FISHER-TYPE PROPERTY OF TRAVELLING WAVES FOR
COMPETITION-DIFFUSION EQUATIONS

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ABSTRACT. We see from Kolmogoroff et al. [9] that the Fisher equation

\[ u_t = u_{xx} + u(1-u), \quad x \in \mathbb{R}, \quad t > 0 \]

has a travelling wave \( u(t,x) = u(\xi), \xi = x - st \) for any \( s < -2 \) which exponentially decays to
0 as \( \xi \to -\infty \) and to 1 as \( \xi \to +\infty \). This suggests the possibility of the existence of infinitely
many travelling waves which connect between two equilibrium points such that one is stable
and the other unstable in the ODE sense. Our purpose in this paper is to show that certain
competition-diffusion equation has a similar property with the Fisher equation. To do it, we
shall employ the comparison principle and the bifurcation theory for heteroclinic orbits.

1. Introduction. One of interesting phenomena in various fields is the appearance of
travelling waves. In order to explain such a phenomenon, we often use reaction-diffusion
equations with the form

\[ u_t = Du_{xx} + f(u), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1} \]

where \( u \) and \( f \) are \( n \)-dimensional vectors, and \( D \) is a diagonal constant \( n \times n \)-matrix.
Travelling waves which we treat in this paper are \( C^2 \)-class functions with the form \( u(t,x) = u(\xi), \xi = x - st \), where \( s \) is the propagation speed. Hence the existence problem of
travelling waves of (1.1) is to find a solution \( u(\xi) \) of the ODE

\[ 0 = Du_{\xi\xi} + s u_\xi + f(u), \quad \xi \in \mathbb{R} \]

for some \( s \in \mathbb{R} \).

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We introduce the Fisher equation
\[ u_t = u_{xx} + u(1 - u), \quad x \in \mathbb{R}, \quad t > 0 \]
as a simple and suggestive example. We see from Kolmogoroff et al. [9] that
\[
\begin{cases}
0 = u_{\xi\xi} + s u_{\xi} + u(1 - u), & \xi \in \mathbb{R}, \\
u(-\infty) = 0, & u(+\infty) = 1
\end{cases}
\]
has a solution for any \( s < -2 \). This suggests the possibility of the existence of infinitely many travelling waves which connect between two equilibrium points such that one is stable and the other unstable in the ODE sense. (We shall call such a property \textit{Fisher-type property}.)

In this paper, we treat the following Lotka-Volterra competition model with diffusion which is one of simple equations in the framework of (1.1) for \( n \geq 2 \):
\[
\begin{cases}
u_t = u_{xx} + f(u, v), \\
v_t = d v_{xx} + g(u, v), \quad x \in \mathbb{R}, \quad t > 0, \\
u(0, x) \geq 0, \quad v(0, x) \geq 0, \quad x \in \mathbb{R},
\end{cases}
\]
(1.2)

where
\[ f(u, v) = u (1 - u - c v), \quad g(u, v) = v (a - bu - v), \]
and \( a, b, c \) and \( d \) are positive constants. When \( u \) and \( v \) are spatially homogeneous, the asymptotic behavior of the solution of (1.2) with the initial conditions \( u(0) > 0 \) and \( v(0) > 0 \) is classified into the following four cases:

(I) If \( a < \min(b, 1/c) \), then \( \lim_{t \to +\infty} (u, v)(t) = (1, 0) \).

(II) If \( b < a < 1/c \), then \( \lim_{t \to +\infty} (u, v)(t) = \left( \frac{1 - a c}{1 - b c}, \frac{a - b}{1 - b c} \right) \).

(III) If \( 1/c < a < b \), then \( (0, a) \) and \( (1, 0) \) are locally stable, and almost solution converges to one of them as \( t \to +\infty \).

(IV) If \( a > \max(b, 1/c) \), then \( \lim_{t \to +\infty} (u, v)(t) = (0, a) \).

We note that \( (1, 0) \) is stable (resp. unstable) and \( (0, a) \) unstable (resp. stable) in the ODE sense for the case (I) (resp. (IV)).

The travelling wave of (1.2) is a \( C^2 \)-class function satisfying the ODE
\[
\begin{cases}
0 = u_{\xi\xi} + s u_{\xi} + f(u, v), \\
0 = d v_{\xi\xi} + s v_{\xi} + g(u, v), \quad \xi \in \mathbb{R},
\end{cases}
\]
(1.3a)

for some \( s \in \mathbb{R} \). We assume the boundary conditions
\[
\begin{align*}
(u, v)(-\infty) &= (0, a), \\
(u, v)(+\infty) &= (1, 0).
\end{align*}
\]
(1.3b)
Okubo et al. [11] showed that (1.3) is reduced to
\[
\begin{align*}
0 &= u_{xx} + s u_x + (1 - c)(1 - u)u, \quad x \in \mathbb{R}, \\
0 &= u(-\infty) = 0, \quad u(+\infty) = 1
\end{align*}
\]
when \(a = 1, b + c = 2\) and \(d = 1\) are satisfied. It is suggested that (1.3) has the Fisher-type property when \((a, b, c)\) satisfies the case (I) or (IV). Our purpose in this paper is to study the parameter region where (1.3) has the Fisher-type property.

There are many interesting studies for the existence of solutions of (1.3). Their results show that (1.3) has a unique positive solution up to the translation for the case (III) (for example, see Gardner [4], Conley and Gardner [2], Mimura and Fife [10] and Kan-on [6]). On the other hand, Tang and Fife [12] considered positive solutions of (1.3a) with the boundary conditions
\[
(u, v)(-\infty) = (0, 0), \quad (u, v)(+\infty) = \left(\frac{1 - ac}{1 - bc}, \frac{a - b}{1 - bc}\right),
\]
and showed that the problem (1.3a), (1.4) has the Fisher-type property for the case (II). By the singular perturbation method, Hosono [5] proved that (1.3) has the Fisher-type property for the case (I) when \(d > 0\) is sufficiently small. In this paper, we shall establish that (1.3) has the Fisher-type property for arbitrary fixed \(d > 0\) when \((a, b, c)\) satisfies the case (I). To do it, we shall employ the comparison principle and the bifurcation theory for heteroclinic orbits.

2. Statement of result. We shall say that \(u(x) = (u, v)(x)\) is monotone if \(u(x)\) is non-decreasing and \(v(x)\) non-increasing. Let \(\mu = (a, b, c)\) and \(\rho = (s, \mu)\), and let \(d > 0\) be an arbitrary fixed constant. We state the following main theorem:

**Theorem 2.1.** There exists a continuous function \(\sigma(\mu)\) defined on
\[
\mathcal{O}_0 = \{ \mu \mid 0 < a < b, a \leq 1/c \}
\]
which satisfies the followings:

(i) There exists a \(C^1\)-class function \(u(\cdot, \rho)\) defined on
\[
\mathcal{O} = \{ \rho \mid \mu \in \text{Int}\mathcal{O}_0, s \leq \sigma(\mu) \}
\]
such that \(u(\xi, \rho)\) is a monotone solution of (1.3) for each \(\rho \in \mathcal{O}\).

(ii) (1.3) has no monotone solution for all \(\mu \in \mathcal{O}_0\) and \(s > \sigma(\mu)\).

(iii) \(\sigma(\mu)\) satisfies \(-2 \leq \sigma(\mu) \leq -2 \sqrt{1 - a} c\) for any \(\mu \in \mathcal{O}_0\).

We shall show the proof of the above theorem in the following sections.
3. Preliminary. The linearized operators of (1.3a) around \((u, v) = (0, a)\) and \((u, v) = (1, 0)\) are represented as

\[
\mathcal{L}_-(u, \rho) = \left( p_u^-(\frac{d}{d\xi}, \rho) u - a b u + p_v^-(\frac{d}{d\xi}, \rho) v \right), \quad \mathcal{L}_+(u, \rho) = \left( p_u^+(\frac{d}{d\xi}, \rho) u - c v \right),
\]

respectively, where \(u = (u, v)\),

\[
p_u^-(\gamma, \rho) = \gamma^2 + s \gamma + 1 - a c, \quad p_v^-(\gamma, \rho) = d \gamma^2 + s \gamma - a, \quad p_u^+(\gamma, \rho) = \gamma^2 + s \gamma - 1, \quad p_v^+(\gamma, \rho) = d \gamma^2 + s \gamma + a - b.
\]

With \(* = u, v\), we denote by \(\lambda_+^\pm(\rho)\) (Re \(\lambda_-^-(\rho) \leq \) Re \(\lambda_+^+(\rho)\)) and \(\sigma_+^\pm(\rho)\) \((\sigma_-^-(\rho) < 0 < \sigma_+^+(\rho)\)) the solutions of \(p_-^*(\gamma, \rho) = 0\) and \(p_+^*(\gamma, \rho) = 0\), respectively, for each \(\rho \in \mathbb{R} \times \mathcal{O}_0\). Clearly we have \(\lambda_-^-(\rho) < 0 < \lambda_+^+(\rho)\) for any \(\rho \in \mathbb{R} \times \mathcal{O}_0\). We put

\[
\Sigma_1^\pm(\rho) = \min\{\sigma_u^\pm(\rho), \sigma_v^\pm(\rho)\}, \quad \Sigma_2^\pm(\rho) = \max\{\sigma_u^\pm(\rho), \sigma_v^\pm(\rho)\},
\]

\[
k_u^0(\rho) = 2 - \#\{\lambda_-^-(\rho), \lambda_+^+(\rho)\}, \quad k_u^-(\rho) = k_u^0(\rho) + 2 - \#\{\lambda_-^-(\rho), \lambda_+^+(\rho)\},
\]

\[
k_u^+(\rho) = 2 - \#\{\lambda_+^-(\rho), \lambda_-^+(\rho)\}, \quad k_v^-(\rho) = 0, \quad \ell_1(\rho) = 0, \quad \ell_2^\pm(\rho) = 2 - \#\{\sigma_u^\pm(\rho), \sigma_v^\pm(\rho)\},
\]

where \(\#A\) is the number of elements of the set \(A\). By \(p_u^+(-s, \rho) < 0\) and \(p_v^+(-s/d, \rho) < 0\) for any \(\rho \in \mathbb{R} \times \mathcal{O}_0\), we obtain

\[
(3.1) \quad \Sigma_1^-(\rho) \leq \sigma_u^-(\rho) < -s, \quad \Sigma_1^-(\rho) \leq \sigma_v^-(\rho) < -s/d
\]

for any \(\rho \in \mathbb{R} \times \mathcal{O}_0\).

Let \(u(\xi) = (u, v)(\xi)\) be an arbitrary monotone solution of (1.3) for \(\rho \in \mathbb{R} \times \mathcal{O}_0\). By the comparison principle, we have \(u(\xi) \in (0, 1) \times (0, a)\) for any \(\xi \in \mathbb{R}\). From the functional form of \((f, g)\), we obtain \(f_u(u(\xi)) < 0\) and \(g_u(u(\xi)) < 0\) for any \(\xi \in \mathbb{R}\). Further we see that \(u_\xi(\xi)\) satisfies \(u_\xi(\xi) \geq 0 \geq v_\xi(\xi)\) for any \(\xi \in \mathbb{R}\) and

\[
\begin{dcases}
0 = u_{\xi\xi\xi} + s u_{\xi\xi} + f_u(u(\xi)) u_\xi(\xi) + f_v(u(\xi)) v_\xi(\xi), \\
0 = d v_{\xi\xi\xi} + s v_{\xi\xi} + g_u(u(\xi)) u_\xi(\xi) + g_v(u(\xi)) v_\xi(\xi), \quad \xi \in \mathbb{R}.
\end{dcases}
\]

We assume \(u_\xi(\xi_1) = 0\) for some \(\xi_1 \in \mathbb{R}\). We have \(u_{\xi\xi}(\xi_1) = 0\) and

\[
0 \leq u_{\xi\xi\xi}(\xi_1) = -f_v(u(\xi_1)) v_\xi(\xi_1) \leq 0.
\]

This shows that \(v_\xi(\xi_1) = 0\) and \(v_{\xi\xi}(\xi_1) = 0\) hold. By uniqueness, we see that \(u(\xi)\) is a constant function. This contradiction implies \(u_\xi(\xi) \geq 0\) for any \(\xi \in \mathbb{R}\). Similarly we can prove \(v_\xi(\xi) < 0\) for any \(\xi \in \mathbb{R}\).
Lemma 3.1a. (Theorem 8.1 in Coddington and Levinson [1]). Suppose that \( \beta(\xi) \) is an arbitrary \( C^1 \)-class function which satisfies \( \beta(-\infty) = 0 \) and \( \int_{-\infty}^{0} |\beta'(\xi)| d\xi < +\infty \). If \( \lambda_u^- (\rho) \neq \lambda_u^+ (\rho) \) holds, then there exists a fundamental set \( \{ \phi^- (\xi), \phi^+ (\xi) \} \) of solutions of \( (p_u^- (\frac{d}{d\xi}, \rho) + \beta(\xi)) u = 0 \) which satisfy \( \phi^\pm (\xi) = e^{\lambda_u^\pm (\rho) \xi} (1 + o(1)) \) as \( \xi \to -\infty \).

Lemma 3.2. Let \( u(\xi) = (u, v)(\xi) \) be an arbitrary monotone solution of (1.3) for \( \rho \in \mathbb{R} \times \mathcal{O}_0 \). Then \( \rho \in \mathcal{O}_1 = \{ \rho | \mu \in \mathcal{O}_0, s \leq -2 \sqrt{1 - a c}, s < 0 \} \) holds. Furthermore if \( \rho \in \mathcal{O}_2 = \{ \rho | \in \mathcal{O}_1 | a = 1/c \} \), then \( \Delta(u, \rho) < +\infty \) holds, where

\[
\Delta(u, \rho) = \limsup_{\xi \to -\infty} \{|u(\xi)|_1 e^{-\lambda_u^\pm (\rho) \xi}\},
\]

and \([u]_j\) is the \( j \)-th element of the vector \( u \).

Proof. We define

\[
\mathcal{O}_3 = \{ \rho | 0 < a = 1/c < b, s > 0 \}, \quad \mathcal{O}_4 = \{ \rho | 0 < a = 1/c < b, s = 0 \},
\]

\[
\mathcal{O}_5 = \{ \rho | \mu \in \text{Int} \mathcal{O}_0, s \geq 0 \}, \quad \mathcal{O}_6 = \{ \rho | \mu \in \text{Int} \mathcal{O}_0, -2 \sqrt{1 - a c} < s < 0 \},
\]

and then have \( \mathbb{R} \times \mathcal{O}_0 = \mathcal{O}_1 \cup (\bigcup_{j=3}^5 \mathcal{O}_j) \).

We consider the case \( \rho \in \mathcal{O}_4 \cup \mathcal{O}_5 \). Putting \( f_1(\xi) = (1 - u + c v)(\xi) \), we have \( f_1(\pm \infty) \geq 0 \). If \( \rho \in \mathcal{O}_4 \) holds and \( f_1(\xi) \) attains non-positive minimum at \( \xi = \xi_2 \in \mathbb{R} \), then we obtain \( f(\xi_2) \leq 0 ,\ g(\xi_2) = 0 \) and \( 0 \leq \frac{df(\xi_2)}{d\xi} + c g(\xi_2) < 0 \). This contradiction implies \( f_1(\xi) > 0 \) for any \( \xi \neq \mathbb{R} \) when \( \rho \in \mathcal{O}_4 \). Since \( f_1(-\infty) = 1 - a c + o(1) > 0 \) as \( \xi \to -\infty \) when \( \rho \in \mathcal{O}_5 \), we see that there exists \( \xi_3 \in \mathbb{R} \) such that \( f(\xi_3) > 0 \) for any \( \xi \leq \xi_3 \). By \( u(\xi) = -s u(\xi) - f(\xi) \) we obtain \( u(\xi) \geq u(\xi_3) > 0 \) for any \( \xi \leq \xi_3 \). This contradicts the fact that \( u(\xi) \) is bounded. Hence we get \( \rho \notin \mathcal{O}_4 \cup \mathcal{O}_5 \).

Next we consider the case \( \rho \in \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_6 \). We can easily check \( \lambda_u^-(\rho) \neq \lambda_u^+(\rho) \), and see that \( f_2(\xi) = a c - u(\xi) - c v(\xi) \) satisfies \( f_2(-\infty) = 0 \) and

\[
\int_{-\infty}^{0} |f_2(\xi)|_1 d\xi \leq \int_{-\infty}^{0} (u(\xi) - c v(\xi)) d\xi \leq 1 + a c.
\]

It follows from Lemma 3.1a that there exists a fundamental set \( \{ \phi^- (\xi), \phi^+ (\xi) \} \) of solutions of \( (p_u^- (\frac{d}{d\xi}, \rho) + f_2(\xi)) u = 0 \) which satisfy \( \phi^\pm (\xi) = e^{\lambda_u^\pm (\rho) \xi} (1 + o(1)) \) as \( \xi \to -\infty \). Since \( u(\xi) \) is a non-trivial solution of \( (p_u^- (\frac{d}{d\xi}, \rho) + f_2(\xi)) u = 0 \), \( u(\xi) \) is represented as \( u(\xi) = C_1 \phi^- (\xi) + C_2 \phi^+ (\xi) \), where \( C_1 \) and \( C_2 \) are some constants and satisfy \( (C_1, C_2) \neq 0 \). We suppose \( \rho \in \mathcal{O}_6 \). Since

\[
\lambda_u^- (\rho) = \lambda_u^+ (\rho) \in \mathbb{C} \setminus \mathbb{R}, \quad \text{Re} \lambda_u^- (\rho) = -s/2 > 0
\]
hold, and since \( u(\xi) \) is a real-valued function, we obtain \( C_2 = \overline{C_1} \neq 0 \) and

\[
0 \leq \liminf_{\xi \to -\infty} \{ u(\xi) e^{-\text{Re} \lambda_+^+(\rho) \xi} \} = \liminf_{\xi \to -\infty} \text{Re} \{ C_1 e^{i \text{Im} \lambda_+^+(\rho) \xi} \} = -|C_1| < 0.
\]

This contradiction implies \( \rho \notin \mathcal{O}_0 \). When \( \rho \in \mathcal{O}_3 \) holds, we have \((C_1, C_2) = 0 \) because of \( u(-\infty) = 0 \) and \( \lambda_u^-(\rho) = -s < 0 = \lambda_u^+(\rho) \). This is a contradiction. Hence we obtain \( \rho \in \mathcal{O}_1 \). When \( \rho \in \mathcal{O}_2 \) holds, we have \( C_1 = 0 < C_2 \) by virtue of \( \lambda_u^-(\rho) = 0 < -s = \lambda_u^+(\rho) \), that is, \( \Delta(u, \rho) < +\infty \). \( \square \)

We can easily check \( 0 < \lambda_u^-(\rho) \leq -s/2 \leq \lambda_u^+(\rho) \) for any \( \rho \in \mathcal{O}_1 \). With \( * = u, v \), by differentiating \( p_u^-(\lambda_u^+(\rho), \rho) = 0 \) and \( p_u^+(\sigma_u^-(\rho), \rho) = 0 \) with respect to \( s \), we obtain

\[
\frac{\partial}{\partial \gamma} \lambda_u^+(\rho) = -\lambda_u^+(\rho) / \frac{\partial}{\partial \gamma} p_u^-(\lambda_u^+(\rho), \rho) < 0,
\]

\[
\frac{\partial}{\partial \gamma} \sigma_u^-(\rho) = -\sigma_u^-(\rho) / \frac{\partial}{\partial \gamma} p_u^+(\sigma_u^-(\rho), \rho) < 0
\]

for any \( \rho \in \text{Int} \mathcal{O}_1 \), that is,

\[
(3.2a) \quad \lambda_u^+(s, \mu) < \lambda_u^+(\tilde{s}, \mu), \quad \sigma_u^-(s, \mu) < \sigma_u^-(\tilde{s}, \mu)
\]

for any \( s > \tilde{s} \) and \( \mu \in \mathcal{O}_0 \). Similarly we have

\[
(3.2b) \quad \lambda_u^-(\tilde{s}, \mu) < \lambda_u^-(s, \mu) \leq \lambda_u^+(s, \mu)
\]

for any \( s > \tilde{s} \) and \( \mu \in \mathcal{O}_0 \).

Let us study precisely the asymptotic behavior of an arbitrary positive solution \( u(\xi) = (u, v)(\xi) \) of (1.3) for \( \rho \in \mathcal{O}_1 \). By

\[
0 = u_{\xi\xi}(\xi) + s u_{\xi}(\xi) + f(u(\xi)) = (p_u^-(\frac{\partial}{\partial \xi}, \rho) + o(1)) u(\xi)
\]

as \( \xi \to -\infty \), we have

\[
u(\xi) = C_3 |\xi|^k_u [\gamma_s(\rho) e^{\lambda_u^-(\rho) \xi} (1 + o(1)) + C_4 e^{\lambda_u^+(\rho) \xi} (1 + o(1))]
\]

as \( \xi \to -\infty \), where \( C_3 \) and \( C_4 \) are suitable constants. By positivity, we obtain either \( C_3 > 0 \) or \( C_3 < 0 < C_4 \). Hence we have \( u(\xi) = C_u^+ |\xi|^{m_u} e^{\gamma_u \xi} (1 + o(1)) \) as \( \xi \to -\infty \), where \( \gamma_u \in \{\lambda_u^-(\rho), \lambda_u^+(\rho)\} \), \( m_u \in \{0, k_u^0(\rho)\} \), and \( C_u^- \) is some positive constant. By

\[
0 = dv_{\xi\xi}(\xi) + s v_{\xi}(\xi) + g(u(\xi))
\]

\[
=(p_v^-(\frac{\partial}{\partial \xi}, \rho) + o(1)) (v(\xi) - a) - ab C_u^- |\xi|^{m_u} e^{\gamma_u \xi} (1 + o(1))
\]
as $\xi \to -\infty$, we have

$$v(\xi) - a = \begin{cases} 
\frac{ab C_u^- |\xi|^m e^{\gamma_u \xi}}{p_v^- (\gamma_u, \rho)} (1 + o(1))(< 0) & \text{if } \gamma_u < \lambda_v^+(\rho), \\
\frac{-ab C_u^- |\xi|^{m_u+1} e^{\gamma_u \xi}}{(m_u + 1) \frac{\partial}{\partial \gamma} p_v^- (\gamma_u, \rho)} (1 + o(1))(< 0) & \text{if } \gamma_u = \lambda_v^+(\rho), \\
\frac{ab C_u^- |\xi|^m e^{\gamma_u \xi}}{p_v^- (\gamma_u, \rho)} (1 + o(1)) & \text{if } \gamma_u > \lambda_v^+(\rho), \\
+C_5 e^{\lambda_v^+(\rho) \xi} (1 + o(1)) & \text{if } \gamma_u > \lambda_v^+(\rho) 
\end{cases}$$

as $\xi \to -\infty$, where $C_5$ is some constant. Since $v(\xi) \in (0, a)$ holds for any $\xi \in \mathbb{R}$, and since $p_v^- (\gamma_u, \rho) > 0$ holds if $\gamma_u > \lambda_v^+(\rho)$, we see that $C_5$ must satisfy $C_5 < 0$. By summarizing the above argument, we get the following asymptotic expansion:

$$(3.3a) \quad \begin{cases} 
u(\xi) = C_u^- |\xi|^m e^{\gamma_u \xi} (1 + o(1)), \\
v(\xi) = a - C_v^- |\xi|^m e^{\gamma_u \xi} (1 + o(1)) 
\end{cases}$$

as $\xi \to -\infty$, where $\gamma_v = \min\{ \gamma_u, \lambda_v^+(\rho) \}$,

$$m_v = \begin{cases} 
m_u & \text{if } \gamma_u < \lambda_v^+(\rho), \\
m_u + 1 & \text{if } \gamma_u = \lambda_v^+(\rho), \\
0 & \text{if } \gamma_u > \lambda_v^+(\rho), 
\end{cases}$$

and $C_v^-$ is a positive constant and satisfies

$$C_v^- = \begin{cases} 
-\frac{ab C_u^-}{p_v^- (\gamma_u, \rho)} & \text{if } \gamma_u < \lambda_v^+(\rho), \\
\frac{-ab C_u^-}{m_v \frac{\partial}{\partial \gamma} p_v^- (\gamma_u, \rho)} & \text{if } \gamma_u = \lambda_v^+(\rho), \\
-C_5 & \text{if } \gamma_u > \lambda_v^+(\rho). 
\end{cases}$$

Analogously we obtain

$$(3.3b) \quad \begin{cases} 
u(\xi) = 1 - C_u^+ |\xi|^{\Sigma_v^- (\rho)} e^{\Sigma_v^- (\rho) \xi} (1 + o(1)), \\
v(\xi) = C_v^+ e^{\Sigma_v^- (\rho) \xi} (1 + o(1)) 
\end{cases}$$

as $\xi \to +\infty$, where $C_u^+$ and $C_v^+$ are some positive constants.
Lemma 3.3. Suppose that \( u(\xi) = (u,v)(\xi) \) is an arbitrary monotone solution of (1.3) for \( \rho \in \mathcal{O}_T = \{ \rho | 0 < a \leq b, c > 0, s < 0 \} \). Then \( \| u \|_{C^0(\mathbb{R}, \mathbb{R}^2)} \leq L(b,c) \) holds, where \( L(b,c) \) is a continuous function in \((b,c)\) which does not depend on \((a,s)\). If \( \rho \in \mathcal{O}_1 \) and \( \Delta(u, \rho) < +\infty \), then \( s > -2 \) holds.

Proof. Putting \( C_6(b,c) = \max_{\alpha \in [0,b]} \| (f, g) \|_{C^0([0,1] \times [0,a], \mathbb{R}^2)} \), we see that \( C_6(b,c) \) is a continuous function in \((b,c)\) which does not depend on \( a \). We easily have \( \| u \|_{C^0(\mathbb{R}, \mathbb{R}^2)} \leq 1 + b \). By

\[
0 = 2 \int_{\xi}^{+\infty} (u_\xi(\tau) + s u(\tau) + f(u(\tau))) u_\xi(\tau) \, d\tau \\
\leq -u_\xi(\xi)^2 + 2 \int_{\xi}^{+\infty} f(u(\tau)) u_\xi(\tau) \, d\tau \
\leq -u_\xi(\xi)^2 + 2 C_6(b,c),
\]

\[
0 = e^{-s \xi} \int_{\xi}^{+\infty} (u_\xi(\tau) + s u(\tau) + f(u(\tau))) e^{s \tau} \, d\tau \\
= -u_\xi(\xi) + e^{-s \xi} \int_{\xi}^{+\infty} f(u(\tau)) e^{s \xi} \, d\tau \
\leq -u_\xi(\xi) + C_6(b,c) / |s|,
\]

for any \( \xi \in \mathbb{R} \), we obtain

\[
\| u_\xi \|_{C^0(\mathbb{R})} \leq \min \{ \sqrt{2 C_6(b,c)}, C_6(b,c) / |s| \},
\]

\[
\| u_\xi \|_{C^0(\mathbb{R})} \leq |s| \| u_\xi \|_{C^0(\mathbb{R})} + \| f(u(\cdot)) \|_{C^0(\mathbb{R})} \leq 2 C_6(b,c).
\]

Similarly we also obtain

\[
\| v_\xi \|_{C^0(\mathbb{R})} \leq \sqrt{2 b C_6(b,c) / d}, \quad \| v_\xi \|_{C^0(\mathbb{R})} \leq 2 C_6(b,c) / d.
\]

By summarizing the above argument, the desired estimate for \( u(\xi) \) is obtained.

By \(-2 < -2 \sqrt{1 - a c} \), we may only consider the case \( s < -2 \sqrt{1 - a c} \). We see from (3.3) and \( \lambda_+^+(\rho) + s / 2 > 0 \) that \( U(\xi) = e^{s \xi / 2} u(\xi) \) is positive and satisfies \( U(\pm \infty) = 0 \). It follows that there exists \( \xi_4 \in \mathbb{R} \) such that \( U(\xi) \) attains positive maximum at \( \xi = \xi_4 \). By

\[
0 \geq U_\xi(\xi_4) = U(\xi_4) \left\{ \left( \frac{s}{2} \right)^2 - 1 + u(\xi_4) + c v(\xi_4) \right\},
\]

we have \( s > -2 \). \( \Box \)

The followings hold by virtue of (3.1-3):
Lemma 3.4. (Lemma 3.5 in [6]). Let $\mu \in \mathcal{O}_0$. Suppose that $u_0(\xi)$ and $u_1(\xi)$ are a monotone solution and an arbitrary positive solution of (1.3) for $s = s_0$ and $s = s_1$, respectively.

(i) If $\Delta(u_0, s_0, \mu) < +\infty$, then $s_0 \geq s_1$ holds.
(ii) If $\Delta(u_0, s_0, \mu) < +\infty$ and $\Delta(u_1, s_1, \mu) < +\infty$, then $s_0 = s_1$ holds and there exists $\tau \in \mathbb{R}$ such that $u_1(\xi) = u_0(\xi + \tau)$ for any $\xi \in \mathbb{R}$.

Lemma 3.5. (Lemma 3.16 in [6]). Let $u(\xi)$ be a monotone non-negative bounded solution of (1.3a) for $\rho \in \mathbb{R} \times \mathbb{R}_+^3$, where $R_+ = (0, +\infty)$. Then $f(u(\pm\infty)) = 0$ and $g(u(\pm\infty)) = 0$ hold.

4. Fundamental solutions. Throughout this section, we assume that (1.3) has a monotone solution $u_0(\xi) = (u_0, v_0)(\xi)$ for $\rho \in \mathcal{O}_1$. We define the linearized operator $\mathcal{L}$ of (1.3a) around $u_0(\xi)$ and its formal adjoint operator $\mathcal{L}^*$ by

$$\mathcal{L}(u, u_0, \rho) = \begin{pmatrix} u_{\xi} + s u_{\xi} + f_u^0(\xi) u + f_v^0(\xi) v \\ d v_{\xi} + s v_{\xi} + g_u^0(\xi) u + g_v^0(\xi) v \end{pmatrix},$$

$$\mathcal{L}^*(u, u_0, \rho) = \begin{pmatrix} u_{\xi} - s u_{\xi} + f_u^0(\xi) u + f_v^0(\xi) v \\ d v_{\xi} - s v_{\xi} + g_u^0(\xi) u + g_v^0(\xi) v \end{pmatrix},$$

respectively, where $f_u^0(\xi) = f_u(u_0(\xi))$, and the other functions $f_v^0$, $g_u^0$ and $g_v^0$ are defined in the same manner with $f_u^0$.

Defining $\varphi(\xi, j, \gamma, \rho)$ and $\psi(\xi, j, \gamma, \rho)$ by

$$\varphi(\xi, j, \gamma, \rho) = (-1)^j \frac{\partial^j}{\partial \gamma^j} \left[ (p^-_\gamma(\gamma, \rho)/(a b), 1) e^{\gamma \xi} \right],$$

$$\psi(\xi, j, \gamma, \rho) = \frac{\partial^j}{\partial \gamma^j} \left[ (1, p^+_\gamma(\gamma, \rho)/c) e^{\gamma \xi} \right],$$

we have

$$\mathcal{L}_-(\varphi(\xi, j, \gamma, \rho), \rho) = (-1)^j \frac{\partial^j}{\partial \gamma^j} \left[ (R^-(\gamma, \rho)/(a b), 0) e^{\gamma \xi} \right],$$

$$\mathcal{L}_+(\psi(\xi, j, \gamma, \rho), \rho) = \frac{\partial^j}{\partial \gamma^j} \left[ (0, R^+(\gamma, \rho)/c) e^{\gamma \xi} \right],$$

where $R^\pm(\gamma, \rho) = p^\pm_\rho(\gamma, \rho) p^\pm_\rho(\gamma, \rho)$. If $\gamma = \gamma_1$ is a zero of $k$-th order for the function $R^-(\gamma, \rho)$ (resp. $R^+(\gamma, \rho)$), then we see that $\varphi(\xi, j, \gamma_1, \rho)$ (resp. $\psi(\xi, j, \gamma_1, \rho)$) is a non-trivial solution of $\mathcal{L}_-(u, \rho) = 0$ (resp. $\mathcal{L}_+(u, \rho) = 0$) for any integer $0 \leq j < k$. We can easily check that $\{ \overline{\varphi}_j(\xi, \rho), \overline{\psi}_j(\xi, \rho) \}$ (resp. $\{ \overline{\varphi}_j(\xi, \rho), \overline{\psi}_j(\xi, \rho) \}$) is a fundamental set of solutions of $\mathcal{L}_-(u, \rho) = 0$ (resp. $\mathcal{L}_+(u, \rho) = 0$), where

$$\overline{\varphi}_j(\xi, \rho) = \varphi(\xi, k_j^\pm(\rho), \lambda_j^\pm(\rho), \rho), \quad \overline{\psi}_j(\xi, \rho) = \psi(\xi, \ell_j^\pm(\rho), \Sigma_j^\pm(\rho), \rho).$$
Since
\[
\begin{pmatrix}
 f_u^0(\xi) \\
g_u^0(\xi)
\end{pmatrix}
= \begin{pmatrix}
 1 - a c & 0 \\
 -a b & -a \\
-1 & -c \\
0 & a - b
\end{pmatrix} + O(\xi^{m_w} e^{\gamma_u \xi}) \quad \text{as } \xi \to -\infty,
\]
holds by virtue of (3.3), we have the following:

**Lemma 3.1b.** (Theorem 8.1 in Coddington and Levinson [1]). There exist fundamental sets \( \{ \Phi_u^\pm(\xi, \rho), \Phi_v^\pm(\xi, \rho) \} \) and \( \{ \Psi_1^\pm(\xi, \rho), \Psi_2^\pm(\xi, \rho) \} \) of solutions of \( \mathcal{L}(u, u_0, \rho) = 0 \) which satisfy

\[
\begin{align*}
\tilde{\Phi}_j^\pm(\xi, \rho) &= \Phi_j^\pm(\xi, \rho) + O(\xi^{k_j^\pm(\rho) - 1} e^{\lambda_j^\pm(\rho) \xi}) \quad \text{as } \xi \to -\infty, \\
\Psi_j^\pm(\xi, \rho) &= \Psi_j^\pm(\xi, \rho) + O(\xi^{\ell_j^\pm(\rho) - 1} e^{\Sigma_j^\pm(\rho) \xi}) \quad \text{as } \xi \to +\infty.
\end{align*}
\]

By
\[
0 = \mathcal{L}(\tilde{\Phi}_u^\pm(\xi, \rho), u_0, \rho) = \mathcal{L}_-(\tilde{\Phi}_u^\pm(\xi, \rho), \rho) + O(|\xi|^{m_u + k_u^\pm(\rho)} e^{(\gamma_u + \lambda_u^\pm(\rho)) \xi})
\]
as \( \xi \to -\infty \), we have
\[
\begin{align*}
\tilde{\Phi}_u^-(\xi, \rho) &= \Phi_u^-(\xi, \rho) + C_7 \Phi_u^+(\xi, \rho) + C_8 \Phi_v^+(\xi, \rho) + o(e^{(\gamma_u + \lambda_u^-(\rho) - \delta) \xi}), \\
\tilde{\Phi}_u^+(\xi, \rho) &= \Phi_u^+(\xi, \rho) + C_9 \Phi_v^+(\xi, \rho) + o(e^{(\gamma_u + \lambda_u^+(\rho) - \delta) \xi})
\end{align*}
\]
as \( \xi \to -\infty \) for any \( \delta > 0 \), where \( C_7, C_8 \) and \( C_9 \) are some constants. Clearly we obtain \( C_8 = 0 \) if \( \lambda_u^-(\rho) > \lambda_u^+(\rho) \), and \( C_9 = 0 \) if \( \lambda_u^-(\rho) > \lambda_u^+(\rho) \). From the definition of \( \Phi_j^\pm(\xi, \rho) \), we have

\[
\lim_{\xi \to -\infty} \left\{ |\xi|^{k_u^\pm(\rho)} e^{-\lambda_u^-(\rho) \xi} [\tilde{\Phi}_u^-(\xi, \rho)]_1 \right\} = \begin{cases}
p_v^-(\lambda_u^-(\rho), \rho)/(ab)(\neq 0) & \text{if } \lambda_u^-(\rho) \neq \lambda_v^+(\rho), \\
-k_u^-(\rho) \frac{\partial}{\partial \xi} p_v^-(\lambda_u^-(\rho), \rho)/(ab)(\neq 0) & \text{if } \lambda_u^-(\rho) = \lambda_v^+(\rho),
\end{cases}
\]

\[
\lim_{\xi \to -\infty} \left\{ |\xi|^{k_u^\pm(\rho)} e^{-\lambda_u^+(\rho) \xi} [\tilde{\Phi}_u^+(\xi, \rho)]_1 \right\} = 0.
\]

By
\[
0 = [\mathcal{L}(\tilde{\Phi}_v^+(\xi, \rho), u_0, \rho)]_1 = (p_v^-(\frac{\partial}{\partial \xi}, \rho) + o(1)) [\tilde{\Phi}_v^+(\xi, \rho)]_1
\]
\[
- c C_u^- |\xi|^{m_w} e^{(\gamma_u + \lambda_u^+(\rho)) \xi} (1 + o(1))
\]
as $\xi \to -\infty$, we have

$$\left[ \Phi^+_v(\xi, \rho) \right]_1 = \left\{ C_{10} \left| \xi \right|^k_+ e^{\lambda^+_u(\rho) \xi} + C_{11} e^{\lambda^+_v(\rho) \xi} \right\} (1 + o(1)) + o(e^{(\gamma_\rho + \lambda^+_u(\rho) - \delta) \xi})$$

as $\xi \to -\infty$ for any $\delta > 0$, where $C_{10}$ and $C_{11}$ are suitable constants. Clearly we obtain $\gamma_\rho \geq \lambda^-_u(\rho)$, and $C_{10} = 0$ if $\lambda^-_u(\rho) \leq \lambda^+_v(\rho)$. Setting

$$C_{12} = \left\{ \begin{array}{ll}
abla \rho C_{10}/p^-_v(\lambda^-_u(\rho), \rho) & \text{if } \lambda^-_u(\rho) > \lambda^+_v(\rho), \\
0 & \text{if } \lambda^-_u(\rho) \leq \lambda^+_v(\rho),
\end{array} \right.$$ 

we have

$$\lim_{\xi \to -\infty} \left\{ \left| \Phi^+_v(\xi, \rho) \right|_1 \left| \xi \right|^{-k_+} e^{-\lambda^+_u(\rho) \xi} \tilde{\Phi}^-_v(\xi, \rho) - C_{12} \tilde{\Phi}^-_u(\xi, \rho) \right\}_1 = 0$$

because of (4.1). We put $\Lambda^-_2(\rho) = \max\{\lambda^+_u(\rho), \lambda^+_v(\rho)\}$, and define

$$\left( \Phi^-_1(\xi, \rho), \Phi^-_2(\xi, \rho) \right) = (\tilde{\Phi}^-_v, \tilde{\Phi}^-_u, \tilde{\Phi}^-_1, \tilde{\Phi}^-_2)(\xi, \rho)$$

$$= \left( \begin{array}{cccc}
I_4 & 0 & 0 & 0 \\
0 & 1 & -C_{12} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{array} \right)$$

We see that $\{\Phi^-_1(\xi, \rho), \Phi^-_2(\xi, \rho)\}$ is a fundamental set of solutions of $L(u, u_0, \rho) = 0$, and that $\Phi^+_2(\xi, \rho) = \left( p^-_v(\lambda^-_u(\rho), \rho)/(ab), 1 \right) e^{\Lambda^-_u(\rho) \xi} + o(e^{\Lambda^-_u(\rho) \xi})$ as $\xi \to -\infty$.

$$\lim_{\xi \to -\infty} \left[ \left| \Phi^-_2(\xi, \rho) \right|_1 \left| \xi \right|^{-k^-} e^{-\lambda^-_u(\rho) \xi} \right] \neq 0, \quad (4.2)$$

$$\lim_{\xi \to -\infty} \left[ \left| \Phi^+_2(\xi, \rho) \right|_1 \left| \xi \right|^{-k^-} e^{-\lambda^-_u(\rho) \xi} \right] = 0 (j = 1, 2)$$

hold. Hence it follows that there exists a non-singular $4 \times 4$-matrix $M = (M_{ij})$ independent of $\xi$ such that $(\Phi^-_1, \Phi^-_2, \Phi^+_1, \Phi^+_2)(\xi, \rho) = (\Psi^-_1, \Psi^-_2, \Psi^+_1, \Psi^+_2)(\xi, \rho) M$ holds for any $\xi \in \mathbb{R}$.

We define the order relations $\succeq_s$ and $\succeq_o$ in the following manner:

$$\left( u_1, v_1 \right) \succeq_s \left( u_2, v_2 \right) \iff u_1 \geq u_2, \quad v_1 \geq v_2,$$

$$\left( u_1, v_1 \right) \succeq_o \left( u_2, v_2 \right) \iff u_1 \geq u_2, \quad v_1 \leq v_2.$$ 

We denote by $\succ_s$ and $\succ_o$ the relations which are defined by replacing $\leq$ with $<$. By

$$0 = [L(\Phi^+_2(\xi, \rho), u_0, \rho)]_1 = (p^-_v(\frac{\partial}{\partial \xi}, \rho) + o(1)) \left[ \Phi^+_2(\xi, \rho) \right]_1$$

$$- c \left| \xi \right|^m e^{(\gamma_\rho + \Lambda^-_u(\rho)) \xi} (1 + o(1))$$
as $\xi \to -\infty$, we have
\[
[\Phi^+_2(\xi, \rho)]_1 e^{-\Lambda^+_{\xi}(\rho)\xi} = \begin{cases} 
\frac{p_u^-(\lambda_u^+(\rho))}{\lambda_u^+(\rho)}(1 + o(1)) > 0 & \text{if } \lambda_u^+(\rho) > \lambda_v^+(\rho), \\
\frac{c C_u^- |\xi|^m e^{\gamma_u^- \xi}}{p_u^-(\gamma_u + \lambda_v^+(\rho), \rho)}(1 + o(1)) > 0 & \text{if } \lambda_v^+(\rho) \leq \lambda_v^+(\rho)
\end{cases}
\]
as $\xi \to -\infty$, that is, $\Phi^+_2(\xi, \rho) \succ_s 0$ near $\xi = -\infty$. Similarly we have
\[
\Psi^\pm_1(\xi, \rho) e^{-\Sigma^\pm_1(\rho)\xi} = \begin{cases} 
\left(1, \frac{p_u^+(\sigma_v^+(\rho), \rho)}{c}\right)(1 + o(1)) & \text{if } \sigma_u^+(\rho) > \sigma_v^+(\rho), \\
\left(1, \frac{b C_v^- e^{\sigma_v^- (\rho)\xi}}{p_v^+(\sigma_u^+(\rho) + \sigma_v^+(\rho), \rho)}(1 + o(1)) & \text{if } \sigma_u^+(\rho) \leq \sigma_v^+(\rho)
\end{cases}
\]
as $\xi \to +\infty$. Since $\sigma_v^-(\rho) < \sigma_u^+(\rho) + \sigma_v^-(\rho) < \sigma_v^+(\rho)$ holds when $\sigma_u^+(\rho) \leq \sigma_v^+(\rho)$, we have
\[
\begin{align*}
p_u^+(\sigma_v^-(\rho), \rho) > 0 & \quad \text{if } \sigma_u^+(\rho) > \sigma_v^+(\rho), \\
p_u^+(\sigma_v^+(\rho), \rho) < 0 & \quad \text{if } \sigma_u^+(\rho) > \sigma_v^+(\rho), \\
p_v^+(\sigma_u^+(\rho) + \sigma_v^-(\rho), \rho) > 0 & \quad \text{if } \sigma_u^-(\rho) \leq \sigma_v^-(\rho), \\
p_v^+(\sigma_u^+(\rho) + \sigma_v^-(\rho), \rho) < 0 & \quad \text{if } \sigma_u^+(\rho) \leq \sigma_v^+(\rho),
\end{align*}
\]
that is, $\Psi_1^-(\xi, \rho) \succ_s 0$ and $\Psi_1^+(\xi, \rho) \succ_o 0$ near $\xi = +\infty$.

**Lemma 4.1.** (Lemma 3.2 in [6]). $\Phi^+_2(\xi, \rho)$ and $\Psi^-_1(\xi, \rho)$ satisfy $\Phi^+_2(\xi, \rho) \succ_s 0$, $\Psi^-_1(\xi, \rho) \succ_s 0$ for any $\xi \in \mathbb{R}$, $\limsup_{\xi \to +\infty} |\Phi^+_2(\xi, \rho)|_2 > 0$ and $\limsup_{\xi \to -\infty} |\Psi^-_1(\xi, \rho)|_1 > 0$.

We assume $M_{44} = 0$. Since $\lim_{\xi \to +\infty} \Phi^+_2(\xi, \rho) = 0$ holds if $M_{34} = 0$, we have $M_{34} \neq 0$. By $|\Psi^+_1(\xi, \rho)|_1 = e^{\Sigma^+_1(\rho)\xi}(1 + o(1))$ as $\xi \to +\infty$, we get
\[
M_{34} [\Phi^+_2(\xi, \rho)]_1 = M_{34}^2 [\Psi^+_1(\xi, \rho)]_1 (1 + o(1)) > 0
\]
as $\xi \to +\infty$. Since $\Phi^+_2(\xi, \rho) \succ_s 0$ for any $\xi \in \mathbb{R}$ and $|\Psi^+_1(\xi, \rho)|_2 < 0$ near $\xi = +\infty$ hold, we have $M_{34} > 0$ and
\[
0 < \limsup_{\xi \to +\infty} \Phi^+_2(\xi, \rho) = M_{34} \limsup_{\xi \to +\infty} |\Psi^+_1(\xi, \rho)|_2 \leq 0.
\]
This contradiction implies that $M_{44} \neq 0$ holds. By Lemma 4.1 and
\[
\sqrt{M}_{21} \Phi^-_2(\xi, \rho) + \sqrt{M}_{31} \Phi^+_1(\xi, \rho) + \sqrt{M}_{41} \Phi^+_2(\xi, \rho) \to 0
\]
as $\xi \to -\infty$, we obtain $\widetilde{M}_{11} \neq 0$, where $\widetilde{M}_{ij}$ is the $(i, j)$-element of $M^{-1}$.

Since $u_{0\xi}(\xi)$ is a bounded solution of $L(u, u_0, \rho) = 0$, $u_{0\xi}(\xi)$ is represented as

$$u_{0\xi}(\xi) = K^-_2 \Phi^-_2(\xi, \rho) + K^-_3 \Phi^-_1(\xi, \rho) + K^+_4 \Phi^+_4(\xi, \rho) = K^+_1 \Psi^+_1(\xi, \rho) + K^+_2 \Psi^+_2(\xi, \rho),$$

where $K^-_2$ and $K^+_3$ are some constants. If $(K^-_2, K^-_3) = 0$ holds, then we have $K^-_4 \neq 0$ and

$$0 = \limsup_{\xi \to +\infty} |u_{0\xi}(\xi)| = |K^-_4| \limsup_{\xi \to +\infty} |\Phi^+_2(\xi, \rho)| > 0$$

because of Lemma 4.1 and the fact that $u_{0\xi}(\xi)$ is non-trivial. This contradiction implies $(K^-_2, K^-_3) \neq 0$. Analogously we can prove $K^+_3 \neq 0$. By (4.2) and

$$K^-_2 \lim_{\xi \to -\infty} \left\{ |\Phi^-_2(\xi, \rho)|_1 |\xi|^{-k^+_u(\rho)} e^{-\lambda^-_u(\rho)\xi} \right\} = \lim_{\xi \to -\infty} \left\{ u_{0\xi}(\xi) |\xi|^{-k^+_u(\rho)} e^{-\lambda^-_u(\rho)\xi} \right\} =$$

$$= \left\{ \begin{array}{ll}
C^-_u \lambda^-_u(\rho) & \text{if } (\gamma_u, m_u) = (\lambda^-_u(\rho), k^0_u(\rho)), \\
0 & \text{otherwise},
\end{array} \right.$$  

we see that $u_0(\xi)$ satisfies $\Delta(u_0, \rho) < +\infty$ if and only if $K^-_2 = 0$ holds.

We define $\{U_j(\xi, \rho)\}_{j=1}^4$ by

$$(U_1, U_2, U_3, U_4)(\xi, \rho) = (\Phi^-_1, \Phi^-_2, \Phi^+_1, \Phi^+_2)(\xi, \rho)$$

(4.3a)

$$\times \left\{ \begin{array}{ccc}
1 & 0 & 0 \\
\frac{M_{21}}{\widetilde{M}_{11}} & 0 & K^-_2 \\
\frac{M_{31}}{\widetilde{M}_{11}} & 1 & K^-_3 \\
\frac{M_{41}}{\widetilde{M}_{11}} & -\frac{M_{43}}{M_{44}} & K^+_4 \\
1 & 0 & 0 \\
\frac{M_{21}}{\widetilde{M}_{11}} & 1 & 0 \\
\frac{M_{31}}{\widetilde{M}_{11}} & 0 & K^-_3 \\
\frac{M_{41}}{\widetilde{M}_{11}} & -\frac{M_{42}}{M_{44}} & K^+_4 \\
\end{array} \right\}$$

if $\Delta(u_0, \rho) = +\infty$, 

if $\Delta(u_0, \rho) < +\infty$.

We easily see $U_3(\xi, \rho) = u_{0\xi}(\xi)$, and that $\{U_j(\xi, \rho)\}_{j=1}^4$ is a fundamental set of solutions of $L(u, u_0, \rho) = 0$ and satisfies

$$(U_1, U_2, U_3, U_4)(\xi, \rho) = (\Psi^-_1, \Psi^-_2, \Psi^+_1, \Psi^+_2)(\xi, \rho)$$

(4.3b)

$$\times \left( \begin{array}{cccc}
\frac{1}{\widetilde{M}_{11}} & N_1 & K^+_1 & M_{14} \\
0 & N_2 & K^+_2 & M_{24} \\
0 & N_3 & 0 & M_{34} \\
0 & 0 & 0 & M_{44} \\
\end{array} \right).$$
where
\[
N_J = \begin{cases} 
(M_{j3} M_{44} - M_{j4} M_{43})/M_{44} & \text{if } \Delta(u_0, \rho) = +\infty, \\
(M_{j2} M_{44} - M_{j4} M_{42})/M_{44} & \text{if } \Delta(u_0, \rho) < +\infty.
\end{cases}
\]

Since \( \{ \Psi_1^\pm(\xi, \rho), \Psi_2^\pm(\xi, \rho) \} \) is a fundamental set of solutions of \( L(u, u_0, \rho) = 0 \), we have \( N_3 \neq 0 \).

We assume \( \Delta(u_0, \rho) < +\infty \) and \( s < -2 \sqrt{1 - a c} \) up to the end of this section. This assumption means \( \lambda_u^-(\rho) < \lambda_v^+(\rho) \) and \( u_0(\xi) = C_u^- e^{\lambda_u^+(\rho) \xi} (1 + o(1)) \) as \( \xi \to -\infty \). We define \( \Phi_u^\pm(\xi, \rho) \) and \( \Phi_v^\pm(\xi, \rho) \) by \( \Phi_u^\pm(\xi, \rho) = (1, 0) e^{-\lambda_u^+(\rho) \xi} \),

\[
\Phi_v^\pm(\xi, \rho) = \frac{\partial k_v^\pm(\rho)}{\partial k_v^\pm(\rho)} \left[ \left( \frac{p_v^- (\gamma, \rho)}{a b} \right) e^{-\gamma \xi} \right]_{\gamma = \lambda_v^\pm(\rho)}.
\]

where \( k_v^- = 0 \) and \( k_v^+(\rho) = 3 - \# \{ \lambda_u^-(\rho), \lambda_v^+(\rho), \lambda_v^-(\rho) \} \). Similarly to the argument of \( \Phi_j^\pm(\xi, \rho) \), we see that \( \{ \Phi_u^\pm(\xi, \rho), \Phi_v^\pm(\xi, \rho) \} \) is a fundamental set of solutions of \( L^*_-(u, \rho) = 0 \), where \( L^*_-(u, \rho) \) is the formal adjoint operator of \( L_-(u, \rho) \). Since Lemma 3.1b is valid even if \( L \) is replaced with \( L^* \), it follows that there exists a fundamental set \( \{ \Phi_u^\pm(\xi, \rho), \Phi_v^\pm(\xi, \rho) \} \) of solutions of \( L^*_-(u, u_0, \rho) = 0 \) which satisfy

\[
\Phi_j^\pm(\xi, \rho) = \Phi_j^\pm(\xi, \rho) + o(\Phi_j^\pm(\xi, \rho)) \text{ as } \xi \to -\infty.
\]

By

\[
0 = [L^*_-(\Phi_u^-\Phi_v^-(\xi, \rho), u_0, \rho)]_2 = \left( p_v^- \left( -\frac{d}{d\xi}, \rho \right) + o(1) \right) [\Phi_u^-\Phi_v^-](\xi, \rho) \right]_2
\]

\[
-c C_u^- e^{(\lambda_u^+(\rho) - \lambda_v^-(\rho)) \xi} (1 + o(1))
\]

as \( \xi \to -\infty \), we have \( [\Phi_u^-\Phi_v^-](\xi, \rho) ]_2 = C_{13} e^{\lambda_u^+(\rho) \xi} (1 + o(1)) + o(\epsilon^2 \xi) \) as \( \xi \to -\infty \), where \( 0 < \gamma < \min\{ -\lambda_u^-(\rho), \lambda_v^+(\rho) - \lambda_v^-(\rho) \} \), and \( C_{13} \) is some constant and satisfies \( C_{13} = 0 \) if \( \lambda_u^- \leq \lambda_v^- \). Since

\[
\Phi_v^+(\xi, \rho) = (1, p_u^- \lambda_v^+(\rho, \rho))/(a b) e^{-\lambda_v^+(\rho) \xi} + o(e^{-\lambda_v^+(\rho) \xi}) \text{ as } \xi \to -\infty
\]

holds if \( \lambda_u^- > \lambda_v^+ \), we see that

\[
\Phi^\pm(\xi, \rho) = \begin{cases} 
\Phi_u^\pm(\xi, \rho) - \frac{a b C_{13}}{p_u^-(\lambda_v^+(\rho, \rho))} \Phi_v^\pm(\xi, \rho) & \text{if } \lambda_u^- > \lambda_v^+, \\
\Phi_u^\pm(\xi, \rho) & \text{if } \lambda_u^- \leq \lambda_v^+.
\end{cases}
\]

satisfies

\[
[\Phi^\pm(\xi, \rho)]_1 = e^{-\lambda_u^-(\rho) \xi} (1 + o(1)), \quad [\Phi^\pm(\xi, \rho)]_2 = o(\epsilon \xi) \text{ as } \xi \to -\infty.
\]

(4.4)
5. Continuation. Let $u_0(\xi) = (u_0, v_0)(\xi)$ be a monotone solution of (1.3) for some $\rho = \rho_0 \in \mathcal{O}_1$. Let $\rho_0 = (s_0, \mu_0)$ and $\mu_0 = (a_0, b_0, c_0)$. Since (1.3) does not depend on $\xi$ explicitly, we may assume $u(0) = u_0(0)$ without loss of generality. Putting $z = t(u, u_\xi, v, v_\xi)$ and

$$f(z, \rho) = \begin{pmatrix} u_\xi \\ -s \ u_\xi - f(u, v) \\ v_\xi \\ -(s \ v_\xi + g(u, v))/d \end{pmatrix},$$

we can rewrite (1.3) as

$$\left\{ \begin{array}{l}
\frac{d}{d\xi} z = f(z, \rho), \quad \xi \in \mathbb{R}, \\
z(-\infty) = t(0, 0, a, 0), \\
z(+\infty) = t(1, 0, 0), \\
z(0) = z_0(0) + t(0, \alpha_1, \alpha_2, \alpha_3),
\end{array} \right.$$  

(5.1)

where $z_0(\xi) = t(u_0, u_0, v_0, v_0)(\xi)$, and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ will be determined later. Further, by the change of variables

$$z = \left\{ \begin{array}{ll}
z_0(\xi) + t(0, 0, a - a_0, 0) + y & \text{for } \xi < 0, \\
z_0(\xi) + y & \text{for } \xi > 0,
\end{array} \right.$$  

(5.1) becomes

$$\left\{ \begin{array}{l}
\frac{d}{d\xi} y = A(\xi) y + N(\xi, y, \rho), \quad \xi \in \mathbb{R} \setminus \{0\}, \\
y(-\infty) = y(+\infty), \\
y(-0) = t(0, \alpha_1, \alpha_2 - a + a_0, \alpha_3), \\
y(+0) = t(0, \alpha_1, \alpha_2, \alpha_3),
\end{array} \right.$$  

(5.2)

where $y(-0) = \lim_{\xi \to 0^-} y(\xi), y(+0) = \lim_{\xi \to 0^+} y(\xi),$

$$A(\xi) = f_z(z_0(\xi), \rho_0), \quad N(\xi, y, \rho) = f(z, \rho) - f(z_0(\xi), \rho_0) - A(\xi) y.$$  

We define the matrix $X(\xi)$ by

$$X(\xi) = \begin{pmatrix} U_2 & U_3 & U_4 & U_1 \\
U_2 \xi & U_3 \xi & U_4 \xi & U_1 \xi \\
V_2 & V_3 & V_4 & V_1 \\
V_2 \xi & V_3 \xi & V_4 \xi & V_1 \xi \end{pmatrix}(\xi, \rho_0),$$

where $\{(U_j, V_j)(\xi, \rho_0)\}_{j=1}^4$ is a fundamental set of solutions of $L(u, u_0, \rho_0) = 0$ constructed in the last section. Clearly it follows that $X(\xi)$ is a fundamental matrix of $\frac{d}{d\xi} y = A(\xi) y$. By (4.3), the following holds:
Lemma 5.1. (Coppel [3, p. 11]). \( \frac{d}{d\xi} y = A(\xi) y \) has an exponential dichotomy on \( R_- = (-\infty, 0) \) (resp. \( R_+ \)) with \( P_- = \text{diag}(0,0,0,1) \) (resp. \( P_+ = \text{diag}(0,1,0,1) \)), that is, there exist positive constants \( C_{14} \) and \( \gamma_3 \) such that for any \( \xi, \eta \in R_- \), \( X(\xi) \) satisfies

\[
|X(\xi)P_{\pm}X(\eta)^{-1}| \leq C_{14} e^{-\gamma_3 (\xi-\eta)} \quad \text{if} \ \xi \geq \eta,
\]

\[
|X(\xi)(I - P_{\pm})X(\eta)^{-1}| \leq C_{14} e^{-\gamma_3 (\eta-\xi)} \quad \text{if} \ \eta \geq \xi.
\]

We know the following (for example, see Lemma 3.2 in Kokubu [8]):

Lemma 5.2. \( y(\xi, \alpha, \rho) \) is a bounded solution of (5.2) if and only if \( y(\xi, \alpha, \rho) \) satisfies \( P_- Y_- (\alpha, \rho) = 0 \) and \( (I - P_+) Y_+ (\alpha, \rho) = 0 \), where

\[
Y^{\pm}(\alpha, \rho) = X(0)^{-1} y(\pm 0, \alpha, \rho) \pm \int_{R_\pm} X(\xi)^{-1} N(\xi, y(\xi, \alpha, \rho), \rho) d\xi.
\]

Let \( x_j(\xi) \) be the \( j \)-th column vector of \( X(\xi) \), and let \( x^*_j(\xi) \) and \( x^*_{ij}(\xi) \) be the \( j \)-th row vector and the \((i, j)\)-element, respectively, of \( X(\xi)^{-1} \). We define the map \( E : \mathbb{R}^7 \to \mathbb{R}^3 \) by \( E(\alpha, \rho) = t([Y_+^{\pm}(\alpha, \rho)]_1, [Y_+^{\pm}(\alpha, \rho)]_3, [Y_-^{\pm}(\alpha, \rho)]_4) \). From \( Y_+^{\pm}(0, \rho_0) = X(0)^{-1} Q \) \( (Q = \text{diag}(0,1,1,1)) \) and \( u_0(0) > 0 \), we obtain \( E(0, \rho_0) = 0 \) and

\[
\det E_\alpha(0, \rho_0) = \det \begin{pmatrix}
    x_{12}^*(0) & x_{13}^*(0) & x_{14}^*(0) \\
    x_{32}^*(0) & x_{33}^*(0) & x_{34}^*(0) \\
    x_{42}^*(0) & x_{43}^*(0) & x_{44}^*(0)
\end{pmatrix} = -\frac{u_0(0)}{\det X(0)} \neq 0
\]

By the implicit function theorem and the comparison principle, we have the following lemma:

Lemma 5.3. There exists a \( C^1 \)-class function \( u(., \rho) \) defined on a neighborhood of \( \rho = \rho_0 \) such that \( u(\xi, \rho) \) is a monotone solution of (1.3) for each \( \rho \) and satisfies \( u(., \rho) \to u_0 \) as \( \rho \to \rho_0 \).

We assume \( \Delta(u_0, \rho_0) < +\infty \) and \( s_0 < -2 \sqrt{1 - a_0 c_0} \) up to the end of this section. It follows from \( 0 < \lambda_u^-(\rho_0) < \lambda_u^+(\rho_0) \) that there exists \( \delta_1 > 0 \) such that

\[
0 < \lambda_u^-(\rho) < \frac{\lambda_u^-(\rho_0) + \lambda_u^+(\rho_0)}{2} < \lambda_u^+(\rho)
\]

for any \( |\rho - \rho_0| < \delta_1 \). Let \( y(\xi, \alpha, \rho) = (y_j(\xi, \alpha, \rho)) \) be a solution of (5.2) for \((\alpha, \rho)\). Putting

\[
u(\xi, \alpha, \rho) (\equiv (u,v)(\xi, \alpha, \rho)) = u_0(\xi) + (y_1, y_3)(\xi, \alpha, \rho) + \begin{cases}
    (0, 0) & \text{for} \ \xi > 0, \\
    (0, a - a_0) & \text{for} \ \xi < 0,
\end{cases}
\]
we see from (3.3) that \( u(\xi, \alpha, \rho) \) is a solution of (1.3) for \( \rho \) and satisfies

\[
\begin{align*}
\begin{cases}
    u(\xi, \alpha, \rho) = C_{15}(\alpha, \rho) e^{\gamma_4(\alpha, \rho) \xi} (1 + o(1)), \\
v(\xi, \alpha, \rho) = a - C_{16}(\alpha, \rho) |\xi|^{m_1(\alpha, \rho)} e^{\gamma_5(\alpha, \rho) \xi} (1 + o(1))
\end{cases}
\end{align*}
\]

as \( \xi \to -\infty \), where \( \gamma_4(\alpha, \rho) \in \{ \lambda_u^-(\rho), \lambda_u^+(\rho) \} \),

\[
\gamma_5 = \min\{ \gamma_4(\alpha, \rho), \lambda_v^+(\rho) \}, \quad m_1(\alpha, \rho) = 2 - \# \{ \gamma_4(\alpha, \rho), \lambda_v^+(\rho) \},
\]

and \( C_{15}(\alpha, \rho) \) and \( C_{16}(\alpha, \rho) \) are positive. Since \( \gamma_4(0, \rho_0) = \lambda_v^+(\rho_0) \) holds by virtue of \( \Delta(u_0, \rho_0) < +\infty \), we have

\[
\begin{align*}
y_1(\xi, \alpha, \rho) &= C_{15}(\alpha, \rho) e^{\gamma_4(\alpha, \rho) \xi} (1 + o(1)) - C_{15}(0, \rho_0) e^{\gamma_4(0, \rho_0) \xi} (1 + o(1)), \\
y_3(\xi, \alpha, \rho) &= -C_{16}(\alpha, \rho) |\xi|^{m_1(\alpha, \rho)} e^{\gamma_5(\alpha, \rho) \xi} (1 + o(1)) + C_{16}(0, \rho_0) |\xi|^{m_1(0, \rho_0)} e^{\gamma_5(0, \rho_0) \xi} (1 + o(1))
\end{align*}
\]

as \( \xi \to -\infty \). Let \( (\tilde{u}^*, \tilde{v}^*)(\xi, \alpha, \rho) \) be a non-trivial solution of \( \mathcal{L}^*(u, u(\cdot, \alpha, \rho), \rho) = 0 \) which satisfies (4.4). We set \( \tilde{x}^*(\xi, \alpha, \rho) = (s \tilde{u}^* - \tilde{u}_\xi^*, \tilde{u}_\xi^*, \tilde{v}_\xi^* - d\tilde{v}_\xi^*, d\tilde{v}_\xi^*)(\xi, \alpha, \rho) \). From (4.4) and \( \lambda_u^-(\rho) + \lambda_u^+(\rho) = -s \), we have

\[
\tilde{x}^*(\xi, \alpha, \rho) y(\xi, \alpha, \rho) (\equiv h(\xi, \alpha, \rho))
\]

\[
= C_{15}(\alpha, \rho) (s + \lambda_u^-(\rho) + \gamma_4(\alpha, \rho)) e^{(\gamma_4(\alpha, \rho) - \lambda_u^-(\rho)) \xi} (1 + o(1)) + o(1)
\]

\[
= \begin{cases} 
\begin{array}{ll}
C_{15}(\alpha, \rho) (\lambda_u^-(\rho) - \lambda_u^+(\rho)) + o(1) < 0 & \text{if } \gamma_4(\alpha, \rho) = \lambda_u^-(\rho), \\
\alpha(1) & \text{if } \gamma_4(\alpha, \rho) = \lambda_u^+(\rho)
\end{array}
\end{cases}
\]

as \( \xi \to -\infty \), that is, \( \Delta(u(\cdot, \alpha, \rho), \rho) < +\infty \) holds if and only if

\[
H(\alpha, \rho) = \lim_{\xi \to -\infty} h(\xi, \alpha, \rho) = 0.
\]

Since \( \tilde{x}^*(\xi, 0, \rho_0) \) is a solution of \( \frac{d}{d\xi} \tilde{x} = -A^* \tilde{x}(\xi) \), we have

\[
\tilde{x}^*(\xi, 0, \rho_0) = (C_1^*, C_2^*, C_3^*, C_4^*) X(\xi)^{-1},
\]

where \( C_j^* \) is some constant. By (4.3), (4.4) and \( (U_3, V_3)(\xi) = u_0(\xi) \), we get

\[
C_2^* \tilde{x}^*(\xi, 0, \rho_0) x_2(\xi) = O(e^{(\lambda_u^+(\rho_0) - \lambda_u^-(\rho_0)) \xi}),
\]

\[
C_3^* \tilde{x}^*(\xi, 0, \rho_0) x_3(\xi) = O(e^{(\Lambda_u^+(\rho_0) - \lambda_u^-(\rho_0)) \xi})
\]
as $\xi \to -\infty$. From $\lambda^*_\alpha(\rho_0) < \lambda^*_\beta(\rho_0) \leq \Lambda^*_2(\rho_0)$, we obtain $C^*_2 = 0$ and $C^*_3 = 0$. We calculate $X(\xi)^{-1}$ directly by using (4.3), and then obtain $x^*_4(\xi) = O(e^{-\lambda^*_\alpha(\rho_0)+\delta}\xi)$ as $\xi \to -\infty$ for any $\delta > 0$. By $|\tilde{x}^*_4(\xi, 0, \rho_0)|_2 = e^{-\lambda^*_\alpha(\rho_0)}\xi(1 + o(1))$ as $\xi \to -\infty$, we have

$$1 = \lim_{\xi \to -\infty} \left\{ |\tilde{x}^*_4(\xi, 0, \rho_0)|_2 e^{\lambda^*_\alpha(\rho_0)}\xi \right\} = C^*_4 \lim_{\xi \to -\infty} \left\{ |x^*_4(\xi)|_2 e^{\lambda^*_\alpha(\rho_0)}\xi \right\} = 0$$

if $C^*_4 \neq 0$. This contradiction implies $C^*_4 \neq 0$.

Putting $(u^*, v^*)(\xi) = (x^*_1, x^*_4/d)(\xi)$, we see that $(u^*, v^*)(\xi)$ is a non-trivial solution of $\mathcal{L}^*(u, u_0, \rho_0) = 0$. By $x^*_1(\xi) = (\tilde{x}^*_4(\xi, 0, \rho_0) - C^*_4 x^*_4(\xi))/C^*_1$, we have $u^*(\xi) = O(e^{-\lambda^-_\alpha(\rho_0)}\xi)$ and $v^*(\xi) = o(1)$ as $\xi \to -\infty$, that is, the functions $(u_0 \xi u^*)(\xi)$ and $(v_0 \xi v^*)(\xi)$ exponentially converge to 0 as $\xi \to -\infty$. By slightly changing the proof of Theorem A.2 in [7], we have the following:

**Lemma 5.4.** $u^*(\xi) v^*(\xi) < 0$ holds for any $\xi \in \mathbb{R}$.

By the variation of constants, we have $y(\xi, \alpha, \rho) = X(\xi) Y(\xi, \alpha, \rho)$ for any $\xi \in \mathbb{R}$, where

$$Y(\xi, \alpha, \rho) = X(0)^{-1} y(-0, \alpha, \rho) - \int_0^\xi X(\tau)^{-1} N(\tau, y(\tau, \alpha, \rho), \rho) d\xi.$$ 

Clearly we obtain $Y(-\infty, \alpha, \rho) = Y^-(-\alpha, \rho)$. Since $y(\xi, 0, \rho_0) = 0$ holds for any $\xi \in \mathbb{R}$, we have $h(\xi, 0, \rho_0) = 0$ and $Y(\xi, 0, \rho_0) = 0$ for any $\xi \in \mathbb{R}$, that is, $H(0, \rho_0) = 0$. By $h(\xi, \alpha, \rho) - h(\xi, 0, \rho_0) = \tilde{x}^*_4(\xi, \alpha, \rho) X(\xi)(Y(\xi, \alpha, \rho) - Y(\xi, 0, \rho_0))$, we obtain $H_0(0, \rho_0) = (C^*_1, 0, 0, C^*_4) Y^-_0(0, \rho_0) = (C^*_1, 0, 0, C^*_4) X(0)^{-1} Q$ and $H_* = (C^*_1, 0, 0, C^*_4) Y^-_0(0, \rho_0)$

$$= - \int_{\mathbb{R}} \tilde{x}^*_4(\xi, 0, \rho_0) f_*(z_0(\xi), \rho_0) d\xi = -C^*_4 I^-_1 - C^*_4 I^-_4,$$

where $I^+_j = \int_{\mathbb{R}} x^*_j(\xi) f_*(z_0(\xi), \rho_0) d\xi$. We define $E(\alpha, \rho)$ by

$$E(\alpha, \rho) = \int (H(\alpha, \rho), [Y^+(\alpha, \rho)]_1, [Y^+(\alpha, \rho)]_3, [Y^-(-\alpha, \rho)]_4).$$

We have $E(0, \rho_0) = 0$ and

$$\det \frac{\partial E}{\partial (\alpha, \rho)}(0, \rho_0) = \det \left[ \begin{array}{cccc} C^*_1 & 0 & 0 & C^*_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \left( \begin{array}{cccc} x^*_{12}(0) & x^*_{13}(0) & x^*_{14}(0) & -I^-_1 \\ x^*_{12}(0) & x^*_{13}(0) & x^*_{14}(0) & I^+_1 \\ x^*_{32}(0) & x^*_{33}(0) & x^*_{34}(0) & I^+_3 \\ x^*_{42}(0) & x^*_{43}(0) & x^*_{44}(0) & -I^-_4 \end{array} \right).$$

$$= - \frac{C^*_1 u_0(0)}{\det X(0)} \int_{\mathbb{R}} x^*_1(\xi) f_*(z_0(\xi), \rho_0) d\xi$$

$$= \frac{C^*_1 u_0(0)}{\det X(0)} \int_{\mathbb{R}} (u_0(\xi) u^*(\xi) + v_0(\xi) v^*(\xi)) d\xi \neq 0$$

because of Lemma 5.4 and $u_0(\xi) \approx_0 0$ for any $\xi \in \mathbb{R}$. We have the following lemma by virtue of the implicit function theorem and the comparison principle:
Lemma 5.5. If $\Delta(u_0, \rho_0) < +\infty$ and $s_0 < \sqrt{1 - a_0 c_0}$ hold, then there exist $C^1$-class functions $\hat{u}(\cdot, \mu)$ and $\hat{s}(\mu)$ defined on a neighborhood of $\mu = \mu_0$ which satisfy the followings:

(i) $\hat{u}(\xi, \mu)$ is a monotone solution of (1.3) with $s = \hat{s}(\mu)$ for each $\mu$ and satisfies $\Delta(\hat{u}(\cdot, \mu), \hat{s}(\mu), \mu) < +\infty$.

(ii) $(\hat{u}(\cdot, \mu), \hat{s}(\mu)) \to (u_0, s_0)$ as $\mu \to \mu_0$ holds.

Lemma 5.6. Suppose that there exist $\{ (s_n, \mu_n) \}_{n=1}^{\infty} \subset O_7$ and $\{ u_n \}_{n=1}^{\infty} \subset C^2(\mathbb{R}, \mathbb{R}^2)$ such that $u_n(\xi)$ is a monotone solution of (1.3) with $\rho = (s_n, \mu_n)$ for each $n$. If the limit $\lim_{n \to \infty} (s_n, \mu_n)(\equiv (s_0, \mu_0))$ exists in $R_+ \times O_0$, then (1.3) has a monotone solution for $\rho = (s_0, \mu_0)$.

Proof. Let $\mu_n = (a_n, b_n, c_n)$ and $\mu_0 = (a_0, b_0, c_0)$. Without loss of generality, we may assume that $a_n > a_0/2$ and $|\mu_n - \mu_0| < |\mu_0|/2$ hold for any $n \in \mathbb{N}$. For each $n$, by virtue of $|u_n(-\infty)|_2 = a_n$, we can take $\tau_n$ as $|u_n(\tau_n)|_2 = a_0/2$, and put $\tilde{u}_n(\xi) = u_n(\xi + \tau_n)$. From Lemma 3.3, we have $\|\tilde{u}_n\|_{C^2(\mathbb{R})} \leq \max_{|\mu - \mu_0| \leq |\mu_0|/2} L(b, c)$ for any $n \in \mathbb{N}$. It follows from Ascoli-Arzela Theorem that for any fixed $\xi_5 > 0$, there exist $\{ \tilde{u}_{n_j}(\xi) \}_{j=1}^{\infty}$ and $\tilde{u}_0(\xi)$ such that $\tilde{u}_0(\xi)$ is a monotone solution of (1.3a) for $\rho = (s_0, \mu_0)$ and satisfies $\lim_{j \to \infty} \|\tilde{u}_{n_j} - \tilde{u}_0\|_{C^2([-\xi_5, \xi_5], \mathbb{R}_+)} = 0$ and $\tilde{u}_0(\xi) \in [0, 1] \times [0, a_0]$ for any $\xi \in \mathbb{R}$. By the normalization of $\tilde{u}_n$ and the comparison principle, we obtain $|\tilde{u}_0(0)|_2 = a_0/2$ and $\tilde{u}_0(\xi) \in [0, 1] \times (0, a_0)$ for any $\xi \in \mathbb{R}$. Since the equilibrium points of (1.2) with $\mu = \mu_0$ are $(0, 0)$, $(0, a_0)$ and $(1, 0)$, we have $u_0(-\infty) = (0, a_0)$ because of Lemma 3.5.

We assume that $\tilde{u}_0(+\infty) = (0, 0)$ holds. Since $\tilde{u}_0(\xi)$ is monotone, we obtain $\tilde{u}_0(\xi) = 0$ for any $\xi \in \mathbb{R}$. Since $\tilde{v}_0(\xi)$ satisfies

$$
\begin{cases}
0 = dv_{\xi \xi} + s_0 v_{\xi} + v(a_0 - v), & \xi \in \mathbb{R}, \\
v(-\infty) = a_0, & v(+\infty) = 0,
\end{cases}
$$

we have

$$
0 = \int_{\mathbb{R}} \{ d \tilde{v}_{0 \xi \xi}(\xi) + s_0 \tilde{v}_0 + \tilde{v}_0(\xi) (a_0 - \tilde{v}_0(\xi)) \} \tilde{v}_0(\xi) d\xi = s_0 \int_{\mathbb{R}} \tilde{v}_0(\xi)^2 d\xi - \frac{a_0^3}{6} < 0.
$$

This contradiction implies $\tilde{u}_0(+\infty) = (1, 0)$, that is, $\tilde{u}_0(\xi)$ is a monotone solution of (1.3) for $\rho = (s_0, \mu_0)$. \qed

6. Proof of Theorem 2.1.

Lemma 6.1. (Theorem 2.1 in [6]). There exist $C^1$-class functions $\hat{u}(\cdot, \mu)$ and $\hat{s}(\mu)$ defined on $\mathcal{P} = \{ (a, b, c) | 0 < 1/c < a < b \}$ such that $\hat{u}(\cdot, \mu)$ is a monotone solution of (1.3) with $s = \hat{s}(\mu)$ for each $\mu$, and $\hat{s}(\mu)$ satisfies $\hat{s}(\mu) \geq -2$ and $\hat{s}_a(\mu) > 0$ for any $\mu$. Furthermore
there exists a $C^1$-class function $\hat{a}(b, c) \in (1/c, b)$ defined on $\mathcal{P}_0 = \{(b, c) | 0 < 1/c < b\}$ such that $\hat{s}(\hat{a}(b, c), b, c) = 0$ for each $(b, c) \in \mathcal{P}_0$.

**Proof of Theorem 2.1.** Let $(b, c) \in \mathcal{P}_0$. We put

$$\mu_n = \left( \frac{b + n}{c(n + 1)}, b, c \right) \in \mathcal{O}_\gamma, \quad u_n(\xi) = \hat{u}(\xi, \mu_n), \quad s_n = \hat{s}(\mu_n)$$

for each $n \in \mathbb{N}$, where $\hat{u}(\xi, \mu)$ and $\hat{s}(\mu)$ are functions given in Lemma 6.1. Clearly we have

$$\lim_{n \to \infty} \mu_n = (1/c, b, c) \in \mathcal{O}_0, \quad \lim_{n \to \infty} s_n (\equiv s_0) \in [-2, 0).$$

We see from Lemmas 3.2 and 5.6 that (1.3) with $(a, s) = (1/c, s_0)$ has a monotone solution which satisfies $\Delta(u, \rho) < +\infty$.

We denote by $\mathcal{E}_0$ the maximal connected set which contains $\{(1/c, b, c) | 0 < 1/c < b\}$ and consists of parameters $\mu \in \mathcal{O}_0$ such that (1.3) with $\Delta(u, \rho) < +\infty$ has a monotone solution for some $s \in -2\sqrt{1-\alpha c}$. From Lemma 3.4, we may regard $\hat{u}(\xi, \mu)$ and $\hat{s}(\mu)$ given in Lemma 5.5 as $C^1$-class functions in $\mu \in \mathcal{E}_0$.

We assume $\lim_{\mu \to \mu_0} \hat{s}(\mu) (\equiv \hat{s}_0) < -2\sqrt{1-a c}$ for some $\mu_0 \in \partial\mathcal{E}_0 \cap \text{Int} \mathcal{O}_0$. By Lemma 3.3, we have $\hat{s}_0 \geq -2$. It follows from Lemmas 5.3 and 5.6 that there exists $\delta_2 > 0$ such that (1.3) has a monotone solutions for any $|\mu - \mu_0| \leq \delta_2$ and $|s - \hat{s}_0| \leq \delta_2$. By Lemma 3.4 and $\Delta(\hat{u}(., \mu), \hat{s}(\mu), \mu) < +\infty$, we have $\hat{s}_0 + \delta_2 \leq \hat{s}(\mu)$ near $\mu = \mu_0$. This contradicts the definition of $\hat{s}_0$. Hence we have $\lim_{\nu \to \mu} \hat{s}(\nu) = -2\sqrt{1-a c}$ for any $\mu \in \partial\mathcal{E}_0 \cap \text{Int} \mathcal{O}_0$. We define $\overline{s}(\mu)$ by

$$\overline{s}(\mu) = \begin{cases} \hat{s}(\mu) & \text{for } \mu \in \mathcal{E}_0, \\ -2\sqrt{1-a c} & \text{for } \mu \in \mathcal{O}_0 \setminus \mathcal{E}_0. \end{cases}$$

Clearly we see that $\overline{s}(\mu)$ is continuous in $\mu \in \mathcal{O}_0$. Let $\mathcal{E}$ be the maximal extended set by Lemma 5.3 which contains $\{(\overline{s}(\mu), \mu) | \mu \in \mathcal{E}_0\}$. We may regard $\overline{u}(., \rho)$ given in Lemma 5.3 as a $C^1$-class function in $\rho \in \mathcal{E}$. If $\rho \in \mathcal{E} \cap \text{Int} \mathcal{O}$, then we have $\rho \in \text{Int} \mathcal{E}$ because of Lemmas 5.3 and 5.6. This means $\mathcal{E} = \mathcal{O}$. Thus the proof is completed. \(\square\)

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