AN APPLICATION OF A POINTWISE
VARIATIONAL PRINCIPLE IN ELASTODYNAMICS

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AN APPLICATION OF A POINTWISE VARIATIONAL PRINCIPLE IN ELASTODYNAMICS

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SUMMARY

A low frequency approximation for the response of the surface of an elastic half-space, due to the vertical oscillation of a rigid body with a flat circular base, is obtained by means of a variational principle which characterizes the displacement field at an arbitrary point on the free surface. A first variational approximation is obtained and checked for accuracy by means of a second variational approximation. This problem has many practical ramifications.
CHAPTER I

INTRODUCTION

We consider a mixed boundary value problem in elastodynamics, which is of interest in connection with seismology and the effect of vibrating machine foundations. This problem can be described physically in terms of a body with a flat circular base oscillating normal to the surface of an elastic half-space. We are interested in the displacement field on the surface of the half-space. Our model will consist of a rigid body with a flat frictionless base oscillating normal to the surface of a linear, homogeneous, isotropic half-space, which is otherwise free from tractions. The oscillations are assumed to have harmonic time dependence $e^{-i\omega t}$.

Our procedure is to construct a pointwise variational principle for the above displacement field, that is, a variational principle which has as its stationary values the components of displacement on the free surface of the half-space. This variational principle, which is similar to that previously constructed in [1], is based on an integral equation formulation of our problem and is discussed in Chapter II. Our results are valid in the "low" range of a certain frequency parameter and for arbitrary points on the free surface of the half-space, except in the immediate vicinity of the edge of the vibrating body.

The variational principle constructed in this thesis is only a
stationary principle, and it is not possible to obtain error estimates using currently available procedures. To motivate the reasons for this difficulty we will compare certain properties associated with Laplace's equation and the Helmholtz equation in a half-space. This will enable us to anticipate the analogous differences to be expected when problems in a half-space associated with the differential equations governing elastostatics and harmonic elastodynamics are considered. For physical problems leading to Laplace's equation one is concerned with real-valued solutions. However, the Helmholtz equation, which arises in connection with harmonic wave phenomena, requires complex-valued solutions in a half-space. This is because of the necessity of imposing the Sommerfeld radiation condition which corresponds physically to the exclusion of waves coming from infinity. Hence an error estimate would necessarily involve error estimates for both the real and imaginary parts of the solution. The standard variational principle for Laplace's equation is Dirichlet's principle. This principle is a minimum principle and is associated with a positive definite metric. Hence there is a corresponding Schwarz inequality which is the basic tool used in obtaining error bounds. The analogous variational principle generating the Helmholtz equation is associated with an indefinite metric, and the corresponding stationary point does not correspond to either a maximum or a minimum of the associated functional. Furthermore, there is no Schwarz inequality, and therefore no bounds are possible. These difficulties are discussed by Dolph [2], [3], [4]. Bramble and Payne [5], based on a knowledge of the smallest eigenvalue of an
operator associated with the governing differential equation, have, by means of a number of ingenious devices, developed a procedure for obtaining bounds in the Dirichlet problem for a class of elliptic second order partial differential equations with an associated metric of indefinite type. However, it should be pointed out that their method is not applicable to exterior problems, for example, half-space problems. A method for obtaining pointwise bounds on the value of the unknown solution of a boundary value problem, or on the derivatives of such a solution, governed by a linear elliptic equation with an associated quadratic form of positive definite type is presented in the work of Synge [6]. The method is motivated by geometry and employs generalized notions of circle, angle and vector projection. As Synge indicates in Part III of [6], for problems associated with an indefinite metric the concepts of generalized angle and vector projection are lost, and the notion of a circle is replaced by that of a hyperbola. Consequently, Synge's technique fails for problems associated with an indefinite metric.

We circumvented the corresponding difficulties in our elasto-dynamics problem by using two variational approximations. In the first such approximation the trial functions used were exact in the limit of zero frequency. To obtain our second variational approximation we took a set of trial functions containing parameters such that the resulting expression reduced to the first variational approximation when these parameters were set equal to zero. The results obtained from the second variational approximation were found to agree extremely well with the first for "low" frequencies, thereby providing a check on the first
variational approximation. In addition the second variational approximation should lead to results valid over a much wider range of frequencies than the first.

The only work located that is related to our problem is contained in articles [7] and [8]. In [7] the authors obtain numerical results for the response of the free surface of an elastic half-space due to the oscillation of a rigid body of the type considered in this thesis, but in various modes of excitation. However, their results are based on a rather restrictive assumption, namely, that the distribution of stress under the oscillating body is the same as that in the corresponding static problem. Our procedure, while bearing some similarity to the method discussed in [7], is a basically different and more accurate approach to the problem under consideration. The authors of [8] also discuss the same problems as those discussed in [7]. Their procedure is based on integral equation formulations of the corresponding boundary value problem. However, only the far-field displacements are obtained.
CHAPTER II

CONSTRUCTION OF A POINTWISE VARIATIONAL PRINCIPLE

We shall construct a pointwise variational principle for the following problem in elastodynamics. Determine the displacement field within a half-space $D$, with boundary $B$, due to a rigid body with a flat base of arbitrary shape oscillating harmonically normal to its surface with zero shear stresses in the area $B_u$ beneath the body and all stresses zero in $B = B - B_u$.

Let the half-space $D = \{x_1, x_2, x_3; x_3 > 0\}$ and take $n = (0,0,-1)$ as the unit normal vector to $B$. Let the components of displacement and stress at the point $P \in D + B$ be given by $u_i(P)$ and $\tau_{ij}(P)$, respectively. Let the time dependent displacement of the body in the $x_3$-direction be $u_3(Q) = de^{-i\omega t}$, $Q \in B_u$. Let $\lambda$ and $\mu$ be the Lamé constants [9]. Then the physical problem described in the preceding paragraph corresponds to the boundary value problem

\[ \mu u_{i,j,j}(P) + (\lambda + \mu)u_{i,j,j}(P) + \rho \omega^2 u_i(P) = 0, \; P \in D, \tag{1a} \]

\[ u_3(Q) = d, \; Q \in B_u, \tag{1b} \]

---

1. Throughout this chapter we employ rectangular Cartesian coordinates.

2. Unless otherwise indicated, here and in the sequel $i, j$ and $\alpha$ each take on the values 1, 2 and 3.

3. The factor $e^{-i\omega t}$ will be omitted in subsequent expressions.
\[ \tau_{13}(Q) = \tau_{23}(Q) = 0 , \quad Q \in B_u , \quad (1c) \]
\[ \tau_{i3}(Q) = 0 , \quad Q \in B_T , \quad (1d) \]

Together with an appropriate radiation condition to assure outgoing waves at infinity. Here repeated subscripts indicate summation over the range 1, 2, 3 and commas denote partial differentiation.

We now proceed to construct a functional which has as its stationary values the three components, \( u_\alpha(P) \), \( P \in D + B_T \), of the displacement field for the problem \((1a,b,c,d)\). The resulting variational principle is a modification of that derived in [1].

In order to construct the required functional, we first develop three integral representations for \( u_\alpha(P) \), \( P \in D + B_T \).

To obtain our first integral representation, we let \( u_1^{\alpha}(P,Q) \) be the components of displacement in the \( x \)-direction \( P \in D + B_T \) due to a unit oscillating force in the \( x^- \)-direction at \( Q \in B_u \), with the boundary \( B \) otherwise free from tractions. In other words the associated traction \( T^\alpha_{\alpha}(P,Q) = 0 \), \( Q \in B, \; P \in D + B - \{Q\} \). According to [1] we have the integral representation

\[ u_\alpha(P) = - \int_{B_u} u_1^{\alpha}(P,Q) \tau_{33}(Q) dA, \quad P \in D + B_T \quad (2) \]

where \( \tau_{33}(Q) \) is the stress normal to \( B_u \) at \( Q \).

We now proceed with the construction of a second integral representation for \( u_\alpha(P) \). Let \( u_1^{\alpha}(Q,P), \; P \in D + B_T, \; Q \in B \), with associated tractions \( T_1^{\alpha}(Q,P) \), be the displacement in the \( x^- \)-direction at \( Q \) due to a unit oscillating point force in the \( x^\alpha \)-direction at \( P \).
with the specifications

$$u^*_\alpha(Q,P) = 0, \; P \in B_T, \; Q \in B_u,$$

(3)

$$T^*_1(Q,P) = T^*_2(Q,P) = 0, \; P \in D + B_T, \; Q \in B_u,$$

and

$$T^*_1(Q,P) = 0, \; P \in D + B_T, \; Q \in B_T - \{Q\}$$

and an appropriate radiation condition at infinity. We note that the functions defined above are not the same as those defined in equation (82) of [1]. Proceeding by analogy with the development in [1]

$$u_\alpha(P) = \int_{B_u} T_1(Q)u^*_3(Q,P) dA_Q + \int_{B_u} T_2(Q)u^*_3(Q,P) dA_Q$$

(4)

$$- \int_{B_u} T_3^*(Q,P)u_3(Q) dA_Q + \int_{B_T} T_1(Q)u^*_3(Q,P) dA_Q$$

$$= - \int_{B_u} T_3^*(Q,P)u_3(Q) dA_Q, \; P \in D + B_T,$$

since $$T_1(Q) = \tau_{1j}(Q)n_j = 0$$ for $$Q \in B_T$$ by (1d). Also, because

$$T^*_3(Q,P) = \tau^*_3(Q,P)n_1 = -\tau^*_3(Q,P),$$

we have

$$u_\alpha(P) = \int_{B_u} \tau^*_3(Q,P) u_3(Q) dA_Q, \; P \in D + B_T.$$

(5)

For future reference we note that

$$u^3_\alpha(P,Q) = \tau^*_3(R,P)u^3_3(R,Q) dA_R, \; P \in D + B_T, \; Q \in B_u.$$

(6)
The derivation of this result is analogous to that for (5).

A third representation for the displacement field \( \alpha(P) \) can be obtained by combining the representations (2) and (6) obtaining

\[
\alpha(P) = - \int \int_{B}^{\alpha_{Q}(P)} \tau_{33}(Q,R) \tau_{33}(R) d\alpha_{R} d\alpha_{Q}, \quad P \in D + B_{T}. \quad (7)
\]

The integral representations (2), (5) and (7) motivate the construction of the stationary functional

\[
\{ \alpha(P) \} = \frac{\int \int_{B}^{\alpha_{Q}(P)} \tau_{33}(Q) d\alpha_{Q} \cdot \int \int_{B}^{\alpha_{Q}(P)} \tau_{33}(Q) d\alpha_{Q}}{\int \int_{B}^{\alpha_{Q}(P)} \tau_{33}(P,Q) \tau_{33}(Q,R) d\alpha_{R} d\alpha_{Q}}, \quad P \in D + B_{T}, \quad (8)
\]

which gives the exact displacement field \( \alpha(P) \), when the admissible (trial) functions \( \tau_{33}(Q) \) and \( \tau_{33}(P,Q) \) are replaced by the exact dynamic stress distributions \( \tau_{33}(Q) \) and \( \tau_{33}(P,Q) \), respectively. The notation \( \{ \alpha(P) \} \) in (8) is used to indicate that the stationary value of the functional is \( \alpha(P) \). We note that the functional (8) is homogeneous with respect to the admissible functions so that multiplicative constants are unessential.

We now proceed to show that the functional (8) is stationary with respect to variations in the vicinity of the exact dynamic stress distributions, \( \tau_{33}(P,Q) \) and \( \tau_{33}(P,Q) \), and that this stationary value is \( \alpha(P) \). For this purpose we will consider the functional
\[ \{u_\alpha(P)\} = - \int_{B_u} \tau_{33}(Q) dA_Q + \int_{B_u} \alpha^{3}(P,Q) \tau_{33}(Q) dA_Q \]

\[ + \int_{B_u} \int_{B_u} \alpha^{3}(Q,P) \tau_{33}(Q) dA_Q \]

and observe that (8) is merely the homogeneous form of (9) and has the same stationary value.

Let

\[ \tau_{33}(Q) = 0(\epsilon) \]  

(10)

and

\[ \alpha^{3}(Q,P) = 0(\epsilon) . \]  

(11)

We therefore require to show that

\[ \{u_\alpha(P)\} - u_\alpha(P) = 0(\epsilon^2), \quad P \in D + B_T . \]  

(12)

We have

\[ \{u_\alpha(P)\} - u_\alpha(P) \]

\[ = \int_{B_u} \tau_{33}(Q) dA_Q + \int_{B_u} \alpha^{3}(P,Q) \tau_{33}(Q) dA_Q \]

\[ + \int_{B_u} \int_{B_u} \alpha^{3}(Q,P) \tau_{33}(Q) dA_Q \]
Using (6) we have
\[
\int_{B_u} u^3(P, Q) \tau_{33}(Q) dA_Q - \int_{B_u} u^3(R, Q) \tau_{33}(R) dA_R dA_Q
\]
\[
= \int_{B_u} \int_{B_u} \tau_{33}(Q, P) u^3(Q, R) \tau_{33}(R) dA_R dA_Q
\]
and (2) implies that
\[
\int_{B_u} \tau_{33}(Q, P) - \tau_{33}(Q, P) u^3(Q, R) \tau_{33}(R) dA_R dA_Q
\]
\[
= - \int_{B_u} \int_{B_u} \tau_{33}(Q, P) - \tau_{33}(Q, P) u^3(Q, R) \tau_{33}(R) dA_R dA_Q.
\]
Combining equations (13), (14) and (15) we have

\[ \{u_\zeta(P)\} - u_\zeta(P) \]

\[ = \int \int_{B_0 B_0} \left[ \tau^{\zeta\zeta}(Q, P) \right. \left. \left( \tau^{\zeta\zeta}(Q, P) \right) U_3(Q, R) \left[ \tau^{\zeta\zeta}(R) - \tau^{\zeta\zeta}(R) \right] \right] dR dQ \]

\[ = 0(e^2) \]

as was required.

We remark that the functional (8) has a saddle point when
\[ \tau^{\zeta\zeta}(Q) = \tau^{\zeta\zeta}(Q) \] and \[ \tau^{\zeta\zeta}(P, Q) = \tau^{\zeta\zeta}(P, Q), \]
rather than a maximum or minimum, so that it is not possible to obtain bounds on approximations obtained by varying the trial functions about their stationary values.

For axisymmetric problems it is convenient to modify the fundamental singularities. For a discussion of this and an explanation of the notation, see Appendix A. With the resulting modifications, we obtain the homogeneous functionals

\[ \{U_{r}(b)\} = \frac{\int_{0}^{a} U_{r}(b, \rho) \tau_{zz}(\rho) \rho \, d\rho \cdot \int_{0}^{a} \sigma_{zz}(\rho, b) U_{z}(\rho) \rho \, d\rho}{\int_{0}^{a} \int_{0}^{a} \sigma_{zz}(\rho, b) U_{z}(\rho, \rho^{'}) \tau_{zz}(\rho, \rho^{'}) \rho \rho^{'} \rho \rho^{'}} \]

and

\[ \{U_{z}(b)\} = \frac{\int_{0}^{a} U_{z}(b, \rho) \tau_{zz}(\rho) \rho \, d\rho \cdot \int_{0}^{a} \sigma_{zz}(\rho, b) U_{z}(\rho) \rho \, d\rho}{\int_{0}^{a} \int_{0}^{a} \sigma_{zz}(\rho, b) U_{z}(\rho, \rho^{'}) \tau_{zz}(\rho, \rho^{'}) \rho \rho^{'} \rho \rho^{'}} \]
which are analogous to (8). Here the superscript $a$ indicates an admissible (trial) function. The homogeneous functionals (17) and (18) will be used to obtain an approximate solution of the boundary value problem (1a,b,c,d) at points $P(b,0,0)$ in the special case where $B_u$ is the circular region $0 \leq \rho \leq a$, $z = 0$. To this end we will choose the trial functions $^a \tau_{zz}(\rho)$, $^a \tau_{zz}(\rho,b)$ and $^a \tau_{zz}(\rho,b)$ so that they have certain known properties of the exact dynamic stress distributions; see Appendix C.
CHAPTER III
CONSTRUCTION OF TRIAL FUNCTIONS
AND VARIATIONAL APPROXIMATIONS

For future reference we introduce the quantities $h$, $k$ and $\gamma$ according to

$$h = \frac{\omega}{c_1}, \quad k = \frac{\omega}{c_2} \quad \text{and} \quad \gamma = \frac{h}{k} < 1,$$

where $c_1$ and $c_2$ are, respectively, the velocity of dilatation and equivoluminal waves and $\omega$ the circular frequency of oscillations, so that $k = 0$ yields the corresponding static problem.

For our first variational approximations we use as our admissible functions the exact static stress distributions corresponding to $\tau_{zz}(\rho)$, $\tau_{zz}^*(\rho, b)$ and $\tau_{zz}^r(\rho, b)$, which will subsequently be labeled $\tau_{zz}^1(\rho)$, $\tau_{zz}^*(\rho, b)$ and $\tau_{zz}^r(\rho, b)$, respectively. These stress distributions have singularities of the correct type at the edge of the disc; see Appendix C.

The trial function $\frac{1}{r_{zz}}(\rho)$ is given in [10] as

$$\frac{1}{\tau_{zz}}(\rho) = \frac{\mu}{\pi} \left( \frac{1}{\tau_{zz}^2} + \frac{1}{\sqrt{a^2 - \rho}} \right), \quad 0 \leq \rho < a.$$  \hspace{1cm} (20)

\footnote{The sign is opposite to that given in [10] due to the choice of the half-space $D$.}
The trial functions \( l_z^{*}z(\rho, b) \) and \( l_z^{*}z(\rho, b) \) can be obtained by solving a pair of Fredholm integral equations. These integral equations are obtained from the integral representations

\[
\hat{U}_z(\rho', \rho) |_{k=0} = \int_0^a \frac{l_z^{*}z(\rho, b)}{l_z^{*}z(\rho, \rho')} \hat{U}_z(\rho, \rho') |_{k=0} \rho d\rho, b > a, \rho' < a, \quad (21)
\]

and

\[
\hat{U}_r(\rho', \rho) |_{k=0} = \int_0^a \frac{l_z^{*}z(\rho, b)}{l_z^{*}z(\rho, \rho')} \hat{U}_r(\rho, \rho') |_{k=0} \rho d\rho, b > a, \rho' < a, \quad (22)
\]

which are adapted from Appendix A, equations (146) and (147), respectively.

From [9] the z-component of the displacement on B, at distance \( r \) from the origin, due to a force of unit strength in the z-direction at the origin is

\[
\frac{1 + 2\mu}{4\mu(\lambda + \mu)} \frac{1}{r}.
\]

Hence the z-component of the displacement on B at distance \( b \) from the origin due to a ring of unit point forces, acting in the z-direction at distance \( \rho' \) from the origin, is

\[
\hat{U}_z(b, \rho') |_{k=0} = \frac{1 + 2\mu}{4\mu(\lambda + \mu)} \int_0^{2\pi} \frac{d\phi}{\sqrt{b^2 + (\rho')^2 - 2b\rho'\cos(\theta - \phi)}} \quad (23)
\]

(See Figure 1). Our integral equation for \( l_z^{*}z(\rho, b) \), obtained from (21) and (23), can now be written
This integral equation for $\tau_{zz}(\rho, b)$ can be solved by a method due to Copson [11]. Using a result in [11], we can write

\begin{equation}
\int_0^a \int_0^{2\pi} \frac{d\varphi}{\sqrt{\rho^2 + (\rho')^2 - 2\rho'\rho\cos(\theta - \varphi)}} \frac{1}{\tau_{zz}(\rho, b)} \rho d\rho d\varphi \quad \text{for} \quad \rho' < a, \quad b > a, \quad (25)
\end{equation}

after a change of the order of integration. The integral equation (24) can now be written

\begin{equation}
\frac{1}{\tau_{zz}(\rho, b)} \rho d\rho \int_0^b \frac{d\varphi}{\sqrt{b^2 + (\rho')^2 - 2b\rho'\cos(\theta - \varphi)}} \quad \text{for} \quad b > a, \quad (26)
\end{equation}

which is equivalent to the system of integral equations.
\[ S(t) = \int_t^\infty \frac{\rho}{\sqrt{\rho^2 - t^2}} \frac{\Gamma_{zz}(\rho, b) \rho d\rho}{T_z(\rho, b)} \]  

(27) and

\[ \frac{1}{4} \int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + (\rho')^2 - 2b\rho' \cos(\theta - \varphi)}} = \int_0^{\rho'} S(t) \frac{dt}{\sqrt{(\rho')^2 - t^2}} \]  

(28)

We first solve equation (28) for \( S(t) \). We can then substitute the result into equation (27) and solve for \( \Gamma_{zz}(\rho, b) \). Equation (28) is solved using \([11]\) obtaining

\[ S(\rho') = \frac{2}{\pi} \frac{d}{d\rho'} \int_0^{\rho'} \frac{t}{\sqrt{\rho'^2 - t^2}} \left[ \frac{1}{4} \int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + t^2 - 2b\cos(\theta - \varphi)}} \right] dt. \]  

(29)

This can be evaluated in closed form. From \([12]\) we have

\[ \int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + t^2 - 2b\cos(\theta - \varphi)}} = \int_0^{\infty} \int_0^{\infty} J_n(b \tau) J_n(t \tau) \cos(n(\theta - \varphi)) \, d\tau \, d\varphi \]  

(30)

Using an addition formula for Bessel functions \([12]\) we have

\[ \int_0^{2\pi} \frac{d\varphi}{\sqrt{b^2 + t^2 - 2b\cos(\theta - \varphi)}} = 2\pi \int_0^\infty J_0(b \tau) J_0(t \tau) \, d\tau \]  

(31)
after term by term integration. Hence

\[
S(p') = \frac{1}{2\pi} \frac{d}{dp} \int_0^{p'} \frac{t}{\sqrt{(p')^2 - t^2}} \left[ \int_0^{2\pi} \frac{d\phi}{\sqrt{b^2 + t^2 - 2bt\cos(\theta - \phi)}} \right] dt (32)
\]

\[
= \frac{d}{dp} \int_0^{p'} \int_0^{\infty} \frac{t}{\sqrt{(p')^2 - t^2}} J_0(\tau) J_0(b\tau) d\tau dt
\]

\[
= \frac{d}{dp} \int_0^{\infty} J_0(b\tau) \left[ \int_0^{p'} \frac{t J_0(\tau) d\tau}{\sqrt{(p')^2 - t^2}} \right] d\tau
\]

\[
= \frac{d}{dp} \int_0^{\infty} \frac{J_0(b\tau) \sin(p'\tau) d\tau}{\tau} = \int_0^{\infty} J_0(b\tau) \cos(p'\tau) d\tau
\]

\[
= \frac{1}{\sqrt{b^2 - (p')^2}}
\]

using results in [12] and [13]. We can now write (27) as

\[
\frac{1}{\sqrt{b^2 - t^2}} = \int_t^a \frac{1}{\sqrt{2 \rho b}} \rho d\rho (33)
\]

From [11] the solution of (33) is

\[
\frac{1}{\tau_{zz}(\rho, b) / \rho d\rho = -\frac{1}{\pi \rho} \frac{d}{d\rho} \int_t^a \frac{t d\tau}{\rho \sqrt{t^2 - \rho^2}} \sqrt{b^2 - t^2}} = \frac{2}{\pi} \sqrt{b^2 - a^2} \frac{\rho}{(b^2 - \rho)} \frac{1}{\sqrt{a^2 - \rho^2}} (34)
\]
Hence

\[ \tau_{zz}^1(\rho, b) = \frac{1}{\sqrt{1 - \left(\frac{\rho}{a}\right)^2 \left[\left(\frac{b}{a}\right)^2 - \left(\frac{\rho}{a}\right)^2\right]}} , \quad 0 < \rho < a, \ b > a \ , \quad (35) \]

except for a nonessential multiplicative constant.

The analogous integral equation for \( \tau_{zz}^1(\rho, b) \), \( \rho < a, \ b > a \), is given in (22). To obtain \( \hat{U}_{r}(b, \rho') \mid_{k=0} \) we use the expression [9]

\[ -\frac{1}{4\pi(\lambda + \mu)} \frac{1}{r} \ , \quad (36) \]

for the radial component of displacement on B, at a distance \( r \) from the origin, due to a force of unit strength in the z-direction at the origin. Referring to Figure 2, we see that \( CE = b - \rho'\cos(\theta - \phi) \) and therefore \( \cos \psi = \frac{b - \rho'\cos(\theta - \phi)}{R} \). Hence the radial component of displacement at \( C(b, \theta, 0) \) due to a unit force in the z-direction at \( A(\rho', \phi, 0) \) is

\[ \hat{U}_{r}(b, \rho') = -\frac{1}{4\pi(\lambda + \mu)} \frac{b - \rho'\cos(\theta - \phi)}{b^2 + (\rho')^2 - 2bp'\cos(\theta - \phi)} \ . \quad (37) \]

Hence the radial component of the displacement on B at distance \( b \) from the origin due to a ring of unit forces in the z-direction on B acting at distance \( \rho' \) from the origin is

\[ \hat{U}_{r}(b, \rho') \mid_{k=0} = -\frac{1}{4\pi(\lambda + \mu)} \int_0^{2\pi} \frac{b - \rho'\cos(\theta - \phi)}{b^2 + (\rho')^2 - 2bp'\cos(\theta - \phi)} \ d\phi \quad (38) \]
Using the results (23) and (38) the integral equation (22) can be written

\[ -\frac{1}{2(\lambda + \mu)b} = \int_0^a \left[ \frac{\lambda}{4\mu(\lambda + \mu)} \int_0^{2\pi} \frac{d\varphi}{\sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\theta - \varphi)}} \right] \] (39)

Since

\[ \int_0^a \left[ \int_0^{2\pi} \frac{d\varphi}{\sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\theta - \varphi)}} \right] \frac{1}{r_{zz}(p,b)} dp \] (40)

using [11] and changing the order of integration, the integral equation (39) is equivalent to

\[ -\frac{\pi}{2} \frac{\mu}{\lambda + 2\mu} \frac{1}{b} = \int_0^{\rho'} \int_t^a \frac{\rho \frac{1}{r_{zz}(\rho,b)} dp dt}{\sqrt{(\rho^2 - t^2)((\rho')^2 - t^2)}} \] (41)
The integral equation (41) is equivalent to the system of integral equations

\[ S(t) = \int_{t}^{a} \frac{a \rho \tau_{zz}(b, \rho) d\rho}{\sqrt{\rho^2 - t^2}}, \quad 0 \leq t \leq a, \quad b > a, \tag{42} \]

and

\[ - \frac{\pi}{2} \frac{1}{b} \frac{\mu}{\lambda + 2\mu} = \int_{0}^{\rho'} \frac{S(t) dt}{\sqrt{(\rho')^2 - t^2}}, \quad 0 \leq \rho' \leq a. \tag{43} \]

Equation (43) has the solution [11]

\[ S(\rho') = \frac{2}{\pi} \frac{d}{d \rho'} \int_{0}^{\rho'} \frac{t}{\sqrt{(\rho')^2 - t^2}} \left[ - \frac{\pi}{2} \frac{1}{b} \frac{\mu}{\lambda + 2\mu} \right] dt = - \frac{1}{b} \frac{\mu}{\lambda + 2\mu}. \tag{44} \]

Using the result (44), equation (42) becomes

\[ - \frac{1}{b} \frac{\mu}{\lambda + 2\mu} = a \rho \tau_{zz}(\rho, b) d\rho. \tag{45} \]

From [11] we have

\[ \tau_{zz}(\rho, b) = - \frac{2}{\pi} \frac{1}{b} \frac{\mu}{\lambda + 2\mu} \frac{1}{\sqrt{a^2 - \rho^2}}, \quad 0 \leq \rho < a, \quad b > a. \tag{46} \]

The trial functions for the second variational approximation were constructed so that they possessed the correct singularities at the edge of the disc and in addition were functions of $\rho^2$; see
Appendix C. A set of functions with these properties is

\[ \frac{2r}{\tau_{zz}}(\rho) = \frac{1}{\tau_{zz}}(\rho)(1 + c_1\rho^2) \]

(47)

\[ \frac{2z}{\tau_{zz}}(\rho) = \frac{1}{\tau_{zz}}(\rho)(1 + c_2\rho^2) \]

(48)

\[ \frac{2^*r}{\tau_{zz}}(\rho, b) = \frac{1}{\tau_{zz}}(\rho, b)(1 + c_3\rho^2) \]

(49)

\[ \frac{2^*z}{\tau_{zz}}(\rho, b) = \frac{1}{\tau_{zz}}(\rho, b)(1 + c_4\rho^2) \]

(50)

where \(c_1, c_2, c_3\) and \(c_4\) are parameters which will eventually be chosen to be functions of the frequency parameter \(k\); \(\frac{2r}{\tau_{zz}}(\rho)\) and \(\frac{2z}{\tau_{zz}}(\rho)\) are the trial functions corresponding to the admissible function \(\frac{a}{\tau_{zz}}(\rho)\) in the second variational approximations for \(U_r(b)\) and \(U_z(b)\), respectively, and \(\frac{2^*r}{\tau_{zz}}(\rho, b)\) and \(\frac{2^*z}{\tau_{zz}}(\rho, b)\) are the trial functions corresponding to the functions \(\frac{a^*r}{\tau_{zz}}(\rho, b)\) and \(\frac{a^*z}{\tau_{zz}}(\rho, b)\), respectively. We note that the trial functions for the first variational approximation are special cases of those constructed above for the second variational approximation.

We now proceed to determine the points \((c_1^S, c_3^S)\) and \((c_2^S, c_4^S)\) which are stationary points for the functionals

\[
\{U_r(b)\} = \frac{\int_0^a \frac{a^z}{\tau_{zz}}(\rho, b)\frac{2r}{\tau_{zz}}(\rho)\rho d\rho \cdot \int_0^a \frac{a^*r}{\tau_{zz}}(\rho, b)\frac{2^*r}{\tau_{zz}}(\rho)\rho d\rho}{\int_0^a \int_0^a \frac{2r}{\tau_{zz}}(\rho, b)\frac{a^z}{\tau_{zz}}(\rho, b')\frac{2r}{\tau_{zz}}(\rho, b')\rho \rho' d\rho d\rho'}
\]

(51)
\[ [U_z(b)] = \frac{\int_0^a \Delta^z_z(p,r)^2\Delta^z_z(p)\rho_1\rho_1 \cdot \int_0^a \Delta^z_z(p,r)U_z(p)\rho_1\rho_1}{\int_0^a \int_0^a \Delta^z_z(p,r)\Delta^z_z(p,r')\rho_1\rho_1\rho_1\rho_1} \] (52)

which are obtained from (17) and (18) respectively. Solving the system of algebraic equations

\[ \frac{\partial}{\partial c_1} [U_z(b)](c_1, c_2, c_3) = 0 \] (53)

\[ \frac{\partial}{\partial c_3} [U_z(b)](c_1, c_2, c_3) = 0 \]

\[ \frac{\partial}{\partial c_2} [U_z(b)](c_2, c_3) = 0 \]

\[ \frac{\partial}{\partial c_4} [U_z(b)](c_2, c_3) = 0 \]

we obtain

\[ c_1^s = \frac{I_{11} I_{13} - I_{19} I_{12}}{I_{11} I_{12} - I_{11} I_{10}} \] (54)

\[ c_2^s = \frac{I_{15} I_{14} - I_{19} I_{16}}{I_{13} I_{16} - I_{13} I_{10}} \] (55)

\[ c_3^s = \frac{I_{13} I_{15} - I_{11} I_{14}}{I_{12} I_{14} - I_{13} I_{16}} \] (56)
and

\[ c_h = \frac{I_{12} - I_{14}}{I_{15} - I_{16}} \]  

(57)

where

\[ I_1 = \int_0^a \frac{l_{rz}(\rho, b)}{r_{zz}(\rho)} u_z(\rho) \rho^3 d\rho \]  

(58)

\[ I_2 = \int_0^a \frac{l_{rz}(\rho, b)}{r_{zz}(\rho)} u_z(\rho) \rho^3 d\rho \]  

(59)

\[ I_3 = \int_0^a \frac{l_{rz}(\rho, b)}{r_{zz}(\rho)} u_z(\rho) \rho^3 d\rho \]  

(60)

\[ I_4 = \int_0^a \frac{l_{rz}(\rho, b)}{r_{zz}(\rho)} u_z(\rho) \rho^3 d\rho \]  

(61)

\[ I_5 = \int_0^a \frac{\delta_z(b, \rho)}{r_{zz}(\rho)} \frac{1}{2} u_z(\rho) \rho^3 d\rho \]  

(62)

\[ I_6 = \int_0^a \frac{\delta_z(b, \rho)}{r_{zz}(\rho)} \frac{1}{2} u_z(\rho) \rho^3 d\rho \]  

(63)

\[ I_7 = \int_0^a \frac{\delta_z(b, \rho)}{r_{zz}(\rho)} \frac{1}{2} u_z(\rho) \rho^3 d\rho \]  

(64)

\[ I_8 = \int_0^a \frac{\delta_z(b, \rho)}{r_{zz}(\rho)} \frac{1}{2} u_z(\rho) \rho^3 d\rho \]  

(65)
The first variational approximations for $U_r(b)$ and $U_z(b)$ are

$$\frac{1}{U_r(b)} = \frac{I_{15}}{I_9}$$  \hspace{1cm} (74)
and

\[ \frac{1}{U_z(b)} = \frac{I_{31}}{I_{11}} \]  \hspace{1cm} (75)

respectively, when the trial functions $1_{zz}(\rho)$, $1_{zz}^*(\rho', b)$ and $1_{zz}^*(\rho', b')$ are substituted into the homogeneous functionals (17) and (18).

The second variational approximations simplify considerably when the expressions (54), (55), (56), and (57) are substituted for $c_1^S$, $c_2^S$, $c_3^S$ and $c_4^S$, respectively. The results are

\[ \frac{2}{\hat{U}_r(b)} = \frac{I_{51}I_{12} - I_{51}I_{10} + I_{61}I_{13} - I_{61}I_{12}}{I_{13}I_{14} - I_{19}I_{10}} \]  \hspace{1cm} (76)

and

\[ \frac{2}{\hat{U}_z(b)} = \frac{I_{71}I_{14} + I_{81}I_{15} - I_{71}I_{16} - I_{11}I_{14}I_{18}}{I_{15}I_{12} - I_{11}I_{16}} \]  \hspace{1cm} (77)
CHAPTER IV

INTEGRALS

We now describe various techniques which were used to reduce the integrals \( I_1 \) to \( I_{16} \) to forms convenient for subsequent numerical evaluation. For purposes of checking, the nonessential multiplicative constants have been retained.

Four of these integrals may be evaluated in closed form and therefore do not require the use of numerical quadrature. They are

\[
I_1 = -\frac{2\gamma}{\pi b}^2, \quad (78)
\]

\[
I_2 = -\frac{ky^2}{3\pi b}^3, \quad (79)
\]

\[
I_3 = \frac{2a}{\pi} \arcsin \left(\frac{a}{b}\right), \quad (80)
\]

and

\[
I_4 = \frac{2da}{\pi} \left[\left(\frac{b}{a}\right)^2 \arcsin \left(\frac{a}{b}\right) - \sqrt{\left(\frac{b}{a}\right)^2 - 1}\right] \quad (81)
\]

where \( \gamma \) is defined in (19).

It is of interest to note that the above expression (78) for \( I_1 \) is exactly the radial displacement \( U_r(b) \) on the plane \( \mathbb{B} \) at \( r = b \) for the problem referred to at the end of Chapter II in the static case. This is indeed to be expected if one compares the integral representation
(143) with the definition of $I_1$ as given by (58). By comparing (132) and (60) a similar statement can be made for $I_3$ given in (80). These observations provide a check on the trial functions $l^*(\rho, b)$ and $l^*_{zz}(\rho, b)$ which are given by (34) and (46), respectively.

We were not able to evaluate the integrals $I_5$ to $I_{16}$ in closed form. Because the fundamental singularities $\Theta_z(b, \rho)$ and $\Theta^z_{\tau}(b, \rho)$ as given by (160) and (165), respectively, are expressed in terms of integrals from zero to infinity and, since the integrands have a variety of singularities, special techniques have been employed to put the integrals in forms convenient for numerical computation.

We let

$$t = \frac{b}{a} > 1,$$

and

$$\xi = k\alpha,$$

where $k$ is defined in (19). We also let

$$f_1(s) = 2s(2s^2 - 1) - 2\sqrt{s^2 - 1} \sqrt{s^2 - y^2} - s^2 \sqrt{s^2 - 1} - \frac{s^3 \sqrt{s^2 - y^2}}{\sqrt{s^2 - y^2}},$$

where $s > 1$ is the location of a Rayleigh pole [14],

$$f_2(x) = \left(x^2 + \frac{1}{2}\right)^2 - x^2 \sqrt{x^2 + y^2} \sqrt{x^2 + 1},$$

$$\hat{s} = k\alpha,$$
and

\[
K(x) = \begin{cases} 
\frac{x\sqrt{2 - x^2}}{(x^2 - \frac{1}{2})^2 + x\sqrt{2 - x^2} \sqrt{1 - x^2}}, & 0 \leq x \leq \gamma \\
\frac{x^3(x^2 - \gamma^2)\sqrt{1 - x^2}}{(x^2 - \frac{1}{2})^2 + x^4(x^2 - \gamma^2)(1 - x^2)}, & \gamma < x \leq 1,
\end{cases}
\tag{87}
\]

with \(\gamma\) as in (19). To avoid possible confusion we note that \(K_0(z)\) is a modified Bessel function. Furthermore, we let

\[
q(\xi) = \frac{1 + \xi^2 + \frac{2}{3}\xi^4 + \frac{4}{15}\xi^6 + (\xi^2 + 2\xi^4 - 1)e^{2i\xi}}{\xi^6}, \tag{88}
\]

\[
q(\xi) = -\frac{i}{2} \left[ \frac{1 - e^{2i\xi}}{\xi} \right] + 2e^{i\xi} \left[ \frac{\sin \frac{\xi}{2} - \cos \frac{\xi}{2}}{\xi^2} \right] - \frac{i}{2} \frac{\xi}{2} q(\xi), \tag{89}
\]

\[
K(x) = \frac{x(x^2 - \frac{1}{2})\sqrt{1 - x^2} \sqrt{x^2 - \gamma^2}}{(x^2 - \frac{1}{2})^2 + x^4(x^2 - \gamma^2)(1 - x^2)}, \gamma \leq x \leq 1, \tag{90}
\]

\[
T(\xi) = \frac{\xi \cos \frac{\xi}{2} + (\xi^2 - 1)\sin \frac{\xi}{2}}{\xi^2}, \tag{91}
\]

\[
T_1(\xi) = \frac{1 - e^{2i\xi}}{\xi}, \tag{92}
\]

\[
V(\xi, u) = \ln[1 - e^{i\xi} J_\nu(ux)], \tag{93}
\]
\[ S(x) = 2e^{\frac{ix}{1}} \left( \frac{2}{x^2} + \frac{2i}{x^3} \right), \]  
\[ S_1(x) = \frac{x \cosh \frac{x}{1} + (\frac{1}{x^2} + 1) \sinh \frac{x}{1}}{x^2}, \]

and

\[ T_2(u,x) = -\frac{2}{ix} \left( \frac{4i}{x^3} - \frac{u^2}{ix} \right) + S(x)J_0(u^2) \]

We also note that \( \lambda, \mu \) and \( \gamma \) obey the identities [9]

\[ \frac{\lambda + \mu}{\lambda + 2\mu} = 1 - \gamma^2 \]  
\[ \frac{\mu}{\lambda + 2\mu} = \gamma^2. \]

The integrals \( I_5 \) to \( I_{13} \) follow, grouped according to the method used to put them in forms suitable for subsequent numerical quadrature. An example of each method is provided in Appendix B.

Method 1:

\[ I_5 = \frac{d}{\pi} (1 - \gamma^2) \left[ -i \int_{\gamma}^{1} k(x) H_1^1(tx) \sin \frac{x}{1} dx \right] \]

\[ = \frac{\pi i s(2s^2 - 1 - 2\sqrt{s^2 - \gamma} \sqrt{s^2 - 1})H_1^1(t^2)\sin \frac{x}{1}}{f_1(s)} \]
\[ I_6 = \frac{d}{\pi k^2} (1 - \gamma^2) \left[ -\int_\gamma^1 \frac{K(x)H_1^{(1)}(t^*)T(s)}{x^2} dx \right. \]
\[ \left. + \frac{mi(2s^2 - 1 - 2\sqrt{s^2 - \gamma^2} \sqrt{s^2 - 1} H_1^{(1)}(ts)T(s)}{r_1(s)} \right] \quad (100) \]

Method 2:

\[ I_7 = -\frac{d}{\pi} (1 - \gamma^2) \left\{ -\frac{2}{\pi} \int_0^\infty \frac{\sqrt{x^4 + \gamma^2} K(tx)\sinh x}{f_2(x)} dx \right. \]
\[ \left. - \frac{1}{\pi} K(x)H_1^{(1)}(tx)\sin x + \frac{mi \sqrt{s^2 - \gamma^2} H_0^{(1)}(tx)\sin x}{f_1(s)} \right\} \quad (101) \]

\[ I_8 = -\frac{d\alpha^2}{\pi} (1 - \gamma^2) \left\{ -\frac{2}{\pi} \int_0^\infty \frac{\sqrt{x^4 + \gamma^2} K(t^*)}{f_2(x)} S_1(s) dx \right. \]
\[ \left. - \frac{1}{\pi} K(x)H_0^{(1)}(tx)T(x) + \frac{mi \sqrt{s^2 - \gamma^2} H_0^{(1)}(ts)T(s)}{f_1(s)} \right\} \quad (102) \]

Method 3:

\[ I_9 = -\frac{d}{\pi^2 t^2} (1 - \gamma^2) \left[ i \int_0^1 \frac{K(x)T_1(s)}{x} dx - \frac{mi \sqrt{s^2 - \gamma^2} T_1(s)}{r_1(s)} \right] \quad (103) \]
Method 4:

\[ I_{10} = \frac{2}{\pi^2} \frac{d^2}{b} \gamma^2 (1 - \gamma^2) \left[ \int_0^1 \frac{K(x)\mathcal{Q}(x)}{x} \; dx - \frac{\pi \sqrt{\beta^2 - \gamma^2} \mathcal{Q}(s)}{f_1(s)} \right] \]  

(104)

\[ - \frac{2\pi}{15} \frac{1}{1 - \gamma} \]

Method 5:

\[ I_{11} = - \frac{2d}{\pi} (1 - \gamma^2) \sqrt{\frac{2}{t^2 - 1}} \int_0^1 \frac{u}{\sqrt{1 - u^2} [t^2 - u^2]} \right] \]

(105)

\[ \left\{ \left[ \frac{1}{\pi} \left[ \int_0^1 \frac{K(x)\mathcal{V}(x,u)}{x} \; dx - \pi \frac{\sqrt{\beta^2 - \gamma^2} \mathcal{V}(s,u)}{f_1(s)} \right] \right] - \frac{\pi}{1 - \gamma} \right\} du \]

\[ I_{12} = - \frac{d^2}{\pi} \sqrt{\frac{2}{t^2 - 1}} \int_0^1 \frac{u}{\sqrt{1 - u^2} [t^2 - u^2]} \]

(106)

\[ \left\{ (1 - \gamma^2) \frac{a k}{2\pi} \left[ - \int_0^\gamma \frac{K(x)\mathcal{T}_2(u,x)}{x} \; dx \right] \right. \]

\[ \left. - \int_0^\gamma \frac{K(x)\mathcal{T}_2(u,x)}{x} \; dx + \frac{\pi s \sqrt{\beta^2 - \gamma^2}}{f_1(s)} \mathcal{T}_2(s) \right] - \left( \frac{1}{2} u^2 \right) \right\} du \]

\[ I_{13} = \frac{d^2}{\gamma^2} \int_0^1 \frac{u}{\sqrt{1 - u^2}} \left\{ (1 - \gamma^2) \frac{a k}{2\pi} \left[ - \int_0^\gamma \frac{K(x)\mathcal{T}_2(u,x)}{x} \; dx \right] \right. \]

(107)
\[ - \int_{\gamma}^{1} K(x)T_2(u, x)dx + \frac{\pi s \sqrt{s^2 - \gamma^2}}{\gamma T_2(u, s)} - (1 + \frac{1}{2} u^2) \int du \]

The remaining three integrals, \( I_{14}, I_{15}, \) and \( I_{16} \), can be expressed in terms of integrals which have previously been considered, as follows:

\[ I_{14} = I_{13} \quad (108) \]

\[ I_{15} = b^2 I_{11} + a^2 b \sqrt{b^2 - a^2} \frac{I_2}{\gamma} \quad (109) \]

\[ I_{16} = b^2 I_{12} + \frac{a^2 b \sqrt{b^2 - a^2}}{\gamma} I_{13} \quad (110) \]
CHAPTER V

NUMERICAL INTEGRATION TECHNIQUES

A number of special numerical quadrature techniques were used in approximating the integrals $I_5$ to $I_{13}$.

In those integrals whose integrands involved $K(x)$, the derivative of the integrand became unbounded in the vicinity of $\gamma$ and $\lambda$. The trapezoid rule was discovered to be more satisfactory computationally in the neighborhood of these points. Hence, the trapezoid rule was applied on the intervals $[\gamma - 0.05, \gamma + 0.05]$ and $[0.95, 1]$. Simpson's Rule was used elsewhere.

The integrals $I_7$ and $I_8$ were reduced to sums of integrals, in each case including an infinite integral with an exponentially decaying integrand. For example, the integral $I_7$ involved the infinite integral

$$\int_0^\infty \frac{K_0(tx) \sinh x}{\left( x^2 + \frac{1}{2} \right)^{3/2}} \frac{\sqrt{x^2 + \gamma^2}}{\sqrt{x^2 + \gamma^2}} \, dx.$$  

If (111) is approximated by

$$\int_0^{50 \text{akt}} \frac{K_0(tx) \sinh x}{\left( x^2 + \frac{1}{2} \right)^{3/2}} \frac{\sqrt{x^2 + \gamma^2}}{\sqrt{x^2 + \gamma^2}} \, dx,$$

(112)
the error is

\[ E = \int_{-\infty}^{\infty} \frac{K_0(tx) \sinh x \sqrt{x^2 + \gamma^2}}{\left( x^2 + \frac{1}{2} \right)^2 - x^2 \sqrt{x^2 + \gamma^2} \sqrt{x^2 + 1}} \, dx. \]  \hfill (113)

Since

\[ \frac{\sqrt{x^2 + \gamma^2}}{\left( x^2 + \frac{1}{2} \right)^2 - x^2 \sqrt{x^2 + \gamma^2} \sqrt{x^2 + 1}} \sim \frac{2}{(1 - \gamma^2)x} \quad \text{as} \quad x \to \infty \]  \hfill (114)

and (see [15])

\[ K_0(u) \sim \left( \frac{\pi}{2u} \right)^{1/2} e^{-u} \quad \text{as} \quad u \to \infty, \]  \hfill (115)

\[ E = \int_{-\infty}^{\infty} \frac{K_0(tx) \sinh x \sqrt{x^2 + \gamma^2}}{\left( x^2 + \frac{1}{2} \right)^2 - x^2 \sqrt{x^2 + \gamma^2} \sqrt{x^2 + 1}} \, dx \]  \hfill (116)

\[ \approx \int_{-\infty}^{\infty} \left( \frac{\pi}{2tx} \right)^{1/2} e^{-tx} \left[ e^{tx} - e^{-tx} \right] \frac{2}{(1 - \gamma^2)x} \, dx \]

\[ = \left( \frac{\pi}{2akt} \right)^{1/2} \frac{1}{1 - \gamma^2} \int_{\frac{1}{50}}^{\infty} \left[ e^{(1-t)x} - e^{-(1+t)x} \right] \, dx \]

\[ \leq \left( \frac{\pi}{2akt} \right)^{1/2} \frac{1}{1 - \gamma^2} \int_{\frac{1}{50}}^{\infty} \left[ e^{(1-t)x} - e^{-(1+t)x} \right] \, dx \]
if akt \leq 50. The final expression in (116) is equal to

\[
\left(\frac{n_0}{2t}\right)^{1/2} \frac{1}{1 - \gamma^2} \{\exp[-50(1-1/t)] - \exp[-50(1+1/t)]\}
\]

which can clearly be neglected if t > 1.1. If akt > 50, the integral

\[
\int_{1}^{\infty} \frac{K_0(tx) \sinh \frac{x}{\sqrt{x^2 + y^2}}}{x^2 + \frac{1}{2}} dx
\]

\[
\frac{1}{x^2 + \frac{1}{2}} - x^2 \frac{x^2 + y^2}{\sqrt{x^2 + 1}}
\]

can be neglected by a similar argument.

Although the integral (112) has a removable singularity at 0, the logarithmic behavior of \(K_0(tx)\) as \(x \to 0\) leads to computational difficulties due to roundoff error. We therefore use the following special numerical technique. If \(f(x)\) is continuous on \([0, \varepsilon]\) and \(c > \varepsilon\) is constant, a form of the Mean Value Theorem for integrals yields

\[
\int_{0}^{6} f(x) \log \left(\frac{x}{c}\right) dx = f(\xi) \int_{0}^{6} \log \left(\frac{x}{c}\right) dx = f(\xi) \varepsilon \log \left(\frac{\varepsilon}{c\varepsilon}\right)
\]

where \(\xi\) is a number in \((0, \varepsilon)\). If \(|f(x)| \leq 0.02, 0 < x \leq 0.02,\) and \(\varepsilon = 0.02,\) the approximation

\[
f(\xi) \varepsilon \log \left(\frac{\varepsilon}{c\varepsilon}\right) \approx \varepsilon f(\xi) \log \left(\frac{\varepsilon}{c\varepsilon}\right)
\]

is sufficiently accurate. Hence, if \(F(x) = f(x) \log \left(\frac{x}{c}\right), x > 0\) we use the approximation
Similarly, computational roundoff error leads to difficulties with such expressions as

\[ q(x) = \frac{1 + x^2 + \frac{2}{3} i x^3 + \frac{4}{15} i x^5 + (x^2 + 2i x - 1)e^{2ix}}{x^6} \]  

as \( x \to 0 \). The remedy for this is to approximate the function by a Taylor polynomial when \( x \) is near 0, i.e.

\[ q(x) \approx 0.8888888 - 0.4571424 i x + 0.177604 x^2 + 0.0564372 i x^3. \]

The integrals \( I_{11} \), \( I_{12} \) and \( I_{13} \) are of the form

\[ \int_0^1 \frac{uf(u)du}{\sqrt{1 - u^2}}, \quad t > 1, \]  

where \( f(u) \) is continuous and bounded on \((0,1]\). The singularity at 1 was eliminated by making the transformation

\[ u = \sin \theta. \]

The resulting integral was approximated using Simpson's Rule.

The above numerical techniques were used to construct the programs UR and UZ20 in Appendix D for the approximation of \( U_r(b) \) and \( U_z(b) \), respectively.

1This function occurs in the treatment of the integral \( I_{10} \).
APPENDIX A

REDEFINITION OF THE FUNDAMENTAL SINGULARITIES FOR OUR AXISYMMETRIC PROBLEM

If $B_{u}$ is a disc of radius $a > 0$ centered at the origin in the plane $B = \{(x,y,z): z = 0\}$, $P = (r,\theta,0)$ and $Q = (\rho,\varphi,0)$ in cylindrical polar coordinates, the representation (2) becomes

$$u_{\alpha}^{*}(r,\theta,0) = -\int_{0}^{2\pi} \int_{0}^{a} u^{3}_{\alpha}(r,\theta,0; \rho,\varphi,0) \tau_{33}(\rho,\varphi,0) \rho d\rho d\varphi .$$  \hspace{1cm} (125)

We note that $\tau_{33}(\rho,\varphi,0)$ is independent of $\varphi$. For this reason we adopt the notation

$$\tau_{zz}(\rho) \equiv \tau_{33}(\rho,\varphi,0)$$  \hspace{1cm} (126)

We shall now use (125) to reformulate definitions for the fundamental singularities.

It will be observed that the component of displacement in the azimuthal direction is zero because of the axisymmetry.

We will now direct our attention to the points $P_{0} = (b,\theta,0)$.

In view of the axisymmetry $u_{z}(b,\theta,0)$ is independent of $\theta$, and hence we may adopt the notation

$$u_{z}(b) \equiv u_{z}(b,\theta,0) .$$  \hspace{1cm} (127)

We let

$$u_{z}^{*}(b,\rho) = \int_{0}^{2\pi} u^{3}_{z}(b,\theta,0; \rho,\varphi,0) d\varphi$$  \hspace{1cm} (128)
which is independent of \( \theta \) because \( u_3^p(b, \theta, 0; \rho, \varphi, 0) \) depends on \( \theta \) and \( \varphi \) only in the combination \( \theta - \varphi \). The definition (128) can be taken as a redefinition of a fundamental singularity for our axisymmetric problem and can be interpreted as the displacement in the z-direction of a point on the ring \( r = b, z = 0 \), due to a ring of unit point forces oscillating in the z-direction around the circle \( r = \rho, z = 0 \). Using (126), (127) and (128), we obtain from (125)

\[
U_z(b) = - \int_0^a u_z(b, \rho) T_z^3(\rho) \rho d\rho .
\] (129)

Taking \( \alpha = 3 \), \( P = P_0 \) and \( Q = (\rho, \varphi, 0) \), equation (4) can be written

\[
u_3(b, \theta, 0) = - \int_0^{2\pi} \int_0^a T^3_3(\rho, \varphi, 0; b, \theta, 0) u_3(\rho, \varphi, 0) \rho d\rho d\varphi .
\] (130)

We define \( T^*_z(\rho, b) \) by

\[
T^*_z(\rho, b) = \int_0^{2\pi} T^3_3(\rho, \varphi, 0; b, \theta, 0) d\varphi, \ b > a, \ 0 \leq \rho < a , \quad (131)
\]

using the observation that \( T^3_3(\rho, \varphi, 0; b, \theta, 0) \) depends on \( \theta \) and \( \varphi \) only in the combination \( \theta - \varphi \). Using definitions (127) and (131), we obtain from (130)

\[
U_z(b) = - \int_0^a T^*_z(\rho, b) U_z(\rho) \rho d\rho .
\] (132)

Employing an argument analogous to the one given for (125), we have
where $u_r(b, \theta, 0)$ is the radial component of displacement at $(b, \theta, 0)$ due to the stress distribution $\tau_{33}(\rho, \varphi, 0) = \tau_{zz}(\rho)$ and $u_r^0(b, \theta, 0; \rho, \varphi, 0)$ is the radial component of displacement at $(b, \theta, 0)$ due to a ring of unit point forces oscillating in the z-direction around the circle $r = \rho$, $z = 0$. The symmetry implies that $u_r(b, \theta, 0)$ is independent of $\theta$, and hence we adopt the notation

$$U_r(b) = u_r(b, \theta, 0).$$  \hfill (134)

We define the fundamental singularity $U^*_r(b, \rho)$ by

$$U^*_r(b, \rho) = \int_0^{2\pi} u_r^0(b, \rho, 0; \rho, \varphi, 0) d\varphi$$  \hfill (135)

with an interpretation analogous to the one given for (128). Using (126), (134) and (136), (133) gives

$$U_r(b) = -\int_0^a U^*_r(b, \rho) \tau_{zz}(\rho) \rho d\rho.$$  \hfill (136)

We now proceed to construct another integral representation for $U_r(b)$. Let $u^*_r(Q, P), P \in B_\rho, Q \in B$, with associated tractions $T^*_r(Q, P)$, be the component of displacement in the $x_1$-direction at $Q$ due to a unit oscillating point force in the radial direction at $P$ with the specifications

$$u^*_3(Q, P) = 0, \ P \in B_\rho, \ Q \in B_u.$$  \hfill (137)
\[ T_{1}^*(Q,P) = T_{2}^*(Q,P) = 0 , \quad P \in B_{T} , \quad Q \in B_{u} \]  

and

\[ T_{1}^*(Q,P) = 0 , \quad P \in B_{T} , \quad Q \in B_{T} - \{P\} . \]

Then an argument analogous to that given for (4) leads to

\[ u_{r}(P) = -\int_{B_{u}} T_{3}^*(Q,P)u_{3}(Q)dA_{Q} . \]  

Taking \( P = (b,0,0) \), \( Q = (p,\varphi,0) \) and the disc \( \rho \leq a \), (140) becomes

\[ u_{r}(b,\theta,0) = -\int_{0}^{a} \int_{0}^{2\pi} T_{3}^*(p,\varphi,0;b,\theta,0)u_{3}(p,\varphi,0)dpd\varphi . \]  

Since \( T_{3}^*(p,\varphi,0;b,\theta,0) \) depends on \( \varphi \) and \( \theta \) only in the combination \( \theta - \varphi \), we can write

\[ T_{3}^*(p,\varphi,0;b,\theta,0) = \int_{0}^{2\pi} T_{3}^*(p,\varphi,0;b,\theta,0)d\varphi . \]

Using (127), (134) and (142), (141) gives

\[ U_{r}(b) = -\int_{0}^{a} \int_{0}^{2\pi} T_{3}^*(p,\varphi)bU_{2}(p)dpd\varphi , \]  

which is a representation analogous to (4).

The representations (129) and (132) can be combined to obtain the representation

\[ U_{z}(b) = \int_{0}^{a} \int_{0}^{a} T_{3}^{*z}(p,b) U_{z}(p,p')\tau_{zz}(p,p')dpdp' , \]  

(144)
and, in a similar manner, we obtain

\[ U_r(b) = \int \int \tau_{zz}^*(\rho, b) \tau_{zz}(\rho', \rho') \rho \rho' \, \text{d}\rho \, \text{d}\rho' \]  

(145)

from (143) and (129).

We note that

\[ \tau_{ZZ}^*(\rho, b) = - \tau_{ZZ}(\rho, b) \]

and

\[ \tau_{r}^*(\rho, b) = - \tau_{zz}(\rho, b) \]

and, hence, that the representations (132), (143), (144) and (145) can be written

\[ U_z(b) = \int \tau_{zz}^*(\rho, b) U_z(\rho) \rho \, \text{d}\rho , \]  

(146)

\[ U_r(b) = \int \tau_{rr}^*(\rho, b) U_z(\rho) \rho \, \text{d}\rho , \]  

(147)

\[ U_z(b) = \int \int \tau_{zz}^*(\rho, b) \tau_{zz}(\rho', \rho') \rho \rho' \, \text{d}\rho \, \text{d}\rho' \]  

(148)

and

\[ U_r(b) = \int \int \tau_{rr}^*(\rho, b) \tau_{zz}(\rho', \rho') \rho \rho' \, \text{d}\rho \, \text{d}\rho' \]  

(149)

respectively.
We now proceed to obtain the resulting expressions for \( u_z(b, \rho) \) and \( u_x(b, \rho) \). From the classical result of Lamb [16], which is quoted in [17], we have

\[
\begin{align*}
\hat{u}^3_3(r, \theta, 0) &= -\frac{k}{8\pi \mu} \int_0^\infty \frac{x\alpha(x)J_0(\kappa r x)}{f(x)} \, dx \\
\hat{u}^3_1(r, \theta, 0) &= \frac{k}{8\pi \mu} \int_0^\infty \frac{x^2[2x^2-1-2\alpha(x)\beta(x)]J_1(\kappa r x)}{f(x)} \, dx
\end{align*}
\]

(150) and

(151)

where

\[
\alpha(x) = \begin{cases} 
\sqrt{x^2 - \gamma^2}, & x > \gamma \\
-\frac{x^2}{\sqrt{\gamma^2 - x^2}}, & -\gamma \leq x \leq \gamma \\
-\sqrt{x^2 - \gamma^2}, & x < -\gamma
\end{cases}
\]

(152)

\[
\beta(x) = \begin{cases} 
\sqrt{x^2 - 1}, & x > 1 \\
-\sqrt{1 - x^2}, & -1 \leq x \leq 1 \\
-\sqrt{x^2 - 1}, & x < -1
\end{cases}
\]

(153)

and

\[
f(x) = (x^2 - \frac{1}{2})^2 - x^2 \alpha(x)\beta(x)
\]

(154)

We note that \( \frac{1}{f(x)} \) has branch points at \( x = \pm \gamma \) and \( x = \pm 1 \) and simple poles at \( x = \pm \sigma \), where \( \sigma > 1 \) is the location of a Rayleigh pole [14].
In view of these singularities the integrals (150) and (151) require interpreting. This interpretation must be in accordance with the radiation condition appropriate for an $e^{-i\omega t}$ time dependence. The contour of integration, in the complex $z$-plane, satisfying this requirement is shown in Figure 4, where we have taken the branch cuts outward along the real axis to infinity from $\pm \gamma$ and $\pm 1$ for the functions $\alpha(z)$ and $\beta(z)$, respectively. We note that the radiation contour has infinitesimal indentations below the points $\gamma$, $1$ and $\alpha$, respectively. For later reference we have also included in (152) and (153) the appropriate definitions of $\alpha(x)$ and $\beta(x)$ on the negative real axis below the corresponding branch cuts.

We now use the expressions (150) and (151) to construct the fundamental singularities $U^Z_z(b, \rho)$ and $U^Z_r(b, \rho)$.

Referring to Figure 3, we have

\[ U^Z_z(b, \rho) = \frac{k}{8\pi\mu} \int_0^{2\pi} \int_0^\infty \frac{x \alpha(x) J_0(kxR)}{r(x)} \, dx \, d\varphi \]  

where

\[ R = \sqrt{b^2 + \rho^2 - 2b\rho \cos(\theta - \varphi)} \]  

Using an addition formula for Bessel functions [12], we can write

\[ U^Z_z(b, \rho) = \frac{k}{8\pi\mu} \int_0^{2\pi} \int_0^\infty \frac{x \alpha(x)}{r(x)} \sum_{m=-\infty}^{\infty} J_m(kbx)J_m(kpx) \cos(m(\theta - \varphi)) \, dx \, d\varphi \]  

(157)
Changing the order of integration and integrating with respect to \( \varphi \), we have

\[
\hat{u}_z(b, \rho) = -\frac{k}{\mu \nu} \int_0^\infty \frac{x\alpha(x)}{f(x)} J_0(kbx)J_0(kpx)dx.
\]  

(158)

With the adoption of the notations

\[
t = \frac{b}{a}
\]

(159a)

and

\[
u = \frac{c}{a}
\]

(159b)

equation (158) can be written

\[
\hat{u}_z(b, \rho) = -\frac{k}{\mu \nu} \int_0^\infty \frac{x\alpha(x)}{f(x)} J_0(t\hat{a}x)J_0(\hat{ux})dx
\]

(160)

where \( \hat{a} \) is given in (83).

To obtain the corresponding expression for \( \hat{u}_r(b, \rho) \), we again refer to Figure 3. Using the notation indicated in this figure and the result (151) the displacement in the radial direction at \( P \) due to a unit force oscillating in the \( z \)-direction at \( Q \) is

\[
\frac{k}{\alpha r} \int_0^\infty \frac{x^2 [2x^2 - 2\alpha(x)\beta(x)]}{f(x)} J_1(kxR)dx \cdot \cos \psi
\]

(161)

where \( R \) is given in (156). Hence

\[
\hat{u}_z(b, \rho) = \frac{k}{\alpha r} \int_0^{2\pi} \int_0^\infty \frac{x^2 [2x^2 - 2\alpha(x)\beta(x)]}{f(x)} J_1(kxR)\cos \psi \, dx \, d\varphi
\]

(162)
Using an addition formula for Bessel functions we have

\[ U^z_r(b, \rho) = \frac{k}{4\pi} \int_0^\infty \frac{x^2 [2x^2 - 1 - 2\alpha(x)\beta(x)]}{f(x)} \sum_{m=-\infty}^{\infty} J_{m+1}(kbx)J_m(kpx)\cos \theta \cos \phi \, dx \, d\omega \]  

(163)

Reversing the order of integration and integrating with respect to \( \phi \), we obtain

\[ U^z_r(b, \rho) = \frac{k}{4\pi} \int_0^\infty \frac{x^2 [2x^2 - 1 - 2\alpha(x)\beta(x)]}{f(x)} J_1(kbx)J_0(kpx) \, dx \]  

(164)

If we use the notations given in (83), (159a) and (159b), we have

\[ U^z_r(b, \rho) = \frac{k}{4\pi} \int_0^\infty \frac{x^2 [2x^2 - 1 - 2\alpha(x)\beta(x)]}{f(x)} J_1(tx^\rho)J_0(ux^\rho) \, dx \]  

(165)
APPENDIX B

METHODS OF ANALYSIS FOR INTEGRALS

We give examples to illustrate the methods used to reduce the integrals of the five types given in Chapter IV to forms suitable for numerical approximation.

As an example of Method I, we consider the integral

\[ I_5 = \int_0^a U^\alpha_I(b, \rho) \frac{1}{T_{zz}(\rho)} \rho d\rho \] (166)

\[ = \frac{d}{\pi} \frac{\lambda + \mu}{\lambda + 2\mu} (ks) \int_0^1 \left[ \int_0^\infty \frac{x^2(2x^2 - 1 - 2\alpha(x)\beta(x))J_1(t^\lambda)J_0(u^\lambda)dx}{f(x)} \right] \frac{udu}{\sqrt{1-u^2}} \]

where \( \frac{1}{T_{zz}(\rho)} \) and \( U^\alpha_I(b, \rho) \) are as given in equations (20) and (165), respectively, and \( u = \frac{2}{a} \). Using the identity [13]

\[ \int_0^1 uJ_0(u^\lambda)du = \frac{\sin \frac{\lambda}{2}}{\frac{\lambda}{2}} \], (167)

and the relationship (97), (166) leads to

\[ I_5 = \frac{d}{\pi} (1 - \gamma^2) \int_0^\infty \frac{x(2x^2 - 1 - 2\alpha(x)\beta(x))J_1(t^\lambda)\sin \frac{\lambda}{x}}{f(x)} dx \] . (168)

Introducing the Hankel functions, we have
\[ I_5 = \frac{d}{2\pi} (1-\gamma^2) \int_0^\infty \frac{x(2x^2-1-2\alpha(x)\beta(x))H_1^{(1)}(tx)\sin x}{f(x)} \, dx \]  

(169)

\[ + \int_0^\infty \frac{x(2x^2-1-2\alpha(x)\beta(x))H_1^{(2)}(tx)\sin x}{f(x)} \, dx \].

The integrals in (169) are understood to be taken along the radiation contour described in Appendix A.

To reduce (169) to a sum of finite integrals we consider the contour integral

\[ \int_\Gamma \frac{z[2z^2-1-2\alpha(z)\beta(z)]H_1^{(2)}(tz)\sin z}{f(z)} \, dz \],

(170)

where \( z = k_0z \) and \( \Gamma \) is the contour shown in Figure 5. We note that \( \alpha(z) \) and \( \beta(z) \) as defined in (152) and (153) have been extended analytically into the lower half-plane with branch cuts outward along the real axis to \( \infty \) for \( +\gamma \) and \( +1 \) for \( \alpha(z) \) and \( \beta(z) \), respectively. The contour \( \Gamma \) has semicircular indentations beneath the branch points and the simple poles \( \pm s \). There is also a branch cut along the negative real axis for the function \( H_1^{(2)}(tz) \).

By Cauchy's Theorem the integral (170) is zero. Also, since [15]

\[ \frac{z[2z^2-1-2\alpha(z)\beta(z)]H_1^{(2)}(tz)\sin z}{f(z)} \sim \frac{M_1}{z^{3/2}} \left[ e^{-i(t-1)\frac{z}{2}} - e^{i(t+1)\frac{z}{2}} \right], \quad z \to \infty, \text{Im}(z) < 0 \],

(171)
where $M_1$ is a complex constant and because $t > 1$, Jordan's Lemma applies, and therefore

$$\lim_{T \to \infty} \int_{-T}^{T} \frac{z^{2e+1} - 2 \alpha(z) \beta(z) H_1^{(2)}(t^2) \sin \frac{z}{f(z)}}{f(z)} \, dz = 0. \quad (172)$$

Letting the radii, $e$, $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$, of the semicircular indentations tend to zero, we obtain

$$\int_{-\infty}^{1} T_2(x) \, dx + \int_{-1}^{0} T_2(x) \, dx + \int_{0}^{1} T_2(x) \, dx + R(-s) \quad (173)$$

$$+ \int_{0}^{\gamma} T_2(x) \, dx + \int_{1}^{\gamma} T_2(x) \, dx + \int_{\gamma}^{\infty} T_2(x) \, dx + R(s) = 0,$$

where $R(s)$ and $R(-s)$ are the contributions from the Rayleigh poles, $s$ and $-s$, respectively, and

$$T_2(x) = \frac{x^{2e+1} - 2 \alpha(x) \beta(x) H_1^{(2)}(t^2) \sin \frac{x}{f(x)}}{f(x)}. \quad (174)$$

Here and in subsequent expressions the notations $\int_{l}^{\infty}$ and $\int_{-\infty}^{-l}$ denote the Cauchy principle value integrals with respect to the poles $s$ and $-s$, respectively. Hence

$$\int_{-\infty}^{\gamma} T_2(x) \, dx + \int_{0}^{\gamma} T_2(x) \, dx + \int_{1}^{1} T_2(x) \, dx + R(s) \quad (175)$$

$$= - \int_{-\infty}^{-1} T_2(x) \, dx - \int_{-1}^{-\gamma} T_2(x) \, dx - \int_{-\gamma}^{0} T_2(x) \, dx - R(-s)$$
\[ \int_1^\infty T_2(-x)dx - \int_1^\infty T_2(-x)dx - \int_0^1 T_2(-x)dx = R(s), \]

where it has been noted that

\[ R(s) = \frac{-s[2s^2-1-2\alpha(-s)]H_1^{(2)}(-t^s)\sin(-s)}{f'(s)} \]

\[ = -\frac{s[2s^2-1-2\alpha(s)]H_1^{(1)}(t^s)\sin s}{f'(s)} = -R(s), \]

since \( f'(s) = f'(s) \) and \( H_1^{(2)}(-t^s) = H_1^{(1)}(t^s) \). Now, since \( f'(s) \) is the same as \( f_1(s) \) in (84), combining the results (169), (174), (175) and (176) and using the notation in (90), we obtain

\[ I_5 = \frac{d}{\pi} (1-\gamma^2) \int_1^\infty \frac{\hat{h}(x)H_1^{(1)}(tx)\sin x}{\sqrt{x}} \, dx \]

\[ + \frac{\pi s[2s^2-1-2\sqrt{s^2-1}]H_1^{(1)}(t^s)\sin s}{f_1(s)}, \]

which is identical to equation (99).

Method 2 is illustrated by the integral

\[ I_7 = \int_0^a \hat{u}_z(b, \rho) \frac{1}{z}\xi_2(\rho) \rho d\rho \]

\[ = -\frac{d}{\pi}(1-\gamma^2)ak \int_0^1 \frac{x\alpha(x)}{f(x)} J_0(tx)J_0(u\hat{x}) \frac{u}{\sqrt{1-u^2}} \, dx du \]
where equations (20), (160) and (159b) have been used. Changing the
order of integration and using (167), we have

\[ I_7 = -\frac{d}{d} (1-\gamma^2) \int_0^\infty \frac{\alpha(x)}{f(x)} J_0(t^* x) \sin x \, dx. \]  

(179)

Introducing Hankel functions, we obtain

\[ I_7 = -\frac{d}{d} (1-\gamma^2) \left[ \int_0^\infty \frac{\alpha(x)}{f(x)} H_0^{(1)}(t^* x) \sin x \, dx \right] \]

\[ + \int_0^\infty \frac{\alpha(x)}{f(x)} H_0^{(2)}(t^* x) \sin x \, dx \]

(180)

where the integrals are taken over the radiation contour as previously
explained in Appendix A. In order to reduce (180) to a form more
suitable for numerical analysis, we consider the contour integral

\[ \int_{\Gamma} \frac{\alpha(z)}{f(z)} H_0^{(2)}(t^* z) \sin z \, dz \]

(181)

where \( \hat{z} = k z \) and \( \Gamma \) is the contour shown in Figure 5. Branch cuts for
\( \alpha(z) \) and \( \beta(z) \) are the same as those for the contour integral (170) with
a branch cut taken along the negative real axis for the function
\( H_0^{(2)}(t^* z) \); \( \alpha(z) \) and \( \beta(z) \) have been extended analytically into the lower
half-plane.

By Cauchy's Theorem the integral (181) is zero. Furthermore,
since [15]
\[
\frac{\alpha(z) H^{(2)}_0(t^2)] \sin \hat{\omega}}{f(z)} \sim \frac{M_2}{z^{3/2}} [e^{-i(t-1)^2} - e^{-i(t+1)^2}]
\]

\[z \to \infty, \text{Im}(z) < 0, \ t > 1,\]

where \(M_2\) is a complex constant, Jordan's Lemma applies, and therefore

\[
\lim_{T \to \infty} \int_{\Gamma_T} \frac{\alpha(z) H^{(2)}_0(t^2) \sin \hat{\omega}}{f(z)} \, dz = 0 \quad (183)
\]

where \(\Gamma_T\) is shown in Figure 5. Hence, upon letting the radii, \(\epsilon, \epsilon_1, \epsilon_2\) and \(\epsilon_3\), of the indentations tend to 0,

\[
\int_{-\infty}^{-1} T_3(x) \, dx + \int_{-1}^{\gamma} T_3(x) \, dx + \int_{-\gamma}^{1} T_3(x) \, dx + R_1(-\gamma) + R_1(s) = 0 \quad (184)
\]

where

\[
T_3(x) = \frac{\alpha(x) H^{(2)}_0(t^2) \sin \hat{\omega}}{f(x)} \quad (185)
\]

and \(R_1(-\gamma)\) and \(R_1(s)\) are the contributions from the Rayleigh poles, \(-\gamma\) and \(s\), respectively. Hence
\begin{align*}
\int_{-\infty}^{\gamma} T_3(x)dx + \int_{-1}^{1} T_3(x)dx + \int_{-1}^{\infty} T_3(x)dx + R_1(s) \\
&= -\int_{-\infty}^{-1} T_3(x)dx - \int_{-1}^{\gamma} T_3(x)dx - \int_{-\infty}^{0} T_3(x)dx - R_1(-s) \\
&= -\int_{-1}^{\infty} T_3(-x)dx - \int_{1}^{1} T_3(-x)dx - \int_{0}^{\gamma} T_3(-x)dx - R_1(s)
\end{align*}

where it has been observed that

\begin{align*}
R_1(-s) &= n! \frac{s^{2/2 - \gamma/2} H_0^{(2)}(-ts)\sin(s)}{f'(s)} \\
&= n! \frac{s^{2/2} H_0^{(1)}(ts)\sin(s)}{f'(s)} = R_1(s),
\end{align*}

since \( f'(-s) = -f'(s) \) and \( H_0^{(2)}(-ts) = -H_0^{(1)}(ts) \). Combining the results (180), (185), (186) and (187), we obtain

\begin{align*}
I_7 &= -\frac{d}{d\gamma} \left[ \int_{-\infty}^{\infty} \int_{1}^{1} H_0^{(1)}(tx)\sin(x) \frac{(x^2 - \frac{1}{2})^2 \sqrt{x^2 - \gamma^2}}{(x^2 - \frac{1}{2}) + x^4(x^2 - \gamma^2)(1-x^2)} \right] dx \\
&+ \frac{\int_{-\infty}^{\infty} H_0^{(1)}(tx)\sin(x) \sqrt{x^2 - \gamma^2}}{(x^2 - \frac{1}{2}) - x^2 \sqrt{x^2 - \gamma^2} \sqrt{x^2 - 1}} \right] dx.
\end{align*}

The Cauchy principle value integral in (188) is not in a form convenient for numerical approximation. In order to obtain a more satisfactory expression, we consider the contour integral...
\[
\int_{\Gamma_1} \frac{\hat{M}(z)}{z} H^{(2)}_{0}(t^2) \sin \hat{z} \, dz
\]  \hspace{1cm} (189)

where

\[
\hat{M}(z) = \frac{z\alpha_2(z)}{(z^2 - \frac{1}{4})^2 - z^2 \alpha_2(z) \beta_2(z)} , \hspace{1cm} (190)
\]

\(\Gamma_1\) is the contour shown in Figure 6; \(\alpha_2(z)\) and \(\beta_2(z)\) are the analytic continuations into the upper right quarter plane of the functions whose values are

\[
\alpha_2(x) = \begin{cases} 
\frac{i}{\sqrt{2}} \sqrt{x^2} , & 0 \leq x \leq \gamma \\
\frac{i}{\sqrt{x^2}} , & x > \gamma 
\end{cases} \hspace{1cm} (191)
\]

and

\[
\beta_2(x) = \begin{cases} 
\frac{i}{\sqrt{1-x^2}} , & 0 \leq x \leq 1 \\
\frac{i}{\sqrt{x^2-1}} , & x > 1 
\end{cases} \hspace{1cm} (192)
\]

on the positive real line with branch cuts on the real line outward to \(\infty\) from \(\gamma\) and \(1\) for \(\alpha_2(z)\) and \(\beta_2(z)\), respectively. \(\Gamma_1\) has semicircular indentations above the branch points \(\gamma\) and \(1\) and the pole \(s\), respectively.

The contour integral (189) is zero by Cauchy's Theorem. Also, since [15]
\[
\frac{\hat{A}(z)}{z} H_0^{(1)}(t^2z) \sin \frac{\theta}{2} \sim \frac{M_3}{z^{3/2}} \left[ e^{i(t^2/2)z} - e^{i(t-1)^2z} \right], \tag{193}
\]

\[ z \to \infty, t > 1, \text{Im}(z) > 0, \]

where \(M_3\) is a complex constant, Jordan's lemma applies on \(\Gamma_\ell T\) as \(T \to \infty\) (see Figure 6), and therefore

\[
\lim_{T \to \infty} \int_{\Gamma_\ell T} \frac{\hat{A}(z)}{z} H_0^{(1)}(t^2z) \sin \frac{\theta}{2} \, dz = 0.
\]

Therefore, upon letting \(T \to \infty\) and \(\epsilon_4, \epsilon_5\) and \(\epsilon_6\) tend to 0, we obtain

\[
\int_Y \hat{A}(x) H_0^{(1)}(t^2x) \sin \frac{x}{2} \, dx + \int_Y \hat{A}(x) H_0^{(1)}(t^2x) \sin \frac{x}{2} \, dx + R_3(s) + \int_Y \frac{\hat{A}(z)}{z} H_0^{(1)}(t^2z) \sin \frac{\theta}{2} \, dz = 0
\]

(194)

where \(R_3(s)\) is the contribution from the Rayleigh pole \(s\) and \(\int_0^\infty\) indicates an integral on the imaginary axis. Solving for the Cauchy principle value integral in (194) and using the identity [15]

\[
H_0^{(1)}(z \sin \theta/2) = \frac{2}{\pi} K_0(z), \quad -\pi < \text{arg} \, z < \frac{\pi}{2},
\]

we have

\[
\int_1^\infty \frac{\hat{A}(x)}{x} H_0^{(1)}(t^2x) \sin \frac{x}{2} \, dx = - \int_Y \hat{A}(x) H_0^{(1)}(t^2x) \sin \frac{x}{2} \, dx - \int_Y \frac{\hat{A}(x)}{x} H_0^{(1)}(t^2x) \sin \frac{x}{2} \, dx
\]

(196)
where $f_2(x)$ is given in (85). Now, since $f'(s)$ is the same as $f_1(s)$ in (84), combining results (188) and (196) we have

$$
I_7 = -\frac{d}{n}(1-\gamma^2)\left[\frac{2}{n} \int_0^\infty \frac{\sqrt{x^2+y^2} K_0(tx) \sinh \frac{x}{t}}{f_2(x)} \, dx \right] (197)
$$

$$
= -i \int_0^\gamma K(x) H_0^{(1)}(tx) \sin \frac{x}{t} \, dx - i \int_\gamma^1 K(x) H_0^{(1)}(tx) \sin \frac{x}{t} \, dx
$$

$$
+ i \pi \frac{\sqrt{s^2-\gamma^2} H_0^{(1)}(ts) \sin \frac{s}{t}}{f_1(s)}
$$

where $K(x)$ is given in (87). We note that this agrees with (101).

As an illustration of Method 3, we consider the integral

$$
I_9 = \int_0^a \int_0^a \frac{1}{r_{zz}(b', \rho')} \partial z(\rho', p) \frac{1}{r_{zz}(\rho)} \rho \, d\rho \, d\rho'
$$

$$
= \frac{2}{\pi} \sqrt{1-\gamma^2} \frac{\alpha k}{b} \int_0^1 \int_0^1 \int_0^\infty \frac{u}{\sqrt{1-u^2}} \frac{v}{\sqrt{1-v^2}} \frac{x_0(x)}{r(x)} \, dJ_0(u) \, dJ_0(v) \, dx
$$

where we have used (20), (46) and (160) and where $u = \frac{\rho}{a}$ and $v = \frac{\rho'}{a}$.

Again the infinite integral is interpreted as in Appendix A. Reversing the order of integration we have
where the symmetry with respect to \( u \) and \( v \) has been employed. From the identity (167) it follows that

\[
I_9 = \frac{2}{n} \gamma^2 (1-\gamma^2) \frac{d}{t} \int_0^\infty \frac{x \alpha(x)}{f(x)} \left[ \int_0^1 \frac{u J_u(u^2) \, du}{\sqrt{1-u^2}} \right]^2 \, dx \tag{199}
\]

where \( t = \frac{b}{a} \). To reduce the final expression in (200) to a sum of finite integrals we consider the contour integral

\[
I_9 = \frac{1}{2} \gamma^2 (1-\gamma^2) \frac{d}{t} \int_0^\infty \frac{\alpha(x)}{f(x)} \frac{1 - \cos \frac{2i}{x}}{x} \, dx
\]

\[
= \frac{1}{2} \frac{\gamma^2 (1-\gamma^2)}{2\pi} \frac{d}{t} \left[ \int_0^\infty \frac{\alpha(x)}{f(x)} \frac{1 - e^{2ix}}{x} \, dx + \int_0^\infty \frac{\alpha(x)}{f(x)} \frac{1 - e^{-2ix}}{x} \, dx \right]
\]

where \( t = \frac{b}{a} \). To reduce the final expression in (200) to a sum of finite integrals we consider the contour integral

\[
\Gamma \frac{\alpha(z)}{f(z)} \frac{1 - e^{-2iz}}{z} \, dz \tag{201}
\]

where \( \Gamma \) is the contour shown in Figure 5, and branch cuts are taken as in the treatment of the contour integral (170) for the functions \( \alpha(z) \) and \( \beta(z) \). Using a procedure similar to the one used for \( I_5 \) we obtain

\[
I_9 = -\frac{1}{2} \gamma^2 (1-\gamma^2) \frac{d}{t} \left[ \int_0^1 \frac{K(x)}{x} \frac{1 - e^{2ix}}{x} \, dx \right)
\]

\[
- \pi \frac{\alpha(s)}{\Gamma(s)} \frac{1 - e^{2is}}{s}
\]

\[
\frac{1}{\Gamma(s)} \left[ \right]
\]
where we note that \( f_1(s) = f'(s) \). Since
\[
T_1(x) = \frac{1-e^{-2ix}}{x} ,
\]
this agrees with (103).

We now consider the integral
\[
I_{10} = \int_0^a \int_0^a T_{zz}(b,p) \delta^2(p',p) \frac{1}{T_{zz}(p)} p^3 \rho^3 \delta \rho \delta p' \tag{203}
\]
\[
= \frac{2}{\pi^2} \gamma^2 (1-\gamma^2) \frac{a^6}{b^6} \int_0^1 \int_0^\infty \frac{u^3}{\sqrt{1-u^2}} \frac{\sqrt{1-v^2}}{x} \frac{\delta(x) J_0(x \delta(x)) J_0(\sqrt{1-v^2})}{\delta(x)} \, dx \, du \, dv ,
\]
where we have used (20), (46) and (160) and where \( u = \frac{2}{a} \) and \( v = \frac{2}{a} \), to illustrate Method 4. Interchanging the order of integration and using the symmetry in \( u \) and \( v \) we have
\[
I_{10} = \frac{2}{\pi} \gamma^2 (1-\gamma^2) \frac{a^6}{b^6} \int_0^1 \int_0^1 \frac{\delta(x) J_0(x \delta(x)) J_0(\sqrt{1-v^2})}{\delta(x)} \, dx \, du \, dv \tag{204}
\]
Using the identity
\[
\int_0^1 \frac{u^3 J_0(u \delta(x)) \delta(x)}{\sqrt{1-u^2}} \, du = \left( \frac{\pi}{2} \right) \left[ \frac{J_{1/2}(x)}{x^{1/2}} - \frac{J_{3/2}(x)}{x^{3/2}} \right] , \tag{205}
\]
which can be derived by making the transformation \( u = \sin t \), referring to [18] and using the identity [15]
\[
J_{5/2}(z) = \frac{2}{z} J_{3/2}(z) - J_{1/2}(z) ,
\]
(204) becomes

\[ I_{10} = \frac{1}{\pi} \int_0^\infty e^{\frac{-2\pi x}{\lambda}} \frac{dx}{b} \left[ \frac{M(x)}{\lambda} \int_0^\infty \frac{J_{1/2}(x)}{\lambda} \frac{dx}{\lambda} \right. \]

\[ \left. - 2 \int_0^\infty M(x) \frac{J_{1/2}(x)J_{3/2}(x)}{\lambda^2} \frac{dx}{\lambda} + \int_0^\infty M(x) \frac{J_{3/2}(x)}{\lambda^3} \frac{dx}{\lambda} \right] \]

where

\[ M(x) = \frac{x\phi(x)}{\iota(x)} \tag{207} \]

Since by [15]

\[ J_{1/2}(x) = \left( \frac{2}{\pi \lambda} \right)^{1/2} \sin \frac{x}{\lambda} \tag{208} \]

\[ \int_0^\infty \frac{x\phi(x)}{\iota(x)} \frac{J_{1/2}(x)}{\lambda} \frac{dx}{\lambda} = \frac{2}{\pi} \int_0^\infty \frac{x\phi(x)}{\iota(x)} \left( \frac{\sin \frac{x}{\lambda}}{\lambda} \right)^2 \frac{dx}{\lambda} \tag{209} \]

and

\[ \int_0^\infty \frac{x\phi(x)}{\iota(x)} \frac{J_{1/2}(x)J_{3/2}(x)}{\lambda^2} \frac{dx}{\lambda^2} \]

\[ = \left( \frac{2}{\pi} \right)^{1/2} \int_0^\infty \frac{x\phi(x)}{\iota(x)} \frac{J_{3/2}(x)\sin \frac{x}{\lambda}}{\lambda^{5/2}} \frac{dx}{\lambda} \tag{210} \]

We observe that the integral (209) has already been considered in connection with \( I_0 \).
The integral

\[ I_{10} = \int_0^\infty M(x) J_{3/2} \left( \frac{x}{A} \right) \sin \frac{x}{A} \, dx \quad (211) \]

was treated by first using the integral representation [15]

\[ J_{3/2}(\frac{x}{A}) = \frac{x^{3/2}}{\sqrt{2\pi}} \int_0^1 (1-v^2) \cos(vx) \, dv \quad (212) \]

obtaining

\[ I_{10}' = \frac{1}{\sqrt{2\pi}} \int_0^1 (1-v^2) \left[ \int_0^\infty M(x) \frac{\sin \frac{x}{A} \cos \frac{vx}{A}}{x} \, dx \right] \, dv \quad (213) \]

The integral

\[ \int_0^\infty M(x) \frac{\sin \frac{x}{A} \cos \frac{vx}{A}}{x} \, dx \quad (214) \]

\[ = \frac{1}{4\pi} \left\{ \int_0^\infty M(x) \left[ e^{i(1+v)\frac{x}{A}} + e^{-i(1-v)\frac{x}{A}} \right] \, dx \right\} - \int_0^\infty M(x) \left[ e^{-i(1+v)\frac{x}{A}} - e^{-i(1-v)\frac{x}{A}} \right] \, dx, \quad v < 1, \]

can be reduced to a sum of finite integrals by considering the contour integral

\[ \int_\gamma M(z) \left[ e^{-i(1+v)\frac{z}{A}} + e^{-i(1-v)\frac{z}{A}} \right] \, dz \quad (215) \]
where $\Gamma$ is the contour in Figure 5. We refer to the explanation following the contour integral (170) for the necessary branch cuts for the functions $\alpha(z)$ and $\beta(z)$. The procedure is similar to the one used for the integral $I_5$, and the result is

$$
\int_{0}^{\infty} M(x) \frac{\sin \frac{\lambda}{x} \cos \frac{\nu}{x}}{\frac{\lambda}{x}} \, dx = \frac{1}{2\pi a} \left\{ \int_{0}^{1} \frac{K(x)}{x} T_{\lambda}(v, \frac{\lambda}{x}) \, dx ight\}
$$

$$
- \frac{\pi \sqrt{2 - \nu}}{f'(s)} T_{\lambda}(v, \frac{\lambda}{x}) \right\}
$$

where $K(x)$ is given in (87) and

$$
T_{\lambda}(v, \frac{\lambda}{x}) = e^{i(1+v)\frac{\lambda}{x}} + e^{i(1-v)\frac{\lambda}{x}}.
$$

Using the result (213), reversing the order of integration, and integrating from 0 to 1 with respect to $v$, we obtain

$$
I_{10}^{1} = - \sqrt{2} \frac{1}{\pi a} \left\{ \int_{0}^{1} \frac{K(x)}{x} e^{i\frac{\lambda}{x}} \left[ \sin \frac{\lambda}{x} - \cos \frac{\lambda}{x} \right] \, dx ight\}
$$

$$
- \pi \frac{\sqrt{2 - \nu}}{f'(s)} e^{i\frac{\lambda}{s}} \left[ \sin \frac{\lambda}{s} - \cos \frac{\lambda}{s} \right] \right\}.
$$

The third integral in (206) is

$$
I_{10}^{\prime} = \int_{0}^{\infty} M(x) \frac{J_{3/2}(\frac{\lambda}{x})}{\frac{\lambda}{x}^3} \, dx
$$

$$
= \int_{0}^{\infty} \left[ M(x) + \frac{2}{1 - \nu} \right] \frac{J_{3/2}(\frac{\lambda}{x})}{\frac{\lambda}{x}^3} \, dx - \frac{2}{1 - \nu} \int_{0}^{\infty} \frac{J_{3/2}(\frac{\lambda}{x})}{\frac{\lambda}{x}^3} \, dx.
$$
From [12] we obtain

$$
\int_0^\infty \frac{J_{3/2}(x)}{x^3} \, dx = \frac{2}{15} \frac{1}{ka} .
$$

(220)

Since by [15]

$$
J_{3/2}(x) = \sqrt{\frac{2}{\pi}} \frac{\hat{x} \cos \hat{x} - \sin \hat{x}}{\hat{x}^{3/2}} ,
$$

(221)

we have

$$
I_{10} = \int_0^\infty \left[ M(x) + \frac{2}{1-Y^2} \right] \frac{J_{3/2}(x)}{x^3} \, dx
$$

(222)

$$
= \frac{2}{\pi} \int_0^\infty \left[ M(x) + \frac{2}{1-Y^2} \right] \left[ \frac{\hat{x}^2 \cos \hat{x}^2 - 2\hat{x} \sin \hat{x} \cos \hat{x} + \sin^2 \hat{x}}{\hat{x}^6} \right] \, dx .
$$

The use of several trigonometric identities yields

$$
I_{10} = \frac{1}{\pi} \left\{ \int_0^\infty \left[ M(x) + \frac{2}{1-Y^2} \right] \left[ \frac{x^2 + 1 + 2i x^3 + 4i x^5 + e^{2i x}(x^2 - 2ix + 1)}{x^6} \right] \, dx \right\}
$$

(223)

$$
+ \int_0^\infty \left[ M(x) + \frac{2}{1-Y^2} \right] \left[ \frac{x^2 + 1 + 2i x^3 + 4i x^5 + e^{-2i x}(x^2 - 2ix + 1)}{x^6} \right] \, dx \right\} .
$$

The integral (222) was broken up in this manner in order to preserve convergence. To reduce (223) to a sum of finite integrals we consider the contour integral.
\[ \int_{\Gamma} \left[ M(z) + \frac{2}{1 - \gamma^2} \right] \left[ 1 + \frac{\Lambda_2}{3} - \frac{\Lambda_3}{15} i^2 \Lambda_5^5 + e^{-2i\frac{\Lambda_2}{2} (z - 2iz - 1)} \right] \, dz \] (224)

where \( \Gamma \) is the contour shown in Figure 5. For associated branch cuts, we refer to the explanation following the contour integral (170). Since

\[ M(z) + \frac{2}{1 - \gamma^2} \sim -\frac{M_4}{z^2}, \quad z \to \infty, \quad \text{Im}(z) < 0, \] (225)

where \( M_4 \) is a complex constant,

\[ \lim_{T \to \infty} \int_{\Gamma_T} \left[ M(z) + \frac{2}{1 - \gamma^2} \right] \left[ 1 + \frac{\Lambda_2}{3} - \frac{\Lambda_3}{15} i^2 \Lambda_5^5 + e^{-2i\frac{\Lambda_2}{2} (z - 2iz - 1)} \right] \, dz (226) \]

\[ = 0. \]

By a procedure similar to the one used for the integral \( I_5 \), we obtain

\[ I_{10}'' = -\frac{2i}{\pi} \left\{ \int_0^\gamma K(x)q(x)dx + \int_0^\gamma K(x)q(x)dx - \frac{\sqrt{\gamma - \gamma_2}}{f'(s)} q'(s) \right\} \] (227)

where \( K(x) \) and \( q(x) \) are given in equations (87) and (88), respectively.

Finally, combining results (206), (218), (219), (220) and (227), we have (see equation (89))

\[ I_{10} = \frac{2}{\pi} \gamma^2 (1 - \gamma^2) \frac{d\gamma}{b} \left[ \int_0^1 K(x) \frac{q(x)}{x} \, dx - \frac{\sqrt{\gamma - \gamma_2}}{f'_1(s)} q(s) - \frac{2\pi}{15} \frac{1}{1 - \gamma^2} \right] \] (228)

where it has been noted that \( f'_1(s) = f'(s) \) and \( t = b/a \). This result agrees with (104).
The integral \( I_{11} \) will be used to illustrate Method 5. Upon using (20), (35) and (160), we have

\[
I_{11} = \int_0^a \int_0^a \frac{1}{\sqrt{1-\rho^2}} \vartheta_{zz}(\rho, \rho') \vartheta_{zz}(\rho, \rho') \rho \rho' \rho'^3 \rho'^3 d\rho d\rho'
\]

\[= - \frac{2dak}{\pi} \frac{1}{\sqrt{1-\rho^2}} \int_0^1 \int_0^1 \frac{u}{\sqrt{1-u^2}} \sqrt{t^2-u^2} \]

\[
\left[ \int_0^\infty (u^4) J_0(u) \right] \frac{v \sqrt{u} dv}{\sqrt{1-v^2}}
\]

where \( M(x) \) is given in (207). Changing the order of integration and using the identity (167) we have

\[
I_{11} = - \frac{2dak}{\pi} (1-\rho^2) \left[ \int_0^1 \frac{u}{\sqrt{1-u^2}} \sqrt{t^2-u^2} \right]
\]

\[
\int_0^\infty (u^4) J_0(u) \frac{\sin \frac{\pi}{x}}{\frac{\pi}{x}} dx \right] du.
\]

We proceed to reduce

\[
I'_{11} = \int_0^\infty M(x) J_0(u^4) \frac{\sin \frac{\pi}{x}}{\frac{\pi}{x}} dx, 0 \leq u \leq 1,
\]

to a sum of finite integrals. We first note that

\[
I'_{11} = \left( \frac{\pi}{2} \right)^{1/2} \int_0^\infty M(x) J_0(u^4) J_{1/2}(4u^4) \frac{1}{x^{1/2}} dx
\]
\[
= (\pi)^{1/2} \left\{ \int_0^\infty \left[ M(x) + \frac{2}{1-y^2} \right] J_0(u\hat{x}) J_{1/2}(\hat{x}) \hat{x}^{-1/2} \, dx \right. \\
- \frac{2}{1-y^2} \int_0^\infty J_0(u\hat{x}) J_{1/2}(\hat{x}) \hat{x}^{-1/2} \, dx \right\} .
\]

From [12] we obtain

\[
\int_0^\infty J_0(u\hat{x}) J_{1/2}(\hat{x}) \hat{x}^{-1/2} \, dx = \left( \frac{\pi}{2} \right)^{1/2} \frac{1}{ak} .
\] (233)

To treat the integral

\[
I_{11}'' = \int_0^\infty \left[ M(x) + \frac{2}{1-y^2} \right] J_0(u\hat{x}) J_{1/2}(\hat{x}) \hat{x}^{-1/2} \, dx ,
\] (234)

we use the integral representation [15]

\[
J_0(u\hat{x}) = \frac{2}{\pi} \int_0^1 \frac{\cos(\eta u\hat{x}) \eta d\eta}{\sqrt{1-\eta^2}} ,
\] (235)

and

\[
J_{1/2}(\hat{x}) = \left( \frac{2\hat{x}}{\pi} \right)^{1/2} \int_0^1 \cos(\eta \hat{x}) d\eta ,
\] (236)

to obtain, after changing the order of integration,

\[
I_{11}'' = \left( \frac{2}{\pi} \right)^{3/2} \int_0^1 \int_0^1 \int_0^\infty \frac{1}{\sqrt{1-\eta^2}} \left[ M(x) + \frac{2}{1-y^2} \right] \cos(\eta u\hat{x}) \cos(\hat{x}d\eta) \, dx \, d\xi .
\] (237)
The integral

\[
I_{11}'' = \int_0^\infty \left[ M(x) + \frac{2}{1-y^2} \right] \cos(u\eta x) \cos(wx) \, dx
\]

(238)

\[
= \frac{1}{4} \left\{ \int_0^\infty \left[ M(x) + \frac{2}{1-y^2} \right] \left[ e^{i(u\eta+w)x} + e^{i|u\eta-w|x} \right] \, dx
\right. \\
+ \int_0^\infty \left[ M(x) + \frac{2}{1-y^2} \right] \left[ e^{-i(u\eta+w)x} + e^{-i|u\eta-w|x} \right] \, dx \right\}
\]

can be reduced to a sum of finite integrals by considering the contour integral

\[
\int_\Gamma \left[ M(z) + \frac{2}{1-y^2} \right] \left[ e^{-i(u\eta+w)x} + e^{-i|u\eta-w|x} \right] \, dz ,
\]

(239)

where branch cuts are taken as in the contour integral (170) for \( \alpha(z) \) and \( \beta(z) \) (see Figure 5). The result is

\[
I_{11}'' = -\frac{i}{2} \left[ \int_0^1 K(x)R(\xi, \eta, w, u) \, dx - \pi \frac{\sqrt{2-2}}{f'(s)} R(s, \eta, w, u) \right] ,
\]

(240)

where

\[
R(\xi, \eta, w, u) = e^{i(u\eta+w)x} + e^{-i|u\eta-w|x} .
\]

(241)
We now note that

\[
\int_0^1 \int_0^1 \frac{R(\hat{x}, \hat{y}, x, u)}{1-\eta^2} \, d\eta \, dw = \int_0^1 \int_0^1 \frac{e^{i(u\eta^2-w)}}{1-\eta^2} \, d\eta \, dw + \int_0^1 \int_0^1 \frac{e^{i\eta \hat{x}^2 + iwx^2}}{1-\eta^2} \, d\eta \, dw
\]

\[
= \left[ \frac{e^{i\hat{x}^2}}{i\hat{x}} \right] \left[ \frac{\pi}{2} J_0(u\hat{x}) + i \frac{\pi}{2} H_0(u\hat{x}) \right] + \frac{i\pi}{\hat{x}}
\]

\[
+ \frac{1}{i\hat{x}} \left[ J_0(u\hat{x}) + i H_0(u\hat{x}) \right] + \frac{e^{i\hat{x}^2}}{i\hat{x}} \left[ \frac{\pi}{2} J_0(u\hat{x}) - i \frac{\pi}{2} H_0(u\hat{x}) \right]
\]

\[
= \frac{\pi}{i\hat{x}} \left[ e^{i\hat{x}^2} J_0(u\hat{x}) - 1 \right],
\]

where \( H_0(u\hat{x}) \) is the Struve function with argument \( u\hat{x} \). [19]

Combining (232), (233), (241) and (242), we have

\[
I_{11} = -\frac{1}{\pi} \left[ \int_0^1 K(x) V(\hat{x}, u) \, dx - \pi \frac{V(\hat{u}, u)}{f'(\hat{u})} \right] - \frac{\pi}{1-\gamma^2} \frac{1}{ak}
\]

(243)

where

\[
V(\hat{x}, u) = i \pi [1 - e^{i\hat{x}^2} J_0(u\hat{x})].
\]
Finally

\[ I_{11} = -\frac{2a}{\pi} (1-\gamma^2) \sqrt{t^2-1} \]  \hspace{1cm} (244)

\[ \cdot \int_0^1 \frac{u}{\sqrt{1-u^2}} \left\{ -\frac{i}{\pi} \int_0^1 \frac{K(x)V(x,u)dx}{x} - \frac{\sqrt{s^2-\gamma^2} V(s,u)}{f''(s)} \right\} \]

\[ -\frac{\pi}{1-\gamma^2} \}

The integrals \( I_{14}, I_{15} \) and \( I_{16} \) are related to the integrals \( I_9, I_{11}, I_{12} \) and \( I_{13} \) as follows.

\[ I_{14} = \int_0^a \int_0^a \int_{-\infty}^{+\infty} \frac{1}{r_{zz}(\rho', b) \partial_z(\rho', \rho) \frac{3}{2} \partial_z(\rho) \rho \rho' d\rho d\rho'} \]  \hspace{1cm} (245)

\[ = A \int_0^1 \int_0^1 \frac{u^3}{\sqrt{1-u^2}} \left[ \int_0^\infty M(x) J_0 (ux) J_0 (vx) dx \right] \frac{v}{\sqrt{1-v^2}} dudv \]

\[ = A \int_0^1 \int_0^1 \frac{u}{\sqrt{1-u^2}} \left[ \int_0^\infty M(x) J_0 (ux) J_0 (vx) dx \right] \frac{v^3}{\sqrt{1-v^2}} dudv \]

\[ = \int_0^a \int_0^a \int_{-\infty}^{+\infty} \frac{1}{r_{zz}(\rho', b) \partial_z(\rho', \rho) \frac{3}{2} \partial_z(\rho) \rho \rho' d\rho d\rho'} = I_{13} \].

(Here \( A \) is a constant.)

\[ I_{15} = \int_0^a \int_0^a \int_{-\infty}^{+\infty} \frac{1}{r_{zz}(\rho', b) \partial_z(\rho', \rho) \frac{3}{2} \partial_z(\rho) \rho (\rho')^3 d\rho d\rho'} \]  \hspace{1cm} (246)
\[= -\frac{2}{\pi \sqrt{b^2-a^2}} \int_0^a \int_0^a \frac{\rho^3}{\sqrt{a^2-\rho^2}} \frac{1}{\tau_{zz}(\rho',\rho)^\frac{1}{2}} \frac{1}{\tau_{zz}(\rho) \rho \rho' \varphi \varphi'} d\rho d\rho'\]

\[= b^2 \int_0^a \int_0^a 1 \frac{1}{\tau_{zz}(\rho',\rho)^\frac{1}{2}} \frac{1}{\tau_{zz}(\rho) \rho \rho' \varphi \varphi'} d\rho d\rho'\]

\[+ \frac{a^2 b \sqrt{b^2-a^2}}{\gamma^2} \int_0^a \int_0^a 1 \frac{1}{\tau_{zz}(\rho',\rho)^\frac{1}{2}} \frac{1}{\tau_{zz}(\rho) \rho \rho' \varphi \varphi'} d\rho d\rho'\]

\[= b^2 I_{11} + \frac{a^2 b \sqrt{b^2-a^2}}{\gamma^2} I_9,\]

which agrees with (109). The result (110) is derived in a similar manner.
APPENDIX C

SOME PROPERTIES OF THE ADMISSIBLE FUNCTIONS

The admissible functions $a_z^a(\rho)$, $a_z^{*a}(\rho, b)$ and $a_z^{*b}(\rho, b)$ were constructed so that they possessed certain properties of the corresponding exact stress distributions.

It is well known that the exact stress distributions $\tau_{zz}(\rho)$, $\tau_{zz}^{*a}(\rho, b)$ and $\tau_{zz}^{*b}(\rho, b)$, $0 \leq \rho < a, b > a$, each have a singularity of the form $\frac{1}{\sqrt{a^2 - \rho^2}}$ at the edge of the disc.

We also claim that the these exact stress distributions are even functions of $\rho$, and we will now verify this.

In order to determine $\tau_{zz}(\rho)$, $0 \leq \rho < a$, we must solve the boundary value problem \((a, b, c, d)\) modified for our axisymmetric problem. That is, we must consider the boundary conditions

\[
\tau_{rz}(\rho) = \frac{1}{\sqrt{a^2 - \rho^2}} \quad \text{at the edge of the disc.}
\]

\[
\tau_{zz}(\rho) = 0, \rho > a
\]

on $B = \{(x, y, z): z = 0\}$ where $\tau_{rz}(\rho)$ is the shear stress at $(\rho, \varphi, 0)$. From [20] these conditions are equivalent to
\[ - \mu k^3 \int_0^\infty \left[ 2A(kx)x^2 + C(kx)(2x^2 - 1) \right] J_1(\rho kx) dx = 0, \quad \rho \geq 0 , \quad (250) \]

\[ k^2 \int_0^\infty x[A(kx) + C(kx)]J_0(\rho kx) dx = d, \quad 0 \leq \rho < a , \quad (251) \]

and

\[ - \mu k^3 \int_0^\infty \frac{A(kx)(2x^2 - 1)}{\alpha(x)} + 2\beta(x)C(kx) J_0(\rho kx) dx = 0 , \quad \rho > a , \quad (252) \]

respectively, for the functions \( A(kx) \) and \( C(kx) \). Equation (250) implies that

\[ 2A(kx)x^2 + C(kx)(2x^2 - 1) = 0 . \quad (253) \]

We obtain from equations (251), (252) and (253) the dual integral equations

\[ k^2 \int_0^\infty \frac{x}{2x^2 - 1} J_0(\rho kx)A(kx) dx = d, \quad 0 \leq \rho < a , \quad (254) \]

and

\[ - \mu k^3 \int_0^\infty x \left[ \frac{2x^2 - 1}{\alpha(x)} - \frac{4x \beta(x)}{2x^2 - 1} \right] J_0(\rho kx)A(kx) dx = 0 , \quad \rho > a . \quad (255) \]

We extract \( \tau_{zz}(\rho) \) [20] as

\[ \tau_{zz}(\rho) = - \mu k^3 \int_0^\infty x \left[ \frac{2x^2 - 1}{\alpha(x)} + \frac{4x \beta(x)}{1 - 2x^2} \right] J_0(\rho kx)A(kx) dx . \quad (256) \]

Since \( J_0(y) \) is an even function of its argument \( y \), \( \tau_{zz}(\rho) \) is an even function of \( \rho \).
The function $\tau_{zz}^*(\rho,b)$ is obtained in a similar manner by first considering the boundary value problem (1a,b,c,d) for axisymmetric problems with the associated axisymmetric boundary conditions

$$\tau_{zz}^*(\rho,b) = 0, \rho \geq 0,$$  \hspace{1cm} (257)

$$\tau_{zz}^*(\rho,b) = f(\rho), \rho > a,$$  \hspace{1cm} (258)

and

$$u_z^*(\rho,b) = 0, 0 \leq \rho < a,$$  \hspace{1cm} (259)

where $f(\rho)$ is a generalized function representing the stress distribution due to a ring of unit point forces applied at $\rho = b$ in the direction. By an analysis similar to the one given for $\tau_{zz}(\rho)$ we obtain

$$\tau_{zz}^*(\rho,b) = -\mu k^3 \int_0^\infty x \left[ \frac{2x^2}{\alpha(x)} - 1 + \frac{4x^2B(x)}{1 - 2x^2} \right] A_1(kx)J_0(\rho kx)dx ,$$  \hspace{1cm} (260)

where the function $A_1(kx)$ is determined by the boundary value problem. Hence again since $J_0(y)$ is an even function of its argument $y$, $\tau_{zz}^*(\rho,b)$ is an even function of $\rho$. By a similar argument, we can also show that $\tau_{zz}^*(\rho,b)$ is an even function of $\rho$. 
APPENDIX D

COMPUTER PROGRAMS USED TO COMPUTE THE

VARIATIONAL APPROXIMATIONS
PROGRAM UZ20(INPUT, OUTPUT, TAPE5=INPUT, TAPE6=OUTPUT)

COMPLEX R5, R6, RG, RH, RR1, RR2, RI, RJ, ANUM1, ANUM2, DEN

COMPLEX VAR1, VAR2

C THIS PROGRAM COMPUTES THE FIRST AND SECOND VARIATIONAL APPROXIMATIONS FOR UZ

G = 0.57735
S = 1.0876653874
READ *, N
WRITE (6, 87)

87 FORMAT(*APPROXIMATION FOR UZ*)

DO 310 K=1, N
READ *, T, AK

PI = 3.1415926536

C COMPUTE THE INTEGRAL -I7
CALL S5(AK, T, G, S, R5)

C COMPUTE THE INTEGRAL -I11
CALL S6(AK, T, G, S, R6)

C COMPUTE THE INTEGRAL -I3
R7 = 2. * ASIN(1. / T) / PI

C COMPUTE THE INTEGRAL I4
RF = (2. / PI) * (T * ASIN(1. / T) - SQRT(T * T - 1.))

C COMPUTE THE INTEGRAL -I8
CALL S8(AK, T, G, S, RG)

C COMPUTE THE INTEGRAL -I12
CALL S12(AK, T, G, S, RR1)
CALL S12(AK, T, G, S, RR2)

C COMPUTE THE INTEGRAL -I15
RI = T * T * R6 - 2. * RR1 * SQRT(T * T - 1.) / PI

C COMPUTE THE INTEGRAL -I16
RJ = T * T * RH - 2. * RR2 * SQRT(T * T - 1.) / PI
ANUM1 = R5 * RH * RF - R5 * R7 * RJ
ANUM2 = R7 * RG * RI - R6 * RF * RG
DEN = RI * RH - R5 * RJ

C COMPUTE THE FIRST VARIATIONAL APPROXIMATION
VAR1 = R5 * R7 / R6

C COMPUTE THE SECOND VARIATIONAL APPROXIMATION
VAR2 = (ANUM1 + ANUM2) / DEN

WRITE (6, 11) AK, T, G

WRITE (6, 12) VAR1, VAR2

12 FORMAT(1X, *VAR1=*, 2F10.5, *VAR2=*, 2F10.5)
A1 = CABS(VAR1)
A2 = CABS(VAR2)
WRITE (6, 147) A1, A2


310 CONTINUE
END
SUBROUTINE S5(AK,T,G,S,F)

COMPLEX AI,S1,S2,S3,S8,H0,RES,F
PI=3.1415926536
AI=CMPLX(0.0,1.0)
CALL SI1(AK,T,G,S1)
CALL SI2(AK,T,G,S2)
CALL SI3(AK,T,G,S3)
X=AK*T*S
X1=(1.0,0.0)*X
CALL BESSI(0,X1,2,BJ,BY,H0,H2)
A=SQRT(S*S-G*G)
B=SQRT(S*S-1.)
DEN=2*S*(2*S**2-1.)-2*S*A*B-S**3*A/B-S**3*B/A
RES=PI*AI*H0*SIN (AK*S)*A/DEN
S8=-S1-AI*S3+RES-AI*S2
F=(1.0-G*G)*S8/PI
RETURN
END
SUBROUTINE SI1(AK,T,G,R)
REAL I11,I12
PI=3.1415926536
M=50
MM=M-1
I11=0.0
P=.02/EXP(1.)
CALL HINT(AK,T,G,P,S1)
R=.02*S1
CALL HINT(AK,T,G,.02,S1)
DO 105 I=1,MM,2
Y2=.02+.98*I/M
Y3=.02+.98*(I+1)/M
CALL HINT(AK,T,G,Y2,S2)
CALL HINT(AK,T,G,Y3,S3)
I11=I11+(S1+4*S2+S3)
S1=S3
105 CONTINUE
R=I11*.98/(3*M)+R
IF (.50.-AK*T) ,LT. 0.0) GO TO 123
F=10./(AK*T)
M=INT(F*10)
MM=M-1
H=(F-1.)/M
DO 205 I=1,MM,2
AM =M
Y2=1.+I*H
Y3=1.+(I+1)*H
CALL HINT(AK, T, G, Y2, S2)
CALL HINT(AK, T, G, Y3, S3)
R = R + (S1 + 4*S2 + S3)*H/3
S1 = S3
205 CONTINUE
123 R = R/PI
RETURN
END

SUBROUTINE HINT(AK, T, G, TAU, BINT)
COMPLEX S2, S21, S22, S23, HO, X1, J1, Y, H2
AI = CMPLX(0.0, 1.0)
PI = 3.1415926536
V = AK*TAU
A = SQRT(TAU*TAU+G*G)
B = SQRT(TAU*TAU+1.0)
DEN = (TAU*TAU+.5)**2 - TAU*TAU*A*B
AEXP = EXP(V) - EXP(-V)
S1 = V*T
S2 = AI*S1
CALL BESSI(0, S2, 2*BJ, BY, H1, H2)
BK = REAL(.5*PI*AI*H1)
ANUM = BK*AEXP*A
BINT = ANUM/DEN
960 RETURN
END

SUBROUTINE S12(AK, T, G, S2)
COMPLEX S2, S21, S22, S23, HO
COMPLEX X1, J1, Y, H2
M = 50
MM = M - 1
S2 = 0.0
S21 = 0.0
DO 80 J = 1, MM, 2
Y2 = G + .05*(.30 - G)*J/M
Y3 = G + .05*(.30 - G)*(J+1)/M
CALL CINT(AK, T, G, Y2, S22)
CALL CINT(AK, T, G, Y3, S23)
S2 = S2 + S21 + 4*S22 + S23
S21 = S22
80 CONTINUE
S2 = S2*(.90 - G)/(3*N)
N = 50
CALL CINT(AK, T, G, G, S21)
DO 180 J = 1, N
Y2 = G + .05*J/N
CALL CINT(AK, T, G, Y2, S22)
S2 = S2 + .025*(S21 + S22)/N
S21 = S22
180 CONTINUE
S21=0.0
DO 195 J=1,N
  Y2=1.0-.05*J/N
  CALL CINT(AK,T,G,Y2,S22)
  S2=S2+.025*(S21+S22)/N
  S21=S22
195 CONTINUE
RETURN
END

SUBROUTINE CINT(AK,T,G,X,S2)
COMPLEX ANUM2,S2,S1,RY,RJ,H0,H2
S1=AK*T*X*(1.0,0.0)
CALL BESSI(0,S1,2,RJ,RY,H0,H2)
ANUM2=H0*SIN(AK*X)*X*X*SQRT(1.-X*X)*(X*X-G*G)
DEN2=(X*X-.5)**4+X**4*(1.-X*X)*(X*X-G*G)
S2=ANUM2/DEN2
RETURN
END

SUBROUTINE S13(AK,T,G,S3)
COMPLEX S31,S32,S33,S3,R,S1
M=50
MM=M-1
S=.02/EXP(1.)
CALL DIM(AK,T,G, S,S1)
R=.02*S1
CALL DINT(AK,T,G,.02,S31)
DO 100 J=1,MM,2
  Y2=(G-.07)*J/MM+.02
  Y3=(G-.07)*(J+1.)/MM+.02
  CALL DINT(AK,T,G,Y3,S33)
  CALL DINT(AK,T,G,Y2,S32)
  R=R+S31+4*S32+S33
  S31=S33
100 CONTINUE
S3=R*(G-.07)*(S31/MM)
200 CONTINUE
S3=R*(G-.07)/(3*M)

200 CONTINUE
M=50
X=G-.05
CALL DINT(AK,T,G,X,S31)
DO 220 I=1,49
  Y2=G-.05+.05*I/M
  CALL DINT(AK,T,G,Y2,S32)
  S3=S3+.025*(S31+S32)/M
  S31=S32
220 CONTINUE
300 CONTINUE
RETURN
SUBROUTINE DINT(AK,T,G,X,S3)
COMPLEX HO,ANUM,S3,Y1,BJ,BY,H2
Y=AK*T*X
Y1=(1.0+0.0)*Y
CALL BESSI(0,Y1,2,BJ,BY,HO,H2)
ANUM=HO*SIN(AK*X)*SQRT(G*G-X*X)
DEN=(.5-X*X)**2+X*X*SQRT(G*G-X*X)
S3=ANUM/DEN
RETURN
END

SUBROUTINE S6(AK,T,G,S,R)
COMPLEX R,Y1,Y2,Y3,Z1,Z2,Z3
PI=3.1415926535
H=PI/100.
R=0.0
X1=0.0
CALL F5(AK,G,X1,S,Z1)
Y1=0.0
DO 50 I=1,50,2
   X2=SIN(I*H)
   X3=SIN((I+1)*H)
   CALL F5(AK,G,X2,S,Z2)
   CALL F5(AK,G,X3,S,Z3)
   Y2=Z2*X2/(T*T-X2*X2)
   Y3=Z3*X3/(T*T-X3*X3)
   R=R*H*(Y1+4*Y2+Y3)/3
   Y1=Y3
50 CONTINUE
R=(2./PI)*SQRT(T*T-1.)*R
RETURN
END

SUBROUTINE F5(AK,G,U,S,R)
COMPLEX AI,R,Y23,Y33,SE,ANUM,RES,Y1,Y2,Y3
PI=3.1415926536
H=.001
Y1=0.0
R=0.0
AI=CMPLX(0.0,1.0)
DO 100 I=1,50
   X2=.001*I
   CALL AX5(G,X2,Y22)
   CALL ES5(AK,X2,U,Y23)
   Y2=Y22*Y23
   R=R+.5*H*(Y1+4*Y2+Y3)/3
   Y1=Y3
100 CONTINUE
H=(G-.1)/50
DO 200 J=1,50,2
X2=.05+J*H
X3=.05+(J+1)*H
CALL AX5(G,X2,Y22)
CALL ES5(AK,X2,U,Y23)
Y2=Y22*Y23
CALL AX5(G,X3,Y32)
CALL ES5(AK,X3,U,Y33)
Y3=Y32*Y33
R=R+(H/3.)*(Y1+Y2+Y3)
Y1=Y3
200 CONTINUE
H=.001
DO 300 I=1,100
X2=G-.05+I*H
X3=G-.05+(I+1)*H
CALL AX5(G,X2,Y22)
CALL ES5(AK,X2,U,Y23)
Y2=Y22*Y23
R=R+.5*H*(Y1+Y2)
Y1=Y2
300 CONTINUE
H=(G-G)/50
DO 400 I=1,50,2
X2=G+.05+I*H
X3=G+.05+(I+1)*H
CALL AX5(G,X2,Y22)
CALL ES5(AK,X2,U,Y23)
Y2=Y22*Y23
CALL AX5(G,X3,Y32)
CALL ES5(AK,X3,U,Y33)
Y3=Y32*Y33
R=R+(H/3.)*(Y1+Y2+Y3)
Y1=Y3
400 CONTINUE
H=.001
DO 500 I=1,50
X2=.95+I*H
CALL AX5(G,X2,Y22)
CALL ES5(AK,X2,U,Y23)
Y2=Y22*Y23
R=R+.5*H*(Y1+Y2)
Y1=Y2
500 CONTINUE
CALL ES5(AK,S,U,SE)
A=SQRT(S*S-G*G)
B=SQRT(S*S-1.)
ANUM=PI*A*SE
ACEN=2.*(S*S-5)*2.*S-2*S*A*B-S**3*A/3-S**3*B/A
RES=ANUM/ADEN
R=(1.-G*G)*AI*(R-RES)/(PI*PI)
R=-R-1.
RETURN
END
SUBROUTINE AX5(G,X,Y)
IF (X .GE. G) GO TO 600
A=SQRT(G*G-X*X)
B=SQRT(1.-X*X)
ANUM=A
ADEN=(X*X-.5)**2+X*X*A*B
Y=ANUM/ADEN
GO TO 700

600 ANUM =X**2*(X*G-G)*SQRT(1.-X*X)
ADEN=(X*X-.5)**4+X**4*(X*G*G)*(1.-X*X)
Y=ANUM/ADEN

700 CONTINUE
RETURN
END
SUBROUTINE ES5(AK,X,U,Y)
COMPLEX UH1,J,8Y,H1,H2
COMPLEX E,Y,AI
PI=3.1415926536
AI=CMPLX(G,0,1,0)
XH=AK*X
E=CEXP(AI*XH)
UH=U*XH
UH1=(1.0,0.0)*UH
CALL BESSI(0,UH1,1,J,BY,H1,H2)
BJ=REAL(J)
Y=AI*PI*(1.-E*BJ)
RETURN
END
SUBROUTINE SG(AK,T,G,S,F)
COMPLEX X1,BJ,8Y,H2
COMPLEX AI,S1,S2,S3,S5,H0,RES,F
PI=3.1415926536
AI=CMPLX(G,0,1,0)
CALL SI4(AK,T,G,S1)
CALL SI5(AK,T,G,S2)
CALL SI6(AK,T,G,S3)
X=AK*T*S
X1=AI*X
CALL BESSI(0,X1,2,BJ,BY,H1,H2)
A=SQRT(S*S-G*G)
B=SQRT(S*S-1.)
DEN=2*S*(2*S**2-1.)-2*S*A*B-S**3*A/B-S**3*B/A
SH=AK*S
Q\((\text{SH} + \text{SH} \cdot \text{SH} - 1.0) \cdot \text{SIN(\text{SH})}) / (\text{SH} \cdot \text{SH})

RES = \pi \cdot \text{AI} \cdot \text{HO} \cdot \text{Q} / \text{AEN}

S5 = -S1 - \text{AI} \cdot S3 + \text{RES} - \text{AI} \cdot S2

F = (1.0 - G \cdot G) \cdot S5 / \pi

RETURN

END

SUBROUTINE SIM AK\(, T, G, R \)

REAL T11, T12

PI = 3.1415926536

M = 50

MM = M - 1

T11 = 0.0

DO 105 I = 1, MM, 2

Y2 = 1.0 * I / M

Y3 = 1.0 * (I + 1) / M

CALL HINT1\(, T, G, Y2, S2 \)

CALL HINT1\(, T, G, Y3, S3 \)

T11 = T11 + S1 * (I + 1) * S2 - S3

S1 = S3

105 CONTINUE

R = T11 / (3 + M)

IF ((50.0 - A \cdot T) \lt 0.00) GO TO 127

F = 10.0 / (A \cdot T)

M = INT\(\cdot F \cdot 10 \)

MM = M - 1

H = (F - 1.0) / M

DO 205 I = 1, MM, 2

AM = M

Y2 = 1.0 + I \cdot H

Y3 = 1.0 + (I + 1) \cdot H

CALL HINT1\(, T, G, Y2, S2 \)

CALL HINT1\(, T, G, Y3, S3 \)

R = R + (S1 + 4.0 * S2 + S3) * H / 3

S1 = S3

205 CONTINUE

127 R = 2.0 * R / \pi

RETURN

END

SUBROUTINE HINT1\(, T, G, TAU, BINT \)

COMPLEX S2, BJ, BY, HO, H2, AI

AI = CMPLX\(0.0, 1.0 \)

PI = 3.1415926536

V = A \cdot TAU

A = SQRT\(\cdot TAU \cdot TAU + G \cdot G \)

B = SQRT\(\cdot TAU \cdot TAU + 1.0 \)

DEN = (TAU \cdot TAU + 5.0) ** 2 - TAU \cdot TAU \cdot A \cdot B

AEXP = (\cdot V \cdot COSH(V) - (V + V + 1.0) \cdot SINH(V)) / (V + V)
S1=V*T
S2=AI*S1
CALL BESSI(0,S2,2,BJ,BY,H0,H2)
BK=REAL(.5*PI*AI*HO)
ANUM=BK*AEXP*A
BINT=ANUM/DEN
960 RETURN
END
SUBROUTINE S15(AK,T,G,S2)
COMPLEX S2,S21,S22,S23
M=50
MM=M-1
S2=0.0
X=G+.05
CALL CIKT1(AK,T,G,X,S21)
DO 80 J=1,MM,2
   Y2=G+.05+(.90-G)*J/M
   Y3=G+(.90-G)*J/(J+1)/M
   CALL CIKT1(AK,T,G,Y2,S22)
   CALL CIKT1(AK,T,G,Y3,S23)
   S2=S2+S21+S22+S23
   S21=S22
80 CONTINUE
S2=S2* (.90-G)/(3*M)
N=50
AI=CMPLX(0.0,1.0)
CALL CIKT1(AK,T,G,G,S21)
DO 180 J=1,N
   Y2=G+.05*J/N
   CALL CIKT1(AK,T,G,Y2,S22)
   S2=S2+S21+S22+N
   S21=S22
180 CONTINUE
S21=0.0
DO 195 J=1,N
   Y2=1.0-.05*J/N
   CALL CIKT1(AK,T,G,Y2,S22)
   S2=S2+S21+S22/N
   S21=S22
195 CONTINUE
RETURN
END
SUBROUTINE CIKT1(AK,T,G,X,S2)
COMPLEX ANUM2,S2,S3,BJ,BY,H0,H2
S1=AK*X
S3=(1.0,0.0)*S1
CALL BESSI(0,S3,2,BJ,BY,H0,H2)
XH=AK*X
Q = XH * COS(XH) + (XH * XH - 1.1) * SIN(XH)
ANUM2 = HO * Q * X * X * SQRT(1. - X * X) * (X * X - G * G)
DEN2 = ((X * X - 5.1) ** 4 + X * X * 4 * (1. - X * X) * (X * X - G * G)) * XH * XH
S2 = ANUM2 / DEN2
RETURN
END
SUBROUTINE SI6(AK, T, G, S3)
COMPLEX S31, S32, S33, S3, R
M = 50
MM = M - 1
R = 0.0
S31 = 0.0
DO 100 J = 1, MM, 2
Y2 = (G -.05) * J / M
Y3 = (G -.05) * (J + 1.) / M
CALL DINT1(AK, T, G, Y3, S33)
CALL DINT1(AK, T, G, Y2, S32)
R = R + 4. * S32 * S33 + S31
S31 = S33
100 CONTINUE
S3 = R * (G -.07) / (3 * M)
200 CONTINUE
M = 50
X = G -.05
CALL DINT1(AK, T, G, X, S31)
DO 220 I = 1, 49
Y2 = G -.05 + .05 * I / M
CALL DINT1(AK, T, G, Y2, S32)
S3 = S3 + .025 * (S31 + S32) / M
S31 = S33
220 CONTINUE
300 CONTINUE
RETURN
END
SUBROUTINE DINT1(AK, T, G, X, S3)
COMPLEX HO, ANUM, S3, Y1, BJ, BY, H2
Y = AK * T * X
Y1 = (1.0, 0.0) * Y
CALL BESSI(0, Y1, 2, BJ, BY, HO, H2)
XH = AK * X
Q = XH * COS(XH) + (XH * XH - 1.1) * SIN(XH)
ANUM = HO * Q * SQRT(G * G - X * X)
DEN = ((.5 - X * X) ** 2 + X * X * SQRT(1. - X * X) * SQRT(G * G - X * X)) * XH * XH
S3 = ANUM / DEN
RETURN
END
SUBROUTINE SH(AK, T, G, S, R)
COMPLEX R, Y1, Y2, Y3, Z1, Z2, Z3
PI = 3.1415926536
H = PI / 100.
R = 0.0
X1 = 0.0
CALL F(AK, G, X1, S, Z1)
Y1 = 0.0
DO 50 I = 1, 50, 2
X2 = SIN(I * H)
X3 = SIN((I + 1) * H)
CALL F(AK, G, X2, S, Z2)
CALL F(AK, G, X3, S, Z3)
Y2 = Z2 * X2 / (T * T - X2 * X2)
Y3 = Z3 * X3 / (T * T - X3 * X3)
R = R + H * (Y1 + 4 * Y2 + Y3) / 3
Y1 = Y3
50 CONTINUE
R = 2.0 * R * SQRT(T * T - 1) / PI
RETURN
END
SUBROUTINE F(AK, G, U, S, R)
COMPLEX AI, R, Y23, Y33, SE, ANUM, RES, Y1, Y2, Y3
PI = 3.1415926536
H = 0.01
Y1 = 0.0
R = 0.0
AI = CMPLX(0.0, 1.0)
DO 100 I = 1, 50
X2 = 0.001 * I
CALL AX(G, X2, Y22)
CALL ES(AK, X2, U, Y23)
Y2 = Y22 * Y23
R = R + 0.5 * H * (Y1 + Y2)
Y1 = Y2
100 CONTINUE
H = (G - L) / 50
DO 200 J = 1, 50, 2
X2 = 0.05 + J * H
X3 = 0.05 + (J + 1) * H
CALL AX(G, X2, Y22)
CALL ES(AK, X2, U, Y23)
Y2 = Y22 * Y23
CALL AX(G, X3, Y32)
CALL ES(AK, X3, U, Y33)
Y3 = Y32 * Y33
R = R + (H / 3) * (Y1 + 4 * Y2 + Y3)
Y1 = Y3
200 CONTINUE
H = 0.001
DO 300 I=1,100
X2=G+.05*I*H
CALL AX(G,X2,Y22)
CALL ES(AK,X2,U,Y23)
Y2=Y22*Y23
R=R+.5*H*(Y1+Y2)
Y1=Y2
300 CONTINUE
H=(.9-G)/50
DO 400 I=1,50,2
X2=G+.05*I*H
X3=G+.05+(I+1)*H
CALL AX(G,X2,Y22)
CALL ES(AK,X2,U,Y23)
Y2=Y22*Y23
CALL AX(G,X3,Y32)
CALL ES(AK,X3,U,Y33)
Y3=Y32*Y33
R=R+(H/3.)*(Y1+Y2+Y3)
Y1=Y3
400 CONTINUE
H=.001
DO 500 I=1,50
X2=1.95+I*H
CALL AX(G,X2,Y22)
CALL ES(AK,X2,U,Y23)
Y2=Y22*Y23
R=R+.5*H*(Y1+Y2)
Y1=Y2
500 CONTINUE
CALL ES(AK,S,U,SE)
A=SQRT(S*S-G*G)
B=SQRT(S*S-1.)
ANUM=PI*S*A*SE
ADEN=(S*S-.5)**2*S-2*S*A*B-S**3*A/B-S**3*B/A
RES=ANUM/ADEN
R=(1.-G*G)*.5*AI*AK*(R-RES)/PI
R=(-R+(1.*.5*U*U))/2.
RETURN
END
SUBROUTINE AX(G,X,Y)
IF (X.GE. G)GO TO 600
A=SQRT(G*G-X*X)
B=SQRT(1.-X*X)
ANUM=X*A
ADEN=(X*X-.5)**2+X*X*A*B
Y=ANUM/ADEN
GO TO 700
ANUM = X**3*(X*X-G*G)*SQRT(1.- X*X)
ADEN = (X*X-.5)**4+X**4*(X*X-G*G)*(1.-X*X)
Y = ANUM/ADEN

CONTINUE
RETURN
END

SUBROUTINE ES(AK,X,U,Y)
COMPLEX UH1,J,BY,H1,H2
COMPLEX AI,Y,A1,A2,A3,A4,Y1
PI = 3.1415926536
AI = CMPLX(0.0,1.0)
XH = AK*X
IF (XH .LE. .0004) GO TO 17
A1 = -2./(AI*XH)-U*U/(AI*XH)
UH = XH*U
UH1 = -1.0*UH
CALL BESSI(0.,UH1,1.,J,BY,H1,H2)
BJ = REAL (J)
A2 = CEXP (AI*XH)
A3 = 2./(AI*XH)+2./(XH*XH)+2.*AI/(XH*XH)
A4 = 2.*A2*A3*B
Y = A1 + A4
GO TO 18
17 CONTINUE
C1 = 8./3.
C2 = -.5+U*U/2.+U**4/16.
C3 = 8./15.-2*U*U/3.
Y = C1+AI*C2*XH+C3*XH**3
GO TO 18
18 CONTINUE
RETURN
END

SUBROUTINE SRKAK(U,G,S,R)
COMPLEX AI,R,Y2,Y3,R2,Y1,Y2,Y3
PI = 3.1415926536
H = .001
Y1 = 0.0
R = 0.0
AI = CMPLX(0.0,1.0)
DO 100 I=1,50
X2 = .001*I
CALL AX2(G,X2,Y22)
CALL ES2(AK,X2,U,Y23)
Y2 = Y22*Y23
R = R+.5*H*(Y1+Y2)
Y1 = Y2
100 CONTINUE
H = (G-.1)/50
DO 200 J=1,50,2

END
X2 = 0.05 + J*H
X3 = 0.05 + (J+1)*H
CALL AX2(G,X2,Y22)
CALL ES2(AK,X2,U,Y23)
Y2 = Y22*Y23
CALL AX2(G,X3,Y32)
CALL ES2(AK,X3,U,Y33)
Y3 = Y32*Y33
R = R + (H/3.)*(Y1 + Y2 + Y3)
Y1 = Y3

200 CONTINUE
H = .001
DO 300 I = 1, 100
X2 = G - 0.05 + I*H
CALL AX2(G,X2,Y22)
CALL ES2(AK,X2,U,Y23)
Y2 = Y22*Y23
R = R + 0.5*H*(Y1 + Y2)
Y1 = Y2

300 CONTINUE
H = (1.9 - G)/50
DO 400 I = 1, 50, 2
X2 = G - 0.05 + I*H
X3 = G - 0.05 + (I+1)*H
CALL AX2(G,X2,Y22)
CALL ES2(AK,X2,U,Y23)
Y2 = Y22*Y23
CALL AX2(G,X3,Y32)
CALL ES2(AK,X3,U,Y33)
Y3 = Y32*Y33
R = R + (H/3.)*(Y1 + Y2 + Y3)
Y1 = Y3

400 CONTINUE
H = .001
DO 500 I = 1, 50
X2 = .95 + I*H
CALL AX2(G,X2,Y22)
CALL ES2(AK,X2,U,Y23)
Y2 = Y22*Y23
R = R + 0.5*H*(Y1 + Y2)
Y1 = Y2

500 CONTINUE
CALL ES2(AK,S,U,SE)
A = SQRT(S*S - G*G)
B = SQRT(S*S - 1.)
ANUM = PI*A*SE
ADEN = 2.0*(S*S - 5)*2.0*S - 2.0*S*A*B - S**3*A/B - S**3*B/A
RES = ANUM/ADEN
R = -1.0 * (1.0 - G * G) * 5.0 * AI * (R = RES) / PI
RETURN
END

SUBROUTINE AX2(G, X, Y)
IF (X .GE. G) GO TO 600
A = SQRT(G * G - X * X)
B = SQRT(1.0 - X * X)
ANUM = A
ADEN = (X * X - .5) ** 2 + X * X * A * B
Y = ANUM / ADEN
GO TO 700
600 ANUM = X ** 2 * (X * X - G * G) * SQRT(1.0 - X * X)
ADEN = (X * X - .5) ** 4 + X ** 4 * (X * X - G * G) * (1.0 - X * X)
Y = ANUM / ADEN
700 CONTINUE
RETURN
END

SUBROUTINE ES2(AK, X, U, Y)
COMPLEX AI, E, Y
AI = CMPLX(0.0, 1.0)
XH = AK * X
E = CEXP(2.0 * AI * XH)
Y = (1.0 - E) / XH
RETURN
END

SUBROUTINE SR2(AK, U, G, S, R)
COMPLEX R, Y1, Y2, Y3, Z1, Z2, Z3
PI = 3.1415926536
H = PI / 100.0
R = 0.0
X1 = 0.0
CALL F(AK, G, X1, S, Z1)
Y1 = 0.0
DO 52 I = 1, 50, 2
X2 = SIN(I * H)
X3 = SIN((I + 1) * H)
CALL F(AK, G, X2, S, Z2)
CALL F(AK, G, X3, S, Z3)
Y2 = Z2 * X2
Y3 = Z3 * X3
R = R + H * (Y1 + 4 * Y2 + Y3) / 3.
Y1 = Y3
52 CONTINUE
R = -1.0 * R
RETURN
END

SUBROUTINE BESSI(ORDER, Z, M, J, Y, H1, H2)
INTEGER ORDER
COMPLEX H1,H2,J,P,Q,T,TO,Z,Z1,A,Y
DIMENSION AJ(2,8),AP(2,6),AQ(2,6),AY(2,9)
DATA AJ/1.0,0.0,99999998,-3.9999998721,-3.9999999710,13.9999973021,2.6666665044,-1.7777560599,-8888839649,2.4443542635,1777582922,-1.0792534928,0.2366167735,3.0767718531,0.0220691555,-1.0005014415,0.0001289769/,4AP/3.9894228793,3.5894228793,-0.0175306200,-0.0292182562,5.0001734300,-0.0022320300,-1.0004876131,0.0005807590,6.0000173565,-0.0000200920,-1.0000370431,0.0000424140/,7AQ/-0.124694414,0.0374008364,-0.045643240,-0.006390400,8.00000869791,-0.001064741,-0.000342468,-0.000397080,9.00000200920,-0.000162200,-1.000032312,-1.000036594/,DATA AS/-5.7721566491,0.0000000004,-1.6911374142,1.6177253972,3.691138879,-10.764572724,-2.3331002234,13.691138879,-10.764572724,2.3331002234,211.6207891416,6.694321484,-4.9105291148,12141.87561,3.6694321484,-4.9105291148,12141.87561,31.1418033012,-0.0146999271,-1691017200,-0.0013508487,4.0148999271,-0.0146999271,-1691017200,-0.0013508487,1699218764,4.0169921876,0.000891322,-0.0010266368,PI=3.14159265359
T=Z/4
IF (CABS(T) .GT. 1.0) GO TO 130
J=(0.0,0.0)
TO=(1.0,0.0)
GO 100 I=1,8
J=J+AJ(ORDER+1,I)*TO
100 TO=TO*T*T
IF (ORDER .EQ. 1) J=T*J
GO TO (125,200,200),M
125 RETURN
130 T=1/T
P=(0.0,0.0)
Q=(0.0,0.0)
TO=(1.0,0.0)
DO 140 I=1,6
P=P+AP(ORDER+1,I)*TO
Q=Q+AQ(ORDER+1,I)*TO
140 TO=TO*T*T
P=SQRT(2*PI)*P
Q=Q*T*SQRT(2*PI)
N=REAL(Z)-PI/4-ORDER*PI/2/(2*PI)
A=CSQRT(2/(PI*ZH)
Z1=Z-PI/4-ORDER*PI/2-2*PI*N
J=A*(P*CCOS(Z1)-Q*CSIN(Z1))
Y=A*(P*CSIN(Z1)+Q*CCOS(Z1))
H1=J+(1.0,1.0)*Y
H2=J-(1.0,1.0)*Y
RETURN
200 Y=(0.0,0.0)
TO=(1.0,0.0)
DO 210 I=1,9
   Y=Y+AY(ORDER+1,I)*TO
210 TO=TO*T*T
   IF (ORDER .EQ.1) Y=Y/Z
   Y=2/PI*(J*CLOG(T*2)-Y)
   H1=J+(0.0,1.0)*Y
   H2=J-(0.0,1.0)*Y
END
PROGRAM UR(INPUT, OUTPUT, TAPE5 = INPUT, TAPEF = OUTPUT)
COMPLEX R1, R2, RB, RC, RE, ANUM1, ANUM2, ADEN2, UR1, UR2
C THIS PROGRAM COMPUTES THE FIRST AND SECOND VARIATIONAL
C APPROXIMATIONS FOR UR
READ *, N, AK
G = .57735
S = 1.8763874
WRITE(6, 29)
29 FORMAT(* APPROXIMATION FOR UR*)
DO 69 K = 1, N
READ *, T
PI = 3.1415926536
C COMPUTE THE INTEGRAL 15
CALL U1(AK, T, G, S, R1)
C COMPUTE THE INTEGRAL 19
CALL U2(AK, T, G, S, P2)
C COMPUTE THE INTEGRAL 11
R4 = -2.*G*G/(PI*T)
C COMPUTE THE INTEGRAL 12
R1 = -4.*G*G/(3.*PI*T)
C COMPUTE THE INTEGRAL 13
CALL U3(AK, T, G, S, RR)
C COMPUTE THE INTEGRAL 16
CALL UC(AK, T, G, S, RC)
C COMPUTE THE INTEGRAL 110
CALL UE(AK, T, G, S, RE)
C COMPUTE THE SECOND VARIATIONAL APPROXIMATION
ANUM1 = R1*RC*RA-R1*R4*RE
ANUM2 = R3*RC*RB-R2*R2*RA
ADEN2 = RC*PC-P2*RE
UR2 = (ANUM1+ANUM2)/ADEN2
C COMPUTE THE FIRST VARIATIONAL APPROXIMATION
UR1 = R1*R4/R2
WRITE(6, 100) AK, T, G
WRITE(6, 200) UR1, UR2
200 FORMAT(1X, *1ST VAR APP=*, 2F10.5, *2ND VAR APP=*, 2F10.5)
A1 = CAR(A1)
A2 = CAR(A2)
WRITE(6, 205) A1, A2
63 CONTINUE
69 CONTINUE
END
SUBROUTINE U1(AK, T, G, S, F)
COMPLEX AI, HS1, RES, Y1, Y2, Y3, ANUM, RES1, F
COMPLEX S2, J1, Y, H2
AT=CMPLX(0,0,1,0)
PI=3.1415926536
RES=0.0
X1=G+.35*(1-G)/50
CALL AIN1(AK,T,G,S,X1,Y1)
DO 101 I=1,50
X2=G+(I)*.05*(1-G)/50
CALL AIN1(AK,T,G,S,X2,Y2)
RES=RES+((.5*.05*(1-G)/50)*(Y1+Y2)
Y1=Y2
101 CONTINUE
DO 104 J=1,101,2
X2=G+.35*(1-G)+J*(.95-G)/100
X3=G+.05*(1-G)+(J+1)*(.95-G)/100
CALL AIN1(AK,T,G,S,X2,Y2)
CALL AIN1(AK,T,G,S,X3,Y3)
RES=RES+((.95-G)/100/3)*(Y1+4*Y2+Y3)
Y1=Y2
104 CONTINUE
A=2*S*S-1
B=SQRT<X*X-G*G>
AEXP=SIN(AK*X)
X1=AK*T*X
X2=(1,J,G)*X1
CALL BESSI(1,X2,2,J1,Y1,H1,H2)
ANUM=(2*S*S-1-2*A*B)*H1*AEXP*S
ADEN=(2*(S*S-.5)*2*S-2*S*A*R-S**3*A/R-S**3*B/A)
RESI=ANUM/ADEN
F=-RES+PI*RES1
F=F*(1-G)*AI/(PI)
RETURN
END
SUBROUTINE AIN1(AK,T,G,S,X,Y)
COMPLEX H,ANUM,Y,X2,Y1,J1,H1,H2
AEXP=SIN(AK*X)
X1=AK*T*X
X2=(1,J,G)*X1
CALL BESSI(1,X2,2,J1,Y1,H1,H2)
A=SORT(X*X-G*G)
B=SORT(1-X*X)
ANUM=(X*X-.5)*A*B*H*AEXP*X
ADEN=((X*X-.5)**4*X**4*(X-X-G*G)*(1-X*X))
Y=ANUM/ADEN
RETURN
END
SUBROUTINE U2(AK,T,G,S,R)
COMPLEX AI,R,Y23,Y33,SE,ANUM,RES,Y1,Y2,Y3
100 CONTINUE
H=(G-.1)/100
DO 200 J=1,100+2
X2=G+.05+J*H
X3=G+.05+(J+1.)*H
CALL AX(G,X2,Y22)
CALL ES(AK,X2,U,Y23)
Y2=Y22*Y23
R=R+.5*H*(Y1+Y2)
Y1=Y2
200 CONTINUE
H=(G-.1)/100
DO 300 I=1,100+2
X2=G+.05+I*H
CALL AX(G,X2,Y22)
CALL ES(AK,X2,U,Y23)
Y2=Y22*Y23
R=R+.5*H*(Y1+Y2)
Y1=Y2
300 CONTINUE
H=(.9-G)/100
DO 400 I=1,100+2
X2=G+.05+I*H
X3=G+.05+(I+1.)*H
CALL AX(G,X2,Y22)
CALL ES(AK,X2,U,Y23)
Y2=Y22*Y23
CALL AX(G,X3,Y32)
CALL ES(AK,X3,U,Y33)
Y3=Y32*Y33
R=R+(H/3.)*(Y1+4*Y2+Y3)
Y1=Y3
CONTINUE
H=3C1
DO 50 I=1,50
X2=.95+I*H
CALL AX(G,X2,Y22)
CALL ES(AK,X2,U,Y23)
Y2=Y22*Y23
R=R+.5*H*(Y1+Y2)
Y1=Y2
50 CONTINUE
CALL ES(AK,S,U,SE)
A=SQRT(S*S-G*G)
3= SQRT(S*S-1.)
ANUM=PT*A*SE
ADEN=2.*(S*S-.5)*2.*S-2*S*A*B-S**3*A/B-S**3*B/A
RES=ANUM/ADEN
R=(1.-G*G)*.5*AI*(R-RES)/PI
R=-2.*G*G*R/(U*PI)
RETURN
END
SUBROUTINE AX(G,X,Y)
IF (X .GE. G) GO TO 600
A=SQRT(G*G-X*X)
B=SQRT(1.-X*X)
ANUM=A
ADEN=(X*X-.5)**2+X*X*A*B
Y=ANUM/ADEN
GOTO 700
600 ANUM=X**2+(X*X-G*G)*SQRT(1.-X*X)
ADEN=(X*X-.5)**4+X**4*(X*X-G*G)*(1.-X*X)
Y=ANUM/ADEN
700 CONTINUE
RETURN
END
SUBROUTINE ES(AK,X,U,Y)
COMPLEX AI,E,Y
AI=CMPLX(0.6,1.0)
XH=AK*X
E=CEXP(2.*AI*XH)
Y=(1.-E)/XH
RETURN
END
SUBROUTINE U8(AK,T,G,S,F)
COMPLEX AI,HS1,RES,Y1,Y2,Y3,ANUM,RES1,F
COMPLEX S2,J1,Y,H2
AI=CMPLX(1.0,1.0)
PI=3.1415926536
RES=0.0
X1=G+.05*(1.-G)/50
CALL AIN2(AK,T,G,S,X1,Y1)
DO 101 I=1,50
X2=G+(I)*.05*(1.-G)/50
CALL AIN2(AK,T,G,S,X2,Y2)
PES=RES+.5*G*(1.-G)/50*(Y1+Y2)
Y1=Y2
101 CONTINUE
DO 104 J=1,100,2
X2=G+J*.05*(1.-G)+J*(.95-G)/100
X3=G+.05*(1.-G)+J*(.95-G)/100
CALL AIN2(AK,T,G,S,X2,Y2)
CALL AIN2(AK,T,G,S,X3,Y3)
PES=RES+(.95-G)/100/(Y1+4*Y2+Y3)
Y1=Y3
104 CONTINUE
A=SQRT(S*S-G*G)
B=SQRT(S*S)
AEXP=AK*S*COS(AK*S)+(AK*AK*S*S-1)*SIN(AK*S)
S1=AK*S*T
S2=(1.0,J,0)*S1
CALL BESXI(1,S2,2,J1,Y1,H1,H2)
ANUM=(2*S*S-1-2*A*B)*H1*AEXP
ADEN=(2*(S*S-.5)**2-2**4*A*R-S**3*A/B/4)**S
RES=ANUM/ADEN
F=-AI*RES+PI*AI*RES1
F=(1,-G*G)*F/(PI*AK*AK)
RETURN
END
SUBROUTINE AIN2(AK,T,G,S,X,Y)
COMPLEX X2,Y1,J1,H2
COMPLEX H,ANUM,Y
AEXP=AK*X*COS(AK*X)+(AK*AK*X*X-1)*SIN(AK*X)
X1=AK*T*X
X2=(1.0,J,0)*X1
CALL BESXI(1,X2,2,J1,Y1,H1,H2)
A=SQRT(X*X-G*G)
B=SQRT(S*S)
ANUM=(X*X-.5)**4*A*B*H*AEXP
ADEN=((X*X-.5)**4*X**4*(X*X-G*G)**4+(1,-X*X)**4)*X
Y=ANUM/ADEN
RETURN
END
SUBROUTINE F1(AK,G,U,S,R)
COMPLEX AI,R,Y23,Y33,SE,ANUM,RES,Y1,Y2,Y3
PI=3.1415926536
H=.001
Y1=0.0
R=0.0
AI=COMPLEX(0.0,1.0)
DO 150 J=1,50
X2=G*1.0
CALL AX1(G,X2,Y22)
CALL ES1(AK,X2,U,Y23)
Y2=Y22*Y23
R=R+.5*H*(Y1+Y2)
Y1=Y2
150 CONTINUE
H=(G-.1)/100
DO 250 J=1,100
X2=.05+J*H
X3=.05+(J+1.0)*H
CALL AX1(G,X2,Y22)
CALL ES1(AK,X2,U,Y23)
Y2=Y22*Y23
CALL AX1(G,X3,Y32)
CALL ES1(AK,X3,U,Y33)
Y3=Y32*Y33
R=R+(H/3.0)*(Y1+4*Y2+Y3)
Y1=Y3
250 CONTINUE
H=.001
DO 350 J=1,100
X2=G+.05+J*H
CALL AX1(G,X2,Y22)
CALL ES1(AK,X2,U,Y23)
Y2=Y22*Y23
R=R+.5*H*(Y1+Y2)
Y1=Y2
350 CONTINUE
H=(.9-G)/100
DO 450 J=1,100
X2=G+.05+J*H
X3=G+.05+(J+1.0)*H
CALL AX1(G,X2,Y22)
CALL ES1(AK,X2,U,Y23)
Y2=Y22*Y23
CALL AX1(G,X3,Y32)
CALL ES1(AK,X3,U,Y33)
Y3=Y32*Y33
R=R+(H/3.0)*(Y1+4*Y2+Y3)
Y1=Y3
450 CONTINUE
H=.001
DO 500 I=1,50
X2=.95+I*H
CALL AX1(G,X2,Y22)
CALL ES1(AK,X2,U,Y23)
Y2=Y22*Y23
*=<+.5*H*(Y1+Y2)
Y1=Y2
500 CONTINUE:
CALL ES1(AK,S,U,SE)
A=SQR(T(S*S-G*G))
B=SQR(T(S*S-1.))
ANUM=PI*S*A*SE
ADEN=2.*((S*S-.5)*2.*S-2*S*A*B-S**3/(A*B-S**3*B/A)
*ES=ANUM/ADEN
R=(1.-G*G)*.5*A1*AK*(R-RES1)/PI
R=(-R-(1.+5*U*U))/2.)
RETURN
END
SUBROUTINE AX1 (G,X,Y)
IF(X .GE. G) GO TO 600
A=SQR(T(G*G-X*X))
B=SQR(T(1.-X*X))
ANUM=X*A
ADEN=(X*X-.5)**2+X*X*A*0
Y=ANUM/ADEN
GOTO 700
600 ANUM=X**3*(X*X-G*G)*SQRT(1.-X*X)
ADEN=(X*X-.5)**2+X*X*A*0
Y=ANUM/ADEN
700 CONTINUE:
RETURN
END
SUBROUTINE ES1(AK,X,U,Y)
COMPLEX UH1,J,Y5,H1,H2
COMPLEX AI,Y,A1,A2,A3,A4,Y1
PI=3.1415926536
AI=CMPLX(0.C,1.Q)
XH=AK*X
IF (XH .LE. .6034) GO TO 17
A1=2./(AI*XH)-4.*AI/(XH**3)-U*U/(AI*XH)
UH=UH*U
UH1=UH**1*0.0
CALL BESSI(0.,UH1,1,J,Y5,H1,H2)
A1=REAL(J)
A2=CEXP(AI*XH)
A3 = 2./(AI*XH) + 2./(XH*XH) + 2.*AI/(XH**3)

A4 = 2.*A2*A3*RJ

Y = A1 + A4

GO TO 18

continue

C1 = 8./3.

C2 = -5 + U*U/2 + (U**4)/16.

C3 = -8./15 - 2*U*U/3.

Y = C1 + A1*C2*XH*C3*XH**3

18 CONTINUE

RETURN

END

SUBROUTINE UC(AK, U, G, S, R)
COMPLEX R, Y1, Y2, Y3, Z1, Z2, Z3
PI = 3.1415926536
H = PI/100.
R = 0.0
X1 = 0.0
CALL F1(AK, G, X1, S, Z1)

Y1 = G, G

DO 52 I = 1, 50, 2
X2 = SIN(I*H)
X3 = SIN((I+1)*H)
CALL F1(AK, G, X2, S, Z2)
CALL F1(AK, G, X3, S, Z3)

Y2 = Z2*X2
Y3 = Z3*X3

R = R + H* (Y1 + 4*Y2 + Y3)/3.

Y1 = Y3

52 CONTINUE

R = 2.*G*G*R/(PI*U)
RETURN

END

SUBROUTINE UE(AK, T, G, S, R)
COMPLEX X1, Y22, Y32, Y1, Y2, Y3, A, ANUM, RES, R
AI = CMPLX(1.0, 1.0)
PI = 3.1415926536
R = 0.0
H = 0.5/50
Y1 = AK*G
DO 105 I = 1, 50
X1 = I*.05/50
CALL AG(M, G, X1, Y21)
CALL AG(AK, G, X1, Y22)

Y2 = Y21 + Y22

P = R + H* (Y1 + Y2)/2

Y1 = Y2
CONTINUE
H = (G - 0.1) / 100
DO 200 I = 1, 100, 2
X1 = 0.5 + I * H
X2 = 0.5 + (I + 1) * H
CALL AM(G, X1, Y21)
CALL AG(AK, G, X1, Y22)
Y2 = Y21 * Y22
CALL AM(G, X2, Y31)
CALL AG(AK, G, X2, Y32)
Y3 = Y31 * Y32
R = R + H * (Y1 + Y2 + Y3) / 3
Y1 = Y3
200 CONTINUE
H = 0.5 / 50
DO 300 I = 1, 50
X1 = (G - 0.5) + I * H
CALL AM(G, X1, Y21)
CALL AG(AK, G, X1, Y22)
Y2 = Y21 * Y22
R = R + H * (Y1 + Y2) / 2
Y1 = Y2
300 CONTINUE
DO 400 I = 1, 50
X1 = G + I * H
CALL AM(G, X1, Y21)
CALL AG(AK, G, X1, Y22)
Y2 = Y21 * Y22
R = R + H * (Y1 + Y2) / 2
Y1 = Y2
400 CONTINUE
H = (G - 0.95) / 100
DO 500 I = 1, 100, 2
X1 = 0.95 + I * H
X2 = 0.95 + (I + 1) * H
CALL AM(G, X1, Y21)
CALL AG(AK, G, X1, Y22)
Y2 = Y21 * Y22
CALL AM(G, X2, Y31)
CALL AG(AK, G, X2, Y32)
Y3 = Y31 * Y32
R = R + H * (Y1 + Y2 + Y3) / 3
Y1 = Y3
500 CONTINUE
H = 0.95 / 50
DO 600 I = 1, 50
X1 = 0.95 + I * H
CALL AM(G,X1,Y21)
CALL AG(AK,G,X1,Y22)
Y2=Y21+Y22
R=R+(Y1+Y2)*H/2
Y1=Y2

CONTINUE
CALL AG(AK,G,S,A)
A NUM=PI*SQRT(S*S-G*G)*A
A1=SQRT(S*S-G*G)
A2=SQRT(S*S-1)
RES=A NUM/ADEN
R=R-RES-2.*PI/(15.*(1.-G*G))
R=-G*G*(1.-G*G)*R/(PI*PI*PI*T)
END

SUBROUTINE AM(G,X,Y)
IF ((X-G).LT.0.) GO TO 800
ANUM=X*X*(X*X-G*G)*SQRT(1.-X*X)
ADEN=(X*X-.5)**2+X*X*SQRT(G*G-X*X)*SQRT(1.-X*X)
Y=ANUM/ADEN
GO TO 85
800 ANUM=SQRT(G*G-X*X)
ADEN=(X*X-.5)**2+X*X*SQRT(G*G-X*X)*SQRT(1.-X*X)
Y=ANUM/ADEN
850 CONTINUE
RETURN
END

SUBROUTINE AG(AK,G,X,Y)
COMPLEX Y,AI,X2,T1,S1,S2,R
AI=CMPLX(0.0,1.0)
X1=AK*X
X2=X1*AI
T1=-.5*AI*(1.-CEXP(2.*X2))
S1=2*CEXP(X2)
CALL B(AK,X1,R)
C=SIN(X1)/(X1**3)-COS(X1)/(X1**2)
Y=TI+S1*C-AI*X1*R/8.
RETURN
END

SUBROUTINE B(AK,X1,R)
COMPLEX AI,R,E,T1,T2,S1,S2,U1,U2
AI=CMPLX(0.0,1.0)
IF (X1.LE.2.393333) GO TO 900
E=CEXP(AI*X1)
T1=-2.*AI/(15.*X1)-.5/(X1*X1)*4*AI/(3*(X1*X1))
T2=(-2*(X1*X1)-2*AI/(X1*X1))*E
S1=T2+AI/X1+2*AI/(X1*X1)
\[ S_2 = T_2 - AI/X_1 - 2.4 AI / (X_1^3) \]
\[ R = T_1 + S_1 \cdot S_2 \]

GO TO 950

900

\[ U_1 = 8.8888888 - 4.5714244 \cdot AI \cdot X_1 + 1.7776 \cdot 4 \cdot X_1 \cdot X_1 \]
\[ U_2 = 0.5643724 \cdot AI \cdot X_1 \cdot X_1 \cdot X_1 \]
\[ P = U_1 + U_2 \]

RETURN

950

END

SUBROUTINE BESSI (ORDER, Z, M, J, Y, H1, H2)
INTEGER ORDER
COMPLEX H1, H2, J, P, Q, T, TO, Z, ZI, A, Y
DIMENSION AJ(2,8), AP(2,6), AQ(2,6), AY(2,9)

DATA AJ/1.0, 1.9999999998, -3.9999998721, -3.9999999711, +3.9999973021, 2.5666663544, -1.7777560599, -8.888839649, +4.4443584253, 1.777582922, -1709253492, -0.236616773, +0.076771853, 0.322569155, -0.055014415, -0.0412897697,
+AP/3.999422793, -3.999422819, -0.017530620, -0.029218256,
+0.001734309, -0.002232035, -0.000487613, -0.000581759, +0.000173565, -0.000209026, -0.000370439, -0.000424417,
+AQ/0.124669441, 3.374008364, 0.034564324, -0.06391433,
++0.0003869791, 0.001064741, 0.076342468, -0.000398708,
++0.000580926, 0.000162226, 0.000032312, -0.000036594,
+AY/-5.772156649, 1.060060034,
+-1.6911374142, -0.6177253972, 3.6911388799,
+-1.7545472724, -2.2331102234, 11.6207891416,
+..0694321484, -4.9105291148, -12.4187561, 1.1418033012,
+..048999971, -16.91081728, -0.013508487, 0.1699218756,
+..030081322, 0.0126833687,
PI = 3.141592653589
T = Z/4
IF (CARST(T) .LT. 1.0) GO TO 130
J = (3.0, 1.0)
TO = (1.0, 1.0)
DO 125 I = 1, M
J = J + AJ(ORDER + 1, I) * TO
110
TO = TO * T
IF (ORDER .EQ. 1) J = T * J
GO TO (125, 220, 220, M)
125 RETURN

130 T = 1/T
P = (0.0, 0.0, 0.0)
Q = (0.0, 0.0, 0.0)
TO = (1.0, 0.0, 0.0)
DO 140 I = 1, M
P = P + AP(ORDER + 1, I) * TO
Q = Q + AQ(ORDER + 1, I) * TO
140 TO = TO * T

P = SQRT(2*PI)*P
Q = Q*T*SQRT(2*PI)
N = (REAL(Z) - PI/4 - ORDER*PI/2)/(2*PI)
A = CSQRT(2/(PI*Z))
Z1 = Z - PI/4 - ORDER*PI/2 - Z*PI*N
J = A*(P*CCOS(Z1) - Q*CSIN(Z1))
Y = A*(P*CSIN(Z1) + Q*CCOS(Z1))
H1 = J + (0, 0, 0, 0) * Y
H2 = J - (0, 0, 1, 0) * Y
RETURN

200  Y = (0, 0, 0, 0)
TO = (1, 0, 0, 0)
DO 210 I = 1, 9
Y = Y + AY(ORDER+1, I) * TO
210  TO = TO * T * T
IF (ORDER.EQ.1) Y = Y / Z
Y = 2/PI*(J*CLOG(T*2)-Y)
H1 = J + (0, 0, 0, 0) * Y
H2 = J - (0, 0, 1, 0) * Y
RETURN
END
APPENDIX E

TABLES
Table 1. First and Second Variational Approximations for $U_r$  
\((a_k = .2, .5, \gamma^2 = 1/3)\)

| $a_k = .2$ | $Re\left(\frac{1}{U_r} \right)$ | $Re\left(\frac{2}{U_r} \right)$ | $Im\left(\frac{1}{U_r} \right)$ | $Im\left(\frac{2}{U_r} \right)$ | $\left|\frac{2}{U_r}\right|$ |
|---|---|---|---|---|---|
| $1.25$ | -.1786 | -.1786 | .0181 | .0181 | .1796 |
| $2.0$ | -.1199 | -.1199 | .0028 | .0028 | .1200 |
| $5.0$ | -.0557 | -.0557 | -.0269 | -.0269 | .0618 |
| $10.0$ | -.0056 | -.0056 | -.0418 | -.0418 | .0422 |
| $20.0$ | .0292 | .0292 | .0114 | .0114 | .0314 |
| $30.0$ | -.0199 | -.0199 | .0172 | .0172 | .0263 |
| $40.0$ | -.0048 | -.0048 | -.0211 | -.0211 | .0216 |
| $50.0$ | .0161 | .0161 | .0053 | .0053 | .0169 |

| $a_k = .5$ | $Re\left(\frac{1}{U_r} \right)$ | $Re\left(\frac{2}{U_r} \right)$ | $Im\left(\frac{1}{U_r} \right)$ | $Im\left(\frac{2}{U_r} \right)$ | $\left|\frac{2}{U_r}\right|$ |
|---|---|---|---|---|---|
| $1.25$ | -.2104 | -.2104 | .0216 | .0216 | .2115 |
| $2.0$ | -.1504 | -.1504 | -.0332 | -.0332 | .1540 |
| $5.0$ | .0143 | .0143 | -.0938 | -.0938 | .0949 |
| $10.0$ | .0280 | .0280 | .0660 | .0660 | .0717 |
| $20.0$ | .0421 | .0421 | .0034 | .0034 | .0422 |
| $30.0$ | .0272 | .0272 | -.0128 | -.0128 | .0301 |
| $40.0$ | .0068 | .0068 | -.0328 | -.0328 | .0335 |
| $50.0$ | -.0167 | -.0167 | -.0193 | -.0193 | .0255 |
Table 2. First and Second Variational Approximations for $U_r$.
($\alpha_k = 1.0, 1.5, \gamma^2 = 1/3$)

<table>
<thead>
<tr>
<th>$t$</th>
<th>$1 \Re (\frac{U_r}{d})$</th>
<th>$2 \Re (\frac{U_r}{d})$</th>
<th>$1 \Im (\frac{U_r}{d})$</th>
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<th>$\frac{2}{U_r/d}$</th>
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Table 3. First and Second Variational Approximations for $U_r$.
($\alpha_k = 2.0, 2.5, \gamma^2 = 1/3$)

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Table 4. First and Second Variational Approximations for $U_r$.
(ak = 3.0, 4.0, $\nu^2 = 1/3$)

| t   | $\text{Re}\left(\frac{1}{U_r} \right)$ | $\text{Re}\left(\frac{2}{U_r} \right)$ | $\text{Im}\left(\frac{1}{U_r} \right)$ | $\text{Im}\left(\frac{2}{U_r} \right)$ | $\left|\frac{2}{U_r} \right|$ |
|-----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|-----------------|
| 1.25| 0.0427                          | 0.0307                          | 0.0709                          | 0.1398                          | 0.1431          |
| 2.0 | 0.0020                          | -0.0330                         | 0.0388                          | 0.0000                          | 0.0330          |
| 5.0 | 0.0120                          | 0.0109                          | 0.0908                          | 0.1190                          | 0.1195          |
| 10.0| -0.0020                         | 0.0085                          | -0.0331                         | -0.0533                         | 0.0540          |
| 20.0| 0.0023                          | 0.0159                          | 0.0144                          | 0.0139                          | 0.0211          |
| 30.0| -0.0024                         | 0.0005                          | 0.0189                          | 0.0301                          | 0.0301          |
| 40.0| -0.0074                         | -0.0152                         | 0.0010                          | 0.0075                          | 0.0169          |
| 50.0| -0.0039                         | -0.0113                         | -0.0120                         | -0.0166                         | 0.0201          |

| t   | $\text{Re}\left(\frac{1}{U_r} \right)$ | $\text{Re}\left(\frac{2}{U_r} \right)$ | $\text{Im}\left(\frac{1}{U_r} \right)$ | $\text{Im}\left(\frac{2}{U_r} \right)$ | $\left|\frac{2}{U_r} \right|$ |
|-----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|-----------------|
| 1.25| -0.3341                         | -0.3555                         | -0.1126                         | -0.1208                         | 0.3755          |
| 2.0 | 0.2140                          | 0.2295                          | 0.1540                          | 0.1562                          | 0.2776          |
| 5.0 | 0.0718                          | 0.0778                          | 0.1362                          | 0.1446                          | 0.1642          |
| 10.0| -0.1054                         | -0.1090                         | -0.0990                         | -0.1032                         | 0.1501          |
| 20.0| -0.0890                         | -0.0932                         | -0.0245                         | -0.0260                         | 0.0968          |
| 30.0| -0.0719                         | -0.0754                         | 0.0120                          | 0.0129                          | 0.0765          |
| 40.0| -0.0534                         | -0.0556                         | 0.0380                          | 0.0400                          | 0.0684          |
| 50.0| -0.0267                         | -0.0277                         | 0.0538                          | 0.0562                          | 0.0627          |
Table 5. First and Second Variational Approximations for $U_r$.
(ak = 5.0, $\gamma^2 = 1/3$)

| ak = 5.0 | \[ \begin{array}{ccccc}
1.25 & \text{Re}(\frac{U}{d}) & \text{Re}(\frac{U}{d}) & \text{Im}(\frac{U}{d}) & \text{Im}(\frac{U}{d}) & \left| \frac{U}{d} \right| \\
2.0 & -0.0756 & -0.0810 & -0.3919 & -0.3986 & 0.4068 \\
5.0 & -0.1982 & -0.1958 & 0.2034 & 0.2057 & 0.2840 \\
10.0 & 0.1988 & 0.1996 & -0.0074 & -0.0076 & 0.1998 \\
20.0 & -0.0531 & -0.0530 & 0.1298 & 0.1308 & 0.1412 \\
30.0 & 0.0959 & 0.0968 & -0.0196 & -0.0199 & 0.0988 \\
40.0 & -0.0568 & -0.0575 & -0.0556 & -0.0561 & 0.0803 \\
50.0 & 0.0559 & 0.0565 & -0.0248 & -0.0251 & 0.0618 \\
\end{array} \right. |
Table 6. First and Second Variational Approximations for $U_z$.
($ak = .2, .5, \gamma^2 = 1/3$)

| $ak = .2$ | $Re\left(\frac{1}{U_z/d}\right)$ | $Re\left(\frac{2}{U_z/d}\right)$ | $Im\left(\frac{1}{U_z/d}\right)$ | $Im\left(\frac{2}{U_z/d}\right)$ | $\left|\frac{2}{U_z/d}\right|$ |
|---|---|---|---|---|---|
| 1.25 | .5829 | .5828 | .0651 | .0650 | .5864 |
| 2.0 | .3169 | .3169 | .1036 | .1036 | .3334 |
| 5.0 | .0623 | .0623 | .1144 | .1144 | .1303 |
| 10.0 | -.0454 | -.0454 | .0510 | .0510 | .0683 |
| 20.0 | -.0020 | -.0020 | -.0395 | -.0395 | .0396 |
| 30.0 | .0238 | .0238 | .0188 | .0188 | .0304 |
| 40.0 | -.0251 | -.0251 | .0056 | .0056 | .0257 |
| 50.0 | .0100 | .0100 | -.0208 | -.0208 | .0230 |

| $ak = .5$ | $Re\left(\frac{1}{U_z/d}\right)$ | $Re\left(\frac{2}{U_z/d}\right)$ | $Im\left(\frac{1}{U_z/d}\right)$ | $Im\left(\frac{2}{U_z/d}\right)$ | $\left|\frac{2}{U_z/d}\right|$ |
|---|---|---|---|---|---|
| 1.25 | .5611 | .5599 | .1611 | .1610 | .5826 |
| 2.0 | .2347 | .2347 | .2380 | .2379 | .3341 |
| 5.0 | -.1241 | -.1241 | .0660 | .0660 | .1406 |
| 10.0 | .0643 | .0643 | -.0550 | -.0550 | .0846 |
| 20.0 | .0121 | .0121 | -.0560 | -.0560 | .0573 |
| 30.0 | -.0271 | -.0271 | -.0423 | -.0423 | .0502 |
| 40.0 | -.0429 | -.0429 | -.0064 | -.0064 | .0434 |
| 50.0 | -.0315 | -.0315 | .0237 | .0237 | .0394 |
Table 7. First and Second Variational Approximations for $U_z$

$\text{ak = 1.0, 1.5, } \gamma^2 = 1/3$)

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<tr>
<th>$t$</th>
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Table 8. First and Second Variational Approximations for $U_z$.
($ak = 2.0, 2.5, \gamma^2 = 1/3$)

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Table 9. First and Second Variational Approximations for $U_\perp$.
$(\alpha_k = 3.0, 4.0, \sqrt{\nu} = 1/3)$

| $t$  | $\text{Re}\left(\frac{U_\perp}{a}\right)$ | $\text{Re}\left(\frac{2U_\perp}{a}\right)$ | $\text{Im}\left(\frac{U_\perp}{a}\right)$ | $\text{Im}\left(\frac{2U_\perp}{a}\right)$ | $\left|\frac{2U_\perp}{a}\right|$ |
|------|-------------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|-----------------|
| 1.25 | 0.2729                             | 0.2765                             | 0.2814                             | 0.2065                             | 0.3451          |
| 2.0  | -0.1559                            | -0.1598                            | 0.0048                             | 0.0006                             | 0.1598          |
| 5.0  | 0.0391                             | 0.0393                             | 0.0025                             | 0.0021                             | 0.0393          |
| 10.0 | -0.0135                            | -0.0135                            | -0.0211                            | -0.0211                            | 0.0251          |
| 20.0 | 0.0166                             | 0.0166                             | -0.0212                            | -0.0212                            | 0.0270          |
| 30.0 | 0.0175                             | 0.0175                             | 0.0023                             | 0.0023                             | 0.0177          |
| 40.0 | -0.0004                            | -0.0004                            | 0.0161                             | 0.0161                             | 0.0161          |
| 50.0 | -0.0137                            | -0.0137                            | 0.0070                             | 0.0070                             | 0.0154          |

| $t$  | $\text{Re}\left(\frac{U_\perp}{a}\right)$ | $\text{Re}\left(\frac{2U_\perp}{a}\right)$ | $\text{Re}\left(\frac{U_\perp}{a}\right)$ | $\text{Re}\left(\frac{2U_\perp}{a}\right)$ | $\left|\frac{2U_\perp}{a}\right|$ |
|------|-------------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|-----------------|
| 1.25 | -0.0168                             | -0.0141                            | -0.5592                             | -0.5656                             | 0.5658          |
| 2.0  | 0.1506                              | 0.1520                             | -0.3621                             | -0.3742                             | 0.4039          |
| 5.0  | 0.2274                              | 0.2279                             | -0.1280                             | -0.1289                             | 0.2618          |
| 10.0 | -0.1368                             | -0.1369                            | 0.1367                              | 0.1370                              | 0.1936          |
| 20.0 | -0.0417                             | -0.0417                            | 0.1268                              | 0.1270                              | 0.1336          |
| 30.0 | 0.0172                              | 0.0173                             | 0.1097                              | 0.1097                              | 0.1110          |
| 40.0 | 0.0578                              | 0.0578                             | 0.0771                              | 0.0771                              | 0.0964          |
| 50.0 | 0.0781                              | 0.0781                             | 0.0371                              | 0.0371                              | 0.0865          |
Table 10. First and Second Variational Approximations for $U_z$.
(ak = 5.0, $\gamma^2 = 1/3$)

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<th>Re($\frac{U_z}{d}$)</th>
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APPENDIX F

FIGURES
Figure 1. Geometry Used in the Derivation of Equation (23).

Figure 2. Geometry Used in the Derivation of Equation (37).
Figure 3. Geometry Used in the Derivation of Equations (155) and (162).

Figure 4. The Radiation Contour.
Figure 5. The Contour $\Gamma$.

Figure 6. The Contour $\Gamma_1$. 
Figure 7. Graphs of $\text{Re} \left( \frac{2 \hat{U}_r}{d} \right)$ and $\text{Im} \left( \frac{2 \hat{U}_r}{d} \right)$, $ak = .2$, $\gamma^2 = \frac{1}{3}$. 
Figure 8. Graphs of $\text{Re} \left( \frac{U_r^2}{d} \right)$ and $\text{Im} \left( \frac{U_r^2}{d} \right)$, $ak = .5$, $\gamma^2 = \frac{1}{3}$.
Figure 9. Graphs of $\text{Re}\left(\frac{\gamma^2}{\text{d}}\right)$ and $\text{Im}\left(\frac{\gamma^2}{\text{d}}\right)$, $\alpha k = .2, \gamma^2 = \frac{1}{3}$. 
Figure 10. Graphs of $\text{Re}\left(\frac{U_z}{d}\right)$ and $\text{Im}\left(\frac{U_z}{d}\right)$, $ak = .5$, $\gamma^2 = \frac{1}{3}$. 


Richard Deane Summers was born January 8, 1946, in Atlanta, Georgia. He attended primary school in Atlanta, Georgia and in Birmingham, Alabama. He completed his secondary education at Sandy Springs High School in Sandy Springs, Georgia in May 1964 and entered the Georgia Institute of Technology in September of that year. He received the degree of Bachelor of Science in Physics in June 1968 and entered graduate school in Physics in September 1968. With the help of a teaching assistantship in Physics, he completed the degree of Master of Science in Physics in December 1969, and the degree was conferred in June 1970. In January 1970 he entered the School of Mathematics as a graduate teaching assistant and received the degree of Master of Science in Applied Mathematics in August 1972.