EIGENVALUE INEQUALITIES FOR RELATIVISTIC HAMILTONIANS AND FRACTIONAL LAPLACIAN

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Selma Yıldırım Yolcu

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Approved by:

Professor Evans M. Harrell II, Advisor
School of Mathematics
Georgia Institute of Technology

Professor Michael Loss
School of Mathematics
Georgia Institute of Technology

Professor Jeffrey Geronimo
School of Mathematics
Georgia Institute of Technology

Professor Shui-Nee Chow
School of Mathematics
Georgia Institute of Technology

Professor Brian Kennedy
School of Physics
Georgia Institute of Technology

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To Türkay, my love...

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SUMMARY

Some eigenvalue inequalities for Klein-Gordon operators $H_{m,\Omega} = \sqrt{-\Delta + m^2}|_\Omega$ and fractional Laplacians $(-\Delta)^s$, $s \in (0,1)$ restricted to a bounded domain Ω in $\mathbb{R}^d$ are proved. Such operators became very popular recently as they arise in many problems ranging from mathematical finance to crystal dislocations, especially relativistic quantum mechanics and α-stable stochastic processes.

Many of the results obtained here are concerned with finding bounds for some functions of the spectrum of these operators. The subject, which is well developed for the Laplacian, is examined from the spectral theory perspective through some of the tools used to prove analogous results for the Laplacian. This work highlights some important results, sparking interest in constructing a similar theory for Klein-Gordon operators. For instance, the Weyl asymptotics and semiclassical bounds for the operator $H_{m,\Omega}$ are developed. As a result, a Berezin-Li-Yau type inequality is derived and an improvement of the bound is proved in a separate chapter.

Other results involving some universal bounds for the Klein-Gordon Hamiltonian with an external interaction $H_{m,\Omega} + V(x)$ are also obtained.

CHAPTER I

INTRODUCTION

In this work, some results in the spectral theory of Klein-Gordon operators and fractional Laplacians are presented. The subject is well developed for the Laplacian, and I shall state many anologous results pertaining to the eigenvalues of the Laplacian. Therefore, in this introductory chapter, several well-known inequalities for the eigenvalues of the Laplacian will be recalled in Section 1.2 after giving the definitions and properties of the Klein-Gordon operator and the fractional Laplacian in Section 1.1. For basic definitions and standard tools, please refer to the Appendix. I will also mention some applications in Section 1.3 to real life problems; for instance, I will mention some results from analysis, stochastic processes and relativistic quantum mechanics, in particular the behavior of the electrons on graphene sheets.

The main tool used here is the Harrell-Stubbe trace inequalities [22], which will be stated and proved in Section 1.4.

1.1 Definitions and Properties of the Klein-Gordon operators and the fractional Laplacian

This section reviews some of the basic definitions and properties of the Klein-Gordon operator and the fractional Laplacian. A reader familiar with this concept can skip this section and directly go to the next section.

1.1.1 Klein-Gordon operator

The Klein-gordon operator is the quantum-mechanical operator corresponding to the Klein-Gordon Hamiltonian. It is a first-order pseudodifferential operator used to model relativistic particles in quantum mechanics and relativistic Brownian motion. On unrestricted space the part representing kinetic energy $\sqrt{|\mathbf{P}|^2 + m^2}$ can be defined as the square root of $-\Delta + m^2$, where m is a nonnegative constant corresponding to the mass, in units where the speed of light is set to 1. The square root can be defined with the spectral theorem for self-adjoint operators. Sometimes we restrict the square root to functions supported within bounded, open domain $\Omega \in \mathbb{R}^d$ and designate the quantum version of $\sqrt{|\mathbf{P}|^2+m^2}\Big|_\Omega$ as $H_{m,\Omega}$. (A full definition of $H_{m,\Omega}$ will be provided later.)

Observe that the operator $H_{m,\Omega}$ is positive definite and has purely discrete spectrum consisting of positive eigenvalues $\{\beta_j\}_{j=1}^{\infty}$ if Ω is bounded. The eigenvalues β_j satisfy

$$0<\beta_1<\beta_2\leq\cdots.$$

When $m=0$, the operator $H_{0,\Omega}$ becomes the generator of the Cauchy stochastic process [52, 6]. Sometimes, we can confine ourselves to the case $m=0$ without loss of generality, because

$$H_{0,\Omega}\leq H_{m,\Omega}\leq H_{0,\Omega}+m. \tag{1.1.1}$$

Klein-Gordon operators can be conveniently defined with the aid of the Fourier transform on the dense subspace of test functions $C_c^\infty(\mathbb{R}^d)$. With the normalization , the Fourier transform and its inverse are defined as follows:

$$\widehat{\varphi}(\xi)=\mathfrak{F}\left[\varphi\right](\xi):=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\exp\left(-i\xi\cdot\mathbf{x}\right)\varphi(\mathbf{x})\,d\mathbf{x},$$

and

$$\mathfrak{F}^{-1}\left[\varphi\right](\mathbf{x}):=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\exp\left(i\xi\cdot\mathbf{x}\right)\varphi(\xi)\,d\xi.$$

Thus, the Laplacian is given by

$$-\Delta\varphi:=\mathfrak{F}^{-1}|\xi|^2\widehat{\varphi}(\xi),$$

and so,

$$\sqrt{-\Delta+m^2}\,\varphi:=\mathfrak{F}^{-1}\sqrt{|\xi|^2+m^2}\,\widehat{\varphi}(\xi). \tag{1.1.2}$$

The semigroup generated on $L^2(\mathbb{R}^d)$ is known explicitly, so that, for instance with $m=0$,

$$\exp\left(-\sqrt{-\Delta}t\right)\left[\varphi\right](\mathbf{x})=p_0(t,\cdot)*\varphi, \tag{1.1.3}$$

where for $t>0$ the transition density (convolution kernel) is

$$p_0(t,\mathbf{x}):=\frac{c_d t}{(t^2+|\mathbf{x}|^2)^{\frac{d+1}{2}}}, \tag{1.1.4}$$

with $c_d := \dfrac{d!}{(4\pi)^{d/2}\Gamma(1+d/2)}$. (Cf. [6]. Note that c_d is the same "semiclassical" constant that appears in the Weyl estimate for the eigenvalues of the Laplacian. It is given in [6] and some other sources as $\pi^{-\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)$, which is equal to c_d by an application of the duplication formula of the gamma function.)

Next, the formal definition of the operator $H_{m,\Omega}$ will be given with the aid of quadratic forms. Consider the quadratic form given by

$$Q(\varphi) = \int_\Omega \overline{\varphi}\sqrt{-\Delta + m^2}\,\varphi, \qquad \varphi \in C_c^\infty(\Omega).$$

Here $\sqrt{-\Delta + m^2}$ is calculated for $\mathbb{R}^d$. Note that the quadratic form Q is defined on a dense subset $C_c^\infty(\Omega)$ of $L^2(\Omega)$. Moreover, notice that Q is positive and symmetric, which can be easily seen by using the Fubini theorem and definition of Fourier transform and its inverse. Since $\Omega \subset \mathbb{R}^d$ is non-empty, bounded and open, $H_{m,\Omega}$ is defined as follows:

Definition 1.1.1 *The Friedrichs extension([2]) of the quadratic form Q on $L^2(\Omega)$ is designated by $H_{m,\Omega}$.*

Note that $H_{m,\Omega}$ is the unique minimal positive operator extending Q. Let $p_{m,\Omega}(t,\mathbf{x},\mathbf{y})$ be the integral kernel of the semigroup $e^{-tH_{m,\Omega}}$. The form of this kernel is typically not known explicitly. However, it is bounded by comparison with the operator $e^{-t\sqrt{-\Delta+m^2}}$ on $L^2(\mathbb{R}^d)$, which is given explicitly in the book Analysis, E. Lieb and M. Loss([41],p.183):

$$e^{-t\sqrt{-\Delta+m^2}}(x,y) = 2\left(\frac{m}{2\pi}\right)^{(d+1)/2} \frac{t}{(t^2+|x-y|^2)^{(d+1)/4}} K_{(d+1)/2}\big(m(t^2+|x-y|^2)^{1/2}\big), \tag{1.1.5}$$

for $x, y \in \mathbb{R}^d$. Here, $K_{(d+1)/2}$ is the modified Bessel function of the third kind. Observe that this kernel is bounded for $t > 0$. Consequently, $e^{-tH_{m,\Omega}}$ is Hilbert-Schmidt and $H_{m,\Omega}$ has purely discrete spectrum.

For the proof of (1.1.5) ([41]), one needs to know that

$$\int_{\mathbb{S}^{d-1}} e^{i\xi\cdot x}\mathrm{d}\xi = (2\pi)^{d/2}|x|^{1-d/2}J_{d/2-1}\big(|x|\big),$$

with $J_{d/2-1}$ denoting the Bessel function of $(d/2-1)$-st order, and

$$\int_0^\infty x^{t+1}J_t(xw)e^{-\alpha\sqrt{x^2+\beta^2}}dx = \sqrt{\frac{2}{\pi}}\,\alpha\beta^{t+3/2}w^t(w^2+\alpha^2)^{-t/2-3/4}K_{t+3/2}\big(\beta\sqrt{w^2+\alpha^2}\big),$$

where J_t denotes the Bessel function of t-th order.

As mentioned in [41], the version of the kernel of $e^{-t\sqrt{-\Delta+m^2}}$ for $d = 3$ was obtained by Erdelyi-Magnus-Oberhettinger-Tricomi in [14], which states that

$$e^{-t\sqrt{-\Delta+m^2}}(x,y) = \frac{m^2}{2\pi^2}\frac{t}{t^2+|x-y|^2}K_2\big(m(t^2+|x-y|^2)^{1/2}\big),$$

where K_2 is the modified Bessel function of the third kind.

Note that the Fourier transform can be more directly applied to $H_{m,\Omega}$ than to the square root of the Dirichlet-Laplacian as defined by the spectral functional calculus, which dominates it in the following sense:

Suppose that $\varphi \in C_c^\infty(\Omega) \subset C_c^\infty(\mathbb{R}^d)$. Since $\mathrm{supp}(\varphi) \in \Omega$ and $-\Delta$ is a local operator,

$$\begin{aligned}
\langle \varphi, H_{m,\Omega}^2\varphi\rangle = \|H_{m,\Omega}\varphi\|^2 &= \int_\Omega \left|\mathfrak{F}^{-1}\left(\sqrt{|\xi|^2+m^2}\hat{\varphi}\right)\right|^2 \\
&= \int_{\mathbb{R}^d} \left|\chi_\Omega\mathfrak{F}^{-1}\left(\sqrt{|\xi|^2+m^2}\hat{\varphi}\right)\right|^2 \\
&\leq \int_{\mathbb{R}^d} \left|\mathfrak{F}^{-1}\left(\sqrt{|\xi|^2+m^2}\hat{\varphi}\right)\right|^2 \\
&= \int_{\mathbb{R}^d} \overline{\varphi}(-\Delta+m^2)\varphi \\
&= \int_\Omega \overline{\varphi}(-\Delta+m^2)\varphi,
\end{aligned}$$

By the spectral mapping theorem, if λ_k is the kth eigenvalue of $-\Delta$, and β_k denotes the kth eigenvalue of $H_{m,\Omega}$,

$$\beta_k \leq \sqrt{\lambda_k+m^2}. \tag{1.1.6}$$

1.1.2 Fractional Laplacian

Consider a function $\varphi : \mathbb{R}^d \to \mathbb{R}$. For $s \in (0,1]$, the fractional Laplacian of φ is defined by a singular integral as (cf., [54],[18])

$$(-\Delta)^s\varphi(x) = C_{d,s}\mathrm{PV}\int_{\mathbb{R}^d}\frac{\varphi(x)-\varphi(y)}{|x-y|^{d+2s}}dy, \tag{1.1.7}$$

where PV denotes the principal value. In fact, the integral becomes nonsingular for the values $1/2 \leq s < 1$ because the singularity around $x = y$ is controlled.

$(-\Delta)^s$ can also be defined as a pseudodifferential operator with the aid of the Fourier transform. The definition is

$$(-\Delta)^s\varphi(\xi) = \mathfrak{F}^{-1}|\xi|^{2s}\widehat{\varphi}(\xi). \tag{1.1.8}$$

Refer to the article [57] or the book [38] for the proof of the equivalence between (1.1.7) and (1.1.8).

1.2 Some eigenvalue inequalities for the Laplacian

This section recalls some known inequalities for the eigenvalues of the Dirichlet Laplacian (i.e., Laplacian $-\Delta$ with the Dirichlet boundary condition) on a bounded domain Ω in $\mathbb{R}^d$. That means that one considers the problem

$$\begin{aligned} -\Delta u &= \lambda u \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.2.1}$$

Let us denote the spectrum and associated orthonormal basis of real eigenfunctions by $\{\lambda_j\}_{j=1}^{\infty}$ and $\{u_j\}_{j=1}^{\infty}$, respectively. The eigenfunctions satisfy

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \lambda_j \leq \cdots \to \infty.$$

The first of such inequalities is the

Rayleigh-Ritz inequality:

$$\lambda_1(\Omega) = \inf_{\substack{\psi \in D(-\Delta) \\ \psi \neq 0 \text{ a.e.}}} \frac{\int_\Omega \psi(-\Delta\psi)}{\int_\Omega \psi^2}, \tag{1.2.2}$$

where $D(-\Delta)$ denotes the domain of $-\Delta$ and ψ is a trial function in $D(-\Delta)$ or a dense core of $D(-\Delta)$.

Higher eigenvalues λ_j can be obtained by using the orthogonality of eigenfunctions:

$$\lambda_j(\Omega) = \inf_{\substack{\psi \in D(-\Delta) \\ \psi \perp u_1, u_2, \ldots u_{j-1}}} \frac{\int_\Omega \psi(-\Delta\psi)}{\int_\Omega \psi^2}. \tag{1.2.3}$$

Faber-Krahn inequality: Faber [16] and Krahn [36] independently proved the following isoperimetric inequality:

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\star) \text{ for } \Omega \in \mathbb{R}^d,$$

where $\Omega^\star$ denotes the spherical rearrangement of the bounded set Ω. The equality is obtained when $\Omega = \Omega^\star$. This inequality was originally conjectured by Rayleigh[50] in 1877, for which reason it is sometimes referred to as Rayleigh-Faber-Krahn inequality. The definition and some elementary properties of the notion of spherical(symmetric) rearrangement are provided in Chapter 4 where the concept of spherical rearrangement of sets and spherically decreasing rearrangement of functions is used to improve a bound that is obtained in Chapter 3.

Next recall some universal inequalities for the eigenvalues of the Dirichlet Laplacian on a bounded set Ω:

Payne, Pólya and Weinberger Inequality: One of the earlier such results goes back to 1955, when Payne, Pólya and Weinberger ([47], [48]) proved that for $\Omega \subset \mathbb{R}^2$,

$$\lambda_{k+1} - \lambda_k \leq \frac{2}{k} \sum_{j=1}^{k} \lambda_j, \quad k = 1, 2, \ldots; \tag{1.2.4}$$

the generalization of this inequality to $\Omega \subset \mathbb{R}^d$ is

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{dk} \sum_{j=1}^{k} \lambda_j, \quad k = 1, 2, \ldots, \tag{1.2.5}$$

The Payne, Pólya and Weinberger Inequality(PPW) inspired many similar inequalities for the eigenvaules of the Laplacian, including the Hile-Protter inequality, Yang's inequalities, the Harrell-Stubbe inequalities, and the Berezin-Li-Yau inequality.

Hile-Protter Inequality: In 1981, Hile and Protter proved in [28] that

$$1 \leq \frac{4}{dk} \sum_{j=1}^{k} \frac{\lambda_j}{\lambda_{k+1} - \lambda_j}. \tag{1.2.6}$$

One can obtain the Payne, Pólya and Weinberger inequality (1.2.5) from (1.2.6) by replacing the λ_j in the denominator on the right side of (1.2.6) by λ_k. So, it can be said that the Hile-Protter inequality (1.2.6) is stronger than the Payne, Pólya and Weinberger inequality (1.2.5).

Yang's Second Inequality: In [61], H.C.Yang proved that

$$\lambda_{k+1} \leq \left(1 + \frac{4}{d}\right) \frac{1}{k} \sum_{j=1}^{k} \lambda_j, \quad m = 1, 2, \ldots, \tag{1.2.7}$$

which follows from

$$\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)^2 \le \frac{4}{d}\sum_{j=1}^{k}\lambda_j(\lambda_{k+1}-\lambda_j), \quad m=1,2,\ldots. \tag{1.2.8}$$

Observe that Yang's Second inequality (1.2.7) implies the Hile-Protter inequality (1.2.6) which consequently implies the Payne, Pólya and Weinberger Inequality (1.2.5). For a detailed discussion of these inequalities, implications and the proofs, one can look at Ashbaugh's article [4].

Harrell-Stubbe Inequalities: In their paper [22], E.M. Harrell and J. Stubbe generalized the inequality (1.2.8) for powers $\sigma \ge 0$ and they proved that for $\sigma \ge 2$,

$$\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)^\sigma \le \frac{2\sigma}{d}\sum_{j=1}^{k}\lambda_j(\lambda_{k+1}-\lambda_j)^{\sigma-1}, \quad m=1,2,\ldots, \tag{1.2.9}$$

and for $0 \le \sigma \le 2$,

$$\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)^\sigma \le \frac{4}{d}\sum_{j=1}^{k}\lambda_j(\lambda_{k+1}-\lambda_j)^{\sigma-1}, \quad m=1,2,\ldots. \tag{1.2.10}$$

In fact, these are special cases of a family of inequalities for traces of functions $f(H)$ of Laplacian and other self-adjoint partial different operators. Next, recall the inequality of Berezin, Li and Yau. This inequality is different from the previous inequalities because it gives a bound for the sum of eigenvalues of the Dirichlet Laplacian in terms of the volume $|\Omega|$ of the bounded open set $\Omega \subset \mathbb{R}^d$.

Berezin-Li-Yau Inequality: In 1983, P. Li and S.-T. Yau proved that

$$\sum_{j=1}^{k}\lambda_j \ge \frac{dC_d}{d+2}|\Omega|^{-2/d}k^{1+2/d}, \tag{1.2.11}$$

where $C_d = 4\pi\Gamma(1+d/2)^{2/d}$.

As mentioned in [39], inequality (1.2.11) can be obtained by a Legendre transform of an earlier result by Berezin [7]. Hence, instead of calling (1.2.11) the Li-Yau inequality, it will be referred to as the Berezin-Li-Yau inequality. Section 3.3 provides a Berezin-Li-Yau type inequality for the eigenvalues of the Klein-Gordon operators. (For a full discussion, see Section 3.3). Moreover, in Chapter 4, the inequality given in Section 3.3 is improved.

1.3 Some Applications

This section provides some applications of the fractional Laplacian $(-\Delta)^s$ for $s \in (0, 1]$ and the Klein-Gordon operator $\sqrt{-\Delta + m^2}$, where m can be regarded as the mass of a relativistic particle. In the first two subsections, these operators are considered over $\mathbb{R}^d$, rather than the bounded domain $\Omega \subset \mathbb{R}^d$. Some applications to analysis and partial differential equations, and the relativistic quantum mechanics will be provided. In the last subsection, the applications to the stochastic processes, especially s-stable processes and Cauchy processes are mentioned.

1.3.1 Analysis and Partial Differential Equations

The operator $(-\Delta)^s$ appears in many applications of partial differential equations and analysis. The article [57] cites many applications of the nonlocal operators, especially fractional operators arising in the areas from obstacle problems [54] to the dislocations of crystals (cf. [30]). Here I will mention only a few examples:

Obstacle problem for the $(-\Delta)^s$ for $s \in (0, 1)$: In [54], L. Silvestre showed that there is a simple maximum principle for the operator $(-\Delta)^s$ by using the integral representation of the $(-\Delta)^s$, $s \in (0, 1)$. His proof relies on the fact that $(-\Delta)^s f$ is a continuous function in an open set $\Omega \subset \mathbb{R}^d$ where for some $\epsilon > 0$, f is a $C^{2s+\epsilon}$ function ($f \in C^{1,2s+\epsilon-1}$ if $s > 1/2$) in the L_s space defined by the norm

$$\|u\|_{L_s} = \int_{\mathbb{R}^d} \frac{|u(x)|}{1 + |x|^{n+2s}} dx.$$

In [54], he proves that for any "obstacle" C_c function $\varphi : \mathbb{R}^d \to \mathbb{R}$ there is a C_0 function u to the problem

$$\begin{aligned} u &> \varphi \text{ in } \mathbb{R}^d, \\ (-\Delta)^s u &\geq 0 \text{ in } \mathbb{R}^d, \\ (-\Delta)^s u(x) &= 0 \text{ for } x\text{'s such that } u(x) > \varphi(x). \end{aligned}$$

He also shows that the proof fails if $n = 1$ and $s = 1/2$ and finds some regularity results for u. For further details and proofs, see [54].

Dislocation Models: As the second application (in continuum mechanics), consider equations

$$\partial_t u + (-\Delta)^{1/2} u + u(u^2 - 1) = 0, \tag{1.3.1}$$

and

$$\partial_t u + (-\Delta)^{s} u + u(u^2 - 1) = 0, \tag{1.3.2}$$

where $s \in (0, 1)$.

Remarks 1.3.1

- *Equations* (1.3.1) *and* (1.3.2) *are considered as examples of a fractional diffusion-reaction equation in [30].*
- *Equation* (1.3.1) *is considered as a model for dislocation dynamics (i.e., line defects in a crystal)([30]).*
- *Equation* (1.3.2) *is called the fractional Allen-Cahn equation. ([30])*

1.3.2 Graphene and Relativistic Quantum Mechanics

Another motivation for this work comes from nanophysics, because relativistic Hamiltonian operators arise when a nonrelativistic particle travels in a two-dimensional hexagonal structure like carbon graphene, a one atom thick allotrope of carbon. Stacks of graphene layers make graphite, a three-dimensional allotrope of carbon, which is found abundantly in nature. Graphene was discovered in 2004 by a group of physicist in Manchester, UK [46]. After that, graphene has been of intense interest recently because of its remarkable electronic and elastic properties [49]. For example, see the most recent article Castro Neto et al.[10] for an extensive review of the electronic properties of graphene, including different type of disorders modifying the Dirac equation. For more details and models pertaining to the graphene, see [58, 56, 53, 31, 44, 45] and references therein.

Due to the special symmetry of the hexagonal lattice, charge carriers behave like *massless* relativistic particles, though with a speed $c/300$ where c is the speed of light. This has been known since 1947, when P.R.Wallace studied the band structure of graphene, though his aim was to study the graphite [58]. Graphene is modelled by a Schrödinger equation at all energy scales but a massless Dirac equation describes the low energy physics around the

Dirac points, i.e., inequivalent corner points in the graphene Brillouin zone. [10, 49]. In [10], it is shown that the effective Hamiltonian equation consists of two massless Diraclike equation. See [10] for details.

On the other hand, there are some works in the literature that uses the Klein-Gordon equation to obtain the energy eigenvalues in triangular graphene quantum dots (flakes). (cf. [26, 3, 42]). This progress is going to be the main part of our further research on the modelling problems of the graphene.

1.3.3 Stochastic Processes

In this section, I discuss the connection of stochastic processes with the Klein-Gordon operators $H_{0,\Omega}$ and with the fractional Laplacian operator $(-\Delta)^{s/2}$, $s \in (0,2)$. I also mention several recent results concerning the fractional Laplacian. For more details about the stochastic processes, please refer to the books [13, 9, 52] and the papers [5, 6, 37] and references therein for more results involving stable processes and Cauchy processes.
Let us begin by recalling some definitions of symmetric s-stable processes, and then mentioning some results regarding these processes.

Definition 1.3.2 *([5],[6]) A symmetric s-stable process of order $s \in (0,2]$ is a stochastic process with stationary and independent increments and with the transition density (i.e., convolution kernel) $p_s(t,x,y)$ given by*

$$\int_{\mathbb{R}^d} e^{-i\xi\cdot y} p_s(t,y)dy = e^{|\xi|^s t},$$

with $t > 0$ and when $x, y \in \mathbb{R}^d$.

Two important examples of symmetric s-stable processes are Brownian motion, which is obtained by setting $s = 2$, and the Cauchy process, which is obtained by setting $s = 1$.

For $t > 0$ and for $x, y \in \mathbb{R}^d$, the transition density in the case of the Brownian motion becomes

$$p_2(t,x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(\frac{-|x-y|^2}{4t}\right),$$

and the transition density in the case of the Cauchy process becomes

$$p_1(t,x,y) = \frac{c_d t}{(t^2 + |x-y|^2)^{(d+1)/2}},$$

where $c_d = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}}$ is the semiclassical constant that appears in the Weyl estimate for the eigenvalues of the Laplacian.

Therefore, the Dirichlet Laplacian is the infinitesimal generator of the Brownian motion for paths that are killed upon leaving the domain Ω, and $H_{0,\Omega}$ is the generator of the Cauchy process with the corresponding killing condition on $\partial\Omega$.

Moreover, the fractional Laplacian operators $(-\Delta)^{s/2}$, $s \in (0, 2]$ are the infinitesimal generators of the symmetric s-stable process.

Several relevant interesting results were obtained in [5, 6, 37]. Please refer to the papers for the proofs and details of the results recalled here:

Theorem 1.3.3 *(Bañuelos and Kulczycki, [6]) Let $\varrho_{s,d}$ be the smallest eigenvalue for the symmetric stable process of order $s \in (0, 2]$, killed off the unit ball $B_1^d(0) \subset \mathbb{R}^d$ with center at the origin. Then*

$$\frac{1}{C_{s,d}} \le \varrho_{s,d} \le \frac{1}{C_{s,d}} \frac{(d+2s)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{d}{2}+s\right)}{2(d+s)\Gamma\left(\frac{d+s}{2}\right)\Gamma(s)}.$$

More precisely, if $\varrho_{1,1}$ is the first eigenvalue of Cauchy process (i.e., $s = 1$) and $d = 1$, then

$$1 \le \varrho_{1,1} \le \frac{3\pi}{8} \backsimeq 1.18,$$

and for $\varrho_{1,2}$ (i.e., d=2),

$$1.57 \backsimeq \frac{\pi}{2} \le \varrho_{1,2} \le \frac{2\pi}{3} \backsimeq 2.09.$$

Chapter 2 contains a result (Corollary 2.2.4) similar to the following theorem regardless of any property of the domain other than boundedness.

Theorem 1.3.4 *(Bañuelos and Kulczycki, [5]) Let D be a bounded convex domain in $\mathbb{R}^d$. Let $\varrho_{1,d}^1$ and $\varrho_{1,d}^2$ be the first two eigenvalues for the generator of the Cauchy process with a killing condition on $\partial\Omega$. Then*

$$\varrho_{1,d}^2 - \varrho_{1,d}^1 \le \frac{\sqrt{\lambda_2^*} - (1/2)\sqrt{\lambda_1^*}}{\mathrm{Inr}(\Omega)}, \tag{1.3.3}$$

where λ_1^* and λ_2^* are the first two eigenvalues for the Dirichlet Laplacian for the unit ball $B_1(0) \subset \mathbb{R}^d$.

Here the inradius of Ω is defined by

$$\mathrm{Inr}(\Omega) = \sup\{d(\mathbf{x}) : \mathbf{x} \in \Omega\},$$

where $d(\mathbf{x}) = \min\{|\mathbf{x}-\mathbf{y}| : \mathbf{y} \notin \Omega\}$. A lower bound for the fundamental spectral gap $\varrho_2 - \varrho_1$ for the eigenvalues ϱ_1 and ϱ_2 of the symmetric s-stable process on a bounded open domain $\Omega \subset \mathbb{R}^d$ that is killed upon exiting Ω is also known from [37]:

Theorem 1.3.5 *(Kwaśnicki, [37]) Let $\varrho^1_{s,d}$ and $\varrho^2_{s,d}$ denote the first two eigenvalues of the symmetric s-stable process on $\Omega \subset \mathbb{R}^d$ with a killing condition on $\partial\Omega$. Then*

$$\varrho^2_{s,d} - \varrho^1_{s,d} \geq \frac{C(s,d)}{(\varrho_1)^{d/s}(\mathrm{diam}(\Omega))^{s+d}},$$

where the constant $C(s,d)$ depends only on the dimension d and the index s.

1.4 Trace Inequality

In [21] universal bounds for spectra of Laplacians were found as consequences of differential inequalities for Riesz means defined on the sequence of eigenvalues. Although the strategy here is the same, as adapted to the eigenvalues $\{\beta_j\}_{j=1}^{\infty}$ of the first-order pseudodifferential operator $H_{m,\Omega}$, the results obtained here and the details of the argument are quite different because the earlier article made heavy use of the fact that the Laplacian is of second order and acts locally, neither of which circumstance applies here.
An essential lemma is an adaptation of a result of [22, 23].

Lemma 1.4.1 (Harrell-Stubbe) Let H be a self-adjoint operator on $L^2(\Omega)$, $\Omega \in \mathbb{R}^d$, with discrete spectrum

$$\beta_1 \leq \beta_2 \leq \cdots < \inf \sigma_{ess}(H),$$

interpreted as $+\infty$ when $\sigma_{ess}(H)$ is empty. Let $\{u_j\}_{j=1}^{\infty}$ be the corresponding normalized eigenfunctions. Assume that for a Cartesian coordinate x_α, the functions $x_\alpha u_j$ and $x_\alpha^2 u_j$ are in the domain of definition of H. Then for any $z < \inf \sigma_{ess}(H)$,

$$\sum_{j:\beta_j \leq z} (z - \beta_j)\langle u_j, [x_\alpha, [H, x_\alpha]]\, u_j\rangle - 2\|\,[H, x_\alpha]\, u_j\|^2 \leq 0, \tag{1.4.1}$$

and

$$\sum_{j:\beta_j \le z} (z-\beta_j)^2 \langle u_j, [x_\alpha, [H, x_\alpha]]\, u_j\rangle - 2(z-\beta_j)\|\, [H, x_\alpha]\, u_j\|^2 \le 0. \tag{1.4.2}$$

Proof. Subject to the domain assumptions made in the statement of the theorem and because $Hu_j = \beta_j u_j$,

$$\begin{aligned} [H, x_\alpha]\, u_j &= (Hx_\alpha - x_\alpha H)u_j \\ &= Hx_\alpha u_j - x_\alpha H u_j \\ &= Hx_\alpha u_j - x_\alpha \beta_j u_j \\ &= (H-\beta_j)\, x_\alpha u_j. \end{aligned}$$

So,

$$\langle u_j, [x_\alpha, [H, x_\alpha]]\, u_j\rangle = 2\langle x_\alpha u_j, (H-\beta_j)\, x_\alpha u_j\rangle.$$

These two identities can be combined and slightly rearranged to yield:

$$\begin{aligned} &(z-\beta_j)\langle u_j, [x_\alpha, [H, x_\alpha]]\, u_j\rangle - 2\|\, [H, x_\alpha]\, u_j\|^2 \\ &= 2\langle((z-\beta_j) - (H-\beta_j))\, x_\alpha u_j, (H-\beta_j)x_\alpha u_j\rangle \\ &= 2\langle(z-H)\, x_\alpha u_j, (H-\beta_j)x_\alpha u_j\rangle. \end{aligned} \tag{1.4.3}$$

Since the eigenfunctions of H are complete,

$$(H-\beta_j)x_\alpha u_j = \sum_k (\beta_k - \beta_j)\langle x_\alpha u_j, u_k\rangle u_k,$$

so the right side of (1.4.3) can be rewritten as

$$\begin{aligned} 2\sum_k (z-\beta_k)\,\langle u_k, x_\alpha u_j\rangle\,(\beta_k-\beta_j)\,\langle x_\alpha u_j, u_k\rangle &= 2\sum_k (z-\beta_k)\,(\beta_k-\beta_j)\,|\langle u_k, x_\alpha u_j\rangle|^2 \\ &\le 2\sum_{k:\beta_k < z} (z-\beta_k)\,(\beta_k-\beta_j)\,|\langle u_k, x_\alpha u_j\rangle|^2, \end{aligned} \tag{1.4.4}$$

provided that $\beta_j \le z$. If we sum (1.4.3) over j with $\beta_j \le z$, i.e., the same values of j as for k in (1.4.4), then after symmetrizing in j, k,

$$\sum_{j:\beta_j \le z} (z-\beta_j)\langle u_j, [x_\alpha, [H, x_\alpha]]\, u_j\rangle - 2\|\, [H, x_\alpha]\, u_j\|^2$$

$$\le \sum_{j,k:\beta_k,\beta_j<z} \left((z-\beta_k)-(z-\beta_j)\right)(\beta_k-\beta_j)\,|\langle u_k, x_\alpha u_j\rangle|^2,$$

which simplifies to

$$-\sum_{j,k:\beta_k,\beta_j<z} (\beta_k-\beta_j)^2\,|\langle u_k, x_\alpha u_j\rangle|^2 \le 0,$$

as claimed in (1.4.1). In order to establish (1.4.2), multiply (1.4.4) by $(z-\beta_j)$ and then sum on j for $\beta_j < z$. Observe that the right side equates to 0 since the summand on the right side is odd in the exchange of j and k. $\square$

Some consequences of more general forms of the Harrell-Stubbe trace inequality are worked out in [23].

1.5 Results

This structure of the thesis is as follows:

- Chapter 2 provides two results for the eigenvalues β_j of $H_{m,\Omega}$ by using a trace inequality for $H_{m,\Omega}$, which will be proved in Section 2.1. The first such result is obtained by rewriting the trace inequality as a quadratic polynomial and comparing the roots of that polynomial. This will lead to a bound for $\dfrac{\beta_2}{\beta_1}$ and a bound for the fundamental spectral gap $\beta_2-\beta_1$. Corollary 2.2.4 then provides a bound for the fundamental gap in terms of the first eigenvalue of the Dirichlet Laplacian on the unit ball of $\mathbb{R}^d$ and the inradius $\mathrm{Inr}(\Omega)$.

  Secondly, the trace inequality is applied to a function related to the Riesz mean $R_\sigma(z) := \sum_j (z-\beta_j)_+^\sigma$ of order σ. This results in a differential inequality, which in turn provides a bound for the Riesz mean $R_1(z)$ of order 1. An application of the Legendre transform to that result results in an upper bound for the ratio $\dfrac{\overline{\beta}_k}{\overline{\beta}_j}$ in terms of the indexes k, j and the dimension d. To my knowledge, it is the first result which gives a bound for that ratio.

- Chapter 3 answers the following question in the vein of Weyl's asymptotic formula: "What is the Weyl formula for the Klein-Gordon operator $H_{m,\Omega}$?" Section 3.2 addresses this question and provides a semiclassical bound for the Klein-Gordon operator

$H_{m,\Omega}$ by utilizing Karamata's Tauberian theorem. By using this bound, a counterpart for $H_{0,\Omega}$ to the Berezin-Li-Yau inequality is obtained in Section 3.3.

- Chapter 4 provides an improved bound for the Berezin-Li-Yau inequality given in Chapter 3 by using symmetric decreasing rearrangenments of functions. The inspiration is the article [43] written by A. Melas in 2002.

- Chapter 5 is devoted to an analogue of the results of Chapter 2 for the case of $H_{m,\Omega}+V(x)$, where V is a real valued locally L^1 function. The result is first proved for $V \geq 0$ by using the fact that the function $\dfrac{1}{x}$ is monotone decreasing and by using the trace inequality for $H_{m,\Omega}$. As for more general potentials V, it is necessary to impose the condition V in some L^s space for $2 \leq d < s < \infty$.

- By using the same strategy in Chapter 2, a trace inequality for $(-\Delta)^s$, $s \in (0,1]$ in Chapter 6 is obtained. A differential inequality in the case of $(-\Delta)^s$ is obtained and implies an upper bound for the Riesz mean $R_1(z)$.

- In Chapter 7, sharp bounds for the spectral function $U_\sigma(z) := \sum_j \dfrac{(z-\beta_j)_+^\sigma}{\beta_j}$ and for the function $\tilde{Z}(t) := \sum_j \dfrac{e^{-\beta_j t}}{\beta_j}$ will be provided. Here, the eigenvalues β_j are the eigenvalues of $H_{m,\Omega}$. Many transformations such as Laplace, the Riesz Iteration Method and the Legendre transform are used to get those results.

CHAPTER II

INEQUALITIES FOR SPECTRA OF $H_{M,\Omega}$

The purpose of the present chapter is twofold. First we follow the strategy of Harrell-Stubbe ([22]) to obtain a universal bound on β_{n+1} in terms of the statistical distribution of the lower eigenvalues.

The second purpose of this chapter is to derive a differential inequality that will be useful for controlling the spectrum of the Klein-Gordon operator $H_{m,\Omega}$.

2.1 A Trace Inequality for $H_{m,\Omega}$

In this chapter, the main tool is the Harrell-Stubbe trace inequality (1.4.2) from [22]:

$$\sum_{j:\beta_j \le z} (z-\beta_j)^2 \langle u_j, [x_\alpha, [H, x_\alpha]]\, u_j\rangle - 2(z-\beta_j)\|\, [H, x_\alpha]\, u_j\|^2 \le 0. \tag{2.1.1}$$

In the case at hand, $H = H_{m,\Omega}$. The Fourier transform of $H_{m,\Omega}$ can be defined on a dense subspace of $L^2(\Omega)$ obtained by the closure of $C_c^\infty(\Omega)$. The Fourier transform is defined on $C_c^\infty(\Omega)$ in the natural way as embedded in $C_c(\mathbb{R}^d)$. Hence,

$$H_{m,\Omega} := \chi_\Omega \mathfrak{F}^{-1}\sqrt{|\xi|^2+m^2}\,\mathfrak{F}.$$

Then the first and second commutators are computed in the following way:

$$\begin{aligned}
[H_{m,\Omega}, x_\alpha]\,\varphi &= (H_{m,\Omega}\, x_\alpha - x_\alpha H_{m,\Omega})\varphi \\
&= \chi_\Omega \mathfrak{F}^{-1}\sqrt{|\xi|^2+m^2}\,\mathfrak{F}[x_\alpha\varphi] - \chi_\Omega x_\alpha \mathfrak{F}^{-1}[\sqrt{|\xi|^2+m^2}\,\hat{\varphi}] \\
&= i\chi_\Omega \mathfrak{F}^{-1}\left[\sqrt{|\xi|^2+m^2}\,\frac{\partial\hat{\varphi}}{\partial\xi_\alpha} - \frac{\partial}{\partial\xi_\alpha}(\sqrt{|\xi|^2+m^2}\,\hat{\varphi})\right] \\
&= -i\chi_\Omega \mathfrak{F}^{-1}\frac{\xi_\alpha}{\sqrt{|\xi|^2+m^2}}\hat{\varphi}.
\end{aligned} \tag{2.1.2}$$

Similarly,

$$[x_\alpha, [H_{m,\Omega}, x_\alpha]]\varphi = \chi_\Omega \mathfrak{F}^{-1}\left[\left(\frac{1}{\sqrt{|\xi|^2+m^2}} - \frac{{\xi_\alpha}^2}{(|\xi|^2+m^2)^{3/2}}\right)\hat{\varphi}\right]. \tag{2.1.3}$$

A summing over α simplifies (2.1.2) and (2.1.3), and thereby yields:

$$\sum_{\alpha=1}^{d} \| \, [H_{m,\Omega}, x_\alpha] \, \varphi\|^2 \leq \left\langle \hat{\varphi}, \frac{|\xi|^2}{|\xi|^2+m^2}\hat{\varphi} \right\rangle \leq 1,$$

and

$$\begin{aligned}\sum_{\alpha=1}^{d}\left(\frac{1}{\sqrt{|\xi|^2+m^2}} - \frac{{\xi_\alpha}^2}{(|\xi|^2+m^2)^{3/2}}\right) &= \frac{(d-1)|\xi|^2 + d\,m^2}{(|\xi|^2+m^2)^{3/2}} \\ &\geq \frac{d-1}{\sqrt{|\xi|^2+m^2}}.\end{aligned}$$

Thus (2.1.1) for $H_{m,\Omega}$ becomes

$$(d-1)\sum_{j=1}^{n}(z-\beta_j)^2\langle u_j, H_{m,\Omega}^{-1}u_j\rangle - 2\sum_{j=1}^{n}(z-\beta_j) \leq 0, \tag{2.1.4}$$

provided $z \in [\beta_n, \beta_{n+1}]$.

Observe that because $\mathrm{supp(u_j)} \in \Omega$,

$$\begin{aligned}\beta_j &= \langle u_j, H_{m,\Omega}u_j\rangle \\ &= \langle u_j, \chi_\Omega \mathfrak{F}^{-1}\sqrt{|\xi|^2+m^2}\hat{u}_j\rangle \\ &= \langle u_j, \mathfrak{F}^{-1}\sqrt{|\xi|^2+m^2}\hat{u}_j\rangle \\ &= \langle \hat{u}_j, \sqrt{|\xi|^2+m^2}\hat{u}_j\rangle.\end{aligned}$$

The eigenfunctions u_j are normalized by assumption. Thus

$$\begin{aligned}1 &= \langle \hat{u}_j, \hat{u}_j\rangle^2 \\ &= \langle (|\xi|^2+m^2)^{1/4}\hat{u}_j, (|\xi|^2+m^2)^{-1/4}\hat{u}_j\rangle^2 \\ &\leq \|(|\xi|^2+m^2)^{1/4}\hat{u}_j\|_2^2 \|(|\xi|^2+m^2)^{-1/4}\hat{u}_j\|_2^2 \\ &= \beta_j\langle \hat{u}_j, (|\xi|^2+m^2)^{-1/2}\hat{u}_j,\end{aligned}$$

which implies that

$$\frac{1}{\beta_j} \leq \langle \hat{u}_j, H_{m,\Omega}^{-1}\hat{u}_j\rangle. \tag{2.1.5}$$

Hence by (2.1.5) together with

$$(z-\beta_j) = -\frac{(z-\beta_j)(z-\beta_j-z)}{\beta_j},$$

(2.1.4) can be rewritten as

$$(d+1)\sum_{j=1}^{n}\frac{(z-\beta_j)^2}{\beta_j} - 2z\sum_{j=1}^{n}\frac{(z-\beta_j)}{\beta_j} \leq 0, \tag{2.1.6}$$

2.2 An Upper Bound for Eigenvalues of $H_{m,\Omega}$

In this section, the trace inequality (2.1.6) is rewritten as an inequality for a quadratic polynomial in z, which implies an upper bound for eigenvalues through analysis of its roots. First, the notation for the normalized moments of the eigenvalues is introduced as follows:

Definition 2.2.1 *Let r be a given real number. For an integer $n > 0$,*

$$\overline{\beta_n^r} := \frac{1}{n}\sum_{j=1}^{n} \beta_j^r.$$

We write $\overline{\beta_n} = \overline{\beta_n^1}$ for $r = 1$.

Theorem 2.2.2 *For each positive integer n, the eigenvalues β_n of $H_{m,\Omega}$ satisfy*

$$\beta_{n+1} \le \frac{1}{(d-1)\overline{\beta_n^{-1}}}\left(d + \sqrt{d^2 - (d^2-1)\overline{\beta_n}\,\overline{\beta_n^{-1}}}\right), \tag{2.2.1}$$

provided that $d \ge 2$.

Remark 2.2.3 *By using the Cauchy-Schwarz inequality,*

$$1 \le \overline{\beta_n}\overline{\beta_n^{-1}},$$

we simplify (2.2.1) and get a bound for β_{n+1} in terms of β_n, which is

$$\beta_{n+1} \le \frac{d+1}{(d-1)\overline{\beta_n^{-1}}} \le \frac{d+1}{d-1}\overline{\beta_n}. \tag{2.2.2}$$

Let $n = 1$. Then (2.2.2) becomes

$$\frac{\beta_2}{\beta_1} \le \frac{d+1}{d-1}. \tag{2.2.3}$$

Equivalently, we get a bound on the fundamental gap

$$\beta_2 - \beta_1 \le \frac{2}{d-1}\beta_1. \tag{2.2.4}$$

Observe that we assumed only that the domain is bounded. Thus, this result is independent of the shape or size of the domain.

Proof of Theorem 2.2.2. Observe that (2.1.6) can be rewritten as

$$\sum_{j=1}^{n} \frac{(d+1)(z^2 - 2z\beta_j + \beta_j^2)}{\beta_j} - 2z \sum_{j=1}^{n} \frac{(z-\beta_j)}{\beta_j} \leq 0,$$

which implies that

$$\sum_{j=1}^{n} \frac{(d-1)z^2 - 2dz\beta_j + (d+1)\beta_j^2}{\beta_j} \leq 0,$$

or,

$$(d-1)z^2 \sum_{j=1}^{n} \frac{1}{\beta_j} - 2dz + (d+1) \sum_{j=1}^{n} \beta_j \leq 0.$$

Using the notation for $\overline{\beta}_n$,

$$(d-1)\overline{\beta_n^{-1}} z^2 - 2dz + (d+1)\overline{\beta_n} \leq 0. \tag{2.2.5}$$

Set $z = \beta_{n+1}$ in (2.2.5). Then β_{n+1} must be less than the larger root of (2.2.5), i.e.,

$$\beta_{n+1} \leq \frac{1}{(d-1)\overline{\beta_n^{-1}}} \left(d + \sqrt{d^2 - (d^2-1)\overline{\beta_n}\,\overline{\beta_n^{-1}}} \right).$$

□

Recall that the inradius $\mathrm{Inr}(\Omega)$ of a region Ω is defined by

$$\mathrm{Inr}(\Omega) = \sup\{d(\mathbf{x}) : \mathbf{x} \in \Omega\},$$

where $d(\mathbf{x}) = \min\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \notin \Omega\}$ [12].

For $m = 0$, and in the case of a bounded convex domain, R. Bañuelos and T. Kulczycki have proved in [5] that the fundamental gap of the Cauchy process is controlled by the inradius $\mathrm{Inr}(\Omega)$,

$$\beta_2 - \beta_1 \leq \frac{\sqrt{\lambda_2^*} - (1/2)\sqrt{\lambda_1^*}}{\mathrm{Inr}(\Omega)},$$

where λ_1^* and λ_2^* are the first and second eigenvalues for the Dirichlet Laplacian for the unit ball, B_1 in $\mathbb{R}^d$.

Corollary 2.2.4 *If β_1^* and λ_1^* denote the fundamental eigenvalues of $H_{0,\Omega}$ and $-\Delta$, respectively, on the unit ball of $\mathbb{R}^d$, then*

$$\beta_2 - \beta_1 \leq \left(\frac{2}{d-1}\right) \frac{\beta_1^*}{\mathrm{Inr}(\Omega)} \leq \left(\frac{2}{d-1}\right) \frac{\sqrt{\lambda_1^*}}{\mathrm{Inr}(\Omega)}. \tag{2.2.6}$$

Proof. Since $H_{0,\Omega}$ is defined by closure from a core of functions in C_c^∞, its fundamental eigenvalue satisfies the principle of domain monotonicity. That is, if $\Omega_1 \supset \Omega_2$, then

$$\beta_1(\Omega_1) \leq \beta_1(\Omega_2).$$

In particular, if Ω is a ball of radius r, then

$$\beta_1(\Omega) \leq \frac{\beta_1^*}{r},$$

which is the fundamental eigenvalue of the unit ball $B_1(0)$ by scaling. The first inequality,

$$\beta_2 - \beta_1 \leq \left(\frac{2}{d-1}\right)\frac{\beta_1^*}{\mathrm{Inr}(\Omega)},$$

follows from (2.2.3), and the second one by (1.1.6), which is

$$\beta_1^* \leq \sqrt{\lambda_1^*}.$$

This completes the proof. □

2.3 A Ratio Bound

The "Riesz mean" of order σ is defined as follows:

$$R_\sigma(z) := \sum_k (z-\beta_k)_+^\sigma,$$

where z is a real variable and $a_+ := \max(0, a)$. In this section, the trace inequality (2.1.1) is applied to a function related to the Riesz mean to obtain an upper bound for the ratio $\frac{\overline{\beta_k}}{\overline{\beta_n}}$ for any $k > n$ where β_n's and β_k's are the eigenvalues of $H_{m,\Omega}$. Let

$$U(z) := \sum_k \frac{(z-\beta_k)_+^2}{\beta_k}, \tag{2.3.1}$$

where z is a real variable.

If $z \in [\beta_n, \beta_{n+1}]$, then

$$U(z) = n(\overline{\beta_n^{-1}}z^2 - 2z + \overline{\beta_n}). \tag{2.3.2}$$

Theorem 2.3.1 *The function $z^{-(d+1)}U(z)$ is nondecreasing in the variable z. Moreover, for $d \geq 2$ and any $n \geq 1$, the "Riesz mean" $R_1(z)$ satisfies*

$$R_1(z) \geq \left(\frac{2n(d-1)^d}{(d+1)^{d+1}\overline{\beta_n}^{\,d}}\right) z^{d+1} \tag{2.3.3}$$

for all $z \geq \left(\frac{d+1}{d-1}\right)\overline{\beta_n}$.

Proof. Eq.(2.1.6) tells us that

$$(d+1)\sum_{k=1}^{n}\frac{(z-\beta_k)_+^2}{\beta_k} - 2z\sum_{k=1}^{n}\frac{(z-\beta_k)_+}{\beta_k} \leq 0.$$

For the function U, this becomes

$$(d+1)U(z) - zU^{'}(z) \leq 0,$$

or, equivalently,

$$\frac{d}{dz}\left\{\frac{U(z)}{z^{d+1}}\right\} \geq 0, \tag{2.3.4}$$

which proves that the function $z^{-(d+1)}U(z)$ is nondecreasing.

On the other hand, Eq. (2.1.4) implies that

$$R_1(z) \geq \frac{d-1}{2}\, U(z). \tag{2.3.5}$$

Since $\frac{U(z)}{z^{d+1}}$ is nondecreasing, for all $z \geq z_{n*} \geq \beta_n$,

$$U(z) \geq \left(\frac{z}{z_{n*}}\right)^{d+1} U(z_{n*}). \tag{2.3.6}$$

Eq. (2.3.2) together with the Cauchy-Schwarz inequality imply that

$$U(z) \geq \frac{n}{\overline{\beta_n}}(z-\overline{\beta_n})^2. \tag{2.3.7}$$

Then, by using (2.3.5), (2.3.6) and (2.3.7), we obtain

$$R_1(z) \geq \frac{(d-1)n}{2\overline{\beta_n}}\left(\frac{z}{z_{n*}}\right)^{d+1}\left(z_{n*}-\overline{\beta_n}\right)^2. \tag{2.3.8}$$

Now choose an optimized value of z_{n*} to maximize the coefficient of z^{d+1}, *viz.*,

$$z_{n*} = \frac{d+1}{d-1}\overline{\beta_n}.$$

Substituting this into (2.3.8) gives

$$\begin{aligned} R_1(z) &\geq \frac{(d-1)n}{2\overline{\beta_n}}\left(\frac{z}{\frac{d+1}{d-1}\overline{\beta_n}}\right)^{d+1}\left(\frac{d+1}{d-1}\overline{\beta_n}-\overline{\beta_n}\right)^2 \\ &= \frac{(d-1)n}{2\overline{\beta_n}}\left(\frac{z(d-1)}{(d+1)\overline{\beta_n}}\right)^{d+1}\overline{\beta_n}^2\left(\frac{2}{d-1}\right)^2 \\ &= \frac{2n(d-1)^d z^{d+1}}{(d+1)^{d+1}\overline{\beta_n}^{d+1}} \end{aligned}$$

for all $z \geq \left(\frac{d+1}{d-1}\right)\overline{\beta_n}$. $\square$

The Legendre transform of $R_1(z)$ is

$$R_1^*(w) = (w - [w])\beta_{[w]+1} + [w]\overline{\beta_{[w]}},$$

where $[w]$ denotes the greatest integer $\leq w$. (The definition, some properties of the Legendre transform and the detailed computation of $R_1^*(w)$ can be found in Appendix B.)

When w approaches an integer value k from below, $R_1^*(k) = k\overline{\beta}_k$. Thus, by taking the Legendre transform of both sides of (2.3.3), obtain

$$k\overline{\beta}_k \leq \frac{d\ \overline{\beta}_n}{2^{1/d}\, n^{1/d}(d-1)} k^{\frac{d+1}{d}}. \tag{2.3.9}$$

This leads us to the following upper bound for ratios of averages of eigenvalues of $H_{m,\Omega}$:

Corollary 2.3.2 *For $k > 2n$, Eq. (2.3.9) implies*

$$\frac{\overline{\beta}_k}{\overline{\beta}_n} \leq \frac{d}{2^{1/d}(d-1)} \left(\frac{k}{n}\right)^{\frac{1}{d}}. \tag{2.3.10}$$

Remark 2.3.3 *The reason for the restriction on k, n is that in Theorem 2.3.1, it was assumed that $z \geq \left(\frac{d+1}{d-1}\right)\overline{\beta_n}$. Since the maximizing value of z_{n*} in the calculation of the Legendre transform of the right side of (2.3.3) depends monotonically on w,*

$$w = 2n\left(\frac{(d-1)z_{n*}}{(d+1)\,\overline{\beta_n}}\right)^d. \tag{2.3.11}$$

Thus the inequality is valid under the assumption that $k > w \geq 2n$.

Now, by using an argument from the article [24] I show that the condition $k > 2n$ can be improved to $k > n$. Define

$$D_n := \left(\frac{d}{d-1}\right)^2 - \left(\frac{d+1}{d-1}\right)\overline{\beta_n^{-1}}\,\overline{\beta_n} \leq \frac{1}{(d-1)^2}, \tag{2.3.12}$$

by the Cauchy-Schwarz inequality.

Theorem 2.3.4 *For all $k > n$,*

$$\overline{\beta_k} \leq \frac{d^{1+1/d}(d-1)^{-1/d}}{(d+1)\overline{\beta_n^{-1}}} \left(\frac{d}{d-1} + \sqrt{D_n}\right)^{1-1/d} \left(\frac{k}{n}\right)^{1/d}. \tag{2.3.13}$$

Observe that the Cauchy-Schwarz inequality $1 \leq \overline{\beta_n^{-1}}\,\overline{\beta_n}$ implies

$$\sqrt{D_n} \leq \frac{1}{d-1}. \tag{2.3.14}$$

Inserting (2.3.14) into (2.3.13) implies that

$$\overline{\beta_k} \leq \frac{d^{1+1/d}(d-1)^{-1/d}}{(d+1)\overline{\beta_n^{-1}}} \left(\frac{d+1}{d-1}\right)^{1-1/d} \left(\frac{k}{n}\right)^{1/d}.$$

Applying Cauchy-Schwarz one more time and after some simplifications the following simpler but slightly weaker inequality is obtained:

Corollary 2.3.5 *For all $k > n$,*

$$\frac{\overline{\beta_k}}{\overline{\beta_n}} \leq \left(\frac{d}{d-1}\right)\left(\frac{d}{d+1}\right)^{1/d}\left(\frac{k}{n}\right)^{1/d}. \tag{2.3.15}$$

Proof. As has been already proven in Section 2.1, the Riesz mean and $U(z)$ satisfy

$$R_1(z) \geq \frac{d-1}{2}\, U(z), \tag{2.3.16}$$

as a result of the Harrell-Stubbe trace inequalities. Observe that

$$U(z) \geq \sum_{k=1}^{n} \frac{(z-\beta_k)_+^2}{\beta_k}.$$

In addition, for $d \geq 2$ and any $j \geq 1$, we know that $\frac{U(z)}{z^{d+1}}$ is nondecreasing in z by Theorem 2.3.1. Then for all $\xi \geq z \geq \beta_n$,

$$\frac{U(\xi)}{\xi^{d+1}} \geq \frac{U(z)}{z^{d+1}} \geq z^{-(d+1)} \sum_{k=1}^{n} \frac{(z-\beta_k)^2}{\beta_k} \tag{2.3.17}$$

We optimize right side of (2.3.17) with respect to z. Since

$$\frac{d}{dz}\left(z^{-(d+1)} \sum_{k=1}^{n} \frac{(z-\beta_k)^2}{\beta_k}\right) = -(d-1)z^{-d-2} \sum_{k=1}^{n} \frac{z-\beta_k}{\beta_k}\left[z - \left(\frac{d+1}{d-1}\right)\beta_k\right].$$

an optimized choice of z_* will be the largest root of

$$\begin{aligned} P(z) &= \sum_{k=1}^{n} \frac{(z-\beta_k)}{\beta_k}\left[z - \left(\frac{d+1}{d-1}\right)\beta_k\right] \\ &= n\left(z^2\overline{\beta_n^{-1}} - \left(\frac{2d}{d-1}\right)z + \left(\frac{d+1}{d-1}\overline{\beta_n}\right)\right). \end{aligned} \tag{2.3.18}$$

Notice that the polynomial $P(z)$ is the same polynomial in (2.2.5) and therefore $P(z) \leq 0$ for all $z \in (\beta_n, \beta_{n+1})$. As a result, $z_* \geq \beta_n$ and given by

$$z_* = \frac{d}{(d-1)\overline{\beta_n^{-1}}} + \frac{\sqrt{D_n}}{\overline{\beta_n^{-1}}}, \tag{2.3.19}$$

where D_n is defined by (2.3.12). Then, equation (2.3.17) becomes

$$\begin{aligned} \frac{U(\xi)}{\xi^{d+1}} &\geq {z_*}^{-(d+1)} \sum_{k=1}^{n} \frac{(z_* - \beta_k)^2}{\beta_k} \\ &= \left(\frac{2}{d+1}\right) {z_*}^{-d} \sum_{k=1}^{n} \frac{z_* - \beta_k}{\beta_k} \end{aligned} \tag{2.3.20}$$

Together, (2.3.16) and (2.3.20) imply that

$$\frac{R_1(\xi)}{\xi^{d+1}} \geq \left(\frac{d-1}{d+1}\right) {z_*}^{-d} \sum_{k=1}^{n} \frac{z_* - \beta_k}{\beta_k},$$

or, equivalently,

$$R(\xi) \geq \left(\frac{(d-1){z_*}^{-d}}{d+1} \sum_{k=1}^{n} \frac{z_* - \beta_k}{\beta_k}\right) \xi^{d+1} \tag{2.3.21}$$

Taking the Legendre transform of both sides we get

$$(w - [w])\beta_{[w]+1} + \sum_{k=1}^{[w]} \beta_k \leq \frac{d(d-1)^{-1/d} z_*}{d+1} \left(\sum_{k=1}^{n} \frac{z_* - \beta_k}{\beta_k}\right)^{-1/d} w^{1+1/d}. \tag{2.3.22}$$

If we write $\displaystyle\sum_{k=1}^{n} \frac{z_* - \beta_k}{\beta_k} = n(z_* \overline{\beta_n^{-1}} - 1)$, then (2.3.22) becomes

$$(w - [w])\beta_{[w]+1} + \sum_{k=1}^{[w]} \beta_k \leq \frac{d}{d+1}(d-1)^{-1/d} z_* (n(z_* \overline{\beta_n^{-1}} - 1))^{-1/d} w^{1+1/d}. \tag{2.3.23}$$

When w approaches k from below, we obtain

$$\overline{\beta_k} \leq \frac{d}{d+1}(d-1)^{-1/d} z_* (z_* \overline{\beta_n^{-1}} - 1)^{-1/d} \left(\frac{k}{n}\right)^{1/d}. \tag{2.3.24}$$

Observe that, by the definition (2.3.19) of z_*, it follows that $z_* \geq \dfrac{d}{(d-1)\overline{\beta_n^{-1}}}$, which , after some simplifications, implies

$$z_* \overline{\beta_n^{-1}} - 1 \geq \frac{1}{d} z_* \overline{\beta_n^{-1}}.$$

When this is inserted into (2.3.24), the result in (2.3.13) follows. $\square$

Remark 2.3.6 *Defining*

$$f(\xi) = \left(\frac{(d-1){z_*}^{-d}}{d+1} \sum_{k=1}^{n} \frac{z_* - \beta_k}{\beta_k} \right) \xi^{d+1},$$

the relation between the maximizing value in the Legendre transform of the right side of (2.3.21) *is given by* $w = f'(\xi^*)$, *i.e.,*

$$w = n(d-1)(z_* \overline{\beta_n^{-1}} - 1) \left(\frac{\xi^*}{z_*} \right)^d .$$

Therefore, the inequality is valid under the assumption that $k > w \geq n$.

CHAPTER III

WEYL ASYMPTOTICS AND SEMICLASSICAL BOUNDS FOR $H_{M,\Omega}$

Throughout this chapter, Ω is assumed to be a bounded domain in $\mathbb{R}^d$, and $|\Omega|$ denotes the volume of Ω. Section 3.1 begins by giving the statements of the Weyl asymptotic formula and Berezin-Li-Yau inequality for the Dirichlet Laplacian on a bounded domain Ω in $\mathbb{R}^d$. In Section 3.2, one of the standard proofs of the aymptotic formula is adapted to give the analogue of the Weyl formula for $H_{m,\Omega}$. Finally, in Section 3.3, a counterpart for $H_{0,\Omega}$ to the Berezin-Li-Yau inequality ([40]) for the Laplacian is proved.

3.1 Weyl Asymptotics and Semiclassical bounds for $-\Delta$

Let us recall the classical estimate of Weyl for the Laplacian. Consider the eigenvalue problem

$$\begin{aligned} -\Delta\psi &= \lambda\psi \text{ on } \Omega \\ \psi &= 0 \text{ on } \partial\Omega \end{aligned}$$

Let λ_k be the kth eigenvalue of the Dirichlet Laplacian and ψ_k be the corresponding eigenfunction. The spectrum of $-\Delta$ is discrete and we its eigenvalues satisfy

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$$

counting multiplicities.

Theorem 3.1.1 *(Weyl's asymptotic formula) As $k \to \infty$,*

$$\lambda_k \sim C_d \left(\frac{k}{|\Omega|}\right)^{2/d}$$

where $C_d = 4\pi^2|B_d|^{-2/d}$ with B_d being the d-dimensional unit ball.

Next, the statement of the Berezin-Li-Yau inequality for the Dirichlet Laplacian is given. Note that the Li-Yau inequality is equivalent to an earlier inequality by Berezin [7] through the Legendre transform, as stated in [39]. (See also [41]).

In their paper [40], P.Li and S-T.Yau proved the foowing theorem:

Theorem 3.1.2 *(Berezin-Li-Yau Inequality for the Dirichlet Laplacian $-\Delta$) Suppose λ_k denotes the kth eigenvalue of the Dirichlet Laplacian $-\Delta$ on a bounded domain Ω in $\mathbb{R}^d$. Then*

$$\sum_{j=1}^{k} \lambda_j \geq \frac{dC_d}{d+2} k^{\frac{d+2}{d}} |\Omega|^{-2/d}$$

where $|\Omega|$ is the volume of Ω.

3.2 Weyl Asymptotics and Semiclassical bounds for $H_{m,\Omega}$

This section considers the eigenvalues β_k of $H_{m,\Omega}$ as $k \to \infty$. Note that R. M. Blumenthal and R. K. Getoor obtained the asymptotic distribution of the eigenvalues for a class of Markov operators for α-stable processes by using Karamata's Tauberian theorem in [8].

Here it suffices to consider the case $m = 0$ because of the inequalities (1.1.1), and the fact that

$$\lim_{|\xi| \to \infty} \frac{\sqrt{|\xi|^2 + m^2}}{|\xi|} = 1.$$

For $t > 0$, the partition function $Z(t)$ is defined as

$$Z(t) := \sum_{j=1}^{\infty} e^{-\beta_j t}.$$

If $N(\beta) := \sum_{\beta_j \leq \beta} 1$ is the counting function, then $Z(t)$ can also be written as

$$Z(t) = \int e^{-\beta t} dN(\beta). \tag{3.2.1}$$

Apart from these definitions, there is an equivalent definition of the partition function $Z(t)$ in terms of the integral kernel $p_\Omega(\mathbf{x}, \mathbf{y}, t)$ which is

$$Z(t) = \int_\Omega p_\Omega(\mathbf{x}, \mathbf{x}, t) d\mathbf{x}. \tag{3.2.2}$$

Here this definition is used to get the results. If we accept that $H_{m,\Omega}$ is well approximated by $\sqrt{-\Delta_\Omega}$ in the "semiclassical limit," then the analogue for $N(\beta)$ of the Weyl asymptotic formula for the Laplacian should be identical to the usual Weyl formula, with the identification of β_k with $\sqrt{\lambda_k}$. This intuition is confirmed by the following:

Proposition 3.2.1 *As $\beta \to \infty$,*

$$N(\beta) \sim \frac{|\Omega|}{(4\pi)^{d/2}\Gamma(1+d/2)}\beta^d. \tag{3.2.3}$$

Equivalently, as $k \to \infty$,

$$\beta_k \sim \sqrt{4\pi}\left(\frac{\Gamma(1+d/2)k}{|\Omega|}\right)^{1/d}. \tag{3.2.4}$$

Moreover, the function U of (2.3.1) satisfies

$$U(z) \sim \frac{|\Omega|}{(4\pi)^{d/2}(d^2-1)\Gamma(1+d/2)}z^{d+1}. \tag{3.2.5}$$

The main tool used here is Karamata's Tauberian Theorem:

Theorem 3.2.2 *(Karamata's Tauberian Theorem, [55]) Let μ be a positive Borel measure on $[0,\infty)$. Suppose that*

$$\int e^{-tx}d\mu < 0$$

for all $t > 0$ and

$$\lim_{t\to 0} t^\gamma \int e^{-tx}d\mu(x) = A$$

for some $\gamma \geq 0$ and $A \geq 0$. Then

$$\lim_{z\to\infty} z^{-\gamma}\mu[0,z) = \frac{A}{\Gamma(\gamma+1)}. \tag{3.2.6}$$

Proof of Proposition 3.2.1. By using the second definition (3.2.1) for the partition function $Z(t)$, we obtain

$$Z(t) = \int e^{-\beta t}dN(\beta),$$

where $N(\beta) := \sum_{\beta_j \leq \beta} 1$ is the counting function. Observe that N is a positive Borel measure on $[0,\infty)$. It is enough to show that

$$\lim_{t\to 0} t^d Z(t) = c_d|\Omega|, \tag{3.2.7}$$

with $c_d := \dfrac{d!}{(4\pi)^{d/2}\Gamma(1+d/2)}$. By using the Karamata's Tauberian theorem, the first claim (3.2.3) is proved. To show (3.2.7), the last definition of $Z(t)$ is used, which is

$$Z(t) = \int_\Omega p_\Omega(\mathbf{x},\mathbf{x},t)d\mathbf{x}.$$

Here, $p_\Omega(\mathbf{x},\mathbf{x},t)$ is the integral kernel of the semigroup $e^{-tH_{0,\Omega}}$.

Recall that the integral kernel of the semigroup $e^{-t\sqrt{-\Delta}}$ is

$$p_0(\mathbf{x},\mathbf{y},t) = p_0(\mathbf{x}-\mathbf{y},t) = \frac{c_d t}{(t^2+|\mathbf{x}-\mathbf{y}|^2)^{\frac{d+1}{2}}}.$$

For all $t>0$ and $\mathbf{x},\mathbf{y}\in\Omega$, we have

$$p_\Omega(\mathbf{x},\mathbf{y},t) < p_0(\mathbf{x}-\mathbf{y},t), \tag{3.2.8}$$

on Ω. Define

$$r_\Omega(\mathbf{x},\mathbf{y},t) := p_0(\mathbf{x}-\mathbf{y},t) - p_\Omega(\mathbf{x},\mathbf{y},t),$$

and let $\delta_\Omega(\mathbf{x}) := \operatorname{dist}(\mathbf{x},\partial\Omega)$. According to [6],

$$0 \le r_\Omega(\mathbf{x},\mathbf{y},t) \le \frac{t}{\delta_\Omega^{d+1}(\mathbf{x})} c_d \mathcal{P}^{\mathbf{y}}(\tau_\Omega < t),$$

where $\mathcal{P}^{\mathbf{y}}(\tau_\Omega<t)$ is the probability that a path originating at $\mathbf{y}$ exits Ω before time t. From (3.2.8),

$$\int_\Omega p_\Omega(\mathbf{x},\mathbf{x},t)d\mathbf{x} \le \int_\Omega p_0(\mathbf{0},t)d\mathbf{x} = c_d\frac{|\Omega|}{t^d}. \tag{3.2.9}$$

Now we will show that the first integral in (3.2.9) is $o(t^{-d})$ when $t\to 0$:

$$\begin{aligned}
\int_\Omega p_\Omega(\mathbf{x},\mathbf{x},t)d\mathbf{x} &= \int_{\{\mathbf{x}:\delta_\Omega(\mathbf{x})<\sqrt{t}\}} p_\Omega(\mathbf{x},\mathbf{x},t)d\mathbf{x} + \int_{\{\mathbf{x}:\delta_\Omega(\mathbf{x})>\sqrt{t}\}} (p_0(\mathbf{0},t) - r_\Omega(\mathbf{x},\mathbf{x},t))d\mathbf{x} \\
&= \int_{\{\mathbf{x}:\delta_\Omega(\mathbf{x})<\sqrt{t}\}} p_\Omega(\mathbf{x},\mathbf{x},t)d\mathbf{x} + |\{x:\delta_\Omega(\mathbf{x})>\sqrt{t}\}|c_d t^{-d} \\
&- \int_{\{\mathbf{x}:\delta_\Omega(\mathbf{x})>\sqrt{t}\}} r_\Omega(\mathbf{x},\mathbf{x},t))d\mathbf{x}.
\end{aligned} \tag{3.2.10}$$

The first integral on the right side of (3.2.10) becomes

$$\begin{aligned}
0 \le \int_{\{\mathbf{x}:\delta_\Omega(\mathbf{x})<\sqrt{t}\}} p_\Omega(\mathbf{x},\mathbf{x},t))d\mathbf{x} &\le \int_{\{\mathbf{x}:\ \delta_\Omega(\mathbf{x})<\sqrt{t}\}} p_0(\mathbf{0},t)d\mathbf{x} \\
&\le c_d t^{-d}|\{x:\delta_\Omega(\mathbf{x})<\sqrt{t}\}| = o(t^{-d})
\end{aligned} \tag{3.2.11}$$

as $t\to 0$. As for the final integral of (3.2.10),

$$\begin{aligned}
0 \le \int_{\{\mathbf{x}:\delta_\Omega(\mathbf{x})>\sqrt{t}\}} r_\Omega(\mathbf{x},\mathbf{x},t))d\mathbf{x} &\le \int_{\{\mathbf{x}:\ \delta_\Omega(\mathbf{x})>\sqrt{t}\}} \frac{t}{\delta_\Omega^{d+1}(\mathbf{x})}d\mathbf{x} \\
&\le \frac{t}{t^{(d+1)/2}}|\Omega| \\
&= 0(t^{(1-d)/2}) = o(t^{-d}).
\end{aligned} \tag{3.2.12}$$

Equations (7.2.8) and (7.2.9) imply that

$$\lim_{t \to 0} t^d Z(t) = c_d |\Omega|.$$

All the conditions in Karamata's Tauberian Theorem are therefore satisfied. Thus, by taking $\gamma = d$ and $A = c_d|\Omega|$ in (3.2.6) we get

$$\lim_{\beta \to \infty} \beta^{-d} N(\beta) = \frac{c_d|\Omega|}{\Gamma(d+1)}.$$

Since $c_d = \dfrac{d!}{(4\pi)^{d/2}\Gamma(1+d/2)}$ and $\Gamma(d+1) = d!$, we get (3.2.3) by using Karamata's Tauberian Theorem.

The claims for β_k and $U(z)$ are easy consequences of (3.2.3). □

3.3 A Counterpart for $H_{0,\Omega}$ to the Berezin-Li-Yau Inequality

Recall, (2.3.7) which states that

$$U(z) \geq \frac{k}{\overline{\beta_k}}(z - \overline{\beta_k})^2.$$

Also,

$$z^{-(d+1)} U(z) \uparrow \frac{2c_d|\Omega|}{d!(d^2-1)}.$$

Because of (2.2.2), a choice of z safely guaranteed to exceed β_k is

$$z = \frac{d+1}{d-1}\overline{\beta_k}.$$

Thus,

$$\frac{2c_d|\Omega|}{d!(d^2-1)} \geq \frac{k}{\overline{\beta_k}} \left(\frac{2}{d-1}\overline{\beta_k}\right)^2 \left(\frac{d+1}{d-1}\overline{\beta_k}\right)^{-(d+1)}.$$

After rearranging terms we get the semiclassical estimate:

$$\overline{\beta_k} \geq \frac{(d-1)2^{1/d}\sqrt{4\pi}}{d+1} \left(\frac{\Gamma(1+d/2)k}{|\Omega|}\right)^{1/d}. \tag{3.3.1}$$

Next the argument of Li and Yau [40] is adapted to get a better estimate. Here the aim is to improve the term $(d-1)2^{1/d}$ in (3.3.1) to d. Begin by generalizing the lemma attributed in [40] to Hörmander:

Lemma 3.3.1 *Let $f : \mathbb{R}^d \to \mathbb{R}$ satisfy $0 \le f(\xi) \le M_1$. Assume that the weight function w is nonnegative and nondecreasing, and that*

$$\int_{\mathbb{R}^d} f(\xi) w(|\xi|) d\xi \le M_2. \tag{3.3.2}$$

Define $R = R(M_1, M_2)$ by the condition that

$$\int_{B_R} w(|\xi|) d\xi = \omega_{d-1} \int_0^R w(r) r^{d-1} dr = \frac{M_2}{M_1}. \tag{3.3.3}$$

Here ω_{d-1} denotes the volume of the d-dimensional unit ball $\mathbb{S}^{d-1}$, i.e.,

$$\omega_{d-1} := \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

Then

$$\int_{\mathbb{R}^d} f(\xi) d\xi \le \frac{\pi^{d/2} M_1}{\Gamma(1+d/2)} R^d. \tag{3.3.4}$$

As a special case, if $w(\xi) = |\xi|^p$, then $R = \left[\frac{M_2(d+p)}{M_1 w_{d-1}}\right]^{\frac{1}{d+p}}$, and so

$$\begin{aligned}\int_{\mathbb{R}^d} f(\xi) d\xi &\le \frac{1}{d}((d+p)M_2)^{\frac{d}{d+p}} (w_{d-1} M_1)^{\frac{-p}{d+p}} \\ &= \left(\frac{d+p}{d} M_2\right)^{\frac{d}{d+p}} \left(\frac{\pi^{d/2} M_1}{\Gamma(1+d/2)}\right)^{\frac{p}{d+p}}.\end{aligned}$$

Proof. Let $g(\xi) := M_1 \chi_{\{|\xi| \le R\}}$ and note that according to the definition of R,

$$\int w(|\xi|) g(\xi) d\xi = M_2. \tag{3.3.5}$$

Observe that

$$(w(|\xi|) - w(R))(f(\xi) - g(\xi)) \ge 0 \tag{3.3.6}$$

for all ξ. Indeed, if $|\xi| > R$, then $g(\xi) = 0$, $f(\xi) \ge 0$, and $w(|\xi|) - w(R) \ge 0$ as w is nondecresing; therefore (3.3.6) follows. If $|\xi| \le R$, then $w(|\xi|) - w(R) \le 0$ and $f(\xi) - g(\xi) = f(\xi) - M_1 \le 0$ as $f \le M_1$, and so (3.3.6) follows again. Hence, (3.3.6) together with (3.3.2) and (3.3.5) yields

$$w(R) \int (f(\xi) - g(\xi)) d\xi \le \int w(|\xi|)(f(\xi) - g(\xi)) \le 0, \tag{3.3.7}$$

and, consequently,

$$\int f(\xi)d\xi \leq \int g(\xi)d\xi = |B_R|M_1 = \frac{\pi^{d/2}M_1}{\Gamma(1+d/2)}R^d. \tag{3.3.8}$$

□

For the application to $H_{0,\Omega}$, note that

$$\beta_\ell = \langle u_\ell, H_{0,\Omega}u_\ell\rangle = \int |\xi||\hat{u}_\ell(\xi)|^2 d\xi. \tag{3.3.9}$$

Choosing $w(|\xi|) = |\xi|$ in the lemma and setting

$$f(\xi) = \sum_{\ell=1}^{k} |\hat{u}_\ell(\xi)|^2,$$

$$k = \int f(\xi)d\xi \leq \left(\|f\|_\infty \frac{\pi^{d/2}}{\Gamma(1+d/2)}\right)^{\frac{1}{d+1}} \left(\left(\sum_{\ell=1}^{k}\beta_\ell\right)\frac{d+1}{d}\right)^{\frac{d}{d+1}}, \tag{3.3.10}$$

or

$$\sum_{\ell=1}^{k}\beta_\ell \geq \frac{d}{d+1}\left(\frac{\Gamma(1+d/2)}{\pi^{d/2}\|f\|_\infty}\right)^{1/d} k^{1+\frac{1}{d}}. \tag{3.3.11}$$

As $\|e^{i\mathbf{x}\cdot\xi}\|_2^2 = |\Omega|$ and by applying Bessel's inequality,

$$\begin{aligned}\sum_{\ell=1}^{k}|\hat{u}_\ell(\xi)|^2 &= \sum_{\ell=1}^{k}\frac{1}{(2\pi)^d}\left|\int_\Omega e^{i\mathbf{x}\cdot\xi}u_\ell(\mathbf{x})d\mathbf{x}\right|^2 \\ &= \frac{1}{(2\pi)^d}\sum_{\ell=1}^{k}\left|\langle e^{i\mathbf{x}\cdot\xi}, u_k\rangle\right|^2 \leq \frac{|\Omega|}{(2\pi)^d}\end{aligned}$$

for $\|f\|_\infty$. In conclusion, we have an analogue of the Li-Yau inequality [40]:

Theorem 3.3.1 *For all $k = 1, \ldots,$ the eigenvalues β_k of $H_{0,\Omega}$ satisfy*

$$\overline{\beta_k} \geq \frac{\sqrt{4\pi}d}{d+1}\left(\frac{\Gamma(1+d/2)k}{|\Omega|}\right)^{1/d}. \tag{3.3.12}$$

Observe that, just like the Li-Yau inequality for the Laplacian, (3.3.12) has the best possible coefficient consistent with the Weyl-type law of Proposition 3.2.1. Moreover, in view of (1.1.6), Theorem 3.3.1 has a corollary for the Dirichlet- Laplacian:

$$\frac{1}{k}\sum_{\ell=1}^{k}\sqrt{\lambda_\ell} \geq \frac{\sqrt{4\pi}d}{d+1}\left(\frac{\Gamma(1+d/2)k}{|\Omega|}\right)^{1/d}, \tag{3.3.13}$$

which is comparable to the Li-Yau inequality, but neither implies it nor is directly implied by it . (For an alternative route to (3.3.13) see Theorem 5.1 of [23].)

CHAPTER IV

AN IMPROVEMENT TO THE BEREZIN-LI-YAU TYPE INEQUALITY FOR THE KLEIN-GORDON OPERATOR

Let Ω be a bounded open domain in $\mathbb{R}^d$ with $|\Omega|$ denoting its volume. The purpose of this chapter is to improve the bound in the Berezin-Li-Yau type inequlality (3.3.12) for the pseudodifferential operator $H_{0,\Omega} := \sqrt{-\Delta}$ restricted to Ω[62]. We denote the eigenvalues by $\{\beta_j\}_{j=1}^{\infty}$ and the corresponding orthonormal basis of real eigenvalues by $\{u_j\}_{j=1}^{\infty}$. Hence,

$$\beta_1 < \beta_2 \leq \beta_2 \leq \ldots \leq \beta_j \leq \ldots.$$

Before stating Theorem 4.3.1 and giving its proof in Section 4.3, we first introduce the notion of symmetric rearrangement of sets and functions in 4.1 and mention some important results involving improved bounds of the Berezin-Li-Yau inequality for the Laplacian in Section 4.2.

4.1 Symmetric Rearrangement of Sets and Functions

This section recalls some facts about rearrangements of sets and functions. Some important properties of rearrangements is also provided. For further details and discussion of the topic, refer to three excellent books; [41] written by E. Lieb and M. Loss, [27] written by A. Henrot and [33] written by B. Kawohl (and the references therein). Note that one of the most important applications of the symmetric (spherical) rearrangements of functions is the Faber-Krahn inequality [16, 36] for the first eigenvalue of the Dirichlet Laplacian $-\Delta$ on a bounded set Ω. Faber-Krahn is an isoperimetric inequality which states that the ball minimizes the first eigenvalue of the Dirichlet Laplacian amongst all bounded sets of the same volume.

Definition 4.1.1 *Let f be a nonnegative measurable function defined on the bounded set $\Omega \subset \mathbb{R}^d$. Assume that f vanishes on the boundary $\partial\Omega$. Then the level sets are defined as*

$$S_f(t) = \{x \in \Omega \mid |f(x)| > t\}.$$

The volume of the level sets define the distribution function μ_f of f [4], that is,

$$\mu_f(t) = |S_f(t)|.$$

Observe that the function $\mu_f(t)$ is nonincreasing. Indeed, if $t < s$, then $|f(x)| > s > t$ and that implies $S_f(t) \supseteq S_f(s)$. Therefore, $\mu_f(t) > \mu_f(s)$.

Definition 4.1.2 *Let Ω be a bounded set in $\mathbb{R}^d$. Then the open ball Ω^* centered at the origin that has the same volume as Ω is called the symmetric (spherical) rearrangement of the set Ω. I.e., it is the ball such that $|\Omega| = |\Omega^*|$.*

If χ_Ω denotes the characteristic function of Ω, then the symmetric decreasing rearrangement of χ_Ω is

$$\chi^*_\Omega = \chi_{\Omega^*}.$$

With the aid of characteristic functions on the level sets $S_f(t)$, we have the following definition:

Definition 4.1.3 *The symmetric (spherical) decreasing rearrangement f^* of a Borel measurable function $f : \mathbb{R}^d \to \mathbb{C}$ vanishing at infinity is defined as*

$$f^*(x) = \int_0^\infty \chi^*_{\{x \in \Omega \,:\, |f(x)| > t\}}(x)dt.$$

Next, some important properties of rearrangements will be pointed out and couple of immediate theorems involving rearrangements are mentioned:

Properties :

- The rearrangement $f^*(x)$ is radially symmetric and nonincreasing as a function of $|x|$.
- $f^*(x)$ is nonnegative.
- The level sets of f^* are the symmetric rearrangements of the level sets of f, i.e., $S_{f^*}(t) = S^*_f(t)$.
- The level sets of f and f^* have the same measure, i.e., $\mu_f(t) = \mu_{f^*}(t)$.

- Suppose f and g are two nonnegative functions on $\mathbb{R}^d$ such that $f(x) \leq g(x)$ for all $x \in \mathbb{R}^d$. Then $f^*(x) \leq g^*(x)$ for all $x \in \mathbb{R}^d$ because the level sets $S_f(t)$ are contained in the level sets $S_g(t)$ for all t. In other words, rearrangement is order preserving.

Although the following theorems are not used in this work, it is worth mentioning them as important applications of symmetric decreasing rearrangements. For further details and the proofs, refer to the books [27] and [41].

Theorem 4.1.4 *If ϕ is any measurable function from $\mathbb{R}^+ \to \mathbb{R}$, then*

$$\int_\Omega \phi \circ f(x)dx = \int_{\Omega^*} \phi \circ f^*(x)dx.$$

An immediate corollary to this theorem is that for $f \in L^p(\mathbb{R}^d)$,

$$\|f\|_p = \|f^*\|_p,$$

where $1 \leq p \leq \infty$.

Theorem 4.1.5 ***(Hardy-Littlewood Inequality [27])*** *If f and g are two nonnegative, measurable functions in $L^2(\Omega)$, then*

$$\int_\Omega f(x)g(x)dx \leq \int_{\Omega^*} f^*(x)g^*(x)dx.$$

Theorem 4.1.6 ***(Pólya's Inequality [27])*** *Let $f \in W_0^{1,p}(\Omega)$ with $1 \leq p < \infty$ for an open set Ω. Then*

$$\int_\Omega |\nabla f(x)|^p dx \geq \int_{\Omega^*} |\nabla f^*(x)|^p dx.$$

Next, a special case of Pólya's inequality, which states that symmetric decreasing rearrangements decrease the kinetic energy, is given:

Theorem 4.1.7 *(Lieb and Loss, [41]) Let f be a nonnegative measurable real-valued function on $\mathbb{R}^d$ vanishing at infinity. Assume that $\|\nabla f\|_2$ is finite. Then $\|\nabla f^*\|_2$ is also finite and*

$$\|\nabla f(x)\|_2 \geq \|\nabla f^*(x)\|_2.$$

Similarly,

$$\langle f, |\mathbf{p}| f\rangle \geq \langle f^*, |\mathbf{p}| f^*\rangle,$$

provided that $\langle f, |\mathbf{p}| f\rangle < \infty$. Here, the operator $|\mathbf{p}|$ is different from the generator of the Caucy process because it is not restricted to a bounded domain $\Omega \subset \mathbb{R}^d$.

4.2 Introduction and Some Classical Results

Before an improvement to the Berezin-Li-Yau type inequality (3.3.12) is given in Section 4.3, some results that improve the bound in the Berezin-Li-Yau inequality for the eigenvalues of the Dirichlet Laplacian is provided. This will lead to see the analogy between the Laplacian and $H_{0,\Omega}$. Moreover, the proof of [43] is adapted, where A.D. Melas utilized symmetric decreasing rearrangement of functions to improve the bound in what is usually termed the Li-Yau inequality ([40]). Begin by recalling this inequality:

Theorem 4.2.1 *Let λ_j be the eigenvalues of the Dirichlet Laplacian on Ω. Then*

$$\sum_{j=1}^{k} \lambda_j \geq \frac{dC_d}{d+2} |\Omega|^{-2/d} k^{1+2/d}, \tag{4.2.1}$$

where $C_d = 4\pi\Gamma(1 + d/2)^{2/d}$.

Note that the inequality (4.2.1) can be obtained by a Legendre transform of an earlier result by Berezin[7] as stated in [39]. Thus, we prefer to call it the "Berezin-Li-Yau inequality" instead of the Li-Yau inequality. In [43], A.D. Melas proved that

$$\sum_{j=1}^{k} \lambda_j \geq \frac{dC_d}{d+2} |\Omega|^{-2/d} k^{1+2/d} + M_d k \frac{|\Omega|}{I(\Omega)}, \tag{4.2.2}$$

where the constant M_d depends only on the dimension. Here $I(\Omega)$ denotes the moment of inertia, i.e.,

$$I(\Omega) = \min_{u \in \mathbb{R}^d} \int_\Omega |x - u|^2 dx.$$

The improvement of the inequality (4.2.2) has recently been studied by many authors, (cf. [35], [59]). More precisely, in [35], H. Kovařík, S. Vugalter and T. Weidl improved (4.2.2) in two dimensions. Their proof for the $d = 2$ case relies upon the geometric properties of the boundary of Ω. There they first state and prove their result in the case of polygons, then in the case of general domains. One immediate difference is that their result has a second term that has the order of k as in the asymptotic behavior of the sum on the left hand side of (4.2.1):

$$\sum_{j=1}^{k} \lambda_j = \frac{dC_d}{d+2}|\Omega|^{-2/d}k^{1+2/d} + \tilde{C}_d\frac{|\partial\Omega|}{|\Omega|^{1+1/d}}k^{1+1/d} + o(k^{1+1/d}) \text{ as } k \to \infty. \tag{4.2.3}$$

As stated in [35], the correction term in (4.2.2) is of larger order than k, which appear in the asymptotics of (4.2.1).

Recall the Riesz mean of order σ:

$$R_\sigma(z) := \sum_j (z - \lambda_j)_+^\sigma.$$

Another analogous result is given in [59], where T. Weidl found a Berezin type bound for the Riesz mean $R_\sigma(z)$ when $\sigma > 3/2$. The second term in this bound is similar to the second term in the asymptotics of $R_\sigma(z)$, up to a constant. His method uses sharp Lieb-Thirring inequalities for operator valued potentials.

To get a similar improvement for $H_{0,\Omega}$, follow the basic strategy of [43], with some important differences of detail.

4.3 Statement and Proof of Theorem

The main result of this chapter is given below:

Theorem 4.3.1 *For $k \geq 1$ and the bounded set Ω,*

$$\sum_{j=1}^{k} \beta_j \geq \frac{d\tilde{C}_d}{d+1}|\Omega|^{-1/d}k^{1+1/d} + \tilde{M}_d\frac{|\Omega|^{1+1/d}}{I(\Omega)}k^{1-1/d}, \tag{4.3.1}$$

where $\tilde{C}_d = \sqrt{4\pi}\,\Gamma(1+d/2)^{1/d}$ and the constant $\tilde{M}_d$ depends only on the dimension d.

There are some differences and similarities between Melas's result for the Laplacian in (4.2.2) and our result (4.3.1) for $H_{0,\Omega}$. First, the power of k in the first term in (4.2.2) is $1+2/d$ while the corresponding power is $1+1/d$ in (4.3.1). This is not surprising because the Klein-Gordon operator can be viewed as the square root of the Laplacian in $\mathbb{R}^d$. Also, the improvement in (4.2.2) consists of $|\Omega|/I(\Omega)$ and in (4.3.1) it is $|\Omega|^{1+1/d}/I(\Omega)$. Moreover, the difference between the powers of the k terms on the right side of (4.3.1) is $2/d$ as in

(4.2.2).

Next, we will state and prove the following lemma, which is the crucial step in proving the theorem.

Lemma 4.3.1 *Let $d \geq 2$ and $\varphi : [0,\infty) \to [0,\infty)$ be a decreasing, absolutely continuous function. Assume that*

$$0 \leq -\varphi'(x) \leq m, \quad x > 0. \tag{4.3.2}$$

Then,

$$\begin{aligned} \int_0^\infty x^d \varphi(x)dx &\geq \frac{1}{d+1}\left(d\int_0^\infty x^{d-1}\varphi(x)dx\right)^{1+1/d}\varphi(0)^{-1/d} \\ &+ \frac{\varphi(0)^{2+1/d}}{6m^2(d^2-1)}\left(d\int_0^\infty x^{d-1}\varphi(x)dx\right)^{1-1/d}. \end{aligned} \tag{4.3.3}$$

Proof. Rescaling the function φ as

$$\eta(x) = \frac{1}{\varphi(0)}\varphi\left(\frac{\varphi(0)}{m}x\right), \tag{4.3.4}$$

implies $\eta(0) = 1$ and $0 \leq -\eta'(x) \leq 1$. To keep the notation simple, let $f(x) := -\eta'(x)$ for $x \geq 0$. Then, $0 < f(x) < 1$ for $x > 0$ and $\int_0^\infty f(x)dx = \eta(0) = 1$. Define

$$A := \int_0^\infty x^{d-1}\eta(x)dx \qquad \text{and} \qquad B := \int_0^\infty x^d \eta(x)dx. \tag{4.3.5}$$

Assume that $B < +\infty$, as otherwise the result is immediate. Thus, there exists a sequence $\{R_j\}$ such that $R_j \to \infty$ and $R_j^{d+1}\eta(R_j) \to 0$ as $j \to \infty$. An application of integration by parts yields

$$\int_0^\infty x^d f(x)dx = Ad, \qquad \text{and} \qquad \int_0^\infty x^{d+1}f(x)dx \leq (d+1)B.$$

By the Initial Value Theorem, we can find a $\alpha \geq 0$ such that

$$\int_\alpha^{\alpha+1} x^{d-1}dx = (Ad)^{1-1/d} \tag{4.3.6}$$

and

$$\int_\alpha^{\alpha+1} x^{d+1}dx \leq \int_0^\infty x^{d+1}f(x)dx \leq (d+1)B. \tag{4.3.7}$$

Next we will prove the following inequality by an induction argument:

$$(d-1)x^{d+1} - (d+1)y^2x^{d-1} + 2y^{d+1} \geq 2y^{d-1}(x-y)^2 \tag{4.3.8}$$

for $y > 0$ and $x \geq 0$. To prove (4.3.8), first divide both sides by y^{d+1}. After setting $\tau = \dfrac{x}{y}$, this is equivalent to show that $g(\tau) \geq 0$, where

$$g(\tau) := (d-1)\tau^{d+1} - (d+1)\tau^{d-1} - 2\tau^2 + 4\tau = (\tau-1)^2\tau\left(\sum_{k=0}^{d-3}(2k+4)\tau^k + (d-1)\tau^{d-2}\right).$$

An induction on d leads to the result. Next, we integrate (4.3.8) from α to $\alpha+1$ and use (4.3.6) and (4.3.7) to get

$$\begin{aligned}(d+1)(d-1)B - (d+1)y^2(Ad)^{1-1/d} + 2y^{d+1} &\geq 2y^{d-1}\int_{\alpha}^{\alpha+1}(x-y)^2dx \\ &\geq 2y^{d-1}\int_{-1/2}^{1/2}s^2ds \\ &= \frac{y^{d-1}}{6}.\end{aligned}$$

Choosing $y = (Ad)^{1/d}$ yields

$$B \geq \frac{1}{d+1}(Ad)^{1+1/d} + \frac{1}{6(d^2-1)}(Ad)^{1-1/d},$$

or, equivalently,

$$\int_0^\infty x^d\eta(x)dx \;\geq\; \frac{1}{d+1}\left(d\int_0^\infty x^{d-1}\eta(x)dx\right)^{1+1/d} + \frac{1}{6(d^2-1)}\left(d\int_0^\infty x^{d-1}\eta(x)dx\right)^{1-1/d},$$

which together with (4.3.4) gives

$$\begin{aligned}\int_0^\infty x^d\varphi(x)dx &\geq \frac{1}{d+1}\left(d\int_0^\infty x^{d-1}\varphi(x)dx\right)^{1+1/d}\varphi(0)^{-1/d} \\ &+ \frac{\varphi(0)^{2+1/d}}{6m^2(d^2-1)}\left(d\int_0^\infty x^{d-1}\varphi(x)dx\right)^{1-1/d}.\end{aligned} \tag{4.3.9}$$

□

Now, it remains to prove the theorem by using the lemma.

Proof of Theorem 4.3.1. Let the Fourier transform of each eigenfunction u_j corresponding to the jth eigenvalue β_j be denoted by

$$\hat{u}_j(\xi) = \frac{1}{(2\pi)^{d/2}}\int_\Omega e^{-ix\cdot\xi}u_j(x)dx.$$

Since the set of eigenfunctions $\{u_j\}_{j=1}^{\infty}$ is orthonormal, the set of $\{\hat{u}_j\}_{j=1}^{\infty}$ is also orthonormal in $\mathbb{R}^d$ by using Plancherel's theorem. Set

$$F(\xi) := \sum_{j=1}^{k} |\hat{u}_j(\xi)|^2.$$

To get the condition in the lemma, we will use the decreasing radial rearrangements and the coarea formula. Let $F^*(\xi) = \varphi(|\xi|)$ be the decreasing radial rearrangement of F. We may assume that φ is absolutely continuous. Let $\mu(t) = |\{F^*(\xi) > t\}| = |\{F(\xi) > t\}|$. Then, $\mu(\varphi(x)) = \omega_d x^d$. By the coarea formula,

$$\mu(t) = \int_t^{|\Omega|/(2\pi)^d} \int_{\{F=x\}} |\nabla F|^{-1} d\sigma_x dx.$$

Then,

$$-\mu'(\varphi(x)) = \int_{\{F=\varphi(x)\}} |\nabla F|^{-1} d\sigma_{\varphi(x)}. \tag{4.3.10}$$

Next we will estimate $|\nabla F|$:

$$\sum_{j=1}^{k} |\nabla \hat{u}_j(\xi)|^2 \leq \frac{1}{(2\pi)^d} \int_{\Omega} |ixe^{-ix\cdot\xi}|^2 dx = \frac{I(\Omega)}{(2\pi)^d},$$

where $I(\Omega)$ is the moment of inertia, which is defined as follows:

$$I(\Omega) = \min_{u\in\mathbb{R}^d} \int_{\Omega} |x-u|^2 dx.$$

After translation, we may assume that

$$I(\Omega) = \int_{\Omega} |x|^2 dx.$$

Observe that for every ξ,

$$|\nabla F(\xi)| \leq 2 \left(\sum_{j=1}^{k} |\hat{u}_j(\xi)|^2\right)^{1/2} \left(\sum_{j=1}^{k} |\nabla \hat{u}_j(\xi)|^2\right)^{1/2} \leq 2(2\pi)^{-d}\sqrt{|\Omega| I(\Omega)}. \tag{4.3.11}$$

Using (4.3.11) in (4.3.10) and setting $m := 2(2\pi)^{-d}\sqrt{|\Omega| I(\Omega)}$ yield

$$\begin{aligned} -\mu'(\varphi(x)) &\geq m^{-1}\mathrm{Vol}_{n-1}(\{F = \varphi(x)\}) \\ &\geq m^{-1} d\omega_d x^{d-1}. \end{aligned}$$

Differentiate $\mu(\varphi(x))$ to obtain $\mu'(\varphi(x))\varphi'(x) = d\omega_d x^{d-1}$. Thus,

$$0 \le -\varphi'(x) \le m, \tag{4.3.12}$$

which is the required condition in the lemma. Observe that

$$\int_{\mathbb{R}^d} F(\xi)d\xi = k. \tag{4.3.13}$$

By an application of Bessel's inequality we obtain

$$0 \le F(\xi) \le \frac{|\Omega|}{(2\pi)^d}, \tag{4.3.14}$$

because the u_j's form an orthonormal set in $L^2(\Omega)$. With the definition of F and

$$\beta_j = \langle u_j, H_{0,\Omega} u_j \rangle = \int_{\mathbb{R}^d} |\xi||\hat{u}_j(\xi)|^2 d\xi,$$

we obtain

$$\int_{\mathbb{R}^d} |\xi| F(\xi)d\xi = \sum_{j=1}^{k} \beta_j. \tag{4.3.15}$$

Hence,

$$k = \int_{\mathbb{R}^d} F(\xi)d\xi = \int_{\mathbb{R}^d} F^*(\xi)d\xi = d\omega_d \int_0^\infty x^{d-1}\varphi(x)dx, \tag{4.3.16}$$

and

$$\sum_{j=1}^{k} \beta_j = \int_{\mathbb{R}^d} |\xi| F(\xi)d\xi = \int_{\mathbb{R}^d} |\xi| F^*(\xi)d\xi = d\omega_d \int_0^\infty x^d \varphi(x)dx, \tag{4.3.17}$$

where ω_d denotes the volume of the d-dimensional unit ball. An application of Lemma 4.3.1 with the equations (4.3.16), (4.3.17) yield

$$\sum_{j=1}^{k} \beta_j \ge \frac{d}{d+1}\omega_d{}^{-1/d}\varphi(0)^{-1/d}k^{1+1/d} + \frac{d}{6m^2(d^2-1)}\omega_d^{1/d}\varphi(0)^{2+1/d}k^{1-1/d}. \tag{4.3.18}$$

Define

$$h(t) = \frac{d}{d+1}\omega_d{}^{-1/d}k^{1+1/d}t^{-1/d} + \frac{Cd}{m^2(d^2-1)}\omega_d^{1/d}k^{1-1/d}t^{2+1/d},$$

where C is a constant to be chosen later. Observe that the function h is decreasing on

$$0 < t \le \left(\frac{m^2(d-1)k^{2/d}}{C(2d+1)\omega_d^{2/d}}\right)^{d/(d+2)}.$$

Let R be the number such that $|\Omega| = \omega_d R^d$. Then,

$$I(\Omega) \geq \int_{B(R)} |x|^2 dx = \frac{d\omega_d R^{d+2}}{d+2},$$

where $B(R)$ is the ball of radius R. Then,

$$m = 2(2\pi)^{-d}\sqrt{|\Omega| I(\Omega)} \geq 2(2\pi)^{-d}\sqrt{\frac{d}{d+2}\omega_d^{-2/d}|\Omega|^{(2d+2)/d}} \geq (2\pi)^{-d}\omega_d^{-1/d}|\Omega|^{(d+1)/d}.$$

Choosing $C = \min\left\{\frac{1}{6}, \frac{m^2(d-1)k^{2/d}(2\pi)^{d+2}}{(2d+1)\omega_d^{2/d}|\Omega|^{1+2/d}}\right\}$ will guarantee that

$$\left(\frac{m^2(d-1)k^{2/d}}{C(2d+1)\omega_d^{2/d}}\right)^{d/(d+2)} \geq (2\pi)^{-d}|\Omega|.$$

Hence, the function h is decreasing on $\left(0, (2\pi)^{-d}|\Omega|\right]$. Since $0 < \varphi(0) \leq (2\pi)^{-d}|\Omega|$, and h is decreasing, we can replace $\varphi(0)$ in (4.3.18) with $(2\pi)^{-d}|\Omega|$. Therefore, (4.3.18) and the fact that $\omega_d = \dfrac{\pi^{d/2}}{\Gamma(1+d/2)}$ result in the following inequality:

$$\sum_{j=1}^{k}\beta_j \geq \frac{\sqrt{4\pi}d}{d+1}\left(\frac{\Gamma(1+d/2)}{|\Omega|}\right)^{1/d} k^{1+1/d} + \frac{Cd}{8\sqrt{\pi}(d^2-1)(\Gamma(1+d/2))^{1/d}}\frac{|\Omega|^{1+1/d}}{I(\Omega)}k^{1-1/d}. \tag{4.3.19}$$

Let $\tilde{M}_d := \dfrac{Cd}{8\sqrt{\pi}(d^2-1)(\Gamma(1+d/2))^{1/d}}$. Then (4.3.19) can be written as

$$\sum_{j=1}^{k}\beta_j \geq \frac{d\tilde{C}_d}{d+1}|\Omega|^{-1/d}k^{1+1/d} + \tilde{M}_d\frac{|\Omega|^{1+1/d}}{I(\Omega)}k^{1-1/d}, \tag{4.3.20}$$

where $\tilde{C}_d = \sqrt{4\pi}\Gamma(1+d/2)^{1/d}$. Recall that the first term on the right of (4.3.20) is same bound as in [25]. $\square$

CHAPTER V

UNIVERSAL BOUNDS FOR $H_{M,\Omega} + V(X)$

This chapter gives an analogue of (2.3.3) for the Klein-Gordon Hamiltonian with an external interaction,

$$H = H_{m,\Omega} + V(\mathbf{x}), \tag{5.0.21}$$

which are used in semi-relativistic quantum mechanics. Moreover, Hamiltonian operators similar to (5.0.21) have been of interest as models of nonrelativistic charge carriers traveling in a two-dimensional hexagonal structure like carbon graphene, a novel material with remarkable electronic properties [31, 58]. Observe that (2.1.4) remains valid for interacting operators H. That will enable us to derive some spectral bounds for (5.0.21). To guarantee that the operators (5.0.21) are self-adjoint with the same domain of definition as for $H_{m,\Omega}$, some conditions on V will be imposed. Throughout this chapter, β_k denotes the kth eigenvalue of (5.0.21) and u_j denotes the corresponding eigenfunction. As in (2.3.1), set $U(z) := \sum_k \frac{(z-\beta_k)_+^2}{\beta_k}$.

5.1 Statement and Proof of the Theorem

Theorem 5.1.1 *Assume that β_k are the eigenvaules of (5.0.21), where V is a real-valued locally L^1 function.*

(a) If $V \geq 0$, then for each k, (2.2.2) holds, i.e.,

$$\beta_{k+1} \leq \frac{d+1}{(d-1)\overline{\beta_k^{-1}}} \leq \frac{d+1}{d-1}\overline{\beta_k}. \tag{5.1.1}$$

Moreover, when $k > 2j$, (2.3.10) holds. That is,

$$\frac{\overline{\beta}_k}{\overline{\beta}_j} \leq \frac{d}{2^{1/d}(d-1)}\left(\frac{k}{j}\right)^{\frac{1}{d}}. \tag{5.1.2}$$

(b) If $V \in L^s$ for some $2 \leq d < s < \infty$, and

$$\alpha := \frac{\|V\|_s (d-2)!(s-1)^{\frac{s-1}{s}}}{\sqrt{\pi}2^{\frac{(d-1)^2}{d}}\Gamma\left(\frac{d}{2}\right)^{\frac{1-2d}{d}}(d|\Omega|)^{\frac{d-s}{sd}}(s-d)^{\frac{s-1}{s}}} < 1,$$

then for each k, the eigenvalues β_k satisfy

$$\frac{\beta_{k+1}}{\overline{\beta_k}} \leq \overline{\beta_k^{-1}}\beta_{k+1} \leq 1 + \frac{2}{(d-1)(1-\alpha)}. \tag{5.1.3}$$

Moreover, $\dfrac{U(z)}{z^{((d+1)-\alpha(d-1))}}$ is a nondecreasing function of $z \in \mathbb{R}$, and for $k > 2j$,

$$\frac{\overline{\beta_k}}{\overline{\beta_j}} \leq \frac{d-\alpha(d-1)}{(d-1)(1-\alpha)2^{1/(d-\alpha(d-1))}} \left(\frac{k}{j}\right)^{1/(d-\alpha(d-1))}. \tag{5.1.4}$$

Proof. Eq. (2.1.4) implies

$$(d-1)\sum_{j=1}^{n}(z-\beta_j)^2\langle u_j, H_{m,\Omega}^{-1}u_j\rangle - 2\sum_{j=1}^{n}(z-\beta_j) \leq 0. \tag{5.1.5}$$

(a) By the fact that the function $f(x) = \dfrac{1}{x}$ is operator monotone decreasing, $V \geq 0$ implies

$$(H_{m,\Omega} + V)^{-1} \leq H_{m,\Omega}^{-1},$$

and hence

$$\frac{1}{\beta_j} \leq \langle u_j, H_{m,\Omega}^{-1}u_j\rangle.$$

The last inequality together with (5.1.5) yields

$$(d-1)\sum_{j=1}^{n}\frac{(z-\beta_j)^2}{\beta_j} - 2\sum_{j=1}^{n}(z-\beta_j) \leq 0.$$

As in the proof of (2.2.2), this implies that

$$(d+1)\sum_{j=1}^{n}\frac{(z-\beta_j)^2}{\beta_j} - 2z\sum_{j=1}^{n}\frac{(z-\beta_j)}{\beta_j} \leq 0,$$

or, equivalently,

$$(d-1)\overline{\beta_n^{-1}}z^2 - 2dz + (d+1)\overline{\beta_n} \leq 0.$$

Thus, setting $z = \beta_{n+1}$ and using the Cauchy-Schwarz inequality $1 \leq \overline{\beta_n}\,\overline{\beta_n^{-1}}$ implies (5.1.1) as in the proof of (2.2.2). Similarly, the proof of (5.1.2) follows in strict analogy with the proof of (2.3.10).

(b) Here, we have to follow a different approach since in this case, we don't know whether the potential V is positive. The resolvent formula states that

$$(H_{m,\Omega} + V)^{-1}VH_{m,\Omega}^{-1} = H_{m,\Omega}^{-1} - (H_{m,\Omega} + V)^{-1}.$$

Since u_j's are the eigenfunctions corresponding to the eigenvalues β_j, we now have

$$\begin{aligned}\frac{1}{\beta_j} &= \langle u_j, (H_{m,\Omega}+V)^{-1}u_j\rangle = \langle u_j, H_{m,\Omega}^{-1}u_j\rangle - \langle u_j, (H_{m,\Omega}+V)^{-1}VH_{m,\Omega}^{-1}u_j\rangle \\ &= \langle u_j, H_{m,\Omega}^{-1}u_j\rangle - \frac{1}{\beta_j}\langle u_j, VH_{m,\Omega}^{-1}u_j\rangle.\end{aligned}$$

This implies that

$$\frac{1}{\beta_j}\left(1+\langle u_j, VH_{m,\Omega}^{-1}u_j\rangle\right) = \langle u_j, H_{m,\Omega}^{-1}u_j\rangle \tag{5.1.6}$$

and

$$\frac{1}{\beta_j}\left(1-\|VH_{m,\Omega}^{-1}u_j\|\right) \leq \langle u_j, H_{m,\Omega}^{-1}u_j\rangle. \tag{5.1.7}$$

Claim: For $2 \leq d \leq s < \infty$,

$$\|VH_{m,\Omega}^{-1}\varphi\|_2 \leq \alpha\|\varphi\|_2 \tag{5.1.8}$$

for any $\varphi \in L^2$.

Granting the claim, with $\varphi = u_j$ in (5.1.7), we get

$$\frac{1-\alpha}{\beta_j} \leq \langle u_j, H_{m,\Omega}^{-1}u_j\rangle. \tag{5.1.9}$$

Note that by Hölder's inequality, the left side of (5.1.8) becomes

$$\|VH_{m,\Omega}^{-1}\varphi\|_2 \leq \|V\|_s\|H_{m,\Omega}^{-1}\varphi\|_{\frac{2s}{s-2}}. \tag{5.1.10}$$

Since $H_{m,\Omega} \geq H_{0,\Omega} > 0$,

$$\|H_{m,\Omega}^{-1}\varphi\|_{\frac{2s}{s-2}} \leq \|H_{0,\Omega}^{-1}\varphi\|_{\frac{2s}{s-2}}. \tag{5.1.11}$$

Now by using the inequality (3.2.8) for the transition density, we obtain

$$e^{-tH_{0,\Omega}}(\mathbf{x},\mathbf{y},t) \leq p_0(\mathbf{x}-\mathbf{y},t) = \frac{-c_d}{d-1}\frac{\partial}{\partial t}\left(t^2+|\mathbf{x}-\mathbf{y}|^2\right)^{-\left(\frac{d-1}{2}\right)}.$$

An application of the Laplace transform yields that the kernel of $H_{0,\Omega}^{-1}$ is less than

$$\int_0^\infty \left(\frac{-c_d}{d-1}\frac{\partial}{\partial t}\left(t^2+|\mathbf{x}-\mathbf{y}|^2\right)^{-\left(\frac{d-1}{2}\right)}\right)dt = \frac{c_d}{d-1}|\mathbf{x}-\mathbf{y}|^{-(d-1)}.$$

Then equations (5.1.10) and (5.1.11) imply that

$$\|VH_{m,\Omega}^{-1}\varphi\|_2 \leq \frac{c_d}{d-1}\|V\|_s\||\mathbf{x}|^{-(d-1)} * \varphi\|_{\frac{2s}{s-2}}.$$

After an application of Young's convolution inequality, we obtain

$$\||\mathbf{x}|^{-(d-1)} * \varphi\|_{\frac{2s}{s-2}} \leq \||\mathbf{x}|^{-(d-1)}\|_{\frac{s}{s-1}} \|\varphi\|_2.$$

Hence,

$$\left\|VH_{m,\Omega}^{-1}\varphi\right\|_2 \leq \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}(d-1)} \|V\|_s \left\||\mathbf{x}|^{-(d-1)}\right\|_{\frac{s}{s-1}} \|\varphi\|_2. \tag{5.1.12}$$

To get an upper bound for $\left\||\mathbf{x}|^{-(d-1)}\right\|_{\frac{s}{s-1}}$, choose R^* as the radius of the ball B_{R^*} centered at the origin having the same volume as Ω. Since by rearrangement,

$$\||\mathbf{x}|^{-(d-1)}\|_{L^{\frac{s}{s-1}}(\Omega)} \leq \||\mathbf{x}|^{-(d-1)}\|_{L^{\frac{s}{s-1}}(B_{R^*})} = \left(\omega_{d-1}\frac{(R^*)^{\frac{s-d}{s-1}}(s-1)}{s-d}\right)^{\frac{s-1}{s}},$$

we get the estimate

$$\||\mathbf{x}|^{-(d-1)}\|_{\frac{s}{s-1}} < 2^{\frac{d-1}{d}} \pi^{\frac{d-1}{2}} \left[\Gamma\left(\frac{d}{2}\right)\right]^{\frac{1-d}{d}} (d|\Omega|)^{\frac{s-d}{sd}} \left(\frac{s-1}{s-d}\right)^{\frac{s-1}{s}}. \tag{5.1.13}$$

The last inequality together with (5.1.12) implies (5.1.8) and consequently (5.1.9).

With the assumption $\alpha < 1$, (5.1.5) and (5.1.9) yield

$$(d-1)\sum_{j=1}^{n} \frac{1-\alpha}{\beta_j}(z-\beta_j)^2 - 2\sum_{j=1}^{n}(z-\beta_j) \leq 0, \tag{5.1.14}$$

or, equivalently,

$$(d-1)(1-\alpha)\overline{\beta_k^{-1}}z^2 - 2[d-\alpha(d-1)]z + [d+1-\alpha(d-1)]\overline{\beta_k} \leq 0. \tag{5.1.15}$$

Observe that β_{k+1} must be smaller than the larger root of (5.1.15) when we set $z = \beta_{k+1}$. That is,

$$\beta_{k+1} \leq \frac{(d-1)(1-\alpha) + 1 + \sqrt{1 - ((d+\alpha-\alpha d)^2 - 1)\left(\overline{\beta_k}\,\overline{\beta_k^{-1}} - 1\right)}}{(d-1)(1-\alpha)\overline{\beta_k^{-1}}}. \tag{5.1.16}$$

By using the same strategy in the proof of (2.2.2), (5.1.16) implies the simpler but slightly weaker inequalities (5.1.3) with the aid of Cauchy-Schwarz inequality in the form $1 \leq \overline{\beta_k}\overline{\beta_k^{-1}}$. Observe that (5.1.14) differs from (2.1.6) only in the extra factor $1-\alpha > 0$, and therefore we can obtain all of the consequences of that inequality by changing some constants accordingly. More precisely, the function $\dfrac{U(z)}{z^{(d+1)-\alpha(d-1)}}$ is nondecreasing, and, therefore,

$$U(z) \geq \left(\frac{z}{z_{j^*}}\right)^{(d+1)-\alpha(d-1)} U(z_{j^*}) \tag{5.1.17}$$

when $z \geq z_{j^*} \geq \beta_j$.

Now, we can rewrite (5.1.14) as

$$\frac{(d-1)(1-\alpha)}{2}U(z) \leq R_1(z). \tag{5.1.18}$$

Eq. (5.1.18) together with the fact that

$$\frac{U(z)}{j} \geq \frac{1}{\overline{\beta}_j}(z-\overline{\beta}_j)^2$$

gives

$$R_1(z) \geq \frac{(d-1)(1-\alpha)j}{2\overline{\beta}_j}\left(\frac{z}{z_{j^*}}\right)^{(d+1)-\alpha(d-1)}(z_{j^*}-\overline{\beta}_j)^2. \tag{5.1.19}$$

To maximize the coefficient of $z^{d+1-\alpha(d-1)}$ we optimize z_{j^*} and get

$$z_{j^*} = \frac{(d+1)-\alpha(d-1)}{(d-1)(1-\alpha)}\overline{\beta_j}.$$

When we substitute this into (5.1.19), we obtain

$$R_1(z) \geq \frac{2j[(d-1)(1-\alpha)]^{d-\alpha(d-1)}}{[(d+1)-\alpha(d-1)]^{(d+1)-\alpha(d-1)}\overline{\beta_j}^{\;d-\alpha(d-1)}}z^{(d+1)-\alpha(d-1)} \tag{5.1.20}$$

for all $z \geq \dfrac{(d+1)-\alpha(d-1)}{(d-1)(1-\alpha)}\overline{\beta_j}$.

Recall that the Legendre transform of $R_1(z)$ is

$$R_1^*(w) = (w-[w])\beta_{[w]+1} + [w]\overline{\beta_{[w]}}\,, \tag{5.1.21}$$

where $[w]$ denotes the greatest integer $\leq w$. When w approaches an integer value k from below,

$$R_1^*(k) = k\overline{\beta}_k.$$

Thus, an application of the Legendre transform to (5.1.20) gives

$$k\overline{\beta_k} \leq \frac{[d-\alpha(d-1)]\overline{\beta_j}}{[(d-1)(1-\alpha)]2^{1/(d-\alpha(d-1))}j^{1/(d-\alpha(d-1))}}k^{1+1/(d-\alpha(d-1))}. \tag{5.1.22}$$

After rearranging terms, we get

$$\frac{\overline{\beta_k}}{\overline{\beta_j}} \leq \frac{d-\alpha(d-1)}{[(d-1)(1-\alpha)]2^{1/(d-\alpha(d-1))}}\left(\frac{k}{j}\right)^{1/(d-\alpha(d-1))}, \tag{5.1.23}$$

which concludes the proof. □

CHAPTER VI

A SPECTRAL INEQUALITY FOR THE FRACTIONAL LAPLACIAN $(-\Delta)^S$

6.1 Introduction

This chapter is about the operator $(-\Delta)^s$ on a bounded domain $\Omega \subset \mathbb{R}^d$. Throughout this chapter, assume that ϱ_j denotes the jth eigenvalue of $(-\Delta)^s$ with the corresponding eigenfunction u_j. Eigenvalues (including multiplicities) satisfy

$$\varrho_1 \leq \varrho_2 \leq \varrho_3 \leq \cdots \leq \varrho_j \leq \cdots \to \infty.$$

Fractional Laplacians can be conveniently defined using the Fourier transform on the dense subspace of test functions $C_c^\infty(\mathbb{R}^d)$. Recall that the Laplacian is given by $-\Delta\varphi := \mathfrak{F}^{-1}|\xi|^2\widehat{\varphi}(\xi)$. Therefore,

$$(-\Delta)^s\varphi := \chi_\Omega \mathfrak{F}^{-1}|\xi|^{2s}\widehat{\varphi}(\xi) \tag{6.1.1}$$

where $s \in [0, 1]$.

In the following, we are interested in proving a theorem like Theorem (2.3.1). So, it is worthwhile to recall some of the basic ingredients of the proof of (2.3.3). First, begin by finding a Harrell-Stubbe type inequality for $(-\Delta)^s$ because it will be the cornerstone of the proof.

6.2 A Trace Inequality for $(-\Delta)^s$

As it was mentioned before, our point of departure is the Harrell-Stubbe Trace inequalities[22]. Recall that for a self adjoint operator H with discrete eigenvalues ϱ_j's,

$$\sum_{j:\varrho_j \leq z} (z-\varrho_j)^2 \langle u_j, [x_\alpha, [H, x_\alpha]]\, u_j\rangle - 2(z-\varrho_j)\|\, [H, x_\alpha]\, u_j\|^2 \leq 0. \tag{6.2.1}$$

Note that here x_α denotes the coordinate function.

By setting $H = (-\Delta)^s$, the first commutator becomes

$$[(-\Delta)^s, x_\alpha]\varphi = -2is\chi_\Omega \mathfrak{F}^{-1}\left(\xi_\alpha|\xi|^{2s-2}\hat{\varphi}\right). \tag{6.2.2}$$

Similarly,

$$[x_\alpha, [(-\Delta)^s, x_\alpha]]\varphi = 2s\chi_\Omega \mathfrak{F}^{-1}\left[\left(|\xi|^{2s-2} + 2(s-1)\xi_\alpha^2 |\xi|^{2s-4}\right)\hat{\varphi}\right]. \tag{6.2.3}$$

Due to (6.2.2) and (6.2.3), there are simplifications when we sum over α:

$$\sum_{\alpha=1}^{d} \| [(-\Delta)^s, x_\alpha]\, \varphi\|^2 = \langle \hat{\varphi}, 4s^2 \xi_\alpha^2 |\xi|^{4s-4}\hat{\varphi}\rangle,$$

and

$$\sum_{\alpha=1}^{d} \langle \varphi, [x_\alpha, [(-\Delta)^s, x_\alpha]\, \varphi\rangle = (2sd + 4s^2 - 4s)\langle \hat{\varphi}, |\xi|^{2s-2}\hat{\varphi}\rangle.$$

In consequence, (6.2.1) yields

$$(2sd + 4s^2 - 4s)\sum_{j=1}^{n}(z - \varrho_j)^2 \langle \hat{u}_j, |\xi|^{2s-2}\hat{\varphi}\rangle - 2\sum_{j=1}^{n}(z - \varrho_j)\langle \hat{u}_j, 4s^2 |\xi|^{4s-2}\hat{u}_j\rangle \leq 0 \tag{6.2.4}$$

provided $z \in [\varrho_n, \varrho_{n+1}]$. Now, to simplify the first term on the right side of (6.2.4) we need the following lemma:

Lemma 6.2.1 *Assume that ϱ_j is the jth eigenvalue of $(-\Delta)^s$ and let u_j be the corresponding eigenfunction. Then*

$$\varrho_j^{1-\frac{1}{s}} \leq \langle \hat{u}_j, |\xi|^{2s-2}\hat{u}_j\rangle. \tag{6.2.5}$$

Proof. Observe that because $\mathrm{supp}(u_j) \in \Omega$, we have

$$\varrho_j \;=\; \langle u_j, (-\Delta)^s u_j\rangle = \langle u_j, \chi_\Omega \mathfrak{F}^{-1}|\xi|^{2s}\hat{u}_j\rangle = \langle u_j, \mathfrak{F}^{-1}|\xi|^{2s}\hat{u}_j\rangle = \langle \hat{u}_j, |\xi|^{2s}\hat{u}_j\rangle$$

Since the eigenfunctions u_j's are normalized, we have

$$1 = \langle \hat{u}_j, \hat{u}_j\rangle^2 = \langle |\xi|^{2s(1-s)}\hat{u}_j^{2(1-s)}, |\xi|^{2s(s-1)}\hat{u}_j^{2s}\rangle. \tag{6.2.6}$$

Recall Hölder's inequality:

$$\int |fg| \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}.$$

Thus, applying Hölder's inequality to (6.2.6) with $p = \frac{1}{1-s} > 1$, $q = \frac{1}{s} > 1$ we get

$$
\begin{aligned}
1 &\leq \left(\int \left[|\xi|^{2s(1-s)}|\hat{u}_j|^{2(1-s)}\right]^{\frac{1}{1-s}} d\xi\right)^{1-s} \left(\int \left[|\xi|^{2s(s-1)}|\hat{u}_j|^{2s}\right]^{\frac{1}{s}} d\xi\right)^{s} \\
1 &\leq \left(\int |\xi|^{2s}\hat{u}_j^2 d\xi\right)^{1-s} \left(\int |\xi|^{2s-2}\hat{u}_j^2 d\xi\right)^{s} \\
1 &\leq \left(\int |\xi|^{2s}\hat{u}_j^2 d\xi\right)^{\frac{1-s}{s}} \left(\int |\xi|^{2s-2}\hat{u}_j^2 d\xi\right) \\
1 &\leq \langle \hat{u}_j, |\xi|^{2s}\hat{u}_j\rangle^{\frac{1-s}{s}} \langle \hat{u}_j, |\xi|^{2s-2}\hat{u}_j\rangle \\
1 &\leq \varrho_j^{\frac{1-s}{s}} \langle \hat{u}_j, |\xi|^{2s-2}\hat{u}_j\rangle \\
\varrho_j^{\frac{s-1}{s}} &\leq \langle \hat{u}_j, |\xi|^{2s-2}\hat{u}_j\rangle
\end{aligned}
$$

□

Lemma 6.2.1 together with Eq. (6.2.4) implies

$$
(2sd + 4s^2 - 4s)\sum_{j=1}^{n}(z - \varrho_j)^2 \varrho_j^{1-\frac{1}{s}} - 2\sum_{j=1}^{n}(z - \varrho_j)\varrho_j^{2-\frac{1}{s}} \leq 0. \tag{6.2.7}
$$

6.3 Main Result

With $a_+ := \max(0, a)$, define

$$
U_s(z) := \sum_k (z - \varrho_k)_+^2 \varrho_k^{1-\frac{1}{s}}, \tag{6.3.1}
$$

where z is a real variable.

Theorem 6.3.1 *The function $z^{\frac{2-d}{2s}-3}U_s(z)$ is nondecreasing in the variable z and*

$$
\sum_{k=1}^{n}(z - \varrho_k)\varrho_k{}^{2-\frac{1}{s}} \geq \frac{8s^2 j(2sd - 4s + 4s^2)^{\frac{2sd-4s+8s^2}{4s^2}}}{(2sd - 4s + 12s^2)^{\frac{2sd-4s+12s^2}{4s^2}}} \frac{\overline{(\varrho_j^{2-\frac{1}{s}})}^{\frac{2sd-4s+12s^2}{4s^2}}}{\overline{(\varrho_j^{3-\frac{1}{s}})}^{\frac{2sd-4s+8s^2}{4s^2}}} z^{\frac{d-2}{2s}+3}. \tag{6.3.2}
$$

Proof. By using

$$
(z - \varrho_j) = -\frac{(z - \varrho_j)(z - \varrho_j - z)}{\varrho_j},
$$

Eq. (6.2.7) can be rewritten as

$$
(2sd + 12s^2 - 4s)\sum_{j=1}^{n}(z - \varrho_j)^2 \varrho_j{}^{1-\frac{1}{s}} - 8s^2 z \sum_{j=1}^{n}(z - \varrho_j)\varrho_j{}^{1-\frac{1}{s}} \leq 0. \tag{6.3.3}
$$

Hence,

$$
(2sd + 12s^2 - 4s)U_s(z) - 4s^2 z U_s'(z) \leq 0,
$$

which implies

$$\frac{d}{dz}\left\{z^{-\frac{(2sd+12s^2-4s)}{4s^2}}U_s(z)\right\} \geq 0,$$

or, equivalently,

$$\frac{d}{dz}\left\{\frac{U_s(z)}{z^{\frac{d-2}{2s}+3}}\right\} \geq 0.$$

This proves the first claim in the theorem.

Since $z^{\frac{2-d}{2s}-3}U_s(z)$ is nondecreasing, for $d > 2s-2$,

$$U_s(z) \geq \left(\frac{z}{z_{j*}}\right)^{\frac{d-2}{2s}+3} U_s(z_{j*}), \tag{6.3.4}$$

when $z \geq z_{j*} \geq \varrho_j$. Now, observe that

$$U_s(z) = \sum_k (z-\varrho_k)^2 \varrho_k^{1-\frac{1}{s}} = z^2 \sum_k \varrho_k^{1-\frac{1}{s}} - 2z \sum_k \varrho_k^{2-\frac{1}{s}} + \sum_k \varrho_k^{3-\frac{1}{s}}. \tag{6.3.5}$$

We apply Cauchy-Schwarz to (6.3.5) to get

$$\frac{U_s(z)}{j} \geq \frac{1}{\overline{\varrho_j^{3-\frac{1}{s}}}}\left[z\overline{\varrho_j^{2-\frac{1}{s}}} - \overline{\varrho_j^{3-\frac{1}{s}}}\right]^2. \tag{6.3.6}$$

On the other hand, Eq. (6.2.7) implies that

$$\frac{(2sd+4s^2-4s)}{8s^2}U(z) \leq \sum_{k=1}^{n}(z-\varrho_k){\varrho_k}^{2-\frac{1}{s}}. \tag{6.3.7}$$

Thus, equations (6.3.4), (6.3.6) together with (6.3.7), imply

$$\sum_{k=1}^{n}(z-\varrho_k){\varrho_k}^{2-\frac{1}{s}} \geq \left(\frac{2sd-4s+4s^2}{8s^2}\right)\left(\frac{z}{z_{j*}}\right)^{\frac{d-2}{2s}+3} \frac{j}{\overline{\varrho_j^{3-\frac{1}{s}}}}\left[z\overline{\varrho_j^{2-\frac{1}{s}}} - \overline{\varrho_j^{3-\frac{1}{s}}}\right]^2. \tag{6.3.8}$$

Now, when we maximize the coefficient of $z^{\frac{d-2}{2s}+3}$, we obtain an optimized value of z_{j*}, which is

$$z_{j*} = \left(\frac{2sd-4s+12s^2}{2sd-4s+4s^2}\right)\frac{\overline{\varrho_j^{3-\frac{1}{s}}}}{\overline{\varrho_j^{2-\frac{1}{s}}}}.$$

After substituting this into (6.3.8), we get

$$\sum_{k=1}^{n}(z-\varrho_k){\varrho_k}^{2-\frac{1}{s}} \geq \frac{8s^2 j(2sd-4s+4s^2)^{\frac{2sd-4s+8s^2}{4s^2}}}{(2sd-4s+12s^2)^{\frac{2sd-4s+12s^2}{4s^2}}} \frac{\left(\overline{\varrho_j^{2-\frac{1}{s}}}\right)^{\frac{2sd-4s+12s^2}{4s^2}}}{\left(\overline{\varrho_j^{3-\frac{1}{s}}}\right)^{\frac{2sd-4s+8s^2}{4s^2}}} z^{\frac{d-2}{2s}+3}. \tag{6.3.9}$$

$\square$

CHAPTER VII

SHARP BOUNDS FOR SPECTRAL FUNCTIONS OF THE KLEIN-GORDON OPERATOR

In what follows, we focus on the Klein-Gordon operators $H_{m,\Omega}$. Some transform techniques are used to obtain estimates for some spectral functions of $H_{m,\Omega}$. The reader interested in applications of these transform techniques to the Laplacian setting can look at, for instance, the article [20] written by E. Harrell and L. Hermi. The proofs given in the following sections follow closely some of the arguments given in [20].

7.1 Introduction

It is worth pointing out some of the familiar properties of the Riesz means which will be used in the present chapter, without comment, in some of the proofs. Foradditional background material, one can look at the book [11] and the article [29].

For $z \geq 0$, the Riesz mean of order σ of a real sequence $\{\beta_j\}$ is defined by

$$R_\sigma(z) = \sum_{j=1}^{\infty} (z - \beta_j)_+^\sigma,$$

where $a_+ = \max\{0, a\}$.
In particular, when $\sigma \to 0^+$, $R_0(z)$ is the same as the counting function

$$\mathcal{N}(z) := \#\{\beta_j : \beta_j \leq z\},$$

or, equivalently,

$$\mathcal{N}(z) = \sum_{j=1}^{\infty} (z - \beta_j)_+^0.$$

Note that the Riesz mean of order σ can be written in terms of $\mathcal{N}$ as

$$\begin{aligned} R_\sigma(z) &= \int_0^\infty (z-t)_+^\sigma \mathrm{d}\mathcal{N}(t) \\ &= \int_0^\infty \sigma (z-t)_+^{\sigma-1} \mathcal{N}(t) \mathrm{d}\,t. \end{aligned}$$

By using this integral representation, a relation between Riesz means of different orders known as the Riesz iteration or the Aizenman-Lieb procedure are obtained. ([1],[39]). Its proof is provided so that this thesis is self-contained:

Theorem 7.1.1 *For $\sigma, \eta > 0$,*

$$R_{\sigma+\eta}(z) = \frac{\Gamma(\sigma+\eta+1)}{\Gamma(\sigma+1)\Gamma(\eta)} \int_0^\infty (z-t)_+^{\eta-1} R_\sigma(t)dt, \tag{7.1.1}$$

Proof. We expand $R_\sigma(z)$ on the right side of (7.1.1) using the integral representation of the Riesz mean and obtain:

$$\begin{aligned}
\int_0^z (z-t)^{\eta-1} R_\sigma(t)dt &= \sigma \int_0^z \int_0^t (z-t)^{\eta-1}(t-w)^{\sigma-1}\mathcal{N}(w)dwdt && (7.1.2)\\
&= \sigma \int_0^z \int_w^z (z-t)^{\eta-1}(t-w)^{\sigma-1}\mathcal{N}(w)dtdw && (7.1.3)\\
&= \sigma \int_0^z \mathcal{N}(w)\left[\int_w^z (z-t)^{\eta-1}(t-w)^{\sigma-1}dt\right]dw && (7.1.4)\\
&= \frac{\sigma\Gamma(\sigma)\Gamma(\eta)}{\Gamma(\sigma+\eta)} \int_0^z (z-w)^{\eta+\sigma-1}\mathcal{N}(w)dw && (7.1.5)\\
&= \frac{\Gamma(\sigma+1)\Gamma(\eta)}{\Gamma(\sigma+\eta+1)} R_{\sigma+\eta}(z). && (7.1.6)
\end{aligned}$$

Now, we justify the steps above. In (7.1.2), we use the integral representation of the Riesz mean to expand the terms in the integral. In (7.1.3), we employ the Fubini-Tonelli Theorem. In (7.1.4), we exploit the fact that

$$\int_0^\infty (1-t)_+^{\eta-1} t^{\sigma-1} dt = \frac{\Gamma(\eta)\Gamma(\sigma)}{\Gamma(\eta+\sigma)} \tag{7.1.7}$$

to evaluate the inner integral. Note that by change of variables (twice) we arrive at the integral in (7.1.7). Finally, the definition of the Riesz mean together with the fact that

$$\sigma\Gamma(\sigma) = \Gamma(\sigma+1), \tag{7.1.8}$$

yields (7.1.6). □

The present chapter is organized as follows: The first section begins with a generalization of (2.1.6) and continues with a sharp bound for $U_\sigma(z)$. Section 7.3 is concerned with results for the partition function $\tilde{Z}(t) = \sum \frac{e^{-\beta_j t}}{\beta_j}$. Moreover, a Berezin-Li-Yau type inequality from a Kac like inequality is obtained in Section 7.3.

7.2 Sharp Bounds for $U_\sigma(z)$

Before continuing, some of the pertinent results proved in the previous sections are briefly recalled for the convenience of the reader. The first such result is the trace inequality (2.1.6) proved in Section 2.1:

$$(d+1)\sum_j \frac{(z-\beta_j)_+^2}{\beta_j} - 2z\sum_j \frac{(z-\beta_j)_+}{\beta_j} \leq 0. \tag{7.2.1}$$

Define the function

$$U_\sigma(z) := \sum_j \frac{(z-\beta_j)_+^\sigma}{\beta_j}.$$

With this notation, (7.2.1) can be written as

$$(d+1)U_2(z) - 2zU_1(z) \leq 0, \tag{7.2.2}$$

or, equivalently,

$$U_2(z) \leq \frac{2}{d+1} zU_1(z). \tag{7.2.3}$$

The following theorem generalizes the last inequality for $\sigma > 0$.

Theorem 7.2.1 *For $\sigma > 0$,*

$$U_\sigma(z) \leq C_\sigma z U_{\sigma-1}(z). \tag{7.2.4}$$

where

$$C_\sigma = \begin{cases} \dfrac{\sigma}{d+1}, & \textit{when } \sigma \geq 2 \\ \dfrac{2}{d+1}, & \textit{when } \sigma \leq 2 \end{cases}$$

Proof. We first prove the case when $\sigma > 2$ by using the Riesz iteration method. The downside of this approach is that it works only when $\sigma > 2$. A way to get around this problem is making use of the reverse Chebyshev inequality from [19]. So, after stating the reverse Chebyshev inequality as a lemma, we provide the proof for $\sigma \leq 2$.

Now, let's prove the first part when $\sigma > 2$: To be able to apply the Riesz iteration method, we begin by rewriting (7.2.1) as

$$\sum_j \frac{(z-\beta_j-t)_+^2}{\beta_j} \leq \frac{2}{d+1} z \sum_j \frac{(z-\beta_j-t)_+}{\beta_j}, \tag{7.2.5}$$

for $t \le z$. Multiplying (7.2.5) by $t^{\sigma-3}$ and integrating from $t=0$ to $t=\infty$ yield

$$\int_0^\infty \sum_j \frac{(z-\beta_j-t)_+^2}{\beta_j} t^{\sigma-3} dt \le \frac{2}{d+1} z \int_0^\infty \sum_j \frac{(z-\beta_j-t)_+}{\beta_j} t^{\sigma-3} dt. \tag{7.2.6}$$

Next, set $t=(z-\beta_j)_+\tau$. Then the integral on the left side of (7.2.6) becomes

$$\begin{aligned}\int_0^\infty \frac{(z-\beta_j-t)_+^2}{\beta_j} t^{\sigma-3} dt &= (z-\beta_j)_+^{2+\sigma-2} \int_0^1 (1-\tau)^2 \tau^{\sigma-3} d\tau \\ &= (z-\beta_j)_+^{\sigma} \frac{\Gamma(3)\Gamma(\sigma-2)}{\Gamma(\sigma+1)}.\end{aligned} \tag{7.2.7}$$

Similarly, by setting $t=(z-\beta_j)_+\tau$, the right side of (7.2.6) reduces to

$$\begin{aligned}\int_0^\infty \frac{(z-\beta_j-t)_+}{\beta_j} t^{\sigma-3} dt &= (z-\beta_j)_+^{\sigma-1} \int_0^1 (1-\tau) \tau^{\sigma-3} d\tau \\ &= (z-\beta_j)_+^{\sigma-1} \frac{\Gamma(2)\Gamma(\sigma-2)}{\Gamma(\sigma)}.\end{aligned} \tag{7.2.8}$$

Note that the equation (7.1.7) is to evaluate these integrals. Thus, (7.2.6) together with (7.2.8), (7.2.9) and (7.1.8) implies that

$$\begin{aligned}\frac{2\Gamma(\sigma-2)}{\Gamma(\sigma+1)} \sum_j \frac{(z-\beta_j)_+^\sigma}{\beta_j} &\le \frac{2z}{d+1} \frac{\Gamma(\sigma-2)}{\Gamma(\sigma)} \sum_j \frac{(z-\beta_j)_+^{\sigma-1}}{\beta_j} \\ \sum_j \frac{(z-\beta_j)_+^\sigma}{\beta_j} &\le \frac{\sigma}{d+1} z \sum_j \frac{(z-\beta_j)_+^{\sigma-1}}{\beta_j} \\ U_\sigma(z) &\le C_\sigma z U_{\sigma-1}(z),\end{aligned}$$

where $C_\sigma = \dfrac{\sigma}{d+1}$ for $\sigma > 2$.

Now, let us turn to the part $\sigma \le 2$. To prove this, the following lemma is required:

Lemma 7.2.1 *For all $\sigma \le \sigma'$,*

$$\frac{U_\sigma(z)}{U_{\sigma-1}(z)} \le \frac{U_{\sigma'}(z)}{U_{\sigma'-1}(z)} \tag{7.2.9}$$

The key point in the proof of Lemma 7.2.1 is the reverse Chebyshev inequality from [19]:

Lemma 7.2.2 *Let $\{a_j\}$ and $\{b_j\}$ be two real sequences, one nonincreasing and the other nondecreasing, and let $\{w_j\}$ be a sequence of nonnegative weights. Then*

$$\sum_{j=1}^n w_j \sum_{j=1}^n w_j a_j b_j \le \sum_{j=1}^n w_j a_j \sum_{j=1}^n w_j b_j \tag{7.2.10}$$

Proof of Lemma 7.2.1. Choose

$$w_j = \frac{(z-\beta_j)_+^{\sigma}}{\beta_j}, \qquad a_j = (z-\beta_j)_+^{\sigma'-\sigma} \qquad \text{and} \qquad b_j = (z-\beta_j)_+^{-1}$$

in (7.2.10). Since

$$0 \le \beta_1 < \beta_2 < \cdots$$

and $\sigma < \sigma'$, the sequence $\{a_j\}$ is decreasing and the sequence $\{\beta_j\}$ is increasing. Then

$$\begin{aligned}
&\left(\sum_{k=1}^{n} \frac{(z-\beta_j)_+^{\sigma}}{\beta_j}\right)\left(\sum_{k=1}^{n} \frac{(z-\beta_j)_+^{\sigma}}{\beta_j}(z-\beta_j)_+^{\sigma'-\sigma}(z-\beta_j)_+^{-1}\right)\\
&\le \left(\sum_{k=1}^{n} \frac{(z-\beta_j)_+^{\sigma}}{\beta_j}(z-\beta_j)_+^{\sigma'-\sigma}\right)\left(\sum_{k=1}^{n} \frac{(z-\beta_j)_+^{\sigma}}{\beta_j}(z-\beta_j)_+^{-1}\right)\\
\Rightarrow\ &\left(\sum_{k=1}^{n} \frac{(z-\beta_j)_+^{\sigma}}{\beta_j}\right)\left(\sum_{k=1}^{n} \frac{(z-\beta_j)_+^{\sigma'-1}}{\beta_j}\right) \le \left(\sum_{k=1}^{n} \frac{(z-\beta_j)_+^{\sigma'}}{\beta_j}\right)\left(\sum_{k=1}^{n} \frac{(z-\beta_j)_+^{\sigma-1}}{\beta_j}\right)\\
\Rightarrow\ & U_\sigma(z)U_{\sigma'-1}(z) \le U_{\sigma'}(z)U_{\sigma-1}(z),
\end{aligned}$$

or, equivalently,

$$\frac{U_\sigma(z)}{U_{\sigma-1}(z)} \le \frac{U_{\sigma'}(z)}{U_{\sigma'-1}(z)},$$

for $\sigma < \sigma'$. □

Now, we are ready to prove the result when $\sigma \le 2$. Choose $\sigma' = 2$ in (7.2.9) to obtain

$$\frac{U_\sigma(z)}{U_{\sigma-1}(z)} \le \frac{U_2(z)}{U_1(z)}, \tag{7.2.11}$$

where $\sigma < 2$. Recall that (7.2.3) imply

$$\frac{U_2(z)}{U_1(z)} \le \frac{2z}{d+1}.$$

After combining the two inequalities above we get

$$U_\sigma(z) \le C_\sigma z U_{\sigma-1}(z),$$

where $C_\sigma = \dfrac{2}{d+1}$ for $\sigma < 2$. □

Next, a bound in terms of $|\Omega|$ for the function U_σ is provided. The tool used here is the following inequality given in [17]:

Lemma 7.2.3 *(Frank, Loss, Weidl, [17]) Let $\rho > \sigma \geq 0$ and $t > z$. Then for all $\beta_j \geq 0$,*

$$(z - \beta_j)^\sigma \leq C(\sigma, \rho)(t - z)^{\sigma - \rho}(t - \beta_j)^\rho,$$

with

$$C(\sigma, \rho) := \begin{cases} 1 & if\ \sigma = 0 \\ \rho^{-\rho}\sigma^\sigma(\rho - \sigma)^{\rho - \sigma} & if\ \rho > \sigma > 0. \end{cases}$$

Proof. Set $f(z) = (z - \beta_j)^\sigma (t - z)^{\rho - \sigma}$ on the interval (β_j, t). When $f(z)$ is maximized with respect to z, the optimized value becomes

$$z_* = \frac{\sigma t + (\rho - \sigma)\beta_j}{\rho}.$$

Thus, $f(z) \leq f(z_*)$ implies

$$\begin{aligned} (z - \beta_j)^\sigma (t - z)^{\rho - \sigma} &\leq \left(\frac{\sigma t + (\rho - \sigma)\beta_j}{\rho} - \beta_j\right)^\sigma \left(t - \frac{\sigma t + (\rho - \sigma)\beta_j}{\rho}\right)^{\rho - \sigma} \\ &= \frac{\sigma^\sigma (t - \beta_j)^\rho (\rho - \sigma)^{\rho - \sigma}}{\rho^\rho}. \end{aligned} \tag{7.2.12}$$

Therefore, the result follows after rearranging terms in (7.2.12). $\square$

Now, a bound for $U_\sigma(z)$ is obtained by using the lemma above:

Theorem 7.2.2 *For $0 \leq \sigma < 2$,*

$$U_\sigma(z) \leq \frac{\sigma^\sigma |\Omega| (d + 1)^d}{2(d - 1)(4\pi)^{d/2} \rho \left(1 + \frac{d}{2}\right) (\sigma + d - 1)^{\sigma + d - 1}}. \tag{7.2.13}$$

Proof. Observe that Lemma 7.2.3 implies

$$\sum_j \frac{(z - \beta_j)^\sigma}{\beta_j} \leq C(\sigma, \rho)(t - z)^{\sigma - \rho} \sum_j \frac{(t - \beta_j)^\rho}{\beta_j}. \tag{7.2.14}$$

Choosing $\rho = 2$ in (7.2.14) yields

$$\sum_j \frac{(z - \beta_j)^\sigma}{\beta_j} \leq \frac{\sigma^\sigma (2 - \sigma)^{2 - \sigma}}{4} (t - z)^{\sigma - 2} \sum_j \frac{(t - \beta_j)^2}{\beta_j}, \tag{7.2.15}$$

for $0 \leq \sigma < 2$ and $t > z$. Optimizing the right side with respect to t gives the result. $\square$

7.3 Sharp Bounds for the Function $\sum_j \frac{e^{-\beta_j t}}{\beta_j}$

This section covers two theorems involving the function

$$\tilde{Z}(t) := \sum_j \frac{e^{-\beta_j t}}{\beta_j}.$$

The Laplace transform is utilized to obtain some bounds for $t^{d-1}\tilde{Z}(t)$. Certain definitions and properties of the Laplace transform used here may be found in Appendix B.f First, the following monotonicity result is obtained:

Theorem 7.3.1 *The function $t^{d-1}\tilde{Z}(t)$ is nonincreasing.*

Proof. Observe that the Laplace transform of $(z-\beta_j)_+^\sigma$ is

$$\mathfrak{L}(z-\beta_j)_+^\sigma) = \frac{\Gamma(\sigma+1)e^{-\beta_j t}}{t^{\sigma+1}}.$$

Then,

$$\mathfrak{L}(U_\sigma(z)) = \frac{\Gamma(\sigma+1)}{t^{\sigma+1}} \sum_j \frac{e^{-\beta_j t}}{\beta_j} = \frac{\Gamma(\sigma+1)}{t^{\sigma+1}}\tilde{Z}(t).$$

An application of the Laplace transform to both sides of (7.2.1) yields

$$\begin{aligned}
\frac{\Gamma(3)}{t^3}\tilde{Z}(t) &\leq \frac{2}{d+1}\left(\frac{\Gamma(3)}{t^3}\tilde{Z}(t) - \frac{\Gamma(2)}{t^2}\tilde{Z}'(t)\right) \\
\frac{d-1}{d+1}\tilde{Z}(t) &\leq \frac{-t}{d+1}\tilde{Z}'(t) \\
(d-1)\tilde{Z}(t) + t\tilde{Z}'(t) &\leq 0 \\
\frac{d}{dt}\left(t^{d-1}\tilde{Z}(t)\right) &\leq 0.
\end{aligned}$$

This proves the theorem. □

Next, an upper bound of the form $\tilde{Z}(t) \leq K(d)\frac{|\Omega|}{t^{d-1}}$ is obtained after an application of the Laplace transform to both sides of (3.2.5). Here, the constant $K(d)$ depends only on the dimension d.

Theorem 7.3.2 *For $d > 1$,*

$$t^{d-1}\tilde{Z}(t) \leq \frac{\Gamma\left(\frac{d+1}{2}\right)}{2(d-1)\pi^{\frac{d+1}{2}}}|\Omega|. \tag{7.3.1}$$

Proof. Recall from Section 3.2 that the Weyl asymptotic formula (3.2.3) implies (3.2.5), which is

$$U_2(z) \leq \frac{|\Omega|}{(4\pi)^{d/2}(d^2-1)\Gamma\left(1+\frac{d}{2}\right)} z^{d+1}.$$

An application of the Laplace transform to both sides yields

$$\begin{aligned}
\mathfrak{L}(U_2(z)) &\leq \frac{|\Omega|}{(4\pi)^{d/2}(d^2-1)\Gamma\left(1+\frac{d}{2}\right)} \mathfrak{L}(z^{d+1}) \\
\frac{\Gamma(3)}{t^3}\sum_{j=1}^{\infty} \frac{e^{-\beta_j t}}{\beta_j} &\leq \frac{|\Omega|}{(4\pi)^{d/2}(d^2-1)\Gamma\left(1+\frac{d}{2}\right)} \frac{\Gamma(d+2)}{t^{d+2}} \\
t^{d-1}\tilde{Z}(t) &\leq \frac{(d+1)\Gamma(d+1)|\Omega|}{2(4\pi)^{d/2}(d^2-1)\Gamma\left(1+\frac{d}{2}\right)}.
\end{aligned}$$

By using

$$\frac{\Gamma(d+1)}{(4\pi)^{d/2}\Gamma\left(1+\frac{d}{2}\right)} = \pi^{-\left(\frac{d+1}{2}\right)}\Gamma\left(\frac{d+1}{2}\right), \tag{7.3.2}$$

and after making some simplifications,

$$t^{d-1}\tilde{Z}(t) \leq \frac{\Gamma\left(\frac{d+1}{2}\right)}{2(d-1)\pi^{\frac{d+1}{2}}}|\Omega|. \tag{7.3.3}$$

□

Observe that the last two theorems imply that as $t \to 0^+$,

$$t^{d-1}\tilde{Z}(t) \to \frac{\Gamma\left(\frac{d+1}{2}\right)}{2(d-1)\pi^{\frac{d+1}{2}}}|\Omega|.$$

This resembles what Kac obtained in [32] for the Laplacian.

The next theorem establish a connection between (7.3.3) and a Berezin-Li-Yau type inequality.

Theorem 7.3.3

$$\frac{\Gamma\left(\frac{d+1}{2}\right)}{2(d-1)(d+1)!\pi^{\frac{d+1}{2}}}|\Omega| \geq \frac{U_2(z_0)}{z_0^{d+1}}. \tag{7.3.4}$$

Proof. We know from Section 2.3 that $z^{-(d+1)}U_2(z)$ is nondecreasing in the variable z. Then for $z > z_0$, we have

$$U_2(z) \geq U_2(z_0)\left(\frac{z}{z_0}\right)^{d+1}. \tag{7.3.5}$$

Let $\tau = z - z_0 > 0$. Then, (7.3.5) implies

$$U_2(z_0+\tau) \geq U_2(z_0)\left(\frac{\tau+z_0}{z_0}\right)^{d+1}. \tag{7.3.6}$$

We will apply Laplace transform to both sides of (7.3.6). Recall that when we apply the Laplace transform to a shifted function we obtain ([51])

$$\mathfrak{L}(f(\tau+z_0)) = e^{z_0 t}\left\{\mathfrak{L}(f) - \int_0^{z_0} e^{-t\tau} f(\tau)d\tau\right\}.$$

By making use of this formula, we obtain

$$\mathfrak{L}((\tau+z_0-\beta_j)_+^2) = e^{(z_0-\beta_j)_+t}\left(\frac{\Gamma(3)}{t^3} - \int_0^{(z_0-\beta_j)_+t} e^{-tu}\, u^2 du\right), \tag{7.3.7}$$

and

$$\mathfrak{L}((\tau+z_0)^{d+1}) = e^{z_0 t}\left(\frac{\Gamma(d+2)}{t^{d+2}} - \int_0^{z_0 t} e^{-tu}\, u^{d+1} du\right). \tag{7.3.8}$$

Inserting these expressions in (7.3.6) we have that

$$\begin{aligned}\sum_j \frac{e^{(z_0-\beta_j)_+t}}{\beta_j}\left(\frac{2}{t^3} - \int_0^{(z_0-\beta_j)_+t} e^{-tu} u^2 du\right)\\ \geq \frac{U_2(z_0)}{z_0^{d+1}} e^{z_0 t}\left(\frac{\Gamma(d+2)}{t^{d+2}} - \int_0^{z_0 t} e^{-tu} u^{d+1} du\right).\end{aligned} \tag{7.3.9}$$

Using the incomplete gamma function

$$\gamma(a,z) = \int_0^z e^{-\tau}\tau^{a-1}d\tau$$

in (7.3.9) simplifies it to

$$t^{d-1}\sum_j \frac{e^{(z_0-\beta_j)_+t}}{\beta_j}\left(2-\gamma(3,(z_0-\beta_j)_+)t^2\right) \geq \frac{U_2(z_0)}{z_0^{d+1}} e^{z_0 t}\left((d+1)! - \gamma(d+2, z_0 t^2)\right). \tag{7.3.10}$$

Observe that

$$\sum_j \frac{e^{(z_0-\beta_j)_+t}}{\beta_j} \geq e^{z_0 t}\sum_j \frac{e^{-\beta_j t}}{\beta_j} = e^{z_0 t}\tilde{Z}(t).$$

With the aid of this observation and after some algebra, (7.3.10) becomes

$$\frac{t^{d-1}}{(d+1)!}\tilde{Z}(t) \geq \frac{U_2(z_0)}{z_0^{d+1}} + R(t), \tag{7.3.11}$$

where the remainder term is

$$R(t) = \frac{t^{d-1}e^{-z_0 t}}{(d+1)!}\sum_j \frac{e^{(z_0-\beta_j)_+t}}{\beta_j}\gamma(3,(z_0-\beta_j)_+t^2) - \frac{U_2(z_0)}{z_0^{d+1}(d+1)!}\gamma(d+2, z_o t^2).$$

Note that $R(t) \to 0$ as $t \to 0^+$. Since

$$t^{d-1}\tilde{Z}(t) \to \frac{\Gamma\left(\frac{d+1}{2}\right)}{2(d-1)\pi^{\frac{d+1}{2}}}|\Omega|,$$

as $t \to 0^+$, Equation (7.3.11) becomes

$$\frac{\Gamma\left(\frac{d+1}{2}\right)}{2(d-1)(d+1)!\pi^{\frac{d+1}{2}}}|\Omega| \geq \frac{U_2(z_0)}{z_0^{d+1}}. \tag{7.3.12}$$

$\square$

APPENDIX A

COMMUTATOR AND ITS PROPERTIES

A.1 Commutator

The commutator of two operators A and B is defined as

$$[A, B] = AB - BA.$$

From this definition, it is obvious that two operators commutes iff their commutator is 0. There are several useful identities such as

- $[A, A] = 0$
- $[A, B] = -[B, A]$ (anticommutativity)
- $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ (Jacobi identity).

APPENDIX B

SOME TRANSFORMS

B.1 Legendre Transform

Definition B.1.1 *The Legendre transform of a function f is*

$$f^*(w) = \sup_{z \in \mathbb{R}^d} \{w \cdot z - f(z)\}$$

where $w \in \mathbb{R}^d$.

Remarks B.1.2 *Among many, the following properties are worth mentioning here:*

- *The mapping $w \mapsto f^*(w)$ is convex.*
- *If $f(z) \leq g(z)$, then $f^*(w) \geq g^*(w)$.*

Example B.1.3 *The Legendre transform of the function $f(z) = \dfrac{z^p}{p}$ is the function*

$$f^*(w) = \frac{w^q}{q} \qquad \text{where} \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Example B.1.4 *The Legendre transform of $R_1(z)$ is a straightforward calculation, to be found explicitly for example in [21, 39]. The result for $k-1 < w < k$ is*

$$R_1^*(w) = (w - [w])\beta_{[w]+1} + [w]\overline{\beta_{[w]}}\,, \tag{B.1.1}$$

where $[w]$ denotes the greatest integer $\leq w$. When w approaches an integer value k from below, $R_1^*(k) = k\overline{\beta}_k$.

B.2 Fourier Transform

The main ingredient of this work is the Fourier transform because it enables us to compute the commutators and inner products in the trace formulae (2.1.1) conveniently.

Definition B.2.1 *The Fourier transform of a function $f(x)$ on the dense subspace of test functions $C_c^\infty(\mathbb{R}^d)$ is defined by*

$$\widehat{f}(\xi) = \mathfrak{F}[f] := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-i\xi \cdot \mathbf{x}) f(\mathbf{x}) d\mathbf{x},$$

For instance, the Laplacian is given by

$$-\Delta\varphi := \mathfrak{F}^{-1}|\xi|^2\widehat{\varphi}(\xi).$$

Similarly, the Fourier transform of the Klein-Gordon operators is

$$\sqrt{-\Delta + m^2}\varphi := \mathfrak{F}^{-1}\sqrt{|\xi|^2 + m^2}\widehat{\varphi}(\xi).$$

and the Fourier transform of the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$ is

$$(-\Delta)^s\varphi = \chi_\Omega\mathfrak{F}^{-1}|\xi|^{2s}\widehat{\varphi}(\xi).$$

B.3 Laplace Transform

Definition B.3.1 *The Laplace transform of a function $f(z)$ is given by*

$$\mathfrak{L}(f(z)) = \int_0^\infty e^{-tz}f(z)dz.$$

The first appearance of the Laplace transform in this thesis is in Section 3.2, where Karamata's Tauberian theorem (3.2.2) is exploited to prove Weyl's asymptotic formula for the Klein-Gordon operators. Note that R. M. Blumenthal and R. K. Getoor obtained the asymptotic distribution of the eigenvalues for a class of Markov operators for α-stable processes by using Karamata's Tauberian theorem in [8].

The Laplace transfrom is extensively used to get the results in Section 7.3. For instance, the Laplace transform of the function $U_\sigma(z)$ yields $\tilde{Z}(t)$. Indeed,

$$\mathfrak{L}(U_\sigma(z)) = \frac{\Gamma(\sigma+1)}{t^{\sigma+1}}\sum_j \frac{e^{-\beta_j t}}{\beta_j} = \frac{\Gamma(\sigma+1)}{t^{\sigma+1}}\tilde{Z}(t).$$

B.4 Riemann-Liouville and Weyl Fractional Integrals

This section briefly discusses two fractional transforms which are pertinent to this thesis, in particular Chapter 7. The second volume [15] would be an excellent source to get into the details of these fractional transforms. As it was stated there, for the relation between fractional integrals and Laplace transforms, refer to the book [60]. For the relation between fractional integrals and Fourier transforms, see [34].

Definition B.4.1 *The Riemann-Liouville transform of the function $f(t)$ or order η is defined as*

$$\mathfrak{R}(\eta, f(t)) = \frac{1}{\Gamma(\eta)} \int_0^z f(t)(z-t)^{\eta-1} dt.$$

Definition B.4.2 *The Weyl transform of the function $f(t)$ or order η is defined as*

$$\mathfrak{W}(\eta, f(t)) = \frac{1}{\Gamma(\eta)} \int_z^\infty f(t)(t-z)^{\eta-1} dt.$$

Next, some of equations that are employed are briefly mentioned. The first such result is the Riemann-Liouville transform of the function $f(t) = t^{\sigma-1}$ for $\sigma > 0$ [15].

$$\mathfrak{R}(\eta, t^{\sigma-1}) = \frac{\Gamma(\sigma)}{\Gamma(\eta+\sigma)} z^{\sigma+\eta-1}.$$

When we use the definition of the Riemann-Liouville transform above we get

$$\int_0^z t^{\sigma-1}(z-t)^{\eta-1} dt = \frac{\Gamma(\eta)\Gamma(\sigma)}{\Gamma(\eta+\sigma)} z^{\sigma+\eta-1}. \tag{B.4.1}$$

By setting $z = 1$ in (B.4.1), Eq. (7.1.7), which was the main ingredient in the Aizenman-Lieb procedure [1], is obtained.

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VITA

Selma Yıldırım Yolcu was born in Turkey in 1980. She received her B.S. degree in Mathematics and B.S. degree in Mathematics Education in 2001 both from Middle East Technical University, Ankara, Turkey. She also received an M.S. degree from Middle East Technical University in 2003, and another M.S. degree from Georgia Institute of Technology in 2008 both in Mathematics. Her research interests include spectral theory, partial differential equations and analysis, in particular eigenvalue inequalities for Klein-Gordon operators and fractional Laplacians.