ON EXACT POISSON STRUCTURES

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ABSTRACT. By studying the exactness of multi-linear vectors on an orientable smooth manifold $\mathbf{M}$, we give some characterizations to exact Poisson structures defined on $\mathbf{M}$ and study general properties of these structures. Following recent works [12, 13, 15], we will pay particular attention to the classification of some special classes of exact Poisson structures such as Jacobian and quasi-homogeneous Poisson structures. A characterization of exact Poisson structures which are invariant under the flow of a class of completely integrable systems will also be given.

1. INTRODUCTION

Let $\mathbf{M}$ be an orientable C^∞ smooth manifold of dimension n and let $C^\infty(\mathbf{M})$ be the space of C^∞ smooth functions defined on $\mathbf{M}$. A *Poisson structure* Λ on $\mathbf{M}$ is an algebra structure on $C^\infty(\mathbf{M})$ satisfying the Leibniz identity, i.e.,

$$\Lambda = \{\cdot,\cdot\} : C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M}) \to C^\infty(\mathbf{M}),$$

is a bilinear map such that for arbitrary $f, g, h \in C^\infty(\mathbf{M})$ the following holds:

(a) (Skew-symmetry) $\{f, g\} = -\{g, f\}$,
(b) (Leibniz rule) $\{f, gh\} = \{f, g\}h + g\{f, h\}$,
(c) (Jacobi identity) $\{\{f, g\}, h\} = \{f, \{g, h\}\} + \{\{f, h\}, g\}$.

With a Poisson structure $\{,\}$, the algebra $(C^\infty(\mathbf{M}), \{,\})$ becomes a Lie algebra (see e.g., [16, 17]). The pair $(\mathbf{M}, \{,\})$ is called a *Poisson Manifold*. In what follows, smooth manifolds always mean orientable C^∞ smooth manifolds.

With respect to a local coordinate system $\{x_i\}$ on $\mathbf{M}$, such a structure can be explicitly defined so that for arbitrary $f, g \in C^\infty(\mathbf{M})$

$$\Lambda(df, dg) = \{f, g\} = \sum_{i,j=1}^{n} w_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \tag{1.1}$$

where $w_{ij} \in C^\infty(\mathbf{M})$, $i, j = 1, \ldots, n$, satisfy the identities

$$w_{ij} + w_{ji} = 0, \qquad \sum_{l=1}^{n} \sum_{\sigma \in A_3} w_{l\sigma(i)} \frac{\partial w_{\sigma(j)\sigma(k)}}{\partial x_l} = 0,$$

here A_3 is the group of cyclic permutations acting on (i, j, k). The matrix $J = (w_{ij})$ is called a *structure matrix* associated to Λ. Since an everywhere non-degenerate

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1991 *Mathematics Subject Classification.* **53D17, 37K10, 70G60, 70G45, 70G65**.

Key words and phrases. **exact Poisson structure, classification, normal form**.

The first author was partially supported by NSF grant DMS0204119.

The second author is partially supported by NSFC grant 10231020 and Shuguang plan of Shanghai.

Poisson structure is necessarily symplectic, Poisson structures are natural extensions to the standard symplectic ones on a smooth manifold.

Let $\mathcal{X}^k(\mathbf{M})$ be the space of smooth k-linear vectors on $\mathbf{M}$ and $\Omega^k(\mathbf{M})$ the space of differential $k-$forms. For a given volume element ω on $\mathbf{M}$ in standard local expression, we consider its induced isomorphism $\Phi : \mathcal{X}^k(\mathbf{M}) \to \Omega^{n-k}(\mathbf{M})$: $u \to \mathbf{i}_u\omega$, where $\mathbf{i}_u\omega$ is the inner product of u and ω. For instance, if $u = f(\mathbf{x})\partial_{i_1} \wedge \partial_{i_2} \wedge \ldots \wedge \partial_{i_k}$ with $1 \le i_1 < i_2 < \ldots < i_k \le n$, and $\omega = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$ under the local coordinate $\{x_i\}$, then $\mathbf{i}_u\omega = (-1)^{i_1-1}(-1)^{i_2-2}\ldots(-1)^{i_k-k}f(\mathbf{x})dx_1 \wedge \widehat{dx_{i_1}} \wedge \ldots \wedge \widehat{dx_{i_2}} \wedge \ldots \wedge \widehat{dx_{i_k}} \wedge \ldots \wedge dx_n$, hereafter $\widehat{x}$ stands for the omission of x. Let

$$D \equiv \Phi^{-1} \circ d \circ \Phi : \mathcal{X}^k(\mathbf{M}) \to \mathcal{X}^{k-1}(\mathbf{M})$$

be the pull-back operator under the isomorphism Φ, where d is the exterior derivative of differential forms. A k-linear vector $\mathbf{X}_k$ is said to be *exact* if $D(\mathbf{X}_k) = 0$. Specifically, a Poisson structure Λ satisfying $D(\Lambda) = 0$ is called an *exact Poisson structure*. We note that symplectic structures are always exact. It is therefore hopeful that an exact Poisson structure resembles a symplectic one to certain extends.

The present paper is devoted to the study of exact Poisson structures with respect to their characterizations and general properties.

In [10], the author showed that the operator D has some important applications. For instance, the operator can be used to compute the Schouten brackets and to varify the Jacobian identity which a Poisson structure should satisfy. In Section 2, we will further study properties of D. In particular, we will show that under the action of D the direct sum $\oplus_{k=0}^n \mathcal{X}^k(\mathbb{R}^n)$ forms a complex. We will also study the homology induced by the complex and its topological structure.

These properties of D and the volume preserving property of exact Poisson structures will be used in Section 3 to study three special classes of Poisson structures: the Lie-Poisson, Jacobian, and the quasi-homogeneous structures. We will also investigate the Hamiltonian flows induced by these Poisson structures, with respect to issues such as normal forms and volume-preservation.

It is known that any Jacobian structure is a Poisson structure ([9, 15]) and any Jacobian structure with constant Jacobian coefficient is exact ([15]). In Section 3, we will give a general sufficient condition for an exact Poisson structure to become Jacobian. We will also study some general properties of the Jacobian structures. Part of our results in this regard generalizes some of those in [15].

As for the quasi-homogeneous Poisson structures, we will give a necessary and sufficient condition for a decomposition of a quasi-homogeneous Poisson structure with respect to exact Poisson structures. This result is an improvement to the corresponding ones in [12, 15]. Restricting to the classical r-Poisson structures - a class of special quadratic Poisson structures, we will obtain a necessary and sufficient condition for a classical r-Poisson structure in $\mathbb{R}^3$ to be exact. Using this result we further prove that any quadratic Jacobian structure is a classical r-Poisson structure in $\mathbb{R}^3$. It is known that any quadratic Poisson structure in $\mathbb{R}^2$ is a classical r-Poisson structure ([12]) but in $\mathbb{R}^3$ the Poisson structure $\overline{\Lambda} = (x_1^2 + \alpha x_2 x_3)\partial_2 \wedge \partial_3$ for $\alpha \neq 0$ is not a classical r-Poisson structure ([13]). However, we note that for $\alpha = 0$ the structure $\overline{\Lambda}$ is a Jacobian structure. This leads to an open problem to *classify all classical r-Poisson structures within the class of quadratic Poisson* *structures*. A more concrete question in dimension 3 is that *whether a quadratic, non-Jacobian Poisson structure is necessarily not a classical r-Poisson structure.*

In Section 4, we will give a characterization of exact Poisson structures which are preserved by a $\mathbb{T}^q$-dense, completely integrable flow. We will also show that such a $\mathbb{T}^q$-dense flow preserving an exact Poisson structure can be non-Hamiltonian, in contrast to the closed 2-form case in which any $\mathbb{T}^q$-dense flow preserving a symplectic structure is necessary a Hamiltonian ([4]).

2. General properties of exact Poisson structures

2.1. **The operator D.** We first recall the following properties of the operator D which will be used later on.

Proposition 2.1. ([10]) *The following holds.*

(i) *For any $\mathbf{X} \in \mathcal{X}(\mathbf{M})$, $D(\mathbf{X}) = \mathrm{div}_\omega \mathbf{X}$, where $\mathrm{div}_\omega \mathbf{X}$ denotes the divergence of the vector field $\mathbf{X}$ with respect to the volume element ω.*

(ii) *For any $\mathbf{X}, \mathbf{Y} \in \mathcal{X}(\mathbf{M})$, $D(\mathbf{X} \wedge \mathbf{Y}) = [\mathbf{Y}, \mathbf{X}] + (\mathrm{div}_\omega \mathbf{Y})\mathbf{X} - (\mathrm{div}_\omega \mathbf{X})\mathbf{Y}$, where $[\cdot,\cdot]$ denotes the usual Lie bracket of two vector fields, and $\wedge$ denotes the wedge product of two vectors.*

(iii) *For any $U \in \mathcal{X}^\mu(\mathbf{M})$ and $V \in \mathcal{X}^\nu(\mathbf{M})$,*

$$[U, V] = (-1)^\mu D(U \wedge V) - D(U) \wedge V - (-1)^\mu U \wedge D(V),$$

where $[U, V]$ denotes the Schouten bracket of the multi-linear vector fields U and V, which is defined as the following: if $U = u_1 \wedge \ldots \wedge u_\mu$ and $V = v_1 \wedge \ldots \wedge v_\nu$, then

$$[U, V] = \sum_{s,t} (-1)^{s+t} u_1 \wedge \ldots \wedge \widehat{u}_s \wedge \ldots \wedge u_\mu \wedge [u_s, v_t] \wedge v_1 \wedge \ldots \wedge \widehat{v}_t \wedge \ldots \wedge v_\nu.$$

(iv) *Let Λ be a Poisson structure and $\mathbf{X}_\Lambda = D(\Lambda)$ be its curl vector field. Then the Lie derivative $L_{\mathbf{X}_\Lambda}\Lambda \equiv [\mathbf{X}_\Lambda, \Lambda] = 0$.*

(v) *A skew-symmetric bilinear vector field Λ is an exact Poisson structure if and only if $D(\Lambda) = 0$ and $D(\Lambda \wedge \Lambda) = 0$.*

Remark 2.1. 1) *Statement* (i) *of the above proposition implies that a vector field is exact if and only if it is divergent free. Hence it follows from* Gauss' Theorem *that the volume is invariant under the flow induced by an exact vector field.*

2) *For a* 3*-dimensional vector fields $\mathbf{v} = a\partial_x + b\partial_y + c\partial_z$, the curl of $\mathbf{v}$ is defined in the usual way as $\nabla \times \mathbf{v} = (c_y - b_z)\partial_x + (a_z - c_x)\partial_y + (b_x - a_y)\partial_z$. This is in fact the curl vector field $\mathbf{X}_\Lambda = D(\Lambda)$ with $\Lambda = a\partial_y \wedge \partial_z + b\partial_z \wedge \partial_x + c\partial_x \wedge \partial_y$. Thus, the operator D unifies the computations for the divergence and the curl of a given vector field.*

Let $\mathcal{X}^*(\mathbb{R}^n) = \oplus_{k=0}^n \mathcal{X}^k(\mathbb{R}^n)$ be the algebra formed by the direct sum of the space of the k-linear vectors. The following result describes certain homological properties induced by the operator D, which will be used later in the classification of exact Poisson structures.

Proposition 2.2. *The following holds.*

(a) $D^2 = 0$.

(b) *The D-homology of $\mathbb{R}^n$ formed by the vector space*

$$H_k(\mathbb{R}^n) = \big((\text{kernel of } D) \cap \mathcal{X}^k(\mathbb{R}^n)\big) \,/\, \big((\text{image of } D) \cap \mathcal{X}^k(\mathbb{R}^n)\big),$$

has the topological structures

$$H_k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n, \\ 0, & 0 \le k < n. \end{cases}$$

Proof. It follows from the definition of D, the de Rham cohomology and Poincaré's Lemma (e.g. [2]) in $\mathbb{R}^n$. □

Remark 2.2. Proposition 2.2 *implies that* $\mathcal{X}^*(\mathbb{R}^n)$ *is a complex induced by* D *and the sequence of vector spaces*

$$0 \longrightarrow \mathcal{X}^n(\mathbb{R}^n) \overset{D}{\longrightarrow} \mathcal{X}^{n-1}(\mathbb{R}^n) \overset{D}{\longrightarrow} \dots \overset{D}{\longrightarrow} \mathcal{X}^0(\mathbb{R}^n) \longrightarrow 0$$

is exact.

2.2. **Characterization of exactness.** The following lemma will be used in the general characterization of exact Poisson structures to be given in this section.

Lemma 2.1 ([8])**.** *Let* Λ *be a smooth Poisson structure in* $\mathbb{R}^n$ *defined by* (1.1)*. Then* Λ *is exact if and only if*

$$\sum_{j=1}^{n} \frac{\partial w_{ij}}{\partial x_j} = 0, \qquad i = 1, \dots, n. \tag{2.1}$$

Proof. It follows from the fact that

$$D(\Lambda) = 2\sum_{i=1}^{n} \left(\sum_{j=1, j\neq i}^{n} \frac{\partial w_{ij}}{\partial x_j} \right) \frac{\partial}{\partial x_i}. \tag{2.2}$$

□

Recall that a smooth function H defined on a smooth manifold $\mathbf{M}$ is a *first integral* of a smooth vector field $\mathbf{X}$ if $\mathbf{X}(H) \equiv 0$ on $\mathbf{M}$.

Our next result gives a characterization of exact Hamiltonian vector fields.

Theorem 2.1. *Let* Λ *be a Poisson structure on a smooth Riemanian manifold* $\mathbf{M}$*. Then a Hamiltonian vector field* $\mathbf{X}_H$ *associated to* Λ *and the Hamilton* H *is exact if and only if the curl vector field* $\mathbf{X}_\Lambda = D(\Lambda)$ *and the gradient vector field* ∇H *are everywhere orthogonal on* $\mathbf{M}$*, i.e.,* $D(\Lambda)$ *belongs to the tangent spaces of the level surfaces of* H*; or equivalently,* H *is a first integral of the curl vector field* $D(\Lambda)$*.*

Proof. On any orientable manifold there is a volume form that locally takes the standard one in $\mathbb{R}^n$, so without loss of generality, we can prove the theorem under a local coordinate system $\{x_i\}_{i=1}^n$ on M by taking the standard volume element $\omega = dx_1 \wedge \ldots \wedge dx_n$, where $n = \dim M$.

Let $J = (w_{ij})$ be the structure matrix of Λ and $\mathbf{X}_H = J\nabla H$ be the associated Hamiltonian vector field. Making use of the skew-symmetry of J, calculations yield that

$$D(\mathbf{X}_H) = -\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial w_{ij}}{\partial x_j} \right) \frac{\partial H}{\partial x_i}. \tag{2.3}$$

By (2.2), we further have

$$D(\mathbf{X}_H) = -\frac{1}{2} D(\Lambda) \cdot \nabla H, \tag{2.4}$$

from which the theorem easily follows. □

Remark 2.3. 1) *From (2.4) and the fact that*

$$D(\mathbf{X}_H) = \Phi^{-1} \circ d \circ \Phi(\mathbf{X}_H) = \Phi^{-1} \circ d \circ \mathbf{i}_{\mathbf{X}_H} \omega = \Phi^{-1}(L_{\mathbf{X}_H} \omega),$$

it follows easily that Λ is exact if and only if any corresponding Hamiltonian vector field is exact, or equivalently, the volume element ω is invariant under any Hamiltonian flow associated to Λ. This result is also a consequence of the theorem given in [18].

2) *By the* Birkhoff Ergodic Theorem*, for any Hamiltonian flow ϕ_t on an orientable manifold $\mathbf{M}$ induced by an exact Poisson structure,*

$$P(f)(\mathbf{x}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f \circ \phi_t(\mathbf{x}) dt,$$

is a well defined L^1 function for any $f \in L^1(\mathbf{M})$.

There are two issues involved in checking whether a skew symmetric, bilinear vector field Λ is an exact Poisson structure, i.e., Λ needs to be exact and satisfy the Jacobian identity. For $n \geq 3$, it is known that Λ is exact if and only if the $(n-2)$-form

$$\Omega_{n-2} = \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} w_{ij} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n$$

is closed, i.e., $d\Omega_{n-2} = 0$ (see for instance [12] or [8]). Moreover, if Λ is exact, then there exists an $(n-3)$-form Ω_{n-3} such that $d\Omega_{n-3} = \Omega_{n-2}$, or equivalently, there exists a 3-linear vector $\mathbf{X}_3$ such that $D\mathbf{X}_3 = \Lambda$. The following proposition establishes certain connections between exactness and the Jacobian identity. In particular, for an exact Poisson structure, the Jacobian identity can be written in a symmetric form.

Proposition 2.3. *Let Λ be a skew symmetric bilinear vector field in $\mathbb{R}^n$ with the structure matrix $J = \{w_{ij}\}$. Then Λ is an exact Poisson structure if and only if*

$$\sum_{j=1}^{n} \frac{\partial w_{ij}}{\partial x_j} = 0, \qquad i = 1, \ldots, n, \tag{2.5}$$

$$\sum_{s=1, s\neq i,j,k}^{n} \left(A^s_{ijk} \cdot \frac{\partial B^s_{ijk}}{\partial x_s} + B^s_{ijk} \cdot \frac{\partial A^s_{ijk}}{\partial x_s} \right) = 0, \qquad 1 \leq i < j < k \leq n, \tag{2.6}$$

where $A^s_{ijk} = (w_{si}, w_{sj}, w_{sk})$, $B^s_{ijk} = (w_{jk}, w_{ki}, w_{ij})$.

Proof. We only prove the necessity, as the sufficiency follows from a similar argument. The condition (2.5) clearly follows from Lemma 2.1. Now for arbitrary integers $1 \le i < j < k \le n$ we have by (2.5) that

$$
\begin{aligned}
& \sum_{s=1}^{n} \sum_{\sigma\in A_3} w_{s\sigma(i)} \frac{\partial w_{\sigma(j)\sigma(k)}}{\partial x_s} \\
= \quad & \sum_{s=1, s\ne i,j,k}^{n} \sum_{\sigma\in A_3} w_{s\sigma(i)} \frac{\partial w_{\sigma(j)\sigma(k)}}{\partial x_s} \\
& + w_{jk}\left(\frac{\partial w_{ij}}{\partial x_j} + \frac{\partial w_{ik}}{\partial x_k}\right) + w_{ki}\left(\frac{\partial w_{ji}}{\partial x_i} + \frac{\partial w_{jk}}{\partial x_k}\right) + w_{ij}\left(\frac{\partial w_{ki}}{\partial x_i} + \frac{\partial w_{kj}}{\partial x_j}\right) \\
= \quad & \sum_{s=1, s\ne i,j,k}^{n} \sum_{\sigma\in A_3} w_{s\sigma(i)} \frac{\partial w_{\sigma(j)\sigma(k)}}{\partial x_s} + \sum_{\sigma\in A_3} w_{\sigma(j)\sigma(k)} \sum_{s=1, s\ne i,j,k}^{n} \frac{\partial w_{s\sigma(i)}}{\partial x_s} \\
= \quad & \sum_{s=1, s\ne i,j,k}^{n} \sum_{\sigma\in A_3} \left(w_{s\sigma(i)} \frac{\partial w_{\sigma(j)\sigma(k)}}{\partial x_s} + w_{\sigma(j)\sigma(k)} \frac{\partial w_{s\sigma(i)}}{\partial x_s} \right).
\end{aligned}
$$

This proves the condition (2.6). □

We note that the condition (2.6) cannot be simplified further under the exactness condition. In dimension 3, the condition (2.6) is trivially satisfied. This means that exactness condition implies the Jacobian identity in dimension 3.

3. Lie-Poisson, Jacobian, and Quasi-homogeneous structures

In this section, we will apply the general characterization of exactness from the previous section to obtain more precise information for three special Poisson structures: Lie-Poisson, Jacobian, and Quasi-homogeneous Poisson structures.

3.1. Lie-Poisson structures.

A *Lie-Poisson structure* in $\mathbb{R}^n$ or $\mathbb{C}^n$ is defined by

$$
L = \sum_{i,j,k=1}^{n} c_{ij}^k x_k \partial_i \wedge \partial_j, \tag{3.1}
$$

with $c_{ij}^k{}'s$ being the structure constants of an n-dimensional Lie algebra.

It is well known that a Lie-Poisson structure is necessary a Poisson structure ([14]). We have the following characterization.

Theorem 3.1. *Let L be the Lie-Poisson structure defined in* (3.1). *The following holds.*

(a) *L is exact if and only if $\sum\limits_{j=1}^{n} c_{ij}^j = 0$, $i = 1, \ldots, n$.*

(b) *An associated Hamiltonian vector field $X_H = L(\cdot, dH)$ is exact if and only if the Hamiltonian H is a first integral of the completely integrable vector fields $\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} c_{ij}^j\right)\partial_i$.*

(c) *If the Lie algebra $\mathfrak{g}$ formed by homogeneous polynomials of degree 1 under the action of the Lie bracket induced by* (3.1) *is nilpotent, then the structure* (3.1) *is affine equivalent to a structure of the type* (3.1) *with $c_{ij}^k = 0$ for $k \ge \min\{i, j\}$, and consequently, the resulting structure is exact.*

Proof. Statements (a) and (b) follow directly from Theorem 2.1. Using Theorem 3.5.4 of [16] and statement (a) above we obtain the statement (c). □

Lie-Poisson structures play important roles in studying normal forms for a class of Poisson structures. Let Λ be an analytic Poisson structure in a neighborhood of the origin in $\mathbb{R}^n$ or $\mathbb{C}^n$ defined by

$$\Lambda = \sum_{i,j,k=1}^{n} (c_{ij}^k x_k + R_{ij}(x))\partial_i \wedge \partial_j, \tag{3.2}$$

where $R_{ij} = O(|x|^2)$, $i,j = 1,\cdots,n$. Using Theorem 2.1, Theorem 4.1 of [5], Theorem 2.1 of [17], and the generalized Darboux Theorem, one easily has the following.

Proposition 3.1. *Let Λ be an analytic Poisson structure of rank $2m$ in a neighborhood of the origin in $\mathbb{R}^n$ or $\mathbb{C}^n$. If the Lie algebra $\mathfrak{g}$ with the structure constants formed by the coefficients $\{c_{ij}^k\}$ of the linear truncation for the singular part of Λ is semi-simple, then Λ is analytically equivalent to*

$$P = \sum_{i=1}^{m} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \sum_{1\le i<j\le n-2m} \left(\sum_{k=1}^{n-2m} c_{ij}^k y_k \right) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}.$$

If, in addition, $\sum_{j=1}^{n-2m} c_{ij}^j = 0$ for $i = 1,\dots,n-2m$, i.e., P is exact, then any Hamiltonian flow associated to Λ is analytically equivalent to a volume-preserving one.

Remark 3.1. *The analyticity in* Proposition 3.1 *can be replaced by smoothness if the Lie algebra $\mathfrak{g}$ is of compact type. The existence of smooth equivalency in the smooth case can be shown by using* Theorem 4.1 *of* [6].

3.2. **Jacobian structures.** Let $n \ge 3$ be an integer. Following [9], a *Jacobian bracket* in $\mathbb{R}^n$ is a bilinear map $\{\cdot,\cdot\} : C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M}) \to C^\infty(\mathbf{M})$, satisfying

$$\{f,g\} = u \det(J(f,g,P_1,\dots,P_{n-2})),$$

where $u, P_i \in C^\infty(\mathbb{R}^n)$, $i = 1,\dots,n-2$, J denotes the usual Jacobian matrix of $f,g,P_1,\dots, P_{n-2}$ with respect to the variables $x_1,\dots,x_n$, and '$\det'$ stands for the determinant of a matrix. With the above setting, the Jacobian bracket is said to be *generated* by $P_1,\dots,P_{n-2}$ with a *Jacobian coefficient* u. It is easy to see that if $P_1,\cdots,P_{n-2}$ are functionally dependent, then the Jacobian bracket is trivial. In what follows, we always assume that the generators $P_1,\dots,P_{n-2}$ of a Jacobian bracket are functionally independent. It was shown in [9] that a *Jacobian structure*, i.e., an algebra structure defined by a Jacobian bracket, is necessary a Poisson structure and in [15] that a Jacobian structure with a constant Jacobian coefficient is always exact.

We now give some conditions under which a Poisson structure becomes Jacobian.

Theorem 3.2. *For Poisson structures in $\mathbb{R}^n$, the following holds.*

(a) *For $n = 2$, a smooth Poisson structure is exact if and only if it is a constant Poisson structure.*
(b) *For $n = 3$, a smooth Poisson structure is exact if and only if it is a Jacobian structure with a constant Jacobian coefficient.*
(c) *For $n > 3$, a smooth Poisson structure is Jacobian with a non-zero constant Jacobian coefficient if and only if it is exact and has rank ≤ 2, and the*

Lebesgue measure of the set of points at which the structure has rank 0 is zero.

Proof. (a) A Poisson structure has the form $\Lambda = \omega(x,y)\partial_x \wedge \partial_y$. By part (b) of Proposition 2.2, $H_2(\mathbb{R}^2) = \mathbb{R}$. It follows that if Λ is exact, then it is a constant Poisson structure. The converse is obvious.

(b) Assume that Λ is an exact Poisson structure, i.e., $D(\Lambda) = 0$. By part (b) of Proposition 2.2, we have that $H_2(\mathbb{R}^3) = 0$, i.e., the kernel of D is equal to the image of D. Hence, there exists a 3-linear vector $\mathbf{X}_3 = P(x,y,z)\partial_x \wedge \partial_y \wedge \partial_z \in \mathcal{X}^3(\mathbb{R}^3)$ such that $D(\mathbf{X}_3) = \Lambda$. Since $D(\mathbf{X}_3) = P_x\partial_y \wedge \partial_z + P_y\partial_z \wedge \partial_x + P_z\partial_x \wedge \partial_y$, we see that Λ is the Jacobian structure generated by P. The sufficient part follows from [15].

(c) The necessary part of (c) is obvious. To prove the sufficient part, we let Λ be an exact Poisson structure of rank ≤ 2. Let $\{(U_i, \phi_i, g_i)\}$ be a partition of unity on $\mathbb{R}^n$, where $\{U_i\}$ is a locally finite open cover of $\mathbb{R}^n$, each ϕ_i is an isomorphism from U_i to $U_i' = \phi_i(U_i) \subset \mathbb{R}^n$ under which the Poisson structure Λ is transformed into $\Lambda_{\phi_i} = w_{12}^{(i)}\partial_{u_1} \wedge \partial_{u_2}$, each $g_i \ge 0$ has support in U_i, and moreover $g(x) = \sum_i g_i(x) = 1$ in $\mathbb{R}^n$, summing over a finite number of i's. The existence of such triples follows from pages 21-22 of [2] and Theorem 2.5.3 of [1].

Since Λ is exact, the structure Λ_{ϕ_i} is exact. By statement (a) Λ_{ϕ_i} is a constant Poisson structure corresponding to the variables u_1 and u_2. Hence $w_{12}^{(i)}$ is a function which is independent of the variables u_1 and u_2. Let

$$p_{1i}(u_3,\ldots,u_n) = \int_{u_{30}}^{u_3} w_{12}^{(i)}(s,u_4,\ldots,u_n)ds, \quad p_{ji} = u_{j+2}, \text{ for } j = 2,\ldots,n-2.$$

Then Λ_{ϕ_i} is a Jacobian structure generated by $p_{1i},\ldots,p_{n-2,i}$ on U_i'. Define

$$P_i(x) = \sum_j P_{ij}(x), \quad x \in \mathbb{R}^n,$$

$i = 1,\ldots,n-2$, where

$$P_{ij}(x) = \begin{cases} g_j(x)(\phi_j^{-1})_* p_{ij}(x), & \text{if } x \in U_j, \\ 0, & \text{if } x \notin U_j. \end{cases}$$

In the above, the subscript '$_*$' denotes the change of function p_{ij} under the coordinate transformation ϕ_j^{-1}. At each point $x \in \mathbb{R}^n$, since $\{(U_i, \phi_i, g_i)\}$ is a partition of unity on $\mathbb{R}^n$, there exists a $j \in \mathbb{N}$ such that $P_i(x) = \left(\phi_j^{-1}\right)_* p_{ij}(x)$. The fact $\Lambda = \left(\phi_j^{-1}\right)_* \left(\Lambda_{\phi_j}\right)$ in U_j implies that Λ is a Jacobian structure generated by $P_1, P_2, \cdots, P_{n-2}$ in some subregion containing x of U_j. Consequently, Λ is a Poisson structure generated by $P_1,\ldots,P_{n-2}$. This proves (c). □

Remark 3.2. 1) *Part* (b) *of the above theorem was stated in* [12] *for quadratic polynomial Poisson structures without a proof, and stated in* [15] *for general polynomial Poisson structures with a proof given for the quadratic case.*

2) *In dimension 2, a skew-symmetric, bilinear vector always satisfies the Jacobian identity. If it is exact, then it is either symplectic or trivial. In dimension 3, by part* (b) *of the above theorem, any Hamiltonian vector field associated to an exact Poisson structure is completely integrable.*

3) *In dimension 2, the set of Poisson structures forms a vector space, which is isomorphic to* $C^\infty(\mathbb{R}^2)$. *The set of exact Poisson structures is a 1-dimensional*

subspace isomorphic to $\mathbb{R}$*, which is therefore closed. In higher dimension, the set of Poisson structures cannot form a vector space. But the set of exact Poisson structures in* $\mathbb{R}^3$ *forms a vector space, which is isomorphic to the quotient space* $C^\infty(\mathbb{R}^3)/\mathbb{R}$.

4) *Using* Proposition 2.3*, an alternative proof of statement* (b) *of* Theorem 3.2 *can be given as follows. Let* $\Omega_1 = w_{23}dx_1 + w_{31}dx_2 + w_{12}dx_3$*. Then* $d\Omega_1 = 0$*, i.e.,* Ω_1 *is closed, hence exact. It follows that there exists a* 0*-form, i.e., a smooth function* P *such that* $dP = \Omega_1$*. This shows that the Poisson structure is Jacobian.*

Following [15], we now give another characterization of Jacobian structures depending on their Casimirs. For a Poisson structure Λ on $\mathbf{M}$, a function $h \in C^\infty(\mathbf{M})$ is called a *Casimir* of Λ if $\Lambda(df, dh) = \{f, h\} = 0$ for arbitrary $f \in C^\infty(\mathbf{M})$, or equivalently, the Hamiltonian vector field $\mathbf{X}_h = L_h\Lambda = \Lambda(\cdot, dh)$ is trivial. In other words, a Casimir of a Poisson structure Λ is a first integral of any Hamiltonian vector field $L_H\Lambda = \Lambda(\cdot, dH)$. The set of Casimirs of Λ is called the *center* of Λ.

The necessary part of the first statement in the following theorem was stated in [15] with a proof given for the case $n = 4$. We note that the proof for the case $n = 4$ in [15] does not extend to the general case.

Theorem 3.3. *A Poisson structure* Λ *in* $\mathbb{R}^n$ *has* $n-2$ *functionally independent Casimirs if and only if it is a Jacobian structure. Consequently, if* Λ *has exactly* $n-2$ *functionally independent Casimirs, then, excluding a set of zero Lebesgue measure, the common level manifold of the Casimirs is symplectic of dimension* 2.

Proof. The sufficient part of the first statement is obvious, because for a Jacobian structure in $\mathbb{R}^n$ its $n-2$ generators are already Casimirs.

We now prove the necessary part of the first statement. Let $\{x_i\}$ be a coordinate system in $\mathbb{R}^n$. Then the Poisson structure Λ has the representation

$$\Lambda(df, dg) \equiv \{f, g\} = \sum_{i,j=1}^{n} w_{ij} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}, \qquad \text{for } f, g \in C^\infty(\mathbb{R}^n),$$

where $w_{ij} \in C^\infty(\mathbb{R}^n)$. Let $P_1, \dots, P_{n-2}$ be a set of functionally independent Casimirs. By definition,

$$\Lambda(dx_i, dP_l) = \sum_{j=1}^{n} w_{ij}\frac{\partial P_l}{\partial x_j} = 0, \qquad i = 1, \dots, n;\ l = 1, \dots, n-2. \tag{3.3}$$

For any $n-2$ distinct elements $i_1, \dots, i_{n-2}$ of $1, \dots, n$, we denote by $\mathcal{J}(x_{i_1}, \dots, x_{i_{n-2}})$ the Jacobian matrix of $P_1, \dots, P_{n-2}$ with respect to $x_{i_1}, \dots, x_{i_{n-2}}$, i.e.,

$$\mathcal{J}(x_{i_1}, \dots, x_{i_{n-2}}) = \begin{pmatrix} \frac{\partial P_1}{\partial x_{i_1}} & \cdots & \frac{\partial P_1}{\partial x_{i_{n-2}}} \\ \frac{\partial P_2}{\partial x_{i_1}} & \cdots & \frac{\partial P_2}{\partial x_{i_{n-2}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial P_{n-2}}{\partial x_{i_1}} & \cdots & \frac{\partial P_{n-2}}{\partial x_{i_{n-2}}} \end{pmatrix}.$$

Since $P_1, \dots, P_{n-2}$ are functionally independent, we can assume without loss of generality that $\det(\mathcal{J}(x_1, \dots, x_{n-2})) \neq 0$.

Note that equation (3.3) with $i = n$ is equivalent to

$$\mathcal{J}(x_1, \ldots, x_{n-2}) \begin{pmatrix} w_{n1} \\ \vdots \\ w_{n,n-2} \end{pmatrix} = -w_{n,n-1} \begin{pmatrix} \frac{\partial P_1}{\partial x_{n-1}} \\ \vdots \\ \frac{\partial P_{n-2}}{\partial x_{n-1}} \end{pmatrix}. \tag{3.4}$$

Since $w_{n,n-1}$ can be arbitrary, if we choose

$$w_{n,n-1} = -u(x_1, \ldots, x_n) \det(\mathcal{J}(x_1, \ldots, x_{n-2})), \tag{3.5}$$

where $u \in C^\infty(\mathbb{R}^n)$ is an arbitrary function, then

$$w_{nj} = u \det(\mathcal{J}(x_1, \ldots, x_{j-1}, x_{n-1}, x_{j+1}, \ldots, x_{n-2})), \quad j = 1, \ldots, n-2. \tag{3.6}$$

Similarly, working with equation (3.3) for $i = n-1$ we have

$$w_{n-1,j} = -u \det(\mathcal{J}(x_1, \ldots, x_{j-1}, x_n, x_{j+1}, \ldots, x_{n-2})), \quad j = 1, \ldots, n-2. \tag{3.7}$$

Since equation (3.3) with $i = n-2$ is equivalent to

$$\begin{aligned} \mathcal{J}(x_1, \ldots, x_{n-2}) \begin{pmatrix} w_{n-2,1} \\ \vdots \\ w_{n-2,n-2} \end{pmatrix} &= -u \det(\mathcal{J}(x_1, \ldots, x_{n-3}, x_n)) \mathcal{J}(x_{n-1}) \\ &\quad + u \det(\mathcal{J}(x_1, \ldots, x_{n-3}, x_{n-1})) \mathcal{J}(x_n), \end{aligned}$$

we have

$$\begin{aligned} w_{n-2,j} &= -u \left(\frac{\det(\mathcal{J}(x_1, \ldots, x_{n-3}, x_n)) \det(\mathcal{J}(x_1, \ldots, x_{j-1}, x_{n-1}, x_{j+1}, \ldots, x_{n-2}))}{\det(\mathcal{J}(x_1, \ldots, x_{n-2}))} \right. \\ &\quad \left. - \frac{\det(\mathcal{J}(x_1, \ldots, x_{n-3}, x_{n-1})) \det(\mathcal{J}(x_1, \ldots, x_{j-1}, x_n, x_{j+1}, \ldots, x_{n-2}))}{\det(\mathcal{J}(x_1, \ldots, x_{n-2}))} \right), \end{aligned} \tag{3.8}$$

for $j = 1, \ldots, n-3$.

We claim that, for any $j = 1, \cdots, n-2$,

$$\begin{aligned} &\det(\mathcal{J}(x_1, \ldots, x_{n-3}, x_n)) \det(\mathcal{J}(x_1, \ldots, x_{j-1}, x_{n-1}, x_{j+1}, \ldots, x_{n-2})) \\ &- \det(\mathcal{J}(x_1, \ldots, x_{n-3}, x_{n-1})) \det(\mathcal{J}(x_1, \ldots, x_{j-1}, x_n, x_{j+1}, \ldots, x_{n-2})) \\ = \; &\det(\mathcal{J}(x_1, \ldots, x_{j-1}, x_{n-1}, x_{j+1}, \ldots, x_{n-3}, x_n)) \det(\mathcal{J}(x_1, \ldots, x_{n-2})). \end{aligned} \tag{3.9}$$

Indeed, let

$$M = \det \begin{pmatrix} \frac{\partial(P_1, \cdots, P_{n-2})}{\partial(x_1, \cdots, x_{n-3})} & \frac{\partial P}{\partial x_{n-2}} & O & \frac{\partial P}{\partial x_{n-1}} & O & \frac{\partial P}{\partial x_n} \\ O & \frac{\partial P}{\partial x_{n-2}} & \frac{\partial(P_1, \cdots, P_{n-2})}{\partial(x_1, \cdots, x_{j-1})} & \frac{\partial P}{\partial x_{n-1}} & \frac{\partial(P_1, \cdots, P_{n-2})}{\partial(x_{j+1}, \cdots, x_{n-3})} & \frac{\partial P}{\partial x_n} \end{pmatrix},$$

where

$$\frac{\partial(P_1, \cdots, P_l)}{\partial(x_1, \cdots, x_k)} = \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \cdots & \frac{\partial P_1}{\partial x_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial P_l}{\partial x_1} & \cdots & \frac{\partial P_l}{\partial x_k} \end{pmatrix},$$

$$\frac{\partial P}{\partial x_k} = \begin{pmatrix} \frac{\partial P_1}{\partial x_k} \\ \vdots \\ \frac{\partial P_{n-2}}{\partial x_k} \end{pmatrix},$$

$l, k = 1, \cdots, n$, and O denotes a zero matrix with appropriate dimension.

By expanding M according to the first $n-2$ rows, we have

$$\begin{aligned} M &= \det(\mathcal{J}(x_1,\ldots,x_{n-3},x_{n-2}))\det(\mathcal{J}(x_1,\ldots,x_{j-1},x_{n-1},x_{j+1},\ldots,x_{n-3},x_n)) \\ &\quad+\det(\mathcal{J}(x_1,\ldots,x_{n-3},x_{n-1}))\det(\mathcal{J}(x_1,\ldots,x_{j-1},x_n,x_{j+1},\ldots,x_{n-2})) \\ &\quad-\det(\mathcal{J}(x_1,\ldots,x_{n-3},x_n))\det(\mathcal{J}(x_1,\ldots,x_{j-1},x_{n-1},x_{j+1},\ldots,x_{n-2})). \end{aligned}$$

Since $M=0$, the claim follows.

From (3.8) and (3.9), we have

$$w_{n-2,j} = -u\det(\mathcal{J}(x_1,\ldots,x_{j-1},x_{n-1},x_{j+1},\ldots,x_{n-3},x_n)), \tag{3.10}$$

$j=1,\ldots,n-3$. Now, equation (3.3) with $1<k<n-2$ reads

$$\begin{aligned} &\mathcal{J}(x_1,\ldots,x_{n-2})\begin{pmatrix} w_{k,1} \\ \vdots \\ w_{k,n-2} \end{pmatrix} = \\ &\qquad -u\det(\mathcal{J}(x_1,\ldots,x_{k-1},x_n,x_{k+1},\ldots,x_{n-2}))\mathcal{J}(x_{n-1}) \\ &\qquad +u\det(\mathcal{J}(x_1,\ldots,x_{k-1},x_{n-1},x_{k+1},\ldots,x_{n-2}))\mathcal{J}(x_n). \end{aligned}$$

Similar to the proof for the case $n-2$ by using a slight modification of (3.9), we have

$$w_{kj} = -u\det(\mathcal{J}(x_1,\ldots,x_{j-1},x_{n-1},x_{j+1},\ldots,x_{k-1},x_n,x_{k+1},\ldots,x_{n-2})), \tag{3.11}$$

for $j=1,\ldots,k-1$.

Combining the formulas (3.5), (3.6), (3.7), (3.10) and (3.11), direct calculation yields that

$$\Lambda(df,dg) = \sum_{i,j=1}^{n} w_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} = u\det(J(f,g,P_1,\ldots,P_{n-2})).$$

This proves the first statement.

To prove the second statement, we denote by $\mathbf{M}$ the common level manifold of the Casimirs of Λ. Then $\dim\mathbf{M}=2$ except perhaps at a subset of zero Lebesgue measure.

From the proof of the first statement, it follows that the $n-2$ functionally independent Casimirs are the generators of the Jacobian structure Λ. In terms of suitable local coordinates $\{x_i\}$, it is easily seen that Λ has the canonical form $\partial_{x_{n-1}}\wedge\partial_{x_n}-\partial_{x_n}\wedge\partial_{x_{n-1}}$ in a neighborhood of each point of the manifold $\mathbf{M}$ except perhaps a subset of zero Lebesgue measure. This means that the structure matrix of Λ has rank 2 in $\mathbb{R}^n$ except perhaps a subset of zero Lebesgue measure, which is equal to the dimension of the manifold. Thus, the second statement follows. □

Part (a) of the following theorem was stated in [15] with a proof given for the case $n=4$.

Theorem 3.4. *Let Λ be a Jacobian structure in $\mathbb{R}^n$ generated by $P_1,\ldots,P_{n-2}$ $\in C^\infty(\mathbb{R}^n)$. The following holds.*

(a) *If $P_1,\ldots,P_{n-2}$ are functionally independent, then the center of Λ is a subalgebra of $C^\infty(\mathbb{R}^n)$, generated by $P_1,\ldots,P_{n-2}$.*

(b) *For any $H\in C^\infty(\mathbb{R}^n)$, the Hamiltonian vector field $\mathbf{X}_H=\Lambda(\cdot,dH)$ has the canonical form*

$$\begin{cases} \dot I = 0, \\ \dot\phi = \omega(I), \end{cases}$$

where $I = (I_1, \ldots, I_{n-1})^\top$ *and* $\omega(I) = 0$ *if* H *is functionally dependent on* $P_1, \ldots, P_{n-2}$.

Proof. (a) Since there are n functionally independent functions in the algebra $C^\infty(\mathbb{R}^n)$, we can choose two functions $f, g \in C^\infty(\mathbb{R}^n)$ such that $f, g, P_1, \ldots, P_{n-2}$ are functionally independent. Assume that Λ is the Jacobian structure generated by $P_1, \ldots, P_{n-2}$. For any $h \in C^\infty(\mathbb{R}^n)$, denote by $\mathbf{X}_h$ as the vector field $\Lambda(\cdot, dh)$. Then $f, P_1, \ldots, P_n$ are functionally independent first integrals of the vector field $\mathbf{X}_f$, and $g, P_1, \ldots, P_n$ are functionally independent first integrals of the vector field $\mathbf{X}_g$. Since a vector field in dimension n has at most $n-1$ functionally independent first integrals, if it has the maximal number of first integrals, then any other first integral is a function of these $n-1$ functionally independent first integrals.

Suppose that $H \in C^\infty(\mathbb{R}^n)$ is a Casimir of the Jacobian structure Λ. Then for any $h \in C^\infty(\mathbb{R}^n)$, we have $\Lambda(dh, dH) = 0$. Consequently, $\mathbf{X}_f(H) = \Lambda(dH, df) = 0$ and $\mathbf{X}_g(H) = \Lambda(dH, dg) = 0$. This means that H is a first integral of both vector fields $\mathbf{X}_f$ and $\mathbf{X}_g$. Hence, H is a smooth function of $f, P_1, \ldots, P_{n-2}$, and also a smooth function of $g, P_1, \ldots, P_{n-2}$. It follows from the functional independency of f and g that H should be a smooth function of $P_1, \ldots, P_{n-2}$. Hence, the center of the Jacobian Poisson structure is a sub-algebra of $C^\infty(\mathbb{R}^n)$, generated by $P_1, \ldots, P_{n-2}$.

(b) If H is functionally dependent on $P_1, \ldots, P_{n-2}$, then the vector field $\mathbf{X}_H$ is trivial. Hence, it has the canonical form with $\omega(I) \equiv 0$.

If H is functionally independent of $P_1, \ldots, P_{n-2}$, then the vector field $\mathbf{X}_H$ is completely integrable with the $n-1$ functionally independent first integrals $H, P_1, \ldots, P_{n-2}$. Hence, the vector field $\mathbf{X}_H$ has the desired normal form. □

Combining Theorem 3.3 and statement (a) of Theorem 3.4, we easily have the following.

Corollary 3.1. (a) *The center of a Poisson structure in* $\mathbb{R}^n$ *having functionally independent Casimirs* $P_1, \ldots, P_{n-2}$ *is a sub-algebra generated by the* $n-2$ *given Casimirs.*

(b) *In* $\mathbb{R}^3$*, a Poisson structure has a Casimir if and only if it is a Jacobian structure. And a Poisson structure is exact if and only if it is a Jacobian structure with a constant Jacobian coefficient.*

Our next result characterizes Poisson structures in $\mathbb{R}^3$ with a Casimir whose level surface is a quadric surface.

Theorem 3.5. *For a Poisson structure* Λ *in* $\mathbb{R}^3$*, the following holds.*

(a) Λ *has no Casimir which defines a torus.*

(b) Λ *has a Casimir which defines a quadric surface if and only if it is affine equivalent to one of the following module a Jacobian coefficient* u*:*
 - $\Lambda_1 = -z\partial_x \wedge \partial_y + y\partial_x \wedge \partial_z - x\partial_y \wedge \partial_z$, *with the Casimir* $C = x^2 + y^2 + z^2$;
 - $\Lambda_2 = z\partial_x \wedge \partial_y + y\partial_x \wedge \partial_z - x\partial_y \wedge \partial_z$, *with the Casimir* $C = x^2 + y^2 - z^2$;
 - $\Lambda_3 = \partial_x \wedge \partial_y + 2y\partial_x \wedge \partial_z - 2x\partial_y \wedge \partial_z$, *with the Casimir* $C = x^2 + y^2 - z$;
 - $\Lambda_4 = \partial_x \wedge \partial_y - 2y\partial_x \wedge \partial_z - 2x\partial_y \wedge \partial_z$, *with the Casimir* $C = x^2 - y^2 - z$;
 - $\Lambda_5 = 2\partial_x \wedge \partial_z + x\partial_y \wedge \partial_z$, *with the Casimir* $C = x^2 - 4y$;
 - $\Lambda_6 = y\partial_x \wedge \partial_z - x\partial_y \wedge \partial_z$, *with the Casimir* $C = x^2 + y^2$;
 - $\Lambda_7 = y\partial_x \wedge \partial_z + x\partial_y \wedge \partial_z$, *with the Casimir* $C = x^2 - y^2$.

Proof. It follows from direct calculations. □

Remark 3.3. 1) *In $\mathbb{R}^3$, the restriction of a Poisson structure to each submanifold defined by a level surface of its Casimir becomes symplectic except at singular points.*

2) *In $\mathbb{R}^3$, normal forms can be obtained for exact Poisson structures around each singular point* ([7]).

3) Theorem 3.5 *can be used to construct completely integrable Hamiltonian systems. In* [3]*, a method of constructing completely integrable Hamiltonian systems starting from a Poisson coalgebra formed by Lie algebra was developed. As an example, using the Casimir associated to Λ_2 above and its deformation, the authors obtained a large class of completely integrable Hamiltonian systems with an arbitrary number of degrees of freedom. In* [11]*, another method of constructing completely integrable Hamiltonian systems was developed, based on the fact that the symplectic manifold defined by the Casimirs of a Poisson structure cannot form a Lie algebra. Applying this method to the Poisson structure Λ_3 above and its associated Casimir, the author also obtained a class of completely integral Hamiltonian systems having arbitrary number of degrees of freedom, including the Calogero system.*

3.3. **Quasi-homogeneous Poisson structures.** A Poisson structure

$$L = \sum_{i,j=1}^{n} w_{ij} \partial_i \wedge \partial_j$$

in $\mathbb{R}^n$ is called *quasi-homogeneous* of weight degree m with respect to a weight $\mathbf{w} = (w_1, \ldots, w_n)$ if every w_{ij} is a quasi-homogeneous polynomial of weight degree $m - 2 + w_i + w_j$ with the same weight $\mathbf{w}$, where m and w_i are positive integers, and $w_1, \ldots, w_n$ do not have common factors. We recall that a polynomial is quasi-homogeneous of weight degree m with respect to the weight $\mathbf{w}$ if each of its monomials, $x_1^{k_1} \ldots x_n^{k_n}$, satisfies $k_1 w_1 + \ldots + k_n w_n = m$. If $w_1 = \ldots = w_n = 1$, a quasi-homogeneous polynomial becomes homogeneous.

The following is an improvement to the results given in Theorem 3.1 of [12] and Theorem 7 of [15].

Theorem 3.6. *Assume that Λ is a quasi-homogeneous Poisson structure of degree m with respect to a weight $\mathbf{w} = (w_1, \ldots, w_n)$. The following holds.*

(a) *The Poisson structure Λ can be decomposed into the form $\Lambda = \Lambda_0 + c\,\mathbf{X}_\Lambda \wedge \mathbf{X}_E$ if and only if the Poisson structure Λ is homogeneous and $c = 1/(m + n - 1)$, where Λ_0 is an exact Poisson structure, $\mathbf{X}_\Lambda = D(\Lambda)$ and $\mathbf{X}_E = \sum_{i=1}^n w_i x_i \partial_i$.*

(b) *The Poisson structure Λ can be decomposed as*

$$\Lambda = \pi + \frac{1}{m - 2 + \sum w_i} \left(\mathbf{X}_\Lambda \wedge \mathbf{X}_E - \Lambda(\mathbf{X}_E) \right),$$

where π is an exact bilinear vector field, and $\mathbf{X}_\Lambda, \mathbf{X}_E$ are as in (a).

Proof. (a) Let $\Lambda_0 = \Lambda - c\mathbf{X}_\Lambda \wedge \mathbf{X}_E$. Using statement (ii) of Lemma 2.1, we have

$$D(\Lambda - c\mathbf{X}_\Lambda \wedge \mathbf{X}_E) = \mathbf{X}_\Lambda - c([\mathbf{X}_E, \mathbf{X}_\Lambda] + (\mathrm{div}_\omega \mathbf{X}_E)\mathbf{X}_\Lambda - (\mathrm{div}_\omega \mathbf{X}_\Lambda)\mathbf{X}_E).$$

For $\mathbf{X}_\Lambda = D(\Lambda)$, combining (2.2) and statement (i) of Lemma 2.1 we have that $\mathrm{div}_\omega \mathbf{X}_\Lambda = 0$. Moreover, $\mathrm{div}_\omega \mathbf{X}_E = \sum_{i=1}^{n} w_i$. Direct calculations using the identities

$\mathbf{X}_E(w_{ij}) = (m-2+w_i+w_j)w_{ij}$ and $\mathbf{X}_E\left(\frac{\partial w_{ij}}{\partial x_j}\right) = \frac{\partial}{\partial x_j}(\mathbf{X}_E w_{ij})$ yield

$$\mathbf{X}_\Lambda(\mathbf{X}_E) = 2\sum_{i=1}^n w_i \left(\sum_{j=1}^n \frac{\partial w_{ij}}{\partial x_j}\right)\partial_i,$$

$$\begin{aligned}\mathbf{X}_E(\mathbf{X}_\Lambda) &= 2\sum_{i=1}^n\left(\sum_{j=1}^n \frac{\partial}{\partial x_j}(\mathbf{X}_E w_{ij})\right)\partial_i = 2\sum_{i=1}^n\left(\sum_{j=1}^n (m-2+w_i+w_j)\frac{\partial w_{ij}}{\partial x_j}\right)\partial_i \\ &= (m-2)\mathbf{X}_\Lambda + 2\sum_{i=1}^n w_i\left(\sum_{j=1}^n \frac{\partial w_{ij}}{\partial x_j}\right)\partial_i + 2\sum_{i=1}^n\left(\sum_{j=1}^n w_j\frac{\partial w_{ij}}{\partial x_j}\right)\partial_i.\end{aligned}$$

Therefore,

$$\begin{aligned}[\mathbf{X}_E,\, \mathbf{X}_\Lambda] &= \mathbf{X}_E(\mathbf{X}_\Lambda) - \mathbf{X}_\Lambda(\mathbf{X}_E) \\ &= (m-2)\mathbf{X}_\Lambda + 2\sum_{i=1}^n\left(\sum_{j=1}^n w_j\frac{\partial w_{ij}}{\partial x_j}\right)\partial_i. \qquad (3.12)\end{aligned}$$

Moreover,

$$\begin{aligned}D(\Lambda - c\mathbf{X}_\Lambda\wedge\mathbf{X}_E) &= \left(1-c\left(m-2+\sum_{i=1}^n w_i\right)\right)\mathbf{X}_\Lambda - 2c\sum_{i=1}^n\left(\sum_{j=1}^n w_j\frac{\partial w_{ij}}{\partial x_j}\right)\partial_i \\ &= 2\sum_{i=1}^n\left(\sum_{j=1}^n\left(1-c\left(m-2+\sum_{l=1}^n w_l + w_j\right)\right)\frac{\partial w_{ij}}{\partial x_j}\right)\partial_i.\end{aligned}$$

In order for $D(\Lambda - c\mathbf{X}_\Lambda\wedge\mathbf{X}_E)\equiv 0$ to hold, we should have $1-c(m-2+\sum\limits_{l=1}^n w_l + w_j) = 0$, $j=1,\ldots,n$. This is equivalent to $w_1 = \ldots = w_n = 1$, because $w_1,\ldots,w_n$ are relatively prime. It follows that $c = 1/(m-1+n)$, and the Poisson structure Λ is homogeneous of degree m.

Now we have by (3.12) that $[\mathbf{X}_E,\, \mathbf{X}_\Lambda] = (m-1)\mathbf{X}_\Lambda$. Therefore, by definition of the Schouten bracket (see (iii) of Lemma 2.1) it is easy to see that

$$[\mathbf{X}_\Lambda\wedge\mathbf{X}_E,\, \mathbf{X}_\Lambda\wedge\mathbf{X}_E] = -\mathbf{X}_E\wedge[\mathbf{X}_\Lambda,\mathbf{X}_E]\wedge\mathbf{X}_\Lambda - \mathbf{X}_\Lambda\wedge[\mathbf{X}_E,\mathbf{X}_\Lambda]\wedge\mathbf{X}_E = 0.$$

This means that $\mathbf{X}_\Lambda\wedge\mathbf{X}_E$ satisfies the Jacobi identity. Consequently, $\Lambda_0 = \Lambda - (1/(m-1+n))\mathbf{X}_\Lambda\wedge\mathbf{X}_E$ is an exact Poisson structure. This proves statement (a).

(b) We let $\pi = \Lambda - (1/(m-2+\sum w_i))\,(\mathbf{X}_\Lambda\wedge\mathbf{X}_E - \Lambda(\mathbf{X}_E))$. Since $\Lambda(\mathbf{X}_E) = \sum_{i,j=1}^n w_j w_{ij}\partial_i\wedge\partial_j$, the statement follows from formula (2.2) and the proof of statement (a). □

We remark that in [15], a decomposition of a homogeneous Poisson structure similar to that in the statement (a) of the above theorem was already obtained but the coefficient c was not correctly computed.

For homogeneous Poisson structures, an important example is related to the quantization of quadratic Poisson structures. It is known that on a real vector space V, one can define a homogeneous, quadratic Poisson structure on V using classical r-matrix on the Lie algebra of linear operators. The converse is true in dimension 2 (see e.g., [12]) but not in general. In [13], the authors gave a counter-example showing that there exists a homogeneous quadratic Poisson structure in dimension 3 which cannot be realized by any classical r-matrix. We recall that a *classical r-matrix* on a Lie algebra $\mathfrak{g}$ is an element r of $\mathfrak{g}\wedge\mathfrak{g}$ for which the Schouten bracket $[r,r]=0$. Let $\mathcal{P}: M_n \to \mathcal{X}(\mathbb{R}^n)$ be a real linear isomorphism from the set of $n\times n$ real matrices to the set of vector fields, defined by

$$\mathcal{P}(E_{ij}) = x_i\partial_j,$$

where E_{ij} is the matrix with entries all vanishing except one equals 1 on the ith row and the jth column. Then for any classical r-matrix $M_1\wedge M_2 \in \mathfrak{g}\wedge\mathfrak{g}$, $\mathcal{P}(r) = \mathcal{P}(M_1)\wedge\mathcal{P}(M_2)$ is a quadratic Poisson structure. A *classical r-Poisson structure* is by definition a quadratic Poisson structure which is the image of a r-matrix under $\mathcal{P}$.

The following results provide a partial answer to the problem we posted toward the end of Section 1.

Theorem 3.7. *Let $\mathfrak{g}$ be the 9-dimensional Lie algebra of linear operators in $\mathbb{R}^3$. Assume that $r = \mathbf{A}\wedge\mathbf{B} \in \mathfrak{g}\wedge\mathfrak{g}$, where $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ are real 3×3 matrices. The following holds.*

(a) *$\mathcal{P}(r)$ is a classical r-Poisson structure and is exact if and only if*

$$\mathbf{AB} + \mathrm{tr}(\mathbf{A})\mathbf{B} = \mathbf{BA} + \mathrm{tr}(\mathbf{B})\mathbf{A}, \tag{3.13}$$

where $\mathrm{tr}(\mathbf{A})$ denotes the trace of the matrix $\mathbf{A}$.

(b) *Any quadratic Jacobian structure in dimension 3 is a classical r-Poisson structure.*

Proof. (a) By definition we have

$$\mathcal{P}(r) = w_{12}\partial_1\wedge\partial_2 + w_{23}\partial_2\wedge\partial_3 + w_{31}\partial_3\wedge\partial_1,$$

where

$$\begin{aligned}
w_{12} &= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3)(b_{12}x_1 + b_{22}x_2 + b_{32}x_3)\\
&\quad -(a_{12}x_1 + a_{22}x_2 + a_{32}x_3)(b_{11}x_1 + b_{21}x_2 + b_{31}x_3),\\
w_{23} &= (a_{12}x_1 + a_{22}x_2 + a_{32}x_3)(b_{13}x_1 + b_{23}x_2 + b_{33}x_3)\\
&\quad -(a_{13}x_1 + a_{23}x_2 + a_{33}x_3)(b_{12}x_1 + b_{22}x_2 + b_{32}x_3),\\
w_{31} &= (a_{13}x_1 + a_{23}x_2 + a_{33}x_3)(b_{11}x_1 + b_{21}x_2 + b_{31}x_3)\\
&\quad -(a_{11}x_1 + a_{21}x_2 + a_{31}x_3)(b_{13}x_1 + b_{23}x_2 + b_{33}x_3).
\end{aligned} \tag{3.14}$$

Again it follows from definition that r is a classical r-matrix if and only if $\mathcal{P}(r)$ is a Poisson structure, i.e.,

$$\sum_{l=1}^{3}\left(w_{l1}\frac{\partial w_{23}}{\partial x_l} + w_{l3}\frac{\partial w_{12}}{\partial x_l} + w_{l2}\frac{\partial w_{31}}{\partial x_l}\right) = 0. \tag{3.15}$$

For the Poisson structure $\mathcal{P}(r)$ to be exact, we should have

$$\frac{\partial w_{12}}{\partial x_2} = \frac{\partial w_{31}}{\partial x_3}, \quad \frac{\partial w_{23}}{\partial x_3} = \frac{\partial w_{12}}{\partial x_1}, \quad \frac{\partial w_{31}}{\partial x_1} = \frac{\partial w_{23}}{\partial x_2}. \tag{3.16}$$

Since (3.16) implies (3.15), $\mathcal{P}(r)$ is an exact classical r-Poisson structure if and only if the condition (3.16) holds.

By comparing coefficients of x_i in (3.16) for $i = 1, 2, 3$, we obtain the condition (3.13).

(b) From the proof of statement (a) and Theorem 3.2, it follows that $\mathcal{P}(r)$ is an exact classical r-Poisson structure if and only if it is a Jacobian structure generated by

$$\begin{aligned} P_r &= \frac{1}{3}(a_{12}b_{13} - a_{13}b_{12})x^3 + (a_{13}b_{11} - a_{11}b_{13})x^2y \\ &+\frac{1}{2}(a_{13}b_{21} + a_{23}b_{11} - a_{11}b_{23} - a_{21}b_{13})xy^2 + \frac{1}{3}(a_{23}b_{21} - a_{21}b_{23})y^3 \\ &+(a_{11}b_{12} - a_{12}b_{11})x^2z + (a_{11}b_{22} + a_{21}b_{12} - a_{12}b_{21} - a_{22}b_{11})xyz \\ &+(a_{21}b_{22} - a_{22}b_{21})y^2z + \frac{1}{2}(a_{11}b_{32} + a_{31}b_{12} - a_{12}b_{31} - a_{32}b_{11})xz^2 \\ &+\frac{1}{2}(a_{21}b_{32} + a_{31}b_{22} - a_{22}b_{31} - a_{32}b_{21})yz^2 + \frac{1}{3}(a_{31}b_{32} - a_{32}b_{31})z^3, \end{aligned}$$

where (a_{ij}) and (b_{ij}) satisfy the condition (3.13).

Now for any quadratic Jacobian structure generated by $P = Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2z + Fxyz + Gy^2z + Hxz^2 + Iyz^2 + Jz^3$ to be a classical r-Poisson structure, it is sufficient to find a $r = (a_{ij}) \wedge (b_{ij}) \in \mathfrak{g} \wedge \mathfrak{g}$ for which the condition (3.13) is satisfied and $P_r = P$. By direct computations, these conditions are equivalent to

(3.17)

$$\begin{aligned} &3Jb_{33} + Ib_{32} + Hb_{31} = 0, \\ &3Ab_{23}b_{33}M_0M_3 - 2Bb_{23}b_{33}M_2M_3 + Cb_{11}b_{33}M_2M_3 - 3Db_{33}^2M_1M_2 \\ &\qquad +2Eb_{23}b_{33}M_1M_3 + 2Gb_{23}b_{33}M_1M_2 - Hb_{13}b_{23}M_1M_3 = Ib_{23}^2M_1M_2, \\ &3Ab_{22}b_{23}(b_{33}M_0M_4 - b_{23}M_2M_3) + 2Bb_{23}M_2(b_{13}b_{22}M_3 \\ &\qquad +b_{23}b_{33}(b_{12}b_{21} - b_{11}b_{22})) + C(b_{12}b_{23}b_{33}M_4 - b_{13}^2b_{22}M_3)M_2 \\ &\qquad +3Db_{33}(b_{12}b_{23} + b_{13}b_{22})M_1M_2 + 2Eb_{22}b_{23}b_{33}M_1M_4 \\ &\qquad +Fb_{23}^2(b_{22} + b_{33})M_1M_2 - 2Gb_{13}b_{22}b_{23}M_1M_2 - Hb_{13}b_{22}b_{23}M_1M_4 = 0, \\ &3AM_0^2 - 2BM_0M_2 + CM_2^2 + 2EM_0M_1 - FM_1M_2 + HM_1^2 = 0, \end{aligned}$$

$a_{21} = 0$, and,

(3.18)

$$\begin{aligned} b_{21}b_{22}b_{23}M_1a_{11} &= -3Ab_{21}^2b_{22}b_{23} + 2Bb_{12}b_{21}^2b_{23} - Cb_{21}(b_{12}b_{13}b_{21} - b_{11}M_1)r \\ &\quad +3D(b_{12}b_{21} + b_{11}b_{22})M_1 + Fb_{21}b_{23}M_1, \\ b_{21}b_{23}M_1a_{12} &= -3Ab_{21}b_{22}b_{23} + 2Bb_{12}b_{21}b_{23} - Cb_{12}b_{13}b_{21} + 3Db_{12}M_1, \\ b_{21}b_{23}M_1a_{13} &= -3Ab_{21}b_{23}^2 + 2Bb_{13}b_{21}b_{23} - Cb_{13}^2b_{21} + 3Db_{13}M_1, \\ b_{21}b_{23}a_{22} &= Cb_{21} + 3Db_{22}, \\ b_{21}a_{23} &= 3D, \\ b_{21}b_{23}^2M_1M_2a_{31} &= -3Ab_{21}^2b_{23}b_{33}M_0 + 2Bb_{21}^2b_{23}b_{33}M_2 - Cb_{13}b_{21}^2b_{33}M_2 \\ &\quad +3D(b_{23}b_{31} + b_{21}b_{33})M_1M_2 - 2Eb_{21}^2b_{23}b_{33}M_1 \\ &\quad -2Gb_{21}b_{23}M_1M_2 + Hb_{13}b_{21}^2b_{23}M_1, \\ b_{21}b_{23}M_1M_2a_{32} &= -3Ab_{21}b_{23}b_{32}M_0 + 2Bb_{21}b_{23}b_{32}M_2 - Cb_{13}b_{21}b_{32}M_2 \\ &\quad +3Db_{32}M_1M_2 - 2Eb_{21}b_{23}b_{32}M_1 + Hb_{12}b_{21}b_{23}M_1, \\ b_{21}b_{23}M_1M_2a_{33} &= -3Ab_{21}b_{23}b_{33}M_0 + 2Bb_{21}b_{23}b_{33}M_2 - Cb_{13}b_{21}b_{33}M_2 \\ &\quad +3Db_{33}M_1M_2 - 2Eb_{21}b_{23}b_{33}M_1 + Hb_{13}b_{21}b_{23}M_1, \end{aligned}$$

where $M_0 = b_{22}b_{33} - b_{23}b_{32}$, $M_1 = b_{12}b_{23} - b_{13}b_{22}$, $M_2 = b_{12}b_{33} - b_{13}b_{32}$, $M_3 = b_{21}b_{33} - b_{23}b_{31}$, and $M_4 = b_{11}b_{23} - b_{13}b_{21}$.

For any given real numbers $A, B, \ldots, J$, system (3.17) has solutions b_{ij}, $1 \le i, j \le 3$ for which system (3.18) also has solutions a_{ij}, $1 \le i, j \le 3$. Corresponding to these $\{b_{ij}\}$ and $\{a_{ij}\}$, $\mathcal{P}(r)$ with $r = (a_{ij}) \wedge (b_{ij})$ is the Jacobian structure generated by the prescribed homogeneous polynomial P. □

4. Invariant exact Poisson structures under a completely integrable flow

According to [4], a smooth vector field in an n-dimensional smooth manifold $\mathbf{M}^n$ is said to be *integrable in the broad sense* if it has p functionally independent first integrals with $0 \le p < n$, and an abelian $(n-p)$-dimensional Lie algebra of symmetries which preserve the p first integrals and are linear independent on the manifold except perhaps at a set of zero Lebesgue measure. This notion of integrability generalizes the classical notion of integrability in the Liouville sense for standard Hamiltonian systems.

For a smooth dynamical system on an n-dimensional smooth manifold which is integrable in the broad sense, we let $\mathbf{M}_l^q$, $q = n - p$, be connected components of the general invariant submanifolds formed by the common level surfaces of the p first integrals, parametrized by $l \in \mathbb{R}^n$. The followings were shown in [4].

- If $\mathbf{M}_l^q$ is compact, then it is a torus $\mathbb{T}^q$.
- If $\mathbf{M}_l^q$ is non-compact, then it is a toroidal cylinder $\mathbb{T}^{q-m} \times \mathbb{R}^m$.
- In a toroidal neighborhood $B_p \times \mathbb{T}^{q-m} \times \mathbb{R}^m$ of the toroidal cylinder, where B_p is a p-dimensional ball, there exist local coordinates $I = (I_1, \ldots, I_p) \in B_p$, $\phi = (\phi_1, \ldots, \phi_{q-m}) \in \mathbb{T}^{q-m}$, and $\rho = (\rho_1, \ldots, \rho_m) \in \mathbb{R}^m$, such that the dynamical system is equivalent to

$$\begin{cases} \dot{I} = 0, \\ \dot{\phi} = \omega(I), \\ \dot{\rho} = Q(I), \end{cases} \tag{4.1}$$

  where $\omega(I) = (\omega_1(I), \cdots, \omega_{q-m}(I))^\top$ and $Q(I) = (q_1(I), \cdots, q_m(I))^\top$ are smooth functions. Moreover, in the case that all $\mathbf{M}_l^q$ are compact, then $m = 0$ and the last equation of (4.1) does not appear.

Consequently, if a smooth dynamical system is integrable in the broad sense, then it is completely integrable in the conventional sense.

We now consider the case that all $\mathbf{M}_l^q$ are compact, i.e., the system (4.1) reduces to

$$\begin{cases} \dot{I} = 0, \\ \dot{\phi} = \omega(I), \end{cases} \tag{4.2}$$

where (I, ϕ) lies in a toroidal domain $B_p \times \mathbb{T}^q \subset \mathbf{M}^n$. We denote by $\mathbf{X}$ the vector field corresponding to (4.2). The dynamical system (4.2) is said to be $\mathbb{T}^q$-*dense at a point* $I_0 \in B_p$ if for any $\phi_0 \in \mathbb{T}^q$, the orbit $\{\phi_0 + \omega(I_0)t\}$ is dense on the torus $\mathbb{T}^q$. Let $\Omega_p \subset B_p$ be the set of points $I \in B_p$ at which the trajectories of system (4.2) are $\mathbb{T}^q$-dense. The dynamical system (4.2) or its induced flow is said to be $\mathbb{T}^q$-*dense* in the toroidal domain if the set Ω_p is everywhere dense in B_p. Clearly, the dynamical system (4.2) is $\mathbb{T}^q$-dense if and only if the frequencies $\{\omega_1(I), \cdots, \omega_q(I)\}$ are rationally independent (i.e., $\omega(I)$ is non-resonant) for almost all $I \in B_p$ and if and only if any first integral is a function of I.

Closed 2-forms which are invariant under the $\mathbb{T}^q$-dense dynamical system (4.2) were characterized in [4]. It was also shown that the system (4.2) preserving a symplectic 2-form must be Hamiltonian. However, if a 2-form is not symplectic, then in general it has no corresponding Poisson structure. We now consider the characterization of the invariant exact Poisson structures under the $\mathbb{T}^q$-dense dynamical system (4.2).

Theorem 4.1. *A Poisson structure* Λ *defined in a toroidal domain* $B_p \times \mathbb{T}^q \subset \mathbf{M}^n$ *is exact and invariant under the* $\mathbb{T}^q$*-dense dynamical system* (4.2) *if and only if*

$$\Lambda = a_{ij}(I)\frac{\partial}{\partial I_i} \wedge \frac{\partial}{\partial I_j} + b_{kl}(I)\frac{\partial}{\partial I_k} \wedge \frac{\partial}{\partial \phi_l} + c_{rs}(I)\frac{\partial}{\partial \phi_r} \wedge \frac{\partial}{\partial \phi_s}, \tag{4.3}$$

with the coefficients satisfying

$$\mathbf{a}_i \cdot \nabla\omega_k = 0, \quad \mathbf{b}_l \cdot \nabla\omega_k - \mathbf{b}_k \cdot \nabla\omega_l = 0, \tag{4.4}$$

$$\operatorname{div} \mathbf{a}_i = 0, \quad \operatorname{div} \mathbf{b}_l = 0, \tag{4.5}$$

where $\mathbf{a}_i = (a_{i1}, \ldots, a_{ip})$, $i = 1, \cdots, p$, *and* $\mathbf{b}_k = (b_{1k}, \ldots, b_{pk})$, $k, l = 1, \cdots, q$.

Proof. Assume that in the toroidal domain, the Poisson structure is of the form

$$\Lambda = a_{ij}(I,\phi)\frac{\partial}{\partial I_i} \wedge \frac{\partial}{\partial I_j} + b_{kl}(I,\phi)\frac{\partial}{\partial I_k} \wedge \frac{\partial}{\partial \phi_l} + c_{rs}(I,\phi)\frac{\partial}{\partial \phi_r} \wedge \frac{\partial}{\partial \phi_s}. \tag{4.6}$$

The Lie derivative of Λ with respect to $\mathbf{X} = \sum\limits_{k=1}^{q} \omega_k(I)\frac{\partial}{\partial \phi_k}$ reads

$$\begin{aligned} L_{\mathbf{X}}\Lambda &= \sum_{1\le i<j\le p} \mathbf{X}(a_{ij})\frac{\partial}{\partial I_i} \wedge \frac{\partial}{\partial I_j} \\ &+ \sum_{1\le i\le p, 1\le k\le q} \left(\mathbf{X}(b_{ik}) + \sum_{r=1, r\ne i}^{p} a_{ri}\frac{\partial \omega_k}{\partial I_r}\right)\frac{\partial}{\partial I_i} \wedge \frac{\partial}{\partial \phi_k} \\ &+ \sum_{1\le k<l\le q} \left(\mathbf{X}(c_{kl}) - \sum_{r=1}^{p} b_{rl}\frac{\partial \omega_k}{\partial I_r} + \sum_{r=1}^{p} b_{rk}\frac{\partial \omega_k}{\partial I_r}\right)\frac{\partial}{\partial \phi_k} \wedge \frac{\partial}{\partial \phi_l}. \end{aligned}$$

In order for Λ to be invariant under the $\mathbb{T}^q$-dense dynamical system (4.2), the above Lie derivative must vanish, i.e.,

$$\begin{aligned} \dot{a}_{ij} &= \mathbf{X}(a_{ij}) = 0, \\ \dot{b}_{ik} &= \mathbf{X}(b_{ik}) = -\sum_{r=1}^{p} a_{ri}\frac{\partial \omega_k}{\partial I_r}, \\ \dot{c}_{kl} &= \mathbf{X}(c_{kl}) = \sum_{r=1}^{p} b_{rl}\frac{\partial \omega_k}{\partial I_r} - \sum_{r=1}^{p} b_{rk}\frac{\partial \omega_l}{\partial I_r}, \end{aligned}$$

$i, j = 1, \cdots, p$; $k, l = 1, \cdots, q$. The first equation in the above means that each a_{ij} is a first integral of the $\mathbb{T}^q$-dense dynamical system (4.2), hence each a_{ij} is a function which is independent of ϕ. Since the coefficients of Λ are all bounded on any compact set, it follows from the last two equations in the above that

$$\sum_{r=1}^{p} a_{ri}\frac{\partial \omega_k}{\partial I_r} = 0, \quad \sum_{r=1}^{p} b_{rl}\frac{\partial \omega_k}{\partial I_r} - \sum_{r=1}^{p} b_{rk}\frac{\partial \omega_l}{\partial I_r} = 0,$$

$r, i = 1, \cdots, p$; $k, l = 1, \cdots, q$. This is the condition (4.4). Consequently, the Poisson structure Λ has the form (4.3).

By Lemma 2.1, the Poisson structure (4.3) is exact if and only if the condition (4.5) holds. This proves the necessary part.

The sufficient part follows easily from the proof for the necessary part. □

Remark 4.1. *Using* (4.5)*, the condition* (4.4) *can be written as*

$$\nabla(\mathbf{a}_i\omega_k) = 0, \qquad \nabla(\mathbf{b}_l\omega_k - \mathbf{b}_k\omega_l) = 0,$$

$i = 1, \cdots, p$*;* $k, l = 1, \cdots, q$*.*

Remark 4.2. *A* $\mathbb{T}^q$*-dense dynamical system* (4.2) *which preserves an exact Poisson structure* (4.3) *can be non-Hamiltonian. We note that if* $\mathbf{X}$ *is a Hamiltonian vector field associated to a Poisson structure* Λ*, then* $\mathbf{X} = \Lambda(\cdot, dH)$ *for some Hamiltonian function* H*, which is equivalent to* $\operatorname{div}(B_1 H) = \omega_1$ *with the compatible conditions*

$$\operatorname{div}(A_i H) = 0, \ \operatorname{div}((B_1\omega_k - B_k\omega_1)H) = 0,$$

where $A_i = (a_{i1}, \ldots, a_{ip}, b_{i1}, \ldots, b_{iq})$ *and* $B_k = (-b_{1k}, \ldots, -b_{pk}, c_{k1}, \ldots, c_{kq})$*,* $i = 1, \cdots, p$*,* $k = 2, \cdots, q$*.*

We now consider the $\mathbb{T}^q$*-dense dynamical system* (4.2) *in dimension 3.*

1) *Let* $p = 1$ *and* $q = 2$*. The Poisson structure* (4.3) *reads*

$$\Lambda = b_{11}(I)\frac{\partial}{\partial I} \wedge \frac{\partial}{\partial \phi_1} + b_{12}(I)\frac{\partial}{\partial I} \wedge \frac{\partial}{\partial \phi_2} + c_{12}(I)\frac{\partial}{\partial \phi_1} \wedge \frac{\partial}{\partial \phi_2}.$$

The conditions (4.4) *and* (4.5) *are reduced to*

$$b_{11}\frac{\partial \omega_2}{\partial I} = b_{12}\frac{\partial \omega_1}{\partial I}, \tag{4.7}$$

and b_{11}*,* b_{12}*=constants, respectively.*

The system (4.2) *is a Hamiltonian if and only if there exists a Hamiltonian function* $H(I, \phi_1, \phi_2)$ *such that*

$$b_{11}\frac{\partial H}{\partial \phi_1} + b_{12}\frac{\partial H}{\partial \phi_2} = 0, \quad -b_{11}\frac{\partial H}{\partial I} + c_{12}\frac{\partial H}{\partial \phi_2} = \omega_1, \quad -b_{12}\frac{\partial H}{\partial I} - c_{12}\frac{\partial H}{\partial \phi_1} = \omega_2. \tag{4.8}$$

So, we must have

$$b_{12}\omega_1 = b_{11}\omega_2. \tag{4.9}$$

But the condition (4.7) *is not sufficient to assure* (4.9) *in general.*

However, if $b_{12}\omega_1(0) = b_{11}\omega_2(0)$*, then the system* (4.8) *always has a solution provided that* (4.7) *holds, consequently the system* (4.2) *is Hamiltonian.*

2) *Let* $p = 2$ *and* $q = 1$*. We claim that the* $\mathbb{T}^q$*-dense dynamical system* (4.2) *is always Hamiltonian. Indeed, the Poisson structure* (4.3) *has the form*

$$\Lambda = a_{12}(I)\frac{\partial}{\partial I_1} \wedge \frac{\partial}{\partial I_2} + b_{11}(I)\frac{\partial}{\partial I_1} \wedge \frac{\partial}{\partial \phi} + b_{21}(I)\frac{\partial}{\partial I_2} \wedge \frac{\partial}{\partial \phi}.$$

The conditions (4.4) *and* (4.5) *are reduced to* $a_{12}\dfrac{\partial \omega}{\partial I_i} = 0$*,* $i = 1, 2$*,* a_{12} *is a constant, and*

$$\frac{\partial b_{11}}{\partial I_1} + \frac{\partial b_{21}}{\partial I_2} = 0. \tag{4.10}$$

Note that the system (4.2) *is Hamiltonian, i.e.,* $\mathbf{X} = \Lambda(\cdot, dH(I))$ *for some Hamiltonian function* H*, if and only if* $a_{12} = 0$*, and*

$$b_{11}(I)\frac{\partial H}{\partial I_1} + b_{21}(I)\frac{\partial H}{\partial I_2} = -\omega(I). \tag{4.11}$$

Since the condition (4.10) *guarantees that the characteristic equation of* (4.11) *has a smooth solution, we can choose suitable* b_{11}, b_{12} *for which* (4.11) *has a global smooth solution* $H(I)$*. This means that we can choose a Poisson structure* Λ *such that the dynamical system* (4.2) *is Hamiltonian induced by* Λ*.*

It is an open problem to *characterize exact Poisson structures invariant under a* $\mathbb{T}^q$*-dense dynamical system such that the dynamical system is Hamiltonian induced by the Poisson structures.*

References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, 2nd edition, The Benjamin/Cummings publishing Company, Reading, Massachusetts, 1978.

[2] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, Spring-Verlag, New York, 1982.

[3] A. Ballesteros and O. Ragnisco, A systematic construction of completely integrable Hamiltonians from coalgebras, *J. Phys. A* **31** (1998), 3791–3813.

[4] O. I. Bogoyavlenskij, Extended integrability and bi-Hamiltonian systems, *Commun. Math. Phys.* **196** (1998), 19–51.

[5] J. F. Conn, Normal forms for analytic Poisson structures, *Ann. Math.* **119** (1984), 577–601.

[6] J. F. Conn, Normal forms for smooth Poisson structures, *Ann. Math.* **121** (1985), 565–593.

[7] I. Cruz and H. Mena-Matos, Normal forms for locally exact Poisson structures in $\mathbb{R}^3$, *J. Geom. Phys.* **43** (2002), 27–32.

[8] I. Cruz and H. Mena-Matos, Normal forms for two classes of exact Poisson structures in dimension 4, preprint.

[9] J. Grabowski, G. Marmo and A. M. Perelomov, Poisson structures: towards a classification, *Mod. Phys. Lett. A* **8** (1993), 1719–1733.

[10] J. L. Koszul, Crochets de Schouten-Nijenhuis et cohomologie, *Astérisque, Soc. Math. de France* hors série (1985), 257–271.

[11] S. P. Kasperczuk, Poisson structures and integrable systems, *Physica A* **284** (2000), 113–123.

[12] Z. Liu and P. Xu, On quadratic Poisson structures, *Lett. Math. Phys.* **26** (1992), 33–42.

[13] D. Manchon, M. Masmoudi and A. Roux, On quantization of quadratic Poisson structures, *Commun. Math. Phys.* **225** (2002), 121–130.

[14] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Spring-Verlag, New York, 1993.

[15] R. Przybysz, On one class of exact Poisson structures, *J. Math. Phys.* **42** (2001), 1913–1920.

[16] V. A. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Spring-Verlag, New York, 1984.

[17] A. Weinstein, The local structure of Poisson manifolds, *J. Diff. Geometry* **18** (1983), 523–557.

[18] A. Weinstein, The modular automorphism group of a Poisson manifold, *J. Geom. Phys.* **23** (1997), 379-394.

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