j+t_i)) + 2 \cos(\lambda(t_j-t_i)) \} d\alpha(\lambda)] a_i a_j \\ &= \sum_{i,j=1}^n [\{ \cos(\lambda(2s)) \cos(t_i+t_j) + \cos(\lambda(2s)) \cos(t_i-t_j) \} \end{aligned}$$

$$\begin{aligned}
& + \cos(t_i + t_j) + \cos(t_i - t_j) \} d\alpha(\lambda) \Big] a_i a_j \\
& = \sum_{i,j=1}^n \left[(1 + \cos(\lambda(2s))) (\cos(\lambda(t_j + t_i))) \right. \\
& \quad \left. + \cos(\lambda(t_j - t_i)) \right) d\alpha(\lambda) \Big] a_i a_j \\
& \leq M_s \sum_{i,j=1}^n \left[\cos(\lambda(t_j + t_i)) + \cos(\lambda(t_j - t_i)) \right] d\alpha(t) \Big] a_i a_j \\
& = M_s \left[\sum_{i,j=1}^n K(t_j, t_i) a_i a_j \right]
\end{aligned}$$

where M_s is an upper bound for the continuous function $1 + \cos(\lambda(2s))$ on the support of α . This proves the necessity of the condition (b).

Theorem 2.1.12 We now consider the unbounded case.

Let $C(t)$ be a family of operators in $B(H)$, and suppose that the following hold,

- (a) $C(0) = I, C(t) = C(-t)$
- (b) $K(s, t) = \frac{1}{2}(C(s+t) + C(s-t))$ is of positive type
- (c) $\omega\text{-}\lim_{t \rightarrow 0} \frac{1}{t} \{C(s+t) - C(s)\}$ exists,

then there is a generalized spectral family E , whose domain is the Borel sets of the complex plane, for which, given $x, y \in H$

$$(2) \quad \langle C(t)x, y \rangle = \int \cos(\lambda t) d\langle E_\lambda x, y \rangle$$

and

$$(3) \quad \|C(t)x\|^2 \leq \int \cos^2(\lambda t) d\langle E_\lambda x, x \rangle < \infty .$$

Proof: If $C(t)$ satisfies (a) and (b), then, by Theorem 2.1.8, we may dilate $C(t)$ to a symmetric cosine operator function $\tilde{C}(t)$ on $\tilde{H} \supset H$. If $G = \{\ell/2^n : \ell, n = 0, \pm 1, \pm 2, \dots\}$, then by Theorem 2.1.3 we may represent, for $t \in G$ and $x \in D(\tilde{C}(t))$

$$(7) \quad \tilde{C}(t)x = \int \phi(\lambda, t) d\tilde{E}_\lambda x ,$$

where \tilde{E} is an orthogonal spectral function and $\phi(\lambda, t)$ is an \tilde{E} -measurable real-valued function. As in (13), substitution of (7) into (a) of Definition 2.1.7 yields, for $t_1, t_2 \in G$

$$\phi(\lambda, t_1)\phi(\lambda, t_2) = \phi(\lambda, t_1+t_2) + \phi(\lambda, t_1-t_2) \quad - \text{ a.e.}$$

Now by Lemma 2.1.1 and condition (c) we have $\omega\text{-}\lim_{t \rightarrow 0^+} \frac{\tilde{C}(t) - I}{t}$ exists. From this we are able to conclude that

$$\left\| \left[\frac{C(2^{-n}) - I}{2^{-n}} \right] x \right\|^2 \leq M(x) \quad t \in G$$

so that

$$\int_{n=1}^{\infty} \left| \phi(\lambda, 2^{-n}) - 1 \right|^2 d\langle E_\lambda x, x \rangle = \sum_{n=1}^{\infty} \left\| \left[\tilde{C}(2^{-n}) - I \right] x \right\|^2 < \infty ,$$

from which it follows that

$$\sum_{n=1}^{\infty} |\phi(\lambda, 2^{-n}) - 1|^2 < \infty \quad \text{a.e.},$$

and from this

$$\phi(\lambda, 2^{-n}) \rightarrow 1 \quad \text{a.e.}$$

From Lemma 4 of (14), we are able to conclude that for almost all λ and each $t \in G$,

$$\phi(\lambda, t) = \int \cos(\xi(\lambda)t).$$

Where $\xi(\lambda)$ is an everywhere finite, \tilde{E} measurable function. By the weak continuity of the family $\tilde{C}(t)$ (Prop. 1) we are now able to conclude that for every $t \in \mathbb{R}$

$$\tilde{C}(t) = \int \cos(\xi(\lambda)t) d\tilde{E}_{\lambda}.$$

By Theorem 2.1.4, we may make a change of variables to obtain

$$\tilde{C}(t) = \int \cos(\lambda t) d\hat{E}_{\lambda}.$$

If we now project onto the subspace H , we obtain for $x, y \in H$

$$(2) \quad \langle C(t)x, y \rangle = \int \cos(\lambda t) d\langle E_{\lambda} x, y \rangle$$

where $\text{pr}\hat{E} = E$.

Similarly,

$$\|C(t)x\|^2 \leq \|\tilde{C}(t)x\|^2 = \int \cos^2(\lambda t) d\langle \hat{E}_\lambda x, y \rangle = \int \cos^2(\lambda t) d\langle E_\lambda x, y \rangle,$$

which concludes the proof.

Theorem 2.1.13 Let $f: \mathbb{R} \rightarrow \mathbb{C}$ satisfy the following conditions:

- (a) $K(s, t) = f(s+t) + f(s-t)$ is of positive type
- (b) $f(0) \neq 0$, $f(t) = f(-t)$
- (c) $f'(t)$ exists for all $t \in \mathbb{R}$,

then there exists a regular probability measure α defined on the Borel sets of the complex plane (with support in $\mathbb{R} \cup i\mathbb{R}$) for which

$$f(t) = \int \cos(\lambda t) d\alpha(\lambda).$$

Proof: Without loss of generality, we may assume that $f(0) = 1$. Our intent is to apply Theorem 2.1.12 with $H = \mathbb{C}$, $\langle a, b \rangle = a \cdot \bar{b}$, and $C(t)(a) = f(t) \cdot a$. Clearly $\bar{K}(s, t) = \frac{1}{2}\{C(s+t) + C(s-t)\}$ is of positive type, $C(0) = 1$, and $C(t) = C(-t)$. In addition, by condition (c) $\omega\text{-}\lim_{t \rightarrow 0} \frac{C(t+s) - C(s)}{t}$ exists. So by Theorem 2.1.12, there is a generalized spectral family E whose domain is the Borel sets in \mathbb{C} for which $\langle C(t)a, b \rangle = \int \cos(\lambda t) d\langle E_\lambda a, b \rangle$ if we set $\alpha(\lambda) = \langle E_\lambda 1, 1 \rangle$. We then have

$$\langle C(t)1, 1 \rangle = f(t) = \int \cos(\lambda t) d\alpha(\lambda).$$

Note: If $\langle C(t)x, y \rangle = \int \cos(\lambda t) d\langle E_\lambda x, y \rangle$ is self-adjoint or if $f(t) = \int \cos(\lambda t) d\alpha(\lambda)$ is real, then conditions (a) and (b) of Theorems 2.1.12 and 2.1.13 are easily seen to be necessary. If $C(t)$ is a weakly measurable cosine operator, then (c) also holds (13). However, condition (c) is not necessary as the following example shows.

Example 2 Let $f(t)$ be the Weienstrass function given by $f(t) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \cos(t5^k)$, then $f(t)$ is no-where differentiable. However if $F(x) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \delta(x-5^k)$, with δ given by

$$\delta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases},$$

then $f(t) = \int_{\mathbb{R}} \cos(\lambda t) dF(\lambda)$.

BIBLIOGRAPHY

1. N. I. Akheiser - I. M. Glazman, The Theory of Linear Operators in Hilbert Space (New York 1963).
2. E. Berkson, "Some Types of Banach Spaces, Hermitian Operators and Bade Functionals," Trans. Amer. Math. Soc., 116 (1965), pp. 376-385.
3. G. Birkhoff, "Orthogonality in linear metric spaces," Duke Math. J., Vol. 1 (1935), pp. 169-172.
4. J. A. Clarkson, "Uniformly convex spaces," Trans. Amer. Math. Soc., Vol. 40 (1936), pp. 396-414.
5. J. Diestel, Geometry of Banach Spaces, Springer Venlag Lecture Notes Number 485, (New York 1975).
6. N. Dunford - J. T. Schwartz, Linear Operator Theory (New York 1967).
7. I. Gelfand - D. Riakov, "Irreducible unitary representations of locally Bicomact groups," Math. Collection, Moscow 13 (1943), pp. 301-316.
8. J. R. Giles, "Classes of Semi-Inner-Product Spaces," Trans. Amer. Math. Soc., 129 (1967), pp. 436-446.
9. R. C. James, "Orthogonality and Linear Functionals in Normed Linear Spaces," Trans. Amer. Math. Soc., 61 (1947), pp. 265-292.
10. R. C. James, "A Characterization of Reflexivity," Studia Math., 23 (1964), 205-216.
11. D. Koehler, "A note on some operator theory in certain semi-inner-product space," Proc. Amer. Math. Soc., Vol. 30 (1971), pp. 363-366.
12. S. Kurepa, "A cosine functional equation in Banach Algebras," Acta. Sci. Math., 23 (1962), pp. 255-267.
13. S. Kurepa, "A cosine functional equation in Hilbert Space," Canadian J. Math., 12 (1960), pp. 45-50.
14. S. Kurepa, "A cosine functional equation in n-dimensional vector space," Glasnik Mat. fiz. ast., 13 (1958), pp. 169-189.

15. G. Lumer, "Semi-Inner-Product Spaces," Trans. Amer. Math. Soc., 100 (1961), pp. 29-43.
16. W. Mlak, "Unitary Dilations of Contraction Operators," Rozprawy Matematyczne XLVI (Warsaw 1965).
17. B. Sz.-Nagy, "On cosine operator functions in Banach spaces," Acta. Sci. Math., 36 (1974), pp. 281-289.
18. B. Sz.-Nagy, Extensions of Linear Transformations in Hilbert Space Which Extend Beyond This Space, (New York 1960).
19. M. Naimark, "Spectral functions of symmetric operators," Izv. A. N. SSSR, 4(#3) (1940), pp. 277-318.
20. M. Naimark, "On a representation of additive operator set functions," C. R. (Doklady) Acad. Sci., URSS, Vol. 41 (1943), pp. 359-361.
21. L. Papini, "Un' osservazione sui prodotti semi-scalari negli spazi di Banach," Boll. U. M. I., 6 (1969), pp. 686-689.
22. F. Riesz - B. Sz.-Nagy, Functional Analysis (New York 1955).
23. R. W. Shonkwiler, "On Generalized Resolvents and an Integral Representation of Nevanlinna," J. Math. Anal. and Appl., 40 (1972), pp. 723-734.
24. M. Sova, "Cosine Operator Functions," Rozprawy Matematyczne XLIX (Warsaw 1966).
25. J. G. Stampfli, "Adjoint Abelian Operators on Banach Spaces," Canadian J. Math., (1969), 505-512.

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