EVOLUTION SYSTEM APPROXIMATIONS OF SOLUTIONS TO CLOSED LINEAR OPERATOR EQUATIONS

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Seaton D. Purdom

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EVOLUTION SYSTEM APPROXIMATIONS OF SOLUTIONS

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Approved:

J. V. Herod, Chairman

N. Chafee

E. R. Immel

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TABLE OF CONTENTS

SUMMARY

With S a linearly ordered set with the least upper bound property, with g a non-increasing real-valued function on S, and with A densely defined dissipative linear operator, an evolution system M is developed to solve the modified Stieltjes integral equation $M(s,t)x = x + A((L)_s\int^t dgM(\cdot,t)x)$. An affine version of this equation is also considered. Under the hypothesis that the evolution system associated with the linear equation is strongly (resp. weakly) asymptotically convergent, an evolution system is used to strongly (resp. weakly) approximate solutions to the closed operator equation $Ay = -z$.

CHAPTER I

INTRODUCTION

If $\underline{\bar{X}}$ is a Banach space, if A is a linear function from D(A) in $\underline{\bar{X}}$ to $\underline{\bar{X}}$, and if g is a function from the real numbers, R, to R which is of bounded variation on each finite interval of R, then the integral equation

$$(1)\quad M(t,0)x = x + \int_t^0 dgAM(\cdot,0)x$$

permits a highly detailed theory. In case A is continuous, the theory for the modified Stieltjes integral equation

$$(2)\quad M(t,0)x = x + (R)\int_t^0 dgAM(\cdot,0)x$$

is subsumed by Mac Nerney in [6]. In case A is continuous and each of $(I-(g(s^-)-g(s))A)^{-1}$ and $(I-(g(s)-g(s^+))A)^{-1}$ exists for each s in R, much of the theory for

$$(3)\quad M(t,0)x = x + (L)\int_t^0 dgAM(\cdot,0)x$$

is subsumed by results due to Herod in [4]. If A is dissipative, linear, and has dense domain, and if g = -I, then equations (1), (2), and (3) are equivalent and the theory has been developed in great detail. Yosida in [11] gives a thorough account. In Chapter III here, under the hypotheses that A is linear, dissipative, and densely defined, and that g is non-increasing, the theory for equation (3), and for an affine version of (3), is offered.

The motivation for the development of the detailed theory of Chapter III is to obtain in Chapter IV an iterative process to solve the equation

$$(4) \quad Ay = z$$

for y. In [1], Browder and Petryshyn consider the related equation

$$(5) \quad y - Ty = z \ .$$

Under the hypotheses that T is a continuous linear operator from $\overline{\underline{X}}$ to $\overline{\underline{X}}$ and that $\lim_{n\to\infty} T^n x$ exists for each x in $\overline{\underline{X}}$, it is established in [1] that if z is in the range of (I-T), then the iterative process $x_{n+1} = z + Tx_n$ converges to a solution y of (5). Contained in [2] is a weak convergence version of [1]. In [9], Martin generalizes the Browder-Petryshyn paper to solve (4) with A continuous, using the product integral techniques of Mac Nerney [6]. What is offered in Chapter IV is a generalization of Martin's results to the present setting in which A is linear, dissipative, and densely defined. Strongly and weakly convergent iterative processes for (4) are discussed. Also, a test is given to determine whether z in (4) is in the range of A.

The results here are most closely parallel to those given by Mac Nerney in [8], by Herod in [4], and by Martin in [9]. What follows is a detailed summary of the results of each.

In Mac Nerney [8], one lets S be a linearly ordered set and lets OA^+ be the collection of functions from H = {(s,t) in S × S such that $s \geq t\}$ to R to which α belongs only in case

(i) if (x,y) is in H, then $\alpha(x,y) \geq 0$, and

(ii) if each of (x,y) and (y,z) is in H, then $\alpha(x,y) + \alpha(y,z) = \alpha(x,z)$.

One lets OM^+ be the collection of functions from H to R to which μ belongs only in case

(i) if (x,y) is in H, then $\mu(x,y) \geq 1$, and

(ii) if each of (x,y) and (y,z) is in H, then $\mu(x,y)\mu(y,z) = \mu(x,z)$.

Established first is that there is a reversible function E^+ from OA^+ to OM^+ such that

(i) if (s,t) is in H, if $\varepsilon > 0$, and if $E^+(\alpha) = \mu$, then there is a subdivision $\{w_p\}_{p=0}^{m}$ of $\{s,t\}$ such that if $\{u_p\}_{p=0}^{n}$ is a refinement of $\{w_p\}_{p=0}^{m}$, then

$$\mu(s,t) - \varepsilon < \prod_{p=1}^{m} (1 + \alpha(w_{p-1},w_p)) \leq \prod_{p=1}^{n} (1 + \alpha(u_{p-1},u_p)) \leq \mu(s,t)$$

and

$$\alpha(s,t) + \varepsilon > \sum_{p=1}^{m} (\mu(w_{p-1},w_p) - 1) \geq \sum_{p=1}^{n} (\mu(u_{p-1},u_p) - 1) \geq \alpha(s,t),$$

and

(ii) if t is in S and if $E^+(\alpha) = \mu$, then $\mu(\cdot,t)$ is the only function $f(\cdot)$ which is of bounded variation on each interval $\{\xi,t\}$ with $\xi \geq t$ and which satisfies the integral equation

$$f(\xi) = 1 + (R)\int_{\xi}^{t} \alpha[f] \quad \text{for each } \xi \geq t \text{ -- the latter}$$

integral being the limit, in the sense of successive refinements, of estimates $\sum_{p=1}^{n} \alpha(u_{p-1},u_p)\, f(u_p)$, based on subdivisions $\{u_p\}_{p=0}^{n}$ of the interval $\{\xi,t\}$.

One next lets OA be the collection of functions from H to the Lipschitz functions from $\underline{\overline{X}}$ to $\underline{\overline{X}}$ to which V belongs only in case

(i) there is an α in OA^+ such that if (s,t) is in H and if each of x and y is in $\underline{\overline{X}}$, then $|V(s,t)x - V(s,t)y| \leq \alpha(s,t)|x - y|$,

(ii) $V(s,t)0 = 0$ for each (s,t) in H, and

(iii) if each of (s,t) and (t,u) is in H, then $V(s,t) + V(t,u) = V(s,u)$;

and one lets OM be the collection of functions from H to the Lipschitz functions from $\underline{\overline{X}}$ to $\underline{\overline{X}}$ to which M belongs only in case

(i) there is a μ in OM^+ such that if (s,t) is in H and if each of x and y is in $\underline{\overline{X}}$, then

$$|(M(s,t)x - x) - (M(s,t)y - y)| \leq (\mu(s,t) - 1)|x - y|,$$

(ii) $M(s,t)0 = 0$ for each (s,t) in H, and

(iii) if each of (s,t) and (t,u) is in H, then $M(s,t)M(t,u) = M(s,u)$.

The main result in [8] is the following:

Theorem. There is a reversible function E from OA to OM such that each of the following is true:

(i) if $E(V) = M$, if (s,t) is in H, if x is in $\underline{\overline{X}}$, and if $\varepsilon > 0$, then there is a subdivision $\{w_p\}_{p=0}^{m}$ of $\{s,t\}$ such that if $\{u_p\}_{p=0}^{n}$ is a refinement of $\{w_p\}_{p=0}^{m}$, then

$$\left|M(s,t)x - \prod_{p=1}^{n} (I + V(u_{p-1},u_p))\right| < \varepsilon \quad \text{and}$$

$$\left|V(s,t)x - \sum_{p=1}^{n} (M(u_{p-1},u_p) - I)x\right| < \varepsilon, \text{ and}$$

(ii) if t is in S and x is in $\underline{\overline{X}}$, then $M(\cdot,t)x$ is the only solution $f(\cdot)$ of the integral equation

$$f(\xi) = x + (R)\int_{\xi}^{t} Vf(\cdot), \text{ each } \xi \geq t,$$

which is of bounded variation on each interval $\{\xi,t\}$ with $\xi \geq t$.

In [4] Herod relaxes the condition that the order-additive function V have Lipschitzian values. One supposes that S is a set with a linear ordering such that $\{S,\geq\}$ has the least upper bound property and that $\{G,+,|\cdot|\}$ is a complete normed Abelian group. If D is a closed subset of G, and if V is a function such that if (x,y) is in H, then V(x,y) is a function from D to G such that

(i) if each of (x,y) and (y,z) is in H, and if P is in D, then $V(x,y)P + V(y,z)P = V(x,z)P$,

(ii) if (a,b) is in H, then there is a non-decreasing function β from S to R such that if P is in D and $\varepsilon > 0$, then there is a number $\delta > 0$ such that $|Q - P| < \delta$ implies $|V(x,y)P - V(x,y)Q| \leq (\beta(x) - \beta(y))\varepsilon$ whenever $a \geq x \geq y \geq b$,

(iii) if $a > b$, then D is contained in the range of $(I - V(a,b))$; and if P and Q are in D, then $|(I - V(a,b))P - (I - V(a,b))Q| \geq |P - Q|$, and

(iv) if $a > b$ and P is in D, then there is a non-decreasing function α from S to R such that if $\{s_p\}_{p=0}^{n}$ is a non-increasing sequence in [b,a] and $a \geq x \geq y \geq b$, then

$$|V(x,y) \prod_{p=1}^{n} (I - V(s_{p-1},s_p))^{-1}P| \leq \alpha(x) - \alpha(y),$$

it then follows from the properties of V that the evolution system M defined by $M(x,y)P = \prod_x^y (I - V)^{-1}P$ exists and that if b is in S and P is in D, then the only function g which is of bounded variation on each finite interval of S and which satisfies

$$g(x) = P + (L)\int_x^b V[g] \quad \text{for each } x \geq b$$

is given by $g(x) = M(x,b)P$. The application of [4] to (3) is immediate provided A is continuous, linear, and dissipative for one may put $V(x,y) = (g(y) - g(x))A$, taking for $\{G,+,|\cdot|\}$ the additive group of continuous linear operators on $\underline{X}$ with the usual norm. In case A is continuous, linear, and $(I - (g(s^-) - g(s^+))A)^{-1}$ exists for each s in R (Harsher restrictions on g are needed if $\{S,\geq\}$ does not have the least upper bound property), let $\beta > 0$ be such that $A - \beta I$ is dissipative. If $\{u,v\}$ is an interval in $\{S,\geq\}$ such that $g(v) - g(u) < \beta^{-1}$, then $(I - (g(v) - g(u))A)^{-1} = (1 - \beta(g(v) - g(u)))^{-1}\left(I - \frac{g(v) - g(u)}{1 - \beta(g(v) - g(u))}(A - \beta I)\right)^{-1}$. Application of [4] with $V(s,t) = \left[\int_s^t dg(1 - \beta dg)^{-1}\right](A - \beta I)$ for each $\{s,t\}$ such that $u \geq s \geq t \geq v$ now gives that $\prod_u^v (I - dgA)^{-1}$ exists. On a given interval $\{x,y\}$ in $\{S,\geq\}$, there are at most finitely many s such that $(g(s^-) - g(s^+)) \geq \beta^{-1}$, leading quickly to the fact that $\prod_x^y (I - dgA)^{-1}$ exists and solves equation (3). A separate proof of the results from [4] to be used here appears in Chapter III below. It should be noted that the least upper bound property on $\{S,\geq\}$ needed in [4] is not needed here.

In [9], Martin generalizes the results of Browder and Petryshyn to solve (4) with A continuous, using the product integral techniques of Mac Nerney [6]. One supposes that g is a non-increasing function from S to R with $\lim_{t\to+\infty} g(t) = -\infty$. The evolution system M defined by $M(s,t)x = \prod_s^t (I + dgA)x$ is said to be strongly asymptotically convergent only in case $\lim_{t \to +\infty} M(t,0)x$ exists for each x in $\overline{\underline{X}}$. Under the hypothesis that M is strongly asymptotically convergent, it is shown that Q, defined by $Qx = \lim_{t \to +\infty} M(t,0)x$, is a continuous projection of $\overline{\underline{X}}$ onto the null space of A and that the closure of the range of A is the null space of Q. Using Q one can then show that if z is in $\overline{\underline{X}}$ and $W(s,t)x = {}_s\Pi^t (I + dg(A + z))x$, then these are equivalent: (i) z is in the range of A; (ii) for each x in $\overline{\underline{X}}$, $\lim_{t \to +\infty} W(t,0)x$ exists and is a solution y_∞ of $Ay = -z$; and (iii) there is an increasing, unbounded sequence $\{t_k\}_{k=1}^{\infty}$ in S and an x in $\overline{\underline{X}}$ such that $\text{w-}\lim_{k \to +\infty} W(t_k,0)x$ exists. It was the desire to extend the results of Martin to discontinuous operators A which led to the study here.

In furtherance of this topic, using techniques developed here, Lovelady in [5] accounts for a class of ergodic methods with application to equation (4) in case A is linear, dissipative, and densely defined. Especially, Lovelady has shown that if $\overline{\underline{X}}$ is reflexive and $C(n,0)x = \int_0^n n^{-1} \exp(tA)x\, dt$, then $\lim_{n \to +\infty} C(n,0)x$ exists and a strongly convergent approximation scheme for (4) is yielded.

CHAPTER II

NOTATION AND PRELIMINARY COMPUTATIONS

Notation. Let $\underline{X}$ be a Banach space with norm $|\cdot|$, and let I be the identity map on $\underline{X}$. The norm of a continuous linear transformation B from $\underline{X}$ to $\underline{X}$ is also denoted by $|B|$, i.e., $|B| = \sup \{|Bx| : x \text{ is in } \underline{X} \text{ and } |x| = 1\}$. If A is a linear function from a subset D(A) of $\underline{X}$ to $\underline{X}$, then A is said to be dissipative only in case for each $\lambda > 0$,

(i) $(I - \lambda A)^{-1}$ exists and has domain all of $\underline{X}$ and

(ii) $|(I - \lambda A)^{-1}| \leq 1$.

Henceforth, A will always denote a dissipative linear function from D(A) in $\underline{X}$ to $\underline{X}$ such that D(A) is dense in $\underline{X}$.

In what follows, elements from $\{(I - \lambda A)^{-1} : \lambda > 0\}$ appear frequently. As in Yosida [11], the notational convention--$J_n = (I - n^{-1}A)^{-1}$, each $n > 0$--is made. The Hille-Yosida Theorem gives that, for $t \geq 0$, exp(tA) exists, is non-expansive, and has many other well-known properties. Results along the same lines using continued products form the first part of the discussion which follows.

Henceforth, S denotes a linearly ordered set with the least upper bound property; R, the real numbers; and g, a non-decreasing function from S to R. (The least upper bound property on S is dropped in a special subsection of Chapter III. The reader wishing to cite Herod's results and thus move quickly to the main results here will wish to keep the least upper bound property.) If F is a function from D(F) in $\underline{\overline{X}}$ to $\underline{\overline{X}}$, and if $\{r_p\}_{p=0}^m$ is a sequence in S, then $\prod_{p=j}^m (I - (g(r_p) - g(r_{p-1}))F)^{-1}$ is defined inductively by

$$\prod_{p=j}^m (I - (g(r_p) - g(r_{p-1}))F)^{-1} =$$

$$(I - (g(r_j) - g(r_{j-1}))F)^{-1} \prod_{p=j+1}^m (I - (g(r_p) - g(r_{p-1}))F)^{-1}$$

with the agreement that $\prod_{p=m+1}^m (I - (g(r_p) - g(r_{p-1}))F)^{-1} = I$. When the sequence $\{r_p\}_{p=0}^m$ is clear, $\prod_{p=j}^m (I - dg_p F)^{-1}$ is written in place of the more cumbersome $\prod_{p=j}^m (I - (g(r_p) - g(r_{p-1}))F)^{-1}$. Occasionally, one writes $\Pi_r(I - dgF)^{-1}$ in place of $\prod_{k=1}^m (I - dg_k F)^{-1}$. If each of s and t is in S with $s \geq t$, and if x is in $\underline{\overline{X}}$, then by ${}_s\Pi^t\ (I - dgF)^{-1}x$, one means the limit, in the sense of successive refinements of subdivisions $\{r_p\}_{p=0}^m$ of $\{s,t\}$, of estimates $\prod_{p=1}^m (I - dg_p F)^{-1}s$. In particular, if z is in $\underline{\overline{X}}$, then $z = {}_s\Pi^t\ (I - dgF)^{-1}x$ only in case for each $\varepsilon > 0$, there is a subdivision $\{r_p\}_{p=0}^m$ of $\{s,t\}$ such that if $\{u_p\}_{p=0}^n$ is a refinement of $\{r_p\}_{p=0}^m$, then $|z - \Pi_u(I - dgF)^{-1}x| < \varepsilon$. In case H is a function from S to $\underline{\overline{X}}$, $(L)\int_s^t dgFH(\cdot)$ refers to the limit, in the sense of successive refinements, of estimates $\sum_{p=1}^{m} dg_p FH(r_{p-1})$; and $(R)\int_s^t dgFH(\cdot)$, of estimates $\sum_{p=1}^{m} dg_p FH(r_p)$.

Preliminary Computations. Three observations needed in Yosida's development of exp(tA) are useful here.

Lemma 0.1. Suppose that B is a continuous linear transformation from $\underline{\overline{X}}$ to $\underline{\overline{X}}$. These are equivalent:

(i) For each $\lambda > 0$, $(I - \lambda B)^{-1}$ exists, has domain all of $\underline{\overline{X}}$, and $|(I - \lambda B)^{-1}| \leq 1$,

(ii) For each $t \geq 0$, $|\exp(tB)| \leq 1$.

Proof. Since B is continuous, one has three representations for exp(tB), i.e.

$$\exp(tB) = \sum_{k=0}^{+\infty} (k!)^{-1} t^k B^k$$

$$= \lim_{n \to +\infty} (I + \frac{t}{n} B)^n$$

$$= \lim_{k \to +\infty} (I - \frac{t}{n} B)^{-k}$$

From the last representation, it is clear that if (i) is true, then $|\exp(tB)| \leq 1$ for each $t \geq 0$.

In case (ii) is true and $\beta > 0$, one may put $H_\beta = \int_0^{+\infty} \beta e^{-\beta s}\exp(sB)\,ds$. Then $|H_\beta| \leq \int_0^{+\infty} \beta e^{-\beta s}|\exp(sB)|\,ds \leq \int_0^{+\infty} (-e^{-\beta s})B\exp(sB)\,ds = 1$. Noting that

$$\begin{aligned}(I - \beta^{-1}B)H_\beta &= H_\beta + \int_0^{+\infty} (-e^{-\beta s})B\exp(sB)\,ds \\ &= H_\beta + [-e^{-\beta s}\exp(sB)]\Big|_0^{+\infty} - \int_0^{\infty} \beta e^{-\beta s}\exp(sB)\,ds \\ &= I\end{aligned}$$

and putting $\beta = \lambda^{-1}$ gives that (ii) implies (i); and the lemma is established.

<u>Lemma 0.2</u>. For each x in $\overline{X}$, $\lim_{n\to+\infty} J_n x = x$.

<u>Proof</u>. Since $J_n = (I - n^{-1}A)^{-1}$ for each $n > 0$, one has $(I - n^{-1}A)(I - n^{-1}A)^{-1} = I$, so

$$(I - n^{-1}A)^{-1}x = x + n^{-1}A(I - n^{-1}A)^{-1}x.$$

If x is in D(A), then A and J_n commute, so $J_n x - x = n^{-1}J_n Ax$. Hence, $|J_n x - x| \leq n^{-1}|Ax|$ from which it follows that $\lim_{n\to+\infty} J_n x = x$ if x is in D(A). If x is not in D(A) and $\varepsilon > 0$, then there is an element y in D(A) such that $|x - y| < \varepsilon/4$. One has

$$\begin{aligned}|(J_n - I)x| &\leq |(J_n - I)(x - y)| + |(J_n - I)y| \\ &\leq (|J_n|+|I|)|x - y| + (J_n - I)y \\ &\leq 2|x - y| + |(J_n - I)y|. \\ &\leq \frac{\varepsilon}{2} + n^{-1}|Ay|.\end{aligned}$$

There exists an N so that $n^{-1}|Ay| < \varepsilon/2$ for each $n > N$. Hence, $|J_n x - x| < \varepsilon$ for each $n > N$, and the result follows at once.

Lemma 0.3. If A is a linear function from D(A) in $\overline{X}$ to $\overline{X}$, and if, for some $\lambda > 0$, $(I - \lambda A)^{-1}$ exists, has domain all of $\overline{X}$, and is continuous, then A is closed.

Proof. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence in D(A), and that $\lim_{n\to\infty} x_n = y$ and $\lim_{n\to\infty} Ax_n = P$. Then

$$y - \lambda P = \lim_{n\to+\infty} x_n - \lim_{n\to+\infty} \lambda A x_n$$

$$= \lim_{n\to+\infty} (I - \lambda A)x_n .$$

Since $(I - \lambda A)^{-1}$ is continuous and has domain all of $\overline{X}$, one has that $(I - \lambda A)^{-1}(y - \lambda P) = \lim_{n\to+\infty} x_n = y$. So y is in $D(I - \lambda A)$ and, hence, is in D(A). One has $(I - \lambda A)y = y - \lambda P$, so $Ay = P$.

Several computations which are needed frequently in the development of $\Pi(I - dgA)^{-1}$ in the present case are now summarized in Lemma 1.

Lemma 1. Suppose that x_o is in D(A), y is in $\overline{X}$, $n > 0$, and $\{\lambda_k\}_{k=1}^{m}$ is a sequence of non-negative numbers. Then for each $k = 1,2,\ldots,m$,

(i) $(I - \lambda_k AJ_n)^{-1}$ exists, has domain all of $\overline{X}$, is continuous, and $|(I - \lambda_k AJ_n)^{-1}| \leq 1$,

(ii) $|(I - \lambda_k AJ_n)^{-1} x_o - (I - \lambda_k A)^{-1} x_o| \leq \lambda_k |(J_n - I)Ax_o|$,

(iii) $|\prod_{k=1}^{m} (I - \lambda_k AJ_n)^{-1} x_o - \prod_{k=1}^{m} (I - \lambda_k A)^{-1} x_o|$

$\leq (\sum_{k=1}^{m} \lambda_k) \ |(J_n - I)Ax_o|$, and

(iv) $|\prod_{k=1}^{m} (I - \lambda_k AJ_n)^{-1} y - \prod_{k=1}^{m} (I - \lambda_k A)^{-1} y|$

$\leq \inf_{x\varepsilon D(A)} \{2|x - y| + |(J_n - I)Ax|\}$

Proof. First, $AJ_n = n(J_n - I)$ so AJ_n is continuous. Hence, $\exp(tAJ_n) = \exp(tn(J_n - I)) = \exp(-nt) \exp(tnJ_n)$. Now $|\exp(tnJ_n)| = |\sum_{k=0}^{\infty} (k!)^{-1}(nt)^k J_n{}^k| \leq \sum_{k=0}^{\infty} (k!)^{-1}(nt)^k = \exp(nt)$, so $|\exp(tAJ_n)| \leq 1$ and the normed estimate of (i) follows from Lemma 0.1.

Since A is linear, the operators A, $(I - \lambda_k AJ_n)^{-1}$, and $(I - \lambda_k A)^{-1}$ all commute. The resultant identity,

$$(I - \lambda_k AJ_n)^{-1}x_o - (I - \lambda_k A)^{-1}x_o =$$

$$(I - \lambda_k AJ_n)^{-1}(I - \lambda_k A)^{-1}\lambda_k (J_n - \lambda I)Ax_o,$$

gives (ii) at once.

For differences of products, one has

$$|\prod_{k=1}^{m}(I - \lambda_k AJ_n)^{-1}x_o - \prod_{k=1}^{m}(I - \lambda_k A)^{-1}x_o| =$$

$$|\sum_{j=1}^{m}\left\{\prod_{k=1}^{j} (I - \lambda_k AJ_n)^{-1} \prod_{k=j+1}^{m} (I - \lambda_k A)^{-1}x_o - \right.$$

$$\prod_{k=1}^{j-1} (I - \lambda_k AJ_n)^{-1} \prod_{k=j}^{m} (I - \lambda_k A)^{-1} x_o\}| \leq$$

$$\sum_{j=1}^{m} |\prod_{k=1}^{j-1} (I - \lambda_k AJ_n)^{-1}|| \prod_{k=j+1}^{m} (I - \lambda_k A)^{-1}||(I - \lambda_j AJ_n)^{-1} x_o - (I - \lambda_j A)^{-1} x_o|$$

$$\leq \sum_{j=1}^{m} \lambda_j |(J_n - I)Ax_o| \text{ from (ii).}$$

Finally, for each x in D(A),

$$|\prod_{k=1}^{m} (I - \lambda_k AJ_n)^{-1} y - \prod_{k=1}^{m} (I - \lambda_k A)^{-1} y| \leq$$

$$|\prod_{k=1}^{m} (I - \lambda_k AJ_n)^{-1} (x - y)| + |\prod_{k=1}^{m} (I - \lambda_k AJ_n)^{-1} (x - y)| +$$

$$|\prod_{k=1}^{m} (I - \lambda_k AJ_n)^{-1} x - \prod_{k=1}^{m} (I - \lambda_k A)^{-1} x| \leq$$

$2|x - y| + (\sum_{k=1}^{m} \lambda_k)|(J_n - I)Ax|$. Since this inequality holds for each x in D(A), (iv) follows at once.

CHAPTER III

DEVELOPMENT OF THE PRODUCT INTEGRALS

The Linear Case. A linear realization of a non-linear result due to Herod [4] is the following, which will be generalized in this paper:

Theorem 2. Suppose that g is a non-increasing function from S to R and that B is a continuous, dissipative, affine transformation from $\underline{\overline{X}}$ to $\underline{\overline{X}}$. If each of s and t is in S with $s \geq t$, then

(i) $M(s,t) = {}_s\Pi^t(I - dgB)^{-1}$ exists,

(ii) If each of x and y is in $\underline{\overline{X}}$, then $|M(s,t)x - M(s,t)y| \leq |x - y|$,

(iii) If r is in S and $s \geq r \geq t$, then $M(s,r)M(r,t) = M(s,t)$,

(iv) If x is in $\underline{\overline{X}}$, then $M(\cdot,t)x$ is the only function F which is of bounded variation on each finite interval of S and which is a solution of the integral equation

$$F(s) = x + (L)\int_s^t dgBF(\cdot).$$

Remark 2.1. It will be seen that the convergence of the estimates to ${}_s\Pi^t(I - dgB)^{-1}$ is uniform in the following sense:

Let each of s and t be in S with $s \geq t$, and let each of ε and c be positive. There exists a subdivision $\{r_p\}_{p=0}^{n}$ of $\{s,t\}$ such that if

x is in $\underline{\overline{X}}$ with $|x| \leq c$, and if $\{v_p\}_{p=0}^{m}$ is a refinement of $\{r_p\}_{p=0}^{n}$, then

$$\left| {}_{v_k}\Pi^t (I - dgB)^{-1}x - \prod_{p=k+1}^{m} (I - (g(v_p) - g(v_{p-1}))B)^{-1}x \right| < \varepsilon$$

and

$$\left| {}_{s}\Pi^{v_k} (I - dgB)^{-1}x - \prod_{p=1}^{k} (I - (g(v_p) - g(v_{p-1}))B)^{-1}x \right| < \varepsilon$$

for $k = 0,1,\ldots,m$.

Remark 2.2. The assertion of uniqueness in (iv) of Theorem 2 is not done here. First, it is not crucial to the proofs of Theorems 3 and 5. Second, the inequalities which establish uniqueness appear at the end of the proof of Theorem 3. Since the assertion of uniqueness, established for discontinuous operators A later, implies that assertion in the continuous case, there is little to be gained from a uniqueness proof here.

Theorem 2 has an exceedingly complex proof. One already familiar with the result may wish to move directly to Theorem 3. The proof given below does have the advantage that it does not require that S have the least upper bound property. Hence, Theorem 2 is here improved. The proof is simplified somewhat by making the following observations first.

Computational Observations. Suppose that L is linear and dissipative, that $\beta \geq \alpha \geq 0$, that $\{\lambda_k\}_{k=1}^{n}$ is a non-negative number sequence, that each of x and z is in $\underline{\overline{X}}$, and that y is in $D(L^2)$. Then

(i) $(I - \beta(L + z))^{-1}x = (I - \beta L)^{-1}(x + \beta z)$,

(ii) $\prod_{k=1}^{n} (I - \lambda_k(L + z))^{-1}x = \prod_{k=1}^{n} (I - \lambda_k L)^{-1}x +$

$$\sum_{k=1}^{n} \prod_{p=1}^{k} (I - \lambda_p L)^{-1}\lambda_k z,$$

(iii) $|(I - \beta L)^{-1}y - \exp(\beta L)y| \leq \frac{1}{2} \beta^2 |L^2 y|$,

(iv) $|\prod_{k=1}^{n} (I - \lambda_k L)^{-1}y - \prod_{k=1}^{n} \exp(\lambda_k L)y| \leq \frac{1}{2} \sum_{k=1}^{n} \lambda_k{}^2 |L^2 y|$

$$\leq \frac{1}{2} (\sum_{k=1}^{n} \lambda_k)^2 |L^2 y|,$$

(v) $|\sum_{k=1}^{n} (\prod_{p=1}^{k} (I - \lambda_p L)^{-1})\lambda_k y - \sum_{k=1}^{n} (\prod_{p=1}^{k} \exp(\lambda_p L))\lambda_k y|$

$$\leq \frac{1}{2} (\sum_{k=1}^{n} \lambda_k)^3 |L^2 y|,$$

(vi) $|\exp(\beta L)y - y| \leq \beta |Ly|$,

(vii) $|\exp(\sum_{k=1}^{n} \lambda_k L)(\sum_{k=1}^{n} \lambda_k)y - \sum_{k=1}^{n} [\prod_{p=1}^{k} \exp(\lambda_p L)]\lambda_k y|$

$\leq (\sum_{k=1}^{n} \lambda_k)^2 |Ly|$, and

(viii) If $\sum_{k=1}^{n} \lambda_k = \beta - \alpha$, then

$$|(I - \beta L)^{-1}y - [\prod_{k=1}^{n} (I - \lambda_k L)^{-1}](I - \alpha L)^{-1}y|$$

$$\leq (\beta - \alpha)\alpha |L^2 y| + (\beta - \alpha)^2 |L^2 y|.$$

Proof. First, if $(I - \beta(L + z))^{-1}x = P$, then $x = (I - \beta L)P - \beta z$ and (i) follows. Repeated application of (i) gives (ii) since

$$\prod_{k=1}^{n} (I - \beta(L + z))^{-1}x = (I - \lambda_1 L)^{-1}\lambda_1 z + (I - \lambda_1 L)^{-1} \prod_{p=2}^{n} (I - \lambda_p(L + z))^{-1}x$$

$$= \prod_{k=1}^{n} (I - \lambda_k L)^{-1}x + \sum_{k=1}^{n} \prod_{p=1}^{k} (I - \lambda_p L)^{-1}\lambda_k z \ .$$

The inequality of (iii) is due to Trotter [10]. One has

$(I - \beta L)^{-1}y - (I - \frac{\beta}{2} L)^{-2}y = (\frac{\beta}{2})^2(I - \frac{\beta}{2} L)^{-2}(I - \beta L)^{-1}L^2y$, so

$|(I - \beta L)^{-1}y - (I - \frac{\beta}{2} L)^{-2}y| \leq (\frac{\beta}{2})^2|L^2y|$. Now,

$$|(I - \beta 2^{-n}L)^{-2^n}y - (I - \beta 2^{-n-1}L)^{-2^{n+1}}y| =$$

$$|\sum_{k=1}^{2^n} (I - \beta 2^{-n}L)^{-k+1}[(I - \beta 2^{-n}L)^{-1} - (I - \beta 2^{-n-1}L)^{-2}] \cdot$$

$$[(I - \beta 2^{-n-1}L)^{-2}]^{2^n-k}y\}| \leq$$

$$\sum_{k=1}^{2n} |(I - \beta 2^{-n}L)^{-1}y - (I - \beta 2^{-n-1}L)^{-2}y| \leq$$

$$(2^n)(\beta 2^{-n-1})^2|L^2y| = \frac{1}{4}\beta^2 2^{-n}|L^2y| \ .$$

Since $\exp(\beta L)y = \lim_{n\to\infty} (I - \beta 2^{-n}L)^{-2^n}y$, one has $|(I - \beta L)^{-1}y - \exp(\beta L)y| \leq \sum_{n=0}^{+\infty} \frac{1}{4}\beta^2 2^{-n}|L^2y| = \frac{1}{2}\beta^2|L^2y|$.

One extends observation (iii) to a non-negative number sequence $\{\lambda_k\}_{k=1}^{n}$ thus:

$$\left|\prod_{k=1}^{n}(I - \lambda_k L)^{-1}y - \prod_{k=1}^{n}\exp(\lambda_k L)^{y}\right| =$$

$$\left|\sum_{k=1}^{n}\left(\prod_{p=1}^{k-1}(I - \lambda_p L)^{-1}\right)(I - \lambda_k L)^{-1} - \exp(\lambda_k L))\prod_{k=k+1}^{n}\exp(\ {}_k L)y\right| \le$$

$$\sum_{k=1}^{n}\left|(I - \lambda_k L)^{-1}y - \exp(\lambda_k L)y\right| \le \sum_{k=1}^{n}\frac{\lambda_k{}^2}{2}|L^2 y| \le \frac{1}{2}\left(\sum_{k=1}^{n}\lambda_k\right)^2|L^2 y|.$$

Observation (v) follows at once since

$$\left|\sum_{k=1}^{n}\prod_{p=1}^{k}(I - \lambda_p L)^{-1})\lambda_k y - \sum_{k=1}^{n}\left(\prod_{p=1}^{k}\exp(\lambda_p L)\right)\lambda_k y\right| \le$$

$$\sum_{k=1}^{n}\lambda_k\left\{\frac{1}{2}\left(\sum_{p=1}^{k}\lambda_p\right)^2|L^2 y|\right\} \le \frac{1}{2}\left(\sum_{k=1}^{n}\lambda_k\right)^3|L^2 y|.$$

To establish (vi), note that $|\exp(\beta L)y - y| = \left|\int_o^{\beta} L\exp(sL)y\,ds\right| \le \beta|Ly|$. From (vi), one has (vii) quickly since

$$\left|\exp\left(\sum_{k=1}^{n}\lambda_k L\right)\left(\sum_{k=1}^{n}\lambda_k\right)y - \sum_{k=1}^{n}\left(\prod_{p=1}^{k}\exp(\lambda_p L)\right)\lambda_k y\right| =$$

$$\left|\sum_{k=1}^{n}\lambda_k\exp\left(\sum_{k=1}^{n}\lambda_k L\right)y - \sum_{k=1}^{k}\lambda_k\left(\prod_{p=1}^{k}\exp(\lambda_p L)\right)^{y}\right| \le$$

$$\sum_{k=1}^{n}\lambda_k\left|\left(\prod_{p=1}^{k}\exp(\lambda_p L)\right)\left[\exp\left(\sum_{p=k+1}^{n}\lambda_p L\right) - I\right]y\right| \le$$

$$\sum_{k=1}^{n}\lambda_k\left(\sum_{p=k+1}^{n}\lambda_p\right)|Ly| \le \left(\sum_{k=1}^{n}\lambda_k\right)^2|Ly|.$$

To establish (viii), recall that $\beta \ge \alpha$ and $\sum_{k=1}^{n}\lambda_k = \beta - \alpha$. One has that

$$|(I - \beta L)^{-1}y - [\prod_{k=1}^{n}(I - \lambda_k L)^{-1}](I - \alpha L)^{-1}y| \leq$$

$$|(I - \beta L)^{-1}y - (I - (\beta - \alpha)L)^{-1}(I - \alpha L)^{-1}y| +$$

$$|(I - \alpha L)^{-1}(I - (\beta - \alpha)L)^{-1}y - (I - \alpha L)^{-1} \prod_{k=1}^{n}(I - \lambda_k L)^{-1}y| \leq$$

$$|(I - \beta L)^{-1}(I - (\beta - \alpha)L)^{-1}(I - \alpha L)^{-1}(\beta - \alpha)\alpha L^2 y| +$$

$$|(I - (\beta - \alpha)L)^{-1}y - \exp((\beta - \alpha)L)y| +$$

$$|\exp((\beta - \alpha)Ly - \sum_{k=1}^{n}(I - \lambda_k L)^{-1}y| \leq$$

$$(\beta - \alpha)\alpha|L^2y| + (\beta - \alpha)^2|L^2y|.$$

Proof of Theorem 2. Let $\{a,b\}$ be an interval in S, and let $c > 0$ be such that $n = c^{-1}(g(b) - g(a))$ is an integer. Consider the intervals $\{[g(a) + (k - 1)c, g(a) + kc]\}_{k=1}^{n}$. Let $k_1,\dots,k_m$ be the integers for each of which there is a z_i in S so that $g(a) + (k_i - 1)c \leq g(z_i) \leq g(a) + k_i c$. Let $d > 0$ and pick $\{\gamma_{p,i} : 1 \leq p \leq m;\ i = 1,2\}$ as follows: For each $i = 2,\dots,m$, let $D_i = \mathrm{Inf}\{g(y) - g(x) : y \text{ in } S,$ $x \text{ in } S,\ g(a) + (k_{i-1} - 1)c \leq g(x) \leq g(a) + k_{i-1}c,\ g(a) + (k_i - 1)c \leq$ $g(y) \leq g(a) + k_i c\}$. Let $\gamma_{i-1,2}$ and $\gamma_{i,1}$ be in S such that $g(a) + (k_{i-1} - 1)c \leq g(\gamma_{i-1,2}) \leq g(a) + k_{i-1}c$, $g(a) + (k_i - 1)c \leq$ $g(\gamma_{i,1}) \leq g(a) + k_i c$, $z_{k_{i-1}} \geq \gamma_{i-1,2}$, $\gamma_{i,1} \geq z_{k_i}$, and $|g(\gamma_{i,1}) - g(\gamma_{i-1,2}) - D_i| < d$. Let $\gamma_{1,1} = a$ and $\gamma_{m,2} = b$.

Define $\{u_k\}_{k=0}^{2m-1}$ by $u_{2(k-1)} = \gamma_{k,1}$ and $u_{2k-1} = \gamma_{k,2}$ for

$k = 1,2,\ldots,m$. Let $\{w_k\}_{k=0}^{r}$ be a subdivision of $\{a,b\}$ which is a refinement of u, and let f be an increasing function from $0,1,2,\ldots,2m-1$ to the non-negative integers such that $w_{f(k)} = u_k$ for $k = 0,1,2,\ldots,2m-1$. If each of x and z is in $\overline{X}$, then

$$\Big| \prod_{k=1}^{2m-1} (I-(g(u_k)-g(u_{k-1}))(L+z))^{-1} x -$$

$$\prod_{k=1}^{2m-1} \Big(\prod_{p=f(k-1)+1}^{f(k)} (I-(g(w_p)-g(w_{p-1}))(L+z))^{-1} \Big) x \Big| =$$

$$\Big| \prod_{k=1}^{2m-1} (I-(g(u_k)-g(u_{k-1}))L)^{-1} x +$$

$$\sum_{k=1}^{2m-1} (g(u_k)-g(u_{k-1})) \prod_{p=1}^{k} (I-(g(u_p)-g(u_{p-1}))L)^{-1} z -$$

$$\prod_{k=1}^{2m-1} \Big(\prod_{p=f(k-1)+1}^{f(k)} (I-(g(w_p)-g(w_{p-1}))L)^{-1} \Big) x -$$

$$\sum_{k=1}^{2m-1} \left[\sum_{p=f(k-1)+1}^{f(k)} \Big(\prod_{j=1}^{p} (I-(g(w_j)-g(w_{j-1}))L)^{-1} \Big) (g(w_p)-g(w_{p-1})) z \right] \Big| \le$$

$$\Big| \prod_{k=1}^{2m-1} (I-(g(u_k)-g(u_{k-1}))L)^{-1} x - \tag{P1}$$

$$\prod_{k=1}^{2m-1} \Big(\prod_{p=f(k-1)+1}^{f(k)} (I-(g(w_p)-g(w_{p-1}))L)^{-1} \Big) x \Big| + \tag{P2}$$

$$\Big| \sum_{k=1}^{2m-1} (g(u_k)-g(u_{k-1})) \prod_{p=1}^{k} (I-(g(u_p)-g(u_{p-1}))L)^{-1} z - \tag{P3}$$

$$\sum_{k=1}^{2m-1}\left[\sum_{p=f(k-1)+1}^{p=f(k)}\left(\prod_{j=1}^{p}(I-(g(w_j)-g(w_{j-1}))L)^{-1}\right)(g(w_p)-g(w_{p-1}))z\right]| \qquad \text{(P4)}$$

<u>Bounding Section I</u>. One uses observations (iii), (iv), and (viii) to bound the quantity whose norm appears on lines (P1), (P2) thus:

$$|\prod_{k=1}^{2m-1}(I-(g(u_k)-g(u_{k-1}))L)^{-1}x -$$

$$\prod_{k=1}^{2m-1}\left(\prod_{p=f(k-1)+1}^{f(k)}(I-(g(w_p)-g(w_{p-1}))L)^{-1}\right)x| =$$

$$|\sum_{k=1}^{2m-1}\left(\prod_{p=1}^{k-1}(I-(g(u_p)-g(u_{p-1}))L)^{-1}\right)[(I-(g(u_k)-g(u_{k-1}))L)^{-1} -$$

$$\prod_{p=f(k-1)+1}^{p(k)}(I-(g(w_p)-g(w_{p-1}))L)^{-1}]\prod_{p=f(k)+1}^{f(2m-1)}(I-(g(w_p)-g(w_{p-1}))L)^{-1}x| \le$$

$$\sum_{k=1}^{2m-1}|(I-(g(u_k)-g(u_{k-1}))L)^{-1}x - \prod_{p=f(k-1)+1}^{f(k)}(I-(g(w_p)-g(w_{p-1}))L)^{-1}x|.$$

If k is odd, then $g(u_k) - g(u_{k-1}) < c$ and observations (iii) and (iv) give that

$$|(I-(g(u_k)-g(u_{k-1}))L)^{-1}x - \prod_{p=f(k-1)+1}^{f(k)}(I-(g(w_p)-g(w_{p-1}))L)^{-1}x| \le$$

$$|(I-(g(u_k)-g(u_{k-1}))L)^{-1}x - \exp((g(u_k)-g(u_{k-1}))L)x| +$$

$$|\exp((g(u_k)-g(u_{k-1}))L)x - \prod_{p=f(k-1)+1}^{f(k)}(I-(g(w_p)-g(w_{p-1}))L)^{-1}x| \le$$

$$(g(u_k)-g(u_{k-1}))^2|L^2x| \leq c(g(u_k)-g(u_k)-g(u_{k-1}))|L^2x|.$$

If k is even, then observation (viii) applies with $\beta = g(u_k) - g(u_{k-1})$ and $D_{\left(\frac{k}{2}\right)} \leq \alpha \leq \beta < D_{\left(\frac{k}{2}\right)} + d$. Hence,

$$|(I-(g(u_k)-g(u_{k-1}))L)^{-1}x - \prod_{p=f(k-1)+1}^{f(k)} (I-(g(w_p)-g(w_{p-1}))L)^{-1}x| \leq$$

$$d(D_{\left(\frac{k}{2}\right)}+d)|L^2x| + d^2|L^2x| .$$

It follows that

$$|\prod_{k=1}^{2m-1}(I-(g(u_k)-g(u_{k-1}))L)^{-1}x - \prod_{k=1}^{f(2m-1)} (I-(g(w_k)-g(w_{k-1}))L)^{-1}x| \leq$$

$$\sum_{k=1}^{2m-1}[c(g(u_k)-g(u_{k-1}))|L^2x|] + [md(g(b)-g(a)+d) +md^2]|L^2x|.$$

<u>Bounding Section II</u>. To bound the quantity whose norm appears on lines (P3) and (P4), note that

$$|(g(u_k)-g(u_{k-1}))\prod_{p=1}^{k}(I-(g(u_p)-g(u_{p-1}))L)^{-1}z -$$

$$\sum_{p=f(k-1)+1}^{f(k)}[\prod_{j=1}^{p}(I-(g(w_j)-g(w_{j-1}))L)^{-1}](g(w_p-g(w_{p-1}))z| \leq$$

$$|(g(u_k)-g(u_{k-1}))\prod_{p=1}^{k}(I-(g(u_p)-g(u_{p-1}))L)^{-1}z - \tag{P5}$$

$$(g(u_k)-g(u_{k-1}))(I-(g(u_k)-g(u_{k-1}))L)^{-1}\prod_{p=1}^{f(k-1)}(I-(g(w_p)-g(w_{p-1}))L)^{-1}z| +$$

(P6)

$$|(g(u_k)-g(u_{k-1}))(I-(g(u_k)-g(u_{k-1}))L)^{-1}\prod_{p=1}^{f(k-1)}(I-(g(w_p)-g(w_{p-1}))L)^{-1}z\ - \tag{P7}$$

$$\sum_{p=f(k-1)+1}^{f(k)}(g(w_p)-g(w_{p-1}))\prod_{j=1}^{p}(I-(g(w_j)-g(w_{j-1}))L)^{-1}z|. \tag{P8}$$

The techniques of bounding section I apply to the quantity whose norm is on lines (P5) and (P6) above. Hence,

$$|(g(u_k)-g(u_{k-1}))\prod_{p=1}^{k}(I-(g(u_p)-g(u_{p-1}))L)^{-1}z\ -$$

$$(g(u_k)-g(u_{k-1}))(I-(g(u_k)-g(u_{k-1}))L)^{-1}\prod_{p=1}^{f(k-1)}(I-(g(w_p)-g(w_{p-1}))L^{-1}z| \leq$$

$$(g(u_k)-g(u_{k-1}))[c(g(b)-g(a))|L^2z| + md(g(b)-g(a)+d)|L^2z| + md^2|L^2z|].$$

To bound the quantity whose norm appears on lines (P7) and (P8) in the event that $g(u_k) - g(u_{k-1}) \leq c$, one might note that if $\{\lambda_p\}_{p=1}^{q}$ is a non-negative number sequence with $\sum_{p=1}^{q}\lambda_p = \beta = g(u_k) - g(u_{k-1})$, and if

$$y = \prod_{p=1}^{f(k-1)}(I - (g(w_p) - g(w_{p-1}))L)^{-1}z ,$$

then

$$|\beta(I-\beta L)^{-1}y - \sum_{p=1}^{q}\lambda_p\prod_{j=1}^{p}(I-\lambda_j L)^{-1}y| \leq$$

$$|\beta(I-\beta L)^{-1}y - \beta\exp(\beta L)y| + |\beta\exp(\beta L)y - \sum_{p=1}^{q}\lambda_p\prod_{j=1}^{p}\exp(\lambda_j L)y| +$$

$$|\sum_{p=1}^{q}\lambda_p\prod_{j=1}^{p}\exp(\lambda_j L)y - \sum_{p=1}^{q}\lambda_p\prod_{j=1}^{p}(I-\lambda_j L)^{-1}y| \leq$$

$$\frac{1}{2}\beta^2|L^2y| + \beta^2|L^2y| + \sum_{p=1}^{q}\lambda_p(\frac{1}{2}\beta^2|L^2y|) \leq$$

$$\frac{3}{2}\beta^2|L^2z| + \frac{1}{2}\beta^3|L^3z| \leq c(\frac{3}{2}\beta|L^2z| + \frac{1}{2}\beta^2|L^2z|) ,$$

using observations (iii), (vii), then (iv), and the fact that $|L^iy| \leq |L^iz|$ for each positive integer i.

If, instead, one has $g(u_k) - g(u_{k-1}) > c$, then one notes that with $\sum_{p=1}^{q}\lambda_p = \beta = g(u_k) - g(u_{k-1})$ so that $\beta - \lambda_{p*} < d$ for some p* in [1,q], one has

$$|\beta(I-\beta L)^{-1}y - \sum_{p=1}^{q}\lambda_p \prod_{j=1}^{p}(I-\lambda_jL)^{-1}y| \leq$$

$$|(\beta-\lambda_{p*})(I-\beta L)^{-1}y - \sum_{p=1}^{p*-1}\lambda_p \prod_{j=1}^{p}(I-\lambda_jL)^{-1}y - \sum_{p=p*+1}^{q}\lambda_p \prod_{j=1}^{p}(I-\lambda_jL)^{-1}y| +$$

$$|\lambda_{p*}(I-\beta L)^{-1}y - \lambda_{p*}\prod_{p=1}^{p*}(I-\lambda_pL)^{-1}y| \leq$$

$$2d|y| + |\lambda_{p*}(I-\beta L)^{-1}y - \lambda_{p*}(I-\lambda_{p*}L)^{-1}(I-(\beta-\lambda_{p*})L)^{-1}y| +$$

$$|\lambda_{p*}(I-\lambda_{p*}L)^{-1}(I-(\beta-\lambda_{p*})L)^{-1}y - \lambda_{p*}(I-\lambda_{p*}L)^{-1}y| +$$

$$|\lambda_{p*}y - \lambda_{p*}\prod_{p=1}^{p*-1}(I-\lambda_pL)^{-1}y| \leq$$

$$2d|y| + \lambda_{p*}(\beta-\lambda_{p*})|L^2y| + |\lambda_{p*}(I-(\beta-\lambda_{p*})L)^{-1}y - \lambda_{p*}y| +$$

$$|\lambda_{p*}y - \lambda_{p*}\exp(\sum_{p=1}^{p*-1}\lambda_pL)y| + |\lambda_{p*}\exp(\sum_{p=1}^{p*-1}\lambda_pL)y - \prod_{p=1}^{p*-1}(I-\lambda_pL)^{-1}y| \leq$$

$$2d|y| + \beta d|L^2y| + (\beta-\lambda_{p*})\lambda_{p*}|Ly| + (\sum_{p=1}^{p*-1}\lambda_p)\lambda_{p*}|Ly| + \lambda_{p*}(\sum_{p=1}^{p*-1}\lambda_p)^2|L^2y| \le$$

$$2d|z| + \beta d|L^2z| + \beta d|Lz| + \beta d|Lz| + \beta d^2|L^2z|.$$

There are at most m points in u such that $g(u_k) - g(u_{k-1}) > c$. It follows that

$$|\sum_{k=1}^{2m-1}(g(u_k)-g(u_{k-1}))\prod_{p=1}^{k}(I-(g(u_p)-g(u_{p-1}))L)^{-1}z -$$

$$\sum_{k=1}^{f(2m-1)}(g(w_k)-g(w_{k-1}))\prod_{p=1}^{k}(I-(g(w_p)-g(w_{p-1}))L)^{-1}z| \le$$

$$\sum_{k=1}^{2m-1}\{(g(u_k)-g(u_{k-1}))|L^2z|(c(g(b)-g(a)) + md(g(b)-g(a)+d) + md^2)\} +$$

$$\sum_{k=1}^{2m-1}\{c(\frac{3}{2}(g(u_k)-g(u_{k-1})) + \frac{1}{2}(g(u_k)-g(u_{k-1})^2)|L^2z|\} +$$

$$md[2|z| + (g(b)-g(a))(2|Lz| + |L^2z| + d|L^2z|)] .$$

Finally, one has that

$$|\prod_{k=1}^{2m-1}(I-(g(u_k)-g(u_{k-1}))(L+z))^{-1}x - \prod_{k=1}^{f(2m-1)}(I-(g(w_k)-g(w_{k-1}))(L+z))^{-1}x| \le$$

$$(c(g(b)-g(a)) + md(g(b)-g(a)+d) + md^2)|L^2x| +$$

$$(g(b)-g(a))(c(g(b)-g(a)) + md(g(b)-g(a)+d) + md^2)|L^2z| +$$

$$c(\frac{3}{2}(g(b)-g(a)) + \frac{1}{2}(g(b)-g(a))^2)|L^2z| +$$

$$md[2|z| + (g(b)-g(a))(2|Lz| + |L^2z| + d|L^2z|)] .$$

Since d is selected after c and c determines m, one has that $_{a}\Pi^{b}(I - dg(L + z))^{-1}x$ exists.

Moreover, it should be noted that, for fixed x and z, the subdivision u is a function of the large discontinuities of g, g(b) - g(a), $|z|$, $|Lz|$, $|L^2z|$, and $|L^2x|$. Especially, Remark 2.1 is established at this point.

Properties (ii) and (iii) follow at once.

In light of Remark 2.1, the fact that

$$\prod_{k=1}^{2m-1} (I-(g(u_k)-g(u_{k-1}))(L+z))^{-1}x =$$

$$\sum_{k=1}^{2m-1} \{ \prod_{p=k}^{2m-1} (I-(g(u_p)-g(u_{p-1}))(L+z))^{-1}x - \prod_{p=k+1}^{2m-1} (I-(g(u_p)-g(u_{p-1}))(L+z))^{-1}x =$$

$$\sum_{k=1}^{2m-1} (g(u_k)-g(u_{k-1}))(L+z) \prod_{p=k}^{2m-1} (I-(g(u_p)-g(u_{p-1}))(L+z))^{-1}x$$

gives the integral equation of (iv) of Theorem 2.

This concludes the proof.

One might note at this point that the proof of Theorem 2 above applies equally well if L is dissipative and if each of x and z is in $D(L^2)$. Hence, the proofs of Theorems 3 and 5 below are somewhat repetitious. They do, however, give an important alternative characterization of $_{s}\Pi^{t}(I - dg(L + z))^{-1}x$. In addition, they make it possible for one already familiar with Herod [4] to move rapidly to the central results here without having to go through the laborious proof above, at the sacrifice of requiring S to have the least upper bound property.

The computational results of Chapter II, combined with Theorem 2, now lead to the development of $M(s,t)x = {}_s\Pi^t(I - dgA)^{-1}x$ in the present case that A is linear, dissipative, and densely defined.

Theorem 3. Let A and g be as before. If each of s and t is in S with $s \geq t$, then

(i) $M(s,t)x = {}_s\Pi^t(I - dgA)^{-1}x$ exists for each x in $\overline{X}$,

(ii) $M(s,t)$ is a continuous linear function from $\overline{X}$ to $\overline{X}$ and $|M(s,t)| \leq 1$,

(iii) If r is in S and $s \geq r \geq t$, then $M(s,r)M(r,t) = M(s,t)$,

(iv) If x_o is in D(A), then $M(s,t)Ax_o = AM(s,t)x_o$,

(v) If x_o is in D(A), then $M(s,t)x_o = x_o + (L)\int_s^t dgAM(\cdot,t)x_o$,

(vi) If x is in $\overline{X}$, then $(L)\int_s^t dgM(\cdot,t)x$ is in D(A), and $M(s,t)x = x + A((L)\int_s^t dgM(\cdot,t)x)$, and

(vii) If x_o is in $D(A^2)$, then $M(\cdot,t)x_o$ is the only function $F(\cdot)$ for which $AF(\cdot)$ is of bounded variation on each finite interval of S and which is a solution of the integral equation $F(s) = x_o + (L)\int_s^s dgAF(\cdot)$.

Proof. If x is in $\overline{X}$, $n > 0$, and each of s and t is in S with $s \geq t$, then $M_n(s,t)x = {}_s\Pi^t(I - dgAJ_n)^{-1}x$ exists by Theorem 2. One shows that $\lim_{n\to+\infty} M_n(s,t)x$ exists as follows:

If $\{r_k\}_{k=0}^{p}$ is a subdivision of $[s,t]$ and x_o is in $D(A)$, then, letting $dg_k = g(r_k) - g(r_{k-1})$ for $k = 1,2,\ldots,p$, one has

$$|M_n(s,t)x_o - M_m(s,t)x_o| \leq |M_n(s,t)x_o - \prod_{k=1}^{p}(I-dg_kAJ_n)^{-1}x_o| +$$

$$|\prod_{k=1}^{p}(I-dg_kAJ_n)^{-1}x_o - \prod_{k=1}^{p}(I-dg_kA)^{-1}x_o| +$$

$$|\prod_{k=1}^{p}(I-dg_kA)^{-1}x_o - \prod_{k=1}^{p}(I-dg_kAJ_m)^{-1}x_o| +$$

$$|\prod_{k=1}^{p}(I-dg_kAJ_m)^{-1}x_o - M_m(s,t)x_o|.$$

Now if $\varepsilon > 0$, there is a subdivision $\{\alpha_k\}_{k=0}^{c}$ such that if $\{\tilde{\alpha}_k\}_{k=0}^{\tilde{c}}$ is a refinement of α, then

$$|M_n(s,t)x_o - \prod_{k=1}^{\tilde{c}}(I-[g(\tilde{\alpha}_k)-g(\tilde{\alpha}_{k-1})]AJ_n)^{-1}x_o| < \frac{\varepsilon}{2} \quad ;$$

and there is a subdivision $\{\beta_k\}_{k=0}^{d}$ so that if $\{\tilde{\beta}_k\}_{k=0}^{\tilde{d}}$ is a refinement of β then

$$|M_m(s,t)x_o - \prod_{k=1}^{\tilde{d}}(I-[g(\tilde{\beta}_k)-g(\tilde{\beta}_{k-1})]A)^{-1}x_o| < \frac{\varepsilon}{2} \quad .$$

Taking a common refinement of α and β for r above, one has that

$$|M_n(s,t)x_o - M_m(s,t)x_o| \leq \frac{\varepsilon}{2} + |\prod_{k=1}^{p}(I-dg_kAJ_n)^{-1}x_o - \prod_{k=1}^{p}(I-dg_kA)^{-1}x_o| +$$

$$|\prod_{k=1}^{p}(I-dg_kA)^{-1}x_o - \prod_{k=1}^{p}(I-dg_kAJ_m)^{-1}x_o| + \frac{\varepsilon}{2} \quad .$$

Application of inequality (iii) of Lemma 1 gives that

$$|M_n(s,t)x_o - M_m(s,t)x_o| \le \varepsilon + (\int_s^t dg)(|(J_n-I)Ax_o| + |(J_m-I)Ax_o|) .$$

The last estimate works for each $\varepsilon > 0$; hence,

$$|M_n(s,t)x_o - M_m(s,t)x_o| \le (\int_s^t dg)(|(J_n-I)Ax_o| + |(J_m-I)Ax_o|) .$$

Lemma 0.2 gives at once that $\lim_{n\to+\infty} M_n(s,t)x_o$ exists for each x_o in D(A) and that the limit is uniform on bounded subsets of S. If y is an element of $\underline{\overline{X}}$ and x_o is in D(A), then

$$|M_n(s,t)y - M_m(s,t)y| \le |M_n(s,t)y - M_n(s,t)x_o| +$$

$$|M_n(s,t)x_o - M_m(s,t)x_o| + |M_m(s,t)x_o - M_m(s,t)y| \le$$

$$2|x_o-y| + |M_n(s,t)x_o - M_m(s,t)x_o| .$$

Now if $\varepsilon > 0$, one finds x_o in D(A) such that $|x_o - y| < \varepsilon/3$. Above, it was shown that there is an N so that $n > N$ and $m > N$ force $|M_n(s,t)x_o - M_m(s,t)x_o| < \varepsilon/3$; moreover, the selection of N can be taken uniformly on bounded subsets of S. Hence, $\lim_{n\to+\infty} M_n(s,t)y$ exists; again the limit is uniform on bounded subsets of S.

Conditions (ii) and (iii) are inherited directly from the corresponding conditions on the M_n. Also, if x is in $\underline{\overline{X}}$ and x_o is in D(A), then

$$|M(s,t)x - \prod_{k=1}^{p}(I-dg_kA)^{-1}x| \le |M(s,t)(x-x_o)| + |M(s,t)x_o - M_n(s,t)x_o| +$$

$$|M_n(s,t)x_o - \prod_{k=1}^{p}(I-dg_kAJ_n)^{-1}x_o| + |\prod_{k=1}^{p}(I-dg_kAJ_n)^{-1}x_o - \prod_{k=1}^{p}(I-dg_kA)^{-1}x_o| +$$

$$|\prod_{k=1}^{p}(I-dg_kA)^{-1}(x-x_o)| \leq |x-x_o| + |M(s,t)x_o - M_n(s,t)x_o| +$$

$$|M_n(s,t)x_o - \prod_{k=1}^{p}(I-dg_kAJ_n)^{-1}x_o| + (\int_s^t dg)|(J_n-I)Ax_o| + |x-x_o| .$$

If, now, $\varepsilon > 0$ is given, one may take x_o in $D(A)$ so that $|x - x_o| < \varepsilon/6$, n large enough so that $|M(s,t)x_o - M_n(s,t)x_o| + ({}_s\int^t dg)|(J_n-I)Ax_o| <$ $\varepsilon/3$, and a subdivision σ of $\{s,t\}$ so that if $\{r_k\}_{k=0}^{p}$ is a refinement of σ, then

$$|M_n(s,t)x_o - \prod_{k=1}^{p}(I - dg_kAJ_n)^{-1}x_o| < \frac{\varepsilon}{3} .$$

The above inequality then gives that for such a subdivision $\{r_k\}_{k=0}^{p}$ one has

$$|M(s,t)x - \prod_{k=1}^{p}(I - dg_kA)^{-1}x| < \varepsilon .$$

Hence, the representation in (i) holds. (A review of the proof of Theorem 2 at this point yields a uniform condition which is the subject of Corollary 3.2.)

Condition (iv) on M follows immediately since A is closed and

$$A\prod_{k=1}^{p}(I - dg_kA)^{-1}x_o = \prod_{k=1}^{p}(I - dg_kA)^{-1}Ax_o .$$

If x_o is in $D(A)$, then

$$|(L)_s\int^t dgM(\cdot,t)Ax_o - x_o - M(s,t)x_o| \leq$$

$$|(L)_s\int^t dgM(\cdot,t)Ax_o - (L)_s\int^t dgM_n(\cdot,t)AJ_nx_o| +$$

$$|M(s,t)x_o - M_n(s,t)x_o| \leq |(L)_s\int^t dg(M_n(\cdot,t) - M(\cdot,t))Ax_o| +$$

$$|(L)_s\int^t dgM_n(s,t)(J_n-I)Ax_o| + |M(s,t)x_o - M_n(s,t)x_o| \leq$$

$$({}_s\int^t dg) \sup_{s\leq z\leq t} \{|M_n(z,t)Ax_o - M(z,t)Ax_o|\} +$$

$$({}_s\int^t dg)|(J_n-I)Ax_o| + ({}_s\int^t dg)|(J_n-I)Ax_o| .$$

From a preceding estimate, one has that

$$\sup_{s\leq z\leq t} \{|M_n(z,t)Ax_o - M(z,t)Ax_o|\} \leq$$

$$\sup_{s\leq z\leq t} \{\inf_{\xi\varepsilon D(A)} \{2|\xi-Ax_o| + ({}_z\int^t dg)|(J_n-I)A\xi|\}\} \leq$$

$$\inf_{\xi\varepsilon D(A)} \{2|\xi-Ax_o| + ({}_s\int^t dg)|(J_n-I)A\xi|\} .$$

If, now, $\varepsilon > 0$ is given, then one may take ξ in $D(A^2)$ so that $({}_s\int^t dg)2|\xi-Ax_o| < \varepsilon/3$ since $D(A^2)$ is dense in $\underline{X}$. One may next take n so large that both $({}_s\int^t dg)^2|(J_n-I)A\xi| < \varepsilon/3$ and $2({}_s\int^t dg)|(J_n-I)Ax_o| < \varepsilon/3$. The above inequality then gives that

$$|(L)_s\int^t dgM(\cdot,t)Ax_o - x_o - M(s,t)x_o| < \varepsilon .$$

Since this holds for each $\varepsilon > 0$, the fact that A commutes with M on D(A) gives that the integral equation of (v) is satisfied. The assertion of (vi) follows at once since A is closed and D(A) is dense.

Finally, suppose that x_o is in $D(A^2)$. Let $\{s_k\}_{k=0}^n$ be a decreasing sequence in S. Since $AM(s_{k-1},s_n)x_o = Ax_o + (L)\int_{s_{k-1}}^{s_n} dgM(\cdot,s_n)A^2x_o$, one has that

$$\sum_{k=1}^{n} |AM(s_{k-1},t)x_o - AM(s_k,t)x_o| =$$

$$\sum_{k=1}^{n} |(L)\int_{s_{k-1}}^{s_n} dgM(\cdot,s_n)A^2x_o - (L)\int_{s_k}^{s_n} dgM(\cdot,s_n)A^2x_o| =$$

$$\sum_{k=1}^{n} |(L)\int_{s_{k-1}}^{s_k} dgM(\cdot,s_n)A^2x_o| \le \sum_{k=1}^{n} (L)\int_{s_{k-1}}^{s_k} dg|A^2x_o| = (\int_{s_o}^{s_n} dg)|A^2x_o| .$$

Hence, the variation of $AM(\cdot,t)x_o$ on $\{s,t\}$ is bounded by $({}_s\int^t dg)|A^2x_o|$. If f is a function from S to D(A) such that $Af(\cdot)$ is of bounded variation and such that the integral equation of (vii) is satisfied, then any subdivision $\{r_k\}_{k=0}^p$ of $\{s,t\}$ which gives that

$$\sum_{k=1}^{p} |dg_k\ Af(r_{k-1}) - (L)\int_{r_{k-1}}^{r_k} dgAf(\cdot)| +$$

$$\sum_{k=1}^{p} |dg_k\ AM(r_{k-1},t)x_o - (L)\int_{r_{k-1}}^{r_k} dgAM(\cdot,t)x_o| < \varepsilon$$

also gives that

$$|f(s) - M(s,t)x_o| \le |f(x) - M(s,t)x_o| +$$

$$\sum_{k=1}^{p} \{|(I - dg_kA)(f(r_{k-1}) - M(r_{k-1},t)x_o| - |f(r_{k-1}) - M(r_{k-1},t)x_o|\} =$$

$$\sum_{k=1}^{p} \{-|f(r_k) - M(r_k,t)x_o| + |(I - dg_kA)(f(r_{k-1}) - M(r_{k-1},t)x_o)|\} \leq$$

$$\sum_{k=1}^{p} |(I - dg_kA)(f(r_{k-1}) - M(r_{k-1},t)x_o) - (f(r_k) - M(r_k,t)x_o)| \leq$$

$$\sum_{k=1}^{p} \{|f(r_k) - f(r_{k-1}) - dg_kAf(r_{k-1})| +$$

$$|M(r_k,t)x_o - M(r_{k-1},t)x_o - dg_kAM(r_{k-1}t)x_o|\} \leq$$

$$\sum_{k=1}^{p} \{|(L)\int_{r_{k-1}}^{r_k} dgAf(\cdot) - dg_kAf(r_{k-1})| +$$

$$|(L)\int_{r_{k-1}}^{r_k} dgAM(\cdot,t)x_o - dg_kAM(r_{k-1},t)x_o|\} < \varepsilon \quad .$$

Hence, $f(x) = M(s,t)x_o$.

<u>Remark</u>. The existence of such a subdivision $\{r_k\}_{k=0}^{p}$ is a standard result in the theory of Stieltjes integration. The two lemmas following indicate how one establishes this result.

<u>Lemma</u>. If g and H are non-increasing functions from S to R, if $\{s,t\}$ is an interval in S with $s \geq t$, and if $\varepsilon > 0$, then there is a subdivision u of $\{s,t\}$ so that if $\{r_p\}_{p=0}^{n}$ is a refinement of u, then

$$\sum_{p=1}^{n} \{(L)\int_{r_{p-1}}^{r_p} dgH(\cdot) - (g(r_p) - g(r_{p-1}))H(r_{p-1})\} < \varepsilon \quad ,$$

the existence of the integrals being conclusion, not hypothesis.

Proof. First note that if each of α, β, and γ is in S with $\alpha \geq \beta \geq \gamma$, then $(g(\beta) - g(\alpha))H(\alpha) + (g(\gamma) - g(\beta))H(\beta) \geq (g(\beta) - g(\alpha))H(\alpha) + (g(\gamma) - g(\beta))H(\alpha) = (g(\gamma) - g(\alpha))H(\alpha)$. Repeated application of this observation gives that if $\{a,b\}$ is an interval in S with $a \geq b$, if $\{u_p\}_{p=0}^{n}$ is a subdivision of $\{a,b\}$, and if $\{r_p\}_{p=0}^{m}$ is a refinement of u, then

$$\sum_{p=0}^{m}(g(r_p) - g(r_{p-1}))H(r_{p-1}) \geq \sum_{p=0}^{n}(g(u_p) - g(u_{p-1}))H(u_{p-1}) .$$

Since $\sum_{p=0}^{m}(g(r_p) - g(r_{p-1}))H(r_{p-1}) \leq (g(b) - g(a))H(b)$, one can put

$Q = \sup\{\sum_{p=1}^{n}(g(u_p) - g(u_{p-1}))H(u_{p-1}) : u \text{ is a subdivision of } \{a,b\}\}$. It follows at once that $(L)_a\int^b dgH(\cdot)$ exists because one has that if u above is a subdivision of $\{a,b\}$ such that $Q - \sum_{p=1}^{n}(g(u_p) - g(u_{p-1}))H(u_{p-1}) < \varepsilon$ and if again r refines u, then $Q - \sum_{p=1}^{m}(g(r_p) - g(r_{p-1}))H(r_{p-1}) < \varepsilon$. The inequality of the lemma now follows if one sets $a = s$ and $b = t$.

Lemma. If g is a non-increasing function from S to R, if H is a function which is of bounded variation on each finite interval of S, if $\{s,t\}$ is an interval in S, then there is a subdivision $\{s_p\}_{p=0}^{k}$ of $\{s,t\}$ so if $\{t_p\}_{p=0}^{n}$ is a refinement of $\{s_p\}_{p=0}^{k}$, then

$$\sum_{p=1}^{n}\left|(L)\int_{t_{p-1}}^{t_p} dgH(\cdot) - (g(t_p) - g(t_{p-1}))H(t_{p-1})\right| < \varepsilon \quad ,$$

the existence of the integrals being conclusion, not hypothesis.

Proof. Let $[a,b]$ be an interval in S with $a \geq b$. For each t in S, let $V(t) = {}_t\int^b |dH|$ if $t \geq b$, and $V(t) = 0$ if $b \geq t$. Then V is non-increasing. If $\{t_p\}_{p=0}^{n}$ is a subdivision of $[a,b]$ and $\{w_p\}_{p=0}^{m}$ is a refinement of $\{t_p\}_{p=0}^{n}$, then there is an increasing function f such that $t_p = w_{f(p)}$ for each $p = 0,1,\ldots,n$. Now,

$$\left| \sum_{p=1}^{n} \left\{ \sum_{j=f(p-1)+1}^{f(p)} (g(w_j)-g(w_{j-1}))H(w_{j-1}) \right\} - \sum_{p=1}^{n} (g(t_p)-g(t_{p-1}))H(t_{p-1}) \right| \leq$$

$$\sum_{p=1}^{n} \left\{ \left| \sum_{j=f(p-1)+1}^{f(p)} \{(g(w_j)-g(w_{j-1}))H(w_{j-1}) - (g(w_j)-g(w_{j-1}))H(w_{f(p-1)})\} \right| \right\} \leq$$

$$\sum_{p=1}^{n} \left\{ \sum_{j=f(p-1)+1}^{f(p)} (g(w_j)-g(w_{j-1})) |H(w_{j-1}) - H(w_{f(p-1)})| \right\} \leq$$

$$\sum_{p=1}^{n} \left\{ \sum_{j=f(p-1)+1}^{f(p)} (g(w_j)-g(w_{j-1}))(V(w_{j-1}) - V(w_{f(p-1)})) \right\} =$$

$$\sum_{p=1}^{n} \left\{ \sum_{j=f(p-1)+1}^{f(p)} (g(w_j)-g(w_{j-1}))V(w_{j-1}) \right\} - \sum_{p=1}^{n} (g(t_p)-g(t_{p-1}))V(t_{p-1}) .$$

By the preceding lemma, the above inequality gives that $(L)_a\int^b dgH(\cdot)$ exists for each $[a,b]$ in S. If $\varepsilon > 0$, then one may take a subdivision $\{s_p\}_{p=0}^{k}$ and $\{w_p\}_{p=0}^{m}$ is a refinement of $\{t_p\}_{p=0}^{n}$ as above, then

$$\sum_{p=1}^{n} \left\{ \sum_{j=f(p-1)+1}^{f(p)} (g(w_j)-g(w_{j-1}))V(w_{j-1}) \right\} - \sum_{p=1}^{n} (g(t_p)-g(t_{p-1}))V(t_{p-1}) < \frac{\varepsilon}{2} .$$

By virtue of the existence of each of $(L)\int_{t_{p-1}}^{t_p} dgH(\cdot)$, one may take the subdivision $\{w_p\}_{p=0}^{m}$ so that

$$\left|(L)\int_{t_{p-1}}^{t_p} dgH(\cdot) - \sum_{j=f(p-1)+1}^{f(p)} (g(w_j) - g(w_{j-1}))H(w_{j-1})\right| < \frac{\varepsilon}{2n}$$

for $p = 1,2,\ldots,n$. It follows at once that

$$\sum_{p=1}^{n} \left|(L)\int_{t_{p-1}}^{t_p} dgH(\cdot) - (g(t_p) - g(t_{p-1}))H(t_{p-1})\right| < \varepsilon \quad .$$

<u>Corollary 3.1</u>. If x is in $\overline{X}$, and if each of s and t is in S with $s \geq t$, then

$$M(s,t)x = \lim_{n\to\infty} M_n(s,t)x = \lim_{n\to\infty} {}_s\Pi^t (I - dgAJ_n)^{-1}x \quad ,$$

uniformly on bounded subsets of S.

Remark 2.1, coupled with the proof that the representation (i) of Theorem 3 holds, gives a version of Remark 2.1 in the present setting.

<u>Corollary 3.2</u>. If x is in $\overline{X}$, and if each of s and t is in S with $s \geq t$, then the limit

$$M(s,t)x = {}_s\Pi^t (I - dgA)^{-1}x$$

is uniform in the following sense:

If $\varepsilon > 0$, then there exists a subdivision $\{r_p\}_{p=0}^{n}$ of $\{s,t\}$ such that if $\{v_p\}_{p=0}^{m}$ is a refinement of $\{r_p\}_{p=0}^{n}$, then

$$\left|{}_{v_k}\Pi^{t}(I - dgA)^{-1}x - \prod_{p=k+1}^{m}(I - (g(v_p) - g(v_{p-1}))A)^{-1}x\right| < \varepsilon$$

and

$$\left|{}_{s}\Pi^{v_k}(I - dgA)^{-1}x - \prod_{p=1}^{k+1}(I - (g(v_p) - g(v_{p-1}))A)^{-1}x\right| < \varepsilon$$

for $k = 0,1,\ldots,m$.

A related integral equation is also satisfied.

Corollary 3.3. If x_o is in $D(A)$, and if each of s and t is in S with $s \geq t$, then

$$M(s,t)x_o = x_o + (R)_s\int^t dgAM(s,\cdot)x_o \quad .$$

Indication of Proof. Let $\{r_p\}_{p=0}^{m}$ be a subdivision of $\{s,t\}$ and let x_o be in $D(A)$. Since

$$\prod_{k=1}^{m}(I - dgA)^{-1}x_o - x_o = \sum_{j=1}^{m} dg_j \prod_{k=1}^{j}(I - dg_kA)^{-1}Ax_o \quad ,$$

Corollary 3.2 gives that $M(s,t)x_o - x_o = (R)_s\int^t dgM(s,\cdot)Ax_o$. Since A is closed, $M(s,t)x - x = A((R)_s\int^t dgM(s,\cdot)x)$ for each x in $\overline{X}$.

The example following indicates the need for a uniqueness condition as cumbersome as (vii) of Theorem 3.

Example. Let $\overline{X}$ = {all continuous $f : R \to R$ such that $\lim_{x\to-\infty} f(x)$

and $\lim_{x \to +\infty} f(x)$ both exist . If f is in $\underline{\overline{X}}$, let $|f| = \sup\{|f(x)| : x \in R\}$. If $(M(s,t)f)(x) = f(x + s - t)$, and if

$$\psi(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x \cos(\frac{\pi}{x}) & \text{if } 0 < x < 2 \\ 0 & \text{if } x \geq 2 \end{cases},$$

then there is a y in $\underline{\overline{X}}$ such that $Ay = d/(dx)y = \psi$. Now $AM(\cdot,0)y = M(\cdot,0)\psi$ and is of bounded variation on no interval of R.

In terms of the evolution system M, the uniqueness condition of Theorem 3 is not vague, for M is a collection of continuous operators and $D(A^2)$ is dense in $\underline{\overline{X}}$.

<u>The Affine Case</u>. With A as in Theorem 3 and z in $\underline{\overline{X}}$, let A + z be the affine transformation defined by $(A + z)x = Ax + z$ for each x in D(A). A few computational results facilitate the development of the affine version of Theorem 3.

<u>Lemma 4</u>. Let A be as before; let $\beta > 0$, $n > 0$, and $\{\lambda_k\}_{k=1}^{m}$ be a sequence of non-negative numbers; let each of u and w be in D(A); and let x, y, and z be in $\underline{\overline{X}}$. Then

(i) $(I - \beta(A + z))^{-1}y = (I - \beta A)^{-1}(y + \beta z)$,

(ii) $(I - \beta(A + z))^{-1}y - (I - \beta A)^{-1}y = \beta(I - \beta A)^{-1}z$,

(iii) $\prod_{k=1}^{m} (I - \lambda_k(A + z))^{-1}y = \prod_{k=1}^{m} (I - \lambda_k A)^{-1}y +$

$\sum_{j=1}^{m} \{ \prod_{k=1}^{j} (I - \lambda_k A)^{-1}\lambda_j z\}$,

(iv) $$\left|\prod_{k=1}^{m}(I - \lambda_k(AJ_n + u))^{-1}w - \prod_{k=1}^{m}(I - \lambda_k(A + u))^{-1}w\right| \le$$

$$(\sum_{k=1}^{m} \lambda_k)|(J_n - I)Aw| + (\sum_{k=1}^{m} \lambda_k)^2|(J_n - I)Au| ,$$

(v) $$\left|\prod_{k=1}^{m}(I - \lambda_k(A + z))^{-1}x - \prod_{k=1}^{m}(I - \lambda_k(A + y))^{-1}x\right| \le$$

$$(\sum_{k=1}^{m} \lambda_k)|z - y| .$$

Proof. Assertions (i) and (iii) are recollections of Observations (i) and (ii) established before the proof of Theorem 2. Assertion (ii) is an immediate consequence of (i).

By (iii) one has that

$$\left|\prod_{k=1}^{m}(I - \lambda_k(AJ_n + u))^{-1}w - \prod_{k=1}^{m}(I - \lambda_k(A+u))^{-1}w\right| \le$$

$$\left|\prod_{k=1}^{m}(I - \lambda_k AJ_n)^{-1}w - \prod_{k=1}^{m}(I - \lambda_k A)^{-1}w\right| +$$

$$\left|\sum_{k=1}^{m}\prod_{p=1}^{k}(I - \lambda_p AJ_n)^{-1}\lambda_k u - \sum_{k=1}^{m}\prod_{p=1}^{k}(I - \lambda_p A)^{-1}\lambda_k u\right| \le$$

$$(\sum_{k=1}^{m} \lambda_k)|(J_n - I)Aw| + \sum_{k=1}^{m} \lambda_k \{\sum_{p=1}^{k} \lambda_p|(J_n - I)Au|\} \le$$

$$(\sum_{k=1}^{m} \lambda_k)|(J_n - I)Aw| + (\sum_{k=1}^{m} \lambda_k)^2|(J_n - I)Au|.$$

Assertion (v) follows at once from (iii).

Estimates obtained in the development of $\Pi(I - dgA)^{-1}$ combined with the above computations now give an affine version of Theorem 3.

Theorem 5. Let A and g be as before and let z be in $\underline{\bar{X}}$. If each of s and t is in S with $s \geq t$, then

(i) $W(z;s,t)x = {}_s\Pi^t(I - dg(A + z))^{-1}x$ exists for each x in $\underline{\bar{X}}$,

(ii) $W(z;s,t)$ is a continuous affine function from $\underline{\bar{X}}$ to $\underline{\bar{X}}$ such that if each of x and y is in $\underline{\bar{X}}$, then $|W(z;s,t)x - W(z;s,t)y| \leq |x - y|$,

(iii) If each of x and y is in $\underline{\bar{X}}$, then $|W(z;s,t)x - W(y;s,t)x| \leq ({}_s\int^t dg)|z - y|$,

(iv) If r is in S and $s \geq r \geq t$, then $W(z;s,r)W(z;r,t) = W(z;s,t)$,

(v) If each of x and z is in $\underline{\bar{X}}$, then $W(z;s,t)x = M(s,t)x + (R)_s\int^t dgM(s,\cdot)z$,

(vi) If x_o is in D(A) and z is in $\underline{\bar{X}}$, then $W(z;s,t)x_o = x_o + (L)_s\int^t dg(A + z)W(z;\cdot,t)x_o$,

(vii) If each of x and z is in $\underline{\bar{X}}$, then $(L)_s\int^t dgW(z;\cdot,t)x$ is in D(A) and $W(z;s,t)x = x + A((L)_s\int^t dgW(z;\cdot,t)x) + {}_s\int^t dgz$, and

(viii) If x is in $D(A^2)$ and z is in D(A), then $W(z;\cdot,t)x$ is the only function $F(\cdot)$ for which $AF(\cdot)$ is of bounded variation on each finite interval of S and which solves the integral equation

$$F(\cdot) = x + (L)_s\int^t dg(A + z)F(\cdot) \quad .$$

Indication of Proof. If $\{r_k\}_{k=0}^{m}$ is a subdivision of $\{s,t\}$, if each of x_o and z is in D(A), and if n is a positive integer, then

$$|\prod_{k=1}^{m}(I - dg_k(AJ_n + z))^{-1}x_o - \prod_{k=1}^{m}(I - dg_k(A + z))^{-1}x_o| =$$

$$|\prod_{k=1}^{m}(I - dg_k AJ_n)^{-1}x_o - \prod_{k=1}^{m}(I - dg_k A)^{-1}x_o +$$

$$\sum_{j=1}^{m}\{\prod_{k=1}^{j}(I - dg_k AJ_n)^{-1}dg_j z - \prod_{k=1}^{j}(I - dg_k A)^{-1}dg_j z\}| \leq$$

$$({}_s\int^t dg)|(J_n - I)Ax_o| + ({}_s\int^t dg)^2|(J_n - I)Az| \ .$$

Hence,

$$|{}_s\Pi^t(I - dg(AJ_n + z))^{-1}x_o - {}_s\Pi^t(I - dg(AJ_p + z))^{-1}x_o| \leq$$

$$({}_s\int^t dg)(|(J_n - I)Ax_o| + |(J_p - I)Ax_o|) +$$

$$({}_s\int^t dg)^2(|(J_n - I)Az| + |(J_p - I)Az| \ ;$$

so $\lim_{n\to\infty} {}_s\Pi^t(I - dg(AJ_n + z))^{-1}x_o$ exists for each of x_o and z in D(A). By (iv) of Lemma 4, $\lim_{n\to\infty} {}_s\Pi^t(I - dg(AJ_n + z))^{-1}x$ exists for x in $\overline{X}$ and z in D(A); and, by (v) of Lemma 4, the limit exists for z in $\overline{X}$ and x in $\overline{X}$. Let $W(z;s,t)x = \lim_{n\to\infty} {}_s\Pi^t(I - dg(AJ_n + z))^{-1}x$. If each of x_o and z_o is in D(A), and if

$$W_n(z;s,t)x = {}_s\Pi^t(I - dg(AJ_n + z))^{-1}x \quad ,$$

then

$$|W(z;s,t)x - \prod_{k=1}^{m} (I - dg(A + z))^{-1}x| \leq |W(z;s,t)x - W_n(z;s,t)x| +$$

$$|W_n(z;s,t)x - W_n(z_o;s,t)x| + |W_n(z_o;s,t)x - W_n(z_o;s,t)x_o| +$$

$$|W_n(z_o;s,t)x_o - \prod_{k=1}^{m} (I - dg_k(AJ_n + z_o))^{-1}x_o| +$$

$$|\prod_{k=1}^{m} (I - dg_k(AJ_n + z_o))^{-1}x_o - \prod_{k=1}^{m} (I - dg_k(A + z_o))^{-1}x_o| +$$

$$|\prod_{k=1}^{m} (I - dg_k(A + z_o))^{-1}x_o - \prod_{k=1}^{m} (I - dg_k(A + z_o))^{-1}x| +$$

$$|\prod_{k=1}^{m} (I - dg_k(A + z_o))^{-1}x - \prod_{k=1}^{m} (I - dg_k(A + z))^{-1}x| \leq$$

$$|W(z;s,t)x - W_n(z;s,t)x| + ({}_s\int^t dg)|z - z_o| + |x - x_o| +$$

$$|W_n(z_o;s,t)x_o - \prod_{k=1}^{m} (I - dg_k(AJ_n + z_o))^{-1}x_o| +$$

$$(({}_s\int^t dg)|(J_n - I)Ax_o| + ({}_s\int^t dg)^2|(J_n - I)Az_o|) +$$

$$|x - x_o| + ({}_s\int^t dg)|z - z_o|.$$

Hence, the representation of W in (i) holds.

Properties (ii) and (iii) follow immediately from the corresponding properties of the approximating products. Property (iv) is inherited from the W_n.

Now, the fact that $\prod_{k=1}^{m} (I - dg_k(A + z))^{-1}x =$

$$\prod_{k=1}^{m} (I - dg_k A)^{-1} x + \sum_{j=1}^{m} dg_j (\prod_{k=1}^{j} (I - dg_k A)^{-1} z)$$

together with Corollary 3.2 gives that

$$W(z;s,t)x = M(s,t)x + (R)_s\int^t dgM(s,\cdot)z \quad ,$$

which is equation (v).

Now if each of x_o and z_o is in $D(A)$, then $AW(z_o;s,t)x_o = W(Az_o;s,t)Ax_o$. As in Theorem 3, one can use $\{W_n\}_{n=1}^{\infty}$ to show that

$$W(z_o;s,t)x_o = x_o + (L)_s\int^t dg(A + z)W(z_o;\cdot,t)x_o \quad .$$

Moreover, if z is in $\underline{\overline{X}}$, then $AW(z;s,t)x_o = AM(s,t)x_o + M(s,t)z - z$ and is integrable. The fact that $|W(z;s,t)x_o - W(z_o;s,t)x_o| \leq ({}_s\int^t dg)|z - z_o|$ then gives that (vi) holds. The integral equation (vii) follows at once since A is closed.

Finally, if z is in $D(A)$ and x is in $D(A^2)$, then $AW(z; ,t)x$ is of bounded variation on each finite interval of S. If F is also, and if satisfies

$$F(s) = x + (L)_s\int^t dg(A + z)F(\cdot) \quad \text{for each } s \geq t \quad ,$$

then

$$F(t) - W(z;t,t)x = 0 \qquad \text{and}$$

$$F(s) - W(z;s,t)x = 0 + (L)_s\int^t dgA(F(\cdot) - W(z;\cdot,t)x) \quad .$$

Theorem 3 gives that $F(s) = W(z;s,t)x$, and the proof is complete.

CHAPTER IV

AN APPLICATION TO CLOSED OPERATOR EQUATIONS

In special circumstances, the results of the preceding section may be used to solve the operator equation

$$(*) \qquad Ay = z$$

for y.

In addition, now let S be an unbounded set of the non-negative numbers with 0 in S, and let g now be such that $\lim_{t\to+\infty} g(t) = -\infty$. Theorems 3 and 5 are now applied in Theorems 7 and 8 to give an iterative procedure to approximate solutions to (*).

<u>Lemma 6</u>. Let B be a linear function from D(B) in $\underline{\overline{X}}$ to $\underline{\overline{X}}$ and let $\lambda > 0$ be such that $(I - \lambda B)^{-1}$ exists, has domain all of $\underline{\overline{X}}$, and is continuous. If $\{x_n\}_{n=1}^{\infty}$ is a sequence in D(B) such that $\text{w-}\lim_{n\to\infty} x_n = y$ and $\text{w-}\lim_{n\to\infty} Bx_n = P$, then y is in D(B) and $By = P$.

<u>Proof</u>. First note that continuous linear functions preserve weak limits. Especially, if $\text{w-}\lim_{n\to\infty} x_n = y$, if ψ is in $\underline{\overline{X}}^*$, and if H is a continuous linear function from $\underline{\overline{X}}$ to $\underline{\overline{X}}$, then ϕ defined by $\phi(\cdot) = \psi(H(\cdot))$ is in $\underline{\overline{X}}^*$; so $\lim_{n\to\infty} \phi(x_n) = \phi(y)$. Hence, $\lim_{n\to\infty} \psi(Hx_n) = \psi(Hy)$ for each ψ in $\underline{\overline{X}}^*$, i.e., $\text{w-}\lim_{n\to\infty} Hx_n = Hy$.

Now let $z_n = x_n - \lambda Bx_n$ for each n. Then $\text{w-}\lim_{n\to\infty} z_n = y - \lambda P$. Since $(I - \lambda B)^{-1}$ is continuous, one has that

$$(I - \lambda B)^{-1}(y - \lambda P) = \underset{n\to\infty}{\text{w-lim}}\ (I - \lambda B)^{-1}(x_n - \lambda B x_n)$$

$$= \underset{n\to\infty}{\text{w-lim}}\ x_n = y\ .$$

Hence, y is in D(B); and $(I - \lambda B)^{-1}(y - \lambda P) = y$ gives that $y - \lambda P = (I - \lambda B)y$ from which $By = P$ follows.

Definition. An evolution system M is strongly (resp., weakly) asymptotically convergent if, and only if,

$$\lim_{t\to+\infty} M(t,0)x \text{ (resp., } \underset{t\to+\infty}{\text{w-lim}}\ M(t,0)x)$$

exists for each x in $\overline{X}$.

Theorem 7. Let A, g, and M be as in Theorem 3 so that $M(s,t)x = {}_s\Pi^t(I - dgA)^{-1}x$ for each x in $\overline{X}$ and for each of s and t in S with $s \geq t$. If M is strongly (resp., weakly) asymptotically convergent, and if $Qx = \lim_{t\to+\infty} M(t,0)x$ (resp., $\underset{t\to+\infty}{\text{w-lim}}\ M(t,0)x$) for each x in $\overline{X}$, then

(i) Q is a continuous projection of $\overline{X}$ onto the null space of A,

(ii) $|Q| \leqq 1$, and

(iii) The null space of Q is the closure of the range of A.

Proof. Case I M is strongly asymptotically convergent.

Let x_o be in D(A) and suppose that $QAx_o = y \neq 0$. Then there is an element T in S such that $|y - M(s,o)Ax_o| < \frac{1}{2}|y|$ for each $s \geq T$. Now

$$|Qx_o - x_o| = |\lim_{s\to\infty}(L)\int_s^o dgM(\cdot,0)Ax_o|$$

$$= |(L)\int_T^o dgM(\cdot,0)Ax_o + \lim_{s\to\infty}(L)\int_s^T dgM(\cdot,0)Ax_o|$$

$$\geq |\lim_{s\to\infty}(L)\int_s^T dg(y + M(\cdot,0)Ax_o - y)| - |(L)\int_T^o dgM(\cdot,0)Ax_o|$$

$$\geq |\lim_{s\to\infty}(L)\int_s^T dgy| - |\lim_{s\to\infty}(L)\int_s^T dg(M(\cdot,0)Ax_o - y)|$$

$$- |(L)\int_T^o dgM(\cdot,0)Ax_o|$$

$$\geq \lim_{s\to\infty}\{(g(T) - g(s))|y| - (g(T) - g(s))\tfrac{1}{2}|y|\}$$

$$- |(L)\int_T^o dgM(\cdot,0)Ax_o|.$$

The last quantity tends to $+\infty$; hence, $QAx_o = 0$. Also, $\lim_{s\to\infty} M(s,0)x_o = Qx_o$ and $\lim_{s\to\infty} AM(s,0)x_o = \lim_{s\to\infty} M(s,0)Ax_o = QAx_o = 0$. Since A is closed, Qx_o is in D(A) and $AQx_o = 0$. Again, since A is closed and D(A) is dense in $\overline{X}$, Qx is in D(A) for each x in $\overline{X}$ and AQx = 0. Hence, one has that Q is a mapping into the null space of A. If z is in D(A) and Az = 0, then $(I - \lambda A)^{-1}z = z$ for each $\lambda > 0$. It follows that $M(s,0)z = z$ for each $s \geq 0$. Thus Qz = z, and one has that Q is a mapping onto the null space of A.

That $|Q| \leq 1$ follows at once from the fact that $|M(t,0)| \leq 1$ for each $t \geq 0$. To see that $Q^2 = Q$, note that if $\lambda > 0$ and x is in $\overline{X}$, then $Q(I - \lambda A)^{-1}x = Q(I + \lambda A(I - \lambda A)^{-1})x = Qx$. By induction, if $\{\lambda_k\}_{k=1}^n$ is a sequence of non-negative numbers, then $Q\prod_{k=1}^n (I - \lambda_k A)^{-1}x = Qx$. Since

Q is continuous, one has that $QM(s,0)x = Qx$ for each $s \geq 0$ and then than $Q^2 = Q$.

Already, one has that if z is in the range of A, then $Qz = 0$. Since Q is continuous, $Qx = 0$ for each x in the closure of the range of A. Since $M(t,0)x - x = A((L)\int_t^0 dgM(\cdot,0)x)$, $Qx = 0$ only in case x is in the closure of the range of A.

The proof in case M is weakly asymptotically convergent uses Lemma 6 and follows much the same lines. Again, one lets x_o be in D(A) and supposes that $QAx_o = y \neq 0$. If ψ is in $\underline{X}^*$ and $\psi(y) \neq 0$, then there is an element T in S such that $|\psi(y) - \psi(M(s,0)Ax_o)| \leq \frac{1}{2}|\psi(y)|$ for $s \geq T$. The fact that $\psi(Qx_o - x_o) = \lim_{s\to\infty} (L)\int_s^0 dg\psi(M(\cdot,0)Ax_o)$ leads to a contradiction; hence, $QAx_o = 0$. As before, the fact that $\text{w-}\lim_{s\to\infty} M(s,0)x_o = Qx_o$ and $\text{w-}\lim_{s\to\infty} AM(s,0)x_o = \text{w-}\lim_{s\to\infty} M(s,0)Ax_o = QAx_o = 0$ coupled with Lemma 6 gives that $AQx_o = 0$. That Qx is in D(A) and $AQx = 0$ for each x in $\underline{X}$ is immediate since D(A) is dense in $\underline{X}$. If, now, $Az = 0$, then again one has $(I - \lambda A)^{-1}z = z$ for each $\lambda > 0$ so $Qz = z$. Hence, (i) is established.

The proof of (ii) follows exactly the same lines as in the strongly convergent case.

The identity $M(t,0)x - x = A((L)\int_t^0 dgM(\cdot,0)x)$ gives at once that $Qx = 0$ only in case x is in the weak closure of the range of A. Since the range of A is a linear subspace of $\underline{X}$, its weak closure is precisely its strong closure; thus, (iii) is established.

<u>Theorem 8</u>. Let A, g, and M be as in Theorem 3 and suppose that $M(\cdot,0)$ is strongly (resp., weakly) asymptotically convergent. If each of s and t is in S with $s \geq t$, if each of x and z is in $\underline{X}$, and if

$W(z;s,t)x = {}_s\Pi^t(I - dg(A + z))^{-1}x$ as in Theorem 5, then these are equivalent:

(i) z is in the range of A,

(ii) For each x in $\underline{X}$, $\lim_{t\to+\infty} W(z;t,0)x$ (resp., $\text{w-}\lim_{t\to+\infty} W(z;t,0)x$) exists and is a solution y of the equation $Ay = -z$,

(iii) There is an x in $\underline{X}$ and an increasing, unbounded sequence $\{t_k\}_{k=1}^{\infty}$ in S such that $\text{w-}\lim_{k\to+\infty} W(z;t_k,0)x$ exists.

Proof. The proof is given first in the case that M is strongly asymptotically convergent.

If $Au = z$, then

$$W(z;t,0)x = M(t,0)x + (R)\int_t^0 dgM(t,\cdot)Au$$

$$= M(t,0)x = M(t,0)u - u.$$

Hence, $\lim_{t\to+\infty} W(z;t,0)x = Qx + Qu - u$ and $AQx + AQu - Au = -z$. Since A is linear, (i) implies (ii).

That (ii) implies (iii) is clear.

Finally, suppose that (iii) holds so that $\text{w-}\lim_{k\to+\infty} W(z;t_k,0)x = u$. If x_o is in $D(A)$, then

$$u + Q(x_o - x) = \text{w-}\lim_{k\to\infty} W(z;t_k,0)x + \lim_{k\to\infty} M(t_k,0)(x_o - x)$$

$$= \text{w-}\lim_{k\to\infty} W(z;t_k,0)x_o$$

and

$$\underset{k\to\infty}{\text{w-lim}}\ AW(z;t_k,0)x_o = \underset{k\to\infty}{\text{w-lim}}\ AW(z;t_k,0)x_o - AM(t_k,0)x_o$$

$$= \underset{k\to\infty}{\text{w-lim}}\ A(R)\int_{tk}^{o} dgM(t_k,\cdot)z$$

$$= \underset{k\to\infty}{\text{w-lim}}\ M(t_k,0)z - z$$

$$= Qz - z.$$

By Lemma 6, one has $Au = Qz - z$.

Now

$$W(z;t_k,0)x = M(t_k,0)x + (R)\int_{t_k}^{o} dgM(t_k,\cdot)z$$

$$= M(t_k,0)x + (R)\int_{t_k}^{o} dgM(t_k,\cdot)(z - Qz)$$

$$+ (R)\int_{t_k}^{o} dgM(t_k,\cdot)Qz$$

$$= M(t_k,0)x + (R)\int_{t_k}^{o} dgM(t_k,\cdot)(-Au) + (R)\int_{t_k}^{o} dgM(t_k,\cdot)Qz$$

$$= M(t_k,0)x + u - M(t_k,0)u + (R)\int_{t_k}^{o} dgQz.$$

Since $\underset{k\to\infty}{\text{w-lim}}\ W(z;t_k,0)x$, $\underset{k\to\infty}{\text{w-lim}}\ M(t_k,0)x$, and $\underset{k\to\infty}{\text{w-lim}}\ M(t_k,0)u$ all exist, it follows that $\underset{k\to\infty}{\text{w-lim}}\ (R)\int_{t_k}^{o} dgQz$ exists. Since $\lim_{s\to\infty} g(s) = +\infty$, $Qz = 0$; so $Au = -z$ and (iii) implies (i).

In case M is weakly asymptotically convergent, again suppose $Au = z$ at first. Again, the variation of parameters formula gives that

$$\underset{t\to+\infty}{\text{w-lim}}\, W(z;t,0)x = Qx + Qu - u\ .$$

Again, note that $A(Qx + Qu - u) = -z$.

To show that (iii) implies (i), again suppose that $\underset{k\to+\infty}{\text{w-lim}}\, W(z;t_k,0)x = u$. One has again that if x_o is in $D(A)$, then

$$u + Q(x_o - x) = \underset{k\to\infty}{\text{w-lim}}\, W(z;t_k,0)x_o$$

and

$$\underset{k\to\infty}{\text{w-lim}}\, AW(z;t_k,0)x_o = Qz - z.$$

Since $W(z;t_k,0)x = M(t_k,0)x + u - M(t_k,0)u + (R)\int_{t_k}^{o} dgQz$, one has $Qz = 0$; so $Au = -z$ and (iii) implies (i).

<u>Remark</u>. The original requirement that A be dissipative can be weakened somewhat. In particular, suppose that A is a linear function from D(A) in $\underline{\overline{X}}$ to $\underline{\overline{X}}$ such that D(A) is dense in $\underline{\overline{X}}$ and that there is a number C such that if $\{\lambda_k\}_{k=1}^{m}$ is a sequence of positive numbers, then $\left|\prod_{k=1}^{m} (I - \lambda_k A)^{-1}\right| \leq C$. It follows that $|\exp(tA)| \leq C$ for each $t \geq 0$. The norm, $|x|_2 = \sup_{t\geq o}|\exp(tA)x|$, is equivalent to the norm $|\cdot|$ on $\underline{\overline{X}}$, and $|\exp(tA)x|_2 \leq |x|_2$. Hence, A is dissipative with respect to $|\cdot|_2$. The previous hypothesis that $|(I - \lambda A)^{-1}| \leq 1$ for each $\lambda > 0$ can thus be weakened.

Another extension of the integral equation theory of Chapter III can be had. If β and λ are numbers, one has the identity

$$(I - \lambda(A + \beta I))^{-1} = (1 - \lambda\beta)^{-1}(I - \lambda(1 - \lambda\beta)^{-1}A)^{-1}$$

provided that $\lambda\beta \neq 1$. If g is a non-increasing function from S to R and if β is negative, then ψ, defined by

$$\psi(t) = \int_t^0 dg(1 - \beta dg)^{-1} ,$$

is non-increasing. If A is dissipative with respect to some norm equivalent to $|\cdot|$, then $A + \beta I$ is dissipative and Theorem 3 already guarantees the existence of ${}_s\Pi^t(I - dg(A + \beta I))^{-1}$ for $s \geq t$ in S. The identity,

$${}_s\Pi^t(I - dg(A + \beta I))^{-1} = ({}_s\Pi^t(1 - \beta dg)^{-1})({}_s\Pi^t(I - d\psi A)^{-1}) ,$$

furnishes better normed estimates in Theorems 3 and 5. The theory of Chapter IV is largely had already as a part of the Hille-Yosida Theorem.

Some passage of the theory of Chapters III and IV to the operator $A + \beta I$, $\beta > 0$, can also be had. One requires of the function g that there exist a number P such that if s is in S, then there exist u and v in S such that $v < s < u$ and $g(v) - g(u) \leq P$. If $\beta P < 1$, one has $\psi(t) = \int_t^0 dg(1 - \beta dg)^{-1}$ is non-increasing,

$${}_s\Pi^t(I - dg(A + \beta I))^{-1} = ({}_s\Pi^t(1 - \beta dg)^{-1})({}_s\Pi^t(I - d\psi A)^{-1}),$$

and the attendant integral equation theory holds. If, as in Chapter IV, $\lim_{t\to+\infty} {}_t\Pi^o(dg(A + \beta I))^{-1}x$ exists (or if the weak limit exists) for each x in $\underline{\overline{X}}$ and is Qx, then the uniform boundedness theorem gives that Q is continuous. Statements (i) and (iii) of Theorem 7 and the iteration description of Theorem 8 follow with A replaced by $A + \beta I$. Even if one has only that $\{|{}_t\Pi^0(I - dg(A + \beta I))^{-1}| : t > 0\}$ is bounded, then the iteration theory of Chapter IV can be developed for the operators $A + \gamma I$, $0 \leq \gamma < \beta$.

REFERENCES

1. F. Browder and W. Petryshyn, The solution by iteration of linear functional equations in Banach spaces, Bull. Amer. Math. Soc., 72 (1966), 566-570.

2. F. Gilfeather, Asymptotic convergence of operators in Hilbert space, Proc. Amer. Math. Soc., 22 (1969), 69-76.

3. J. V. Herod, A pairing of a class of evolution systems with a class of generators, Trans. Amer. Math. Soc., 157 (1971), 247-260.

4. J. V. Herod, Generators for evolution systems with quasicontinuous trajectories, Pacific J. Math., 53 (1974), 153-161.

5. D. L. Lovelady, Ergodic methods for closed linear operator equations, to appear.

6. J. S. Mac Nerney, Integral equations and semigroups, Illinois J. Math., 7 (1963), 148-173.

7. J. S. Mac Nerney, A linear initial-value problem, Bull. Amer. Math. Soc., 69 (1963), 314-329.

8. J. S. Mac Nerney, A non-linear integral operation, Illinois J. Math., 8 (1964), 621-638.

9. R. H. Martin, Jr., Product integral approximations of solutions to linear operator equations, Proc. Amer. Math. Soc., 41 (1973), 506-512.

10. H. F. Trotter, Approximation of semigroups of operators, Pacific J. Math., 8 (1958), 887-919.

11. K. Yosida, Functional Analysis, Springer-Verlag, New York, 1968.

VITA

Seaton Driskell Purdom was born June 9, 1948 in Birmingham, Alabama, the son of Thomas T. and Louise D. Purdom.

He attended Westminster Boys' High in Atlanta, Georgia, graduating in June, 1965.

In 1969, he obtained a B.S. in Applied Mathematics from the Georgia Institute of Technology; in 1971 a Masters. From 1968 to 1975, he was employed as a Graduate Teaching Assistant in the School of Mathematics.

He has presented papers at one national and two regional meetings of the American Mathematical Society. A paper based upon the results in this work has been accepted for publication by the *Transactions of the American Mathematical Society*.