PROOF AND GENERALIZATION OF KAPLAN-YORKE’S
CONJECTURE ON PERIODIC SOLUTION
OF DIFFERENTIAL DELAY EQUATIONS

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ABSTRACT. In this paper, using the theory of existence of periodic solutions of Hamiltonian systems, we show that infinitely many periodic solutions of differential delay equations can be yielded from a family of periodic solutions of the coupled generalized Hamiltonian systems. Some sufficient conditions on the existence of periodic solutions of differential delay equations are obtained. As a corollary of our results, we show that the conjecture of Kaplan-Yorke on the search for periodic solutions for certain special classes of scalar differential delay equations is true.

1. Introduction.

In 1974, Kaplan and Yorke [8] studied and introduced a new technique for establishing the existence of periodic solutions for certain special classes of scalar differential delay equations of the forms

\[ y'(t) = -f(y(t - 1)) \quad (1.1) \]

and

\[ y'(t) = -[f(y(t - 1)) + f(y(t - 2))] \quad (1.2) \]

when \( f \) is an odd function. They successfully reduced the search for periodic solutions of (1.1) and (1.2) to the problem of finding periodic solutions of associated ordinary differential systems and obtained some precise conditions under which the equations (1.1) and (1.2) have nontrivial periodic solutions of period 4 and 6 respectively. When the scalar differential equations contain more than two delays, like the equation (1.3) below, they remarked with the following conjecture (Remark 2.1 in Kaplan and Yorke [8]):

“Consider the differential delay equation

\[ x'(t) = -[f(x(t - 1)) + f(x(t - 2)) + \cdots + f(x(t - (n - 1)))] \quad (1.3) \]
where \( f(0) = 0 \), \( f \) is odd and \( xf(x) > 0 \) for \( x \neq 0 \). Let \( X = (x_1, x_2, \cdots, x_n)^T \) and \( \Psi(x) = (f(x_1), f(x_2), \cdots, f(x_n))^T \), where “\( T \)” denotes the transpose. Let \( A_n \) denote the \( n \times n \) antisymmetric matrix

\[
A_n = \begin{pmatrix}
0 & -1 & -1 & \cdots & -1 & -1 \\
1 & 0 & -1 & \cdots & -1 & -1 \\
1 & 1 & 0 & \cdots & -1 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 0 & -1 \\
1 & 1 & 1 & \cdots & 1 & 0 \\
\end{pmatrix}
\]

It is readily verified that given any periodic solution \( x(t) \) of (1.3) with period \( 2n \), such that \( x(t) = -x(t-n) \) for all \( t \), it must satisfy the system of ordinary differential equations

\[
X'(t) = A_n \Psi(X),
\]

where \( x_1(t) = x(t), \ x_2(t) = x(t-1), \cdots, x_n(t) = x(t-(n-1)) \). Theorems 1.1 and 1.2 show that if \( n = 2 \) or \( n = 3 \) and \( f \) is suitably well behaved near 0 and \( \infty \), then converse is also true. Hopefully, such a converse should be true for (1.3) and (1.4) \( \cdots \). Unfortunately, we have been unable to carry through the details of the proof in this more general setting”.

Existence of periodic solutions of delay differential equations has also been investigated by many other authors using different techniques. In 1978, using a different idea from the conjecture of Kaplan and Yorke [8] and some general fixed point principles of nonlinear functional analysis, Nussbaum [15] proved that there exists a periodic solution of (1.3) with period \( 2n \). The questions and results given by Nussbaum [15] are more general. In our recent paper Gopalsamy et al [7], we have shown how to construct the periodic solution of Kaplan-Yorke type for some more general differential delay equations with two or three delays. For other related works, readers can refer to the references cited in Gopalsamy et al [7]. However, to the best of our knowledge, the conjecture of Kaplan-Yorke “in this more general setting” for (1.3) and (1.4) is still open.

In this paper, we consider differential delay equations of the form

\[
x'(t) = -\sum_{i=1}^{n-1} f(x(t-r_i)), \tag{1.5}
\]

which is slightly more general than (1.3), where \( f \) is a suitable odd function and \( r_i \ (i = 1, 2, \cdots, n-1) \) are constant delays. We show that the coupled system (1.4) of (1.5) is actually a classical Hamiltonian system (for \( n = 2k \)) or a generalized Hamiltonian system (for \( n = 2k + 1 \)). Using the relationship of periodic solutions between (1.4) and (1.5) and the theory of periodic solutions of Hamiltonian systems, we derive some interesting results on the existence of infinitely many periodic solutions of (1.5). As a corollary of our results, we show that the conjecture of Kaplan-Yorke is true.
The organization of this paper is as follows. In the next section we first establish the equivalence on the existence of periodic solutions of (1.5) and the associated coupled ordinary differential systems and hence we are able to reduce the search for periodic solutions of (1.5) to the problem of finding periodic solutions of the associated ordinary differential systems. In section 3, using the theory of existence of periodic solutions of Hamiltonian systems, we show the existence of periodic solutions of the coupled systems. In the last section, sufficient conditions on the existence of periodic solutions of (1.5) are given and, applying our results to (1.3), we give a positive answer to the conjecture of Kaplan-Yorke. We conclude the paper with an application to an interesting example from neural networks.

2. Equivalence on the existence of periodic solutions.

Throughout this section we assume that

\((H_1)\) the function \(f(\cdot) \in C^r(\mathbb{R}), r \geq 1, f(-x) = -f(x), f(0) = 0, xf(x) > 0\) for \(x \neq 0\) and \(0 < x < a\); where \(a\) is a constant.

The function

\[ H(X) = H(x_1, x_2, \cdots, x_n) = F(x_1) + F(x_2) + \cdots + F(x_n) = h \]

(2.1)
is called a Hamiltonian, where \(F(x) = \int_0^x f(s) ds\) for \(x \in \mathbb{R}\) and \(h > 0\) is a constant. Denote \(\Psi(X) = \nabla H(X)\) in which \(\nabla H(X)\) is the gradient of \(H(X)\) and hence \(\Psi(X) = \nabla H(X) = (f(x_1), f(x_2), \cdots, f(x_n))^T\).

It is known from the theory of generalized Hamiltonian systems (see Li et al [9], Meyer and Hall [12] and Nussbaum [15]) that the system (1.4) can be rewritten as

\[
\frac{dX}{dt} = A_n \nabla H(X),
\]

(2.2)

where \(A_n\) is the \(n \times n\) antisymmetric matrix defined in the previous section. Clearly, when \(n = 2k\), (2.2) is a classical 2\(k\)-dimensional Hamiltonian system since \(A_n^T J + JA_n = 0\), where \(J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\) is a \(2k \times 2k\) matrix and \(I\) is \(k \times k\) identity matrix; that is \(A_n\) is a classical Hamiltonian (or symplectic) matrix. When \(n = 2k + 1\), \(A_n\) is called a structure matrix of the system (2.2) and the system (2.2) is called a generalized Hamiltonian system. In this case, there is an invariant function

\[ C(X) = C(x_1, x_2, \cdots, x_n) = x_1 + \sum_{i=1}^{k} (x_{2i+1} - x_{2i}) = c, \]

(2.3)

where \(c\) is a constant. \(C(X)\) is called a Casimir function of (2.2). The level set defined by (2.3) is called a symplectic leaf. We know from the theory of the generalized Hamiltonian systems that the system (2.2) can be reduced to a 2\(k\)-dimensional Hamiltonian system on the symplectic leaf \(x_1 + \sum_{i=1}^{k} (x_{2i+1} - x_{2i}) = c\). In fact, by
choosing $c = 0$, the symplectic leaf is given by $x_{2k+1} = \sum_{i=1}^{k} (x_{2i} - x_{2i-1})$ and on this leaf we have

$$
\begin{align*}
\frac{dx_1}{dt} &= -f(x_2) - f(x_3) - \cdots - f(x_{2k}) - f\left(\sum_{i=1}^{k} (x_{2i} - x_{2i-1})\right), \\
\frac{dx_2}{dt} &= f(x_1) - f(x_3) - \cdots - f(x_{2k}) - f\left(\sum_{i=1}^{k} (x_{2i} - x_{2i-1})\right), \\
&\vdots \\
\frac{dx_{2k}}{dt} &= f(x_1) + f(x_2) + \cdots + f(x_{2k-1}) - f\left(\sum_{i=1}^{k} (x_{2i} - x_{2i-1})\right)
\end{align*}
$$

with the new Hamiltonian

$$
H^\ast(X) \equiv H^\ast(x_1, x_2, \cdots x_{2k}) = F(x_1) + F(x_2) + \cdots + F(x_{2k}) + F\left(\sum_{i=1}^{k} (x_{2i} - x_{2i-1})\right) = h. \quad (2.4)
$$

The previous system can be rewritten as the following

$$
\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2k-1} \\ x_{2k} \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & -1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} f(x_1) - f\left(\sum_{i=1}^{k} (x_{2i} - x_{2i-1})\right) \\ f(x_2) + f\left(\sum_{i=1}^{k} (x_{2i} - x_{2i-1})\right) \\ \vdots \\ f(x_{2k-1}) - f\left(\sum_{i=1}^{k} (x_{2i} - x_{2i-1})\right) \\ f(x_{2k}) + f\left(\sum_{i=1}^{k} (x_{2i} - x_{2i-1})\right) \end{pmatrix}
$$

i.e.,

$$
\frac{dX}{dt} = A_{2k} \triangledown H^\ast(X), \quad X = (x_1, x_2, \cdots, x_{2k})^T. \quad (2.5)
$$

Clearly, $A_{2k}$ is a $2k \times 2k$-symplectic matrix. It means that (2.5) is also a classical Hamiltonian system.

To establish the relationship on the existence of periodic solutions of (1.5) and the associated coupled systems (2.2) (for $n = 2k$) and (2.5) (for $n = 2k + 1$), we define a map $T_n : \mathbb{R}^n \to \mathbb{R}^n$ by the $n \times n$ matrix

$$
T_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n}. \quad (2.6)
$$
It is easy to verify that the characteristic equation of (2.6) is \( \lambda^n + 1 = 0 \). Hence we have \((T_n^n) = -I_n\), \((T_n^{2n}) = I_n\), and \((T_n^{\ell}) = T_n^{2^{n-\ell}}\), \( \ell = 1, 2, \ldots, n \), where \( I_n \) is the \( n \times n \) identity matrix. It follows that the set \( G_T = \{ g | g = T_n^m, m = 1, 2, \ldots, 2n \} \) is a group of transformations acting on \( \mathbb{R}^n \). If we rewrite the right side of (2.2) as \( \dot{X} = \Phi(X) \), then it is easy to see that \( \Phi(gX) = g\Phi(X) \) for all \( g \in G_T \) and hence the group \( G_T \) is a symmetry group (see Olver [16]) of the system (2.2). The mapping: \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) is called \( G_T \)-equivalent. Moreover, the Hamiltonian \( H(X) \) defined by (2.1) is a \( G_T \)-invariant function, that is, \( H(X(t)) = H(gX(t)) \).

We note that \((T_n^m)^T \cdot T_n^m = I_n\), thus the group \( G_T \) is a closed compact subgroup of the orthogonal matrix group \( O(n) \) with the order \( 2n \) and \( O(n) \) is also a Lie group. \( G_T \) has also a closed subgroup \( G_T^{sub} = \{ g | g = T_n^m, m = 1, 2, \ldots, n \} \) with the order \( n \) and \( \det(T_n^m) = 1 \) for \( m = 1, 2, \ldots, n \). Hence \( G_T^{sub} \) is an \( n \)-order closed subgroup of the special orthogonal group \( SO(n) \). Each matrix in \( SO(n) \) represents a rotation in \( \mathbb{R}^n \) about an axis through the origin and the group \( SO(n) \) is often called \( n \)-dimensional rotation group. On the other hand, the action \( g = T_n \) on \( \mathbb{R}^n \) can be represented by the product of a reflection \( x_1 \to -x_1 \) and a rotation in \( \mathbb{R}^n \), that is

\[
T_n = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix} \begin{pmatrix}
-1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix} \equiv (T_n)_{rot}(T_n)_{ref}.
\]

It follows from the theory of linear algebra that for a standard basis \((\epsilon_1, \epsilon_2, \ldots, \epsilon_n)\) in \( \mathbb{R}^n \), a rotation by \( \theta \) on the plane \((\epsilon_i, \epsilon_j)\) has the following matrix representation

\[
R_{ij} = \begin{pmatrix}
I_{i-1} \\
S_{j-i+1} \\
I_{j-1}
\end{pmatrix},
\]

where \( I_k \) is the \( k \times k \) identity matrix and

\[
S_{j-i+1} = \begin{pmatrix}
cos \theta & 0 & \cdots & 0 & \sin \theta \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
-sin \theta & 0 & \cdots & 0 & \cos \theta
\end{pmatrix}_{(j-i+1) \times (j-i+1)};
\]

\( R_{ij} \) is called an elementary rotation matrix. We know that a rotation matrix \((T_n)_{rot} \) can be decomposed into a product of finitely many elementary rotation matrices. Furthermore it is well known from the theory of groups and symmetry (see Armstrong [2]) that if \( g \in O(n) \), then the invertible linear transformation \( gx : \mathbb{R}^n \to \mathbb{R}^n \), \( x \in \mathbb{R}^n \) preserves the distance and orthogonality; that is, \( \|gx - gy\| = \|x - y\| \) and \( <gx, gy> = <x, y> \), where \( \| \cdot \| \) is the Euclidean norm and \( < \cdot , \cdot > \) is the inner product in \( \mathbb{R}^n \). Hence, if \( X(t) \) is a periodic solution of (2.2) and (2.5), then \( g^m X(t) \)
describes the same trajectory as $X(t)$ for $g \in G_T$ and $m = 1, 2, \cdots, 2n$. In fact, $g^{2n}$ leaves each point of $X(t)$ fixed individually. $g^{m}(m \neq 2n)$ must leave the set of the periodic orbit $\{X(t)| t \in \mathbb{R}\}$ fixed, as a set, possibly phase-shifting the individual points on it (see Fiedler [5] and Golubitsky et al [6]).

**Theorem 2.1.** Suppose that the condition $(H_1)$ holds and

$(H_2)$ there exist nonnegative integers $m_1, m_2, \cdots, m_{n-1}$ (not necessary distinct) such that the delays of (1.5) satisfy

$$\frac{r_1}{1+2nm_1} = \frac{r_2}{2+2nm_2} = \cdots = \frac{r_{n-1}}{(n-1)+2nm_{n-1}} = \mu; \quad (2.7)$$

$(H_3)$ on an energy hypersurface $S$ of $H(X) = h$ ($H^*(X) = h$), the system (2.2) ((2.5)) has a nontrivial periodic solution $X_o(t) = (x_1(t), x_2(t), \cdots, x_{2k}(t))$ of period $p = 2n\mu$ with initial condition

$$\begin{align*}
X_o(t)|_{t=0} &= X_o(0) = (x_1(0), x_2(0), \cdots, x_{2k}(0)) \\
&= (0, \rho, x_2(0), \cdots, x_{2k}(0)), \quad a > \rho > 0. \quad (2.8)
\end{align*}$$

Then $x(t) = x_1(t)$ is a nontrivial periodic solution of (1.5) with period $p = 2n\mu$.

**Remark.** It is well known that for the autonomous Hamiltonian systems (2.2) and (2.5), every nontrivial periodic solution around the origin is oscillating. Without loss of the generality, we can assume in (2.8) of Theorem 2.1 that $x_1(0) = 0, x_2(0) = \rho$ and $H(X_o(0)) = h$.

**Proof of Theorem 2.1.** It follows from the condition $(H_1)$ and the oddness of $f(x)$ that if $X_o(t)$ is a periodic solution of (2.2) or (2.5), then $T^n_mX_o(t)$ ($m = 1, 2, \cdots, 2n - 1$) are also the periodic solutions of (2.2) or (2.5) with the same period as $X_o(t)$. Moreover, by the $G_T$-invariance of $H$ and $H^*$, these solutions have the same Hamiltonian energy, i.e.

$$H(X_o(t)) = H(T^n_mX_o(t)) \quad \text{or} \quad H^*(X_o(t)) = H^*(T^n_mX_o(t)), \quad m = 1, 2, \cdots, 2n.$$  

Hence $X_o(t)$ and all of the $T^n_mX_o(t)$ ($m = 1, 2, \cdots, 2n$) lie on the same level set of $H$ or $H^*$ and, by the hypothesis $(H_3)$ and the discussion preceding this theorem, they have the common orbit as $X_o(t)$ in the $2k$-dimensional phase space. Therefore $X_o(t)$ will meet other solutions $T^n_mX_o(t)$ after some translation, say $\tau$, $0 < \tau < p$.

(a) The case $n = 2k$. In this case, we consider the system (2.2). It is noticed that $T^n_n(X_o(t)) = -X_o(t)$ is also a periodic solution of (2.2) and hence there exists $\tau_1, 0 < \tau_1 < p$ such that

$$X_o(t) = -X_o(t + \tau_1) = X_o(t + 2\tau_1), \quad (2.9)$$

which implies that $2\tau_1 = p$, that is, $\tau_1 = \frac{p}{2}$. On the other hand, there is a $\tau_2, 0 < \tau_2 < p$ such that

$$\begin{align*}
X_o(t) &= T^n_nX_o(t + \tau_2) = T^n_2X_o(t + 2\tau_2) = \cdots = T^n_{n}X_o(t + n\tau_2) \\
&= T^n_{n+1}X_o(t + (n + 1)\tau_2) = \cdots = T^n_{2n}X_o(t + 2n\tau_2) \\
&= X_o(t + 2n\tau_2). \quad (2.10)
\end{align*}$$
Since all points in the set defined by \( P = \{ T_n^m X_0(0); m = 0, 1, 2, \cdots, 2n - 1 \} \) are pairwise distinct, we have \( m \tau_2 \not\equiv 0 \pmod{2n} \) for \( m = 1, 2, \cdots, 2n - 1 \). From (2.10) we see that there exists an integer \( \ell \) such that \( 2n \tau_2 = \ell p \), or \( \tau_2 = \frac{\ell p}{2n} \). It follows from (2.9) and (2.10) that

\[
X_0(t) = -X_0(t + \frac{p}{2}) = -X_0(t - \frac{p}{2}) = T_n^n X_0(t + n \tau_2) = -X_0(t + n \tau_2)
\]

and which implies that \( \tau_2 = \frac{p}{2n} = \mu \) and hence \( \ell = 1 \). Again, using (2.10) and the definition of \( T_n \), we have

\[
x_1(t) = x_2(t + \tau_2) = x_3(t + 2 \tau_2) = \cdots = x_1(t + 2n \tau_2) = x_2(t + (2n + 1) \tau_2) = \cdots = x_2(t + (1 + 2m_1) \tau_2); \\
x_1(t) = x_3(t + 2 \tau_2) = x_4(t + 3 \tau_2) = \cdots = x_1(t + 2n \tau_2) = x_3(t + (2n + 2) \tau_2) = \cdots = x_3(t + (2 + 2m_2) \tau_2); \\
\vdots \\
x_1(t) = x_n(t + (n - 1) \tau_2) = -x_1(t + n \tau_2) = \cdots = x_1(t + 2n \tau_2) = x_n(t + (2n + (n - 1)) \tau_2) = \cdots = x_n(t + ((n - 1) + 2nm_{n-1}) \tau_2);
\]

where \( m_1, m_2, \cdots, m_{n-1} \) are defined by (2.7). We derive from the previous relationship and the hypothesis \((H_2)\) that

\[
\begin{align*}
x_2(t) &= x_1(t - (1 + 2m_1) \tau_2) = x_1(t - r_1) \\
x_3(t) &= x_2(t - (2 + 2m_2) \tau_2) = x_2(t - r_2) \\
\vdots \\
x_n(t) &= x_{n-1}(t - ((n - 1) + 2nm_{n-1}) \tau_2) = x_{n-1}(t - r_{n-1})
\end{align*}
\]

\hspace{1cm} (2.11)

Hence it follows from (2.11) and the first equation in the system (2.2) that \( x(t) = x_1(t) \) is a nontrivial periodic solution of (1.5) with period \( p = 2n \mu = \frac{2n \tau_1}{1 + 2m_1} \).

(b) The case \( n = 2k + 1 \). In this case, let \( Y_0(t) = (X_0^T(t), x_{2k+1}(t))^T \), where \( X_0(t) = (x_1(t), x_2(t), \ldots, x_{2k}(t))^T \) is the periodic solution of (2.5) and \( x_{2k+1} = \sum_{i=1}^{k} [x_{2i}(t) - x_{2i-1}(t)] \). We also notice that \( T^n Y_0(t) = -Y_0(t) \) is a periodic solution of (2.2). Treating \( Y_0(t) \) as \( X_0(t) \) of the case (a), we can show similarly that the result is also true. This completes the proof. \( \blacksquare \)

3. Periodic solutions of the coupled systems.

We now turn to the existence of periodic solutions of the coupled systems (2.2) (for \( n = 2k \)) and (2.5) (for \( n = 2k + 1 \)) of (1.5). We assume that the function \( f \) satisfies \((H_1)\) and \( f'(0) = \omega > 0 \).
Lemma 3.1. Let $\lambda$ be an eigenvalue of the linearized systems of (2.2) and (2.5) at the origin $X = (0, 0, \cdots, 0)_{2k}$ and $\lambda_0 = \frac{1}{2}$, then the coefficient matrices $\omega A_{2k}$ and $\omega A_{2k}^*$ of the linearized systems of (2.2) and (2.5) have respectively the following characteristic equations:

$$|\lambda_0 I - A_{2k}| = \frac{1}{2} \left[ (1 + \lambda_0)^{2k} + (1 - \lambda_0)^{2k} \right] = 0,$$

$$|\lambda_0 I - A_{2k}^*| = \frac{1}{2\lambda_0} \left[ (1 + \lambda_0)^{2k+1} + (1 - \lambda_0)^{2k+1} \right] = 0.$$

Therefore, $A_{2k}$ and $A_{2k}^*$ have respectively $k$ pairs of purely imaginary eigenvalues as follows:

$$\lambda_A = \pm i \frac{\sin \left( \frac{(2\ell+1)\pi}{2k} \right)}{1 + \cos \left( \frac{(2\ell+1)\pi}{2k} \right)}, \quad \ell = 0, 1, 2, \cdots, k - 1;$$

$$\lambda_A^* = \pm i \frac{\sin \left( \frac{2\pi}{2k+1} \right)}{1 + \cos \left( \frac{2\pi}{2k+1} \right)}, \quad \ell = 1, 2, \cdots, k.$$

Proof. It is easy to calculate that the coefficient matrices of the linearized systems of (2.2) and (2.5) at the origin have the following forms

$$
\begin{pmatrix}
0 & -\omega & -\omega & \cdots & -\omega & -\omega \\
\omega & 0 & -\omega & \cdots & -\omega & -\omega \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\omega & \omega & \omega & \cdots & 0 & -\omega \\
\omega & \omega & \omega & \cdots & \omega & 0
\end{pmatrix}
= \omega A_{2k}
$$

and

$$
\begin{pmatrix}
\omega & -2\omega & 0 & -2\omega & \cdots & 0 & -2\omega \\
2\omega & -\omega & 0 & -2\omega & \cdots & 0 & -2\omega \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2\omega & 0 & 2\omega & 0 & \cdots & \omega & 2\omega \\
2\omega & 0 & 2\omega & 0 & \cdots & 2\omega & -\omega
\end{pmatrix}
= \omega A_{2k}^*
$$

respectively. It follows that

$$|\lambda_0 I - A_{2k}| = \det \begin{pmatrix} \lambda_0 & 1 & 1 & \cdots & 1 & 1 \\
-1 & \lambda_0 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & -1 & \cdots & \lambda_0 & 1 \\
-1 & -1 & -1 & \cdots & -1 & \lambda_0 \end{pmatrix}$$

and

$$|\lambda_0 I - A_{2k}^*| = \det \begin{pmatrix} \lambda_0 & -1 & 2 & 0 & 2 & \cdots & 0 & 2 \\
-2 & \lambda_0 & 1 & 0 & 2 & \cdots & 0 & 2 \\
-2 & 0 & \lambda_0 & -1 & 2 & \cdots & 0 & 2 \\
-2 & 0 & -2 & \lambda_0 & 1 & \cdots & 0 & 2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-2 & 0 & -2 & 0 & \cdots & \lambda_0 & -1 & 2 \\
-2 & 0 & -2 & 0 & \cdots & -2 & \lambda_0 & 1 \end{pmatrix}.$$
Simplifying these determinants by row transformations and expanding them along the last row, we obtain (3.1) and (3.2), which can be written as

$$\left| \lambda_o I - A_{2k} \right| = \frac{(1 + \lambda_o)^{2k}}{2} \left[ 1 + \left( \frac{1 - \lambda_o}{1 + \lambda_o} \right)^{2k} \right] = \frac{(1 + \lambda_o)^{2k}}{2} \left[ 1 + \mu^{2k} \right] = 0 \quad (3.3)$$

and

$$\left| \lambda_o I - A_{2k}^* \right| = \frac{(1 + \lambda_o)^{2k+1}}{2\lambda_o} \left[ 1 - \left( \frac{1 - \lambda_o}{1 + \lambda_o} \right)^{2k+1} \right] = \frac{(1 + \lambda_o)^{2k}}{2\lambda_o} \left[ 1 - \mu^{2k+1} \right] = 0, \quad (3.4)$$

where \( \mu = \frac{1 - \lambda_o}{1 + \lambda_o} \) and hence \( \lambda_o = \frac{1 - \mu}{1 + \mu} \). One notices that \( \lambda_o = -1 \) is not a characteristic root of \( A_{2k} \) and \( A_{2k}^* \). It follows from (3.3) and (3.4) that \( \mu^{2k} = -1 \) and \( \mu^{2k+1} = 1 \) and hence \( A_{2k} \) and \( A_{2k}^* \) have respectively \( k \)-pairs of distinct purely imaginary eigenvalues given by the Lemma.

Remark 3.1. For the matrix \( \omega A_{2k} \), the conclusion of Lemma 3.1 can also be followed from the existence of Hamiltonian integral with a positive definite Hessian matrix at the origin (see Lemma 3.2 below). Based on the theory of Hamiltonian systems, we know that if \( \lambda \) is an eigenvalue of a real Hamiltonian matrix, then \(-\lambda, \lambda, -\lambda \) and \( -\lambda^{-1}, \lambda^{-1} \) are also the eigenvalues of the matrix. Hence the set of the eigenvalues of \( A_{2k} \) can be arranged according to the order of their moduli of the eigenvalues as follows:

$$|\lambda_A| : \quad \gamma_1 = \frac{\sin \frac{\pi}{2k}}{1 + \cos \frac{\pi}{2k}}, \quad \gamma_2 = \frac{\sin \frac{3\pi}{2k}}{1 + \cos \frac{3\pi}{2k}}, \ldots, \quad \gamma_{\left[ \frac{k-1}{2} \right]} = \frac{\sin \left( \frac{(k-1)\pi}{2k} \right)}{1 + \cos \left( \frac{(k-1)\pi}{2k} \right)},$$

and

$$\gamma_{\left[ \frac{k-1}{2} \right]}, \ldots, \gamma_2^{-1}, \gamma_1^{-1};$$

similarly, the set of the eigenvalues of \( A_{2k}^* \) can be arranged as follows:

$$|\lambda_A^*| : \quad \tilde{\gamma}_1 = \frac{\sin \frac{2\pi}{2k+1}}{1 + \cos \frac{2\pi}{2k+1}}, \quad \tilde{\gamma}_2 = \frac{\sin \frac{4\pi}{2k+1}}{1 + \cos \frac{4\pi}{2k+1}}, \ldots, \quad \tilde{\gamma}_{\left[ \frac{k}{2} \right]} = \frac{\sin \left( \frac{(k+1)\pi}{2k+1} \right)}{1 + \cos \left( \frac{(k+1)\pi}{2k+1} \right)},$$

and

$$\tilde{\gamma}_{\left[ \frac{k}{2} \right]}, \ldots, \tilde{\gamma}_2^{-1}, \tilde{\gamma}_1^{-1},$$

where \( \left[ \frac{k}{2} \right] \) and \( \left[ \frac{k-1}{2} \right] \) denote the integer part of \( \frac{k}{2} \) and \( \frac{k-1}{2} \) respectively.

We know from the theory of convex analysis (see Rockafellar [17]) that, if the function \( H \) is a twice continuously differentiable real-valued function on an open convex set \( S \) in \( \mathbb{R}^n \), then \( H \) is convex on \( S \) if and only if its Hessian matrix

$$H_{xx} = (h_{ij}(X))_{2k \times 2k}, \quad h_{ij}(X) = \frac{\partial^2 H}{\partial x_i \partial x_j}(x_1, x_2, \ldots, x_{2k}), \quad (i, j = 1, 2, \ldots, n)$$

is positive semi-definite for every \( X \in S \) (see p. 27 in Rockafellar [17]). This result leads to the following conclusion.
Lemma 3.2. Suppose that \( f(x) \in C^1(\mathbb{R}) \), \( f(-x) = -f(x) \), \( f(0) = 0 \), \( xf(x) > 0 \) for all \( x \neq 0 \). If \( f'(x) > 0 \) for all \( x \neq 0 \) then the Hamiltonian functions \( H \) and \( H^* \) defined by (2.1) and (2.4) are convex functions in \( \mathbb{R}^{2k} \).

Proof. Let \( H_{xx}(X) \) and \( H_{xx}^*(X) \) be the Hessian matrix of \( H \) and \( H^* \) at the point \( X = (x_1, x_2, \ldots, x_{2k}) \in \mathbb{R}^{2k} \) respectively. Then it is easy to see that

\[
H_{xx}(X) = \begin{pmatrix}
  f'(x_1) & 0 & 0 & \cdots & 0 & 0 \\
  f'(x_2) & 0 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & f'(x_{2k-1}) & 0 \\
  0 & 0 & 0 & \cdots & 0 & f'(x_{2k})
\end{pmatrix}
\]

and \( H_{xx}^*(X) \) is given by

\[
\begin{pmatrix}
  f'(x_1) + f'(x_{2k+1}) & -f'(x_{2k+1}) & \cdots & \cdots & f'(x_{2k+1}) & -f'(x_{2k+1}) \\
  -f'(x_{2k+1}) & f'(x_2) + f'(x_{2k+1}) & \cdots & \cdots & f'(x_{2k+1}) & f'(x_{2k+1}) \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  f'(x_{2k+1}) & -f'(x_{2k+1}) & \cdots & f'(x_{2k}) + f'(x_{2k+1}) & -f'(x_{2k+1}) \\
  -f'(x_{2k+1}) & f'(x_{2k+1}) & \cdots & -f'(x_{2k+1}) & f'(x_{2k+1}) + f'(x_{2k+1})
\end{pmatrix}
\]

in which \( x_{2k+1} = \sum_{i=1}^{k} (x_2 - x_{2i-1}) \). Clearly, the condition \( f'(x) > 0 \) for all \( x \neq 0 \) implies that all the principal minors of \( H_{xx}(X) \) and \( H_{xx}^*(X) \) are positive for all \( X = (x_1, x_2, \ldots, x_{2k}) \neq 0 \) in \( \mathbb{R}^{2k} \) and hence the proof is completed.

We now consider the canonical Hamiltonian system

\[
\frac{dx_i}{dt} = -\frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = \frac{\partial H}{\partial x_i}, \quad (i = 1, 2, \ldots, k)
\]

which can be written as

\[
\dot{z} = J \nabla H(z), \quad z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad J = J_{2k} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\]

where 0 is the \( k \times k \) zero matrix, \( I \) is the \( k \times k \) identity matrix and \( H(z) = H(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k) \equiv H(x, y) \) is independent of \( t \) and vanishes together with its first partial derivatives at \( x = 0, y = 0 \); that is \( H(0) = H_z(0) = 0 \).

In the following Lemma, sufficient conditions for the existence of periodic solution of Hamiltonian system (3.7) having the minimal period \( p \) are given.

Lemma 3.3. (Clarke and Ekeland [4], pp. 105-106) Let \( H \) be non-negative and convex function vanishing only at the origin.

(i) Suppose \( H \) is differentiable except at the origin and satisfies two conditions:

\[
H(x, y)/(x^2 + y^2) \to 0 \quad \text{as} \quad (x, y) \to \infty,
\]

(ii) Suppose \( H \) is differentiable except at the origin and satisfies two conditions:
\[
H(x,y)/(x^2 + y^2) \to \infty \quad \text{as} \quad (x,y) \to 0. \tag{3.10}
\]

Then, for any \( p > 0 \), there is a periodic solution with minimal period \( p \) to Hamiltonian system (3.7).

(ii) Suppose that \( H \) is locally Lipschitz and \( H(x,y) \to \infty \) as \( (x,y) \to \infty \). Then for any \( h > 0 \), there is a nontrivial absolutely continuous and periodic solution \( z(t) = (x(t),y(t)) \) which satisfies \((-\dot{y}(t), \dot{x}(t)) \in \partial H(x(t),y(t)) \) and

\[
H(x(t),y(t)) = h \quad \text{for all} \quad t, \tag{3.11}
\]

where \( \partial H(x(t),y(t)) \) denotes the subdifferential of \( H \) at \( (x,y) \) in the sense of convex analysis (see Rockafellar [17]).

In Lemma 3.3, the behavior of \( H \) near 0 and at infinity are assumed. In the case without the assumption of \( H \) near 0, we will use the following Lemma in our further discussion.

**Lemma 3.4.** (Breis and Coron [3], p. 566 and Mawhin and Willem [11], p. 53) Let \( H : \mathbb{R}^{2k} \to \mathbb{R} \) be a \( C^1 \) convex function such that \( H(0) = H_z(0) = 0 \). Assume that

\[
\lim_{|z| \to \infty} \frac{H(z)}{|z|^2} = 0, \quad \lim_{|z| \to \infty} H(z) = \infty. \tag{3.12}
\]

Then there exists \( p_0 > 0 \) such that for every \( p > p_0 \), the system (3.7) possesses a periodic solution \( z_p(t) \) with minimal period \( p \). Moreover,

\[
\min_{t \in \mathbb{R}} |z_p(t)| \to \infty \quad \text{as} \quad p \to +\infty.
\]

By using the theory of Hamiltonian systems (see for example Meyer and Hall [12]), we know that there exists a linear transformation preserving the Hamiltonian form, under which the system (2.2) and (2.5) can be transformed into the canonical form (3.8). Thus we have from Lemmas 3.2, 3.3 and 3.4 the following result with the assumption of convexity.

**Theorem 3.5.** Suppose that the function \( f(x) \) satisfies the following hypothesis:

\( (H_1^*) : \quad f(x) \in C^1(\mathbb{R}), f(-x) = -f(x), f(0) = 0, xf(x) > 0 \)

and \( f'(x) > 0 \) for all \( x \neq 0 \);

(i) If \( \lim_{x \to 0} \frac{f(x)}{x} = f'(0) = +\infty, \lim_{x \to -\infty} \frac{f(x)}{x} = 0 \), then for any \( p > 0 \), the 2k-dimensional Hamiltonian system (2.2) ((2.5)) has nontrivial periodic solution \( X_p(t) \) with the minimal period \( p \) and this periodic solution lies on an energy hypersurface \( H(X_p(t)) = h \).

(ii) If \( \lim_{x \to -\infty} \frac{f(x)}{x} = 0 \), then there is \( p_0 > 0 \) such that for every \( p > p_0 \), the system (2.2) ((2.5)) has a nontrivial periodic solution \( X_p(t) \) with minimal period \( p \) and
there exists \( h > 0 \) such that \( H(X_p(t)) = h(H^*(X_p(t)) = h) \).
In both of cases (i) and (ii), \( \min_{t \in \mathbb{R}} |X_p(t)| \to +\infty \) as \( p \to +\infty \).

Proof. Obviously, the condition \((H_1^*)\) implies that both \( H \) and \( H^* \) defined by (2.1)
and (2.4) are nonnegative, convex and satisfying \( H(0) = H^*(0) = 0, H_z(0) = H_z^*(0) = 0 \). In addition,
\[
\lim_{x \to -\infty} \int_0^x f(s) \, ds = \lim_{x \to -\infty} F(x) = \infty
\]
and hence
\[
\lim_{z \to -\infty} H(z) = \lim_{z \to -\infty} H^*(z) = \infty.
\]
It follows from the assumptions of \( \lim_{x \to -\infty} f(x) / x = 0 \) and \( \lim_{x \to 0} f(x) / x = \infty \)
that the conditions (3.9), (3.10) and (3.12) in Lemmas 3.3 and 3.4 hold. Thus by
using these Lemmas, we have the conclusion of Theorem 3.5. \( \square \)

In Lemmas 3.3 and 3.4, the Hamiltonian \( H \) has been assumed to be subquadratic
in the sense that \( H \) satisfies the conditions (3.9), (3.10) and (3.12). In the case of \( H \)
is superquadratic, Ambrose and Mancini [1] have shown that if \( H \) is strictly convex
and super-quadratic at both zero and infinity, then for all \( p > 0 \), the system (3.8)
has a nontrivial \( p \)-periodic solution having \( p \) as minimal period. In particular,
in our case, the following result will be used.

Lemma 3.6. (see Ambrose and Mancini [1]) (i) Suppose that \( H(z) \) is strictly convex and homogeneous of degree \( \beta > 2 \), that is, \( H \) satisfies \( H(sz) = s^\beta H(z) \) for
any \( s > 0 \). Then for all \( p > 0 \), the system (3.8) has a periodic solution having \( p \) as minimal period.
(ii) Assume that the Hamiltonian \( H \) of (3.8) has the form \( H(z) = \bar{H}(z) + R(z) \),
where \( \bar{H}(z) \), \( R(z) \) \( \in \mathcal{C}^2 \), \( \bar{H}(z) \) is strictly convex and homogeneous of degree \( \beta > 2 \);
\( R(0) = R'(0) = 0 \) and \( \|R''(z)\| = o(|z|^{\beta-2}) \) as \( z \to 0 \). Then there exists \( p_0 > 0 \) such
that for all \( p > p_0 \), the system (3.8) has a periodic solution having \( p \) as minimal period.
The amplitude of such a solution tends to zero as \( p \to \infty \).

Based on Lemma 3.6, we can derive the following result immediately.

Theorem 3.7. (i) If \( f(x) = \omega x^{2k+1} \) for some integer \( k \geq 1 \) and \( \omega > 0 \), then for all
\( p > 0 \), the system (2.2) ((2.5)) has a periodic solution \( X_p(t) \) having \( p \) as minimal
period.
(ii) If \( f(x) = \omega x^{2k+1} + R(x) \), where \( R(0) = 0, xR(x) > 0 \) for \( x \neq 0 \), \( R'(x) > 0 \) and
\( R(x) = o(x^{\beta-2}) \) (as \( x \to 0 \)) for \( \beta > 2 \), then there exists \( p_0 > 0 \) such that for all
\( p > p_0 \), the system (2.2) ((2.5)) has a periodic solution \( X_p(t) \) having \( p \) as minimal
period and \( \max_{t \in \mathbb{R}} |X_p(t)| \to 0 \) as \( p \to \infty \).

We next consider the situation without assuming the convexity of \( H \).

Lemma 3.8. (Lyapunov center Theorem, see Nemitskii and Stepanov [14] and
Roels [18]) Suppose that the Hamiltonian \( H \) is holomorphic in a neighborhood
of the origin and the expansion of a power series of \( H \) at the origin lacks linear and
constant terms but the coefficients of second degree terms do not all vanish. If the matrix $A$ formed by the linear approximation have $k$ pairs of purely imaginary roots $\pm \gamma_1 i, \cdots, \pm \gamma_k i$ and that no ratio $\gamma_j/\gamma_s (j \neq s)$ is an integer (i.e., nonresonance condition holds), then the system (3.8) admits $k$ distinct one-parameter families of periodic solutions, depending on real parameters $\epsilon_j, j = 1, 2, \cdots, k$. If $\epsilon_j$ tends to zero, then the corresponding orbits of each family tend to the equilibrium (the origin) and the periods of the orbits tend to $2\pi/|\gamma_j| (j = 1, 2, \cdots, k)$.

Different from the Lyapunov theorem, Weinstein [19] and Moser [13] obtained a very remarkable theorem which shows that the nonresonance condition and the assumption of holomorphism of $H$ in Lemma 3.8 are, in fact, not necessary. In our case, we state their result as follows.

**Lemma 3.9.** (Weinstein [19] and Moser [13]) If $H \in C^2$ near $z = 0$ and the Hessian matrix $H_{zz}(0)$ is positive definite, then for sufficiently small $\epsilon$ any energy surface of (3.8)

$$H(z) = H(0) + \epsilon^2$$

(3.13)

contains at least $k$ periodic orbits of (3.8) whose periods are close to those of the linearized system of (3.8).

We continue our discussion on the systems (2.2) and (2.5). Instead of imposing the global conditions $xf(x) > 0$ for all $x \neq 0$ and $f'(x) > 0$ for all $x \in \mathbb{R}$, we only consider the local behavior of $f(x)$ at some neighborhood of the origin. We know from Lemma 3.1 that the linearized systems of (2.2) and (2.5) at the origin $X = 0$ satisfy the nonresonance condition of Lyapunov center theorem, provided $f'(0) = \omega > 0$. By the proof of Lemma 3.2, $H_{xx}(0)$ and $H^*_{xx}(0)$ are positive definite. Hence both of Lemmas 3.8 and 3.9 can be used to derive the following result.

**Theorem 3.10.** Suppose that the condition ($H_1$) holds. If $f'(0) = \omega > 0$ then the system (2.2) ((2.5)) has $k$ distinct families of periodic solutions in a neighborhood of the origin and each family of periodic solutions depends on one parameter $\epsilon_j, j = 1, \cdots, k$. If $\epsilon_j \to 0$, then the corresponding orbits tend to the origin and the periods of the orbits tend to $2\pi/|\gamma_j|$ for $n = 2k$ ( $2\pi/|\gamma_j|$ for $n = 2k + 1$).

**Remark 3.2.** One can see from the proof of Lyapunov center theorem (see Nemitskii and Stepanov [14]) that the periods of $k$ families of periodic solutions have the following expansion:

$$p_j = \frac{2\pi}{|\gamma_j|} \left( 1 + h_1^{(j)} \epsilon_j + h_2^{(j)} \epsilon_j^2 + \cdots \right).$$

(3.14)

Therefore, if for every $j = 1, 2, \cdots, k$, one of $h_i^{(j)}$ is not zero, then $p_j$ is varying.

4. **Periodic solution of differential delay equations.**

This section is devoted to the study on the existence of periodic solutions of differential delay equation (1.5).
Theorem 4.1. Suppose that the conditions \((H_2)\) of Theorem 2.1 and \((H^*_1)\) of Theorem 3.5 hold.

(i) If

\[
\lim_{x \to 0} \frac{f(x)}{x} = f'(0) = +\infty \quad \text{and} \quad \lim_{x \to \infty} \frac{f(x)}{x} = 0 \quad \text{or} \quad f(x) = \omega x^{2k+1}
\]

for some integers \(k \geq 1\), then the system (1.5) has infinitely many periodic solutions with the period \(p = \prod_{i=1}^{N} \frac{2n^i}{(1+2n^i)(1+2n^i)}\), where \(N > 0\) and \(\ell_i \geq 0\) are any integers.

(ii) If

\[
\lim_{x \to -\infty} \frac{f(x)}{x} = 0 \quad \text{or} \quad f(x) = \omega x^{2k+1} + R(x),
\]

where \(R(x)\) satisfies the condition (ii) in Theorem 3.7. Moreover, if \(p_0 < 2n\mu < +\infty\), where \(p_0\) is defined by Lemmas 3.4 and 3.6 (ii), \(\mu\) is defined by (2.7). Then there exists a nontrivial periodic solution of the system (1.5) with period \(p = 2n\mu\).

Proof. (i) It follows from Theorem 3.5 (i) and Theorem 3.7 (i) that for any \(p > 0\), the coupled systems (2.2) and (2.5) of (1.5) have a periodic solution of period \(p\). Specially, there exists a periodic solution of (2.2) and (2.5) with period \(p = 2n\mu\). Then it follows from Theorem 2.1 that (1.5) has a solution \(x(t)\) having the period \(p = 2n\mu\). Note that

\[
x'(t) = -\sum_{i=1}^{n-1} f(x(t - r_i)) = -\sum_{i=1}^{n-1} f \left( x \left( t - \frac{i + 2nm_i}{i + 2nm_i} \right) \right)
\]

\[
= -\sum_{i=1}^{n-1} f(x(t - m_ip - i\mu)) = -\sum_{i=1}^{n-1} f(x(t - i\mu))
\]

\[
= -\sum_{i=1}^{n-1} f(x(t - r_i^{(1)})),
\]

where \(r_i^{(1)} = i\mu\) satisfying

\[
\frac{r_1^{(1)}}{1 + 2n\ell_1} = \frac{r_2^{(1)}}{2(1 + 2n\ell_1)} = \cdots = \frac{r_i^{(1)}}{i(1 + 2n\ell_1)} = \cdots
\]

\[
= \frac{\mu}{1 + 2n\ell_1} = \mu_1,
\]

where \(\ell_1\) is any integer. It follows that the condition \((H_2)\) of Theorem 2.1 holds. To the “new equation” \((4.1)\), one can use the previous method and derive that there exists a periodic solution of period of \((4.1)\) with period \(p_1 = 2n\mu_1\). Repeating the same deduction, we can obtain the periodic solutions of the equation

\[
x'(t) = -\sum_{i=1}^{n-1} f(x(t - r_i^{(j)})),
\]

with period \(p_j = 2n\mu_j\), where \(r_i^{(j)} = i\mu_j\), \(\mu_j = \prod_{i=1}^{j} \frac{\mu_1^{i-1}}{(1 + 2n\ell_j^i)}\). To sum up, the system (1.5) has infinitely many periodic solutions with periods \(p = \...\)
\[
\frac{2\pi}{(1+2nm_i)} \prod_{i=1}^{N} \frac{1}{1+2n\ell_i} \] for all integers \( N > 0 \) and \( \ell_i \geq 0, i = 1, 2, \cdots \).

(ii) The conclusion of Theorem 4.1 (ii) can be derived from Theorems 3.5 (ii), 3.7 (ii) and Theorem 2.1.

**Remark 4.1.** It is easy to see that if we let \( r_1 = 1, r_2 = 2, \cdots r_{n-1} = n - 1 \) and \( m_i = 0 \) in (2.7), then Theorem 4.1 implies immediately that there exists a periodic solution of the equation (1.3) with period \( 2n \) and hence the conjecture of Kaplan-Yorke is proved. Moreover, we have a more complete result: under the conditions of Theorem 4.1 (i), the equation (1.3) has countable infinitely many periodic solutions having periods \( p = \prod_{i=1}^{N} \frac{2n}{1+2n\ell_i} \) for any positive integers \( N \) and \( \ell_i \).

We finally consider the case when the assumption on the convexity of \( H \) does not hold. In this case, one can use Lemmas 3.8, 3.9 and Theorem 2.1 and obtain the following result.

**Theorem 4.2.** Suppose that the conditions \( (H_1) \) and \( (H_2) \) hold. Assume \( f'(0) = \omega > 0 \) and there exists at least one of \( h_i^{(j)} \) in (3.14) which is not zero, and \( p_j \) is monotonously increasing from \( 2\pi/|\gamma_j| \). If

\[
\frac{2\pi}{|\gamma_j|} < 2n\mu < \beta
\] (4.4)

for some \( j \) and \( \beta > 0 \), then the equation (1.5) has a nontrivial periodic solution of period \( p = 2n\mu \), where \( \mu \) is defined by (2.7).

As an example of application, we consider an equation of analog neural networks as follows (see Marcus and Westervelt [10])

\[
x'(t) = -\alpha x(t) + \sum_{j=1}^{n-1} \tanh x(t - r_j).
\] (4.5)

When \( \alpha = 0 \), (4.5) has the form of (1.5). Taking into account \( f(x) = \tanh x \), it is easy to see that \( f(-x) = -f(x), f'(x) = 1/\cosh^2 x > 0, f'(0) = 1, \lim_{x \to \infty} \frac{f(x)}{x} = 0. \) Applying Theorems 4.1 and 4.2, we can conclude that if \( r_j (j = 1, 2, \cdots, n-1) \) satisfy the condition \( (H_2) \) and

\[
2\pi \frac{\sin \frac{\pi}{2k}}{1 + \cos \frac{\pi}{2k}} < 2n\mu < \infty \quad \text{or} \quad 2\pi \frac{\sin \frac{\pi}{2k+1}}{1 + \cos \frac{\pi}{2k+1}} < 2n\mu < \infty
\]

then the equation (4.5) has a periodic solution of period \( 2n\mu \) for both \( n = 2k \) and \( n = 2k + 1 \).

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