Generalization of a Theorem of Malta and Palis

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Abstract
We study the saddle-node bifurcation of a partially hyperbolic fixed point in a Lipschitz family of \( C^\infty \) diffeomorphisms on an n-dimensional manifold, in the case that the stable set and unstable set of the fixed point intersect the stable and unstable manifolds of other invariant sets in a ‘critical’ manner. Sufficient conditions are found which guarantee realization of only codimension one bifurcations in a family. A diffeomorphism for which the conditions are not satisfied is shown to be in the closure of the set of codimension 2 bifurcation surfaces. These results are a generalization of a one-dimensional result of Malta and Palis.

1 Introduction

One of the most important problems of bifurcation theory is the study of the boundary of the Morse-Smale diffeomorphisms and neighboring systems. An extensive part of the boundary is a codimension one, smooth submanifold of the Banach manifold \( \text{Diff}^k(M) \) of all diffeomorphisms of a manifold \( M \), the so-called bifurcation surface. An essential part of the surface, call it \( B \), is formed by diffeomorphisms having fixed (or periodic) points of saddle-node type. The surface \( B \) separates each small ball in \( \text{Diff}^k(M) \) centered at a point of \( B \) into two parts: one of these contains only Morse-Smale systems, the other can, in principal, contain systems with nontrivial basic sets, strange attractors, etc. (for example [3]). In the simplest case, all structurally stable systems in this second half of the small ball are of the Morse-Smale type; the bifurcation does not lead out of the Morse-Smale systems. But, even for that case the structurally unstable systems can form very complex structures depending of the behavior of heteroclinic orbits in the bifurcation moment. We restrict ourselves here to the study of this situation.

We are interested in the following question: Assume that a bifurcation does not
lead out of the Morse-Smale systems. Under which conditions will a family $T_\mu$ of
diffeomorphisms, $T_0 \in B$, contain structurally unstable diffeomorphisms of only 'the
first degree of instability', i.e. belonging to only codimension one (not more!) bifur-
cation sets? If this is the case, then the piece of $B$ in the ball can be imagined as a
limit of pieces of bifurcation surfaces related to the existence of critical heteroclinic
orbits (see below).

For flows on two-dimensional surfaces (diffeomorphisms of one-dimensional man-
ifolds), this problem was studied by I.P. Malta and J. Palis [7]. They found suffi-
cient conditions for the realization of only codimension one bifurcations in families
transversal to $B$. These conditions single out an open everywhere dense subset of $B$
- the 'good' set $B_g$ - such that every family $T_\mu$, $T_0 \in B_g$, transversal to $B$ encounters
only codimension one bifurcations. The necessity of these conditions was proved later
[6].

In our work we generalize the Malta-Palis result in the following sense. First, we
consider diffeomorphisms on $n$-dimensional manifolds, $n > 1$. Among other things,
this introduces the complication that critical heteroclinic orbits may become count-
able infinitely and have limit points. Secondly, we suggest a new tool, standard co-
dinates (see below), to study this bifurcation. The Malta-Palis conditions were
formulated in terms of embedding a diffeomorphism into a flow. It seems to us that
the standard coordinate language is more constructive.

![Figure 1. A semistable limit cycle with heteroclinic orbits. The Poincare map $\Pi$ on the
section $\Sigma$ transverse to the limit cycle has the form of a saddle-node and the intersection
of the heteroclinic orbits with the section forms critical points. This case was studied
by Malta and Palis.](image-url)
2 Local structure of saddle-node fixed points

Hyp. 1 Let $T_\mu$ be a one-parameter family of $C^k$-diffeomorphisms, $k \geq 2$, of an $n$-manifold $M$, $n \geq 2$, which is transversal to $B$ and Lipschitz continuous in $\mu$ in the $C^k$ topology, for $\mu \in [-\mu_0, \mu_0]$. In particular, for $\mu < 0$, $T_\mu$ is a Morse-Smale diffeomorphism. For $\mu = 0$, $T_\mu$ has a nonhyperbolic fixed point $\bar{0}$ of quadratic saddle-node type, which is a node along hyperbolic directions. All other periodic orbits are hyperbolic. For $\mu > 0$, $T_\mu$ has no fixed point in a neighborhood of $\bar{0}$.

With these assumptions, there is a neighborhood $U$ of $\bar{0}$ and $C^k$-smooth coordinates $(x, z)$ on $U$, $x \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$, for which $T_\mu$ has the local form $\Phi_\mu$ given by:

\[
\begin{align*}
\tilde{x} &= A x + f(x, z, \mu) \\
\tilde{z} &= z + R(z, \mu),
\end{align*}
\]

where the spectrum of $A$ satisfies $|\sigma(A)| < q_0$, for some $q_0 < 1$. Without loss of generality, we will assume hereout that $|A| < q_1$, for $q_1 < 1$. The function $f$ is $C^k$-smooth in $(x, z)$, and Lipschitz continuous in $\mu$. The function $f$ and its first derivatives vanish for $(x, z, \mu) = (0, 0, 0)$. This form is obtained by changing variables to rectify the stable fibers over the local center unstable manifold of the parameterized map [8] [2] [11]. In these local coordinates it can be seen that all the points in the half space, $z \leq 0$, limit to $\bar{0}$ under forward iterations of $T_0$. This set is known as the local stable set of $\bar{0}$. The set of all points which limit to $\bar{0}$ under forward iteration for $\mu = 0$, which is an extension of the local stable set, is called the stable set and is denoted by $S^s$. Similarly, the half line, $x = 0$, $z \geq 0$ is known as the local unstable set. Its extension, $S^u$, is known as the unstable set of $\bar{0}$.

By a quadratic saddle-node we mean that $R''(\bar{0}, 0) \neq 0$. We can assume without loss of generality that $R$ in (1) has the following form

\[
R(z, \mu) = \mu + z^2 + o(z^2). \tag{2}
\]

Let us restrict our attention for a moment to the stable set $S^s$. Observe that $S^s$ has a stable foliation in which each leaf is represented by its $z$-coordinate. An important fact is that for $\mu = 0$ this foliation is unique (See for instance [4]). The evolution of the $z$-coordinate, (and thus, that of the leaves) is determined by the equation

\[
\tilde{z} = \phi(z, 0) = z + R(z, 0). \tag{3}
\]

Choose $a_0 < 0$ and let $a_i = \phi^i(a_0, 0)$ and $I_i = [a_i, a_{i+1})$. Then given $z_0 \in I_0$ let $z_i = \phi^i(z_0, 0)$ and for each $i$ let $u_i(z_0)$ be the ratio

\[
u_i(z_0) = \frac{z_i - a_i}{a_{i+1} - a_i},
\]
and consider the limit
\[ u(z_0) = \lim_{i \to \infty} u_i(z_0). \]

**Lemma 1** The limit \( u(z_0) \) exists and is independent of the choice of local coordinates. When considered as functions on \( I_0 \), the sequence \( \{u_i\} \) converges in the \( C^2\)-topology. Finally, if \( u_{a_0}(z_0) \) and \( u_{\tilde{a}_0}(z_0) \) are defined as above for \( a_0 \) and \( \tilde{a}_0 \) respectively, then there exists a constant \( c(a_0, \tilde{a}_0) \) independent of \( z_0 \) such that \( u_{a_0}(z_0) = u_{\tilde{a}_0}(z_0) + c(a_0, \tilde{a}_0) \).

See [4] for the proof. Let \( \pi \) denote the projection of points in \( S^* \) onto \( I_0 \) which maps a point \( \bar{P} \in S^* \) to the base point of the unique fiber with base point in \( I_0 \) which intersects the orbit of \( \bar{P} \).

**Definition 1** Given \( a_0 \), the function \( u_{a_0} \circ \pi : S^* \to I = [0,1] \), is called the **standard coordinate map** of the leaves of \( S^* \).

It follows from Lemma 1 that this function is \( C^2 \) on \( S^* \). By considering the inverse of the local diffeomorphism we may similarly define standard coordinates for the leaves of \( S^* \) (in this case each leaf is a single point).

Now consider the local diffeomorphism for parameter values \( \mu > 0 \). The fixed point disappears and the graph of \( \phi_\mu(z) = z + R(z, \mu) \) forms a narrow ‘funnel’ with the diagonal line. Let \( \{z_i\} \) be any orbit on the \( z \)-axis under \( \phi_\mu \) for \( \mu > 0 \) and for \( \ell > 0 \) denote
\[ \cdots < z_{i-1}^- < -\ell \leq z_i^- < \cdots < z_i^+ < \ell \leq z_{i+1}^+ < \cdots. \]

Consider the following lemma which first appeared in [1] and [8]. A detailed proof is contained in [4].

**Lemma 2** As \( \ell \) and \( \mu \) approach zero,
\[ | \prod_{i=i^-}^{i^+} \phi_\ell(z_i, \mu_j) - 1 | = O(\ell + \sqrt{\mu}). \]

This lemma says in effect that transition through the ‘funnel’ is nearly a rigid translation.

### 3 Description of critical heteroclinic orbits

#### 3.1 The order of heteroclinic orbits. Critical points.

In order to describe behavior of heteroclinic orbits of the diffeomorphism \( T_0 \) we need to recall some facts from the theory of Morse-Smale diffeomorphisms [9] [10]. Let
\( \mathcal{L} = \{ L_i \} \) (\( \mathcal{R} = \{ R_k \} \)) be the set of hyperbolic periodic orbits for the diffeomorphism \( T_0 \) such that \( W_{L_i}^u \cap S^* \neq \emptyset \) (\( W_{R_k}^s \cap S^u \neq \emptyset \)), where \( W_{L_i}^u \) (\( W_{R_k}^s \)) is the unstable (stable) set of the periodic orbit \( L_i \) (\( R_k \)): it contains \( l_i \) (\( r_k \)) unstable (stable) manifolds of the \( l_i \)-periodic (\( r_k \)-periodic) points belonging to the orbit \( L_i \) (\( R_k \)) where \( l_i \) (\( r_k \)) is the minimal period of \( L_i \) (\( R_k \)).

**Definition 2** [8] A point \( P \in W_{L_i}^u \cap S^* \) (\( P \in W_{R_k}^s \cap S^u \)) is said to belong to a critical heteroclinic orbit if \( W_{L_i}^u \) (\( W_{R_k}^s \)) intersects a leaf of the stable (unstable) foliation on \( S^* \) (\( S^u \)) at \( P \) nontransversally. The point \( P \) itself in this case is called critical. See figure 2.

Existence of the critical heteroclinic orbits for \( \mu = 0 \) is a cause of bifurcations for \( \mu > 0 \). We restrict ourselves in this work to the following partial but important case.

**Hyp. 2** If \( \dim W_{L_i}^u \neq 1 \) (\( \dim W_{R_k}^s \neq n - 1 \)) then \( W_{L_i}^u \) (\( W_{R_k}^s \)) does not contain critical points.

Denote by \( \mathcal{L}^1 \) (\( \mathcal{R}^{n-1} \)) the subset of \( \mathcal{L} \) (\( \mathcal{R} \)) with \( \dim W_{L_i}^u = 1 \) (\( \dim W_{R_k}^s = n - 1 \)). The set \( \mathcal{L}^1 \) (\( \mathcal{R}^{n-1} \)) admits a partial order. We write \( L_i \prec L_k \), if \( W_{L_i}^u \cap W_{L_k}^s \neq \emptyset \) and \( L_i \prec L_k \), if \( L_i \prec L_k \) and \( W_{L_i}^u \cap W_{L_k}^s \) contains a finite (nonempty) set of heteroclinic orbits. It follows from [9] and [10] (see also [5]) that if \( L_i \prec L_k \) and \( W_{L_i}^u \cap W_{L_k}^s \) contains infinitely many orbits then there exists a set \( \{ L_{k_1}, L_{k_2}, \ldots, L_{k_m} \} \subset \mathcal{L} \), \( m \geq 1 \), such that

\[
L_i < L_{k_1} < L_{k_2} < \cdots < L_{k_m} < L_k.
\]

Among all such chains joining \( L_i \) and \( L_k \) we can choose a chain of maximal length \( m_0 \). In this case all heteroclinic orbits in the intersection \( W_{L_i}^u \cap W_{L_k}^s \) are said to be of order \( m_0 \) [5]. The following result is contained in [10] (Some details of the proof can be found also in [5]).

**Lemma 3** Let \( H_m \) be the set of all heteroclinic points of order \( m \). For an arbitrarily small \( \epsilon > 0 \), only a finite number of points of \( H_{m+1} \) may lie outside an \( \epsilon \) neighborhood of the closure of \( H_m \). In other words, all accumulation points of the set \( H_{m+1} \) are contained in \( \text{clos}(H_m) \), \( m = 0, 1, \ldots \).

In addition we need the following assumption.

**Hyp. 3** Assume that \( \mathcal{L} \cap \mathcal{R} = \emptyset \).

Hypothesis 3 states that \( \emptyset \) has no homoclinic orbits or homoclinic contours. This ensures that only the original periodic points belong to the nonwandering set for \( \mu > 0 \). With Hyp. 3, it is simple to see that the definitions above can be extended to
the sets $L \cup 0$ and $R \cup 0$; the point $0$ can occupy only the last (right) place for every chain of ordered orbits in $L \cup 0$ and the first place in $R \cup 0$.

![Diagram](image)

Figure 2. An example where critical points occur on both sides of a saddle-node point in a 2 dimensional diffeomorphism. The stable fibers of the stable set $S^u$ are shown in lighter lines.

3.2 Accumulation rates of critical points

First of all, we assume that critical points satisfy the following assumptions.

**Hyp. 4** A manifold $W^s_{L_n}$ has quadratic tangency with a leaf of the stable (unstable) foliation on $S^s$ at every critical point. A manifold $W^s_{R_n}$ has only transversal intersections with $S^u$.

Note that each intersection of $W^s_{R_n}$ and $S^u$ consists of isolated points and each of these points is necessarily critical by our definition. (See figure 2.)

**Hyp. 5** Each of the hyperbolic periodic orbits has a leading direction of multiplicity one and the heteroclinic points joining periodic points in $L^1 \cup 0$ and $R^{n-1} \cup 0$ approach periodic points only along leading directions.

By leading direction we mean the direction associated with the multiplier of the periodic orbit closest to the unit circle. Hypothesis 5 is needed because an arbitrarily small perturbation can cause an orbit on a nonleading direction to jump to the leading direction. The assumptions Hyp. 4 and Hyp. 5 single out an open everywhere dense set on the bifurcation surface, and we can use them without loss of generality.
By assumption, critical heteroclinic points may belong only to either $S^* \cap W^u(L^1)$ or to $S^u \cap W^u(R^{n-1})$. Denote by $C^*$ ($C^u$) the set of all critical points belonging to $S^* \cap W^u(L^1)$ ($S^u \cap W^u(R^{n-1})$). For the sake of definiteness, let us consider points in $C^*$. Since $C^*$ is $T_0$-invariant then the notion of order can be applied to points in it. Denote by $B_r(0)$ the closed ball of radius $r > 0$ and let $C^*_r = C^* \cap B_r(0)$. The following result follows from Hyp. 2 and 3, the definitions of criticality, order, the $C^2$-Lambda Lemma, and Lemma 3.

**Lemma 4** If $B_r(0) \cap L^1 = \emptyset$, then

1. The set $C^*_r$ is closed.

2. All accumulation points of the set $H_{m+1} \cap C^*_r$ belong to the set $(\bigcup_{i=0}^m H_i) \cap C^*_r$.

Remark. It is simple to express conditions of the existence of infinitely many critical heteroclinic points in terms of the partial order. Indeed, it can be proved using the $C^2$-Lambda Lemma and our assumptions that if $L_i \prec L_j$ and $W_{L_i}^u \cap C^* \neq \emptyset$ then $W_{L_j}^u \cap C^* \neq \emptyset$ and $W_{L_i}^u$ contains infinitely many critical heteroclinic orbits.

Let us emphasize the following important circumstance. If $P_0 = \lim_{n \to \infty} P_n$, where $P_n \in C^*_r$, then $W_{L_i}^u = \tilde{r}(\partial W_{L_i}^u)$. Here $L(P_0)$ and $L(P_n)$ are periodic points in $L^1$, $\tilde{r}$ denotes the upper topological limit, and $\partial W_{L_i}^u$ is the set of limiting points of $W_{L_i}^u$ ($W_{L_i}^u$ is considered as an injective immersion in $M$). The number of periodic orbits in $L^1$ is finite. Therefore, $W_{L_i}^u = \tilde{r}(\partial W_{L_i}^u)$ for some fixed $i$ and the sequence $P_n$ contains a subsequence $P_{n_k}$ with $L(P_{n_k}) = L_i$. This implies that the rate of accumulation of the subsequence $P_{n_k}$ to $P_0$ can be expressed in terms of the multiplier of the point $L(P_0)$.

More precisely, it can be formulated as follows. Consider an arbitrary sequence $P_n \in C^*_r$, $\lim_{n \to \infty} P_n = P_0$. For any fixed $n$, let $L_{i_n}$ be the periodic point in a periodic orbit $L_i$ such that $P_n \in W_{L_{i_n}}^u$, $L_0$ be a periodic point in a periodic orbit $L_0$ such that $P_0 \in W_{L_0}^u$. Assume that $W_{L_{i_n}}^u$ is countinuously parameterized by a parameter $t \in \mathbb{R}$ in such a way that $t(L_{i_n}) = 0$, and $t(L_0) > 0$. Since $L_0 \prec L_i$ and $W_{L_0}^u \cap W_{L_{i_n}}^u \neq \emptyset$, there exists a point $Q_n \in W_{L_0}^u \cap W_{L_{i_n}}^u$ such that $t(Q_n) < t(P_n)$ and $Q_n$ is the closest point to $P_n$ on $W_{L_{i_n}}^u$, i.e. there is no point $Q' \in W_{L_0}^u \cap W_{L_{i_n}}^u$ such that $t(Q_n) < t(Q') < t(P_n)$. Let $J_0 = [L_0, P_0]$ be the arc on $W_{L_0}^u$ with the end points $L_0$, $P_0$, and $J_n = [Q_n, P_n]$ be the arc on $W_{L_n}^u$ with endpoints $Q_n$, $P_n$. The $C^2$-Lambda Lemma implies that $T^{l_0,j}(J_n) \cap \epsilon(J_0)$ goes to $J_0$ as $j \to \infty$ in the $C^2$-metric, where $l_0$ is the period of $L_0$ and $\epsilon(J_0)$ is an $\epsilon$-neighborhood of $J_0$. Therefore, $T^{l_0,j}(J_n)$ has to contain a critical point in the $\epsilon$-neighborhood of $P_0$. It allows us to give the following definition.

**Definition 3** A sequence $\{P_n\}$, $P_0 = \lim_{n \to \infty} P_n$, is called representative if there exists $n_0 > 0$ and an arc $J_{n_0}$ as above, such that for any $n > n_0$ the arc $J_n$ contains
a point $P_n$, where $J_n = T^{k_n} J_{n-1} \cap \epsilon(J_0)$, $n = n_0 + 1, n_0 + 2, \ldots$. 

The rate of accumulation of the arcs $T^{k_n} J_n \cap \epsilon(J_0)$ to $J_0$ is governed by a multiplier of $L_0$. In other words the following lemma holds.

**Lemma 5** If a sequence $P_n \subset C_r^*$ is representative then

$$
\frac{d(P_n, P_{n+1})}{|\lambda|} \to 1, \quad \text{as} \quad k \to \infty,
$$

where $|\lambda| < 1$ is the modulus of a multiplier of the periodic orbit $L(P_0)$, such that $P_0 \in W^u(L(P_0))$.

Lemma 5 implies that the rates of accumulations of sequences in $C_r^*$ are completely determined by 'stable' multipliers of points in $\mathcal{L}^1$.

It can be shown in the same way that the accumulation rates of the sequences in $C_r^u$ are determined by the reciprocals of 'unstable' multipliers of points in $\mathcal{R}^{n-1}$.

## 4 Main Results

### 4.1 Formulation of theorems

Recall that $B$ denotes the part of the bifurcation surface on which the diffeomorphisms have fixed points of saddle-node type. Let $\{\alpha_n\}$ and $\{\beta_m\}$ be the set of accumulation rates of critical points on $S^*$ and $S^u$ respectively. In other words, $\alpha_n$ is the modulus of a 'leading' multiplier of some periodic orbit $L_i \in \mathcal{L}^1$ and $1/\beta_m$ is the modulus of the leading multiplier of $R_k \in \mathcal{R}^{n-1}$.

We denote by $\sim$ the following relation on $(0, 1)$: $\alpha \sim \beta$ if and only if there exist positive integers $p$ and $q$ such that

$$
\alpha^p = \beta^q. \quad (4)
$$

It is clear that $\sim$ is an equivalence relation.

Suppose that there are $M$ rates of accumulation on $S^*$ and $N$ on $S^u$. (M and/or $N$ may be zero). Then we say that an element of $B$ belongs to the subset $B^*$ if

$$
\alpha_m \sim \beta_n
$$

for all $1 \leq m \leq M$ and $1 \leq n \leq N$. (An element of $B$ also belongs to $B^*$ if the condition is satisfied vacuously.) For every pair $\sigma = (\tilde{p}, \tilde{q})$ where $\tilde{p} = (p_1, p_2, \ldots, p_M) \in \mathbb{N}^M$ and $\tilde{q} \in \mathbb{N}^N$, denote by $B_\sigma$ the subset of $B^*$ such that the accumulation rates satisfy

$$
\alpha_{1}^{p_1} = \alpha_{2}^{p_2} = \ldots = \alpha_{M}^{p_M} = \beta_{1}^{p_1} = \ldots = \beta_{N}^{p_N}, \quad (5)
$$
and the subset of $B^*$ for which at least one side has no accumulations we denote by $B_{(0,0)}$. It is easy to see that $B \setminus B^*$ is a residual subset of $B$, since it is the intersection of the complements of the sets $\{B_\sigma\}$.

The main results are the following. Let the standard coordinates of the critical points of $S^*$, including limit points, be given by $\{a_i\}$ and the critical points of $S^u$ be given by $\{b_k\}$. Let $B_g$ be the subset of $B^*$ for which

$$a_i - a_j \neq b_k - b_l$$

for all $i \neq j$ and $k \neq l$. It is not hard to see that $B_g$ is dense in $B^*$ and that $B_g \cap B_\sigma$ is open in $B_\sigma$ for each $\sigma$ (and thus $B_g$ is residual in $B^*$).

**Theorem 1** If $T_0 \in B_g$ then every transversal family $T_\mu$ through $T_0$ encounters only codimension 1 bifurcations in some neighborhood of $\mu = 0$.

**Theorem 2** If $T_0$ belongs to $B \setminus B_g$ then every neighborhood of $T_0$ in the $C^k$-topology contains a codimension 2 bifurcation point.

We will address each of these theorems by subcases. The first subcase is that for which either $S^*$ or $S^u$ has no accumulation of critical points. Theorem 1 will be implied by Propositions 1 and 3 below and Theorem 2 is implied by Propositions 2 and 4.

### 4.2 No accumulation on one side

Suppose that $S^*$ and $S^u$ of $T_0$ each have a finite set of critical points, or that at most one of them, say $S^*$, has a countable set. Let the standard coordinates of the critical points of $S^*$, including limit points, be given by $\{a_i\}$ and the critical points of $S^u$ be given by $\{b_k\}$.

**Proposition 1** Suppose that Eq. 6 holds for all $i \neq j$ and $k \neq l$. Then any transversal family $T_\mu$ through $T_0$ encounters only codimension 1 bifurcations in some neighborhood of $\mu = 0$.

**Proposition 2** Suppose that Eq. 6 is not satisfied for some $i \neq j$ and $k \neq l$. Then every neighborhood of $T_0$ in the $C^k$-topology contains a codimension 2 bifurcation point.

Note: Although Eq. 6 represents a countably infinite number of conditions, these conditions can be verified by checking only a finite number of conditions. That is, we need only consider those $i$ and $j$ for which

$$|a_i - a_j| \geq \min |b_k - b_l|.$$
And, since \( \{a_i\} \) can only have a finite number of contraction rates, this can easily be reduced to checking those conditions only for \( i, j \leq i_0 \) for some finite \( i_0 \).

### 4.3 Accumulation on both sides

Next suppose that \( S^s \) and \( S^u \) each have a countable number of critical heteroclinic orbits. Denote by \( \{a_i\} \) and \( \{b_k\} \) the standard coordinates of these critical orbits, including limit points. Let \( \{\alpha_m\}_{i=1}^M \) and \( \{\beta_n\}_{i=1}^N \) be the set of accumulation rates of critical point on \( S^s \) and \( S^u \) respectively.

**Proposition 3** Suppose for \( T_0 \in B \) that \( \alpha \sim \beta \) for all \( \alpha \in \{\alpha_n\} \) and \( \beta \in \{\beta_m\} \), and that Eq. 6 holds for all \( i \neq j \) and \( k \neq l \) \((T_0 \in B_g)\). Then any transversal family \( T_\mu \) through \( T_0 \) encounters only codimension 1 bifurcations in some neighborhood of \( \mu = 0 \).

**Proposition 4** Suppose that \( \alpha_m \not\sim \beta_n \) for some \( m \) and \( n \) \((T_0 \in B \setminus B^*)\). Then every \( C^k \) neighborhood of \( T_0 \) contains a codimension 2 bifurcation point.

### 5 Proofs of propositions

#### 5.1 Proof of Proposition 1

Let

\[
\delta = \inf \left[ \frac{a_i - a_j}{b_k - b_l} - 1 \right].
\]

(7)

Since \( \{a_i\} \) and \( \{b_k\} \) are compact in \((0, 1)\), the minimum is attained (consider the quantity in Eq. 7 as a function \( \{a_i\} \times \{b_k\} \rightarrow \mathbb{R} \)) and by Eq. 6 it is greater than zero.

Choose \( \ell \) and \( \mu_0 \) small enough that the quantity in Lemma 2 is less than \( \delta/3 \). Let \( I^-_\mu = [\phi^-_\mu(-\ell), \phi^-_\mu(\ell)] \) and \( I^+_\mu = [\phi^-_\mu(\ell), \ell] \). Denote by \( p^i_\mu \) and \( q^k_\mu \) the coordinates in \( I^-_\mu \) and \( I^+_\mu \) of the critical points corresponding to \( a_i \) and \( b_k \).

From Lemma 1 we may also choose \( \ell \) and \( \mu_0 \) small enough that

\[
|a_i - a_j - \frac{p^i_\mu - p^j_\mu}{\phi^-_\mu(-\ell) + \ell}| < \frac{\delta}{3},
\]

and similarly for the quantities on \( S^u \). We have then that

\[
|a_i - a_j - \frac{p^i_\mu - p^j_\mu}{q^k_\mu - q^l_\mu}| < \frac{\delta}{3}
\]

or

\[
\frac{p^i_\mu - p^j_\mu}{q^k_\mu - q^l_\mu} - 1 > \frac{2\delta}{3}
\]

(8)
for all \(i \neq j\) and \(k \neq l\).

In order to encounter a codimension 2 bifurcation we would to have some \(\mu\) and some integer \(n\) for which

\[
\phi^n_\mu(p^i) = q^k_\mu \quad \text{and} \quad \phi^n_\mu(p^j_\mu) = q^l_\mu.
\]

This would imply that

\[
\frac{\phi^n_\mu(p^i) - \phi^n_\mu(p^j_\mu)}{q^k_\mu - q^l_\mu} = 1.
\]

By the Mean Value Theorem, there would exist \(p^* \in (p^i_\mu, p^j_\mu)\) such that

\[
\frac{(\phi^n_\mu)'(p^*)(p^i_\mu - p^j_\mu)}{q^k_\mu - q^l_\mu} = 1.
\]

Equation 8 then would imply that

\[
|\left(\phi^n_\mu\right)'(p^*) - 1| > \frac{2\delta}{3}.
\]

But by assumption

\[
|\left(\phi^n_\mu\right)'(p^*) - 1| < \frac{\delta}{3}.
\]

Finally, a simple application of the \(C^k\) dependence on parameters of invariant manifolds on compact sets shows that no codimension 2 bifurcation may occur.

5.2 Proof of Proposition 2

We have that \(a_i - a_j = b_k - b_l\) for some \(i \neq j\) and \(k \neq l\). Suppose first that \(a_i\) and \(a_j\) correspond to actual critical points (not limit points). Let \(p^i_\mu, p^j_\mu, q^k_\mu,\) and \(q^l_\mu\) be the \(z\) coordinates of these critical orbits in some fundamental intervals.

For \(n \geq n_0\), for some \(n_0\), there exists a sequence \(\{\mu_n\}\) converging to zero such that

\[
\phi^n_{\mu_n}(p^i_{\mu_n}) = q^k_{\mu_n}.
\]

Next choose a sequence of quadruples \(\{(p^i_n, p^j_n, q^k_n, q^l_n)\}\) and a sequence of pairs of integers \(\{(s_n, t_n)\}\) that satisfy the following

\[
p^i_n = \phi^{s_n}_{\mu_n}(p^i_{\mu_n}),
\]

\[
p^j_n = \phi^{s_n}_{\mu_n}(p^j_{\mu_n}),
\]

\[
q^k_n = \phi^{-t_n}_{\mu_n}(q^k_{\mu_n}),
\]

\[
q^l_n = \phi^{-t_n}_{\mu_n}(q^l_{\mu_n}),
\]
\[
\lim_{n \to \infty} q_n^i = 0,
\]

and

\[-p_n^i \in (\phi_{\mu_n}^{-1}(q_n^i), q_n^i].\]

It is easy to see that such a sequence exists. Let

\[u_n = n - s_n - t_n.\]

Then

\[\phi_{\mu_n}^{u_n}(p_n^i) = q_n^k.\]

Now by Lemma 1 and Lemma 2

\[
\lim_{n \to \infty} \frac{\phi_{\mu_n}^{u_n}(p_n^i) - q_n^k}{q_n^k - q_n^j} = \lim_{n \to \infty} \frac{p_n^i - p_n^j}{q_n^k - q_n^j} = 1.
\]

Thus a \(C^k\) perturbation of \(\phi_{\mu_n}\) will make \(\phi_{\mu_n}^{u_n}(p_n^i) = q_n^j\), the size of the perturbation going to zero as \(n \to \infty\). Now applying the \(C^k\) dependence of invariant manifolds, correspondingly small perturbations of \(T_0\) result in codimension 2 or higher bifurcations.

Now suppose that \(a_i\) is the limit point of critical points and not a critical point itself. Then we can repeat the above argument but replace \(a_i\) by an approximating sequence.

5.3 Proof of Proposition 3

We begin by proving the case where \(\{a_i\}\) and \(\{b_k\}\) each corresponds a single representative sequence of \(C^s\) and \(C^u\). Let \(a_\infty\) and \(b_\infty\) denote the limit points of these sequences respectively. Let \(\alpha\) and \(\beta\) be the contraction rates of the points. We have that \(\alpha^p = \beta^q\) with fixed positive integers \(p\) and \(q\).

Since the standard coordinates are \(C^2\) and because of Lemma 5, there exist \(g, h \in C^2\) such that \(g(0) = h(0) = 0\) and \(g'(0) = h'(0) = 1\) and

\[d_i \equiv a_i - a_\infty = g(\alpha^i d_0)\]
\[e_i \equiv b_i - b_\infty = h(\beta^i e_0)\]

Thus,

\[
\frac{a_i - a_j}{b_k - b_l} = \frac{g(\alpha^i d_0) - g(\alpha^j d_0)}{h(\beta^k e_0) - h(\beta^l e_0)}
\]

\[= \frac{(\alpha^i - \alpha^j)d_0 + O((\alpha^i - \alpha^j)^2)}{(\beta^k - \beta^l)e_0 + O((\beta^k - \beta^l)^2)}
\]

\[= \frac{\alpha^i(1 - \alpha^i - \alpha^j)d_0 + O(\alpha^{2i}(1 - \alpha^i - \alpha^j))}{\beta^k(1 - \beta^{-k})e_0 + O(\beta^{2k}(1 - \beta^{-k}))}\]
\[ = \alpha^{i-k} \frac{(1 - \alpha^{i-j})d_0 + O(\alpha^{2i})}{(1 - \beta^{l-k})e_0 + O(\beta^{2k})}. \]

Since Eq. 6 holds, if we show that this last expression does not accumulate at 1, this will imply that \( \delta \) in Eq. 7 is greater than 0, and the steps of the proof of Proposition 1 may be repeated. It is easy to see that we need only check that

\[
\alpha^{i-k} \frac{d_0}{e_0}
\]

does not accumulate at 1. This cannot happen because \((i - k)\) mod 1 takes on only a finite set of values. These finite set of values do not cause difficulties because we assumed that Eq. 6 holds for all \( i \neq j \) and \( k \neq l \), including limit points. 

If \( \{a_i\} \) and \( \{b_k\} \) have multiple accumulations then we may use Lemma 5 and apply the steps of the preceding proof to subsequences governed by each pair \((\alpha_i, \beta_j)\) of accumulation rates.

### 5.4 Proof of Proposition 4

If Eq. 6 is disobeyed for some \( i \neq j \) and \( k \neq l \), then the proof of Proposition 2 suffices and so we assume Eq. 6 is satisfied for all \( i \neq j \) and \( k \neq l \). Let \( \alpha \in \{\alpha_n\} \) and \( \beta \in \{\beta_m\} \) be such that \( \alpha \neq \beta \). As in the proof of Proposition 3 we have

\[
\frac{a_i - a_j}{b_k - b_l} = \frac{\alpha^i(1 - \alpha^{j-i})d_0 + O(\alpha^{2i})}{\beta^k(1 - \beta^{l-k})e_0 + O(\beta^{2k})}.
\]

Thus, we need only show that

\[
\frac{\alpha^i d_0}{\beta^k e_0}
\]

approaches 1 for some subsequence of \( i \) and \( k \). Because of the existence of representative sequences in \( C^* \) and \( C^u \), we can assume without loss of generality that \( i \) and \( k \) may take arbitrary sufficiently large integer values. Assume that \( \alpha \neq \beta \) and consider the expression

\[
\log_\alpha \left( \frac{\alpha^i}{\beta^k} \right) = i - k \log_\alpha \beta. \quad (9)
\]

If \( \log_\alpha \beta \) is irrational (it is if \( \alpha \neq \beta \)), then it is well known that the set of values attained by the right hand side of Eq. 9 is everywhere dense in the real numbers.

Now, having obtained a sequence of critical values for which the ratio

\[
\frac{a_i - a_j}{b_k - b_l}
\]

approaches 1 we repeat the proof of Proposition 2 using this sequence.
6 Conclusions

Theorem 1 describes behavior of neighboring diffeomorphisms of $T_0$ when $T_0$ belongs to the 'good' set $B_g$. It implies that the codimension 2 or higher bifurcation points might reach $T_0$ only in the directions nontransversal to $B$ at a point $T_0 \in B_g$. As concerned neighboring diffeomorphisms of 'bad' points (in $B \setminus B_g$), Theorem 2 only says that codimension 2 points do exist. Some preliminary considerations show that codimension 2 diffeomorphisms might reach $T_0 \in B \setminus B_g$ in directions transversal to $B$, but we have no time (and space) to include this possible result in the present work.

Hypothesis 2 singles out an open but relatively small set among all possible situations. In fact, one needs to consider all possible variants of behavior of critical heteroclinic orbits for diffeomorphisms on the boundary of the Morse-Smale set, including those with a saddle-node point which is a saddle along hyperbolic directions. We hope to study this elsewhere.

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References


