A STUDY OF SIGMA-MONOGENIC FUNCTIONS

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George Lee Cain, Jr.

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Approved:

Robert H. Kasriel, Chairman

James M. Osborn

Harold A. Gersch

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CHAPTER I

INTRODUCTION

There arise frequently, in problems of mechanics of continuous media, systems of partial differential equations of the form

$$\begin{aligned} s_1(x)\ u_x(x,y) &= t_1(y)\ v_y(x,y) \\ s_2(x)\ u_y(x,y) &= -t_2(y)\ v_x(x,y) \end{aligned} \tag{0}$$

and the associated second order equations obtained from this system by eliminating one unknown function:

$$\left[\frac{t_1(y)}{s_1(x)}\ v_y(x,y)\right]_y + \left[\frac{t_2(y)}{s_2(x)}\ v_x(x,y)\right]_x = 0$$

$$\left[\frac{s_1(x)}{t_1(y)}\ u_x(x,y)\right]_x + \left[\frac{s_2(x)}{t_2(y)}\ u_y(x,y)\right]_y = 0 .$$

For example, the rotationally symmetric potential flow of an incompressible perfect fluid gives rise to the system

$$u_z(z,r) = \frac{1}{r}\ v_r(z,r)$$

$$u_r(z,r) = -\frac{1}{r}\ v_z(z,r) ,$$

where u is the velocity potential, v is Stokes' stream function, and r, θ, and z are the usual cylindrical coordinates.

When $s_1 \equiv t_1 \equiv s_2 \equiv t_2 \equiv 1$, the equations (0) are the familiar Cauchy-Riemann equations, and the associated second order equations become Laplace's equation. This result suggests considering those functions of the complex variable $z = x + iy$ which are of the form

$$f(z) = u(x,y) + i\, v(x,y) ,$$

where u and v are related by the equations (0). These functions, called sigma-monogenic functions, were first investigated by Bers and Gelbart [1] in 1943.

The theory of sigma-monogenic functions is, as might be expected, very similar to parts of the classical theory of analytic functions. Analytic functions are, of course, a special case ($s_1 \equiv t_1 \equiv s_2 \equiv t_2 \equiv 1$) of sigma-monogenic functions. Sigma-monogenic functions are in turn a subclass of a more general class of complex functions called pseudoanalytic functions, first introduced by Bers [3] in 1950. The theory of pseudoanalytic functions developed from considerations of the general second order elliptic partial differential equation (that is, those in which the variables are not separable).

This study deals exclusively with sigma-monogenic functions. In Chapter II, the class of such functions is defined to be those complex functions for which the limit of a certain generalized difference quotient exists and is continuous. An immediate consequence of this definition is that the real and imaginary parts of such a function are related by equations (0). It was suggested by Bers and Gelbart [2] that the continuity requirement is unnecessary. They did not, however, offer a proof.

A generalized integral is introduced in Chapter II and sigma-monogenic versions of several classical complex integral theorems are obtained (Cauchy's Theorem, Morera's Theorem, etc.).

Chapter III is devoted to showing that a function sigma-monogenic at a point can be expanded in a series of so-called formal powers in a neighborhood of the point.

In Chapter IV, it is shown that a sigma-monogenic function has isolated zeros. These functions thus have the so-called unique continuation property; that is, a sigma-monogenic function is uniquely determined by its values on every infinite sequence of points which has a limit point at which the function is sigma-monogenic.

It is also proved in this chapter that sigma-monogenic functions are open. Hence they are a subclass of the class of light open functions. It follows from this that a sigma-monogenic function is topologically equivalent to some analytic function (Whyburn [7]).

In Chapter V, a well-known problem in the mathematical theory of elasticity, the torsion problem for a conical shaft, is solved using the theory of sigma-monogenic functions. This is an illustration of how the theory can be applied to physical problems in the mechanics of continua.

CHAPTER II

SIGMA-MONOGENIC FUNCTIONS

Sigma-Differentiation.--In all that follows, it will be assumed that the functions $s_1(x)$, $s_2(x)$, $t_1(y)$, and $t_2(y)$ are positive, real valued, analytic functions of a single real variable. The matrix $\Sigma(x,y)$ is defined by

$$\Sigma(x,y) = \begin{pmatrix} s_1(x) & t_1(y) \\ s_2(x) & t_2(y) \end{pmatrix}.$$

The matrix $\Sigma'(x,y)$ is defined by

$$\Sigma'(x,y) = \begin{pmatrix} \frac{1}{s_2(x)} & t_1(y) \\ \frac{1}{s_1(x)} & t_2(y) \end{pmatrix}.$$

Hence it follows that $(\Sigma')' = \Sigma$.

Define the function $m_\Sigma(a)$, for nonnegative a, as follows:

$$m_\Sigma(a) = \sup_{|t|\leq a, i=1,2} \left\{s_i(t), \frac{1}{s_i(t)}, t_i(t), \frac{1}{t_i(t)}\right\}$$

It is readily seen that $m_\Sigma(a)$ is a nondecreasing function and that $m_\Sigma(a) = m_{\Sigma'}(a)$.

Let $f(z) = u(x,y) + i\,v(x,y)$ be a complex function (not necessarily analytic) of the complex variable $z = x + iy$. The sigma-difference quotient, $D_\Sigma f(z)$, of f at z is defined by

$$D_\Sigma(f(z) = \mathrm{Re}\left\{\frac{s_1(x)\Delta_1 u + it_1(y)\Delta_2 v}{\Delta z}\right\} + i\mathrm{Im}\left\{\frac{\Delta_2 u/t_2(y) + i\Delta_1 v/s_2(x)}{\Delta z}\right\},$$

where

$$\Delta_1 u = \tfrac{1}{2}[u(z + \Delta z) + u(z + \overline{\Delta z}) - 2u(z)] ,$$

$$\Delta_2 u = \tfrac{1}{2}[u(z + \Delta z) + u(z - \overline{\Delta z}) - 2u(z)] ,$$

and $\Delta_1 v$ and $\Delta_2 v$ are defined similarly.

<u>Definition 1</u>: A function f is said to be <u>sigma-differentiable</u> at z if the limit $\lim_{\Delta z \to o} D_\Sigma f(z)$ exists and is finite. The function $f^{`}(z) = \lim_{\Delta z \to o} D_\Sigma f(z)$ is called the <u>sigma-derivative</u> of f.

<u>Theorem 1</u>: If $f = u + iv$ has a sigma-derivative at z, then u and v have first partial derivatives at z and

$$f^{`}(z) = \lim_{\Delta z \to o} D_\Sigma f(z) = s_1(x)u_x + iv_x/s_2(x)$$

$$= t_1(y)v_y - iu_y/t_2(y) .$$

Proof: Let Δz approach zero along the real axis. Then $\Delta z = \Delta x$, and

$$\Delta_1 u = u(x + \Delta x, y) - u(x,y) ,$$

$$\Delta_1 v = v(x + \Delta x, y) - v(x,y) .$$

The sigma-difference quotient in this case becomes

$$D_{\Sigma}f(z) = s_1(x)\{u[x + \Delta x, y] - u[x,y]\}/\Delta x$$
$$+ i[1/s_2(x)]\{v[x + \Delta x, y] - v[x,y]\}/\Delta x .$$

By hypothesis the limit $\lim_{\Delta z \to 0} D_{\Sigma}f(z)$ exists, so each of the partial derivatives u_x and v_x exist and

$$f'(z) = s_1(x)u_x + iv_x/s_2(x) . \qquad (1)$$

Next let Δz approach zero along the imaginary axis. In this case $\Delta z = i\Delta y$ and

$$\Delta_2 u = u(x,y + \Delta y) - u(x,y) ,$$

$$\Delta_2 v = v(x,y + \Delta y) - v(x,y) .$$

Hence

$$D_{\Sigma}f(z) = t_1(y)\{v[x,y + \Delta y] - v[x,y]\}/\Delta y$$
$$- i[1/t_2(y)]\{u[x,y + \Delta y] - u[x,y]\}/\Delta y .$$

It follows that v_y and u_y exist and

$$f'(z) = t_1(y)\, v_y - iu_y/t_2(y) . \qquad (2)$$

This proves the theorem.

If a function f has a sigma-derivative at z, then the real and imaginary parts of f have first partial derivatives, which because of equations (1) and (2) are related by the generalized Cauchy-Riemann equations:

$$s_1(x)u_x = t_1(y)v_y$$
$$s_2(x)u_y = -t_2(y)v_x \ . \qquad (3)$$

The converse is in general not true. Consider the function $f = u + iv$, where

$$u(x,y) = \begin{cases} \dfrac{x^3 - y^3}{s_1(0)x^2 + s_2(0)y^2} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$$

$$v(x,y) = \begin{cases} \dfrac{x^3 + y^3}{t_2(0)x^2 + t_1(0)y^2} , & (x,y) \neq (0,0) \\ 0 , & (s,y) = (0,0) \ . \end{cases}$$

The partial derivatives at $z = 0$ are then given by

$$u_x(0) = 1/s_1(0), \quad u_y(0) = -1/s_2(0) \ ,$$
$$v_x(0) = 1/t_2(0), \quad v_y(0) = 1/t_1(0) \ .$$

At $z = 0$ the first partial derivatives satisfy the generalized Cauchy-Riemann equations (3). It is now shown that f does not have a sigma-derivative at $z = 0$.

For $\Delta z = iy$, $z = 0$,

$$\Delta_2 v = v(0,y) = y/t_1(0) \ ,$$
$$\Delta_2 u = u(0,y) = -y/s_2(0) \ ,$$

and the sigma-difference quotient becomes

$$D_\Sigma f(0) = t_1(0)(\Delta_2 v/y) - i\Delta_2 u/t_2(0)y$$
$$= 1 + i/[s_2(0)\ t_2(0)] \ . \tag{4}$$

For $\Delta z = x + ix$, $z = 0$,

$$\Delta_2 u = u(x,x) = 0$$

$$\Delta_1 v = v(x,x) = \frac{x}{t_2(0) + t_1(0)}$$

and the imaginary part of the sigma-difference quotient is given by

$$\text{Im}[D_\Sigma f(0)] = \frac{1}{s_2(0)[t_1(0) + t_2(0)]} \ .$$

Comparison of this expression with equation (4) shows that the limit of the sigma-difference quotient at $z = 0$ will not exist.

The generalized Cauchy-Riemann equations (3) do, however, suffice to establish the existence of the sigma-derivative if the functions u and v have continuous first partial derivatives.

Theorem 2: Let $u(x,y)$ and $v(x,y)$ be two real valued functions defined on some domain S, and assume that the four partial derivatives u_x, u_y, v_x, and v_y exist and are continuous on S. If for some point z_o in S, $s_1(x)u_x = t_1(y)v_y$ and $s_2(x)u_y = -t_2(y)v_x$, then the complex function $f = u + iv$ has a sigma-derivative at z_o.

Proof: Let $\Delta z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ and form the sigma-difference quotient at $z_o = x_o + iy_o$.

$$D_\Sigma f(z_o) = \mathrm{Re}\left\{\frac{s_1(x_o)\ \Delta_1 u + it_1(y_o)\Delta_2 v}{r(\cos\theta\ i\ \sin\theta)}\right\}$$

$$+ i\,\mathrm{Im}\left\{\frac{\Delta_2 u/t_2(y_o) + i\Delta_1 v/s_2(x_o)}{r(\cos\theta + i\sin\theta)}\right\},$$

$$D_\Sigma f(z_o) = \left\{s_1(x_o)\ \frac{\Delta_1 u}{r}\cos\theta + t_1(y_o)\ \frac{\Delta_2 v}{r}\sin\theta\right\}$$

$$+ i\left\{\frac{1}{s_2(x_o)}\ \frac{\Delta_1 v}{r}\cos\theta - \frac{1}{t_2(y_o)}\ \frac{\Delta_2 u}{r}\sin\theta\right\}. \qquad (5)$$

Now

$$2\Delta_1 u = u(z_o + re^{i\theta}) + u(z_o + re^{-i\theta}) - 2u(z_o)$$

$$= u(z_o + re^{i\theta}) - u(z_o) + u(z_o + re^{-i\theta}) - u(z_o)$$

$$= r\ u_\theta(z_o + r_1 e^{i\theta}) + r\ u_{-\theta}(z_o + r_2 e^{-i\theta}), \qquad (6)$$

by the Mean Value Theorem, where u_θ is directional derivative in θ direction and r_1 and r_2 are such that $0 \le r_1, r_2 \le r$. The directional derivatives are given by

$$u_\theta = u_x\cos\theta + u_y\sin\theta,$$

$$u_{-\theta} = u_x\cos(-\theta) + u_y\sin(-\theta) = u_x\cos\theta - u_y\sin\theta.$$

Substitution of these expressions in equation (6) gives

$$\frac{2\Delta_1 u}{r} = [u_x(z_o + r_1 e^{i\theta}) + u_x(z_o + r_2 e^{-i\theta})]\cos\theta$$

$$+ [u_y(z_o + r_1 e^{i\theta}) - u_y(z_o + r_2 e^{-i\theta})]\sin\theta.$$

By similar applications of the Mean Value Theorem, the following expressions are obtained:

$$\frac{2\Delta_2 u}{r} = [u_x(z_o + r_1 e^{i\theta}) - u_x(z_o - r_3 e^{-i\theta})] \cos\theta$$

$$+ [u_y(z_o + r_1 e^{i\theta}) + u_y(z_o - r_3 e^{-i\theta})] \sin\theta ,$$

$$\frac{2\Delta_1 v}{r} = [v_x(z_o + r_4 e^{i\theta}) + v_x(z_o + r_5 e^{-i\theta})] \cos\theta$$

$$+ [v_y(z_o + r_4 e^{i\theta}) - v_y(z_o + r_5 e^{-i\theta})] \sin\theta ,$$

$$\frac{2\Delta_2 v}{r} = [v_x(z_o + r_4 e^{i\theta}) - v_x(z_o + r_6 e^{-i\theta})] \cos\theta$$

$$+ [v_y(z_o + r_4 e^{i\theta}) + v_y(z_o - r_6 e^{-i\theta})] \sin\theta ,$$

where $0 \leq r_j \leq r$, $j = 1, \ldots, 6$.

Substitution of these expressions into equation (5) yields

$$2D_\Sigma f(z_o) = \{s_1(x_o)[u_x(z_o + r_1 e^{i\theta}) + u_x(z_o + r_2 e^{-i\theta})] \cos^2\theta$$

$$+ t_1(y_o)[v_y(z_o + r_4 e^{i\theta}) + v_y(z_o - r_6 e^{-i\theta})] \sin^2\theta\}$$

$$+ \{s_1(x_o)[u_y(z_o + r_1 e^{i\theta}) - u_y(z_o + r_2 e^{-i\theta})]$$

$$+ t_1(y_o)[v_x(z_o + r_4 e^{i\theta}) - v_x(z_o - r_6 e^{-i\theta})]\} \sin\theta \cos\theta$$

$$+ i\{1/t_2(y_o)[-u_y(z_o + r_1 e^{i\theta}) - u_y(z_o - r_3 e^{-i\theta})] \sin^2\theta +$$

$$+ 1/s_2(x_o)[v_x(z_o + r_4e^{i\theta}) + v_x(z_o + r_5e^{-i\theta})] \cos^2\theta\}$$

$$+i\{1/t_2(y_o)[-u_x(z_o + r_1e^{i\theta}) + u_x(z_o - r_3e^{-i\theta})]$$

$$+ 1/s_2(x_o)[v_y(z_o + r_4e^{i\theta}) - v_y(z_o + r_5e^{-i\theta})]\} \sin\theta\cos\theta$$

$$= \{A\} + \{B\}\sin\theta\cos\theta + i\{C\} + i\{D\}\sin\theta\cos\theta , \quad (7)$$

where A, B, C, and D are the expressions in the brackets.

The partial derivatives u_x, u_y, v_x, and v_y are continuous, so given any positive ε, $r = |\Delta z|$ can be taken small enough to insure that

$$|u_y(z_o + r_1e^{i\theta}) - u_y(z_o + r_2e^{-i\theta})| < \varepsilon/2s_1(x_o) ,$$

$$|v_x(z_o + r_4e^{i\theta}) - v_x(z_o - r_6e^{-i\theta})| < \varepsilon/2t_1(y_o) .$$

For this choice of r, it follows that

$$|\{B\}\sin\theta\cos\theta| \le |\{B\}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

It can be shown similarly that for sufficiently small r

$$|\{D\}\sin\theta\cos\theta| < \varepsilon .$$

Hence

$$\lim_{\Delta z \to o} \{B\}\sin\theta\cos\theta = \lim_{\Delta z \to o} \{D\}\sin\theta\cos\theta = 0 .$$

Because of the continuity of the partial derivatives, $\{A\}$ can be written as

$$\{A\} = s_1(x_o)[2u_x(z_o) + \varepsilon_1)] \cos^2\theta + t_1(y_o)[2v_y(z_o) + \varepsilon_2] \sin^2\theta ,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $r \to 0$. Thus

$$\{A\} = 2s_1(x_o)u_x(z_o) \cos^2\theta + 2\, t_1(y_o)v_y(z_o) \sin^2\theta$$

$$+ \varepsilon_1 s_1(x_o)\cos^2\theta + \varepsilon_2 t_1(y_o) \sin^2\theta$$

$$= 2s_1(x_o)u_x(z_o) + \varepsilon_1 s_1(x_o) \cos^2\theta + \varepsilon_2 t_1(y_o) \sin^2\theta ,$$

since $s_1(x_o)u_x(z_o) = t_1(y_o)v_y(z_o)$ by hypothesis.

So $\lim_{\Delta z \to o} \{A\}$ exists. In the same manner it is proved that $\lim_{\Delta z \to o} \{B\}$ also exists. Reference to equation (7) shows that $\lim_{\Delta z \to o} D_\Sigma f(z_o)$ exists, and the theorem is proved.

<u>Theorem 3</u>: Let $u(x,y)$ and $v(x,y)$ be two real valued functions defined on some domain S, and assume that the partial derivatives u_x, u_y, u_{xy}, v_x, v_y, and v_{xy} exist and are continuous on S. If u and v satisfy the generalized Cauchy-Reimann equations (3) everywhere in S, then the partial derivatives u_{xx}, v_{xx}, u_{yy}, and v_{yy} also exist and are related by the second order equations

$$[1/s_1(x)][t_1(y)v_y(x,y)]_y + t_2(y)[v_x(x,y)/s_2(x)]_x = 0 \qquad (8)$$

$$(1/t_1)(s_1 u_x)_x + s_2(u_y/t_2)_y = 0 .$$

Proof: From the equations (3),

$$u_x = (t_1/s_1)v_y ,$$

$$u_y = -(t_2/s_2)v_x . \qquad (9)$$

The derivative u_{xy} exists and is continuous, so u_{yx} also exists and is in fact equal to u_{xy}. Similarly, the other mixed partials can be shown to exist. Thus the equation (8) follows from differentiation of equations (9). Equation (9) can be obtained in the same way.

Definition 2: A function $f(z)$ possessing a continuous sigma-derivative at a point z and at all points of some neighborhood of z is called sigma-monogenic at z. A function $f(z)$ is said to be sigma-monogenic in a region D if it is sigma-monogenic at every z in D. The function $f(z)$ is said to be sigma-monogenic on an arbitrary set A if it is sigma-monogenic in some region containing A.

Theorem 4: A sigma-monogenic function is continuous.

Proof: Let $f = u + iv$ be sigma-monogenic at $z = x + iy$, and let $\Delta z = re^{i\theta}$. Then

$$r[\mathrm{Re}(D_\Sigma f)] = s_1(x)\Delta_1 u \cos\theta + t_1(y)\Delta_2 v \sin\theta .$$

f is sigma-monogenic, so $\lim\limits_{\Delta z \to 0} r[\mathrm{Re}(D_\Sigma f)] = 0$ independent of the direction θ. Take $\theta = 0$. Then

$$\lim_{\Delta z \to 0} \frac{r[\mathrm{Re}(D_\Sigma f)]}{s_1(x)} = \lim_{\Delta z \to 0} \Delta_1 u = 0 ,$$

and so u is continuous. v is shown to be continuous by taking $\theta = \pi/2$. This proves the theorem.

Some examples.--For $s_1 = t_1 = s_2 = t_2 = 1$ the matrix is

$$\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and the equations (3) are in this case the usual Cauchy-Riemann equations, and the sigma-monogenic functions are the analytic functions.

Let a be a real constant > -1 and consider

$$\Sigma = \begin{pmatrix} 1 & \frac{1}{1+a} \\ 1 & \frac{1}{1+a} \end{pmatrix}$$

If g is a function analytic in some region D, then the function f defined by

$$f(z) = g(z) + ia\ \mathrm{Im}\,g(z)$$

is sigma-monogenic in D. Letting $g = U + iV$ gives

$$f(z) = u + iv = U + i(V + aV) = U + i(1 + a)V .$$

It follows that

$$u_y = U_y, \quad u_x = U_x, \quad v_x = (1 + a)V_x, \quad \text{and} \quad v_y = (1 + a)V_y .$$

g is analytic so

$$V_y = U_x, \quad \text{and} \quad V_x = - U_y ;$$

therefore

$$\frac{1}{1+a} v_y = V_y = U_x = u_x ,$$

$$\frac{1}{1+a} v_x = V_x = -U_y = -u_y .$$

This shows that f is sigma-monogenic. In fact the sigma-derivative of f is given by

$$\grave{f}(z) = [1/(1+a)]v_y - i(1+a)u_y$$

$$= V_y - i(1+a)U_y$$

$$= g'(z) + ia\,\mathrm{Im}\,g'(z) .$$

Let

$$\Sigma = \begin{pmatrix} 1 & y^{-3} \\ 1 & y^{-3} \end{pmatrix} .$$

Consider the function

$$f(z) = \frac{e^x}{y} J_1(y) + i\, e^x y^2 J_2(y) ,$$

where $J_n(y)$ denotes the n^{th} order Bessel function of the first kind. Since

$$\frac{d}{dy} y^n J_n(y) = y^n J_{n-1}(y) ,$$

$$\frac{d}{dy} y^{-n} J_n(y) = -y^{-n} J_{n+1}(y) ,$$

it follows that

$$u_x = \frac{e^x}{y} J_1(y), \qquad v_x = e^x y^2 J_2(y) ,$$

$$u_y = -\frac{e^x}{y} J_2(y), \qquad v_y = e^x y^2 J_1(y) .$$

Thus

$$u_x = y^{-3} v_y ,$$

$$u_y = -y^{-3} v_x ,$$

and so f is sigma-monogenic.

Notice that Theorem 3 says that $\frac{e^x}{y} J_1(y)$ and $e^x y^2 J_2(y)$ are particular solutions of the second order partial differential equation

$$w_{xx} + w_{yy} - 3y^{-1} w_y = 0.$$

Sigma-Integration.--Let $f = u + iv$ be a continuous function on domain D and let C be a rectifiable curve in D. The sigma-integral $\int_C f(z)\, d_\Sigma z$ of f over C is defined by

$$\int_C f(z)\, d_\Sigma z = \int_C s_2(x)\, u(x,y)\, dx - t_2(y)\, v(x,y)\, dy$$

$$+ i \int_C [v(x,y)/s_1(x)]\, dx + [u(x,y)[t_1(y)]\, dy,$$

where the integrals on the right are the usual line integrals. It follows, of course, that

$$\int_C f(z)\, d_{\Sigma'} z = \int_C [u(x,y)/s_1(x)]\, dx - t_2(y)\, v(x,y)\, dy$$

$$+ i \int_C s_2(x)\, v(x,y)\, dx + [u(x,y)/t_1(y)]\, dy\, .$$

If a and b are real constants and f and g are continuous functions, then

$$\int_C [af(z) + bg(z)]\, d_\Sigma z + b \int_C g(z)\, d_\Sigma z + b \int_C g(z)\, d_\Sigma z\, .$$

This property does not hold in general if a and b are complex:

$$\int_C i d_\Sigma z = - \int_C t_2(y)\, dy + i \int_C [1/s_1(x)]\, dx$$

$$\int_C d_\Sigma z = - \int_C [1/t_1(y)]\, dy + i \int_C s_2(x)\, dx\, .$$

It follows from the corresponding property of line integrals, that for points a, b, and c on the Curve C,

$$\int_a^c f(z)\ dz = \int_a^b f(z)\ dz + \int_b^c f(z)\ dz \qquad \text{(on C)} .$$

Theorem 5: If f is a function sigma-monogenic in a simply connected region D and C is a simple closed curve contained in D, then

$$\int_C f(z)\ d_\Sigma z = 0 .$$

Proof: f is sigma-monogenic, and therefore the partial derivatives u_x, u_y, v_x, and v_y exist and are continuous in D. Moreover these partials satisfy the generalized Cauchy-Riemann equations (3) in D. This implies that the differentials

$$s_2(x)\ u(x,y)\ dx - t_2(y)\ v(x,y)\ dy$$

and

$$[v(x,y)/s_1(x)]\ dx + [u(x,y)/t_1(y)]\ dy$$

are exact. The conclusion of the theorem follows immediately.

From this theorem it follows that in a simply connected region the sigma-integral of a Σ-monogenic function between two points is independent of the path of integration. Thus for a fixed point a in a simply connected region D and a Σ-monogenic function f, the Σ-integral $\int_a^z f d_\Sigma \xi$ defines a function $F(z)$ for $z \varepsilon D$:

$$F(Z) = \int_a^z f(\xi)\ d_\Sigma \xi .$$

<u>Theorem 6</u>: Let f be Σ-monogenic in a simply connected retion D. If a is a point in D, then the function

$$F(z) = \int_a^z f(\xi)\, d_\Sigma \xi$$

is Σ'-monogenic in D.

Proof:

$$F(z) = \int_a^z f(\xi)\, d_\Sigma \xi = U(x,y) + iV(x,y) ,$$

where $U(x,y)$ and $V(x,y)$ are functions such that

$$U_x(x,y) = s_2(x)\, u(x,y), \qquad U_y(x,y) = - t_2(y)\, v(x,y) ,$$

$$V_x(x,y) = v(x,y)/s_1(x), \qquad V_y(x,y) = u(x,y)/t_1(y) .$$

Hence,

$$U_x(x,y)/s_2(x) = u(x,y) = t_1(y)\, V_y(x,y) ,$$

$$U_y(x,y)/s_1(x) = - [t_2(y)/s_1(x)]\, v(x,y) = - t_2(y)\, V_x(x,y) .$$

The partials U_x, U_y, V_x, V_y, are continuous because of the continuity of u and v; thus the conclusion of the theorem follows from Theorem 2.

<u>Theorem 7</u>: Let $f(z) = u(x,y) + i\, v(x,y)$ be Σ-monogenic at $z = x + iy$. Then the functions $u(x,y)$ and $v(x,y)$ are analytic functions of the real variables x and y at z.

Proof: Let

$$F(z) = \int_a^z f(\xi)\, d_\Sigma \xi .$$

Then, as in the proof of Theorem 6,

$$U_x(x,y)/s_2(x) \;=\; u(x,y) \;=\; t_1(y)\; V_y(x,y) \;,$$

$$U_y(x,y)/s_1(x) \;=\; -\;[t_2(y)/s_1(x)]\; v(x,y) = -\; t_2(y)\; V_x(x,y) \;,$$

where

$$F(z) \;=\; U(x,y) + i\; V(x,y) \;.$$

$f(z)$ is sigma-monogenic by hypothesis, so the partial derivatives u_x, u_y, v_x, v_y, exist and are continuous. Thus the derivatives

$$[U_x(x,y)/s_2(x)\;\; t_1(y)]_x$$

$$[U_y(x,y)/s_1(x)\;\; t_2(y)]_y$$

exist and are related by

$$\left[\frac{U_x(x,y)}{s_2(x)\;\; t_1(y)}\right]_x - \left[\frac{U_y(x,y)}{s_1(x)\;\; t_2(y)}\right]_y \;=\; 0 \;.$$

This is an elliptic equation with analytic coefficients, so it follows from the theory of partial differential equations that $U(x,y)$ is an analytic function (See, for instance, John [4]).

It follows immediately that $u(x,y)$ is also an analytic function. A similar argument holds for $v(x,y)$.

<u>Theorem 8</u>: Let $f(z) = u + iv$ be Σ-monogenic. Then the Σ-derivative, $f'(z)$, of f is Σ'-monogenic.

Proof:

$$f^{\backprime}(z) = s_1(x)\ u_x(x,y) + iv_x(x,y)/s_2(x)$$

$$= t_1(y)\ v_y(x,y) - iu_y(x,y)/t_2(y)$$

$$= \varphi + i\eta$$

from Theorem 1.

By Theorem 7, u and v are analytic; hence φ and η are analytic functions, and so have continuous first partial derivatives.

$$\varphi_x(x,y) = t_1(y)\ v_{yx}(x,y), \qquad \varphi_y(x,y) = s_1(x)\ u_{xy}(x,y)\ ,$$

$$\eta_x(x,y) = -\ u_{yx}(x,y)/t_2(y), \qquad \eta_y(x,y) = v_{xy}(x,y)/s_2(x)\ .$$

Hence,

$$\varphi_x/s_2 = (t_1/s_2)\ v_{yx} = t_1\eta_y$$

$$\varphi_y/s_1 = u_{xy} = -\ t_2\eta_x\ ,$$

and the theorem is proved.

Thus since $(\Sigma')' = \Sigma$, unlimited Σ and Σ'-differentiation is possible.

<u>Theorem 9</u>: Let f be Σ-monogenic at a.

$$\left(\int_a^z f(\xi)\ d_\Sigma\xi\right)^{\backprime} = f(z)\ ,$$

and

$$\int_a^z f^{\backprime}(\xi)\ d_{\Sigma'}\xi = f(z) - f(a)\ ,$$

for z in some neighborhood of a.

Proof: Let

$$F(Z) = \int_a^z f(\xi)\, d_\Sigma \xi = U(x,y) + iV(x,y) .$$

Then, as pointed out in the proof of Theroem 6,

$$U_x(x,y) = s_2(x)\, u(x,y), \qquad V_x(x,y) = v(x,y)/s_1(x) ,$$

where $f(z) = u(x,y) + iv(x,y)$.

Thus

$$d_{\Sigma'} F(z)/d_{\Sigma'} z = U_x(x,y)/s_2(x) + i\, V_x(x,y)\, s_1(x)$$

$$= u(x,y) = iv(x,y)$$

$$= f(z) .$$

To prove the second part of the theorem, let $a = a_1 + ia_2$ and $b = b_1 + ib_2$. Then for $f^{\backprime}(z) = u^*(x,y) + iv^*(x,y)$.

$$\int_a^b f^{\backprime}(z)\, d_{\Sigma'} z = \int_a^b [u^*(x,y)/s_1(x)]dx - [t_2(y)\; v^*(x,y)\; dy]$$

$$+ i \int_a^b [s_2(x)\; v^*(x,y)]dx + [u^*(x,y)/t_1(y)dy]$$

$$= \int_{a_1}^{b_1} [u^*(x,a_2)/s_1(x)]dx - \int_{a_2}^{b_2} t_2(y)\; v^*(b_1,y)dy$$

$$+ i \int_{a_1}^{b_1} s_2(x)\; v^*(x,a_2)dx + i \int_{a_2}^{b_2} [u^*(b_1,y)/t_1(y)]dy .$$

From Theorem 1

$$f^{\backprime}(z) = s_1(x)\; u_x(x,y) + iv_x(x,y)/s_2(x)$$

$$= t_1(y)\; v_y(x,y) - iu_y(x,y)/t_2(y) .$$

Or, in other words,

$$u^*(x,y) = s_1(x)\, u_x(x,y) = t_1(y)\, v_y(x,y) ,$$

$$v^*(x,y) = v_x(x,y)/s_2(x) = -\, u_y(x,y)/t_2(y) .$$

Hence,

$$\int_a^b f'(z)\, d_{\Sigma'} z = \int_{a_1}^{b_1} u_x(x,a_2)\, dx + \int_{a_2}^{b_2} u_y(b_1,y)\, dy$$

$$+ i \int_{a_1}^{b_1} v_x(x,a_2)\, dx + i \int_{a_2}^{b_2} v_y(b_1,y)\, dy$$

$$= u(b_1,a_2) - u(a_1,a_2) + u(b_1,b_2) - u(b_1,a_2)$$

$$+ i[v(b_1,a_2) - v(a_1,a_2) + v(b_1,b_2) - v(b_1,a_2)]$$

$$= u(b) - u(a) + i[v(b) - v(a)]$$

$$= f(b) - f(a) ,$$

which proves the theorem.

<u>Theorem 10</u>: Let f be continuous in a simply connected region D. If for any simple closed curve C in D

$$\int_C f(z)\, d_\Sigma z = 0,$$

then f is Σ-monogenic in D.

Proof: The function $F(z)$ defined in D by

$$F(z) = \int_a^z f(\xi)\, d_\Sigma \xi, \qquad a \in D ,$$

is well defined by virtue of the hypothesis. If $F(z) = U + iV$, then, as in the proof of Theorem 6,

$$U_x(x,y) = s_2(x)\, u(x,y), \qquad U_y(x,y) = -\, t_2(y)\, v(x,y),$$

$$V_x(x,y) = v(x,y)/s_1(x), \qquad V_y(x,y) = u(x,y)/t_1(y),$$

where $f(z) = u(x,y) + iv(x,y)$.

f is continuous so the partial derivatives U_x, U_y, V_x, and V_y are continuous. Theorem 2 then implies that the function $F(z)$ is Σ'-monogenic in D. From Theorem 9 it follows that the Σ'-derivative of $F(z)$ is $f(z)$. The conclusion of the theorem then follows immediately from Theorem 8.

CHAPTER III

EXPANSION THEOREM

Formal Powers.--A complex constant is Σ-monogenic for any Σ; therefore Σ-monogenic functions can be constructed by repeated Σ- and Σ'-integration of a constant. The functions so obtained are called formal powers, and in the analytic case $[m_\Sigma(z) \equiv 1]$ reduce to the usual powers of the independent variable z.

For $n \geq 0$, a and z_o any complex constants, the n^{th} formal powers are defined inductively as follows:

$$Z^{(o)}(a,z_o:z) = \tilde{Z}^{(o)}(a,z_o:z) = a$$

$$Z^{(n)}(a,z_o:z) = n\int_{z_o}^{z} \tilde{Z}^{(n-1)}(a,z_o:\zeta)\,d_{\Sigma'}\zeta$$

$$\tilde{Z}^{(n)}(a,z_o:z) = n\int_{z_o}^{z} Z^{(n-1)}(a,z_o:\zeta)\,d_{\Sigma}\zeta \,.$$

It follows immediately from the definition that the functions $Z^{(n)}(a,z_o:z)$ are Σ-monogenic and that $\tilde{Z}^{(n)}(a,z_o:z)$ are Σ'-monogenic. Theorem 9 implies that

$$\frac{d_\Sigma}{d_\Sigma z} Z^{(n)}(a,z_o:z) = n\,\tilde{Z}^{(n-1)}(a,z_o:z)$$

$$\frac{d_{\Sigma'}}{d_{\Sigma'} z} Z^{(n)}(a,z_o:z) = n\,Z^{(n-1)}(a,z_o:z) \,.$$

If $a = a_1 + a_2 i$, a_1, a_2 real, then

$$\int_{z_0}^{z} a d_\Sigma \zeta = \int_{z_0}^{z} (a_1 + ia_2) d_\Sigma \zeta = a_1 \int_{z_0}^{z} 1 d_\Sigma \zeta + a_2 \int_{z_0}^{z} i d_\Sigma \zeta$$

which implies that

$$Z^{(n)}(a_1 + ia_2, z_0 : z) = a_1 Z^{(n)}(1, z_0 : z) + a_2 Z^{(n)}(i, z_0 : z) \tag{10}$$

The same relation of course holds for the $\tilde{Z}^{(n)}(a, z_0 : z)$.

The following real functions are introduced to facilitate the study of formal powers.

$$X^{(o)}(x_0 : x) = \tilde{X}^{(o)}(x_0 : x) = Y^{(o)}(y_0 : y) = \tilde{Y}^{(o)}(y_0 : y) = 1$$

$$X^{(n)}(x_0 : x) = \begin{cases} n \int_{x_0}^{x} \frac{1}{s_1(\xi)} X^{(n-1)}(x_0 : \xi) d\xi, & n \text{ odd} \\ n \int_{x_0}^{x} s_2(\xi) X^{(n-1)}(x_0 : \xi) d\xi, & n \text{ even} \end{cases}$$

$$\tilde{X}^{(n)}(x_0 : x) = \begin{cases} n \int_{x_0}^{x} s_2(\xi) \tilde{X}^{(n-1)}(x_0 : \xi) d\xi, & n \text{ odd} \\ n \int_{x_0}^{x} \frac{1}{s_1(\xi)} \tilde{X}^{(n-1)}(x_0 : \xi) d\xi, & n \text{ even} \end{cases}$$

$$Y^{(n)}(y_0 : y) = \begin{cases} n \int_{y_0}^{y} \frac{1}{t_1(\eta)} Y^{(n-1)}(y_0 : \eta) d\eta, & n \text{ odd} \\ n \int_{y_0}^{y} t_2(\eta) Y^{(n-1)}(y_0 : \eta) d\eta, & n \text{ even} \end{cases}$$

$$\tilde{Y}^{(n)}(y_o:y) = \begin{cases} n \int_{y_o}^{y} t_2(\eta)\, \tilde{Y}^{(n-1)}(y_o:\eta)\, d\eta, & n \text{ odd} \\ n \int_{y_o}^{y} \frac{1}{t_1(\eta)}\, \tilde{Y}^{(n-1)}(y_o:\eta)\, d\eta, & n \text{ even} \end{cases}$$

$Z^{(n)}(1;z_o:z)$ and $\tilde{Z}^{(n)}(1;z_o:z)$ will be denoted by $Z^{(n)}(z_o:z)$ and $\tilde{Z}^{(n)}(z_o:z)$, respectively, and $Z^{(n)}(0:z)$ and $\tilde{Z}^{(n)}(0:z)$ will sometimes be denoted by $Z^{(n)}(z)$ and $\tilde{Z}^{(n)}(z)$. Also, $X^{(n)}(0:x)$ will be denoted by $X^{(n)}(x)$, etc.

Theorem 11:

$$Z^{(n)}(z_o:z) = \begin{cases} \sum_{v=0}^{n} \binom{n}{v} i^{n-v} X^{(v)}(x_o:x)\; Y^{(n-v)}(y_o:y), & n \text{ odd} \\ \sum_{v=0}^{n} \binom{n}{v} i^{n-v} \tilde{X}^{(v)}(x_o:x)\; Y^{(n-v)}(y_o:y), & n \text{ even} \end{cases}$$

$$\tilde{Z}^{(n)}(z_o:z) = \begin{cases} \sum_{v=0}^{n} \binom{n}{v} i^{n-v} \tilde{X}^{(v)}(x_o:x)\; Y^{(n-v)}(y_o:y), & n \text{ odd} \\ \sum_{v=0}^{n} \binom{n}{v} i^{n-v} X^{(v)}(x_o:x)\; Y^{(n-v)}(y_o:y), & n \text{ even} \end{cases}$$

$$Z^{(n)}(i;\, z_o:z) = \begin{cases} i \sum_{v=0}^{n} \binom{n}{v} i^{n-v} \tilde{X}^{(v)}(x_o:x)\; \tilde{Y}^{(n-v)}(y_o:y), & n \text{ odd} \\ i \sum_{v=0}^{n} \binom{n}{v} i^{n-v} X^{(v)}(x_o:x)\; \tilde{Y}^{(n-v)}(y_o:y), & n \text{ even} \end{cases}$$

$$Z^{(n)}(i;\ z_o:z) = \begin{cases} i\sum\limits_{v=o}^{n} \binom{n}{v} i^{n-v}\ X^{(v)}(x_o:x)\ \tilde{Y}^{(n-v)}(y_o:y), & n \text{ odd} \\ i\sum\limits_{v=o}^{n} \binom{n}{v} i^{n-v}\ \tilde{X}^{(v)}(x_o:x)\ \tilde{Y}^{(n-v)}(y_o:y), & n \text{ even} \end{cases}$$

Proof: The formulas are established by induction. They are clearly valid for $n = 0$.

Assume n is even and that

$$Z^{(n)}(z_o:z) = \sum_{v=o}^{n} \binom{n}{v} i^{n-v}\ \tilde{X}^{(v)}(x_o:x)\ y^{(n-v)},$$

$$Z^{n}(z_o:z) = i^{n}\Bigg[\sum_{k=o}^{n/2}(-1)^{k}\binom{n}{2k}\ \tilde{X}^{(2k)}(x_o:x)\ y^{(n-2k)}(y_o:y)$$
$$+\ i\sum_{k=1}^{n/2}(-1)\binom{n}{2k-1}\ \tilde{X}^{(2k-1)}(x_o:x)\ Y^{(n-2k+1)}(y_o:y)\Bigg].$$

By definition,

$$\tilde{Z}^{(n+1)}(z_o:z) = (n+1)\int_{z_o}^{z} Z^{(n)}(z_o:\zeta)\,d_{\Sigma}\zeta\ ,$$

$$\tilde{Z}^{(n+1)}(z_o:z) = (n+1)\ \Big\{\int_{z_o}^{z}\{s_2(\xi)\mathrm{Re}[Z^{(n)}(z_o:\zeta)]d\xi - t_2(\eta)\ \mathrm{Im}[Z^{(n)}(z_o:\zeta)]d\eta\}$$

$$+\ i\int_{z_o}^{z}\{\frac{1}{s_1(\xi)}\ \mathrm{Im}[Z^{(n)}(z_o:\zeta)]d\xi + \frac{1}{t_1(\eta)}\ \mathrm{Re}[Z^{(n)}(z_o:\zeta)]d\eta\}\Big\}\ .$$

$$(n+1)\mathrm{Re}[Z^{n}(z_o:\zeta)] = i^{n}\sum_{k=o}^{n/2}(-1)^{k}\binom{n+1}{2k}(n+1-2k)\,\tilde{X}^{(2k)}(x_o:\xi)\,Y^{(n-2k)}(y_o:\eta)\ ,$$

and

$$(n+1)\,\mathrm{Im}[Z^{n}(z_o:\zeta)] = i^{n}\sum_{k=o}^{n/2}(-1)^{k}\binom{n+1}{2k-1}(n+2-2k)\tilde{X}^{(2k-1)}(x_o:\xi)\ Y^{(n-2k+1)}(y_o:\eta)\ ,$$

since

$$(n+1)\binom{n}{p} = \frac{(n+1)n!}{p!(n-p)!} = \frac{(n+1)!}{p!(n-p)!}$$

$$= (n-p+1)\frac{(n+1)!}{p!(n-p+1)!}$$

$$= (n-p+1)\binom{n+1}{p}\ .$$

Hence

$$\mathrm{Re}[\tilde{Z}^{(n+1)}(z_o:z)] = \int_{z_o}^{z}\Big\{\Big[\sum_{k=o}^{n/2} i^{n-2k}\binom{n+1}{2k}(n+1-2k)s_2(\xi)\ \tilde{X}^{(2k)}(x_o:\xi)\ Y^{(n-2k+1)}(y_o:\eta)\Big]d\xi$$

$$-\Big[\sum_{k=o}^{n/2} i^{n-2k}\binom{n+1}{2k-1}\ (n+2-2k)\ \ t_2(\eta)\ \tilde{X}^{(2k-1)}(x_o:\xi)\ Y^{(n-2k+1)}(y_o:\eta)\Big]d\eta\Big\},$$

$$= \int_{x_o}^{x} s_2(\xi)\ \tilde{X}^{(n)}(x_o:\xi)\,d\xi$$

$$-\int_{y_o}^{y}\Big[\sum_{k=1}^{n/2} i^{n-2k}\binom{n+1}{2k-1}(n+2-2k)t_1(\eta)\tilde{X}^{(2k-1)}(x_o:x)\ Y^{(n-2k+1)}(y_o:\eta)\Big]d\eta,$$

since

$$Y^{(n)}(y_o:y_o) \equiv 0 \quad \text{for} \quad n \geq 1 \quad \text{and} \quad Y^{o}(y_o:y_o) \equiv 1\ .$$

$$\mathrm{Re}[\tilde{Z}^{(n+1)}(z_o:z)] = \tilde{X}^{(n+1)}(x_o:x) + \sum_{k=1}^{n/2} i^{n-2k+2}\binom{n+1}{2k-1}\tilde{X}^{(2k-1)}(x_o:x)\,[(n+2-2k)\int_{y_o}^{y} t_2(\eta)\,Y^{(n-2k+1)}(y_o:\eta)d\eta]$$

$$= \tilde{X}^{(n+1)}(x_o:x) + \sum_{k=1}^{n/2} i^{n-2k+2}\binom{n+1}{2k-1}\tilde{X}^{(2k-1)}(x_o:x)\;Y^{(n+1-2k+1)}(y_o:y)\ .$$

Similarly is obtained

$$\mathrm{Im}[\tilde{Z}^{(n+1)}(z_o:z)] = \sum_{k=0}^{n/2} i^{n-2k}\binom{n+1}{2k}\tilde{X}^{2k}(x_o:x)\;Y^{(n+1-2k)}(y_o:y)\ ,$$

so that

$$\tilde{Z}^{(n+1)}(z_o:z) = \sum_{\nu=0}^{n+1} i^{(n+1-\nu)}\binom{n+1}{\nu}\tilde{X}^{(\nu)}(x_o:x)\;Y^{(n+1-\nu)}(y_o:y)\ .$$

The other induction arguments are similar.

<u>Lemma 1</u>:

$$m_\Sigma^{-n}(|x_o| + |x|) \le \frac{X^{(n)}(x_o:x)}{(x-x_o)^n} \le m_\Sigma^{n}(|x_o| + |x|)$$

$$m_\Sigma^{-n}(|x_o| + |x|) \le \frac{\tilde{X}^{(n)}(x_o:x)}{(x-x_o)^n} \le m_\Sigma^{n}(|x_o| + |x|)$$

$$m_\Sigma^{-n}(|y_o| + |y|) \le \frac{Y^{(n)}(y_o:y)}{(y-y_o)^n} \le m_\Sigma^{n}(|y_o| + |y|)$$

$$m_\Sigma^{-n}(|y_o| + |y|) \le \frac{\tilde{Y}^{(n)}(y_o:y)}{(y-y_o)^n} \le m_\Sigma^{n}(|y_o| + |y|)\ .$$

Proof:

Case 1; $x > x_o$:

The proof is by induction on n.

$$X^{(1)}(x_o:x) = \int_{x_o}^{x}[1/s_1(\xi)]d\xi \leq \sup_{\xi\varepsilon[x_o,x]}[1/s_1(\xi)](x - x_o)$$

$$\leq m_\Sigma(|x_o| + |x|)(x - x_o) .$$

Similarly

$$X^{(1)}(x_o:x) \geq m_\Sigma^{-1}(|x_o| + |x|)(x - x_o) .$$

Assume the estimate holds for n. If n is even, then

$$X^{(n+1)}(x_o:x) = (n + 1) \int_{x_o}^{x} \frac{1}{s_1(\xi)} X^{(n)} d\xi$$

$$\leq (n + 1) \int_{x_o}^{x} m_\Sigma^{n+1} (|x_o| + |\xi|)(\xi - x_o)^n d\xi$$

by induction hypothesis.

m_Σ is nondecreasing, so

$$X^{(n+1)}(x_o:x) \leq (n + 1) m_\Sigma^{n+1}(|x_o| + |x|) \int_{x_o}^{x} (\xi - x_o)^n d\xi$$

$$\leq m_\Sigma^{n+1}(|x_o| + |x|)(x - x_o)^{n+1} .$$

Similarly,

$$X^{(n+1)}(x_0:x) \geq m_\Sigma^{-n-1}(|x_0| + |x|)(x - x_0)^{n+1} .$$

Case 2; $x_0 > x$:

$$- X^{(1)}(x_0:x) = \int_x^{x_0} \frac{1}{s_1(\xi)} d\xi$$

$$\leq m_\Sigma(|x_0| + |x|)(x_0 - x)$$

$$\frac{X^{(1)}(x_0:x)}{x - x_0} \leq m_\Sigma(|x_0| + |x|) .$$

Assume estimate holds for n.

$$- X^{(n+1)}(x_0:x) = (n + 1) \int_x^{x_0} \frac{1}{s_1(\xi)} X^{(n)}(x_0:\xi) d\xi$$

$$\leq m_\Sigma^{n+1}(|x_0| + |x|)(x_0 - x)^{n+1}$$

$$\frac{X^{(n)}(x_0:x)}{(x - x_0)^{n+1}} \leq m_\Sigma^{n+1}(|x_0| + |x|) .$$

The other estimate proceeds similarly.

The argument for n odd is virtually the same. The results for $\tilde{X}^{(n)}(x_0:x)$, $Y^{(n)}(y_0:y)$, and $\tilde{Y}^{(n)}(y_0:y)$ follow in the same fashion.

Lemma 2:

$$|Z^{(n)}(a;z_0:z)| \leq 2^{\frac{1}{2}} |a| \{2^{\frac{1}{2}} m_\Sigma(|z_0| + |z|) |z - z_0|\}^n .$$

Proof: Assume n is odd; the proof is virtually unchanged for even n.

$$|Z^{(n)}(1;z_o:z)| = \left|\sum_{\nu=1}^{n}\binom{n}{\nu} i^{n-\nu} X^{(\nu)}(x_o:x)\ Y^{(n-\nu)}(y_o:y)\right|$$

$$\leq \sum_{\nu=1}^{n}\binom{n}{\nu} |X^{(\nu)}(x_o:x)\ Y^{(n-\nu)}(y_o:y)|$$

$$\leq \sum_{\nu=1}^{n}\binom{n}{\nu} m_\Sigma^{\nu}(|x_o| + |x|)\ |x - x_o|^{\nu} m_\Sigma^{n-\nu}(|y_o| + |y|) \cdot |y - y_o|^{n-\nu} \tag{11}$$

from the triangle inequality and Lemma 1.

$$|x_o| \leq |z_o|, \quad |y_o| \leq |z_o|, \quad |x| \leq |z|, \quad |y| \leq |z| ,$$

so that

$$|x_o| + |x| \leq |z_o| + |z| ,$$

and

$$|y_o| + |y| \leq |z_o| + |z| ;$$

hence, since m_Σ is nondecreasing,

$$m_\Sigma(|x_o| + |x|) \leq m_\Sigma(|z_o| + |z|) ,$$

$$m_\Sigma(|y_o| + |y|) \leq m_\Sigma(|z_o| + |z|) .$$

Thus estimate (11) becomes

$$|Z^{(n)}(1;z_o:z)| \leq \sum_{\nu=1}^{n}\binom{n}{\nu} m_\Sigma^{n}(|z_o| + |z|)\ |\dot{x} - x_o|^{\nu}\ (y - y_o|^{n-\nu}$$

$$\leq m_\Sigma^{n}(|z_o| + |z|) \sum_{\nu=1}^{n}\binom{n}{\nu} |x - x_o|^{\nu}\ |y - y_o|^{n-\nu} \leq$$

$$\leq \; m_\Sigma^n(|z_o| + |z|)(|x - x_o| + |y - y_o|)^n$$

$$\leq \; m_\Sigma^n(|z_o| + |z|)(2^{\frac{1}{2}}|z - z_o|)^n$$

$$|Z^{(n)}(1;z_o:z)| \; \leq \; (2^{\frac{1}{2}} m_\Sigma(|z_o| + |z|) \; |z - z_o|)^n \; .$$

In the same manner is obtained

$$|Z^{(n)}(i;z_o:z)| \leq \; (2^{\frac{1}{2}} m_\Sigma(|z_o| + |z|) \; |z - z_o)^n \; .$$

For $a = \alpha + i\beta$, relation (11) gives

$$|Z^{(n)}(a;z_o:z)| \; = \; |\alpha Z^{(n)}(1;z_o:z) + \beta Z^{(n)}(i;z_o:z)|$$

$$|Z^{(n)}(a;z_o:z)| \; \leq \; |\alpha| \; |Z^{(n)}(1;z_o:z)| + |\beta| \; |Z^{(n)}(i,z_o:z)|$$

$$\leq (|\alpha| + |\beta|\{2^{\frac{1}{2}} m_\Sigma(|z_o| + |z|) \; |z - z_o|\}^n$$

$$\leq \; 2^{\frac{1}{2}}|a|\{2^{\frac{1}{2}} m_\Sigma(|z_o| + |z|) \; |z - z_o|\}^n \; ,$$

which proves the lemma.

Higher Order Sigma-Derivatives.--Higher order $\Sigma - \Sigma'$ derivatives of a Σ-monogenic function f are defined by

$$f^{[0]}(z) \; = \; f(z)$$

$$f^{[n]}(z) \; = \; \begin{cases} d_\Sigma f^{[n-1]}(z)/d_\Sigma z, & n \text{ odd} \\ d_{\Sigma'} f^{[n-1]}(z)/d_{\Sigma'} z, & n \text{ even} \end{cases}$$

It is evident that $f^{[n]}(z)$ is Σ-monogenic for even n and Σ'-monogenic for odd n.

<u>Lemma 3</u>: Let $f(z) = u(x,y) + iv(x,y)$ be a Σ-monogenic function. Let D_1 and D_2 be operators defined by

$$D_1 u(x,y) = s_1(x)\, u_x(x,y), \qquad D_2 u(x,y) = u_x(x,y)/s_2(x).$$

If $f^{[n]}(z) = u^{[n]}(x,y) + iv^{[n]}(x,y)$, then

$$u^{[2n+1]}(x,y) = D_1(D_2\, D_1)^n\, u(x,y)$$

$$u^{[2n]}(x,y) = (D_2\, D_1)^n\, u(x,y)$$

$$v^{[2n+1]}(x,y) = D_2(D_1\, D_2)^n\, v(x,y)$$

$$v^{[2n]}(x,y) = (D_1\, D_2)^n\, v(x,y)\ .$$

Proof: The proof is by induction on n.

$$f^{[1]}(z) = s_1(x)\, u_x(x,y) + iv_x(x,y)/s_2(x)$$

$$= D_1\, u(x,y) + iD_2\, v(x,y)$$

$$f^{[2]}(z) = \frac{1}{s_2(x)}\, [D_1\, u(x,y)]_x + i\, s_1(x)[D_2\, v)_x$$

$$= D_2\, D_1\, u(x,y) + i\, D_1\, D_2\, v(x,y)\ .$$

Assume the lemma is true for n. Then

$$f^{[2n]}(z) = (D_2 D_1)^n u(x,y) + i(D_1 D_2)^n v(x,y) ,$$

$$f^{[2n+1]}(z) = D_1(D_2 D_1)^n u(x,y) + iD_2(D_1 D_2)^n v(x,y) .$$

Hence

$$f^{[2(n+1)]}(z) = \frac{1}{s_2(x)} [D_1(D_2 D_1)^n u(x,y)]_x$$

$$+ i s_1(x) [D_2(D_1 D_2)^n v(x,y)]_x$$

$$= D_2[D_1(D_2 D_1]^n u(x,y) + i D_1 D_2[(D_1 D_2)^n] v(x,y)$$

$$= (D_2 D_1)^{n+1} u(x,y) + i (D_1 D_2)^{n+1} v(x,y) .$$

$$f^{[2(n+1)+1]}(z) = s_1(x) [(D_2 D_1)^{n+1} u(x,y)]_x$$

$$+ i \frac{1}{s_2(x)} [(D_1 D_2)^{n+1} v(x,y)]_x$$

$$= D_1(D_2 D_1)^{n+1} u(x,y) + i D_2(D_1 D_2)^{n+1} v(x,y) .$$

Theorem 12: Let f be Σ-monogenic at $z = z_o$. If

$$f^{[n]}(z_o) = 0, \quad n = 0, 1, 2, \ldots,$$

then $f(z) \equiv 0$.

Proof: Without loss of generality, it can be assumed that $z_o = 0$.

The following relation is first proved:

$$u^{[n]} = A\left(x, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^{n-1}u}{\partial x^{n-1}}\right) + B(x) \frac{\partial^n u}{\partial x^n} ,$$

where A and B are functions such that $B(x) \neq 0$ and

$$A = \sum_{i=1}^{n=1} C_i(x) \frac{\partial^i u}{\partial x^i}$$

for some analytic functions $C_i(x)$.

$$u^{[1]} = s_1(x) \frac{\partial u}{\partial x} ;$$

so for $n = 1$, the relation holds trivially. Assume true for n even. Then by the preceeding Lemma,

$$u^{[n+1]} = s_1(x) \frac{\partial}{\partial x} (u^{[n]})$$

$$= s_1(x) \frac{\partial}{\partial x} \left[A(x, \ldots, \frac{\partial^{n-1} u}{\partial x^n}\right]$$

$$= s_1(x) \left[\frac{\partial A}{\partial x} + \frac{\partial B}{\partial x} \frac{\partial^n u}{\partial x^n} + B \frac{\partial^{n+1} u}{\partial x^{n+1}}\right] .$$

$$\frac{\partial A}{\partial x} = \sum_{i=1}^{n-1} \{C_i \frac{\partial^{i-1} u}{\partial x^{i+1}} + C_i \frac{\partial^i u}{\partial x^i}\}$$

$$= C_1' \frac{\partial u}{\partial x} + \sum_{i=2}^{n-1} \{C_{i-1} + C_i'\} \frac{\partial^i u}{\partial x^i} + C_{n-1} \frac{\partial^n u}{\partial x^n} .$$

Hence,

$$u^{[n+1]} = s_1 C_1' \frac{\partial u}{\partial x} + \sum_{i=2}^{n-1} s_1\{C_{i-1} + C_i'\} \frac{\partial^i u}{\partial x^i} + s_1\left[C_{n-1} + \frac{\partial B}{\partial x}\right] \frac{\partial^n u}{\partial x^n} + B \frac{\partial^{n+1} u}{\partial x^{n+1}} s_1 .$$

For n odd, replace s_1 by $1/s_2$.

Now it is shown that $\left(\frac{\partial^n u}{\partial x^n}\right)_{x=y=o} = 0, \quad n = 0,1,\ldots$. This is obviously true for $n = 0,1$. Assume true for $n - 1$.

$$u^{[n]} = 0 \quad \text{by hypothesis, so}$$

$$u^{[n]} = A + B\frac{\partial^n u}{\partial x^n} .$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=y=o} = \left(\frac{\partial^2 u}{\partial x^2}\right)_{x=y=o} = \cdots = \left(\frac{\partial^{n-1} u}{\partial x^{n-1}}\right) = 0$$

by the induction hypothesis. Hence

$$A\left(x, \ldots, \frac{\partial^{n-1} u}{\partial x^{n-1}}\right)_{x=y=o} = 0 .$$

Thus

$$B(0)\left(\frac{\partial^n u}{\partial x^n}\right)_{x=y=o} = 0 .$$

But $B(0) \neq 0$, so

$$\left(\frac{\partial^n u}{\partial x^n}\right)_{x=y=o} = 0 .$$

Theorem 7 implies that $u(x,0)$ is an analytic function of x, so

$$u(x,0) \equiv 0 .$$

Similarly is obtained $v(x,0) \equiv 0$. Therefore

$$\frac{d}{dx}[v(x,0)] = \left(\frac{\partial v}{\partial x}\right)_{y=o} \equiv 0$$

$$\frac{\partial u}{\partial y} = -\frac{t_2(y)}{s_2(x)} \frac{\partial v}{\partial x} ,$$

since f is Σ-monogenic. So

$$\left(\frac{\partial u}{\partial y}\right)_{y=o} = -\frac{t_2(0)}{s_2(x)} \left(\frac{\partial v}{\partial x}\right)_{y=o} \equiv 0 .$$

u is a solution of a second order partial differential equation with analytic coefficients (Theorem 3). For $y = 0$, $u = 0$ and $\frac{\partial u}{\partial y} = 0$, so from the Cauchy-Kowaleski theorem (Petrovsky, [5]), $u \equiv 0$ is the unique solution of the equation (8). $u \equiv 0$, so v is constant, since $\frac{\partial u}{\partial x} = \frac{t_1}{s_1} \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{t_2}{s_2} \frac{\partial v}{\partial x}$.

But $f(0) = u(0) + i\, v(0) = 0$; hence $v \equiv 0$. Thus $f \equiv 0$.

<u>Lemma 4</u>: Let $F(z)$, $G(z)$, and $H(z)$ be functions analytic at $z = 0$. Let the sequence $\{F_n(z)\}$ be defined by

$$F_o(z) = F(z)$$

$$F_n(z) = \begin{cases} G(z)\ F'_{n-1}(z), & n \text{ odd} \\ H(z)\ F'_{n-1}(z), & n \text{ even} \end{cases}$$

Then there exists a positive constant C such that

$$|F_n(0)| \leq n!\ C^n$$

for all $n = 0, 1, 2, \ldots$.

Proof: Set

$$F(z) = \sum_{n=o}^{\infty} a_n z^n$$

$$G(z) = \sum_{n=o}^{\infty} b_n z^n$$

$$H(z) = \sum_{n=o}^{\infty} c_n z^n .$$

Let $d > 0$ be chosen such that F, G, and H are all analytic for $|z| \leq d$. Let M be a bound for F, G, and H on the circle $|z| = d$. Then from Cauchy's estimate

$$|F^{(n)}(0)| \leq M\, n!\, d^{-n}$$

$$|G^{(n)}(0)| \leq M\, n!\, d^{-n}$$

$$|H^{(n)}(0)| \leq M\, n!\, d^{-n} .$$

Hence

$$|a_n| \leq M\, d^{-n}, \quad |b_n| \leq M\, d^{-n}, \quad |c_n| \leq M\, d^{-n} .$$

Let the function K(z) be defined as follows:

$$K(z) = \sum_{n=o}^{\infty} M\, d^{-n} z^n$$

$$= \frac{M}{1 - z/d} = \frac{Md}{d - z}, \quad |z| < d .$$

K(z) is analytic for $|z| < d$. Let the sequence of functions $\{K_n(z)\}$ be defined as follows:

$$K_o(z) = K(z)$$

$$K_n(z) = K(z)\ K'_{n-1}(z)\ .$$

The $K_n(z)$ are constructed by the same formal process used to construct the $F_n(z)$ by taking $F(z) = G(z) = H(z) = K(z)$. Thus because of the estimates above on $|a_n|$, $|b_n|$, and $|c_n|$,

$$|F_n(0)| \leq |K_n(0)|\ .$$

By definition

$$K_o(z) = \frac{Md}{d - z}\ ,$$

$$K_1(z) = \frac{Md}{d - z}\left[\frac{Md}{(d - z)^2}\right] = \frac{M^2 d^2}{(d - z)^3}\ ,$$

$$K_2(z) = \frac{Md}{d - z}\left[\frac{3M^2 d^2}{(d - z)^4}\right] = \frac{3M^3 d^3}{(d - z)^5}$$

$$\vdots$$

$$K_n(z) = 1 \cdot 3 \cdot \ldots \cdot (2n - 1)\ \frac{M^{n+1}\ d^{n+1}}{(d - z)^{2n+1}}\ .$$

Since

$$1 \cdot 3 \cdot \ldots \cdot (2n - 1) < 2 \cdot 4 \cdot \ldots \cdot 2n = 2^n\ n!\ ,$$

it follows that

$$K_n(0) = 1 \cdot 3 \cdot \ldots \cdot (2n - 1)\ M^{n+1}/d^n$$

$$< 2^n\ n!\ M^{n+1}/d^n$$

$$\leq n!\ (2M^2/d)^n\ ,$$

if M is taken > 1. Hence

$$|F_n(0)| < |K_n(0)| < n!(2M^2/d)^n ,$$

which proves the lemma.

<u>Lemma 5</u>: If $f(z) = u + iv$ is Σ-monogenic at $z = z_o$, then there exists a constant C such that

$$|f^{[n]}(z_o)| < n!\, C^n.$$

Proof: Without loss of generality z_o can be taken to be 0. It is known from Theorem 7 that $u(x,y)$ is an analytic function of x and y. Thus $u(x,0)$ is an analytic function of the single variable x, and may be extended to an analytic function of the complex variable z. That is, the function $u(z,0)$ obtained by formally replacing x by z is analytic in some neighborhood of the origin. Similarly the functions $s_1(z)$ and $s_2(z)$ are analytic functions of z near the origin.

Apply Lemma 4 to the case $F(z) = u(z,0)$, $G(z) = s_1(z)$, and and $H(z) = 1/[s_2(z)]$. Then

$$F_n(z) = \begin{cases} s_1(z)\, F'_{n-1}(z), & n \text{ odd} \\ 1/[s_2(z)]\, F'_{n-1}(z), & n \text{ even} \end{cases}$$

Thus there is a constant C_1 such that

$$|F_n(0)| < n!\, C_1^n .$$

Since Lemma 3 implies that $F_n(0) = u^{[n]}(0,0)$,

$$|F_n(0)| = |u^{[n]}(0,0)| < n!\, C_1^n .$$

In the same way it is shown that

$$|v^{[n]}(0,0)| < n!\, C_2^n$$

for some constant C_2. Thus

$$|F^{[n]}(0)| = |u^{[n]}(0,0) + i\, v^n(0,0)|$$

$$\leq |u^{[n]}(0,0)| + |v^{[n]}(0,0)|$$

$$< n!(C_1^n + C_2^n) \leq n!(C_1 + C_2)^n .$$

Expansion Theorem.--Consider the infinite series

$$\sum_{n=o}^{\infty} Z^{(n)}(a_n;z_o:z) .$$

If this series converges uniformly on every compact subset of some simply connected region D, the limit function will be Σ-monogenic in D, as can be shown by making use of Theorem 10.

The problem in this section is to decide under what conditions a function Σ-monogenic at a point z_o can be expanded in such a power series.

Theorem 13: If the ordinary power series $\sum_{n=o}^{\infty} a_n(z - z_o)^n$ has a positive radius of convergence r, then there is a neighborhood of

z_o, $N(z_o)$, such that the formal power series

$$\sum_{n=o}^{\infty} Z^{(n)}(a_n;z_o:z)$$

converges uniformly and absolutely for all z in $N(z_o)$.

Proof: Let $N(z_o)$ be defined by

$$N(z_o) = \{z \mid |z - z_o| < \rho/[2^{\frac{1}{2}}m_{\Sigma}(|z_o| + r)]\}$$

where ρ is such that $0 < \rho < r$. From Lemma 2,

$$|Z^{(n)}|(a_n;z_o:z) \leq 2^{\frac{1}{2}} |a_n| \rho^n, \qquad z \in N(z_o) .$$

$\sum_{n=o}^{\infty} |a_n| \rho^n$ converges since $\rho < r$ and $\sum_{n=o}^{\infty} a_n(z - z_o)^n$ has radius of convergence r by hypothesis. Hence

$$\sum_{n=o}^{\infty} Z^{(n)}(a_n;z_o:z)$$

converges uniformly on $N(z_o)$, by the Weierstass M test.

<u>Corollary</u>: The formal power series

$$\sum_{n=o}^{\infty} n \tilde{Z}^{(n-1)}(a_n;z_o:z) ,$$

obtained from Σ-differentiating the original series term-by-term, also converges uniformly and absolutely in $N(z_o)$.

Proof: The conclusion is immediate from the fact that $\sum_{n=o}^{\infty} n\, a_n(z - z_o)^{n-1}$ has the same radius of convergence as the series $\sum_{n=o}^{\infty} a_n(z - z_o)^n$.

<u>Theorem 14</u>: If

$$f(z) = \sum_{n=o}^{\infty} Z^{(n)}(a_n;z_o:z) ,$$

then

$$n!\, a_n = f^{[n]}(z_o) .$$

Proof:

$$f(z) = \sum_{k=o}^{\infty} Z^{(k)}(a_k;z_o:z)$$

by hypothesis. This series can be Σ - Σ'-differentiated term-by-term n times to yield

$$f^{[n]}(z) = \begin{cases} \sum_{k=n}^{\infty} k(k-1)\ldots(k-n+1)\, Z^{(k-n)}(a_k;z_o:z), & n \text{ even} \\ \sum_{k=n}^{\infty} k(k-1)\ldots(k-n+1)\, \tilde{Z}^{(k-n)}(a_k;z_o:z), & n \text{ odd} \end{cases}$$

The conclusion of the theorem follows at once from the fact that

$$Z^{(k)}(a_k;z_o:z_o) = \tilde{Z}^{(k)}(a_k;z_o:z_o) = 0$$

for all $k \geq 1$, and

$$Z^{(o)}(a_n;z_oz) = Z^{(o)}(a_n;z_o:z) = a_n .$$

Theorem 15: A function $f(z)$ which is Σ-monogenic at $a = z_o$ can be expanded in a formal power series

$$f(z) = \sum_{n=o}^{\infty} Z^{(n)}(a_n;z_o:z)$$

valid in some neighborhood of z_o .

Proof: Define

$$a_n = f^{[n]}(z_o)/n! \ .$$

From Lemma 5, there is a constant C such that

$$|a_n| = \left|\frac{f^{(n)}(z_o)}{n!}\right| < C^n \ .$$

Then

$$\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} \leq C ,$$

or

$$\frac{1}{\lim\limits_{n\to\infty} \sup \sqrt[n]{|a_n|}} \geq \frac{1}{C} > 0 \ .$$

Thus the ordinary power series

$$\sum_{n=o}^{\infty} a_n(z - z_o)^n$$

has a positive radius of convergence. Theorem 13 then implies that the formal power series

$$F(z) = \sum_{n=o}^{\infty} Z^{(n)}(a_n;z_o:z)$$

converges in some neighborhood of z_o. It follows from the previous theorem that

$$F^{[n]}(z_o) = n!\ a_n .$$

Set

$$G(z) = F(z) - f(z) .$$

Then

$$G^{[n]}(z) = F^{[n]}(z) - f^{[n]}(z)$$

$$= n!\ a_n - n!\ a_n$$

$$= 0 .$$

Hence from Theorem 12, $G(z) \equiv 0$; that is,

$$F(z) = f(z) = \sum_{n=o}^{\infty} Z^{(n)}(a_n;z_o:z) .$$

CHAPTER IV

MAPPINGS DEFINED BY SIGMA-MONOGENIC FUNCTIONS

The major results of this chapter will be to show that a mapping $w = f(z)$ from the complex plane into the complex plane, for $f(z)$ sigma-monogenic, is an open mapping, and its zeros are isolated. This means, of course, that a sigma-monogenic function is light open, and hence is locally topologically equivalent to some analytic function. (Whyburn [7]).

Zeros of Sigma-Monogenic Functions.--In this section will be proved that the zeros of a sigma-monogenic function are isolated.

Lemma 6: Let $s_1(0) = s_2(0) = t_1(0) = t_2(0) = 1$. Given two constants $a \neq 0$ and z_0, and a positive integer n, there exists a number $\delta > 0$ and a number $\varepsilon > 0$ such that

$$|Z^{(n)}(a;z_0:z) > \delta|z - z_0|^n, \quad \text{for } |z - z_0| \leq \varepsilon .$$

Proof: Without loss of generality it can be assumed that $z_0 = 0$.

Let ε_1 be any positive number. Since $s_1(x)$, $s_2(x)$, $t_1(y)$, and $t_2(y)$ are continuous and never zero, there exists a number ε^* such that

$$\left.\begin{array}{l} |1 - s_j^k(x)| < \varepsilon_1 , \\ |1 - t_j^k(y)| < \varepsilon_1 , \end{array}\right. \quad k = \pm 1, \quad j = 1, 2$$

whenever $|x|$, $|y| < \varepsilon^*$. Choose $\varepsilon_2 < \inf(\varepsilon_1, \varepsilon^*)$. It then follows at once from the definition of m_Σ and from the inequalities above that

$$m_\Sigma(\varepsilon_2) < 1 + \varepsilon_1 . \tag{12}$$

Define a sequence of positive integers $\{c_n\}$ as follows:

$$c_1 = 1 ,$$

$$c_n = 1 + 2n\, c_{n-1} , \quad n > 1 .$$

It is now shown by induction that

$$|X^{(n)}(x) - x^n| < \varepsilon_1 c_n |x|^n, \quad \text{for } |x| < \varepsilon_2 . \tag{13}$$

For $n = 1$,

$$|X(x) - x| = |\int_0^x \frac{d\xi}{s_1(\xi)} - x| = |\int_0^x [\frac{1}{s_1(\xi)} - 1]\, d\xi|$$

$$\leq \int_0^x |\frac{1}{s_1(\xi)} - 1|\, d\xi$$

$$\leq \varepsilon_1 |x| .$$

Assume

$$|X^{(n-1)}(x) - x^{n-1}| < \varepsilon_1 c_{n-1} |x|^{n-1}, \quad |x| < \varepsilon .$$

For even n,

$$|X^{(n)}(x) - x^n| = |n \int_0^x s_2(\xi)\, X^{(n-1)}(\xi)\, d\xi - x^n|$$

$$= |n \int_0^x [s_2(\xi)\, X^{(n-1)}(\xi) - \xi^{n-1}]\, d\xi| =$$

$$= \left| n \int_0^x [s_2(\xi)\ X^{(n-1)}(\xi) - s_2(\xi)\ \xi^{n-1} + s_2(\xi)\ \xi^{n-1} - \xi^{n-1}]\ d\xi \right|$$

$$\leq n \int_0^x |s_2(\xi)\ [X^{(n-1)}(\xi) - \xi^{n-1}]|\ d\xi + n \int_0^x |\xi^{n-1}|\ |s_2(\xi) - 1|\ d\xi$$

$$\leq n(1 + \varepsilon_1)\ \varepsilon_1 c_{n-1}\ |x|^{n-1}\ |x| + \varepsilon_1 |x|^n$$

$$\leq \varepsilon_1 |x|^n\ [n(1 + \varepsilon_1)\ c_{n-1} + 1]$$

$$|X^{(n)}(x) - x^n| \leq \varepsilon_1\ c_n\ |x|^n ,$$

since ε_1 can be taken to be less than 1.

For n odd the proof proceeds in exactly the same manner with s_2 everywhere replaced by $1/s_1$. Thus the estimate (13) is established for all n.

The same estimate is valid for the functions $\tilde{X}^{(n)}(x)$, $Y^{(n)}(y)$, and $\tilde{Y}^{(n)}(z)$.

From Theorem 11,

$$|Z^{(n)}(z) - z^n| = \left| \sum_{\nu=0}^{n} \binom{n}{\nu}\ i^{n-\nu}\ X^{(\nu)}(x)\ Y^{(n-\nu)}(y) - z^n \right|$$

where n is odd; the argument is virtually unchanged in case n is even.

$$z^n = (x + iy)^n = \sum_{\nu=0}^{n} \binom{n}{\nu}\ i^{n-\nu}\ x^\nu\ y^{n-\nu} .$$

Hence,

$$|Z^{(n)}(z) - z^n| = |\sum_{\nu=0}^{n} \binom{n}{\nu} i^{n-\nu} [X^{(\nu)}(x)\ Y^{(n-\nu)}(y) - x^\nu\ y^{n-\nu}]$$

$$\leq \sum_{\nu=0}^{n} \binom{n}{\nu} |X^{(\nu)}(x)\ Y^{(n-\nu)}(y) - Y^{(n-\nu)}(y)\ x^\nu$$

$$+ Y^{(n-\nu)}(y)\ x^\nu - x^\nu\ y^{n-\nu}|$$

$$\sum_{\nu=0}^{n} \binom{n}{\nu} |Y^{(n-\nu)}(y)|\ |X^{(\nu)}(x) - x^\nu|$$

$$+ \sum_{\nu=0}^{n} \binom{n}{\nu} |x^\nu|\ |Y^{(n-\nu)}(y) - y^{n-\nu}| . \qquad (14)$$

If $|z| < \varepsilon_2$, then $|x|$, $|y| < \varepsilon_2$ and the estimate (13) implies that

$$\sum_{\nu=0}^{n} \binom{n}{\nu} |Y^{(n-\nu)}(y)|\ |X^{(\nu)}(x) - x^\nu| < \sum_{\nu=0}^{n} \binom{n}{\nu} \varepsilon_1\ c_\nu |x|^\nu\ |Y^{(n-\nu)}(y)| .$$

But from Lemma 1,

$$|Y^{(n-\nu)}(y)| \leq m_\Sigma^{n-\nu}(\varepsilon_2)\ |y|^{n-\nu} .$$

Hence

$$\sum_{\nu=0}^{n} \binom{n}{\nu} |Y^{(n-\nu)}(y)|\ |X^{(\nu)}(x) - x^\nu| < \varepsilon_1 c_n \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu\ m_\Sigma^{n-\nu}(\varepsilon_2) |y|^{n-\nu}$$

$$< \varepsilon_1 c_n (|x| + m_\Sigma(\varepsilon_2)\ |y|)^n \quad \text{(Binomial Theorem)}$$

$$< \varepsilon_1 c_n [|x| + (1 + \varepsilon_1)\ |y|]^n$$

$$< 2^{2n}\ c_n\ \varepsilon_1 |z|^n, \qquad \text{for } |z| < \varepsilon_2 .$$

Also,

$$\sum_{\nu=0}^{n} \binom{n}{\nu} |x|^{\nu} |Y^{(n-\nu)}(y) - y^{n-\nu}| < \varepsilon_1 c_n \sum_{\nu=0}^{n} \binom{n}{\nu} |x|^{\nu} |y^{n-\nu}|$$

$$< \varepsilon_1 c_n (|x| + |y|)^n$$

$$< 2^{2n} \varepsilon_1 c_n |z|^n .$$

Estimate (14) now becomes

$$|Z^{(n)}(z) - z^n| < 2^{2n} c_n \varepsilon_1 |z|^n + 2^{2n} c_n \varepsilon_1 |z|^n$$

$$< 2^{2n+1} c_n \varepsilon_1 |z|^n .$$

In the same manner can be obtained

$$|Z^{(n)}(i:z) - i z^n| < 2^{2n+1} c_n \varepsilon_1 |z|^n .$$

For $a = \alpha + \beta i$,

$$Z^{(n)}(a:z) = \alpha Z^{(n)}(z) + \beta Z^{(n)}(i:z)$$

$$|Z^{(n)}(a:z) - a z^n| \leq |\alpha| |Z^{(n)}(z) - z^n| + |\beta| |Z^{(n)}(i:z) - i z^n|$$

$$< 2^{2n+1} c_n \varepsilon_1 |z|^n(|\alpha| + |\beta|)$$

$$< 2^{2n+1} c_n \varepsilon_1 |a| |z|^n .$$

Or

$$|a z^n - Z^{(n)}(a:z)| < 2^{2n+2} c_n \varepsilon_1 |a| |z|^n ,$$

$$|a| |z|^n - Z^{(n)}(a:z)| < 2^{2n+2} c_n \varepsilon_1 |a| |z|^n .$$

Thus

$$|Z^{(n)}(a:z)| > (1 + 2^{2n+2} c_n \varepsilon_1) |a| |z|^n .$$

If ε_1 is taken so that

$$\varepsilon_1 < 1/2^{2n+2} c_n$$

then

$$\delta = |a|(1 - 2^{2n+2} c_n \varepsilon_1) > 0 ;$$

and

$$|Z^{(n)}(a:z)| > \delta |z|^n, \quad \text{for } |z| < \varepsilon_2 ,$$

which proves the lemma.

Lemma 7: Let $s_1(0) = s_2(0) = s_o$ and $t_1(0) = t_2(0) = t_o$. Then the conclusion of Lemma 6 follows.

Proof: Define the matrix Σ^* as follows

$$\Sigma^* = \begin{pmatrix} s_1(x)/s_o & t_1(y)/t_o \\ s_2(x)/s_o & t_2(y)/t_o \end{pmatrix}$$

Let $Z^{(n)}(a:z) = U(x,y) + i\,V(x,y)$. Let the function $g(z)$ be defined by

$$g(z) = s_o\, U(x,y) + i\, t_o\, V(x,y) .$$

Let

$$u^*(x,y) = s_o\, U(x,y)$$

$$v^*(x,y) = t_o\, V(x,y) .$$

Then

$$u^*_x(x,y) = s_o\, U_x(x,y), \qquad u^*_y(x,y) = s_o\, U_y(x,y)\ ,$$

$$v^*_x(x,y) = t_o\, V_x(x,y), \qquad v^*_y(x,y) = t_o\, V_y(x,y)\ .$$

$Z^{(n)}(a:z)$ is Σ-monogenic, which implies

$$s_1(x)/s_o\ u^*_y(x,y) = s_1(x)\ U_x(x,y) = t_1(y)\ V_y(x,y) = t_1(y)/t_o\ v^*_y(x,y)$$

$$s_2(x)/s_o\ u^*_y(x,y) = s_2(x)\ U_y(x,y) = -t_2(y)\ V_x(x,y) = -t_2(y)/t_o\ v^*_x(x,y)\ .$$

So, by Theorem 2, $g(z)$ is Σ*-monogenic.

It will now be shown that

$$g(z) = Z^{(n)}_*(a^*:z)\ ,$$

for some constant a^*, where $Z^{(n)}_*(a^*:z)$ is the formal power obtained from Σ*.

Assume n is even. The argument for odd n is only slightly different. Let m be any positive integer such that $m \leq n$. For odd m, $m = 2k + 1$, and Lemma 3 gives

$$[Z^{(n)}(a:z)]^{[m]} = D_1(D_2\ D_1)^k\ U + i\ D_2(D_1\ D_2)^k\ V\ .$$

For even m, $m = 2k$, then

$$[Z^{(n)}(a:z)]^{[m]} = (D_2\ D_1)^k\ U + i\ (D_1\ D_2)^k\ V\ .$$

Consider now the m^{th} order Σ*-Σ*′-derivative of $g(z)$. Lemma 3 gives

$$g^{[m]}(z) = \begin{cases} D_1^* \, (D_2^* \, D_1^*)^k \, s_o \, U + i \, D_2^*(D_1^* \, D_2^*)^k \, t_o \, V, & m \text{ odd} \\ (D_2^* \, D_1^*)^k \, s_o \, U \; + \; i \, (D_1^* \, D_2^*)^k \, t_o \, V, & m \text{ even} \, , \end{cases}$$

where

$$D_1^* = (1/s_o) D_1, \qquad D_2^* = s_o \, D_2 \, .$$

Hence

$$g^{[m]}(z) = \begin{cases} D_1 (D_2 \, D_1)^k \, U + i \, D_2 (D_1 \, D_2)^k \, V \, t_o \, s_o \, , & m \text{ odd} \\ s_o (D_2 \, D_1)^k \, U + i \, t_o (D_1 D_2)^k \, V, & m \text{ even} \, . \end{cases}$$

For all $m < n$

$$[Z^{(n)}(a{:}0)]^{[m]} = 0 \, ,$$

and for $m = n$

$$[Z^{(n)}(a{:}z)]^{[m]} \equiv a = \alpha + i \beta \, .$$

Hence

$$g^{[m]}(0) = 0, \qquad \text{for } 0 \leq m < n$$

$$g^{[n]}(z) \equiv s_o \, \alpha + i \, t_o \, \beta \, .$$

It follows from Theorem 12 applied to $Z_*^{(n)}(a^*{:}z) - g(z)$ that

$$g(z) = Z_*^{(n)}(a^*{:}z) \, ,$$

where

$$a^* = s_o \, \alpha + i \, t_o \, \beta \, .$$

From the definition of $g(z)$

$$|g(z)|^2 = s_o^2 U^2 + t_o^2 V^2$$

$$\leq m_\Sigma^2(0) (U^2 + V^2)$$

$$\leq m_\Sigma^2(0) |Z^{(n)}(a:z)|^2 .$$

Hence

$$m_\Sigma^{-1}(0) |g(z)| \leq |Z^{(n)}(a:z)| .$$

The matrix Σ^* satisfies the hypothesis of Lemma 6, so there exist positive numbers δ' and ε such that

$$|Z_*^{(n)}(a^*:z)| > \delta'|z|^n, \qquad \text{for } |z| < \varepsilon .$$

This, combined with estimate (15), implies

$$|Z^{(n)}(a:z)| \geq m_\Sigma^{-1}(0) |g(z)| > m_\Sigma^{-1}(0) \delta'|z|^n, \qquad |z| < \varepsilon ,$$

which proves the lemma.

Lemma 8: The conclusion of Lemma 6 holds for arbitrary Σ .

Proof: Let the following change of variable be made:

$$z = x + iy = A\xi + i B\eta$$

where

$$A = [s_1(0) \ t_2(0)]^{\frac{1}{2}}$$

$$B = [s_2(0) \ t_1(0)]^{\frac{1}{2}} ,$$

and $\zeta = \xi + i\eta$ is a new independent variable.

Let the matrix $\Sigma*$ be defined as follows:

$$\Sigma* = \begin{pmatrix} [s_2(0)/s_1(0)]^{\frac{1}{2}} s_1(A\xi) & [t_2(0)/t_1(0)]^{\frac{1}{2}} t_1(B\eta) \\ [s_1(0)/s_2(0)]^{\frac{1}{2}} s_2(A\xi) & [t_1(0)/t_2(0)]^{\frac{1}{2}} t_2(B\eta) \end{pmatrix}$$

Next it is shown that a sigma-monogenic function $f(z)$ is a $\Sigma*$-monogenic function of the variable ζ.

$$f(z) = u(x,y) + i\, v(x,y)$$

$$f(\zeta) = u(A\xi, B\eta) + i\, v(A\xi, B\eta) .$$

Also

$$u_\xi = u_x A, \qquad u_\eta = u_y B ,$$

$$v\xi = v_x A, \qquad v_\eta = v_y B .$$

Since f is a Σ-monogenic function of z,

$$s_1(x)\, u_x(x,y) = t_1(y)\, v_y(x,y) ,$$

$$s_2(x)\, u_y(x,y) = - t_2(y)\, v_x(x,y) .$$

Making the above defined change of variable results in

$$[s_1(A\xi)/A]\, u_\xi\, (A\xi, B\eta) = [t_1(B\eta)/B]\, v_\eta(A\xi, B\eta) ,$$

$$[s_2(A\xi)/B]\, u_\eta\, (A\xi, B\eta) = - [t_2(B\eta)/A]\, v_\xi(A\xi, B\eta) ;$$

or

$$\{1/[s_1(0)\ t_2(0)]^{\frac{1}{2}}\}\, s_1(A\xi)\, u_\xi = \{1/[s_2(0)\ t_1(0)]^{\frac{1}{2}}\}\, t_1(B\eta)\, v\eta ,$$

$$\{1/[s_2(0)\ t_1(0)]^{\frac{1}{2}}\}\, s_2(A\xi)\, u_\eta = - \{1/[s_1(0)\ t_2(0)]^{\frac{1}{2}}\}\, t_2(B\xi)\, v\xi .$$

Hence

$$[s_2(0)/s_1(0)]^{\frac{1}{2}}\ s_1(A\xi)\ u_\xi \;=\; [t_2(0)/t_1(0)]^{\frac{1}{2}}\ t_1(B\eta)\ v_\eta\ ,$$

$$[s_1(0)/s_2(0)]^{\frac{1}{2}}\ s_2(A\xi)\ u_\eta \;=\; -[t_1(0)/t_2(0)]^{\frac{1}{2}}\ t_2(B\eta)\ v_\xi\ .$$

It can be shown similarly that a Σ'-monogenic function of z is a $\Sigma^{*\prime}$-monogenic function of ζ.

Consider the relation between the Σ-derivative of a function and the Σ^*-derivative of the same function treated as a function of ζ.

$$d_\Sigma\ f(z)/d_\Sigma z \;=\; s_1(x)\ u_x + [i/s_2(x)]\ v_x$$

$$d_{\Sigma*}f(\zeta)/d_{\Sigma*}\zeta \;=\; s_1(A\xi)[s_2(0)/s_1(0)]^{\frac{1}{2}}\ Au_x$$

$$+\ Ai/[s_1(0)/s_2(0)]^{\frac{1}{2}}\ s_2(A\xi)\ v_x$$

$$=\; s_1(A\xi)[t_2(0)\ s_2(0)]^{\frac{1}{2}}\ u_x$$

$$+\ i[t_2(0)\ s_2(0)]^{\frac{1}{2}}\ v_x/s_2(A\xi)\ ,$$

from which follows

$$d_\Sigma\ f(z)d_\Sigma z \;=\; [s_2(0)\ t_2(0)]^{-\frac{1}{2}}\ d_{\Sigma*}f(\zeta)/d_{\Sigma*}\zeta\ .$$

The same result holds for Σ'- and $\Sigma^{*\prime}$-derivatives, so that

$$f^{[m]*}(\zeta) \;=\; [s_2(0)/t_2(0)]^{m/2}\ f^{[m]}(z)\ , \tag{16}$$

where $f^{[m]*}(\zeta)$ denotes the mth order Σ^*- $\Sigma^{*\prime}$-derivative .

For $0 \leq m < n$,

$$[Z^{(n)}(a:0)]^{[m]} = 0$$

$$[Z^{(n)}(a:z)]^{[n]} \equiv a .$$

Relation (16) therefore implies that

$$[Z^{(n)}(a:0)]^{[m]*} = 0, \quad 0 \leq m < n$$

$$[Z^{(n)}(a:\zeta)]^{[n]*} \equiv [s_2(0)/t_2(0)]^{n/2} a .$$

From Theorem 12 it follows that

$$Z_*^{(n)}(a^*:\zeta) = Z^{(n)}(a:z) , \tag{17}$$

where

$$a^* = [s_2(0)/ t_2(0)]^{n/2} .$$

The elements of the matrix Σ^* satisfy the hypothesis of Lemma 7, so there exist positive constants δ and ε such that

$$|Z_*^{(n)}(a^*:\zeta)| > \delta|\zeta|^n, \qquad |\zeta| < \varepsilon .$$

From the definition of ζ,

$$|\zeta| \geq m_\Sigma^{-1}(0) \; |z|$$

so that

$$|Z^{(n)}(a:z)| = |Z_*^{(n)}(a^*:\zeta)| > \delta|\zeta|^n < \frac{\delta}{[m(0)]^n} |z|^n$$

whenever $|z| < \varepsilon \, m_\Sigma(0)$.

Theorem 16: Let f be a function Σ-monogenic in some domain D, and suppose f is not identically zero on D. Let $a \varepsilon D$ be a point such that $f(a) = 0$. Then there is a neighborhood $N(a)$ of a such that for all $z \varepsilon N(a)$, $f(z) \neq 0$, $(z \neq a)$.

Proof:

$$f(z) = \sum_{n=o}^{\infty} Z^{(n)}(a_n;a:z)$$

for $|z - a|$ sufficiently small.

$$Z^{(n)}(a_n;a:z) = 0, \qquad n \geq 1$$

$$Z^{(o)}(a_o;a:z) \equiv a_o ,$$

so $f(a) = 0$ implies that $a_o = 0$. Since f is not identically zero, there must be at least one $a_n \neq 0$. Let k be the smallest integer for which $a_k \neq 0$. Then

$$f(z) = \sum_{n=k}^{\infty} Z^{(n)}(a_n;a:z)$$

$$= Z^{(k)}(a_k;a:z) + \sum_{n=k+1}^{\infty} Z^{(n)}(a_n;a:z) .$$

$$|f(z)| \geq |Z^{(k)}(a_k;a:z)| - \sum_{n=k+1}^{\infty} |Z^{(n)}(a_n;a:z)| \tag{18}$$

since the formal power expansion is absolutely convergent.

Consider the ordinary power series

$$\sum_{n=k+1}^{\infty} a_n[2^{\frac{1}{2}} m_{\Sigma}(2|a| + r)]^n (z - a)^n \tag{19}$$

for any $r > 0$.

From Lemma 5, there is some positive constant C so that

$$|a_n| = \frac{|f^{[n]}(a)|}{n!} < C .$$

Thus

$$|a_n| [2^{\frac{1}{2}} m_{\Sigma}(2|a| + r)]^n \leq [C\, 2^{\frac{1}{2}} m(2|a| + r)]^n$$

$$\limsup_{n \to \infty} \sqrt[n]{|a_n|[2^{\frac{1}{2}} m_{\Sigma}(2|a| + r)]^n} \leq C\, 2^{\frac{1}{2}} m_{\Sigma}(2|a| + r) .$$

This implies that the power series (19) has a positive radius of convergence, say ε. From Lemma 2,

$$|Z^{(n)}(a_n;a:z)| \leq 2^{\frac{1}{2}}|a_n| [2^{\frac{1}{2}} m_{\Sigma}(|a| + |z|)]^n |z - a|^n$$

$$\leq 2^{\frac{1}{2}}|a_n| [2^{\frac{1}{2}} m_{\Sigma}(2|a| + \varepsilon)]^n |z - a|^n$$

for $|z - a| < \varepsilon$. Therefore

$$\sum_{n=k+1}^{\infty} |Z^{(n)}(a_n;a:z)| \leq 2^{\frac{1}{2}} \sum_{n=k+1}^{\infty} |a_n| [2^{\frac{1}{2}}m_{\Sigma}(2|a| + \varepsilon)]^n|z - a|^n \tag{20}$$

Let the function $G(z)$ be defined as follows:

$$G(z) = 2^{\frac{1}{2}} \sum_{n=k+1}^{\infty} |a_n|[2^{\frac{1}{2}}m_{\Sigma}(2|a| + \varepsilon)]^n |z - a|^n, \qquad |z - a) \leq \varepsilon/2 .$$

Estimates (20) and (18) then yield

$$|f(z)| \geq |Z^{(k)}(a_k;a:z)| - G(z) \tag{21}$$

for $|z - a|$ small enough.

Lemma 8 gives

$$|Z^{(k)}(a_k;a:z)| > \delta|z - a|^k$$

for some positive δ and $|z - a|$ sufficiently small. This, together with the estimate (21), implies that $|z - a|$ can be taken small enough to insure that

$$|f(z)| > \delta|z - a|^k - G(z) .$$

$G(z)$ is continuous in $G(a) = 0$, so $|z - a|$ can certainly be taken small enough to insure that

$$\frac{G(z)}{|z - a|^k} < \frac{\delta}{2} .$$

Then

$$|f(z) > \delta|z - a|^k - \frac{\delta}{2}|z - a|^k = \frac{\delta}{2}|z - a|^k ,$$

which proves the theorem.

<u>Corollary</u>: If f is Σ-monogenic in D and $f(z) = 0$ for all z of some set that has limit point in D, then $f(z) \equiv 0$ in D.

<u>Corollary</u>: If f and g are Σ-monogenic in D, and if $f(z) = g(z)$ for all z in some set that has a limit point in D, then $f(z) = g(z)$ for all z in D.

<u>Openness of Sigma-Monogenic Functions</u>.--In this section is proved that sigma-monogenic functions are open. First, two purely topological lemmas are proved.

<u>Lemma 9</u>: Let O be a bounded open set in E_2. Let f be a complex valued function continuous on the closure of O, $\bar{O}$, and open on O. If W is a component of the complement of $f(\bar{O} - O)$ such that $f(O) \cap W \neq \emptyset$, then $W \subset f(O)$.

Proof: f is open on O by hypothesis, so $W - f(O)$ is closed relative to W.

$\bar{O}$ is compact; therefore $f(\bar{O})$ is compact. Hence $f(\bar{O}) \cap W$ is closed relative to W.

$$f(\bar{O}) \cap W = [f(\bar{O} - O) \cup f(O)] \cap W$$

$$= [f(\bar{O} - O) \cap W] \cup [f(O) \cap W]$$

$$= f(O) \cap W ,$$

since $f(\bar{O} - O) \cap W = \emptyset$. Therefore $f(O) \cap W$ is closed relative to W.

$$W = [f(O) \cap W] \cup [W - f(O)] ,$$

so W is the union of two disjoint sets, each closed relative to W. W in connected and $f(O) \cap W \neq \emptyset$; therefore $W - f(O)$ must be empty. Thus $W \subset f(O)$, which proves the lemma.

<u>Lemma 10</u>: Suppose V is an open set in E_2 and $p \in V$. If f is a complex valued function continuous on V and open on $V - p$, then f is open on V.

Proof: Let G be an open set of V containing p. It will be shown that $f(p)$ is in the interior of $f(G)$.

G is open so there is an $r > 0$ so that the set

$$C = \{z \mid |z - p| = r\}$$

is contained in G. Let C^* denote the interior of C; that is,

$$C^* = \{z \mid |z - p| < r\} .$$

$f(C)$ is contained in the interior of $f(G)$ since f is by hypothesis open on $V - p$. Thus if $f(p) \varepsilon f(C)$, then $f(p)$ is in the interior of $f(G)$ and the lemma is proved.

Consider the case where $f(p) \not\varepsilon f(C)$. There is an $r' > 0$ so that the set H

$$H = \{w \mid |w - f(p)| < r'\}$$

is contained in the complement of $f(C)$ since the complement of $f(C) \cup f(p)$ is open. Since f is continuous at p, $f(G - C - p) \cap [H - f(p)]$ is not empty.

Let W be the component of the complement of $f(C + p)$ containing $H - f(p)$. Apply the previous lemma with $O = G - C - p$. Then

$$H - f(p) \subset W \subset f(O) = f(G - C - p) .$$

Hence,

$$H \subset f(G - C) ,$$

and H and $f(C)$ are disjoint, so $H \subset f(g)$, and the lemma is proved.

Theorem 17: A function, f, sigma-monogenic in some set D is open in D.

Proof: Let f = u + iv. The Jacobian of f is

$$J_f = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - v_x u_y$$

$$= (s_1/t_1)\, u_x^2 + (s_2/t_2)\, u_y^2$$

s_1, s_2, t_1, and t_2 are positive, so the Jacobian can vanish only where u_x and u_y are zero simultaneously. This together with Theorem 1 implies that the Jacobian is zero at exactly those points where the sigma-derivative $f^{\backprime}(z)$ of f is zero. The Σ-derivative is Σ′-monogenic, so the zeros are isolated (Theorem 16).

Hence the Jacobian, J_f, has isolated zeros, so that the function f is open except at isolated points. Let p be one of these points at which the Jacobian is zero. Let G be a neighborhood of p which does not contain any other zeros of J_f. Lemma 10 then implies that f is open on G. In other words, f is actually open at p. Since p was arbitrary zero of J_f, this shows f is open everywhere in D.

CHAPTER V

APPLICATIONS

As mentioned in the Introduction, there are numerous physical problems, particularly in the mechanics of continua, that give rise to systems of the form

$$
\begin{aligned}
s_1(x)\ u_x(x,y) &= t_1(y)\ v_y(x,y) \\
s_2(x)\ u_y(x,y) &= -t_2(y)\ v_x(x,y)
\end{aligned}
\tag{22}
$$

and the associated second order equations

$$
\begin{aligned}
[(t_1/s_1)\ v_y]_y + [(t_2/s_2)\ v_x]_x &= 0 \\
[(s_1/t_1)\ u_x]_x + [(s_2/t_2)\ u_y]_y &= 0\ .
\end{aligned}
\tag{23}
$$

It has been shown that any continuously differentiable solutions of the system (22) determines a Σ-monogenic function, and conversely, the real and imaginary parts of any Σ-monogenic function satisfy the system (22). Thus the Expansion Theorem (Theorem 15) gives a series respresentation for all solutions of equations (22) and (23). This can be very useful in actually finding solutions or properties of solutions of these equations.

Also important for applications is the fact that the Σ- Σ'-integral of a Σ'- Σ-monogenic function is itself Σ- Σ'-monogenic. The process of Σ-Σ'-integration can thus be used to generate particular solutions of (22) and (23) from a known solution.

<u>The Torsion Problem</u>.--To illustrate ways in which the properties of Σ-monogenic functions can be used in solving a particular applied problem, the so-called torsion problem for an elastic solid of revolution will be considered.

Consider a circular shaft of varying radius. Let r, θ, and z be the usual cylindrical coordinates, with the axis of the shaft coinciding with the z-axis. In other words, the shaft is the solid of revolution obtained by rotating a curve $r = f(z)$ about the z-axis. The problem of finding the displacements in the r, θ, and z directions when couples are applied to the ends of the shaft is the torsion problem.

Let the radial, tangential, and axial displacements be denoted by u_1, u_2, and u_3 respectively. It is known that $u_1 = u_3 = 0$, and

$$\frac{\partial}{\partial z}\left(\frac{u_2(z,r)}{r}\right) = \frac{1}{r^3}\frac{\partial F(z,r)}{\partial r}$$

$$\frac{\partial}{\partial r}\left(\frac{u_2(z,r)}{r}\right) = -\frac{1}{r^3}\frac{\partial F(z,r)}{\partial z} \tag{24}$$

where $F(z,r)$ is a so-called stress function. Because of axial symmetry, F and u_2 are independent of θ. The torsion problem consists of integrating the system (24) subject to the boundary condition $F(z,r) = 0$ on the curve $r = f(z)$. (Sokolnikoff [6])

It will now be seen how for a particular configuration the theory of sigma-monogenic functions provides a straightforward solution of this problem. The special case where $r = f(z)$ is a straight line will be considered. This, of course, means that the shaft is a right circular cone, or a frustum thereof.

There is no problem in finding particular solutions of the system (24). The difficulty is finding a solution such that $F(z,r) = 0$ on the boundary $r = f(z)$.

For the complex variable $\zeta = \xi + i\eta$, the first formal power $Z^{(1)}(1;\zeta_o:\zeta)$ has as its imaginary part a function which depends only on η:

$$Z^{(1)}(1;\zeta_o:\zeta) = \int_{\xi_o}^{\xi} [1/s_1(\xi)] + i \int_{\eta_o}^{\eta} \frac{1}{t_1(y)}\, dy \, .$$

This imaginary part will thus be constant along the line η = constant. This observation inspires one to seek a change of variables such that if the new variables are ρ and φ, then the boundary becomes either ρ = constant or φ = constant.

When the boundary is a straight line through the origin, this is easy. Let ρ and φ be the usual polar coordinates:

$$z = \rho \cos \varphi$$

$$r = \rho \sin \varphi \, .$$

In this case, φ = constant along the boundary $r = f(z)$.

Introduce the new function $v = u_2/r$. Let

$$\tilde{F}(\rho,\varphi) = F(\rho \cos \varphi, \rho \sin \varphi)$$

$$\tilde{v}(\rho,\varphi) = v(\rho \cos \varphi, \rho \sin \varphi) .$$

Then

$$\tilde{F}_\varphi = F_z \rho \sin \varphi + F_r \rho \cos \varphi$$

$$\tilde{v}_\rho = v_z \cos \varphi + v_r \sin \varphi .$$

Using equations (24) gives

$$\tilde{F}_\varphi = (\rho^3 \sin^3 \varphi\, v_r) \rho \sin \varphi + (\rho^3 \sin^3 \varphi)^{V_z} \rho \cos \varphi$$

$$\left(\frac{1}{\sin^3 \varphi}\right) \tilde{F}_\varphi = \rho^4 \sin \varphi\, v_r + \rho^4 \cos \varphi\, v_z$$

$$= \rho^4 \tilde{v}_\rho .$$

Similarly

$$\left(\frac{1}{\sin^3 \varphi}\right) \tilde{F}_\rho = - \rho^2 \tilde{v}_\varphi .$$

Hence the function $G(\zeta) = v(\rho,\varphi) + i F(\rho,\varphi)$ is a sigma-monogenic function of $\zeta = \rho + i \varphi$ with

$$\Sigma = \begin{pmatrix} \rho^4 & \frac{1}{\sin^3 \varphi} \\ \rho^2 & \frac{1}{\sin^3 \varphi} \end{pmatrix} .$$

As was pointed out above, the imaginary part of the first Σ formal power $Z^{(1)}(1;\zeta_o:\zeta)$ will be constant on φ = constant and will give the solution to the torsion problem for the conical shaft. It remains only to calculate $\operatorname{Im} Z^{(1)}(1;\zeta_o:\zeta)$. For convenience take $\eta_o = \pi/2$.

$$\begin{aligned}\operatorname{Im} Z^{(1)}(1;\zeta_o:\zeta) &= \int_{\pi/2}^{\varphi} \frac{1}{t_1(y)}\,dy \\ &= \int_{\pi/2}^{\varphi} \sin^3 y\,dy \\ &= -\cos\varphi + \frac{1}{3}\cos^3\varphi .\end{aligned}$$

Any constant multiple of $\operatorname{Im} Z^{(1)}(1;\zeta_o:\zeta)$ will thus be an acceptable stress function F. Returning to z-r coordinates gives

$$F(z,r) = k\left\{\frac{z}{(r^2+z^2)^{\frac{1}{2}}} - \frac{1}{3}\left[\frac{z}{(r^2+z^2)^{\frac{1}{2}}}\right]^3\right\}$$

as the solution desired.

BIBLIOGRAPHY

1. Bers, L., and Gelbart, A., "On a Class of Differential Equations in Mechanics of Continua," Quarterly of Applied Mathematics, Volume 1, 1943, pp. 168-188.

2. Bers, L., and Gelbart, A., "On a Class of Functions Defined by Partial Differential Equations," Transactions of the American Mathematical Society, Volume 56, 1944, pp. 67-93.

3. Bers, L., "An Outline of the Theory of Pseudoanalytic Functions," Bulletin of the American Mathematical Society, Volume 62, 1956, pp. 291-331.

4. John, F., "On Linear Partial Differential Equations with Analytic Coefficients," Communications on Pure and Applied Mathematics, Volume 2, 1949, pp. 209-253.

5. Petrovsky, I. G., Lectures on Partial Differential Equations. New York, New York: Interscience Publishers, Inc., 1954.

6. Sokolnikoff, I. S., Mathematical Theory of Elasticity. New York, New York: McGraw-Hill Book Company, Inc., 1956.

7. Whyburn, G. T., Topological Analysis. Princeton, New Jersey: Princeton University Press, 1958.