Some conditions on periodicity for sum-free sets

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Abstract

Cameron has introduced a natural bijection between the set of one way infinite binary sequences and the set of sum-free sets (of positive integers), and observed that a sum-free set is ultimately periodic only if the corresponding binary sequence is ultimately periodic. He asked if the converse also holds. In this paper we present necessary and sufficient conditions for a sum-free set to be ultimately periodic, and show how these conditions can be used to test specific sets; these tests produce the first evidence of a positive nature that certain sets are, in fact, not ultimately periodic.

1 Introduction

There is a natural bijection between the set of one way infinite binary sequences and the set of sum-free sets of positive integers. In [1] Cameron observed that a sum-free set is ultimately periodic only if the corresponding binary sequence is ultimately periodic, and asked whether the converse is also true. This question is still open; however there do exist relatively simple sets for which it would appear that the answer is no; that is, the sets correspond to ultimately periodic binary sequences, but the sets themselves are apparently aperiodic. A major difficulty is that while it is a relatively simple matter to determine that a set *is* ultimately periodic (requiring only a finite number of terms) no method is presently known that will show that a sum-free set is not ultimately periodic, from a consideration of only finitely many elements of the set.

In this paper we introduce two new functions $g_S(n)$ and $\bar{g}_S(n)$ defined on the positive integers, and we show that the behaviour of these functions determines whether a set is ultimately periodic or not. More precisely, we prove that, if its corresponding binary sequence is ultimately periodic, then a sum-free set S is ultimately periodic if and only if $g_S(n)$ is bounded, and that if it is not bounded then $\bar{g}_S(n)$ grows at least as fast as $\log n$.

2 Definitions

A set S of positive integers is said to be **sum-free** if there do not exist $x, y, z \in S$ such that $x + y = z$. Observe that we do not require x, y to be distinct. We shall denote the set of sum-free sets of positive integers by $\mathcal{S}$.

A sum-free set is said to be **ultimately complete** if for all sufficiently large n, either $n \in S$ or there exist $x, y \in S$ such that $x + y = n$. A sum-free set is **periodic** if there exists a positive integer m such that for all $n \geq 1$, $n \in S$ if and only if $n + m \in S$. A sum-free set is said to be **ultimately periodic** if there exist positive integers m, n_0 such that for all $n > n_0$, $n \in S$ if and only if $n + m \in S$.

If S is ultimately periodic, then there is a unique minimum period; indeed, if m_1, m_2 are both periods for the elements of S greater than n_0 then the greatest common divisor of m_1, m_2 is also a period for S.

If S is a periodic subset of $\mathbb{N}$, the set of positive integers, let $\bar{S}$ denote the set

$$\bar{S} = \{s_1, s_2, \ldots, s_k\} \pmod{m}$$

where m is the least modulus for which $s \in S$ if and only if $s \equiv s_i \pmod m$ for some $s_i \in \bar{S}$. If S is ultimately periodic, let $\bar{S}$ be defined similarly, in that for some n_0 for all $s \geq n_0, s \in S$ if and only if $s \equiv s_i$ $\pmod m$ for some $s_i \in \bar{S}$.

For example,

$$S = \{1, 3, 5, 7, 9, \ldots\} \text{ is periodic, } \bar{S} \equiv \{1\} \pmod 2$$

$$S = \{1, 5, 7, 9, \ldots\} = \{1, 3, 5, 7, 9, \ldots\} \setminus \{3\} \text{ is ultimately periodic, } \bar{S} \equiv \{1\} \pmod 2$$

$$S = \{1\} \cup \{5, 8, 11, 14, 17, 20, \ldots\} \text{ is ultimately periodic, } \bar{S} \equiv \{2\} \pmod 3$$

3 The Bijection.

Define the bijection θ between the set $2^{\mathbb{N}}$ of binary sequences and the set $\mathcal{S}$ of sum-free sets as follows.

Let σ be an element of $2^{\mathbb{N}}$, say $\sigma_1\sigma_2\sigma_3\ldots$ where $\sigma_i \in \{0, 1\}$ for every i. We now construct sets S_i, T_i, U_i; start with $S_0 = T_0 = U_0 = \emptyset$.

For $i = 1, 2, 3, \ldots$ perform the following operations. Let n_i be the least element of $\mathbb{N} \setminus (S_{i-1} \cup T_{i-1} \cup U_{i-1})$. Then

$$\text{if } \sigma_i = 1, \text{ put } \begin{cases} S_i &= S_{i-1} \cup \{n_i\} \\ T_i &= S_i + S_i \\ U_i &= U_{i-1} \end{cases}$$
$$\text{if } \sigma_i = 0, \text{ put } \begin{cases} S_i &= S_{i-1} \\ T_i &= T_{i-i} \\ U_i &= U_{i-1} \cup \{n_i\}. \end{cases}$$

Let $S = \bigcup_i S_i$; then, since each S_i is sum-free, and since $S_i \subset S_{i+1}$, S is also sum-free. Let θ be the mapping from $2^{\mathbb{N}}$ to $\mathcal{S}$ defined by these operations, so that, for example

$$\begin{array}{lll} \theta: & 11111111\ldots & \mapsto \{1, 3, 5, 7, 9, 11, 13, 15, \ldots\} \\ \theta: & 01010101\ldots & \mapsto \{2, 5, 8, 11, \ldots\} \\ \theta: & 10101010\ldots & \mapsto \{1, 4, 7, 10, \ldots\} \\ \theta: & 1010010101\ldots & \mapsto \{1, 4, 8, 11, 14, \ldots\}. \end{array}$$

It is natural now to ask whether θ is invertible; in essence, since each entry in a binary sequence σ corresponds to exactly one element of $S \cup (\mathbb{N} \setminus (S + S))$, this is easily seen to be the case; indeed, let S be a sum-free set, and construct an infinite binary sequence as follows: define a ternary sequence τ by setting

$$\tau_n = \begin{cases} 1 & \text{if } n \in S \\ * & \text{if } n \in S + S \\ 0 & \text{otherwise} \end{cases}$$

Convert this sequence to a binary sequence by deleting all *'s. It is an easy exercise to check that this is the inverse of the mapping from $2^{\mathbb{N}}$ to $\mathcal{S}$ defined above. We have thus defined a bijection θ from $2^{\mathbb{N}}$ to $\mathcal{S}$.

We shall now make some observations about this bijection:

1. *It is very natural*: if asked to construct a sum-free set, we would be quite likely to do it in an element by element fashion, making a choice whether or not to include each element of $\mathbb{N}$. An element would be considered for inclusion only if it didn't cause a violation of the condition that S be sum-free. If we now consider the elements of $\mathbb{N}$ in the obvious order (1, then 2, then 3, then 4,...) we obtain exactly the bijection θ between lists of choices made (binary sequences) and sum-free sets.

2. *There is a natural metric on the set of sequences* $2^{\mathbb{N}}$: two sequences are at distance 2^{-k} if they differ for the first time at the $(k + 1)$-st place; there is also a natural metric on the set of sum-free sets: two sets are at a distance 2^{-k} if $k + 1$ is the least element in $S_1 \triangle S_2 = (S_1 \cup S_2) \setminus (S_1 \cap S_2)$.

The bijection θ is clearly bicontinuous with respect to the induced topologies, so we have a homeomorphism between $2^{\mathbb{N}}$ and $\mathcal{S}$. An open ball of radius 2^{-k} about σ in $2^{\mathbb{N}}$ consists of all sequences whose initial segment of length k agrees with σ. An open ball of radius 2^{-k} in $\mathcal{S}$ about S_0 consists of all sum-free sets S such that $S \cap \{1, 2, 3, \ldots, k\} = S_0 \cap \{1, 2, 3, \ldots, k\}$.

3. *If S is ultimately periodic then $\theta^{-1}(S)$ is also ultimately periodic*: further, the period of $\theta^{-1}(S)$ divides (period of S – no. of elements in a period which are ultimately sums of smaller elements of S).

We defer the proof of this to Lemma 4.1

4. *S is ultimately complete if and only if the sequence $\theta^{-1}(S)$ contains only finitely many zeroes.* Indeed, in the construction given for a sum-free set from a binary sequence, an element is not included if and only if it is either a sum of smaller elements already in the set, or the corresponding term in the binary sequence is zero. Thus if S is ultimately complete then we can only have finitely many elements excluded because of zeroes in $\theta^{-1}(S)$.

This immediately implies that the set of ultimately complete sum-free sets is countable. By way of a contrast, we have

Proposition 3.1 *The set of maximal sum-free sets (i.e. those sum-free sets for which for every $n \notin S$ there exist $x, y \in S$ such that either $x + y = n$ or $x + n = y$) is uncountable.*

Proof Consider the set

$$\{9, 11, 14, 16, 19, 21, 24, 26, 29, \ldots\} = \{n | n = 5k \pm 1, k = 2, 3, \ldots\}.$$

This set is clearly sum-free. Further, if we add to this set the element 2, we find that the only solutions to the equation $x + y = z$ are of the form $2 + 5k - 1 = 5k + 1$. Consider now an arbitrary partition of $\{2, 3, 4, 5, \ldots\}$ into two parts, say N_1, N_2. Then the set S_{N_1,N_2} given by

$$\{2\} \cup \{5k - 1 | k \in N_1\} \cup \{5k + 1 | k \in N_2\}$$

is sum-free, since by definition $N_1 \cap N_2 = \emptyset$. Then none of the integers $5k - 1, k \in N_2$ or $5k + 1, k \in N_1$ can be added to the set S_{N_1,N_2}, as they are respectively differences or sums of pairs of elements in S_{N_1,N_2}. Now extend S_{N_1,N_2} to a maximal sum-free set, say T_{N_1,N_2}: it is immediate from the preceding comments that the sets T_{N_1,N_2}, T_{M_1,M_2} are distinct if $N_1 \neq M_1$: as there are uncountably many partitions of $\{2, 3, 4, \ldots\}$ we have proven the proposition. ∎

Corollary 3.1 *There exist uncountably many aperiodic maximal sum-free sets of positive lower density.*

Indeed, the lower asymptotic density of T_{N_1,N_2} is at least $\frac{1}{5}$. This answers a question of Stewart (personal communication), regarding the existence of aperiodic maximal sum-free sets of positive density.

4 Periodicity of Sum-free Sets

We shall now consider one of the most intriguing questions regarding sum-free sets, namely the relationship between the periodicity of a binary string σ, and the periodicity of the associated sum-free set $\theta(\sigma)$: Cameron [1] asked whether it is true that that $\theta^{-1}(S)$ is ultimately periodic if and only if S is ultimately periodic.

In Lemma 4.1 we prove that if a sum-free set S is ultimately periodic, then so is $\theta^{-1}(S)$. In Lemma 4.2 we show, essentially, that if a set S appears to be ultimately periodic for long enough, and if it has an ultimately periodic input sequence $\theta^{-1}(S)$ then S is ultimately periodic.

We introduce new functions, $g(n) = g_S(n)$ and $\bar{g}_S(n)$: in Theorem 4.1 we show that if $\theta^{-1}(S)$ is ultimately periodic, then S is ultimately periodic if and only if $g(n)$ is bounded. In Theorem 4.2 we show that if $\theta^{-1}(S)$ is ultimately periodic, and $g(n)$ is not bounded, then for $n > n_0$, $\bar{g}_S(n) > c \log n$.

4.1 When is a sum-free set periodic?

Cameron(personal communication) has asked whether any of the following statements are true:
(i) A binary string σ is ultimately periodic if and only if $\theta(\sigma)$ is ultimately periodic.
(ii) σ has only finitely many zeroes if and only if $\theta(\sigma)$ is ultimately periodic and ultimately complete.
Clearly (i) $\Longrightarrow$ (ii), but not necessarily vice versa.

Each of these questions is still open; however, since they were first suggested, we have found evidence to suggest that (i) is false, and Cameron [1] has found evidence that (ii) may also be false.

Before presenting this evidence, we shall prove the following lemmata: Lemma 4.1 shows that in each of the questions, the "if" part holds, and Lemma 4.2 shows that in order to prove that a sum-free set is ultimately periodic, we need only consider a finite prefix of the set.

Lemma 4.1 (Cameron[1]) *If $\theta(\sigma)$ is ultimately periodic then σ is also ultimately periodic.*

Proof. Suppose that $\theta(\sigma) = S$, and that the periodic part of S is $\bar{S} \pmod m$. Then

$$S = T \cup \{s_1 + km, s_2 + km, \ldots, s_i + km | k \geq k_0\}$$

where

$$\bar{S} = \{s_1, s_2, \ldots, s_i\} \pmod m,\ 0 < s_j < m$$

and

$$T = S \cap \{1, 2, \ldots, k_0 m\}$$

For every $n \geq 1$ construct $\tau(n)$ as follows:

$$\tau(n) = \begin{cases} 1 & \text{if} \quad n \in S \\ 0 & \text{if} \quad n \notin S \quad \text{and } \nexists x, y \in S, x + y = n \\ * & \text{if} \quad n \notin S \quad \text{and } \exists x, y \in S, x + y = n. \end{cases}$$

Then the sequence $\tau = \tau(1)\tau(2)\tau(3)\ldots$ is an infinite ternary sequence; further, if we erase the *'s in τ then we obtain exactly $\sigma = \theta^{-1}(S)$. (We note that the sequence obtained from τ by replacing each * with a 0 is exactly the characteristic function of the set S.) Thus, if we prove that τ is ultimately periodic, then it will follow immediately that σ is ultimately periodic.

Consider an element $n > 3k_0 m$. Then

$$\tau(n) = \begin{cases} 1 & \text{if} \quad n \equiv s_1, s_2, \ldots, \text{ or } s_i \pmod m \\ * & \text{if} \quad \exists t \in T, s_j \in \{s_1, \ldots, s_i\} \text{ such that } n \equiv t + s_j \pmod m \\ & \quad \text{or if } \exists s_{j_1}, s_{j_2} \in \{s_1, \ldots, s_i\} \text{ such that } n \equiv s_{j_1} + s_{j_2} \pmod m \\ 0 & \quad \text{otherwise.} \end{cases}$$

Observe that the value of $\tau(n)$ depends solely upon the congruence class of n mod m, since T, $\{s_1, \ldots, s_i\}$ are both finite sets. Thus $\tau(n)$ is periodic (mod m) for $n > 3k_0 m$, and we deduce that σ is also ultimately periodic. ∎

This lemma proves exactly the "if" direction.

Where will we run into difficulties when we try to reverse this proof? The crucial step involves the erasing of the *'s in τ: given a periodic sequence σ it is easy to insert *'s in such a way that the resulting ternary sequence is most definitely aperiodic (for example, insert a * after every p_kth 1, where p_k is the kth prime). Of course, it is unlikely that such insertions would leave a sum-free set: statement (i) states essentially that only by inserting in a periodic manner is it possible to ensure that S is sum-free.

In order to prove the "only if" direction, it would be necessary to show that certain sets are ultimately periodic; in certain circumstances this is possible. The following Lemma shows that if a set S **is** ultimately periodic, then we need only consider a finite prefix of S, along with the binary sequence $\theta^{-1}(S)$ in order to **prove** that S is ultimately periodic.

Lemma 4.2 *Suppose that $\sigma = \theta^{-1}(S)$ is ultimately periodic, and*

$$S \cap \{1, 2, \ldots, 4n\} = T \cup S_1 \cup S_2 \cup S_3$$

where

$$\begin{aligned} T &= S \cap \{1, 2, \ldots, n\} \\ S_1 &= S \cap \{n+1, n+2, \ldots, 2n\} \\ S_2 &= S \cap \{2n+1, 2n+2, \ldots, 3n\} \\ S_3 &= S \cap \{3n+1, 3n+2, \ldots, 4n\}. \end{aligned}$$

Suppose further that

$$\begin{aligned} S_2 &= \{s+n \mid s \in S_1\} \\ S_3 &= \{s+2n \mid s \in S_1\} \end{aligned}$$

and that

$$\tau(\min_{s\in S_1} s), \tau(\min_{s\in S_2} s), \tau(\min_{s\in S_3} s)$$

each correspond to the same point in a period in $\theta^{-1}(S)$. Then S is ultimately periodic, and the period of S divides n.

Proof. We shall show by induction that $4n+k \in S$ if and only if $k \equiv s \pmod n$ for some $s \in S_3$.

First, $4n+1 \in S$ if and only if $3n+1 \in S$; indeed, $3n+1 \in S$ if and only if

$$\nexists t \in T,\ s \in S_2 \text{ such that } t+s = 3n+1$$

$$\text{and } \nexists s_1 \in S_1,\ s_2 \in S_1 \text{ such that } s_1+s_2 = 3n+1$$

$$\text{and the corresponding bit of } \theta^{-1}(S) \text{ is a 1.}$$

Similarly, $4n+1 \in S$ if and only if

$$\nexists t \in T,\ s \in S_3 \text{ such that } t+s = 4n+1$$

$$\text{and } \nexists s_1 \in S_1,\ s_2 \in S_2 \text{ such that } s_1+s_2 = 4n+1$$

$$\text{and the corresponding bit of } \theta^{-1}(S) \text{ is a 1.}$$

It is clear that these three conditions are equivalent, since S_i is constant $\pmod n$.

Exactly the same argument may now be used to prove that if $4n+i \in S$ if and only if $i \equiv s \pmod n$ for some $s \in S_3$ for each $i < k$, then $4n+k \in S$ if and only if $k \equiv s \pmod n$ for some $s \in S_3$. ∎

In order to test Cameron's conjectures, we generated the sum-free sets corresponding to periodic binary inputs, with period at most 7. For all inputs with periods of length at most 4, the corresponding sum-free set was ultimately periodic, with a small (usually fewer than 10 terms) non-periodic part, and a small period (always less than 25). Of the 30 inputs with periods of length 5 (all strings of length five except for 00000 and 11111, which have period 1), all but 3 inputs were quickly periodic; the ones which were not are 01001, 01010, 10010. Other potential counterexamples to Cameron's conjecture will be exhibited in section 4.2. The first of these, the set $\theta(\dot{0}100\dot{1}) = \{2, 6, 9, 14, 19, 26, 29, 36, 39, 47, 54, 64, 69, 79, 84, 91, \ldots\}$ certainly appears to be aperiodic; for example, considering the sequence of differences between consecutive elements of the set up to 10^7, this exhibits long strings which are repeated, separated by short "glitches" which seem to show no sign of settling down to be periodic. This, of course, is all evidence of a rather flimsy type: it is essentially of the form "we looked, but we couldn't find anything"; we shall now present a theorem which gives evidence which is more concrete in nature that certain sum-free sets are aperiodic. It may also be used to show that a sum-free set is ultimately periodic without actually having to find the period. Using the functions $g_S(n)$, $\bar{g}_S(n)$, we will provide evidence of a positive nature that $\sigma(\dot{0}100\dot{1})$ is aperiodic.

Define functions $g_S(n)$, $\bar{g}_S(n)$ as follows:

$$g_S(n) = \begin{cases} 0 & \text{if } \nexists x, y \in S \text{ such that } x+y=n \\ \min x & \text{such that } x+y=n,\ x,y \in S \text{ if there exist } x,y \in S \text{ such that } x+y=n \end{cases}$$

and

$$\bar{g}_S(n) = \max_{k \le n} g_S(k)$$

Theorem 4.1 *S is ultimately periodic if and only if σ is ultimately periodic and $\bar{g}_S(n)$ is ultimately constant, i. e. $g_S(n)$ is bounded.*

Proof. Suppose that S is ultimately periodic. Let

$$S = T \cup S_1 \cup S_2 \cup S_3 \cup \ldots$$

where $T = S \cap \{1, 2, \ldots, n\}$, $S_1 = S \cap \{n+1, n+2, \ldots, 2n\}$, and $S_{i+1} = S_i + n = \{s + n | s \in S_i\}$ for every i.

If $g_S(n) \geq 1$, then $\exists x, y \in S$ such that $x + y = n$. Thus, either

$$x \in T, y \in S_i \text{ for some } i,$$

$$\text{or } x \in S_i, y \in S_j \text{ for some } i, j.$$

If the former holds, then $g_S(n) \leq \max_{t \in T} t$.
If the latter holds, then some $x \in S_1$, $y \in S_{j-i+1}$ also satisfy $x + y = n$. Thus, if $g_S(n) \geq 1$ then $g_S(n) \leq max_{s \in S_1} s$, and we have shown that if S is ultimately periodic then $g_S(n)$ is bounded.

To prove the converse, suppose that $g_S(n) \leq k \; \forall n$. Let

$$T = S \cap \{1, 2, \ldots, k\}.$$

Then, for every n, n is expressible as a sum $x + y = n$, $x, y \in S$ if and only if n is expressible as a sum $t + y' = n$, $t \in T$, $y' \in S$. Let the input sequence $\theta^{-1}(S)$ have ultimate period p, and suppose n_0 is sufficiently large that n_0 corresponds to the periodic part of $\theta^{-1}(S)$.

Define

$$S_n = S \cap \{n, n+1, \ldots, n+k-1\}$$

Then, for $n > n_0$, S_{n+1} is determined by the triple (T, S_n, i_n) where i_n is our current position in a period.

Now let

$$T_n = \{s - n + 1 | s \in S_n\}$$

There are at most 2^k possibilities for the set T_n for each n, and there are p possibilities for the integer i_n; thus, since there are infinitely many values of n, there must exist n, j such that

$$(T_n, i_n) = (T_{n+j}, i_{n+j})$$

Then, since T_{n+1} is determined by (T, T_n, i_n), it is clear that then $(T_{n+1}, i_{n+1}) = (T_{n+j+1}, i_{n+j+1})$, and similarly that for all $m \geq n$, $(T_m, i_m) = (T_{m+j}, i_{m+j})$ Thus, from n onwards, S is periodic, with period dividing j. ∎

Thus, if we have a set for which $g_S(n)$ is not bounded then we know that this set cannot be periodic.

As a simple, but useful, extension of this theorem, we have

Theorem 4.2 *If, for sufficiently large n, $\bar{g}_S(n) < \log_2\left(\frac{n}{6p}\right)$ where p is the length of a period in the input string $\theta^{-1}(S)$, then S is periodic.*

(Here "sufficiently large" means
(i) $n > 2s$ where s is the smallest element of S (to ensure that $\bar{g}_S(n) > 0$) and
(ii) n is large enough that we are in the periodic part of the string $\theta^{-1}(S)$.)

Proof. Observe that since there are at most $2^k p$ choices for the pair (T_n, i_n) we will be able to find $n, n+j$ such that $n \geq n_0$, $n + j \leq 2^k p$
Thus, as in the proof of Theorem 4.1 we see that

$$S \cap \{1, 2, \ldots, \} = T \cup S_1 \cup S_2 \cup S_3$$

where $S_2 = S_1 + j = \{s + j | s \in S_1\}$, and $S_3 = S_1 + 2j$. and where the least element of S_1 is at most $n/3$. Then this is sufficient to ensure that S is ultimately periodic; indeed, it is enough to ensure that S is periodic from S_1 onwards. ∎

Computing the values of $\bar{g}_S(n)$ for the set $\theta(\dot{0}100\dot{1})$, for all $n \leq 200000$, we find that $\bar{g}$ appears to be very far from bounded: in fact it seems to increase in a roughly linear fashion; the following are the values of $\bar{g}_S(n)$ for which $\bar{g}_S(n) > \bar{g}_S(n-1)$ (since the function is weakly increasing, these values determine the function).

n	$\bar{g}_S(n)$	n	$\bar{g}_S(n)$	n	$\bar{g}_S(n)$	n	$\bar{g}_S(n)$
4	2	885	430	6411	3034	47437	23304
12	6	1288	445	6674	3297	49313	24133
18	9	1457	577	6709	3332	50678	25180
33	14	1820	597	6754	3377	50996	25498
52	26	1850	627	10360	4014	65250	28709
72	36	2028	805	11144	4798	68410	30974
94	47	2058	835	12692	6346	75499	37613
133	54	2103	880	14779	7104	82800	38422
182	91	2356	1133	16129	7675	88756	44378
192	96	2371	1148	19678	9839	111332	54455
227	106	2401	1178	22914	11457	112419	55542
242	121	2446	1223	24624	12312	121318	57969
274	137	3650	1522	27324	13394	126698	63349
322	161	4394	1795	30140	14127	137806	65796
348	174	4632	2068	40677	15179	142928	71464
362	181	4945	2381	43908	16281	171101	81091
637	237	5128	2564	43948	21974	188656	82178
647	247	6053	2676	46355	22222	199466	99733
690	345						

This behaviour continues for much larger n: indeed,

$$g(1211692) = 605846$$

(this is the largest value of $n < 10^7$ for which $g(n) = \frac{n}{2}$) and

$$g(9662060) = 4621889$$

(this is the largest value of $n < 10^7$ for which $\bar{g}$ increases).

If it could be shown for such a set S that such behaviour continues, that is that there exist an infinite number of n such that $g_S(n)/n$ is close to $\frac{1}{2}$, then it would follow immediately from Theorem 4.1 that S is aperiodic; it does not, however, appear that it is a simple matter to prove this.

4.2 Computational Evidence

If $\theta(\dot{0}100\dot{1})$ could be proven to be aperiodic, then there would be no need to list further potential counterexamples to Cameron's conjecture. Since a proof of this cannot presently be found, there is some value to testing periodicity over large classes of sum-free sets, in the hope that a recognizable pattern to the counterexamples might eventually emerge.

Table A exhibits all potentially aperiodic (up to 10^7) incomplete sum-free sets of the form $\theta(\sigma)$ with periodic binary inputs σ of periods 5, 6 or 7. This includes the three potential counterexamples mentioned earlier.

TABLE A: INCOMPLETE SUM-FREE SETS $\theta(\sigma)$ APERIODICITY CHECKED UP TO 10^7		
period(σ)=5	period(σ)=6	period(σ)=7
$\theta(\dot{0}100\dot{1})$	$\theta(\dot{0}1000\dot{1})$	$\theta(\dot{0}01000\dot{1})$
$\theta(\dot{0}101\dot{0})$	$\theta(\dot{0}1100\dot{1})$	$\theta(\dot{0}01001\dot{0})$
$\theta(\dot{1}001\dot{0})$	$\theta(\dot{0}1110\dot{0})$	$\theta(\dot{0}10000\dot{1})$
	$\theta(\dot{1}0001\dot{0})$	$\theta(\dot{0}10001\dot{0})$
	$\theta(\dot{1}0100\dot{1})$	$\theta(\dot{0}10010\dot{0})$
	$\theta(\dot{1}0101\dot{1})$	$\theta(\dot{0}10010\dot{1})$
		$\theta(\dot{0}10101\dot{0})$
		$\theta(\dot{0}10101\dot{1})$
		$\theta(\dot{0}10110\dot{1})$
		$\theta(\dot{0}11000\dot{1})$
		$\theta(\dot{1}00001\dot{0})$
		$\theta(\dot{1}00010\dot{0})$
		$\theta(\dot{1}00011\dot{0})$
		$\theta(\dot{1}01010\dot{0})$

We mention that periodicity in sum-free sets need not arrive quickly. The periodic sum-free set $S = \theta(\dot{0}11001\dot{1})$ has period 10710, after a transient phase of approximately 89,000 terms. Moreover,

the largest integer n in S for which n+10710 is *not* in S is n=489115

and

the largest integer n *not* in S for which n+10710 *is* in S is n=489108.

Table B lists all potentially aperiodic sum-free sets of the form $\theta(uv\dot{x}y\dot{z})$ or $\theta(abcde\dot{x}\dot{y})$. These are the simplest such cases, i.e., the binary inputs simultaneously have minimal preperiod and minimal period.

TABLE B: INCOMPLETE SUM-FREE SETS $\theta(uv\dot{x}y\dot{z})$ OR $\theta(abcde\dot{x}\dot{y})$ APERIODICITY CHECKED UP TO 10^7	
$\theta(uv\dot{x}y\dot{z})$	$\theta(abcde\dot{x}\dot{y})$
$\theta(00\dot{0}0\dot{1})$	$\theta(00001\dot{1}\dot{0})$
	$\theta(11000\dot{0}\dot{1})$
	$\theta(00110\dot{0}\dot{1})$

Cameron [1] found the first potentially aperiodic complete sum-free set, which we will indicate below using different notation from above. The existence of such a set suggests that Dickson's problem [2, 4] may have a negative solution. Queneau [6] and Finch [3] have studied a variation of this problem involving what are known as 0-additive sequences; an update on this direction of research appears in [5].

By the "base" of an ultimately complete sum-free set $S = \{s_1 < s_2 < \ldots < s_n < \ldots\}$, we mean the minimal set of S-elements,

$$B = \{s_1, s_2, \ldots, s_n\}$$

such that recursive application of the greedy algorithm, starting with B, gives the sum-free set S.

By the phrase "all sum-free bases up to p", we mean the collection of all sets B which are bases of ultimately complete sum-free sets S and whose largest element is at most p. For example, the collection of all sum-free bases up to 7 is:

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{2,5\}, \{2,6\}, \{2,7\},$$
$$\{3,5\}, \{3,7\}, \{4,6\}, \{4,7\}, \{5,7\}, \{1,3,7\}, \{1,4,7\}, \{4,5,7\}.$$

We examined each of the 76,080 sum-free bases up to 27 and determined whether each of the corresponding complete sum-free sets were periodic (checked up to 10^7). All aperiodic cases are listed in Table C.

We also took interest in sum-free bases with three or fewer elements and examined these bases up to 35. These are also included in Table C.

Table C is divided into two parts: the "aperiodic" part [for which g-values appear to be unbounded and no pattern is seen] and the "tentatively periodic" part [for which g-values are bounded above by $3.2 * 10^6$]. For the latter we also indicate the largest observed g-value and our best estimate of the period, if possible.

TABLE C: COMPLETE SUM-FREE SETS LISTED BY BASE
APERIODICITY CHECKED UP TO 10^7

tentatively periodic cases	aperiodic cases
base	base
{3, 4, 13, 18, 24 } [1]	{10, 13, 15, 16, 17, 24 }
max g-val = 2937317	{12, 15, 17, 18, 19, 25 }
est period = 3274006	{11, 16, 17, 26 }
	{1, 3, 8, 20, 26 }
{8, 14, 15, 17, 26 }	{9, 21, 24, 27 }
max g-val = 2898098	{2, 15, 16, 23, 27 }
est period = ?	{5, 6, 14, 23, 27 }
	{3, 12, 17, 19, 21, 27 }
{14, 15, 16, 18, 21, 26 }	{10, 15, 16, 18, 22, 27 }
max g-val = 1349528	{14, 16, 17, 18, 21, 27 }
est period = ?	{6, 14, 17, 18, 22, 25, 27 }
	{9, 16, 29 }
{14, 15, 18, 20, 22, 24, 26 }	{8, 18, 30 }
max g-val = 1424518	{8, 27, 32 }
est period = 1291498	{9, 26, 32 }
	{10, 18, 34 }
{4, 17, 18, 19, 24, 27 }	{9, 28, 35 }
max g-val = 3132839	{11, 26, 35 }
est period = 1022104	{12, 21, 35 }
{15, 16, 18, 22, 24, 27 } [2]	
max g-val = 2330099	
est period = 2673770	
{4, 21, 32 } [3]	
max g-val = 770538	
est period = ?	

We reiterate that periodicity need not arrive quickly. For example, the periodic complete sum-free set S based on $\{10, 14, 15, 17, 22\}$ has period = 2,875,722 after a transient phase of approximately 584,000 terms. Moreover,

the largest integer n in S for which $n + 2875722$ is *not* in S is $n = 4,562,648$

and

the largest integer n *not* in S for which $n + 2875722$ *is* in S is $n = 4,453,256$.

[1] This is Cameron's example

[2] Evidently the same (minus one term) as {15, 16, 18, 21, 22, 24, 27 }, which is not listed (to avoid duplicity).

[3] This is quite unexpected - the maximum g-value is quite small but no clear signs of periodicity are apparent.

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