INITIAL-BOUNDARY VALUE PROBLEMS IN FLUID DYNAMICS

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To my parents and my wife.

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SUMMARY

This thesis is devoted to studies of initial-boundary value problems (IBVPs) for systems of partial differential equations (PDEs) arising from fluid mechanics modeling, especially for the compressible Euler equations with frictional damping, the Boussinesq equations, the Cahn-Hilliard equations and the incompressible density-dependent Navier-Stokes equations. The emphasis of this thesis is to understand the influences to the qualitative behavior of solutions caused by boundary effects and various dissipative mechanisms including damping, viscosity and heat diffusion. We will present results concerning global existence and large-time asymptotic behavior of solutions to miscellaneous initial-boundary value problems. The results obtained consist of three parts.

The Part 1, containing Chapters II–III, is concerned with the study of compressible Euler equations with frictional damping. In Chapter II, we first construct global L^∞ entropy weak solutions to the IBVP for one-dimensional damped compressible Euler equations on bounded domains with physical boundaries. Time asymptotically, the density is conjectured to satisfy the porous medium equation (PME) and the momentum obeys to the classical Darcy's law. Based on entropy principle, we show that the physical solution converges to steady states exponentially fast in time. We also prove that the same is true for the related IBVP of porous medium equation provided that the two systems carry the same initial mass and thus justify the validity of Darcy's law in large time. In Chapter III, we continue the study of damped compressible Euler equations on bounded domains. We prove global existence and uniqueness of classical solutions to the IBVP for three-dimensional damped compressible Euler equations on bounded domains with the slip boundary condition when the initial data is near its

equilibrium. Furthermore, based on energy estimate, we show that the classical solution is captured by that of the porous medium equation exponentially fast as time tends to infinity and justify Darcy's law in large time.

In Part 2, we study the two-dimensional Boussinesq equations with partial viscosity. In Chapter IV, we first prove global existence of smooth solutions to the IBVP for the viscous non-heat-conductive Boussinesq equations on bounded domains with arbitrary smooth initial data and the no-slip boundary condition. In addition, the uniform bound of the kinetic energy is obtained as a by product. Then we study the IBVP for another type of 2D Boussinesq equations with partial viscosity which is inviscid and heat-conductive. We show that there exists a unique global smooth solution to the IBVP for arbitrary smooth initial data and for physical boundary conditions. Furthermore, due to dissipation and boundary effects, we prove that the kinetic energy is uniformly bounded in time and the temperature converges exponentially to a constant state which is the value of the temperature on the boundary of the domain. The results obtained in this part suggest that the partial dissipative mechanism is indeed strong enough to compensate the effects of gravitational force and nonlinear convection in order to prevent the development of singularity in the systems.

Part 3 is contributed to the mathematical analysis of multi-phase/mixing flows. In Chapter V, we first study the IBVP for a system of PDEs obtained by coupling the Cahn-Hilliard equation and the two-dimensional Boussinesq equations which stands for a model of a multi-phase flow under shear and the influence of gravitational force. Then we study a model of a two-component mixture, with a diffusive mass exchange among the medium particles of various density accounted for, which is closely related to the 2D incompressible density-dependent Navier-Stokes equations. For both systems of equations, we prove global existence of smooth solutions to the IBVPs with arbitrary smooth initial data and physical boundary conditions.

CHAPTER I

INTRODUCTION

Mathematical analysis of fluid mechanics was initiated more than one century ago. Although enormous efforts have been made on this subject since then, the resolution of some basic issues is still missing. There are a number of fundamental problems still remaining unsolved.

Mathematically speaking, most of the modeling equations in fluid mechanics can be formulated into the general quasilinear systems of partial differential equations taking the following form:

$$\partial_t \mathbf{U} + \mathrm{div}_{\mathbf{x}} F(\mathbf{U}) = Q_\varepsilon(\mathbf{U}, D\mathbf{U}, D^2\mathbf{U}), \quad \mathbf{x} \in \mathbb{R}^n,\ t \geq 0,\ \mathbf{U} \in \mathbb{R}^m, \tag{1.0.1}$$

where $\mathrm{div}_{\mathbf{x}}$ is the divergence operator, D^s stands for spatial derivatives, $F(\mathbf{U}) \in \mathbb{R}^{m\times n}$ is a smooth vector-valued function, $Q_\varepsilon \in \mathbb{R}^m$ is related to dissipation, which may be viscosity, heat diffusion, damping, relaxation and etc. Important examples include Euler equations, Navier-Stokes equations, Boltzmann equations and Boussinesq equations for compressible and incompressible flows, the equations of magnetohydrodynamics (MHD) for electrically conducting compressible fluids and the equations of nonlinear thermoviscoelastic materials. Topics on (1.0.1) are physically important and mathematically challenging. One major challenge in this field is the question of global existence and large time asymptotic behavior of solutions to certain initial value (Cauchy) problem or initial-boundary value problem (IBVP) for modeling equations. Definite answers to this question will undoubtedly shed light on understanding of basic issues in fluid dynamics.

In real world, flows often move in bounded domains with constraints from boundaries, where initial-boundary value problems appear. Solutions of IBVPs usually exhibit different behaviors and much richer phenomena comparing with Cauchy problems. One major difference between IBVPs and Cauchy problems is the lack of information of spatial derivatives of solutions on boundaries of domains, which makes the analysis of IBVPs significantly different from that of Cauchy problems. Another feature is the availability of Poincaré's inequality on bounded domains which, together with dissipative mechanisms, usually leads to exponential decay of solutions to IBVPs, instead of algebraic decay of solutions to Cauchy problems. Therefore, when problems are set on bounded domains, IBVPs distinguish themselves from Cauchy problems significantly.

This thesis is contributed to studies of IBVPs for several systems of partial differential equations arising from fluid dynamics, including the compressible Euler equations with frictional damping, the Boussinesq equations, the Cahn-Hilliard equations and the incompressible density-dependent Navier-Stokes equations. The emphasis of this thesis is to understand the influences to the qualitative behavior of solutions caused by boundary effects and various dissipative mechanisms such as damping, viscosity and heat diffusion. The background and main results obtained on these topics will be presented in Sections 1.2–1.4.

1.1 Notations

Before introducing the background and main results, we list some notations which will be used later.

Throughout this thesis, $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{W^{s,p}}$ denote the norms of the usual Lebesgue measurable function spaces L^p $(1 \le p < \infty)$, L^∞ and the usual Sobolev space $W^{s,p}$ respectively, i.e.,

$$\|f\|_{L^p} \equiv \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p d\mathbf{x}\right)^{1/p}, \quad \text{for } f \in L^p(\Omega),\ 1 \le p < \infty,$$

$$\|f\|_{L^\infty} \equiv \|f\|_{L^\infty(\Omega)} = \text{ess}\ \sup_\Omega |f|, \quad \text{for } f \in L^\infty(\Omega),$$

$$\|f\|_{W^{s,p}} \equiv \|f\|_{W^{s,p}(\Omega)} = \left(\sum_{|\alpha|\le s} \int_\Omega |D^\alpha f|^p d\mathbf{x} \right)^{1/p}, \quad \text{for } f \in W^{s,p}(\Omega),\ 1 \le p < \infty,$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is any multi-index with order $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$. For $p = 2$, we denote the norm $\|\cdot\|_{L^2}$ by $\|\cdot\|$ and $\|\cdot\|_{W^{s,2}}$ by $\|\cdot\|_{H^s}$ respectively. For simplicity, we use the following notation:

$$\|(f_1, f_2, ..., f_m)\|_V^2 \equiv \sum_{i=1}^m \|f_i\|_V^2,$$

where V denotes various function spaces.

We also need some function spaces of Hölder continuous functions. $C^\alpha(\Omega)$ stands for the Banach space of functions on Ω which are uniformly Hölder continuous with exponent α, while $C^{\alpha,\alpha/2}(\Omega_T)$ denotes the Banach space of functions on $\Omega_T = \Omega \times [0, T]$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t.

In the *a priori* estimates, the generic constant will be denoted by C.

There are also some other notations which we will explain later in specific chapters.

1.2 Damped Euler Equations

1.2.1 Background

Damping is the effect that tends to reduce the amplitude of oscillations of an oscillatory system. The phenomenon of damping is observed everywhere and in everyday life. Typical examples include the mass-spring-damper, the RLC circuit and the harmonic oscillator. There are lots of real world applications of damping such as generation of vibrations of specific frequencies, audio system measurements, active mass damper, design of vehicle suspension and thrust damping in aerospace engineering.

In fluid mechanics, the damping effect is observed in mathematical modeling of flow and transport through porous media which plays an important role in environmental studies as well as in reservoir engineering. Applications include the spread of pollutants from a landfill through the soil system and of oil spills in the subsurface, the intrusion of seawater in coastal aquifers, and new methods for enhanced oil recovery and underground gas storage.

In the mathematical study of flow through porous media, one of the most commonly used modeling systems is the damped compressible Euler equations:

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0, \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) + \nabla P = -\alpha \rho U, \quad x \in \mathbb{R}^n, \quad t \geq 0, \end{cases} \tag{1.2.1}$$

which occur in the modeling of compressible isentropic flow through a porous medium. The medium induces a friction force, proportional to the linear momentum in the opposite direction. Therefore, system (1.2.1) expresses the conservation of mass and the momentum balance. Here ρ, $U = (u_1, u_2, ..., u_n)$ and P denote the density, velocity and pressure respectively; the constant $\alpha > 0$ models friction. Assuming the flow is a polytropic perfect gas, then $P(\rho) = P_0 \rho^\gamma$, with P_0 a positive constant, and $\gamma > 1$ the adiabatic gas exponent. Without loss of generality, we take $P_0 = \frac{1}{\gamma}$, $\alpha = 1$ throughout this thesis.

1.2.1.1 The 1D Model

When $n = 1$, system (1.2.1) turns out to be

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\alpha \rho u, \quad x \in \mathbb{R}, \quad t \geq 0, \end{cases} \tag{1.2.2}$$

which is hyperbolic with two characteristic speeds $\lambda_1 = u - \sqrt{P'(\rho)}$ and $\lambda_2 = u + \sqrt{P'(\rho)}$. Furthermore, by definition, system (1.2.2) is strictly hyperbolic at the point away from vacuum where $\rho = 0$.

In experiments, Darcy's law is observed in the same process. Thus, we have another model:

$$\begin{cases} \rho_t = P(\rho)_{xx}, \\ m = -P(\rho)_x, \end{cases} \tag{1.2.3}$$

where m stands for the momentum. The first equation is the well-known porous medium equation (PME) and the second equation states Darcy's law. So, it is natural to expect that system (1.2.2) and system (1.2.3) are equivalent in long time. Actually, the following conjecture was proposed by Tai-ping Liu in [64]:

Conjecture. *As $t \to \infty$, system(1.2.2) is equivalent to system (1.2.3).*

Due to strong physical background and significant mathematical challenge, system (1.2.2) and its time-asymptotic behavior have received considerable attentions, and investigations have been carried on for decades since the pioneer work of Nishida [78].

In the case of small smooth solutions away from vacuum, system (1.2.2) can be transformed to the p-system with damping by changing to Lagrangian coordinates (c.f. [94]) and the problem has been well understood. Extensive literature is available in this field. For Cauchy problem, the readers are referred to [42, 43, 44, 45, 46, 68, 74, 79, 80, 105]. For initial-boundary value problems, see [47, 48, 71, 81, 83]. For more references on the p-system with damping, we refer to [25, 27, 49, 69, 72, 73, 83, 91, 108].

When the solution is large, rough and contains vacuum, the difficulty of the problem is significantly increased. The main difficulties come from the interaction of three mechanisms: nonlinear convection, lower-order dissipation of damping and resonance due to vacuum where two characteristics coincide. First of all, it has been shown in [107] that when the initial data is large or rough, shock will develop in finite time for system (1.2.2) due to nonlinearity. Second, although the damping prevents formation of singularity if initial data is small and smooth, it breaks the self-similarity of the system, which is crucial for large solutions; see [50, 51]. Third, when the solution contains vacuum, it has been shown by Liu & Yang [65, 66] that local smooth solutions of (1.2.2) blow up in finite time *before shock formation* due to resonance caused by vacuum. Therefore, it is suitable to consider weak solutions in order to study global existence and large time behavior of (1.2.2).

The phenomena mentioned above make both analytical and numerical studies for system (1.2.2) highly non-trivial problems. Indeed, the only global weak solutions to (1.2.2) are constructed in L^∞ space by using the method of compensated compactness in the direction of Cauchy problem; see [29] for $1 < \gamma \leq 5/3$ and [55] for $1 \leq \gamma < 3$. Concerning large time behavior of L^∞ weak solutions to (1.2.2) for Cauchy problem, some essential progress was made recently in [55], where the authors justified the conjecture by using the rescaling argument in [91]. Later, in [53] and [54], the authors developed some new technique based on the conservation of mass and entropy analysis to attack the problem. They showed that L^∞ weak solutions with vacuum converge, strongly in $L^p(\mathbb{R})$ with decay rates, to the similarity solution and the Barenblatt solution of the porous medium equation.

However, when the problem is set on bounded domains, the story changes dramatically. The key approach used in [42]–[54] for Cauchy problem is to compare the solution of (1.2.2) with that of (1.2.3) directly via energy estimates. Unfortunately, this approach does not work for the initial-boundary value problem mainly due to the boundary effects. Therefore, the global existence and large-time behavior of L^∞ weak solutions to initial-boundary value problem for damped compressible Euler equations remains as an important open problem. This problem has particular interest since, as we mentioned in the beginning of the Introduction, in real world, flows often move in bounded domains with constraints from boundaries. We will give a definite answer to this problem in this thesis.

1.2.1.2 The 3D Case

Now we turn our attention to the multi-dimensional case of (1.2.1). From the physical point of view, the multi-dimensional model describes more realistic phenomena. Furthermore, besides the features and difficulties mentioned in the one-dimensional case, the multi-dimensional model presents some unique features which are totally absent in the 1D case. When the problem is set in multi-dimensional spaces, one of the most significant phenomena appears, which is the effect of vorticity. It is known that the cumulation of vorticity determines the global existence/finite-time blow up of smooth solutions for the 3D incompressible Euler or Navier-Stokes equations. Indeed, Beale, Kato & Majda [8] proved that if a solution of the 3D Euler or Navier-Stokes equations is initially smooth and loses its regularity at some later time, then the maximum vorticity necessarily grows without bound as the critical time approaches; equivalently, if the vorticity remains bounded, a smooth solution persists. Therefore, due to strong physical background and significant mathematical challenge, the multi-dimensional damped Euler equations is much less understood than its 1D companion. For Cauchy problem, investigations were carried out among small smooth solutions. In [93, 99], based on energy estimates, the authors proved global existence, uniqueness and large-time behavior of smooth solutions to the Cauchy problem for three-dimensional damped compressible Euler equations when the initial data is near its equilibrium. Recently, by applying the combination of the Green function on estimating the lower order derivatives and the energy method for the higher order derivatives, Wang & Yang [100] proved global existence and L^2 convergence rate of solutions for 2D damped compressible Euler equations when the initial data is a small perturbation of a planar diffusion wave. However, in the direction of the initial-boundary value problem, even the global existence of small smooth solutions is still an important open problem. In this thesis, we will give a definite answer to this problem.

1.2.2 Analysis of Damping Effect

1.2.2.1 The 1D Model

Let us consider the one-dimensional compressible Euler equation with frictional damping. After introducing the momentum $m = \rho u$, we can rewrite (1.2.2) as follows:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \Big(\dfrac{m^2}{\rho} + P(\rho)\Big)_x = -m. \end{cases} \tag{1.2.4}$$

System (1.2.4) is supplemented by the following initial and boundary conditions:

$$\begin{cases} \rho(x,0) = \rho_0(x), \;\; m(x,0) = m_0(x), \quad 0 < x < 1, \\ m|_{x=0} = 0, \quad m|_{x=1} = 0, \quad t \geq 0, \\ \displaystyle\int_0^1 \rho_0(x)\; dx = \rho_* > 0. \end{cases} \tag{1.2.5}$$

Where, the last condition is imposed to avoid the trivial case, $\rho \equiv 0$.

As mentioned above, time asymptotically, we expect that the solution to (1.2.4)–(1.2.5) will be captured by that of the following problem:

$$\begin{cases} \rho_t = P(\rho)_{xx}, \\ \rho(x,0) = \rho_0(x), \quad 0 < x < 1, \\ P_x|_{x=0} = P_x|_{x=1} = 0, \quad t \geq 0. \end{cases} \tag{1.2.6}$$

In this thesis, we continue the study of [47] and [48] on bounded domains with typical physical boundary condition (1.2.5). We will study the global existence and large time behavior of L^∞ weak solutions. The following two theorems are the main results of this part. For the global existence of L^∞ weak solutions we have

Theorem 1.2.1. *Suppose that the initial data (ρ_0, m_0) satisfy the conditions*

$$0 \leq \rho_0(x) \leq M_1, \quad \rho_0 \not\equiv 0, \quad |m_0(x)| \leq M_2 \rho_0(x),$$

for some positive constants $M_i (i = 1, 2)$. Then, for $\gamma > 1$, the initial-boundary value problem (1.2.4)-(1.2.5) has a global weak solution $(\rho(x,t), m(x,t))$, as defined *in Definition 2.2.1 which will be given in Section 2.2, satisfying the following estimates and entropy condition:*

$$\begin{aligned}
&0 \le \rho \le C, \quad |m| \le C\rho \ \text{ a.e. } \textit{for a constant } C > 0 \textit{ independent of } t, \textit{ and} \\
&\int_0^T \int_0^1 \left(\eta(\rho, m)\tilde{\psi}_t + q(\rho, m)\tilde{\psi}_x\right) dx\ dt - \int_0^T \int_0^1 \eta(\rho, m) m \tilde{\psi}\ dx\ dt \ge 0,
\end{aligned} \tag{1.2.7}$$

for all weak and convex entropy pairs (η, q) *and for all nonnegative smooth functions* $\tilde{\psi} \in C_0^1(I_T)$.

Concerning the large-time behavior of the solution obtained in the above theorem we have

Theorem 1.2.2. *Suppose* $\int_0^1 \rho_0(x)dx = \rho_*$. *Let* (ρ, m) *be any* L^∞ *entropy weak solution of the initial-boundary problem (1.2.4)-(1.2.5) defined in Definition 2.2.1, satisfying the estimates*

$$0 \le \rho(x,t) \le \Lambda < \infty, \quad |m(x,t)| \le M_3 \rho(x,t),$$

where M_3, Λ *are positive constants and let* $(\tilde{\rho}, \tilde{m})$ *be the weak solution of (1.2.6) with mass* ρ_* *and* $\tilde{m} = -P(\tilde{\rho})_x$. *Then, there exist constants* $C,\ \delta > 0$ *depending only on* $\gamma,\ \rho_*, \Lambda$ *and initial data such that*

$$\left\| \left((\rho - \tilde{\rho}),\ (m - \tilde{m})\right)(\cdot, t) \right\|_{L^2([0,1])}^2 \le C \exp\{-\delta t\}. \tag{1.2.8}$$

The existence of entropy weak solutions will be achieved by means of Godunov scheme [36] and the compensated compactness frameworks established by [29], [30], [62], [63], [77] and [96]. The proof of Theorem 1.2.1 is in the spirit of [104] and [88]. For the large time behavior, we adopt the new framework introduced by [53] and [55] based on entropy dissipation. The exponential decay rates are obtained in this case on bounded domains.

1.2.2.2 The 3D Case

Continuing the study of the damped compressible Euler equations, we consider the 3D model:

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0, \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) + \nabla P = -\rho U, \end{cases} \tag{1.2.9}$$

with the following initial and boundary conditions:

$$\begin{cases} (\rho, U)(\mathbf{x}, 0) = (\rho_0, U_0)(\mathbf{x}), \quad \mathbf{x} = (x, y, z) \in \Omega, \\ U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0, \\ \int_\Omega \rho_0(\mathbf{x}) d\mathbf{x} = \bar{\rho} > 0, \end{cases} \tag{1.2.10}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal vector on the boundary of Ω. The boundary condition is the so-called *slip boundary condition*.

Due to the dissipation in the momentum equations and the boundary effect, the kinetic energy is expected to vanish as time tends to infinity while the potential energy will converge to a constant. Furthermore, it is easy to see that

$$\int_\Omega \rho(\mathbf{x}, t) d\mathbf{x} = \int_\Omega \rho_0(\mathbf{x}) d\mathbf{x} = \bar{\rho},$$

due to the conservation of total mass. This suggests that the asymptotic state of the solution should be $(\rho, U)|_{t\to\infty} = (\bar{\rho}/|\Omega|, \mathbf{0})$. In this thesis, we will prove, under the assumption that the initial perturbation around the equilibrium state is small, there exists a unique global classical solution to (1.2.9)–(1.2.10) and the solution converges exponentially to the equilibrium state.

As in the preceding section, we will show that the classical solution of (1.2.9)–(1.2.10) is captured by that of the decoupled system

$$\begin{cases} \tilde{\rho}_t = \Delta P(\tilde{\rho}), \\ \tilde{M} = -\nabla P(\tilde{\rho}) \end{cases} \tag{1.2.11}$$

with the initial-boundary conditions

$$\begin{cases} \tilde{\rho}(\mathbf{x}, 0) = \tilde{\rho}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ \nabla P \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{1.2.12}$$

exponentially in time provided that

$$\int_\Omega \tilde{\rho}_0(\mathbf{x}) d\mathbf{x} = \int_\Omega \rho_0(\mathbf{x}) d\mathbf{x}. \tag{1.2.13}$$

The following theorem is our main result in this direction, which generalizes the study of [93] and [99] on bounded domain with typical physical boundary condition $(1.2.10)_2$.

Theorem 1.2.3. *Suppose that the initial data satisfy the compatibility condition, i.e.,* $\partial_t^l U(0) \cdot \mathbf{n}|_{\partial\Omega} = 0, 0 \leq l \leq 2$, *where* $\partial_t^l U(0)$ *is the* l^{th} *time derivative at* $t = 0$ *of any solution of (1.2.9)–(1.2.10), as calculated from (1.2.9) to yield an expression in terms of* ρ_0 *and* U_0*. Then there exists a constant* ε *such that if* $(\rho_0 - \bar{\rho}/|\Omega|, U_0) \in H^3(\Omega)$ *and* $\|(\rho_0 - \bar{\rho}/|\Omega|, U_0)\|_{H^3} \leq \varepsilon$, *then there exists a unique global solution* (ρ, U) *of the initial-boundary value problem (1.2.9)–(1.2.10) in* $C^1(\bar{\Omega} \times [0, \infty)) \cap X_3([0, \infty), \Omega)$, *where the exact definition of* $X_3([0, \infty), \Omega)$ *will be given in Section 3.1. Moreover, there exist positive constants* $C > 0, \eta > 0$, *which are independent of* t, *such that*

$$|||(\rho - \bar{\rho}/|\Omega|)(\cdot, t)||| + |||U(\cdot, t)||| \leq C\|(\rho_0 - \bar{\rho}/|\Omega|, U_0)\|_{H^3} \exp\{-\eta t\}. \tag{1.2.14}$$

Concerning the relationship between the solutions of (1.2.9)–(1.2.10) and (1.2.11)–(1.2.12), we have

Theorem 1.2.4. *Let* (ρ, U) *be the unique global classical solution of (1.2.9)–(1.2.10) and let* $M = \rho U$*. Let* $(\tilde{\rho}, \tilde{M})$ *be the global solution of (1.2.11)–(1.2.12) with* $\tilde{\rho}_0 \in L^\infty(\Omega)$, *and* $0 \leq \tilde{\rho}_0 \leq \rho^*$ *for some constant* ρ^* *satisfying* $\bar{\rho}/|\Omega| < \rho^* < \infty$*. Then, there exist constants* $C,\ \delta > 0$ *independent of* t *such that*

$$\|(\rho - \tilde{\rho})(\cdot, t)\|_{H^1} + \|(M - \tilde{M})(\cdot, t)\| \leq C \exp\{-\delta t\}, \quad \text{as } t \to \infty. \tag{1.2.15}$$

Remark 1.2.1. *The results obtained in above theorems suggest that damping effect is strong enough to compensate nonlinear convection in order to prevent development of singularity in the system, provided that the initial perturbation near the equilibrium is small.*

1.3 2D Boussinesq Equations

1.3.1 Background

For decades, the question of global existence/finite time blow-up of smooth solutions for the three-dimensional incompressible Euler or Navier-Stokes equations has been one of the most outstanding open problems for both engineers and mathematicians. The answer to this question will undoubtedly play a key role in understanding core problems in fluid dynamics such as the onset of turbulence. Enormous efforts have been made on this challenging problem, but the resolution of some basic issues is still missing. The main difficulty is to understand the vortex stretching effect in 3D flows, which is absent in the two-dimensional case. There are a great amount of literatures concerning partial answers to this question. We refer the reader to [22, 61, 97] and the references therein for detailed discussions on this subject.

As part of the effort to understand the vortex stretching effect in 3D flows, various simplified model equations have been proposed. Among these models, the 2D Boussinesq system is known to be one of the most commonly used because it is analogous to the 3D incompressible Euler or Navier-Stokes equations for axisymmetric swirling flow, and it shares a similar vortex stretching effect as that in the 3D incompressible flow. In fact, in cylindrical coordinates, the vortex formulation of the Euler equations describing 3D incompressible axisymmetric swirling flow can be written as (c.f. [70]):

$$\begin{cases} \dfrac{D}{Dt}\left[(rv^\alpha)^2\right] = 0, \\ \dfrac{D}{Dt}\left(\dfrac{\omega^\alpha}{r}\right) = -\dfrac{1}{r^4}\left[(rv^\alpha)^2\right]_{x_3}, \\ \dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + v^r\dfrac{\partial}{\partial r} + v^3\dfrac{\partial}{\partial x_3}, \end{cases} \tag{1.3.1}$$

where (v^r, v^α, v^3) are the velocity components with respect to the cylindrical coordinates (r, α, x_3); $\omega^\alpha = v^r_{x_3} - v^3_r$ is the second component of the vorticity and $\frac{D}{Dt}$ stands for the material derivative.

On the other hand, the vortex formulation of the 2D Boussinesq equations in Cartesian coordinates reads:

$$\begin{cases} \dfrac{D}{Dt}\theta = 0, \\ \dfrac{D}{Dt}\omega = -\theta_x, \\ \dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y}, \end{cases} \tag{1.3.2}$$

where θ is the scalar temperature and $\omega = v_x - u_y$ is the 2D vorticity. Therefore, we have the following correspondence between the two sets of equations in (1.3.1) and (1.3.2):

$$\begin{aligned} x_3 &\longleftrightarrow x, & r &\longleftrightarrow y, \\ \omega^\alpha &\longleftrightarrow \omega, & (rv^\alpha)^2 &\longleftrightarrow \theta, \\ v^r &\longleftrightarrow v, & v^3 &\longleftrightarrow u. \end{aligned}$$

With this correspondence, we see that (1.3.1) is formally identical to (1.3.2) provided that all external variable coefficients in (1.3.1) are evaluated at $r = 1$. Thus, away from the axis of singularity $r = 0$ for swirling flows, the qualitative behavior of solutions for the two systems of equations are expected to be identical. Better understanding of the 2D Boussinesq system will certainly shed light on the understanding of 3D flows.

In this thesis, we consider the two-dimensional Boussinesq equations with dissipations:

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = \nu \Delta U + \theta \mathbf{e}_2, \quad \mathbf{x} \in \mathbb{R}^2, \quad t \geq 0, \\ \theta_t + U \cdot \nabla \theta = \kappa \Delta \theta, \\ \nabla \cdot U = 0, \end{cases} \tag{1.3.3}$$

where $U = (u, v)$ is the velocity vector field, P is the scalar pressure, θ is the scalar temperature, the constants $\nu \geq 0, \kappa \geq 0$ model viscosity and thermal diffusion respectively, and $\mathbf{e}_2 = (0, 1)^{\mathrm{T}}$.

System (1.3.3) is potentially relevant to the study of atmospheric and oceanographic turbulence, as well as other astrophysical situations where rotation and stratification play a dominant role (see e.g. [87]). In fluid mechanics, system (1.3.3) is used in the field of buoyancy-driven flow. It describes the flow of a viscous incompressible fluid subject to convective heat transfer under the influence of gravitational force (c.f. [70]).

In recent years, the 2D Boussinesq equations (1.3.3) have attracted significant attention. When $\Omega = \mathbb{R}^2$, the Cauchy problem for (1.3.3) with full viscosity (i.e., $\nu > 0$, $\kappa > 0$) has been well studied. In [13], Cannon & DiBenedetto studied the Cauchy problem for the 2D Boussinesq equations with full viscosity. They found a unique, global in time, weak solution. Furthermore, they improved the regularity of the solution when initial data is smooth. Recently, the result of global existence of smooth solutions to (1.3.3) is generalized to the cases of "partial viscosity" (i.e., either $\nu > 0$, $\kappa = 0$, or $\nu = 0$, $\kappa > 0$) by Hou-Li [41] and Chae [15] independently. In [41], Hou & Li proved the global well-posedness of the Cauchy problem for the viscous Boussinesq equations (i.e., $\nu > 0$, $\kappa = 0$). They showed that solutions with initial data in H^m ($m \geq 3$) do not develop finite-time singularities. In [15], Chae considered the Boussinesq system for incompressible fluid in $\mathbb{R}^2$ with either zero diffusion ($\kappa = 0$) or zero viscosity($\nu = 0$). He proved global-in-time regularity in both cases. The key approach used in the proof of the Cauchy problem is to combine the vortex formulation of the equations with an inequality of logarithmic growth rate which takes the following form:

$$\|f\|_{L^\infty}^2 \leq C_0\big(\|\nabla f\|_{L^2} + \|f\|_{L^2} + 1\big) \log\big(\|\Delta f\|_{L^2} + \|f\|_{L^2} + e\big).$$

By combining this inequality with Gronwall's inequality the authors in [41, 15] get the estimate of the maximum gradient of the velocity which leads to the global regularity of the solution.

On the other hand, the global regularity/singularity question for the case of (1.3.3) with zero viscosity and zero diffusion (i.e., $\nu = \kappa = 0$) still remains as an outstanding open problem in mathematical fluid mechanics due to nonlinear convection and lack of dissipation. Previous investigations on this subject are primarily concerned with numerical simulations. In [32], E & Shu systematically studied the nonlinear development of potential singularities in the 2D Boussinesq equations with smooth initial data and they found no evidence for singular solutions in their numerical solutions. Although there is still no rigorous mathematical proof, this work of E & Shu provides convincing evidence that 3D swirling flows do not become singular in finite time. We refer the readers to [18], [19], [23], [95] for more studies in this direction.

When the problem is set on bounded domains, the case of $\nu > 0$, $\kappa > 0$ has been analyzed in great extent (see e.g. [67] and references therein). Recently, the local existence and blow-up criterion of smooth solutions for the inviscid case ($\nu = \kappa = 0$) is established in [52], see also [16].

However, as mentioned in the Introduction, due to the lack of information of spatial derivatives of the solution on the boundary of the domain, the key approach used for the Cauchy problem does not apply to the initial-boundary value problem. Furthermore, when either $\kappa = 0$ or $\nu = 0$, the problem distinguishes itself significantly from the one with full viscosity. The reasons are as follows: When $\kappa = 0$, the Boussinesq system (1.3.3) turns out to be

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = \nu \Delta U + \theta \mathbf{e}_2, \\ \theta_t + U \cdot \nabla \theta = 0, \\ \nabla \cdot U = 0, \end{cases} \tag{1.3.4}$$

where the temperature equation is a pure transport equation. To establish the global regularity of the temperature, one has to gain the smoothness of the particle path in the first place, in other words, one has to achieve the smoothness of the velocity field before obtaining any regularity of the temperature. But, this is far from easy mainly due to nonlinear convection and coupling between the equations for the velocity and the temperature and gravitational force.

In the direction of $\nu = 0$, the situation is more complicated comparing with (1.3.4). In this case, (1.3.3) becomes

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = \theta \mathbf{e}_2, \\ \theta_t + U \cdot \nabla \theta = \kappa \Delta \theta, \\ \nabla \cdot U = 0, \end{cases} \tag{1.3.5}$$

in which the velocity equation becomes the 2D non-homogeneous Euler equations. From standard results [57] we know that the regularity of the velocity can be built up only after the C^1 estimate of the non-homogeneous term ($\theta \mathbf{e}_2$ in this case) is achieved. Again, it is highly non-trivial to establish the C^1 estimate of the temperature due to nonlinear convection and coupling between the equations.

Therefore, the questions of global regularity/finite time singularity for the initial-boundary value problems for the 2D Boussinesq equations with partial viscosity still remain as important open problems. We will give definite results to these problems in this thesis.

1.3.2 Global Existence of Buoyancy Driven Flow

We study the 2D Boussinesq equations with partial viscosity on bounded domains.

1.3.2.1 Case $\nu > 0, \kappa = 0$

In this case, we consider the following IBVP

$$\begin{cases} (1.3.4), \\ (U, \theta)(\mathbf{x}, 0) = (U_0, \theta_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ U|_{\partial\Omega} = 0, \end{cases} \tag{1.3.6}$$

where the boundary condition is the so-called *no-slip boundary condition.*

In this thesis, we will generalize the study of [15] and [41] to bounded domains with typical physical boundary condition $(1.3.6)_3$. For global existence of smooth solutions, we require the following compatibility conditions

$$\begin{cases} \nabla \cdot U_0 = 0, \quad U_0|_{\partial\Omega} = 0, \\ \nu \Delta U_0 + \theta_0 \mathbf{e}_2 - \nabla P_0 = 0, \quad \mathbf{x} \in \partial\Omega, \ t = 0, \end{cases} \tag{1.3.7}$$

where $P_0(\mathbf{x}) = P(\mathbf{x}, 0)$ is the solution to the Neumann boundary problem

$$\begin{cases} \Delta P_0 = \nabla \cdot [\theta_0 \mathbf{e}_2 - U_0 \cdot \nabla U_0], \quad \mathbf{x} \in \Omega, \\ \nabla P_0 \cdot \mathbf{n}|_{\partial\Omega} = [\nu \Delta U_0 + \theta_0 \mathbf{e}_2] \cdot \mathbf{n}|_{\partial\Omega}, \end{cases} \tag{1.3.8}$$

with $\mathbf{n}$ the unit outward normal to $\partial\Omega$.

Our main results are stated in the following theorem.

Theorem 1.3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. If $(\theta_0(\mathbf{x}), U_0(\mathbf{x})) \in H^3(\Omega)$ satisfies the compatibility conditions (1.3.7)–(1.3.8), then there exists a unique solution (θ, U) of (1.3.6) globally in time such that $\theta(\mathbf{x}, t) \in C([0,T]; H^3(\Omega))$ and $U(\mathbf{x}, t) \in C([0,T]; H^3(\Omega)) \cap L^2([0,T]; H^4(\Omega))$ for any $T > 0$. Moreover, there exists a constant $\bar{C} > 0$ independent of t such that*

$$\|U(\cdot, t)\|_{L^2}^2 \le \max\left\{ \|U(\cdot, 0)\|_{L^2}^2 \ , \ \frac{\bar{C}^2}{\nu^2} \|\theta(\cdot, 0)\|_{L^2}^2 \right\}, \quad \forall\, t \ge 0. \tag{1.3.9}$$

1.3.2.2 $\nu = 0, \kappa > 0$

In this case, the IBVP becomes

$$\begin{cases} (1.3.5), \\ (U, \theta)(\mathbf{x}, 0) = (U_0, \theta_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = \bar{\theta}, \end{cases} \tag{1.3.10}$$

where $\bar{\theta}$ is a constant.

Due to the dissipation in the temperature equation and boundary effects, the temperature is expected to converge to its boundary value. This suggests that the equilibrium state of the temperature should be $\bar{\theta}$. In this thesis, we will prove that there exists a unique global smooth solution to (1.3.10) for smooth initial data. Moreover, we will show that the temperature converges exponentially to its boundary value as time goes to infinity, and the velocity is uniformly bounded in time.

For the global existence of smooth solutions, the following compatibility conditions are required:

$$\begin{aligned} &U_0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \nabla \cdot U_0 = 0, \\ &\theta_0|_{\partial\Omega} = \bar{\theta}, \quad U_0 \cdot \nabla\theta_0 - \kappa\Delta\theta_0|_{\partial\Omega} = 0. \end{aligned} \tag{1.3.11}$$

Our main result is stated in the following theorem.

Theorem 1.3.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. If $(U_0(\mathbf{x}), \theta_0(\mathbf{x})) \in H^3(\Omega)$ satisfies the compatibility conditions (1.3.11), then there exists a unique solution (U, θ) of (1.3.10) globally in time such that $U \in C([0,T]; H^3(\Omega))$ and $\theta \in C([0,T]; H^3(\Omega)) \cap L^2([0,T]; H^4(\Omega))$ for any $T > 0$. Moreover, there exist constants $\eta > 0, \bar{C} > 0, C(p) > 0, \tilde{C} > 0$, which are independent of t such that for any fixed $p \in [2, \infty)$,*

$$\|(\theta - \bar{\theta})(\cdot, t)\|_{H^3} \leq \bar{C} \exp\{-\eta t\}, \quad \forall\ t \geq 0; \tag{1.3.12}$$

$$\|U(\cdot, t)\|_{W^{1,p}} \leq C(p), \quad \|\omega(\cdot, t)\|_{L^\infty} \leq \tilde{C}, \quad \forall\ t \geq 0, \tag{1.3.13}$$

where $\omega = v_x - u_y$ is the 2D vorticity

Remark 1.3.1. *The results obtained in Theorems 1.3.1–1.3.2 suggest that either the viscous dissipation or the thermal diffusion is strong enough to compensate the effects of gravitational force and nonlinear convection in order to prevent the development of singularity of the system. It should be pointed out that, in the theorems obtained above, no smallness restriction is put upon the initial data which is significantly different from Theorem 1.2.3.*

1.4 Multi-phase/Mixing Flows

1.4.1 Background

This part of the thesis is devoted to the mathematical analysis of multi-phase flows and mixing flows which are generalizations of studies of 2D Boussinesq equations. We first study the system of partial differential equations obtained by coupling the Cahn-Hilliard equation to the 2D Boussinesq equations, which stands for a model of a two-phase flow under shear and the influence of gravitational force. Then, a mathematical model of a two-component mixture with a diffusive mass exchange among the medium particles of various density accounted for (c.f. [5]) is investigated.

1.4.1.1 Multi-Phase Flow

In fluid dynamics, two-phase flow occurs in a system containing, for example, gas and liquid with a meniscus separating the two phases. Two-phase flow has been commonly-studied in areas such as large-scale power systems, pump cavitation, climate systems and groundwater flow. Several features make two-phase flow an interesting and challenging branch of fluid dynamics such as surface tension, significant density difference and dramatic change in sound speed which introduces compressible effects into the problem.

In the field of mathematical analysis of two-phase flow, the Cahn-Hilliard equation

$$\phi_t = \Delta\big(F'(\phi) - \alpha\Delta\phi\big)$$

is commonly used, which describes the process of phase separation, by which the two components of a binary fluid spontaneously separate and form domains pure in each component. We refer the readers to [3, 10, 11, 12, 7, 28, 33, 76, 101, 102, 103] for studies on the Cahn-Hilliard equation, and the coupling of Cahn-Hilliard equation and Navier-Stokes equations.

In this thesis, we consider the coupling of the Cahn-Hilliard and the 2D Boussinesq equations:

$$\begin{cases} \phi_t + U \cdot \nabla \phi = \Delta \mu, \\ \mu = -\alpha \Delta \phi + F'(\phi), \\ U_t + U \cdot \nabla U + \nabla P = \nu \Delta U + \mu \nabla \phi + \theta \overrightarrow{\mathbf{e}_2}, \\ \theta_t + U \cdot \nabla \theta = 0, \\ \nabla \cdot U = 0, \end{cases} \tag{1.4.1}$$

where ϕ is the order parameter and μ is a chemical potential derived from a coarse-grained study of the free energy of the fluid (c.f. [37]). System (1.4.1) stands for a model of a two-phase flow under shear and the influence of gravitational force.

We are interested in the global existence and large-time behavior of smooth solutions to the initial-boundary value problem of (1.4.1). As mentioned in the preceding section that the main difficulty encountered in the analysis of the Boussinesq equations is the regularity of the velocity field priori to the regularity of the temperature. Therefore, when the Cahn-Hilliard equation is coupled to the Boussinesq equations, the complexity of the problem will significantly increase. More detailed analysis, comparing with the IBVP for Boussinesq equations, is required in order to answer the question of global existence and large time behavior of smooth solutions to (1.4.1). The study of this part of the thesis is a generalization of the result obtained in [10] in the sense that we take additionally the effect of gravitational force into consideration.

1.4.1.2 Mixing Flow

The last part of this thesis is concerned with the study of the following system of equations:

$$\begin{cases} (\rho U)_t + \nabla \cdot (\rho U \otimes U) + \nabla P = \nabla\big(\nabla \cdot (\lambda \rho U)\big) + \rho \vec{f} + \\ \qquad \nabla \cdot \big(\mu \nabla U - \lambda \rho [(\nabla U) + (\nabla U)^{\mathrm{T}}] + \nabla(\lambda \rho U)\big), \\ \rho_t + \nabla \cdot (\rho U) = \lambda \Delta \rho, \\ \nabla \cdot U = 0, \end{cases} \tag{MF}$$

which describes the motion of a two-component mixture, with a diffusive mass exchange among the medium particles of various density accounted for (c.f. [5, 56, 58]). Here, U is the mean velocity, ρ is the mixture density, $\mu > 0$ is the viscosity and $\lambda > 0$ is the diffusive coefficient.

For smooth solutions, system (MF) can be simplified by using the density equation as:

$$\begin{cases} \rho(U_t + U \cdot \nabla U) + \nabla P = \lambda\big[\nabla\rho \cdot \nabla U + U \cdot \nabla(\nabla\rho)\big] + \mu\Delta U + \rho\vec{f}, \\ \rho_t + U \cdot \nabla\rho = \lambda\Delta\rho, \\ \nabla \cdot U = 0, \end{cases} \tag{1.4.2}$$

System (1.4.2) is immediately transformed to the incompressible density-dependent Navier-Stokes equations by setting $\lambda = 0$:

$$\begin{cases} \rho(U_t + U \cdot \nabla U) + \nabla P = \mu\Delta U + \rho\vec{f}, \\ \rho_t + U \cdot \nabla\rho = 0, \\ \nabla \cdot U = 0, \end{cases} \tag{1.4.3}$$

which generalizes the standard incompressible Navier-Stokes equations for a homogeneous fluid to the case of a non-homogeneous fluid. System (1.4.3) is used in applied fields of fluid dynamics such as oceanology and hydrology and has been well-studied; see [5, 61].

Concerning (1.4.2), previous investigations were carried out for weak solutions when the problem is set on a bounded domain $\Omega \subset \mathbb{R}^2$ with physical boundary conditions; see [5]. In this thesis, our goal is to improve the regularity of the global weak solution constructed in [5] via the method of energy estimate.

1.4.2 Summary of Main Results

The results obtained in this chapter are as follows.

1.4.2.1 Multi-phase Flow

For the IBVP of multi-phase flow:

$$\begin{cases} (1.4.1), \\ (\phi, \mu, \theta, U)(\mathbf{x}, 0) = (\phi_0, \mu_0, \theta_0, U_0)(\mathbf{x}); \\ \nabla \phi \cdot \mathbf{n}|_{\partial\Omega} = \nabla \mu \cdot \mathbf{n}|_{\partial\Omega} = 0,\ U|_{\partial\Omega} = 0, \end{cases} \tag{1.4.4}$$

we require the following compatibility conditions in order to study smooth solutions:

$$\begin{cases} \nabla \cdot U_0 = 0,\ \nabla \phi_0 \cdot \mathbf{n}|_{\partial\Omega} = \nabla \mu_0 \cdot \mathbf{n}|_{\partial\Omega} = U_0|_{\partial\Omega} = 0, \\ \nu \Delta U_0 + \mu_0 \nabla \phi_0 + \theta_0 \mathbf{e}_2 - \nabla P_0 = 0, \quad \mathbf{x} \in \partial\Omega, \\ \mu_0 = -\alpha \Delta \phi_0 + F'(\phi_0), \end{cases} \tag{1.4.5}$$

where $P_0(\mathbf{x}) = P(\mathbf{x}, 0)$ is the solution to the Neumann boundary problem

$$\begin{cases} \Delta P_0 = \nabla \cdot [\theta_0 \mathbf{e}_2 + \mu_0 \nabla \phi_0 - U_0 \cdot \nabla U_0], \\ \nabla P_0 \cdot \mathbf{n}|_{\partial\Omega} = [\nu \Delta U_0 + \theta_0 \mathbf{e}_2] \cdot \mathbf{n}|_{\partial\Omega}. \end{cases} \tag{1.4.6}$$

Our main results are stated in the following theorem.

Theorem 1.4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and suppose that $F(\cdot)$ satisfies the following conditions:*

- (H_1) *F is of C^5 class and $F \geq 0$.;*
- (H_2) *There exist constants $C_1, C_2 > 0$ such that $|F^{(n)}(\phi)| \leq C_1 |\phi|^{p-n} + C_2$, $n = 1, \cdots, 5,\ \forall\ 5 \leq p < \infty$ and $\phi \in \mathbb{R}$;*
- (H_3) *There exists a constant $F_1 > 0$ such that $F'' \geq -F_1$.*

If $\phi_0(\mathbf{x}) \in H^4(\Omega)$, $\mu_0(\mathbf{x}) \in H^2(\Omega)$, $(\theta_0(\mathbf{x}), U_0(\mathbf{x})) \in H^3(\Omega)$ satisfy the compatibility conditions (1.4.5)–(1.4.6), then there exists a unique solution (ϕ, μ, θ, U) of (1.4.4) globally in time such that $\phi(\mathbf{x}, t) \in C([0,T); H^4(\Omega)) \cap L^2([0,T); H^6(\Omega))$, $\mu(\mathbf{x}, t) \in C([0,T); H^2(\Omega)) \cap L^2([0,T); H^4(\Omega))$, $U(\mathbf{x}, t) \in C([0,T); H^3(\Omega)) \cap L^2([0,T); H^4(\Omega))$, and $\theta(\mathbf{x}, t) \in C([0,T); H^3(\Omega))$ for any $T > 0$.

Remark 1.4.1. *The results obtained in Theorem 1.4.1 can be generalized to the case of parameter dependent viscosity. In other words, we can replace* $\Delta\mu$ *by* $\nabla\cdot(\beta(\phi)\nabla\mu)$ *and* $\nu\Delta U$ *by* $\nabla\cdot(\nu(\phi)\nabla U)$ *respectively. In this case, the modeling equations describe more realistic phenomena comparing with (1.4.1). By imposing appropriate conditions on* $\beta(\phi)$ *and* $\nu(\phi)$ *we can study the global existence of smooth solutions to the more complicated system. However, the proof will be in the same spirit of that for Theorem 1.4.1. Therefore, to illustrate the main ideas, we only present the simple case in this thesis. In the theorem obtained above, no smallness assumption is put upon the initial data.*

1.4.2.2 Mixing Flow

In the direction of mixing flow, we consider the IBVP

$$\begin{cases} (1.4.2), \\ (U,\rho)(\mathbf{x},0) = (U_0,\rho_0)(\mathbf{x}), \quad m \le \rho_0(\mathbf{x}) \le M; \\ U|_{\partial\Omega} = 0, \quad \nabla\rho\cdot\mathbf{n}|_{\partial\Omega} = 0. \end{cases} \tag{1.4.7}$$

In order to build up the regularity constructed in [5], we need the following compatibility conditions:

$$\begin{cases} \nabla\cdot U_0 = 0,\ U_0|_{\partial\Omega} = 0, \quad \nabla\rho_0\cdot\mathbf{n}|_{\partial\Omega} = 0, \\ \lambda\nabla\rho_0\cdot\nabla U_0 + \mu\Delta U_0 + \vec{f}_0\rho_0 - \nabla P_0 = 0, \quad \mathbf{x}\in\partial\Omega, \end{cases} \tag{1.4.8}$$

where $P_0(\mathbf{x})$ is the solution to the Neumann boundary problem

$$\begin{cases} \nabla\cdot\left(\dfrac{\nabla P_0}{\rho_0}\right) = \nabla\cdot\left(\dfrac{\lambda}{\rho_0}\big(\nabla\rho_0\cdot\nabla U_0 + U_0\cdot\nabla(\nabla\rho_0)\big) - U_0\cdot\nabla U_0 + \dfrac{\mu}{\rho_0}\Delta U_0 + \vec{f}_0\right), \\ \nabla P_0\cdot\mathbf{n}|_{\partial\Omega} = [\lambda\nabla\rho_0\cdot\nabla U_0 + \mu\Delta U_0 + \vec{f}_0\rho_0]\cdot\mathbf{n}|_{\partial\Omega}. \end{cases} \tag{1.4.9}$$

Our main results are stated in the following theorem.

Theorem 1.4.2. *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded domain with smooth boundary and suppose that the constant* $\mu_1 = 2\mu - \lambda(M-m) > 0$. *If* $(\rho_0(\mathbf{x}), U_0(\mathbf{x}))\in H^3(\Omega)$ *satisfy the* *compatibility conditions (1.4.8)–(1.4.9), then there exists a unique solution* (ρ, U) *of (1.4.7) globally in time such that* $(\rho, U)(\mathbf{x}, t) \in C([0, T]; H^3(\Omega)) \cap L^2([0, T]; H^4(\Omega))$ *for any* $T > 0$.

Remark 1.4.2. *The condition* $2\mu - \lambda(M - m) > 0$ *can be roughly seen through the stress tensor in the momentum equation in* (MF)*, where competition between viscous dissipation and mass exchange happens. Therefore, the rate of mass exchange must not exceed a threshold in order to guarantee the exsitence of global solutions. Still, in the theorem obtained above, no smallness assumption is put upon the initial data.*

Concluding Remark. *Theorems 1.2.1–1.2.2 are taken from [85]. Theorems 1.3.1–1.3.2 come from [86] and the results on Boussinesq system can be found in [60] and [106] respectively.*

CHAPTER II

1D COMPRESSIBLE EULER EQUATIONS WITH DAMPING

2.1 Introduction

In this chapter, we consider the one-dimensional compressible Euler equations with frictional damping:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\alpha \rho u. \end{cases} \tag{2.1.1}$$

Such a system occurs in the mathematical modeling of compressible flow through a porous medium. Here ρ, u, and P denote the density, velocity and pressure respectively; the constant $\alpha > 0$ models friction. Assuming the flow is a polytropic perfect gas, then $P(\rho) = P_0\rho^\gamma$, with P_0 a positive constant, and $\gamma > 1$ the adiabatic gas exponent. Without loss of generality, we take $P_0 = \frac{1}{\gamma}$, $\alpha = 1$ throughout this chapter.

After introducing the momentum $m = \rho u$, we can rewrite (2.1.1) as follows:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \Big(\dfrac{m^2}{\rho} + P(\rho)\Big)_x = -m. \end{cases} \tag{2.1.2}$$

System (2.1.2) is supplemented by the following initial value and boundary conditions:

$$\begin{cases} \rho(x,0) = \rho_0(x),\ m(x,0) = m_0(x), \quad 0 < x < 1, \\ m|_{x=0} = 0, \quad m|_{x=1} = 0, \quad t \geq 0, \\ \displaystyle\int_0^1 \rho_0(x)\ dx = \rho_* > 0. \end{cases} \tag{2.1.3}$$

Where, the last condition is imposed to avoid the trivial case, $\rho \equiv 0$.

For large time, it is conjectured that Darcy's law is valid and (2.1.2) is well approximated by the decoupled system

$$\begin{cases} \tilde{\rho}_t = P(\tilde{\rho})_{xx}, \\ \tilde{m} = -P(\tilde{\rho})_x. \end{cases} \tag{2.1.4}$$

Where, the first equation is the well-known porous medium equation while the second equation states Darcy's law. The initial boundary conditions turn into

$$\begin{cases} \tilde{\rho}(x,0) = \tilde{\rho}_0(x), \quad 0 < x < 1, \\ P_x|_{x=0} = 0, \quad P_x|_{x=1} = 0, \quad t \geq 0. \end{cases} \tag{2.1.5}$$

When the initial data is small smooth and is away from vacuum, the global existence and large time behavior of the solutions to (2.1.2)–(2.1.3) were established in [47] and [48]. However, when initial data is large or rough, shock will develop in finite time [107], and one has to consider weak entropy solutions. One of the main difficulties is that the weak solution may contain the vacuum state, where the system (2.1.2) experiences resonance since two family of characteristics coincide, [64], [65] and [66]. In this chapter, we will first construct L^∞ weak entropy solutions to (2.1.2)–(2.1.3) for physical initial data, and then prove that any L^∞ entropy weak solution of (2.1.2)–(2.1.3) converges exponentially to equilibrium state. We then prove that the solutions of the related diffusion problem (2.1.4)–(2.1.5) tend to the same equilibrium state exponentially fast in time provided that

$$\int_0^1 \tilde{\rho}_0(x)dx = \int_0^1 \rho_0(x)dx. \tag{2.1.6}$$

We thus justified the validity of Darcy's law in large time.

Notation 2.1.1. *Unless specified, throughout this chapter, C and C_i will denote generic constants which are independent of ρ, m and t.*

2.2 Preliminaries and Main Results

We first introduce some basic facts about system (2.1.2) and the homogeneous compressible Euler equations. For more details, see [26] and [94]. It is convenient to use vector form of the system. Set

$$v = (\rho, m)^T,\ f(v) = \Big(m, \frac{m^2}{\rho} + \frac{\rho^\gamma}{\gamma}\Big)^T,\ g(v) = (0, -m)^T, \tag{2.2.1}$$

we rewrite (2.1.2)–(2.1.3) as

$$\begin{cases} v_t + f(v)_x = g(v), \\ v(x,0) = v_0(x),\ x \in (0,1), \\ m(0,t) = m(1,t) = 0. \end{cases} \tag{2.2.2}$$

Clearly, the Jacobian matrix of flux f is

$$\nabla f = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + \rho^{\gamma-1} & \frac{2m}{\rho} \end{pmatrix}, \tag{2.2.3}$$

which has eigenvalues

$$\lambda_1 = \frac{m}{\rho} - \rho^\theta,\quad \lambda_2 = \frac{m}{\rho} + \rho^\theta, \tag{2.2.4}$$

and the so-called *Riemann invariants* are

$$w = \frac{m}{\rho} + \frac{\rho^\theta}{\theta},\quad z = \frac{m}{\rho} - \frac{\rho^\theta}{\theta}, \tag{2.2.5}$$

where $\theta = \frac{\gamma-1}{2}$.

We now give the definition of weak solution to (2.1.2)–(2.1.3).

Definition 2.2.1. *For every $T > 0$, we define a weak solution of (2.1.2)–(2.1.3) to be a pair of bounded measurable functions $v(x,t) = \big(\rho(x,t), m(x,t)\big)$ satisfying the following pair of integral identities:*

$$\int_0^T\int_0^1 (\rho\psi_t + m\psi_x)\ dx\ dt + \int_{t=0} \rho_0\psi\ dx = 0, \tag{2.2.6}$$

$$\int_0^T\int_0^1 \Big(m\psi_t + \Big(\frac{m^2}{\rho} + P(\rho)\Big)\psi_x\Big)\ dx\ dt - \int_0^T\int_0^1 m\psi\ dx\ dt + \int_{t=0} m_0\psi\ dx = 0, \tag{2.2.7}$$

for all $\psi \in C_0^\infty(I_T)$ satisfying $\psi(x,T) = 0$ for $0 \le x \le 1$ and $\psi(0,t) = \psi(1,t) = 0$ for $t \ge 0$, where $I_T = (0,1) \times (0,T)$, and $\frac{m}{\rho}$ vanishes when $\rho = 0$. Moreover, (ρ, m) satisfy the initial boundary conditions (2.1.3) in the sense of trace, defined in (2.4.8)–(2.4.9) below.

An interesting feature of nonlinear hyperbolic balance laws is that when weak solution is concerned, the uniqueness is lost. In order to select the physical relevant solutions, one often imposes entropy admissible conditions. We now define the entropy and entropy flux pairs.

Definition 2.2.2. *A pair of mappings $\eta: \mathbb{R}^2 \to \mathbb{R}$ and $q: \mathbb{R}^2 \to \mathbb{R}$ is called an entropy-entropy flux pair if it satisfies the following equation*

$$\nabla q = \nabla \eta \nabla f.$$

Let $\tilde{\eta}(\rho, m/\rho) = \eta(\rho, m)$. If $\tilde{\eta}(0, u) = 0$, then η is called a weak entropy.

Among all entropies, the most natural entropy is the mechanical energy

$$\eta_e(\rho, m) = \frac{m^2}{2\rho} + \frac{\rho^\gamma}{\gamma(\gamma-1)}, \tag{2.2.8}$$

which plays a very important role in estimates for entropy dissipation measures. It is easy to check that η_e is a weak and convex entropy.

Definition 2.2.3. *The weak solution $v(x,t) = \big(\rho(x,t), m(x,t)\big)$ defined in Definition 2.2.1 is said to be entropy admissible if for any convex entropy η and associated entropy flux q, the following entropy inequality holds*

$$\eta_t + q_x + \eta_m m \leq 0, \tag{2.2.9}$$

in the sense of distribution.

Typically, in order to construct approximate solutions to non-homogeneous hyperbolic systems, fractional step scheme (operator splitting) is applied. In each time step, one first solves the associated homogeneous system, then apply the ODE correction ignoring fluxes. In this chapter, we will use many results of the homogeneous compressible Euler equations:

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \left(\dfrac{m^2}{\rho} + P(\rho)\right)_x = 0, \end{cases}$$

or equivalently,

$$v_t + f(v)_x = 0. \tag{2.2.10}$$

One of the building blocks is the Riemann problem

$$\begin{cases} (2.2.10), \quad t > 0, \quad x \in \mathbb{R}, \\ (\rho, m)|_{t=0} = \begin{cases} (\rho_l, m_l), \ x < 0, \\ (\rho_r, m_r), \ x > 0, \end{cases} \end{cases} \tag{2.2.11}$$

where $\rho_l,\ \rho_r,\ m_l$, and m_r are constants satisfying $0 \leq \rho_l,\ \rho_r,\ |m_l/\rho_l|,\ |m_r/\rho_r| < \infty$. There are two distinct types of rarefaction waves and shock waves, called *elementary waves* , which are labeled 1-rarefaction or 2-rarefaction waves and 1-shock or 2-shock waves, respectively.

Lemma 2.2.1. *There exists a global weak entropy solution of (2.2.11) which is piece-wise smooth function satisfying*

$$\begin{aligned} &w(x,t) = w(\frac{x}{t}) \leq \max\{w(\rho_l, m_l), w(\rho_r, m_r)\}, \\ &z(x,t) = z(\frac{x}{t}) \geq \min\{z(\rho_l, m_l), z(\rho_r, m_r)\}, \\ &w(x,t) - z(x,t) \geq 0. \end{aligned}$$

It follows that the region $\Lambda = \{(\rho, m) : \ w \leq w_0,\ z \geq z_0,\ w - z \geq 0\}$ is an *invariant region* for the Riemann problem (2.2.11). More precisely, if the Riemann data lies in Λ, then the solution of (2.2.11) lies in Λ, too.

Lemma 2.2.2. *If* $\{(\rho, m) : \ a \leq x \leq b\} \subset \Lambda$, *then*

$$\left(\frac{1}{b-a} \int_a^b \rho \ dx,\ \frac{1}{b-a} \int_a^b m \ dx \right) \in \Lambda. \tag{2.2.12}$$

Concerning the IBVP, the boundary Riemann solver is applied.

Lemma 2.2.3. *For the mixed problem*

$$\begin{cases} (2.2.10),\ t > 0, \quad x > 0, \\ (\rho, m)|_{t=0} = (\rho_0, m_0),\ x > 0, \\ m|_{x=0} = 0,\ t \geq 0, \end{cases} \tag{2.2.13}$$

where (ρ_0, m_0) *are constants, there exists a weak entropy solution in the region* $\{(x,t): x \geq 0,\ t \geq 0\}$ *satisfying the following estimates*

$$w(x,t) \leq \max\{w(\rho_0,m_0), -z(\rho_0,m_0)\},$$
$$z(x,t) \geq z(\rho_0,m_0), \quad \text{and} \quad w(x,t) - z(x,t) \geq 0.$$

The term $-z(\rho_0, m_0)$ is new to the mixed problem because of the shock waves reflecting off or coming out at the boundary $x = 0$. Similar to (2.2.13), we can solve the following mixed problem in the region $\{(x,t):\ x \leq 1,\ t \geq 0\}$:

$$\begin{cases} (2.2.10),\ t > 0, \quad x < 1, \\ (\rho, m)|_{t=0} = (\rho_0, m_0),\ x < 1, \\ m|_{x=1} = 0,\ t \geq 0, \end{cases} \tag{2.2.14}$$

The weak entropy solution of (2.2.14) satisfies the following estimates:

$$z(x,t) \geq \min\{z(\rho_0,m_0), -w(\rho_0,m_0)\},$$
$$w(x,t) \leq w(\rho_0,m_0), \quad \text{and} \quad w(x,t) - z(x,t) \geq 0.$$

Lemma 2.2.4. *Suppose that* $(\rho(x,t), m(x,t))$ *is a solution of (2.2.11) or (2.2.13) and or (2.2.14). Then, the jump strength of* $m(x,t)$ *across an elementary wave can be dominated by that of* $(\rho(x,t))$ *across the same elementary wave, i.e.,*

$$\text{across a shock wave}: \qquad |m_r - m_l| \leq C|\rho_r - \rho_l|,$$
$$\text{across a rarefaction wave}: \quad |m - m_l| \leq C|\rho - \rho_l| \leq C|\rho_r - \rho_l|,$$

where C *depends only on the bounds of* ρ *and* $|m|$.

Lemma 2.2.5. *For any* $\varepsilon > 0$, *there exist constants* $h > 0$ *and* $k > 0$ *such that the solution of (2.2.11) in the region* $\{(x,t):\ |x| < h,\ 0 \leq t < k\}$ *satisfies*

$$\int_{-h}^{h} |\rho(x,t) - \rho(x,0)|\, dx \ \leq\ Ch\varepsilon, \quad 0 \leq t \leq k, \tag{2.2.15}$$

where C *depends only on the bounds of* ρ *and* $|m|$, *and the mesh lengths* h *and* k *satisfy* $\max_{i=1,2}\ \sup|\lambda_i(\rho,m)| < \frac{h}{2k}$.

The following two theorems are the main results of this chapter.

Theorem 2.2.1. *Suppose that the initial data* (ρ_0, m_0) *satisfy the conditions*

$$0 \le \rho_0(x) \le M_1, \quad \rho_0 \not\equiv 0, \quad |m_0(x)| \le M_2 \rho_0(x),$$

for some positive constants $M_i (i = 1, 2)$. *Then, for* $\gamma > 1$, *the IBVP (2.1.2)–(2.1.3) has a global weak solution* $(\rho(x,t), m(x,t))$, *as defined in Definition 2.2.1, satisfying the following estimates and entropy condition:*

$$\begin{aligned} &0 \le \rho \le C, \quad |m| \le C\rho \text{ a.e. for a constant } C > 0 \text{ independent of } t, \text{ and} \\ &\int_0^T \int_0^1 \left(\eta(\rho,m)\tilde{\psi}_t + q(\rho,m)\tilde{\psi}_x \right) dx\ dt - \int_0^T \int_0^1 \eta(\rho,m) m \tilde{\psi}\ dx\ dt \ge 0, \end{aligned} \tag{2.2.16}$$

for all weak and convex entropy pairs (η, q) *for (2.1.2)–(2.1.3) and for all nonnegative smooth functions* $\tilde{\psi} \in C_0^1(I_T)$.

Theorem 2.2.2. *Suppose* $\int_0^1 \rho_0(x) dx = \rho_*$. *Let* (ρ, m) *be any* L^∞ *entropy weak solution of (2.1.2)–(2.1.3) defined in Definition 2.2.1, satisfying the estimates*

$$0 \le \rho(x,t) \le \Lambda < \infty, \quad |m(x,t)| \le M_1 \rho(x,t),$$

where M_1, Λ *are positive constants and let* $(\tilde{\rho}, \tilde{m})$ *be the weak solution of (2.1.4)–(2.1.5) with mass* ρ_* *and* $\tilde{m} = -P(\tilde{\rho})_x$. *Then, there exist constants* C, $\delta > 0$ *depending on* γ, ρ_*, Λ, *and initial data such that*

$$\left\| (\rho - \tilde{\rho}, m - \tilde{m})(\cdot, t) \right\|^2_{L^2([0,1])} \le C e^{-\delta t}. \tag{2.2.17}$$

The proof of Theorem 2.2.1 is in the spirit of [104] and [88]. We construct the approximate solutions v_h derived by the Godunov scheme [36]. The L^∞ norm of approximate solutions is established. The compensated compactness framework is then applied to the sequence of approximate solutions to obtain a global weak entropy solution. The boundary conditions are verified in the sense of trace.

We then prove the exponential decay rate of the L^2-norm of the difference between solutions of (2.1.2)–(2.1.3) and (2.1.4)–(2.1.5). The proof involves the introduction of an antiderivative through mass conservation law, accurate estimation on the dissipation of entropy and and the application of the theory of divergence-measure fields [21] and Poincaré's inequality. We will see that an easy lemma plays an important role in the control of singularity near vacuum. It should be pointed out that the key approach used for Cauchy problem is to compare the solution of (2.1.2)–(2.1.3) with the similarity solution of (2.1.4)–(2.1.5) via energy estimates. Unfortunately, the exponential decay rate cannot be achieved by this approach, due to the boundary effects. Instead of comparing two solutions directly, we first show that the large time asymptotic state for both solutions is a constant state $(\rho_*, 0)$ and both solutions tend to the constant state exponentially fast. Hence by the triangular inequality we can see that the solution of (2.1.2)–(2.1.3) tends to that of (2.1.4)–(2.1.5) exponentially fast as time goes to infinity.

2.3 Approximate Solutions

The approximate solutions will be constructed by Godunov scheme [36] with operator splitting. We choose the space mesh length $h = \frac{1}{N}$, where N is a positive integer. The time mesh length $k = k(h)$ will be chosen later so that the Courant-Friedrich-Levy condition

$$\max_{i=1,2} \left(\sup |\lambda_i(v)|\right) < \frac{h}{2k} \tag{2.3.1}$$

holds for a given $T > 0$. We partition the interval $[0,1]$ into cells, with the j^{th} cell centered at $x_j = jh,\ j = 1, \cdots, N-1$. Set $x_0 = 0$ and $x_N = 1$. We now use the Godunov scheme to construct a sequence of approximate solutions of (2.2.2). Namely, we solve the Riemann problems (2.2.11) in the region $R_j^1 \equiv \big\{(x,t) :\ x_{j-\frac{1}{2}} \le x < x_{j+\frac{1}{2}},\ 0 \le t < k\big\}$:

$$\begin{cases} \dfrac{\partial}{\partial t}\, \underline{v}_h \ + \ \dfrac{\partial}{\partial x}\, f(\underline{v}_h) \ = \ 0, \\ \underline{v}_h|_{t=0} = \begin{cases} (\rho_j^0, m_j^0), & x < x_j, \\ (\rho_{j+1}^0, m_{j+1}^0), & x > x_j, \quad j = 1, \cdots, N-1, \end{cases} \end{cases}$$

where

$$\rho_j^0 = \frac{1}{h} \int_{x_{j-1}}^{x_j} \rho_0(x)\ dx, \qquad m_j^0 = \frac{1}{h} \int_{x_{j-1}}^{x_j} m_0(x)\ dx, \quad \text{for } j = 1, \cdots,\ N.$$

We also solve the mixed problems (2.2.13) and (2.2.14) with (ρ_1^0, m_1^0) and (ρ_N^0, m_N^0), in regions $\big\{(x,t):\ 0 \le x < x_{\frac{1}{2}},\ 0 \le t < k\big\}$ and $\big\{(x,t):\ x_{N-\frac{1}{2}} \le x < 1,\ 0 \le t < k\big\}$, respectively. Then we set

$$v_h(x,t) = \underline{v}_h(x,t)\ +\ V(\underline{v}_h(x,t))t, \quad 0 \le x \le 1, \quad 0 \le t < k, \tag{2.3.2}$$

where $V(v) = \big(V_1(v), V_2(v)\big) \equiv (0, -m)$, and

$$v_j^1 = \frac{1}{h} \int_{x_{j-1}}^{x_j} v_h(x, t_1 - 0)\ dx, \quad j = 1, \cdots, N. \tag{2.3.3}$$

Suppose that we have defined approximate solutions $v_h(x,t)$ for $0 \le t < t_i$. We then define

$$v_h(x,t) = \underline{v}_h(x,t) + V(\underline{v}_h(x,t))(t - t_i), \quad t_i \le t < t_{i+1}, \tag{2.3.4}$$

where $\underline{v}_h(x,t)$ are piecewise smooth functions defined as solutions of the Riemann problems in the region $R_j^i\big\{(x,t):\ x_{j-\frac{1}{2}} \le x < x_{j+\frac{1}{2}},\ t_i \le t < t_{i+1}\big\}$

$$\begin{cases} (2.2.10), \\ \underline{v}_h(x,t)|_{t=t_i} = \begin{cases} v_j^i, & x < x_j, \\ v_{j+1}^i, & x > x_j, \quad j = 1, \cdots, N-1, \end{cases} \end{cases} \tag{2.3.5}$$

and as solutions of mixed problems in the two regions R_0^i and R_N^i:

$$R_0^i = \{(x,t):\ 0 \le x < x_{\frac{1}{2}},\ t_i \le t < t_{i+1}\},$$
$$\begin{cases} (2.2.10), & x > 0,\ t > t_i, \\ \underline{v}_h(x,t)|_{t=t_i} = v_1^i, & x > 0, \\ \underline{m}_h|_{x=0} = 0. \end{cases}$$
$$R_N^i = \{(x,t):\ x_{N-\frac{1}{2}} \le x < 1,\ t_i \le t < t_{i+1}\}, \tag{2.3.6}$$
$$\begin{cases} (2.2.10), & x < 1,\ t > t_i, \\ \underline{v}_h(x,t)|_{t=t_i} = v_1^i, & x < 1, \\ \underline{m}_h|_{x=1} = 0. \end{cases}$$

Next, we set

$$v_j^{i+1} = \frac{1}{h}\int_{x_{j-1}}^{x_j} v_h(x, t_{i+1} - 0)dx, \quad 1 \le j \le N. \tag{2.3.7}$$

Therefore, inductively, the approximate solutions $v_h = (\rho_h, m_h) \equiv (\underline{\rho}_h, m_h)$ are well-defined, since $\underline{\rho}_h \ge 0$. We summarize the above process as follows:

$$v^{i+1} = A_h \circ R \circ E_k(\cdot, v^i), \tag{2.3.8}$$

where A_h is the cell-averaging operator (2.3.7), $E_k(x, v^i)$ is the Riemann solver (3.5) (or boundary Riemann solver (2.3.6)), and R is the reconstruction step (2.3.4).

For $t_i \le t < t_{i+1}$, we set

$$w_h(x,t) = \underline{w}_h(x,t) - \frac{\underline{w}_h(x,t) + \underline{z}_h(x,t)}{2}(t - t_i), \tag{2.3.9}$$

$$z_h(x,t) = \underline{z}_h(x,t) - \frac{\underline{w}_h(x,t) + \underline{z}_h(x,t)}{2}(t - t_i), \tag{2.3.10}$$

where $\underline{w}_h$ and $\underline{z}_h$ are Riemann invariants corresponding to the Riemann solutions $\underline{v}_h$.

With the help of $w_h(x,t)$ and $z_h(x,t)$ defined by (2.3.9) and (2.3.10), we prove the following uniform bound for the approximate solutions.

Theorem 2.3.1. *Suppose that the initial data (ρ_0, m_0) satisfy the following conditions:*

$$0 \le \rho_0(x) \le M_1, \quad \rho_0(x) \not\equiv 0, \quad |m_0(x)| \le M_2 \rho_0(x). \tag{2.3.11}$$

Then, the approximate solutions (ρ_h, m_h) derived by the Godunov scheme are uniformly bounded in the region $\overline{I}_T \equiv \big\{(x,t): \ 0 \le x \le 1, \ 0 \le t \le T\big\}$ for any $T > 0$; that is, there is a constant $C > 0$ independent of t such that

$$0 \le \rho_h(x,t) \le C, \quad |m_h(x,t)| \le C\rho_h(x,t). \tag{2.3.12}$$

Proof. Assume that $0 < k < 1$. For $t_i \le t < t_{i+1}$ ($i \ge 0$ integers), the Riemann invariant properties imply that

$$\begin{aligned} w_h(x,t) &= \underline{w}_h(x,t)\Big(1 - \frac{t-t_i}{2}\Big) - \underline{z}_h(x,t)\frac{t-t_i}{2} \\ &\le \sup_x \underline{w}_h(x,t_i+0)\Big(1 - \frac{t-t_i}{2}\Big) - \inf_x \underline{z}_h(x,t_i+0)\frac{t-t_i}{2}, \\ z_h(x,t) &= \underline{z}_h(x,t)\Big(1 - \frac{t-t_i}{2}\Big) - \frac{\underline{w}_h(x,t)}{2}(t-t_i) \\ &\ge \inf_x \underline{z}_h(x,t_i+0)\Big(1 - \frac{t-t_i}{2}\Big) - \sup_x \underline{w}_h(x,t_i+0)\frac{t-t_i}{2}. \end{aligned}$$

In particular, we obtain

$$\begin{aligned} \sup_x w_h(x,t_{i+1}-0) &\le \sup_x \underline{w}_h(x,t_i+0)\Big(1-\frac{k}{2}\Big) - \inf_x \underline{z}_h(x,t_i+0)\frac{k}{2}, \\ \inf_x z_h(x,t_{i+1}-0) &\ge \inf_x \underline{z}_h(x,t_i+0)\Big(1-\frac{k}{2}\Big) - \sup_x \underline{w}_h(x,t_i+0)\frac{k}{2}. \end{aligned}$$

Let $\alpha_i = \max\big\{\sup_x \underline{w}_h(x,t_i+0), \ -\inf_x \underline{z}_h(x,t_i+0)\big\}$. Then

$$\max\Big\{\sup_x w_h(x,t_{i+1}-0), \ -\inf_x z_h(x,t_{i+1}-0)\Big\} \le \alpha_i. \tag{2.3.13}$$

By (2.3.7) we know that

$$\begin{aligned} \sup_x w_h(x,t_{i+1}+0) &\le \sup_x w_h(x,t_{i+1}-0), \\ \inf_x z_h(x,t_{i+1}+0) &\ge \inf_x z_h(x,t_{i+1}-0). \end{aligned} \tag{2.3.14}$$

Therefore

$$\alpha_{i+1} \le \alpha_i, \quad \text{and} \quad \alpha_i \le \alpha_0, \quad 0 \le i \le n, \tag{2.3.15}$$

where $\alpha_0 = \max\Big\{\sup_x w_0(x),\ -\inf_x z_0(x)\Big\}$. Then, from (2.3.15) and Lemma 2.2.1 and Lemma 2.2.3 we have

$$w_h(x,t) \le \alpha_0, \quad z_h(x,t) \ge -\alpha_0, \quad \text{and}$$
$$w_h(x,t) - z_h(x,t) \ge 0.$$

Then there is a constant $C > 0$ independent of h, k and t such that

$$0 \le \rho_h(x,t) \le C, \quad |m_h(x,t)| \le C\rho_h(x,t).$$

This completes the proof the Theorem 2.3.1.

Now, we can choose the time mesh length $k = k(h)$. Let

$$\lambda = \max_{i=1,2}\Bigg\{\sup_{0\le\rho\le C,\ |m|\le C\rho} |\lambda_i(\rho,m)|\Bigg\},$$

then we take

$$k = \frac{T}{n}, \quad \text{where} \quad n = \max\Bigg\{\Bigg[\frac{4\lambda T}{h}\Bigg] + 1,\ \Bigg[\frac{T}{2}\Bigg] + 1\Bigg\}. \tag{2.3.16}$$

For this k, both the CFL condition and $0 < k < 1$ hold.

2.4 Global Existence of Weak Solutions

With the uniform L^∞ estimates given in Theorem 2.3.1, and the specific structure of system (2.1.2), now it is standard to apply the compensated compactness framework ([29], [30], [62], [63]) to the approximate solution $\{v_h\}$, to conclude that there exists a convergent subsequence $\{v_{h_j}\}_{j=1}^\infty$ such that $h_j \to 0$ as $j \to \infty$ and

$$\big(\rho_{h_j}(x,t), m_{h_j}(x,t)\big) \to \big(\rho(x,t), m(x,t)\big) \ \text{ a.e.} \tag{2.4.1}$$

Furthermore, such a limit $(\rho, m)(x,t)$ satisfies (2.2.6) and (2.2.7) for any test function $\psi(x,t) \in C_0^\infty(I_T)$ for any $T > 0$. Also, the entropy inequality holds in the sense of distribution. The proof is in the same spirit of [88] and [104], we omit the details here. Clearly, there is a constant $C > 0$ such that

$$0 \le \rho(x,t) \le C, \quad |m(x,t)| \le C\rho(x,t) \ \text{ a.e..} \tag{2.4.2}$$

Now we turn to the initial and boundary conditions of weak solutions. First, we need to determine the traces of weak solutions whose exact meaning will be stated below. Let $v = (\rho, m)$ be a weak solution of (2.1.2) obtained in (2.4.1). We introduce the generalized function $\mathcal{A}: \ C_0^1(\mathbb{R}^2) \longrightarrow \mathbb{R}^2$ as follows: for $\psi \in C_0^1(\mathbb{R}^2)$,

$$\mathcal{A}(\psi) = -\int_0^T \int_0^1 \big(v\psi_t + f(v)\psi_x + g(v)\psi\big)dxdt. \tag{2.4.3}$$

We take smooth $\zeta_0(t), \zeta_T(t), \xi_0(x), \xi_1(x)$ with

$$\begin{aligned} &\zeta_0(0) = 1, \quad \zeta_0(T) = 0; \quad \zeta_T(0) = 0, \quad \zeta_T(T) = 1; \\ &\xi_0(0) = 1, \quad \xi_0(1) = 0; \quad \xi_1(0) = 0, \quad \xi_1(1) = 1. \end{aligned} \tag{2.4.4}$$

For any $\chi(x)$, we define the generalized functions:

$$\begin{aligned} &v^\star(\cdot, 0)(\chi) = \mathcal{A}(\chi \cdot \zeta_0) - \chi(0)\mathcal{A}(\xi_0 \cdot \zeta_0) - \chi(1)\mathcal{A}(\xi_1 \cdot \zeta_0), \\ &v^\star(\cdot, T)(\chi) = -\mathcal{A}(\chi \cdot \zeta_T) + \chi(0)\mathcal{A}(\xi_0 \cdot \zeta_T) + \chi(1)\mathcal{A}(\xi_1 \cdot \zeta_T), \\ &f^\star(v)(0, \cdot)(\chi) = \mathcal{A}(\xi_0 \cdot \chi), \\ &f^\star(v)(1, \cdot)(\chi) = -\mathcal{A}(\xi_1 \cdot \chi), \end{aligned} \tag{2.4.5}$$

where $(\chi \cdot \zeta_0)(x,t) = \chi(x)\zeta_0(t)$ and so on mean the tensor product.

Then we can define the trace of v along the segments $(0,1) \times \{0\}$ and $(0,1) \times \{T\}$, and the trace of $f(v)$ along the segments $\{0\} \times (0,T)$ and $\{1\} \times (0,T)$ respectively as $v^\star(\cdot,0)$, $v^\star(\cdot,T)$, $f^\star(v)(0,\cdot)$ and $f^\star(v)(1,\cdot)$. Similarly, for any $t \in (0,T)$, we can also define $v^\star(\cdot,t)$ as the trace of v along the segment $(0,1) \times \{t\}$. For any $x \in (0,1)$, define $f^\star(v)(x,\cdot)$ as the trace of $f(v)$ along the segment $\{x\} \times (0,T)$.

Similar to [39], we have

Lemma 2.4.1. *Let v satisfy (2.1.2) in distributional sense, then,*

$$\begin{aligned} v^\star(\cdot,0)|_{(0,1)}, \quad v^\star(\cdot,T)|_{(0,1)} \ &\in \ L^\infty_{loc}(0,1); \\ f^\star(v)(0,\cdot)|_{(0,T)}, \quad f^\star(v)(1,\cdot)|_{(0,T)} \ &\in \ L^\infty_{loc}(0,T), \end{aligned} \tag{2.4.6}$$

and for any $\psi \in C_0^1(\mathbb{R}^2)$,

$$
\begin{aligned}
&\int_0^T \int_0^1 \big(v\psi_t + f(v)\psi_x + g(v)\psi\big) dxdt \\
&= \int_0^1 v^\star(x,T)\psi(x,T)dx - \int_0^1 v^\star(x,0)\psi(x,0)dx \\
&\quad + \int_0^T f^\star(v)(1,t)\psi(1,t)dt - \int_0^T f^\star(v)(0,t)\psi(0,t)dt.
\end{aligned}
\tag{2.4.7}
$$

Theorem 2.4.1. *Let* $v_{h_j} = (\rho_{h_j}, m_{h_j})$ *be the convergent sequence of approximate solutions of (2.1.2)–(2.1.3) constructed in Section 3 and* $v = (\rho, m)$ *is the limit function obtained in (2.4.1). Then* $v(x,t)$ *satisfies the initial-boundary conditions:*

$$m^\star(0,t) = m^\star(1,t) = 0, \quad t \in (0,T); \tag{2.4.8}$$

$$v^\star(x,0) = v_0(x), \qquad x \in (0,1). \tag{2.4.9}$$

Proof. From (2.2.6)–(2.2.7), it is easy to see, for any $\psi \in C_0^1(\mathbb{R}^2)$, that

$$\lim_{j\to+\infty} \left[\int_0^T \int_0^1 (v_{h_j}\psi_t + f(v_{h_j})\psi_x + g(v_{h_j})\psi) dxdt + \int_{t=0} v_{h_j}\psi dx - \int_{t=T} v_{h_j}\psi dx \right] = 0, \tag{2.4.10}$$

which implies

$$\int_0^T \int_0^1 (v\psi_t + f(v)\psi_x + g(v)\psi) dxdt + \lim_{j\to+\infty} [\int_{t=0} v_{h_j}\psi dx - \int_{t=T} v_{h_j}\psi dx] = 0. \tag{2.4.11}$$

Therefore, (2.4.7) and (2.4.11) give

$$
\begin{aligned}
&\lim_{j\to+\infty} (\int_{t=T} v_{h_j}\psi dx - \int_{t=0} v_{h_j}\psi dx) \\
&= \int_0^1 v^\star(x,T)\psi(x,T)dx - \int_0^1 v^\star(x,0)\psi(x,0)dx \\
&\quad + \int_0^T f^\star(v)(1,t)\psi(1,t)dt - \int_0^T f^\star(v)(0,t)\psi(0,t)dt.
\end{aligned}
\tag{2.4.12}
$$

The first component of (2.4.12) reads

$$
\begin{aligned}
&\int_0^1 \rho^\star(x,T)\psi(x,T)dx - \int_0^1 \rho^\star(x,0)\psi(x,0)dx + \int_0^T m^\star(1,t)\psi(1,t)dt \\
&\quad - \int_0^T m^\star(0,t)\psi(0,t)dt - (\int_{t=T} \rho\psi dx - \int_{t=0} \rho\psi \ dx) = 0.
\end{aligned}
\tag{2.4.13}
$$

Taking $\psi(x,t) = \zeta(x)\chi(t) \in C_0^1(\mathbb{R}^2)$ with $\zeta, \chi \in C_0^1(\mathbb{R})$, and $\chi(0) = 1$, $\chi(T) = 0$, $\zeta(1) = \zeta(0) = 0$ in (2.4.13), we get

$$\int_0^1 \rho^\star(x,0)\zeta(x)dx = \int_0^1 \rho_0(x)\zeta(x)dx,$$

which implies $\rho^\star(x,0) = \rho_0(x)$ on $(0,1)$.

Taking $\psi(x,t) = \zeta(x)\chi(t) \in C_0^1(\mathbb{R}^2)$ with $\zeta, \chi \in C_0^1(\mathbb{R})$, and $\chi(0) = \chi(T) = 0$, $\zeta(1) = 0$, $\zeta(0) = 1$ in (2.4.13), we get

$$\int_0^T m^\star(0,t)\chi(t)dx = 0.$$

Thus $m^\star(0,t) = 0$ on $(0,T)$. It is similar to show that $m^\star(1,t) = 0$ on $(0,T)$. Using the second component of (2.4.12), it is easy to show $m^\star(x,0) = m_0(x)$ on $(0,1)$. This completes the proof of Theorem 2.4.1.

Collecting all results obtained above, we thus conclude the proof of Theorem 2.3.1. However, we remark that Theorem 2.4.1 might not apply to all weak solutions which satisfy (2.2.6)–(2.2.7). However, in the same spirit, one could show that the weak solutions obtained as vanishing viscosity limit with the same boundary condition (2.1.3) verifies (2.4.8) and (2.4.9). This explains the last line in Definition 2.2.1.

2.5 Large Time Behavior of Weak Solution

Now we investigate the large time asymptotic behavior of any entropy weak solution of (2.1.2)–(2.1.3), including the one obtained in the preceding section.

Theorem 2.5.1. *Let (ρ, m) be any L^∞ entropy weak solution of the initial boundary problem (2.1.2)–(2.1.3), satisfying $\int_0^1 \rho_0(x)dx = \rho_*$ and*

$$0 \le \rho(x,t) \le \Lambda < \infty, \quad |m(x,t)| \le M_1\rho(x,t), \tag{2.5.1}$$

where M_1, Λ are positive constants. Then, there exist constants C, $\delta > 0$ depending on γ, ρ_*, Λ, and initial data such that

$$\big\|\big(\rho - \rho_*,\ m\big)(\cdot,t)\big\|^2_{L^2([0,1])} \le Ce^{-\delta t}. \tag{2.5.2}$$

To prove Theorem 2.5.1, we first give a lemma which will play an important role in controlling the singularity near vacuum.

Lemma 2.5.1. *Let $0 \leq \rho \leq \Lambda < \infty$. There is a positive constant C_1 such that*

$$\big[P(\rho) - P(\rho_*) - P'(\rho_*)(\rho - \rho_*)\big] \leq C_1\big[P(\rho) - P(\rho_*)\big](\rho - \rho_*). \tag{2.5.3}$$

Proof. Consider

$$\Gamma(\rho) = \frac{\gamma}{\rho_*}(P(\rho) - P(\rho_*))(\rho - \rho_*) - \big[P(\rho) - P(\rho_*) - P'(\rho_*)(\rho - \rho_*)\big]. \tag{2.5.4}$$

Clearly, $\Gamma(\rho)$ is continuous for $\rho \geq 0$. Since

$$\Gamma(0) = P(\rho_*) > 0, \tag{2.5.5}$$

there exists $d \in (0, \rho_*)$ such that

$$\Gamma(\rho) > \frac{1}{2}P(\rho_*) > 0, \quad for\ \rho \in [0, d]. \tag{2.5.6}$$

For $\rho > d > 0$, we can see that

$$P'(d)(\rho - \rho_*)^2 \leq \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*), \tag{2.5.7}$$

and

$$P(\rho) - P(\rho_*) - P'(\rho_*)(\rho - \rho_*) \leq \begin{cases} \dfrac{P''(d)}{2}(\rho - \rho_*)^2, & 1 < \gamma \leq 2, \\ \dfrac{P''(\Lambda)}{2}(\rho - \rho_*)^2, & \gamma > 2. \end{cases} \tag{2.5.8}$$

Choosing

$$C_1 = \max\{\frac{\gamma}{\rho_*}, \ \frac{P''(d)}{2P'(d)}, \ \frac{P(\Lambda)}{2P'(d)}\}, \tag{2.5.9}$$

we thus have

$$P(\rho) - P(\rho_*) - P'(\rho_*)(\rho - \rho_*) \leq C_1\big[P(\rho) - P(\rho_*)\big](\rho - \rho_*). \tag{2.5.10}$$

This completes the proof of Lemma 2.5.1.

We then set

$$w = \rho - \rho_*, \quad z = m, \tag{2.5.11}$$

which satisfy

$$\begin{cases} w_t + z_x = 0 \\ z_t + \left(\frac{m^2}{\rho}\right)_x + \big[P(\rho) - P(\rho_*)\big]_x + z = 0, \end{cases} \tag{2.5.12}$$

and

$$\int_0^1 w(x,t)dx = 0. \tag{2.5.13}$$

Define

$$y = -\int_0^x w(\sigma,t)d\sigma. \tag{2.5.14}$$

which implies that

$$y_x = -w = \rho_* - \rho, \quad y_t = z. \tag{2.5.15}$$

Since

$$\int_0^1 \rho(x,t)dx = \int_0^1 \rho_0(x)dx = \rho_*,$$

we have

$$y(0) = y(1) = 0. \tag{2.5.16}$$

Therefore the second equation of (2.5.12) turns into

$$y_{tt} + \left(\frac{m^2}{\rho}\right)_x + \big[P(\rho) - P(\rho_*)\big]_x + y_t = 0. \tag{2.5.17}$$

Multiplying y with (2.5.17) and integrating over $[0,1]$ using the theory of divergence-measure fields [21], we have

$$\frac{d}{dt}\int_0^1 \Big(y_t y + \frac{1}{2}y^2\Big)dx - \int_0^1 y_t^2 dx + \int_0^1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*)dx = \int_0^1 \frac{m^2}{\rho} y_x dx. \tag{2.5.18}$$

Since $\rho, u = m/\rho, m = y_t \in L^\infty[0,1]$, we get

$$\frac{d}{dt}\int_0^1 \Big(y_t y + \frac{1}{2}y^2\Big)dx - \int_0^1 y_t^2 dx + \int_0^1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*)dx = \int_0^1 \frac{\rho_*}{\rho} y_t^2 dx - \int_0^1 y_t^2 dx, \tag{2.5.19}$$

i.e.

$$\frac{d}{dt}\int_0^1 \Big(y_t y + \frac{1}{2}y^2\Big)dx + \int_0^1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*)dx = \int_0^1 y_t^2 \frac{\rho_*}{\rho}dx. \qquad (2.5.20)$$

In order to deal with the nonlinearity, we now use the entropy inequality, rather than the usual energy method. Let

$$\eta_e = \frac{m^2}{2\rho} + \frac{P(\rho)}{\gamma - 1}, \qquad q_e = \frac{m^3}{2\rho^2} + \frac{\rho^{\gamma-1}m}{\gamma - 1}$$

be the mechanical energy and related flux. We define

$$\eta_* = \eta_e - \frac{1}{\gamma - 1}P'(\rho_*)(\rho - \rho_*) - \frac{1}{\gamma - 1}P(\rho_*). \qquad (2.5.21)$$

Thus, by the definition of weak entropy solution, the following entropy inequality holds in the sense of distribution:

$$\eta_{*t} + \frac{1}{\gamma - 1}[P'(\rho_*)(\rho - \rho_*)]_t + q_{ex} + \frac{m^2}{\rho} \le 0. \qquad (2.5.22)$$

Since ρ_* is a constant, we get

$$\eta_{*t} + \frac{P'(\rho_*)}{\gamma - 1}(\rho - \rho_*)_t + q_{ex} + \frac{m^2}{\rho} \le 0. \qquad (2.5.23)$$

By the conservation of mass and theory of divergence-measure fields [21], we have

$$\frac{d}{dt}\int_0^1 \eta_* dx + \int_0^1 \frac{m^2}{\rho}dx \le 0,$$

i.e.,

$$\frac{d}{dt}\int_0^1 \eta_* dx + \int_0^1 \frac{y_t^2}{\rho}dx \le 0. \qquad (2.5.24)$$

Choosing $K = \max\{2, \ \ 2\Lambda + \rho_*\}$, we add (2.5.20) to (2.5.24) $\times\, K$,

$$\frac{d}{dt}\int_0^1 \Big(K\eta_* + yy_t + \frac{1}{2}y^2\Big)dx + \int_0^1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*)dx + \int_0^1 \frac{K - \rho_*}{\rho}y_t^2 dx \le 0, \qquad (2.5.25)$$

Using the expression of η_* we get

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \Big(\frac{K}{2\rho}y_t^2 + yy_t + \frac{1}{2}y^2 + \frac{K}{\gamma - 1}\big[P(\rho) - P(\rho_*) - P'(\rho_*)(\rho - \rho_*)\big]\Big)dx \\ &+ \int_0^1 \big[P(\rho) - P(\rho_*)\big](\rho - \rho_*)dx + \int_0^1 \frac{K - \rho_*}{\rho}y_t^2 dx \le 0. \end{aligned} \qquad (2.5.26)$$

Clearly, Lemma 2.5.1 implies

$$\int_0^1 \frac{K}{\gamma-1}\big[P(\rho)-P(\rho_*)-P'(\rho_*)(\rho-\rho_*)\big]dx \le \frac{C_1K}{\gamma-1}\int_0^1 \big[P(\rho)-P(\rho_*)\big](\rho-\rho_*)dx. \tag{2.5.27}$$

On the other hand, since P is a convex function, the Lemma 4.1 of [55] and Poincaré's inequality imply that there are positive constants C_2 and C_3 such that

$$\begin{aligned}\int_0^1 \Big(\frac{K}{2\rho}y_t^2+yy_t+\frac{1}{2}y^2\Big)dx &\le \int_0^1 \Big(\frac{K}{2\rho}y_t^2+\frac{1}{2}y_t^2+y^2\Big)dx\\ &\le C_2\int_0^1 \frac{K-\rho_*}{\rho}y_t^2dx+\int_0^1 y^2dx\\ &\le C_2\int_0^1 \frac{K-\rho_*}{\rho}y_t^2dx+\int_0^1 y_x^2dx\\ &\le C_2\int_0^1 \frac{K-\rho_*}{\rho}y_t^2dx+C_3\int_0^1 \big[P(\rho)-P(\rho_*)\big](\rho-\rho_*)dx.\end{aligned} \tag{2.5.28}$$

Therefore, for $C_4=\max\{C_2,C_3\}$, it holds

$$\int_0^1 \Big(K\eta_*+yy_t+\frac{1}{2}y^2\Big)dx \le C_4\Big(\int_0^1 \big[P(\rho)-P(\rho_*)\big](\rho-\rho_*)dx+\int_0^1 \frac{K-\rho_*}{\rho}y_t^2dx\Big). \tag{2.5.29}$$

Therefore, from (2.5.26)–(2.5.29), we conclude that there is a positive constant C_5 such that

$$\frac{d}{dt}\int_0^1 \Big(K\eta_*+yy_t+\frac{1}{2}y^2\Big)dx+C_5\int_0^1 \Big(K\eta_*+yy_t+\frac{1}{2}y^2\Big)dx \le 0. \tag{2.5.30}$$

Furthermore, since $K>2\Lambda\ge 2\rho$, we know that

$$\begin{aligned}&K\eta_*+yy_t+\frac{1}{2}y^2\\ &\ge 2y_t^2+yy_t+\frac{1}{2}y^2+\frac{K}{\gamma-1}\big[P(\rho)-P(\rho_*)-P'(\rho_*)(\rho-\rho_*)\big]\\ &\ge y_t^2+C_6(\rho-\rho_*)^2,\end{aligned} \tag{2.5.31}$$

where C_6 is a positive constant. Hence, (2.5.30) implies that

$$\int_0^1 \Big(K\eta_*+yy_t+\frac{1}{2}y^2\Big)dx \le C_7\exp\{-C_5t\}, \tag{2.5.32}$$

and

$$\int_0^1 y_t^2 + (\rho - \rho_*)^2 dx \le C_8 \exp\{-C_5 t\}. \tag{2.5.33}$$

This completes the proof of Theorem 2.5.1.

As indicated in introduction, we also expect that (2.1.2)–(2.1.3) is captured by (2.1.4)–(2.1.5) time asymptotically if

$$\int_0^1 \tilde{\rho}_0(x)\ dx = \rho_*. \tag{2.5.34}$$

In view of Theorem 2.5.1, we will show that the large time asymptotic state of (2.1.4)–(2.1.5) is also the constant state $(\rho_*,\ 0)$. Then by applying the triangle inequality we can prove Theorem 2.2.2.

Consider

$$\begin{cases} \tilde{\rho}_t - \tilde{P}_{xx} = 0, \\ \tilde{\rho}(x,0) = \tilde{\rho}_0(x), \qquad 0 \le x \le 1, \\ \tilde{P}_x(0,t) = \tilde{P}_x(1,t) = 0, \quad t \ge 0, \end{cases} \tag{2.5.35}$$

where $\tilde{P} = P(\tilde{\rho})$, and $\tilde{P}_0'(0) = \tilde{P}_0'(1) = 0$, for $\tilde{P}_0(x) = P(\tilde{\rho}_0(x))$. The initial data $\tilde{\rho}_0$ satisfies

$$0 \le \tilde{\rho}_0(x) \le \Lambda, \quad \text{and} \ \int_0^1 \tilde{\rho}_0(x) dx = \int_0^1 \rho_0(x) dx = \rho_*. \tag{2.5.36}$$

The global existence and large time behavior of weak solutions of (2.5.35) has been established in [4], see also [98]. Here, we give a proof in different version including the decay of momentum.

Theorem 2.5.2. *Let $\tilde{\rho}_0(x)$ satisfy (2.5.36). Then for the global weak solution $\tilde{\rho}(x,t)$ of (2.5.35) and $\tilde{m} = -\tilde{P}_x$, there exist positive constants c_1 and $\delta_1 > 0$ such that*

$$\int_0^1 \big((\tilde{\rho} - \rho_*)^2 + \tilde{m}^2\big) dx \le c_1 \exp\{-\delta_1 t\}, \qquad \text{as} \ \ t \to +\infty. \tag{2.5.37}$$

Proof. First, we note that $0 \le \tilde{\rho}(x,t) \le \Lambda$ due to the comparison principle [98]. Second, there is a $T > 0$ such that $\rho(x,t) > 0$ is a classical solution for $t > T$, see [4].

Then, for $t > T$, we consider the equation

$$(\tilde{\rho} - \rho_*)_t = (\tilde{P} - P_*)_{xx}, \tag{2.5.38}$$

which is equivalent to $(2.5.35)_1$, where $\tilde{P} = P(\tilde{\rho}), P_* = P(\rho_*)$. Let

$$\psi(x,t) = \tilde{\rho}(x,t) - \rho_*, \tag{2.5.39}$$

and

$$\phi = \int_0^x \psi(r,t)\ dr, \tag{2.5.40}$$

then

$$\phi_x = \psi = \tilde{\rho} - \rho_*. \tag{2.5.41}$$

Due to the conservation of mass we have

$$\phi(0) = \phi(1) = 0. \tag{2.5.42}$$

Integrating (2.5.38) over $[0,x]$ and use the boundary condition we get

$$\phi_t = (\tilde{P} - P_*)_x. \tag{2.5.43}$$

Multiplying (2.5.43) by ϕ and integrating over $[0,1]$ we get

$$\frac{d}{dt}\int_0^1 \frac{1}{2}\phi^2 dx + \int_0^1 (\tilde{P} - P_*)(\tilde{\rho} - \rho_*)dx = 0. \tag{2.5.44}$$

Multiplying (2.5.38) by $\tilde{\rho} - \rho_*$ and integrating over $[0,1]$ we get

$$\frac{d}{dt}\int_0^1 \frac{1}{2}(\tilde{\rho} - \rho_*)^2 dx + \int_0^1 (\tilde{P} - P_*)_x(\rho - \rho_*)_x dx = 0. \tag{2.5.45}$$

Since $(\tilde{P} - P_*)_x = \tilde{P}_x = P'(\tilde{\rho})\tilde{\rho}_x = P'(\tilde{\rho})(\tilde{\rho} - \rho_*)_x$, one has

$$\frac{d}{dt}\int_0^1 \frac{1}{2}(\tilde{\rho} - \rho_*)^2 dx + \int_0^1 P'(\tilde{\rho})(\tilde{\rho} - \rho_*)_x(\tilde{\rho} - \rho_*)_x dx = 0,$$

i.e.

$$\frac{d}{dt}\int_0^1 \frac{1}{2}(\tilde{\rho} - \rho_*)^2 dx + \int_0^1 P'(\tilde{\rho})[(\tilde{\rho} - \rho_*)_x]^2 dx = 0. \tag{2.5.46}$$

Multiplying (2.5.38) by $(\tilde{P}-P_*)$ and integrating over $[0,1]$ we get

$$\int_0^1 [P(\tilde{\rho})-P(\rho_*)](\tilde{\rho}-\rho_*)_t dx + \int_0^1 [(\tilde{P}-P_*)_x]^2 dx = 0. \tag{2.5.47}$$

Now, we define

$$F(\tilde{\rho}-\rho_*) = \int_0^{\tilde{\rho}-\rho_*} [P(\rho_*+\xi)-P(\rho_*)]d\xi, \tag{2.5.48}$$

then we have

$$F_t = [P(\tilde{\rho})-P(\rho_*)](\tilde{\rho}-\rho_*)_t.$$

So (2.5.47) turns out to be

$$\frac{d}{dt}\int_0^1 F dx + \int_0^1 [(\tilde{P}-P_*)_x]^2 dx = 0. \tag{2.5.49}$$

From the definition of F, we know

$$F = \int_0^{\tilde{\rho}-\rho_*} P'(\zeta)\xi d\xi,$$

where ζ is between $\tilde{\rho}$ and ρ_*. Since $0 \le \tilde{\rho}, \rho_* \le \Lambda$ we know that

$$0 \le F \le \frac{P'(\Lambda)}{2}(\tilde{\rho}-\rho_*)^2. \tag{2.5.50}$$

Since $P(\tilde{\rho}) = \tilde{\rho}^\gamma/\gamma$, then $\tilde{\rho} = (\gamma\tilde{P})^{\frac{1}{\gamma}}$, and so $\tilde{\rho}_t = (\gamma\tilde{P})^{\frac{1}{\gamma}-1}\tilde{P}_t$. Then we consider the equation of $\tilde{P}$

$$\tilde{P}_t = (\gamma\tilde{P})^{1-\frac{1}{\gamma}}\tilde{P}_{xx},$$

i.e.

$$(\tilde{P}-P_*)_t = (\gamma\tilde{P})^{1-\frac{1}{\gamma}}(\tilde{P}-P_*)_{xx}. \tag{2.5.51}$$

Multiplying (2.5.51) by $(\tilde{P}-P_*)_{xx}$ and integrating over $[0,1]$ we get

$$\frac{d}{dt}\int_0^1 \frac{1}{2}[(\tilde{P}-P_*)_x]^2 dx + \int_0^1 (\gamma\tilde{P})^{1-\frac{1}{\gamma}}[(\tilde{P}-P_*)_{xx}]^2 dx = 0. \tag{2.5.52}$$

Coupling (2.5.45), (2.5.46), and (2.5.52), adding the results to (2.5.49), and notice that $P'(\tilde{\rho}) \ge 0$ and $(\gamma\tilde{P})^{1-\frac{1}{\gamma}} \ge 0$, we arrive at

$$\frac{d}{dt}\int_0^1 \left\{\phi^2+(\tilde{\rho}-\rho_*)^2+F+[(\tilde{P}-P_*)_x]^2\right\}dx+\int_0^1 \left\{(\tilde{P}-P_*)(\tilde{\rho}-\rho_*)+[(\tilde{P}-P_*)_x]^2\right\}dx \le 0, \tag{2.5.53}$$

where we have thrown some non-negative terms (in the second part of the LHS) away. Since $\phi_x = \tilde{\rho} - \rho_*$, by Poincaré's inequality and (2.5.50) we obtain

$$\int_0^1 \Big\{\phi^2+(\tilde{\rho}-\rho_*)^2+F+[(\tilde{P}-P_*)_x]^2\Big\}dx \le \int_0^1 \Big\{\Big(2+\frac{P'(\Lambda)}{2}\Big)(\tilde{\rho}-\rho_*)^2+[(\tilde{P}-P_*)_x]^2\Big\}dx. \tag{2.5.54}$$

Now, from Lemma 4.1 in [55], we know that

$$C_9(\tilde{\rho}-\rho_*)^2 \le (\tilde{P}-P_*)(\tilde{\rho}-\rho_*), \tag{2.5.55}$$

where C_9 is a constant. Combining (2.5.54) and (2.5.55) we obtain

$$\int_0^1 \Big\{\phi^2+(\tilde{\rho}-\rho_*)^2+F+[(\tilde{P}-P_*)_x]^2\Big\}dx \le \int_0^1 \Big\{C_{10}(\tilde{P}-P_*)(\tilde{\rho}-\rho_*)+[(\tilde{P}-P_*)_x]^2\Big\}dx, \tag{2.5.56}$$

where $C_{10} = \Big(2+\frac{P'(\Lambda)}{2}\Big)/C_9$. Therefore, (2.5.56) implies that

$$\int_0^1 \Big\{\phi^2+(\tilde{\rho}-\rho_*)^2+F+[(\tilde{P}-P_*)_x]^2\Big\}dx \le C_{11}\int_0^1 \Big\{(\tilde{P}-P_*)(\tilde{\rho}-\rho_*)+[(\tilde{P}-P_*)_x]^2\Big\}dx, \tag{2.5.57}$$

where $C_{11} = \max\{C_{10}, 1\}$. Combining (2.5.53) and (2.5.57) we get

$$\frac{d}{dt}\int_0^1 \Big\{\phi^2+(\tilde{\rho}-\rho_*)^2+F+[(\tilde{P}-P_*)_x]^2\Big\}dx+\frac{1}{C_{11}}\int_0^1 \Big\{\phi^2+(\tilde{\rho}-\rho_*)^2+F+[(\tilde{P}-P_*)_x]^2\Big\}dx \le 0,$$

which implies that

$$\int_0^1 \Big\{\phi^2+(\tilde{\rho}-\rho_*)^2+F+[(\tilde{P}-P_*)_x]^2\Big\}dx \le C_{12}\exp\Big\{-\frac{t}{C_{11}}\Big\}, \tag{2.5.58}$$

where C_{12} is a constant depending on the initial data.

Since $F \ge 0$, we obtain

$$\int_0^1 \Big\{\phi^2+(\tilde{\rho}-\rho_*)^2+[(\tilde{P}-P_*)_x]^2\Big\}dx \le C_{12}\exp\Big\{-\frac{t}{C_{11}}\Big\}, \tag{2.5.59}$$

i.e.

$$\int_0^1 \Big\{\phi^2+(\tilde{\rho}-\rho_*)^2+\tilde{m}^2\Big\}dx \le C_{12}\exp\Big\{-\frac{t}{C_{11}}\Big\}. \tag{2.5.60}$$

This completes the proof of Theorem 2.5.2.

Theorem 2.2.2 is a immediate consequence of Theorem 2.5.1 and Theorem 2.5.2.

CHAPTER III

3D DAMPED COMPRESSIBLE EULER EQUATIONS

3.1 Introduction

In this chapter we continue the study of the damped compressible Euler equations on bounded domains. We consider the 3D compressible Euler equations with frictional damping:

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0 \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) + \nabla P = -\rho U, \end{cases} \tag{3.1.1}$$

where ρ, U, $M = \rho U$ and P denote the density, velocity, momentum and pressure respectively; the constant $\alpha > 0$ models friction and $P(\rho) = \frac{1}{\gamma}\rho^\gamma,\ 1 < \gamma$. System (3.1.1) is supplemented by the following initial and boundary conditions:

$$\begin{cases} (\rho, U)(\mathbf{x}, 0) = (\rho_0, U_0)(\mathbf{x}), \quad \mathbf{x} = (x, y, z) \in \Omega, \\ U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0, \\ \displaystyle\int_\Omega \rho_0 d\mathbf{x} = \bar{\rho} > 0, \end{cases} \tag{3.1.2}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outward normal vector on the boundary of Ω.

Due to the dissipation in the momentum equations and the boundary effect, the kinetic energy is expected to vanish as time tends to infinity while the potential energy will converge to a constant. Furthermore, it is easy to see that

$$\int_\Omega \rho(\mathbf{x}, t) d\mathbf{x} = \int_\Omega \rho_0(\mathbf{x}) d\mathbf{x} = \bar{\rho}$$

due to the conservation of total mass. This suggests that the asymptotic state of the solution should be $(\rho, U)|_{t\to\infty} = (\bar{\rho}/|\Omega|, \mathbf{0})$. In this chapter, we will prove, under the assumption that the initial perturbation around the equilibrium state is small, there exists a unique global classical solution to (3.1.1)–(3.1.2) and the solution converges exponentially to the equilibrium state. We also prove the same is true for the solution of the decoupled system

$$\begin{cases} \tilde{\rho}_t = \Delta P(\tilde{\rho}), \\ \tilde{M} = -\nabla P(\tilde{\rho}), \end{cases} \tag{3.1.3}$$

with the initial and boundary conditions

$$\begin{cases} \tilde{\rho}(\mathbf{x}, 0) = \tilde{\rho}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ (\nabla P(\tilde{\rho})) \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{3.1.4}$$

provided that

$$\int_\Omega \tilde{\rho}_0 d\mathbf{x} = \int_\Omega \rho_0 d\mathbf{x}. \tag{3.1.5}$$

Notation 3.1.1. *Throughout this chapter, the energy space under consideration is:*

$$X_3([0,T],\Omega) \equiv \left\{F : \Omega \times [0,T] \to \mathbb{R}^3(\text{or } \mathbb{R}) \;\middle|\; \partial_t^l F \in L^\infty([0,T]; H^{3-l}(\Omega)), l = 0,1,2,3\right\},$$

equipped with norm

$$\|F\|_{3,T} \equiv \operatorname{ess} \sup_{0 \leq t \leq T} |||F(\cdot,t)||| \equiv \operatorname{ess} \sup_{0 \leq t \leq T} \Big[\sum_{l=0}^{3} \|\partial_t^l F(\cdot,t)\|_{H^{3-l}}^2\Big]^{1/2},$$

for any $F \in X_3([0,T],\Omega)$. Unless specified, throughout this chapter, C and C_i will denote generic constants which are independent of ρ, U and t. The values of the constants are different from those in previous chapter.

In this chapter, we generalize the study of [93] on bounded domains with the slip boundary condition $(3.1.2)_2$. For the global existence and large time behavior of classical solutions, we have the following

Theorem 3.1.1. *Suppose that the initial data satisfy the compatibility condition, i.e., $\partial_t^l U(0) \cdot \mathbf{n}|_{\partial\Omega} = 0, 0 \leq l \leq 2$, where $\partial_t^l U(0)$ is the l^{th} time derivative at $t = 0$ of any solution of (3.1.1)–(3.1.2), as calculated from (3.1.1) to yield an expression in terms of ρ_0 and U_0. Then there exists a constant ε such that if $(\rho_0 - \bar{\rho}/|\Omega|, U_0) \in H^3(\Omega)$*

and $\|(\rho_0 - \bar{\rho}/|\Omega|, U_0)\|_{H^3} \le \varepsilon$, *then there exists a unique global solution* (ρ, U) *of the initial-boundary value problem (3.1.1)–(3.1.2) in* $C^1(\bar{\Omega} \times [0,\infty)) \cap X_3([0,\infty), \Omega)$. *Moreover, there exist positive constants* $C > 0, \eta > 0$, *which are independent of* t, *such that*

$$|||(\rho - \bar{\rho}/|\Omega|)(\cdot,t)||| + |||U(\cdot,t)||| \le C\|(\rho_0 - \bar{\rho}/|\Omega|, U_0)\|_{H^3} \exp\{-\eta t\}. \qquad (3.1.6)$$

Concerning the relationship between the solutions of (3.1.1)–(3.1.2) and (3.1.3)–(3.1.5), we have

Theorem 3.1.2. *Let* (ρ, U) *be the unique global classical solution of (3.1.1)–(3.1.2) and define* $M = \rho U$. *Let* $(\tilde{\rho}, \tilde{M})$ *be the global solution of (3.1.3)–(3.1.5) with* $\tilde{\rho}_0 \in L^\infty(\Omega)$, *and* $0 \le \tilde{\rho}_0 \le \rho^*$ *for some constant* ρ^* *satisfying* $\bar{\rho}/|\Omega| < \rho^* < \infty$. *Then, there exist constants* $C,\ \delta > 0$ *independent of* t *such that*

$$\|(\rho - \tilde{\rho})(\cdot,t)\|_{H^1} + \|(M - \tilde{M})(\cdot,t)\| \le C\exp\{-\delta t\}, \quad \text{as } t \to \infty. \qquad (3.1.7)$$

We prove Theorem 3.1.1 by showing the global existence and large time behavior of classical solutions to the IBVP for the perturbation $(\rho - \bar{\rho}/|\Omega|, U - \mathbf{0})$. Due to the slip boundary condition, the classical energy estimates can not be applied directly to spatial derivatives. The proof of Theorem 3.1.1 is based on some special energy estimates which strongly depend on the estimate of ∇U by $\nabla \times U$ and $\nabla \cdot U$, see Lemma 3.3.2 below. Using the special structure of (3.1.1) together with an induction on the number of spatial derivatives, the estimate of total energy is reduced to those for the vorticity and temporal derivatives. And the proof is completed by showing that (3.1.6) is true for the vorticity and temporal derivatives. Compared with the classical energy estimate for 3D initial-boundary value problems, which requires the localization of $\partial\Omega$, see for example [75, 90], our approach is short and neat. This idea has also been used for the incompressible Euler equations in a free boundary problem, see [24].

Theorem 3.1.2 is proved in a similar fashion as Theorem 2.2.2. We prove that both solutions of (3.1.1)–(3.1.2) and (3.1.3)–(3.1.5) tend to the same equilibrium state exponentially fast. Thus, Theorem 3.1.2 is an easy consequence of the triangle inequality. Moreover, the proof of the asymptotic behavior of the solution of (3.1.3)–(3.1.5), see Theorem 3.4.1 below, requires neither smoothness nor smallness condition on the initial data, i.e., the initial perturbation around the asymptotic state could be rough and large, which is a significant difference from the proof of Theorem 3.1.1. The argument is somewhat delicate mainly due to the nonlinearity in the diffusion. It should be pointed out that, the global existence and large time behavior of solutions of (3.1.3)–(3.1.5) have been studied in [4] based on dynamical system approach, see also [98]. In this chapter, we give a different proof on the asymptotic behavior of the solution based on the method of energy estimate. The decay in momentum is also achieved.

3.2 Reformulation and Local Existence

In order to carry out standard energy estimate, we first reformulate the IBVP (3.1.1)–(3.1.2). Without any loss of generality, we assume $\bar{\rho}/|\Omega| = 1$. First we reformulate (3.1.1) to get a symmetric hyperbolic system. Introducing the nonlinear transformation $\tilde{\sigma} = \rho^\theta/\theta$ with $\theta = (\gamma - 1)/2$ (ρ^θ is called *sound speed*) we get from the original system that

$$\begin{cases} \tilde{\sigma}_t + U \cdot \nabla\tilde{\sigma} + \theta\tilde{\sigma}\nabla \cdot U = 0, \\ U_t + U \cdot \nabla U + \theta\tilde{\sigma}\nabla\tilde{\sigma} = -U. \end{cases}$$

Since the equilibrium density is conjectured to be $\bar{\rho}/|\Omega| = 1$, we let $\sigma = \tilde{\sigma} - 1/\theta$ and get the desired symmetric system for the perturbation

$$\begin{cases} \sigma_t + U \cdot \nabla\sigma + \theta\sigma\nabla \cdot U + \nabla \cdot U = 0, \\ U_t + U \cdot \nabla U + \theta\sigma\nabla\sigma + \nabla\sigma = -U. \end{cases} \tag{3.2.1}$$

The initial and boundary conditions become

$$\begin{cases} (\sigma, U)(\mathbf{x}, 0) = (\sigma_0, U_0)(\mathbf{x}), \\ U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{3.2.2}$$

with

$$\sigma_0 = \frac{\rho_0^\theta}{\theta} - \frac{1}{\theta}.$$

The following lemmas are consequences of regularity and could be proved using the same idea in [93].

Lemma 3.2.1. *For any $T > 0$, if $(\rho, U) \in C^1(\bar{\Omega} \times [0, T])$ is a solution of (3.1.1) with $\rho > 0$, then $(\sigma, U) \in C^1(\bar{\Omega} \times [0, T])$ is a solution of (3.2.1) with $((\gamma - 1)/2)\sigma + 1 > 0$. Conversely, if $(\sigma, U) \in C^1(\bar{\Omega} \times [0, T])$ is a solution of (3.2.1) with $((\gamma-1)/2)\sigma+1 > 0$ and $\rho = (((\gamma-1)/2)\sigma+1)^{2/(\gamma-1)}$, then $(\rho, U) \in C^1(\bar{\Omega} \times [0, T])$ is a solution of (3.1.1) with $\rho > 0$.*

Lemma 3.2.2. *If $(\rho, U) \in C^1(\bar{\Omega} \times [0, T])$ is a uniformly bounded solution of (3.1.1) with $\rho(x, 0) > 0$, then $\rho(x, t) > 0$ on $\bar{\Omega} \times [0, T]$. If $(\sigma, U) \in C^1(\bar{\Omega} \times [0, T])$ is a uniformly bounded solution of (3.2.1) with $((\gamma-1)/2)\sigma(x, 0)+1 > 0$, then $((\gamma-1)/2)\sigma(x, t)+1 > 0$ on $\bar{\Omega} \times [0, T]$.*

The following local existence result can be established using the arguments in [90].

Lemma 3.2.3. *If $(\sigma_0, U_0) \in H^3(\Omega)$ and satisfy the compatibility condition, i.e., $\partial_t^l U(0) \cdot \mathbf{n}|_{\partial\Omega} = 0, 0 \leq l \leq 2$, then there exists a unique local solution (σ, U) of the initial-boundary value problem (3.2.1)–(3.2.2) in $C^1(\bar{\Omega} \times [0, T]) \cap X_3([0, T], \Omega)$ for some finite $T > 0$. Moreover, there exist positive constants $\varepsilon_0, C_0(T)$ such that if $\|\sigma(\cdot, 0)\|_{H^3} + \|U(\cdot, 0)\|_{H^3} \leq \varepsilon_0$, then $\|\sigma\|_{3,T} + \|U\|_{3,T} \leq C_0\big(\|\sigma(\cdot, 0)\|_{H^3} + \|U(\cdot, 0)\|_{H^3}\big)$.*

3.3 Global Existence and Large Time Behavior

We now prove the global existence and the large time behavior of the solution of (3.2.1)–(3.2.2). For convenience, we let

$$W(t) \equiv |||\sigma(t)|||^2 + |||U(t)|||^2 = \sum_{l=0}^{3} \big(\|\partial_t^l \sigma(t)\|_{H^{3-l}}^2 + \|\partial_t^l U(t)\|_{H^{3-l}}^2\big). \tag{3.3.1}$$

Theorem 3.3.1. *There exists $\varepsilon > 0$ such that if $W(0) \le \varepsilon^2$, then there is a unique global classical solution of (3.2.1)–(3.2.2) such that there exist positive constants $C > 0, \eta > 0$, which are independent of t, such that*

$$W(t) \le CW(0)e^{-\eta t}. \tag{3.3.2}$$

The proof of Theorem 3.3.1 is based on several steps of careful energy estimates which are stated as a sequence of lemmas. First we recall some inequalities of Sobolev type (c.f. [97]).

Lemma 3.3.1. *Let Ω be any bounded domain in $\mathbb{R}^3$ with smooth boundary. Then*

$$\text{(i)}\ \|f\|_{L^\infty(\Omega)} \le C\|f\|_{H^2(\Omega)},$$

$$\text{(ii)}\ \|f\|_{L^p(\Omega)} \le C\|f\|_{H^1(\Omega)}, \quad 2 \le p \le 6,$$

for some constant $C > 0$ depending only on Ω.

Due to the slip boundary condition, the spatial derivatives are unknown on the boundary. Following the standard procedure, see for example [75, 90], one can establish the energy estimates for the spatial derivatives by using cutoff functions and localizations of $\partial\Omega$, and Theorem 3.3.1 could be established in this fashion. However, we notice that the proof is long and tedious. Here we give another version of the proof which is short and neat. The proof will strongly depend on the following lemma (see [9]), which gives the estimate of ∇U by $\nabla \cdot U$ and $\nabla \times U$.

Lemma 3.3.2. *Let $U \in H^s(\Omega)$ be a vector-valued function satisfying $U \cdot \mathbf{n}|_{\partial\Omega} = 0$, where $\mathbf{n}$ is the unit outer normal of $\partial\Omega$. Then*

$$\|U\|_{H^s} \le C(\|\nabla \times U\|_{H^{s-1}} + \|\nabla \cdot U\|_{H^{s-1}} + \|U\|_{H^{s-1}}), \tag{3.3.3}$$

for $s \geq 1$, and the constant C depends only on s and Ω.

The next lemma is an application of Lemma 3.3.2, which plays an important role in the proof of Theorem 3.3.1. Indeed, the lemma states that the spatial derivatives are bounded by the temporal derivatives and the vorticity. Let $\omega = \nabla \times U$ and define

$$E(t) \equiv \sum_{l=0}^{3} \big(\|\partial_t^l \sigma\|^2 + \|\partial_t^l U\|^2\big), \quad \text{and} \quad V(t) \equiv \sum_{l=0}^{2} \|\partial_t^l \omega\|^2_{H^{2-l}}, \tag{3.3.4}$$

Lemma 3.3.3. *Let (σ, U) be the solution of (3.2.1)–(3.2.2). There is a small constant $\bar{\delta}$ such that if $W(t) \leq \bar{\delta}$, then there exists a constant $C_1 > 0$ such that*

$$W(t) \leq C_1 \Big(V(t) + E(t)\Big).$$

Proof. From the velocity equation $(3.2.1)_2$ we have

$$\nabla \sigma = -\frac{1}{\theta\sigma + 1}(U + U_t + U \cdot \nabla U). \tag{3.3.5}$$

Taking the L^2 inner product of (3.3.5) with $\nabla\sigma$, we get

$$\|\nabla \sigma\|^2 = \int_\Omega -\frac{1}{\theta\sigma + 1}(U + U_t + U \cdot \nabla U) \cdot \nabla \sigma \; d\mathbf{x},$$

using the smallness of $W(t)$, Lemma 3.3.1 (i), and Cauchy-Schwartz inequality, we easily get

$$\begin{aligned} \|\nabla \sigma\|^2 &\leq C\big(\|U\|^2 + \|U_t\|^2\big) + C\|U\|^2_{L^\infty}\|\nabla U\|^2 \\ &\leq C\big(\|U\|^2 + \|U_t\|^2\big) + CW(t)^{\frac{3}{2}}. \end{aligned} \tag{3.3.6}$$

The continuity equation $(3.2.1)_1$ implies

$$\nabla \cdot U = -\frac{1}{\theta\sigma + 1}(\sigma_t + U \cdot \nabla \sigma). \tag{3.3.7}$$

Therefore, we obtain

$$\|\nabla \cdot U\|^2 \leq C\Big(\|\sigma_t\|^2 + W(t)^{\frac{3}{2}}\Big). \tag{3.3.8}$$

Using Lemma 3.3.2 with $s = 1$ and (3.3.8) we have

$$\begin{aligned} \|U\|_1^2 &\leq C(\|\omega\|^2 + \|\nabla \cdot U\|^2 + \|U\|^2) \\ &\leq C(\|\omega\|^2 + \|\sigma_t\|^2 + \|U\|^2 + W(t)^{\frac{3}{2}}). \end{aligned} \tag{3.3.9}$$

Next, we take time derivatives of (3.3.5) and (3.3.7). It is clear that every time derivative up to order two of $\nabla\sigma$ and $\nabla \cdot U$ is again bounded by $E(t)$. Furthermore, together with an induction on the number of spatial derivatives, the same is true for any derivative up to order two of $\nabla\sigma$ and $\nabla \cdot U$. By applying Lemma 3.3.2 with $s = 1, 2, 3$ respectively we finally deduce the lemma. This completes the proof of Lemma 3.3.3.

Lemma 3.3.3 reduced the estimate of $W(t)$ to those for $E(t)$ and $V(t)$. Our next goal is to deal with the estimates of $E(t)$ and $V(t)$.

Lemma 3.3.4. *There is a constant $C > 0$ such that*

$$\frac{d}{dt}E(t) + 2\sum_{l=0}^{3} \|\partial_t^l U\|^2 \leq CW(t)^{\frac{3}{2}}. \tag{3.3.10}$$

Proof. *Zero order estimate:* We calculate $\sigma(3.2.1)_1 + U \cdot (3.2.1)_2$ and get

$$\frac{1}{2}\frac{d}{dt}(\sigma^2+|U|^2)+|U|^2 = -(1+\theta)\sigma(U\cdot\nabla\sigma)-\theta\sigma^2(\nabla\cdot U)-U\cdot(U\cdot\nabla U)-\nabla\cdot(\sigma U). \tag{3.3.11}$$

Integrating (3.3.11) over Ω using the Divergence Theorem and the boundary condition we get

$$\frac{1}{2}\frac{d}{dt}\big(\|\sigma\|^2 + \|U\|^2\big) + \|U\|^2 \leq C\big(\|\nabla\sigma\|_{L^\infty} + \|\nabla U)\|_{L^\infty}\big)\big(\|\sigma\|^2 + \|U\|^2\big). \tag{3.3.12}$$

Applying Lemma 3.3.1 (i) to (3.3.12) we get

$$\frac{d}{dt}\big(\|\sigma\|^2 + \|U\|^2\big) + 2\|U\|^2 \leq CW(t)^{\frac{3}{2}}. \tag{3.3.13}$$

First order estimate: Differentiating (3.2.1) with respect to t, multiplying the resulting equations by σ_t, U_t respectively, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\sigma_t^2 + |U_t|^2) + |U_t|^2 \\
=&(1/2-\theta)\sigma_t^2(\nabla\cdot U) - \sigma_t(U_t\cdot\nabla\sigma) - U_t\cdot(U_t\cdot\nabla U) - \frac{1}{2}|U_t|^2(\nabla\cdot U) \\
&-\nabla\cdot\left(\frac{(\sigma_t^2+|U_t|^2)}{2}U + (\theta\sigma+1)\sigma_t U_t\right).
\end{aligned}$$

Integrating the above equation over Ω using the boundary conditions $U \cdot \mathbf{n}|_{\partial\Omega} = 0$ and $U_t \cdot \mathbf{n}|_{\partial\Omega} = 0$ we get

$$\frac{d}{dt}\big(\|\sigma_t\|^2 + \|U_t\|^2\big) + 2\|U_t\|^2 \leq C\big(\|\nabla\sigma\|_{L^\infty} + \|\nabla \cdot U\|_{L^\infty} + \|\nabla U\|_{L^\infty}\big)W(t), \quad (3.3.14)$$

for some constant $C > 0$. From Lemma 3.3.1 (i) we get

$$\frac{d}{dt}\big(\|\sigma_t\|^2 + \|U_t\|^2\big) + 2\|U_t\|^2 \leq CW(t)^{\frac{3}{2}}. \quad (3.3.15)$$

Second order estimate: Repeating the above procedure again for 2nd order time derivatives we get the following

$$\frac{d}{dt}\big(\|\sigma_{tt}\|^2 + \|U_{tt}\|^2\big) + 2\|U_{tt}\|^2 \leq CW(t)^{\frac{3}{2}}. \quad (3.3.16)$$

Third order estimate: If we repeat the above procedure to the 3rd order estimates, we find that the 4th order estimates will be needed due to the Sobolev inequality in Lemma 3.3.1 (i). However, this issue could be resolved by Lemma 3.3.1 (ii). We calculate $\partial_t^3\sigma\partial_t^3(3.2.1)_1 + \partial_t^3 U \cdot \partial_t^3(3.2.1)_2$ and get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\sigma_{ttt}^2 + |U_{ttt}|^2) + |U_{ttt}|^2 = &\Big[\frac{1}{2}(\nabla \cdot U)(|U_{ttt}|^2) + (\frac{1}{2} - \theta)(\nabla \cdot U)\sigma_{ttt}^2 - \\ &(U_{ttt} \cdot \nabla\sigma + 3U_t \cdot \nabla\sigma_{tt} + 3\theta\sigma_t\nabla \cdot U_{tt})\sigma_{ttt} - \\ &(U_{ttt} \cdot \nabla U + 3U_t \cdot \nabla U_{tt} + 3\theta\sigma_t\nabla\sigma_{tt}) \cdot U_{ttt}\Big] - \\ &\Big\{3(U_{tt} \cdot \nabla\sigma_t + \theta\sigma_{tt}\nabla \cdot U_t)\sigma_{ttt} + \\ &3(U_{tt} \cdot \nabla U_t + \theta\sigma_{tt}\nabla\sigma_t) \cdot U_{ttt}\Big\} - \\ &\nabla \cdot \Big(\frac{\sigma_{ttt}^2}{2}U + \frac{|U_{ttt}|^2}{2}U + (\theta\sigma)\sigma_{ttt}U_{ttt}\Big).\end{aligned}$$

Integrating the above equation over Ω, applying Lemma 3.3.1 (i) to the terms inside the [ ] we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\sigma_{ttt}\|^2 + \|U_{ttt}\|^2) + \|U_{ttt}\|^2 \leq CW(t)^{\frac{3}{2}} &+ 3\Big|\int_\Omega \sigma_{ttt}(U_{tt} \cdot \nabla\sigma_t + \theta\sigma_{tt} \cdot \nabla U_t)d\mathbf{x}\Big| \\ &+ 3\Big|\int_\Omega U_{ttt} \cdot (U_{tt} \cdot \nabla U_t + \theta\sigma_{tt}\nabla\sigma_t)d\mathbf{x}\Big|.\end{aligned}$$

Using Hölder's inequality, Lemma 3.3.1 (ii), and Cauchy-Schwartz inequality we can estimate the second term on the RHS above as follows:

$$
\begin{aligned}
&\Big|\int_\Omega \sigma_{ttt}(U_{tt}\cdot\nabla\sigma_t+\theta\sigma_{tt}\cdot\nabla U_t)d\mathbf{x}\Big|\\
&\le \|\sigma_{ttt}\|_{L^2}\big(\|U_{tt}\|_{L^4}\|D\sigma_t\|_{L^4}+\theta\|\sigma_{tt}\|_{L^4}\|DU_t\|_{L^4}\big)\\
&\le C\|\sigma_{ttt}\|_{L^2}\big(\|U_{tt}\|_{H^1}\|D\sigma_t\|_{H^1}+\theta\|\sigma_{tt}\|_{H^1}\|DU_t\|_{H^1}\big)\\
&\le C\|\sigma_{ttt}\|_{L^2}\big(\|U_{tt}\|_{H^1}^2+\|D\sigma_t\|_{H^1}^2+\|\sigma_{tt}\|_{H^1}^2+\|DU_t\|_{H^1}^2\big)\\
&\le CW(t)^{\frac{3}{2}}.
\end{aligned}
\tag{3.3.17}
$$

The third term can be estimated in the same way. Then we get the 3rd order estimate:

$$
\frac{d}{dt}\big(\|\sigma_{ttt}\|^2+\|U_{ttt}\|^2\big)+2\|U_{ttt}\|^2\le CW(t)^{\frac{3}{2}}. \tag{3.3.18}
$$

Therefore, (3.3.10) follows from (3.3.13), (3.3.15)–(3.3.16) and (3.3.18). This completes the proof of Lemma 3.3.4.

Lemma 3.3.4 contains the dissipation in velocity. In the next lemma we are going to explore the dissipation in density due to nonlinearity.

Lemma 3.3.5. *There exist constants $c_0, C>0$ such that*

$$
\frac{d}{dt}\Bigg(\sum_{l=1}^{3}\int_\Omega\big(-\partial_t^{l-1}\sigma\partial_t^l\sigma\big)d\mathbf{x}\Bigg)+\sum_{l=0}^{3}\|\partial_t^l\sigma\|^2\le CW(t)^{\frac{3}{2}}+c_0\sum_{l=0}^{3}\|\partial_t^lU\|^2. \tag{3.3.19}
$$

Proof. First of all, due to the conservation of total mass we know $\int_\Omega(\rho-1)dx=0$, where ρ is the solution of (3.1.1) and $1=\bar{\rho}/|\Omega|$ is the equilibrium state of ρ. Letting $\hat{\rho}=\rho-1$, then Poincaré's inequality (c.f. [34]) implies that $\|\hat{\rho}\|^2\le C\|\nabla\hat{\rho}\|^2$. By definition, $\sigma=(\tau\hat{\rho}+1)\hat{\rho}$ for some $\tau\in[0,1]$ and $\nabla\sigma=(\hat{\rho}+1)^{\theta-1}\nabla\hat{\rho}$. So that for $W(t)$ small, $\|\sigma\|^2\le C\|\nabla\sigma\|^2$. Using (3.3.6) we obtain

$$
\|\sigma\|^2\le C\big(W(t)^{\frac{3}{2}}+\|U\|^2+\|U_t\|^2\big). \tag{3.3.20}
$$

Calculating $\partial_t(3.2.1)_1-(\theta\sigma+1)\nabla\cdot(3.2.1)_2$ we get

$$
\sigma_{tt}+(U\cdot\nabla\sigma)_t+\theta\sigma_t(\nabla\cdot U)-(\theta\sigma+1)\nabla\cdot\Big[U\cdot\nabla U+(\theta\sigma+1)\nabla\sigma+U\Big]=0. \tag{3.3.21}
$$

Multiplying (3.3.21) by σ we obtain

$$
\begin{aligned}
&\sigma_{tt}\sigma + (U\cdot\nabla\sigma)_t\sigma + \theta\sigma_t(\nabla\cdot U)\sigma - (\theta\sigma+1)\nabla\cdot\Big[U\cdot\nabla U + (\theta\sigma+1)\nabla\sigma + U\Big]\sigma \\
=&(\sigma\sigma_t)_t - \sigma_t^2 + (U\cdot\nabla\sigma)_t\sigma + \theta\sigma\sigma_t(\nabla\cdot U) + (\theta\sigma+1)\sigma\nabla\cdot U_t \\
=&(\sigma\sigma_t)_t - \sigma_t^2 + (U_t\cdot\nabla\sigma)\sigma + (U\cdot\nabla\sigma_t)\sigma + \theta\sigma\sigma_t(\nabla\cdot U) + \\
&\nabla\cdot\big[(\theta\sigma^2+\sigma)U_t\big] - U_t\cdot\nabla(\theta\sigma^2+\sigma) \\
=&(\sigma\sigma_t)_t - \sigma_t^2 + (U_t\cdot\nabla\sigma)\sigma + \nabla\cdot(\sigma\sigma_t U) - \sigma\sigma_t(\nabla\cdot U) - \sigma_t(U\cdot\nabla\sigma) + \theta\sigma\sigma_t(\nabla\cdot U) + \\
&\nabla\cdot\big[(\theta\sigma^2+\sigma)U_t\big] - U_t\cdot\nabla(\theta\sigma^2+\sigma) \\
=&(\sigma\sigma_t)_t - \sigma_t^2 + (U_t\cdot\nabla\sigma)\sigma + (\theta-1)\sigma\sigma_t(\nabla\cdot U) - \sigma_t(U\cdot\nabla\sigma) - 2\theta\sigma U_t\cdot\nabla\sigma - \\
&U_t\cdot\nabla\sigma + \nabla\cdot\big[(\theta\sigma^2+\sigma)U_t + \sigma\sigma_t U\big] \\
=&0,
\end{aligned}
\tag{3.3.22}
$$

where we used the equation $U\cdot\nabla U + (\theta\sigma+1)\nabla\sigma + U = -U_t$. Integrating (3.3.22) over Ω and using Cauchy-Schwartz inequality we get

$$
-\frac{d}{dt}\Big(\int_\Omega \sigma\sigma_t d\mathbf{x}\Big) + \|\sigma_t\|^2 \le C\big(W(t)^{\frac{3}{2}} + \|\nabla\sigma\|^2 + \|U\|^2\big), \tag{3.3.23}
$$

which together with (3.3.6) gives

$$
-\frac{d}{dt}\Big(\int_\Omega \sigma\sigma_t d\mathbf{x}\Big) + \|\sigma_t\|^2 \le C\big(W(t)^{\frac{3}{2}} + \|U\|^2 + \|U_t\|^2\big). \tag{3.3.24}
$$

Next, we take time derivatives of (3.3.21). Similar derivations show that

$$
\begin{aligned}
&-\frac{d}{dt}\Big(\int_\Omega \sigma_t\sigma_{tt} d\mathbf{x}\Big) + \|\sigma_{tt}\|^2 \le C\big(W(t)^{\frac{3}{2}} + \|U_t\|^2 + \|U_{tt}\|^2\big), \\
&-\frac{d}{dt}\Big(\int_\Omega \sigma_{tt}\sigma_{ttt} d\mathbf{x}\Big) + \|\sigma_{ttt}\|^2 \le C\big(W(t)^{\frac{3}{2}} + \|U_{tt}\|^2 + \|U_{ttt}\|^2\big),
\end{aligned}
$$

which together with (3.3.20) and (3.3.24) deduce (3.3.19). This completes the proof of Lemma 3.3.5.

Now, we are ready to combine Lemma 3.3.4 and 3.3.5 to characterize the total dissipation. For this purpose, we let $C_2 \equiv \max\{2, c_0\}$, and define

$$\begin{aligned} E_1(t) &\equiv C_2 E(t) - \sum_{l=1}^{3} \int_\Omega \left(\partial_t^{l-1}\sigma \partial_t^l \sigma\right) d\mathbf{x} \\ &= C_2 \sum_{l=0}^{3} \left(\|\partial_t^l \sigma\|^2 + \|\partial_t^l U\|^2\right) - \sum_{l=1}^{3} \int_\Omega \left(\partial_t^{l-1}\sigma \partial_t^l \sigma\right) d\mathbf{x}. \end{aligned} \tag{3.3.25}$$

It is easy to see that $E_1(t) \geq 0$ for any $t \geq 0$. Then we have

Lemma 3.3.6. *There exist constants $C_3, C > 0$ such that*

$$\frac{d}{dt} E_1(t) + C_3 E(t) \leq C W(t)^{\frac{3}{2}}. \tag{3.3.26}$$

Proof. $C_2 \times (3.3.10) + (3.3.19)$ yields

$$\frac{d}{dt} E_1(t) + c_0 \sum_{l=0}^{3} \|\partial_t^l U\|^2 + \sum_{l=0}^{3} \|\partial_t^l \sigma\|^2 \leq C W(t)^{\frac{3}{2}}. \tag{3.3.27}$$

Let $C_3 = \min\{c_0, 1\}$, then (3.3.26) follows directly from (3.3.27).

The next lemma is contributed to the estimate of $V(t)$ defined in Lemma 3.3.3.

Lemma 3.3.7. *For $V(t)$ defined in Lemma 3.3.3, there exists a constant $C > 0$ such that*

$$\frac{d}{dt} V(t) + 2V(t) \leq C W(t)^{\frac{3}{2}}. \tag{3.3.28}$$

Proof. Taking the curl of the velocity equation of (3.2.1) we get

$$\omega_t + \omega = -U \cdot \nabla\omega + \omega \cdot \nabla U - \omega(\nabla \cdot U).$$

Let ∂ denote any mixed time and spatial derivative of order $0 \leq |\partial| \leq 2$, then by taking any mixed derivative of the above equation, we get

$$\partial\omega_t + \partial\omega = \partial\{-U \cdot \nabla\omega + \omega \cdot \nabla U - \omega(\nabla \cdot U)\}.$$

Multiplying the above equation by $\partial\omega$ and integrating the resulting equation by using the boundary condition, together with the standard energy estimate used in deriving (3.3.17), we get

$$\frac{1}{2}\frac{d}{dt}\|\partial\omega(t)\|^2 + \|\partial\omega(t)\|^2 \leq C W(t)^{\frac{3}{2}}.$$

Finally, we deduce the lemma by summing up the above inequality for all $0 \le |\partial| \le 2$. This completes the proof of Lemma 3.3.7.

Proof of Theorem 3.3.1. From (3.3.3), (3.3.25), and the definition of C_2 we can easily see that $E(t)$ and $E_1(t)$ are equivalent, i.e., there exist constants $c_1, c_2 > 0$ such that

$$c_1 E_1(t) \le E(t) \le c_2 E_1(t). \tag{3.3.29}$$

Then, by (3.3.26) and (3.3.29) we have

$$\frac{d}{dt} E_1(t) + c_1 C_3 E_1(t) \le C W(t)^{\frac{3}{2}}. \tag{3.3.30}$$

Combining (3.3.28) and (3.3.30) we get

$$\frac{d}{dt}\Big(V(t) + E_1(t)\Big) + \Big(2V(t) + c_1 C_3 E_1(t)\Big) \le C W(t)^{\frac{3}{2}}. \tag{3.3.31}$$

Let $C_4 \equiv \min\{2, c_1 C_3\}$, then we get from (3.3.31) that

$$\frac{d}{dt}\Big(V(t) + E_1(t)\Big) + C_4\Big(V(t) + E_1(t)\Big) \le C W(t)^{\frac{3}{2}}. \tag{3.3.32}$$

On the other hand, from (3.3.4) and (3.3.29) we see that

$$W(t) \le C_1\Big(V(t) + c_2 E_1(t)\Big). \tag{3.3.33}$$

Let $C_5 \equiv \max\{C_1, c_2 C_1\}$, then we get

$$W(t) \le C_5\Big(V(t) + E_1(t)\Big). \tag{3.3.34}$$

For $W(t)$ sufficiently small, (3.3.32) and (3.3.34) yield

$$\frac{d}{dt}\Big(V(t) + E_1(t)\Big) + C_4\Big(V(t) + E_1(t)\Big) \le \frac{C_4}{2}\Big(V(t) + E_1(t)\Big). \tag{3.3.35}$$

Thus, we get

$$\frac{d}{dt}\Big(V(t) + E_1(t)\Big) + \frac{C_4}{2}\Big(V(t) + E_1(t)\Big) \le 0, \tag{3.3.36}$$

which yields the exponential decaying of $V(t) + E_1(t)$. Finally, the exponential decay of $W(t)$ follows from (3.3.34). This completes the proof of Theorem 3.3.1.

3.4 Asymptotic Behavior and Porous Medium Equation.

We turn to the investigation of the large time behavior of classical solutions of (3.1.3)–(3.1.4). As indicated in the introduction, we expect that (3.1.1)–(3.1.2) is captured by (3.1.3)–(3.1.4) time asymptotically if

$$\int_\Omega \tilde{\rho}_0 d\mathbf{x} = \int_\Omega \rho_0 d\mathbf{x} = \bar{\rho}.$$

In view of Theorem 3.3.1, we will show that the large time asymptotic state of (3.1.3)–(3.1.4) is also the constant state $(\bar{\rho}/|\Omega|, \mathbf{0})$. Then, by applying the triangle inequality we can prove Theorem 3.1.2. Without loss of generality, we assume $\bar{\rho}/|\Omega| = 1$.

Consider

$$\begin{cases} \tilde{\rho}_t = \Delta P(\tilde{\rho}), \\ \tilde{\rho}(\mathbf{x}, 0) = \tilde{\rho}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ (\nabla P(\tilde{\rho})) \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{3.4.1}$$

where the initial data satisfy

$$\begin{cases} \displaystyle\int_\Omega \tilde{\rho}_0 d\mathbf{x} = \bar{\rho}, \quad \tilde{\rho}_0(\mathbf{x}) \in L^\infty(\Omega), \\ 0 \leq \tilde{\rho}_0(\mathbf{x}) \leq \rho^* \ \text{ for some constant } 1 < \rho^* < \infty. \end{cases} \tag{3.4.2}$$

The global existence of solutions to (3.4.1)–(3.4.2) has been established in [4], see also [98]. It is also shown in there that $\|(\tilde{\rho} - 1)\|_{L^\infty}$ tends to zero exponentially as time goes to infinity. Here, we give a different proof based on the method of energy estimate including the decay in momentum.

Theorem 3.4.1. *Let $\tilde{\rho}$ be the global solution of (3.4.1)–(3.4.2) with $\tilde{M} = -\nabla P(\tilde{\rho})$. Then, there exist positive constants $C > 0, \eta > 0$ independent of t such that*

$$\|(\tilde{\rho} - 1)\|_{H^1} + \|\tilde{M}(\cdot, t)\| \leq Ce^{-\eta t}, \quad \text{as } \ t \to \infty.$$

Proof. First, we observe that due to the comparison principle (c.f. [98]),

$$0 \leq \tilde{\rho}(\mathbf{x}, t) \leq \rho^*, \quad \forall\ (\mathbf{x}, t) \in \bar{\Omega} \times [0, \infty). \tag{3.4.3}$$

Second, there is a $T > 0$ such that $\tilde{\rho}(\mathbf{x}, t)$ is a classical solution and $\tilde{\rho}(\mathbf{x}, t) > \frac{1}{2}$ for $t > T$ and $\mathbf{x} \in \bar{\Omega}$, see [98]. Then, for $t > T$, we consider the equation

$$(\tilde{\rho} - 1)_t = \Delta(P(\tilde{\rho}) - P(1)). \tag{3.4.4}$$

Taking L^2 inner product of (3.4.4) with $(\tilde{\rho} - 1)$ we obtain, after integration by parts

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(\tilde{\rho} - 1)\|^2 - \int_\Omega \Delta[P(\tilde{\rho}) - P(1)](\tilde{\rho} - 1)d\mathbf{x} \\ =&\frac{1}{2}\frac{d}{dt}\|(\tilde{\rho} - 1)\|^2 + \int_\Omega \frac{|\nabla(P(\tilde{\rho}) - P(1))|^2}{P'(\tilde{\rho})}d\mathbf{x} \\ =&0. \end{aligned} \tag{3.4.5}$$

Using (3.4.3) we get from (3.4.5) that

$$\frac{1}{2}\frac{d}{dt}\|(\tilde{\rho} - 1)\|^2 + \frac{1}{P'(\rho^*)}\|\nabla(P(\tilde{\rho}) - P(1))\|^2 \leq 0. \tag{3.4.6}$$

Since $\tilde{\rho} = \gamma^{1/\gamma}\tilde{P}^{1/\gamma}$, for smooth solutions, $(3.4.1)_1$ is equivalent to

$$\tilde{P}_t - \gamma^{1-1/\gamma}\tilde{P}^{1-1/\gamma}\Delta\tilde{P} = 0, \tag{3.4.7}$$

where $\tilde{P} = P(\tilde{\rho})$. Now, we define

$$\Phi \equiv \tilde{P} - \bar{P} = P(\tilde{\rho}) - P(1),$$

then we get

$$\Phi_t - a\tilde{P}^{1-1/\gamma}(\Delta\Phi) = 0, \tag{3.4.8}$$

where $a = \gamma^{1-1/\gamma}$. Taking L^2 inner product of (3.4.8) with $\Delta\Phi$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\Phi\|^2 + aP(1/2)^{1-1/\gamma}\|\Delta\Phi\|^2 \leq 0. \tag{3.4.9}$$

Combining (3.4.6) and (3.4.9) we deduce

$$\frac{1}{2}\frac{d}{dt}\Big(\|(\tilde{\rho} - 1)\|^2 + \|\nabla\Phi\|^2\Big) + C_1\Big(\|\nabla\Phi\|^2 + \|\Delta\Phi\|^2\Big) \leq 0, \tag{3.4.10}$$

for $C_6 = \min\{1/P'(\rho^*), aP(1/2)^{1-1/\gamma}\}$.

To explore the secret of (3.4.10), we observe that since

$$\Phi = P(\tilde{\rho}) - P(1) = P'(\varrho)(\tilde{\rho} - 1)$$

for some $\varrho \in [1/2, \rho^*]$, then

$$\begin{aligned} &\|\Phi\|^2 \leq P'(\rho^*)^2 \|(\tilde{\rho} - 1)\|^2, \\ &\|\nabla\Phi\|^2 \geq P'(1/2)^2 \|\nabla(\tilde{\rho} - 1)\|^2. \end{aligned} \tag{3.4.11}$$

Due to the conservation of total mass, i.e. $\int_\Omega (\tilde{\rho} - 1) d\mathbf{x} = 0$, and Poincaré's inequality we get

$$\begin{aligned} \|\Phi\|^2 &\leq P'(\rho^*)^2 \|(\tilde{\rho} - 1)\|^2 \\ &\leq CP'(\rho^*)^2 \|\nabla(\tilde{\rho} - 1)\|^2 \\ &\leq C\left(\frac{P'(\rho^*)}{P'(1/2)}\right)^2 \|\nabla\Phi\|^2. \end{aligned} \tag{3.4.12}$$

Combining (3.4.10)–(3.4.12) we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|(\tilde{\rho} - 1)\|^2 + \|\nabla\Phi\|^2\Big) + C_7\Big(\|(\tilde{\rho} - 1)\|^2 + \|\nabla\Phi\|^2 + \|\Delta\Phi\|^2\Big) \leq 0, \tag{3.4.13}$$

for some constant $C_7 > 0$ depending on ρ^*. Finally, we deduce the theorem by (3.4.11), (3.4.13) and noticing that $\tilde{M} = -\nabla\Phi$. This completes the proof of Theorem 3.4.1.

Theorem 3.1.2 in Section 1 is an immediate consequence of Theorem 3.3.1 and Theorem 3.4.1.

CHAPTER IV

2D BOUSSINESQ EQUATIONS

4.1 Introduction

In this chapter we consider the 2D viscous Boussinesq equations

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = \nu \Delta U + \theta \mathbf{e}_2, \\ \theta_t + U \cdot \nabla \theta = \kappa \Delta \theta, \\ \nabla \cdot U = 0, \end{cases} \tag{4.1.1}$$

where $U = (u, v)$ is the velocity vector field, P is the scalar pressure, θ is the scalar density, the constant $\nu, \kappa > 0$ model viscous dissipation and heat diffusion respectively, and $\mathbf{e}_2 = (0,1)^{\mathrm{T}}$. In this chapter, we consider (4.1.1) in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$ and with partial viscosity (i.e., either $\nu > 0, \kappa = 0$ or $\nu = 0, \kappa > 0$). The system is supplemented by the following initial and boundary conditions:

For $\nu > 0, \kappa = 0$:

$$\begin{cases} (U, \theta)(\mathbf{x}, 0) = (U_0, \theta_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega; \\ U|_{\partial\Omega} = 0. \end{cases} \tag{4.1.2}$$

For $\nu = 0, \kappa > 0$:

$$\begin{cases} (U, \theta)(\mathbf{x}, 0) = (U_0, \theta_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega; \\ U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = \bar{\theta}, \end{cases} \tag{4.1.3}$$

where $\bar{\theta}$ is a constant and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$..

Notation 4.1.1. *Throughout this chapter, the function spaces under consideration are:*

$$C([0,T]; H^3(\Omega)) \quad \text{and} \quad L^2([0,T]; H^4(\Omega)),$$

equipped with norms

$$\sup_{0\le t\le T}\|\Psi(\cdot,t)\|_{H^3},\quad \text{for}\quad \Psi\in C([0,T];H^3(\Omega)),$$
$$\left(\int_0^T\|\Psi(\cdot,\tau)\|_{H^4}^2d\tau\right)^{1/2},\quad \text{for}\quad \Psi\in L^2([0,T];H^4(\Omega)).$$

Unless specified, throughout this chapter, C will denote various generic constants which are independent of U and θ, but may depend on the time T. Moreover, the values of the constants are different from those in previous chapters.

In this chapter, we will generalize the study of [15] and [41] to bounded domains with typical physical boundary conditions. For the global existence of smooth solutions, we require the following compatibility conditions:

For $\nu>0,\kappa=0$:

$$\begin{cases}\nabla\cdot U_0=0,\quad U_0|_{\partial\Omega}=0,\\ \nu\Delta U_0+\theta_0\mathbf{e}_2-\nabla P_0=0,\quad \mathbf{x}\in\partial\Omega,\ t=0,\end{cases}\tag{4.1.4}$$

where $P_0(\mathbf{x})=P(\mathbf{x},0)$ is the solution to the Neumann boundary problem

$$\begin{cases}\Delta P_0=\nabla\cdot[\theta_0\mathbf{e}_2-U_0\cdot\nabla U_0],\quad \mathbf{x}\in\Omega,\\ \nabla P_0\cdot\mathbf{n}|_{\partial\Omega}=[\nu\Delta U_0+\theta_0\mathbf{e}_2]\cdot\mathbf{n}|_{\partial\Omega}.\end{cases}\tag{4.1.5}$$

For $\nu=0,\kappa>0$:

$$\begin{cases}U_0\cdot\mathbf{n}|_{\partial\Omega}=0,\quad \nabla\cdot U_0=0,\\ \theta_0|_{\partial\Omega}=\bar\theta,\quad U_0\cdot\nabla\theta_0-\kappa\Delta\theta_0|_{\partial\Omega}=0.\end{cases}\tag{4.1.6}$$

Our main results are stated in the following theorems.

Theorem 4.1.1 (For $\nu>0,\kappa=0$). *Let $\Omega\subset\mathbf{R}^2$ be a bounded domain with smooth boundary. If $(\theta_0(\mathbf{x}),U_0(\mathbf{x}))\in H^3(\Omega)$ satisfies the compatibility conditions (4.1.4)–(4.1.5), then there exists a unique solution (θ,U) of (4.1.1)–(4.1.2) globally in time such that $\theta(\mathbf{x},t)\in C([0,T);H^3(\Omega))$ and $U(\mathbf{x},t)\in C([0,T);H^3(\Omega))\cap L^2([0,T);H^4(\Omega))$ for any $T>0$. Moreover, there exists a constant $\bar C>0$ independent of t such that*

$$\|U(\cdot,t)\|_{L^2}^2\le\max\left\{\|U(\cdot,0)\|_{L^2}^2\ ,\ \frac{\bar C^2}{\nu^2}\|\theta(\cdot,0)\|_{L^2}^2\right\},\quad \forall\ t\ge0.\tag{4.1.7}$$

Theorem 4.1.2 (For $\nu = 0, \kappa > 0$)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. If $(U_0(\mathbf{x}), \theta_0(\mathbf{x})) \in H^3(\Omega)$ satisfies the compatibility conditions (4.1.6), then there exists a unique solution (U, θ) of (4.1.1) and (4.1.3) globally in time such that $U \in C([0,T); H^3(\Omega))$ and $\theta \in C([0,T); H^3(\Omega)) \cap L^2([0,T); H^4(\Omega))$ for any $T > 0$. Moreover, there exist constants $\eta > 0, \bar{C} > 0, C(p) > 0$, which are independent of t such that for any fixed $p \in [2, \infty)$,*

$$\|(\theta - \bar{\theta})(\cdot, t)\|_{H^3} \leq \bar{C} \exp\{-\eta t\}, \quad \|U(\cdot, t)\|_{W^{1,p}} \leq C(p), \quad \forall\ t \geq 0. \tag{4.1.8}$$

The proofs of the above theorems mainly consist of two parts. First, we show the global existence of weak solutions, i.e., solutions satisfying the following definitions:

Definition 4.1.1. *(θ, U) is said to be a global weak solution of (4.1.1)–(4.1.2), if for any $T > 0$, $U \in C([0,T); L^2(\Omega)) \cap L^2([0,T); H_0^1(\Omega))$, $\theta \in C([0,T); L^p(\Omega)), \forall\ 1 \leq p < \infty$, and it holds that*

$$\begin{aligned}
\int_\Omega U_0 \cdot \Phi(\mathbf{x}, 0) d\mathbf{x} + \int_0^T \int_\Omega \big(U \cdot \Phi_t + U \cdot (U \cdot \nabla\Phi) + \theta\phi_2& \\
- \nu\nabla\phi_1 \cdot \nabla u - \nu\nabla\phi_2 \cdot \nabla v \big) d\mathbf{x}dt &= 0, \\
\int_\Omega \theta_0 \psi(\mathbf{x}, 0) d\mathbf{x} + \int_0^T \int_\Omega \big(\theta\psi_t + \theta U \cdot \nabla\psi \big) d\mathbf{x}dt &= 0,
\end{aligned}$$

for any $\Phi = (\phi_1, \phi_2) \in C_0^\infty(\Omega \times [0,T])^2$ satisfying $\Phi(\mathbf{x}, T) = 0$ and $\nabla \cdot \Phi = 0$, and for any $\psi \in C_0^\infty(\Omega \times [0,T])$ satisfying $\psi(\mathbf{x}, T) = 0$.

Definition 4.1.2. *(U, θ) is said to be a global weak solution of (4.1.1) and (4.1.3), if for any $T > 0$, $U \in C([0,T); H^1(\Omega))$, $\theta \in C([0,T); L^2(\Omega)) \cap L^2([0,T); H^1(\Omega))$, and it holds that*

$$\begin{aligned}
\int_\Omega U_0 \cdot \Phi(\mathbf{x}, 0) d\mathbf{x} + \int_0^T \int_\Omega \big(U \cdot \Phi_t + U \cdot (U \cdot \nabla\Phi) + \theta\mathbf{e}_2 \cdot \Phi \big) d\mathbf{x}dt = 0, \\
\int_\Omega \theta_0 \psi(\mathbf{x}, 0) d\mathbf{x} + \int_0^T \int_\Omega \big(\theta\psi_t + \theta U \cdot \nabla\psi - \nabla\theta \cdot \nabla\psi \big) d\mathbf{x}dt = 0,
\end{aligned}$$

for any $\Phi = (\phi_1, \phi_2) \in C^\infty(\Omega \times [0,T])^2$ satisfying $\Phi(\mathbf{x}, T) = 0$, $\nabla \cdot \Phi = 0$ and $\Phi \cdot \mathbf{n}|_{\partial\Omega} = 0$, and for any $\psi \in C_0^\infty(\Omega \times [0,T])$ satisfying $\psi(\mathbf{x}, T) = 0$.

We then build up the regularity of the solution by energy estimate under the initial and boundary conditions. The energy estimate is somewhat delicate mainly due to the coupling between the velocity and density equations by convection and gravitational force and the boundary effects. Great efforts have been made to simplify the proof. Current proof involves intensive applications of Sobolev embeddings and we will see that the Ladyzhenskaya's inequalities play a crucial role in the estimation of the solutions. The results on Stokes equations by Temam [97] and classical results on elliptic equations [2] are important in our energy framework. These are mainly due to the problem is set on the bounded domain, distinguishing itself from the Cauchy problem in [41] and [15]. Roughly speaking, because of the lack of the spatial derivatives of the solution at the boundary, our energy framework proceed as follows: We first apply the standard energy estimate on the solution and the temporal derivatives of the solution. We then apply the Temam's results on Stokes equation and the results on elliptic equations to obtain the spatial derivatives. Such a process will be repeated up to third order, and then the carefully coupled estimates will be composed into a desired estimate leading to global regularity, large-time behavior and uniqueness of the solutions. These results suggest that either the viscous dissipation or the heat diffusion is strong enough to compensate the effects of gravitational force and nonlinear convection in order to prevent the development of singularity of the system. It should be pointed out that in the theorems obtained above, no smallness restriction is put upon the initial data which is a major difference from Theorem 3.1.1.

4.2 Preliminaries and Weak Solutions

We first list several facts which will be used in the proofs of Theorems 4.1.1 and 4.1.2. Then we prove the global existence of weak solutions. First we recall some Sobolev and Ladyzhenskaya type inequalities which are well-known and standard (c.f. [97]).

Lemma 4.2.1. *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with C^1 smooth boundary. Then the following embeddings and inequalities hold:*

(i) $H^1(\Omega) \hookrightarrow L^p(\Omega), \quad \forall\ 1 < p < \infty;$

(ii) $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega), \quad \forall\ 2 < p < \infty;$

(iii) $\|f\|_{L^4}^2 \le 2\|f\|\|\nabla f\|, \quad \forall\ f : \Omega \to \mathbb{R}$ and $f \in H_0^1(\Omega);$

(iv) $\|f\|_{L^4}^2 \le C\big(\|f\|\|\nabla f\| + \|f\|^2\big),\ \forall\ f : \Omega \to \mathbb{R}$ and $f \in H^1(\Omega);$

(v) $\|f\|_{L^8}^2 \le C\big(\|f\|\|\nabla f\|_{L^4} + \|f\|^2\big),\ \forall\ f : \Omega \to \mathbb{R}$ and $f \in W^{1,4}(\Omega).$

We then recall some useful results from [97] on Stokes equations which will be used in the proof of Theorem 4.1.1.

Lemma 4.2.2. *Let Ω be any open bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$. Consider the Stokes problem*

$$\begin{cases} -\nu\Delta U + \nabla P = f \quad in\ \Omega \\ \nabla \cdot U = 0 \quad in\ \Omega \\ U = 0 \quad on\ \partial\Omega. \end{cases}$$

If $f \in W^{m,p}$, then $U \in W^{m+2,p}, P \in W^{m+1,p}$ and there exists a constant $c_0 = c_0(p, \nu, m, \Omega)$ such that

$$\|U\|_{W^{m+2,p}} + \|P\|_{W^{m+1,p}} \le c_0\|f\|_{W^{m,p}}$$

for any $p \in (1, \infty)$ and the integer $m \ge -1$.

Now we collect several facts which will be used in the proof of Theorem 4.1.2. First, we recall some classical result on elliptic equations (c.f. [2]).

Lemma 4.2.3. *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Consider the Dirichlet problem:*

$$\begin{cases} \kappa\Delta\Theta = f \quad in\ \Omega, \\ \Theta = 0 \quad on\ \partial\Omega. \end{cases}$$

If $f \in W^{m,p}$, *then* $\Theta \in W^{m+2,p}$ *and there exists a constant* $C = C(p, \kappa, m, \Omega)$ *such that*

$$\|\Theta\|_{W^{m+2,p}} \leq C\|f\|_{W^{m,p}}$$

for any $p \in (1, \infty)$ *and the integer* $m \geq -1$.

The next three lemmas are useful in the estimation of the velocity field.

Lemma 4.2.4. *Let* $\Omega \subset \mathbb{R}^2$ *be any bounded domain with smooth boundary* $\partial\Omega$, *and let* $U \in W^{s,p}(\Omega)$ *be a vector-valued function satisfying* $\nabla \cdot U = 0$ *and* $U \cdot \mathbf{n}|_{\partial\Omega} = 0$, *where* $\mathbf{n}$ *is the unit outward normal to* $\partial\Omega$. *Then there exists a constant* $C = C(s, p, \Omega)$ *such that*

$$\|U\|_{W^{s,p}} \leq C(\|\nabla \times U\|_{W^{s-1,p}} + \|U\|_{L^p})$$

for any $s \geq 1$ *and* $p \in (1, \infty)$.

The following lemma is standard and can be found in [70].

Lemma 4.2.5. *Let* $\Omega \subset \mathbb{R}^2$ *be any open bounded domain with smooth boundary* $\partial\Omega$. *Then for any multiindex* β *with order* $|\beta| \geq 3$ *and any functions* $f \in H^{|\beta|}(\Omega), g \in H^{|\beta|-1}(\Omega)$, *it holds that*

$$\|D^\beta(fg) - fD^\beta g\| \leq C\big(\|\nabla f\|_{L^\infty}\|g\|_{H^{|\beta|-1}} + \|f\|_{H^{|\beta|}}\|g\|_{L^\infty}\big), \tag{4.2.1}$$

for some constant $C = C(|\beta|, \Omega)$.

Concerning the 2D incompressible Euler equations, the following lemma can be found in [57] and [61].

Lemma 4.2.6. *Let* $\Omega \subset \mathbb{R}^2$ *be any bounded domain with smooth boundary* $\partial\Omega$. *Consider the initial-boundary value problem:*

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = G, \\ \nabla \cdot U = 0, \\ U(\mathbf{x}, 0) = U_0(\mathbf{x}), \quad U \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{4.2.2}$$

where $\mathbf{n}$ *is the unit outward normal to* $\partial\Omega$*. For any fixed* $T > 0$*, let* $U_0(\mathbf{x}) \in C^{1+\gamma}(\bar{\Omega})$*,* $\nabla \cdot U_0(\mathbf{x}) = 0$*,* $U_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$*, and let* $G \in C([0,T]; C^{1+\gamma}(\bar{\Omega}))$*, where* $0 < \gamma < 1$*. Then there exists a solution* (U, P) *to (4.2.2) such that* $(U, P) \in C^1(\bar{\Omega} \times [0,T])$*.*

Now, we establish the global existence of weak solutions of (4.1.1)–(4.1.2) and (4.1.1) and (4.1.3). Indeed, we have

Lemma 4.2.7. *Under the assumptions in Theorem 4.1.1, there exists a global weak solution* (U, θ) *of (4.1.1)–(4.1.2) such that, for any* $T > 0$*,* $U \in C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega))$*, and* $\theta \in C([0,T]; L^p(\Omega)), \forall\ 1 \le p < \infty$*.*

Lemma 4.2.8. *Under the assumptions of Theorem 4.1.2, there exists a global weak solution* (U, Θ) *of (4.1.1) and (4.1.3) such that, for any* $T > 0$*,* $U \in C([0,T]; H^1(\Omega))$*, and* $\Theta \in C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega))$*.*

We prove the above lemmas by a fixed point argument and the method of energy estimate. To explain the heart of the matter, we only give the proof of Lemma 4.2.7. Lemma 4.2.8 can be proved in a similar fashion.

Proof of Lemma 4.2.7. Following [61], we prove the lemma by a fixed point argument. To do so, we fix any $T \in [0, \infty)$ and consider the problem (4.1.1)–(4.1.2) in $\Omega \times [0,T]$. Let B be the closed convex set in $C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega))$ defined by

$$\begin{aligned} B = \big\{ & V = (v_1, v_2) \in C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega)) \big| \\ & \nabla \cdot V = 0, \text{ a.e. on } \Omega \times (0,T),\ \|V\|^2_{C([0,T];L^2(\Omega))} + \|V\|^2_{L^2([0,T];H_0^1(\Omega))} \le R_0 \big\}, \end{aligned} \tag{4.2.3}$$

where R_0 will be determined later. For fixed $\varepsilon \in (0,1)$ and any $V \in B$, we first mollify V using the standard procedure (c.f. [61]) to get

$$V_\varepsilon = \overline{V}_\varepsilon * \eta_{\varepsilon/2},$$

where $\overline{V}_\varepsilon$ is the truncation of V in $\Omega_\varepsilon = \{\mathbf{x} \in \Omega \mid \text{dist}(\mathbf{x}, \partial\Omega) > \varepsilon\}$ (extended by 0 to

Ω), and $\eta_{\varepsilon/2}$ is the standard mollifier. Then V_ε satisfies

$$
\begin{aligned}
&V_\varepsilon \in C([0,T]; C_0^\infty(\bar{\Omega})),\ \nabla \cdot V_\varepsilon = 0,\\
&\|V_\varepsilon\|_{C([0,T];L^2(\Omega))} \le C\|V\|_{C([0,T];L^2(\Omega))},\\
&\|V_\varepsilon\|_{L^2([0,T];H_0^1(\Omega))} \le C\|V\|_{L^2([0,T];H_0^1(\Omega))},
\end{aligned}
\tag{4.2.4}
$$

for some constant $C > 0$ which is independent of ε. Similarly, we regularize the initial data to obtain the smooth approximation $\theta_0^\varepsilon(\mathbf{x})$ for $\theta_0(\mathbf{x})$ and $U_0^\varepsilon(\mathbf{x})$ for $U_0(\mathbf{x})$ respectively, such that

$$
\begin{aligned}
&\theta_0^\varepsilon(\mathbf{x}) \in C_0^\infty(\bar{\Omega}), \quad \|\theta_0^\varepsilon(\mathbf{x}) - \theta_0(\mathbf{x})\|_{H^1(\Omega)} < \varepsilon,\\
&U_0^\varepsilon(\mathbf{x}) \in C_0^\infty(\bar{\Omega}), \quad \nabla \cdot U_0^\varepsilon(\mathbf{x}) = 0 \text{ and } \|U_0^\varepsilon(\mathbf{x}) - U_0(\mathbf{x})\|_{H^1(\Omega)} < \varepsilon.
\end{aligned}
$$

Then we solve the transport equation with smooth initial data

$$
\begin{cases}
\theta_t + V_\varepsilon \cdot \nabla\theta = 0,\\
\theta(\mathbf{x}, 0) = \theta_0^\varepsilon(\mathbf{x}),
\end{cases}
\tag{4.2.5}
$$

and we denote the solution by θ^ε. Next, we solve the nonhomogeneous (linearized) Navier-Stokes equation with smooth initial data

$$
\begin{cases}
\nabla \cdot U = 0\\
U_t + V_\varepsilon \cdot \nabla U + \nabla P = \nu\Delta U + \theta^\varepsilon \mathbf{e}_2,\\
U|_{\partial\Omega} = 0, \quad U(\mathbf{x}, 0) = U_0^\varepsilon(\mathbf{x}),
\end{cases}
\tag{4.2.6}
$$

and denote the solution by U^ε and the corresponding pressure by P^ε. Then we define the mapping $F_\varepsilon(V) = U^\varepsilon$. The solvabilities of (4.2.5) and (4.2.6) follow easily from [61]. Next, we prove that F_ε satisfies the conditions of Schauder fixed point theorem, i.e., $F_\varepsilon : B \to B$ is continuous and compact. These will be achieved by the method of energy estimate.

We start from (4.2.5). For any $2 \le p < \infty$, multiplying $(4.2.5)_1$ by $\theta|\theta|^{p-2}$ and integrating the resulting equation over Ω by parts, we get

$$
\|\theta(\cdot, t)\|_{L^p} = \|\theta_0^\varepsilon\|_{L^p} \le \|\theta_0\|_{L^p} + \varepsilon c(\Omega, p), \quad \forall\ 0 \le t \le T,\ \forall\ 0 < \varepsilon < 1,
$$

i.e.,

$$\|\theta^\varepsilon(\cdot,t)\|_{L^p} = \|\theta_0^\varepsilon\|_{L^p} \le \|\theta_0\|_{L^p} + \varepsilon c(\Omega,p), \quad \forall\ 0 \le t \le T,\ \forall\ 0 < \varepsilon < 1, \tag{4.2.7}$$

where $c(\Omega,p)$ is a constant depending only on Ω and p. We then estimate $\|U^\varepsilon\|^2_{L^2([0,T];H^1_0(\Omega))}$. Taking L^2 inner product of $(4.2.6)_2$ with U, after integrating by parts and using Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \nu\|\nabla U\|^2 \le C(\delta)\|\theta^\varepsilon\|^2 + \delta\|U\|^2, \tag{4.2.8}$$

where δ is a constant to be determined. Since U satisfies the no-slip boundary condition, Poincaré's inequality implies that $\|U\| \le C\|\nabla U\|$ for some constant C depending only on Ω. Choosing $\delta = \nu/2C$ in (12) we obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \frac{\nu}{2}\|\nabla U\|^2 \le C\|\theta^\varepsilon\|^2, \tag{4.2.9}$$

which together with (4.2.7) yields, after integration over $[0,T]$, that

$$\|U\|^2_{C([0,T];L^2(\Omega))} + \nu\|\nabla U\|^2_{L^2([0,T];L^2(\Omega))} \le CT(\|\theta_0\|^2 + \varepsilon) + (\|U_0\|^2 + \varepsilon).$$

Since $0 < \varepsilon < 1$, we have

$$\|U\|^2_{C([0,T];L^2(\Omega))} + \|U\|^2_{L^2([0,T];H^1_0(\Omega))} \le C(T,\theta_0,U_0,\nu,\Omega),$$

i.e.,

$$\|U^\varepsilon\|^2_{C([0,T];L^2(\Omega))} + \|U^\varepsilon\|^2_{L^2([0,T];H^1_0(\Omega))} \le C(T,\theta_0,U_0,\nu,\Omega). \tag{4.2.10}$$

Choosing R_0 such that $R_0 \ge C(T,\theta_0,U_0,\nu,\Omega)$ we see that F_ε maps B into B for any $0 < \varepsilon < 1$. We remark that the constant $C(T,\theta_0,U_0,\nu,\Omega)$ in (4.2.10) does not depend on ε.

Next we prove the compactness of F_ε. For this purpose, we continue to find estimates of $\|\nabla U^\varepsilon\|^2_{C([0,T];L^2(\Omega))}$ and $\|U^\varepsilon_t\|^2_{L^2([0,T];L^2(\Omega))}$. Taking L^2 inner product of

$(4.2.6)_2$ with U_t, one has

$$\begin{aligned}\frac{\nu}{2}\frac{d}{dt}\|\nabla U\|^2+\|U_t\|^2 &\le \int_\Omega |V_\varepsilon||U_t||\nabla U|d\mathbf{x}+\int_\Omega \theta\mathbf{e}_2\cdot U_t d\mathbf{x}\\ &\le \frac{1}{4}\|U_t\|^2+\|V_\varepsilon\nabla U\|^2+\frac{1}{4}\|U_t\|^2+\|\theta\|^2\\ &\le \frac{1}{2}\|U_t\|^2+\|V_\varepsilon\|_{L^\infty}^2\|\nabla U\|^2+C\end{aligned}$$

which implies that

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla U\|^2+\frac{1}{2}\|U_t\|^2\le\|V_\varepsilon\|_{L^\infty}^2\|\nabla U\|^2+C. \tag{4.2.11}$$

Applying Gronwall's inequality to (4.2.11) and using (4.2.4) we have

$$\|\nabla U\|^2_{C([0,T];L^2(\Omega))}+\|U_t\|^2_{L^2([0,T];L^2(\Omega))}\le C. \tag{4.2.12}$$

By Lemma 4.2.2 we know that

$$\begin{aligned}\|U\|_{H^2}&\le C\big(\|U_t\|+\|\theta\|+\|V_\varepsilon\cdot\nabla U\|\big)\\ &\le C\big(\|U_t\|+C+C\|V_\varepsilon\|_{L^\infty}\|\nabla U\|\big),\end{aligned} \tag{4.2.13}$$

which together with (4.2.12) yields

$$\|U^\varepsilon\|^2_{L^2([0,T];H^2(\Omega))}\le C. \tag{4.2.14}$$

From (4.2.12) and (4.2.14) we know that F_ε is compact by Sobolev embedding theorem.

Now we prove the continuity of F_ε. Let $F_\varepsilon(V_i)=U_i^\varepsilon$, by definition we know

$$\begin{cases}\theta_{it}^\varepsilon+V_{i\varepsilon}\cdot\nabla\theta_i^\varepsilon=0,\\ U_{it}^\varepsilon+V_{i\varepsilon}\cdot\nabla U_i^\varepsilon+\nabla P_i^\varepsilon=\nu\Delta U_i^\varepsilon+\theta_i^\varepsilon\mathbf{e}_2,\\ \nabla\cdot U_i^\varepsilon=0,\quad U_i^\varepsilon|_{\partial\Omega}=0,\\ (\theta_i^\varepsilon,U_i^\varepsilon)(\mathbf{x},0)=(\theta_0^\varepsilon,U_0^\varepsilon)(\mathbf{x}),\quad i=1,2.\end{cases}$$

Subtracting the equation for $i = 2$ from the one for $i = 1$ we have

$$\begin{cases} \varrho^\varepsilon_t + V_{1\varepsilon} \cdot \nabla \varrho^\varepsilon + W_\varepsilon \cdot \nabla \theta^\varepsilon_2 = 0, \\ \chi^\varepsilon_t + V_{1\varepsilon} \cdot \nabla \chi^\varepsilon + W_\varepsilon \cdot \nabla U^\varepsilon_2 + \nabla Q^\varepsilon = \nu \Delta \chi^\varepsilon + \varrho^\varepsilon \mathbf{e}_2, \\ \nabla \cdot \chi^\varepsilon = 0, \quad \chi^\varepsilon|_{\partial\Omega} = 0, \\ (\varrho^\varepsilon, \chi^\varepsilon)(\mathbf{x}, 0) = \mathbf{0}, \end{cases} \tag{4.2.15}$$

where $\varrho^\varepsilon = \theta^\varepsilon_1 - \theta^\varepsilon_2$, $W_\varepsilon = V_{1\varepsilon} - V_{2\varepsilon}$, $\chi^\varepsilon = U^\varepsilon_1 - U^\varepsilon_2$, and $Q^\varepsilon = P^\varepsilon_1 - P^\varepsilon_2$. Taking the L^2 inner products of $(4.2.15)_1$ with ϱ^ε and $(4.2.15)_2$ with χ^ε we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\varrho^\varepsilon\|^2 &= -\int_\Omega (W_\varepsilon \cdot \nabla \theta^\varepsilon_2)\varrho^\varepsilon d\mathbf{x}, \\ \frac{1}{2}\frac{d}{dt}\|\chi^\varepsilon\|^2 + \nu\|\nabla\chi^\varepsilon\|^2 &= -\int_\Omega (W_\varepsilon \cdot \nabla U^\varepsilon_2)\chi^\varepsilon d\mathbf{x} + \int_\Omega \varrho^\varepsilon \mathbf{e}_2 \cdot \chi^\varepsilon d\mathbf{x}. \end{aligned} \tag{4.2.16}$$

Since $\theta^\varepsilon_2 \in C([0,T]; C^\infty(\bar{\Omega}))$, we get from $(4.2.16)_1$ that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\varrho^\varepsilon\|^2 &\le \|\nabla\theta^\varepsilon_2\|_{L^\infty}\|W_\varepsilon\|\|\varrho^\varepsilon\| \\ &\le C(\|W_\varepsilon\|^2 + \|\varrho^\varepsilon\|^2), \end{aligned}$$

from which we get

$$\begin{aligned} \|\varrho^\varepsilon\|^2 &\le e^{CT}\int_0^T \|W_\varepsilon\|^2 d\tau \\ &\le C\|W_\varepsilon\|^2_{C([0,T];L^2(\Omega))}. \end{aligned} \tag{4.2.17}$$

Since $U^\varepsilon_2 \in L^2([0,T]; H^2(\Omega))$, we derive from $(4.2.16)_2$:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\chi^\varepsilon\|^2 + \nu\|\nabla\chi^\varepsilon\|^2 &\le \|W_\varepsilon\|\|\nabla U^\varepsilon_2\|_{L^4}\|\chi^\varepsilon\|_{L^4} + \|\varrho^\varepsilon\|\|\chi^\varepsilon\| \\ &\le C\|W_\varepsilon\|\|U^\varepsilon_2\|_{H^2}\|\chi^\varepsilon\|_{H^1} + \|\varrho^\varepsilon\|\|\chi^\varepsilon\| \\ &\le C\|W_\varepsilon\|\|U^\varepsilon_2\|_{H^2}\|\nabla\chi^\varepsilon\| + \|\varrho^\varepsilon\|\|\chi^\varepsilon\| \\ &\le C\|W_\varepsilon\|^2\|U^\varepsilon_2\|^2_{H^2} + \frac{\nu}{2}\|\nabla\chi^\varepsilon\|^2 + \frac{1}{2}\|\varrho^\varepsilon\|^2 + \frac{1}{2}\|\chi^\varepsilon\|^2 \\ &\le C(t)\|W_\varepsilon\|^2_{C([0,T];L^2(\Omega))} + \frac{\nu}{2}\|\nabla\chi^\varepsilon\|^2 + \frac{1}{2}\|\chi^\varepsilon\|^2, \end{aligned} \tag{4.2.18}$$

where $\int_0^T C(\tau)d\tau \le C$ and we have used (4.2.17). From (4.2.18) we get

$$\frac{1}{2}\frac{d}{dt}\|\chi^\varepsilon\|^2 + \frac{\nu}{2}\|\nabla\chi^\varepsilon\|^2 \le C(t)\|W_\varepsilon\|^2_{C([0,T];L^2(\Omega))} + \frac{1}{2}\|\chi^\varepsilon\|^2, \tag{4.2.19}$$

which implies, after applying Gronwall's inequality, that

$$\|\chi^\varepsilon\|^2 \le C\|W_\varepsilon\|^2_{C([0,T];L^2(\Omega))}. \tag{4.2.20}$$

Integrating (4.2.19) over $[0,T]$ using (4.2.20) we have

$$\int_0^T \|\nabla\chi^\varepsilon\|^2 d\tau \le C\|W_\varepsilon\|^2_{C([0,T];L^2(\Omega))}. \tag{4.2.21}$$

Combining (4.2.20) and (4.2.21) we get

$$\|\chi^\varepsilon\|^2_{C([0,T];L^2(\Omega))} + \|\chi^\varepsilon\|^2_{L^2([0,T];H^1_0(\Omega))} \le C\|V_1 - V_2\|^2_{C([0,T];L^2(\Omega))},$$

i.e.,

$$\|U_1^\varepsilon - U_2^\varepsilon\|^2_B \le C\|V_1 - V_2\|^2_B,$$

where $\|\cdot\|^2_B = \|\cdot\|^2_{C([0,T];L^2(\Omega))} + \|\cdot\|^2_{L^2([0,T];H^1_0(\Omega))}$. By definition we know

$$\|F_\varepsilon(V_1) - F_\varepsilon(V_2)\|^2_B \le C\|V_1 - V_2\|^2_B,$$

which implies that $F_\varepsilon : B \to B$ is continuous.

Therefore, Schauder theorem implies that for any fixed $\varepsilon \in (0,1)$, there exists $U^\varepsilon \in B$ such that $F_\varepsilon(U^\varepsilon) = U^\varepsilon$, namely,

$$\begin{cases} \theta^\varepsilon + U_\varepsilon \cdot \nabla\theta^\varepsilon = 0 \\ U_t^\varepsilon + U_\varepsilon \cdot \nabla U^\varepsilon + \nabla P^\varepsilon = \nu\Delta U^\varepsilon + \theta^\varepsilon \mathbf{e}_2, \\ \nabla \cdot U^\varepsilon = 0, \\ U^\varepsilon|_{\partial\Omega} = 0, \quad (\theta^\varepsilon, U^\varepsilon)(\mathbf{x},0) = (\theta_0^\varepsilon, U_0^\varepsilon)(\mathbf{x}), \end{cases}$$

where U_ε is the regularization of U^ε. By a bootstrap argument (c.f. [61]) we know that $(\theta^\varepsilon, U^\varepsilon) \in C^\infty(\bar\Omega \times [0,T])$. Then it is obvious that $(\theta^\varepsilon, U^\varepsilon)$ satisfy the integral identities, i.e.,

$$\begin{aligned} 0 &= \int_\Omega U_0^\varepsilon \cdot \Phi(\mathbf{x},0)d\mathbf{x} \\ &\quad + \int_0^T \int_\Omega \big(U^\varepsilon \cdot \Phi_t + U_\varepsilon \cdot (U^\varepsilon \cdot \nabla\Phi) + \theta^\varepsilon \mathbf{e}_2 \cdot \Phi - \nu\nabla\phi_1 \cdot \nabla u^\varepsilon - \nu\nabla\phi_2 \cdot \nabla v^\varepsilon\big)d\mathbf{x}dt, \\ 0 &= \int_\Omega \theta_0^\varepsilon \psi(\mathbf{x},0)d\mathbf{x} + \int_0^T \int_\Omega \big(\theta^\varepsilon\psi_t + \theta^\varepsilon U_\varepsilon \cdot \nabla\psi\big)d\mathbf{x}dt, \end{aligned} \tag{4.2.22}$$

for any $\varepsilon > 0$, $\Phi = (\phi_1, \phi_2) \in C_0^\infty(\bar{\Omega} \times [0,T])^2$ satisfying $\Phi(\mathbf{x}, T) = 0$ and $\nabla \cdot \Phi = 0$, and for any $\psi \in C^\infty(\bar{\Omega} \times [0,T])$ satisfying $\psi(\mathbf{x}, T) = 0$.

In view of (4.2.7), (4.2.10) and from the definition of U_ε we know that there exist functions $U \in B$ and $\theta \in C([0,T]; L^p(\Omega)),\ \forall\ 2 \leq p < \infty$ such that as $\varepsilon \to 0$,

$$\begin{aligned}
&U_\varepsilon \rightharpoonup U \quad \text{weakly in } C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega)),\\
&U^\varepsilon \rightharpoonup U \quad \text{weakly in } C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega)),\\
&\theta^\varepsilon \rightharpoonup \theta \quad \text{weakly in } C([0,T]; L^p(\Omega)),\ \forall\ 2 \leq p < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\|U\|^2_{C([0,T];L^2(\Omega))} + \|U\|^2_{L^2([0,T];H_0^1(\Omega))} \leq C(T, \theta_0, U_0, \nu, \Omega),\\
&\|\theta\|_{C([0,T];L^p(\Omega)} \leq \|\theta_0\|_{C([0,T];L^p(\Omega)}, \quad \forall\ 2 \leq p < \infty.
\end{aligned} \tag{4.2.23}$$

Since

$$U \cdot \nabla\psi \in C([0,T]; L^2(\Omega)),$$

we have

$$\begin{aligned}
&\left| \int_0^T \int_\Omega (\theta^\varepsilon U^\varepsilon \cdot \nabla\psi - \theta U \cdot \nabla\psi)\ d\mathbf{x}dt \right|\\
\leq& C\|\theta^\varepsilon\|_{L^2([0,T];L^2(\Omega))} \|U^\varepsilon - U\|_{L^2([0,T];L^2(\Omega))} + \left| \int_0^T \int_\Omega (\theta^\varepsilon U \cdot \nabla\psi - \theta U \cdot \nabla\psi)\ d\mathbf{x}dt \right|\\
\leq& C\|U^\varepsilon - U\|_{L^2([0,T];L^2(\Omega))} + \left| \int_0^T \int_\Omega (\theta^\varepsilon - \theta) U \cdot \nabla\psi d\mathbf{x}dt \right|\\
&\to 0, \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Moreover, since

$$\begin{aligned}
&\left| \int_0^T \int_\Omega [U_\varepsilon \cdot (U^\varepsilon \cdot \nabla\Phi) - U \cdot (U \cdot \nabla\Phi)]\ d\mathbf{x}dt \right|\\
=& \left| \int_0^T \int_\Omega [U_\varepsilon \cdot (U^\varepsilon \cdot \nabla\Phi) - U_\varepsilon \cdot (U \cdot \nabla\Phi) + U_\varepsilon \cdot (U \cdot \nabla\Phi) - U \cdot (U \cdot \nabla\Phi)]\ d\mathbf{x}dt \right|\\
\leq& C \int_0^T \int_\Omega (|U_\varepsilon||U^\varepsilon - U| + |U||U_\varepsilon - U|)\ d\mathbf{x}dt\\
\leq& C\big(\|U_\varepsilon\|_{L^2([0,T];L^2(\Omega))} \|U_\varepsilon - U\|_{L^2([0,T];L^2(\Omega))} + \|U\|_{L^2([0,T];L^2(\Omega))} \|U_\varepsilon - U\|_{L^2([0,T];L^2(\Omega))}\big)\\
\leq& C\|U_\varepsilon - U\|_{L^2([0,T];L^2(\Omega))} \to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

letting $\varepsilon \to 0$ in (4.2.22) we verified that (θ, U) is a weak solution to (4.1.1)–(4.1.2) in $\Omega \times [0,T]$. We conclude the argument by noticing that T is arbitrary. This combining with (4.2.23) completes the proof of Lemma 4.2.7.

4.3 Viscous Boussinesq Equations

Now we build up the regularity and uniqueness of the solution obtained in Lemma 4.2.7, and therefore give proof of Theorems 4.1.1. The following theorem gives the key estimates.

Theorem 4.3.1. *Under the assumption of Theorem 4.1.1, the solution obtained in Lemma 4.2.7 satisfies the following estimates:*

$$\|U\|_{C([0,T);H^3(\Omega))} + \|U\|_{L^2([0,T);H^4(\Omega))} + \|\theta\|_{C([0,T);H^3(\Omega))} \le C,$$

for any $T > 0$. Moreover, there exists a constant $\bar{C} > 0$ independent of t such that

$$\|U(\cdot,t)\|^2 \le \max\left\{\|U(\cdot,0)\|^2\ ,\ \frac{\bar{C}^2}{\nu^2}\|\theta(\cdot,0)\|^2\right\}, \quad \forall\ t \ge 0. \tag{4.3.1}$$

Remark 4.3.1. *The constant $\bar{C}$ in the theorem is actually the constant of Poincaré's inequality on the domain Ω. Therefore, it depends only on Ω. See the proof of Lemma 4.3.9 below for details.*

The proof of Theorem 4.3.1 is based on several steps of careful energy estimates which are stated as a sequence of lemmas. First, we observe that the same method used to derive (4.2.7) can be applied to $(4.1.1)_2$ if V_ε is replaced by U in (4.2.5). Therefore, we have the conservation of L^p norm for θ, i.e., for any $p \in [2,\infty)$, it holds that

$$\|\theta(\cdot,t)\|_{L^p} = \|\theta_0\|_{L^p}, \quad \forall\ t \ge 0.$$

Furthermore, by letting $p \to \infty$ in the above estimate, one has

$$\|\theta(\cdot,t)\|_{L^\infty} = \|\theta_0\|_{L^\infty}, \quad \forall\ t \ge 0.$$

Fix any $T > 0$. In the rest part of this section, the time is restricted to be within the interval $[0, T]$ until specified otherwise. Then we start with estimates of $\|U\|^2_{C([0,T];L^2(\Omega))}$ and $\|\nabla U\|^2_{L^2([0,T];L^2(\Omega))}$.

Lemma 4.3.1. *Under the assumptions of Theorem 4.1.1, it holds that*

$$\|U\|^2_{C([0,T];L^2(\Omega))} \le C \quad \text{and} \quad \|\nabla U\|^2_{L^2([0,T];L^2(\Omega))} \le C. \tag{4.3.2}$$

Proof. Taking L^2 inner product of $(4.1.1)_1$ with U, we obtain, after integration by parts, that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|U\|^2 + \nu\|\nabla U\|^2 &= -\int_\Omega (U\cdot\nabla U)\cdot U d\mathbf{x} + \int_\Omega \theta\mathbf{e}_2\cdot U d\mathbf{x} \\
&= -\frac{1}{2}\int_\Omega U\cdot\nabla(|U|^2)d\mathbf{x} + \int_\Omega \theta\mathbf{e}_2\cdot U d\mathbf{x} \\
&= -\frac{1}{2}\int_\Omega \nabla\cdot(U|U|^2)d\mathbf{x} + \int_\Omega \theta\mathbf{e}_2\cdot U d\mathbf{x} \\
&= \int_\Omega \theta\mathbf{e}_2\cdot U d\mathbf{x}.
\end{aligned}$$

Applying Cauchy-Schwartz inequality to the RHS of the above equality, we get

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \nu\|\nabla U\|^2 \le \frac{1}{2}\|\theta\|^2 + \frac{1}{2}\|U\|^2. \tag{4.3.3}$$

By dropping $\nu\|\nabla U\|^2$ from (4.3.3) and then applying Gronwall's inequality to the resulting inequality, we find that

$$\begin{aligned}
\|U(\cdot,t)\|^2 &\le e^t\Big(\|U_0\|^2 + \int_0^t \|\theta_0\|^2 d\tau\Big) \\
&\qquad \le e^T\big(\|U_0\|^2 + T\|\theta_0\|^2\big) \le C, \quad \forall\ t\in[0,T],
\end{aligned}$$

which also implies, after integrating (4.3.3) over $[0, T]$, that

$$\nu\int_0^T \|\nabla U(\cdot,\tau)\|^2 d\tau \le C.$$

This completes the proof of Lemma 4.3.1.

The next Lemma is dealing with $\|\nabla U\|^2_{C([0,T];L^2(\Omega))}$ and $\|U_t\|^2_{L^2([0,T];L^2(\Omega))}$.

Lemma 4.3.2. *Under the assumptions of Theorem 4.1.1, it holds that*

$$\|\nabla U\|^2_{C([0,T];L^2(\Omega))} \leq C \quad and \quad \|U_t\|^2_{L^2([0,T];L^2(\Omega))} \leq C.$$

Proof. Taking L^2 inner product of $(4.1.1)_1$ with U_t, integrating the resulting equations over Ω by parts, we get

$$\begin{aligned}\frac{\nu}{2}\frac{d}{dt}\|\nabla U\|^2 + \|U_t\|^2 &\leq \int_\Omega |U||U_t||\nabla U| \, d\mathbf{x} + \int_\Omega \theta v_t \, d\mathbf{x} \\ &\leq C\|U\|^2_{L^4}\|\nabla U\|^2_{L^4} + \frac{1}{4}\|U_t\|^2 + C\|\theta_0\|^2,\end{aligned} \tag{4.3.4}$$

where we have used Hölder's inequality and Cauchy-Schwartz inequality as follows:

$$\int_\Omega |U||U_t||\nabla U| \, d\mathbf{x} \leq C\|U\|^2_{L^4}\|\nabla U\|^2_{L^4} + \frac{1}{8}\|U_t\|^2,$$

and

$$\int_\Omega \theta v_t \, d\mathbf{x} \leq \frac{1}{8}\|U_t\|^2 + C\|\theta_0\|^2.$$

We now apply the Ladyzhenskaya's inequality to estimate $\|U\|^2_{L^4}\|\nabla U\|^2_{L^4}$. Applying Lemma 4.2.1 (iii) on U and (iv) on ∇U, we have

$$\begin{aligned}\|U\|^2_{L^4}\|\nabla U\|^2_{L^4} &\leq C(\|U\|\|\nabla U\|)\big(\|\nabla U\|\|\nabla^2 U\| + \|\nabla U\|^2\big) \\ &\leq C\|\nabla U\|^2\|\nabla^2 U\| + C\|\nabla U\|^3 \\ &\leq C(\delta)\|\nabla U\|^4 + C\|\nabla U\|^3 + \delta\|U\|^2_{H^2},\end{aligned} \tag{4.3.5}$$

where we have used Lemma 4.3.1 and $\delta > 0$ is a small number to be determined. Therefore, we update (4.3.4) as

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla U\|^2 + \frac{3}{4}\|U_t\|^2 \leq C + C(\delta)\|\nabla U\|^4 + C\|\nabla U\|^3 + \delta\|U\|^2_{H^2}, \tag{4.3.6}$$

We now rewrite the equation $(4.1.1)_1$ as

$$-\nu\Delta U + \nabla P = -U_t - U\cdot\nabla U + \theta\mathbf{e}_2.$$

Lemma 4.2.2 with $m = 0$ and $p = 2$ implies that

$$\begin{aligned}\|U\|^2_{H^2} &\leq C\big(\|U_t\|^2 + \|\theta\|^2 + \|U\cdot\nabla U\|^2\big) \\ &\leq C\big(\|U_t\|^2 + C\big) + C\|U\|^2_{L^4}\|\nabla U\|^2_{L^4} \\ &\leq \widetilde{C}\big(C + \|U_t\|^2 + \|\nabla U\|^4 + \|\nabla U\|^3\big),\end{aligned} \tag{4.3.7}$$

where we have used (4.3.5). Now, choosing $\delta = 1/(4\widetilde{C})$ and combining (4.3.6) and (4.3.7), we get

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla U\|^2 + \frac{1}{2}\|U_t\|^2 \le C\big(\|\nabla U\|^4 + \|\nabla U\|^3\big) + C.$$

Therefore, Young's inequality yields

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla U\|^2 + \frac{1}{2}\|U_t\|^2 \le C\|\nabla U\|^2\|\nabla U\|^2 + C. \tag{4.3.8}$$

By dropping $\frac{1}{2}\|U_t\|^2$ from (4.3.8) we obtain

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla U\|^2 \le C\big(\|\nabla U\|^2\|\nabla U\|^2 + C\big). \tag{4.3.9}$$

Then using Lemma 4.3.1, Gronwall's inequality implies that

$$\|\nabla U(\cdot,t)\|^2 \le C, \quad \forall\, t \in [0,T]. \tag{4.3.10}$$

Using (4.3.10), after integrating (4.3.8) over $[0,T]$ we obtain

$$\int_0^T \|U_t(\cdot,\tau)\|^2 d\tau \le C, \tag{4.3.11}$$

which completes the proof of Lemma 4.3.2.

Next, we estimate $\|U_t\|^2_{C([0,T];L^2(\Omega))}$ and $\|\nabla U_t\|^2_{L^2([0,T];L^2(\Omega))}$.

Lemma 4.3.3. *Under the assumptions of Theorem 4.1.1, it holds that*

$$\|U_t\|^2_{C([0,T];L^2(\Omega))} \le C \quad \textit{and} \quad \|\nabla U_t\|^2_{L^2([0,T];L^2(\Omega))} \le C. \tag{4.3.12}$$

Proof. We take the temporal derivative of $(4.1.1)_1$ to get

$$U_{tt} + U_t \cdot \nabla U + U \cdot \nabla U_t + \nabla P_t = \nu \Delta U_t + \theta_t \vec{\mathbf{e}}_2. \tag{4.3.13}$$

Taking L^2 inner product of (4.3.13) with U_t we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U_t\|^2 + \nu\|\nabla U_t\|^2 &= -\int_\Omega (U_t\cdot\nabla U)\cdot U_t d\mathbf{x} + \int_\Omega \theta_t v_t d\mathbf{x}\\ &= -\int_\Omega (U_t\cdot\nabla U)\cdot U_t d\mathbf{x} - \int_\Omega (U\cdot\nabla\theta) v_t d\mathbf{x} \\ &\le \|U_t\|^2_{L^4}\|\nabla U\| + \int_\Omega \theta(U\cdot\nabla v_t) d\mathbf{x}.\end{aligned} \tag{4.3.14}$$

With the help of Lemma 4.3.1 and 4.3.2, and Lemma 4.2.1 (iii) on U_t, we note that

$$\begin{aligned}\|U_t\|_{L^4}^2\|\nabla U\| &\le C\|U_t\|_{L^4}^2\\ &\le C\|U_t\|\|\nabla U_t\|\\ &\le \frac{\nu}{4}\|\nabla U_t\|^2+C\|U_t\|^2.\end{aligned} \tag{4.3.15}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega \theta(U\cdot\nabla v_t)d\mathbf{x} &\le \|\theta\|_{L^\infty}\|U\|\|\nabla U_t\|\\ &\le \frac{\nu}{4}\|\nabla U_t\|^2+C.\end{aligned} \tag{4.3.16}$$

Therefore, combining (4.3.14)–(4.3.16), we arrive at

$$\frac{1}{2}\frac{d}{dt}\|U_t\|^2+\frac{\nu}{2}\|\nabla U_t\|^2\le C(\|U_t\|^2+1). \tag{4.3.17}$$

Using Gronwall's inequality, and Lemma 4.3.2, we obtain (4.3.12). This completes the proof of Lemma 4.3.3.

As an immediate consequence of Lemma 4.3.3 and Lemma 4.2.1 (i), one has

Lemma 4.3.4. *Under the assumptions of Theorem 4.1.1, it holds that*

$$\int_0^T \|U_t(\cdot,\tau)\|_{L^p}^2 d\tau\le C,\quad \forall\ 1\le p<\infty. \tag{4.3.18}$$

This lemma will play an important role on the estimations of the maximum norms of U and ∇U in the following Lemma.

Lemma 4.3.5. *Under the assumptions of Theorem 4.1.1, it holds that*

$$\|U\|_{C([0,T];L^\infty(\Omega))}^2\le C\quad and\quad \|\nabla U\|_{L^2([0,T];L^\infty(\Omega))}^2\le C. \tag{4.3.19}$$

Proof. We see that $\|U_t\|$ and $\|\nabla U\|$ are bounded by Lemmas 4.3.2 and 4.3.3. Therefore, one reads from (4.3.7) that

$$\|U\|_{H^2}^2\le C\big(\|U_t\|^2+\|\nabla U\|^3+\|\nabla U\|^4+C\big)\le C, \tag{4.3.20}$$

which implies, by Sobolev embedding,

$$\|U(\cdot,t)\|_{L^\infty}^2 \le C, \quad \forall\ t \in [0,T]. \tag{4.3.21}$$

As an immediate consequence of (4.3.20)–(4.3.21) we see that

$$\|U \cdot \nabla U\|_{H^1}^2 \le C\big(\|U\|_{L^\infty}^2 + \|U\|_{H^2}^2\big)\|U\|_{H^2}^2 \le C, \quad \forall\ t \in [0,T], \tag{4.3.22}$$

which implies by Lemma 4.2.1 (i) that

$$\|U \cdot \nabla U\|_{L^p}^2 \le C, \ \ \forall\ 1 \le p < \infty, \quad \forall\ t \in [0,T] \tag{4.3.23}$$

Therefore, using Lemma 4.2.2, (4.3.18) and (4.3.23) we obtain

$$\begin{aligned} \int_0^T \|U\|_{W^{2,p}}^2\ d\tau &\le C\int_0^T (\|U_t\|_{L^p}^2 + \|U\cdot\nabla U\|_{L^p}^2 + \|\theta\|_{L^p}^2)\ d\tau \\ &\le C, \ \ \forall\ 1 \le p < \infty. \end{aligned} \tag{4.3.24}$$

Applying Lemma 4.2.1 (ii) to ∇U we get the second half of (4.3.19) from (4.3.24) immediately. This completes the proof of Lemma 4.3.5.

In order to improve the regularity of U, the problem will involve the spatial derivatives of θ. We now establish the following lemma to estimate $\nabla\theta$.

Lemma 4.3.6. *Under the assumptions of Theorem 4.1.1, it holds that*

$$\|\nabla\theta(\cdot,t)\|_{L^\infty} \le C, \quad \forall\ t \in [0,T]. \tag{4.3.25}$$

Proof. For any $p \ge 2$, taking ∇ of $(4.1.1)_2$, dot multiplying the resulting equation with $|\nabla\theta|^{p-2}\nabla\theta$, after integration by parts we get

$$\frac{1}{p}\frac{d}{dt}\Big(\|\nabla\theta\|_{L^p}^p\Big) \le \|\nabla U\|_{L^\infty}\|\nabla\theta\|_{L^p}^p, \tag{4.3.26}$$

which yields

$$\frac{d}{dt}\Big(\|\nabla\theta\|_{L^p}\Big) \le \|\nabla U\|_{L^\infty}\|\nabla\theta\|_{L^p}. \tag{4.3.27}$$

Gronwall's inequality yields

$$\begin{aligned}\|\nabla\theta(\cdot,t)\|_{L^p} &\le \|\nabla\theta_0\|_{L^p}\exp\Big\{\int_0^T \|\nabla U\|_{L^\infty}d\tau\Big\} \\ &\le C, \quad \forall\ p\ge 2,\ and\ \forall\ t\in[0,T].\end{aligned} \tag{4.3.28}$$

Letting $p\to\infty$ we obtain (4.3.25). This completes the proof of Lemma 4.3.6.

The estimates of $\|\nabla U_t\|^2_{C([0,T];L^2(\Omega))}$ and $\|U_{tt}\|^2_{L^2([0,T];L^2(\Omega))}$ will be given in the next lemma, based on which we will establish the desired regularity stated in Theorem 4.3.1.

Lemma 4.3.7. *Under the assumptions of Theorem 4.1.1, it holds that*

$$\|\nabla U_t\|^2_{C([0,T];L^2(\Omega))}\le C, \quad and \quad \|U_{tt}\|^2_{L^2([0,T];L^2(\Omega))}\le C. \tag{4.3.29}$$

Proof. Taking L^2 inner product of (4.3.13) with U_{tt} we get

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla U_t\|^2+\|U_{tt}\|^2\le\int_\Omega(|U_{tt}||U_t||\nabla U|+|U_{tt}||U||\nabla U_t|+\theta_t v_{tt})\ d\mathbf{x}. \tag{4.3.30}$$

We now estimate the RHS term by term. First of all, we apply the Hölder inequality and Lemma 4.3.3 to obtain

$$\begin{aligned}\int_\Omega|U_{tt}||U_t||\nabla U|\ d\mathbf{x} &\le \frac{1}{6}\|U_{tt}\|^2+C\|\nabla U\|^2_{L^\infty}\|U_t\|^2 \\ &\le \frac{1}{6}\|U_{tt}\|^2+C\|\nabla U\|^2_{L^\infty},\ \forall\ t\in[0,T].\end{aligned} \tag{4.3.31}$$

Similarly, using Hölder inequality and Lemmas 4.3.5 and 4.3.6, we have the following estimates

$$\begin{aligned}\int_\Omega|U_{tt}||U||\nabla U_t|\ d\mathbf{x} &\le \frac{1}{6}\|U_{tt}\|^2+C\|U\|^2_{L^\infty}\|\nabla U_t\|^2 \\ &\le \frac{1}{6}\|U_{tt}\|^2+C\|\nabla U_t\|^2,\ \forall\ t\in[0,T],\end{aligned} \tag{4.3.32}$$

and

$$\begin{aligned}\int_\Omega|\theta_t v_{tt}|\ d\mathbf{x} &\le \frac{1}{6}\|U_{tt}\|^2+C\|\theta_t\|^2 \\ &\le \frac{1}{6}\|U_{tt}\|^2+C\|U\cdot\nabla\theta\|^2 \\ &\le \frac{1}{6}\|U_{tt}\|^2+C\|\nabla\theta\|^2_{L^\infty}\|U\|^2 \\ &\le \frac{1}{6}\|U_{tt}\|^2+C,\ \forall\ t\in[0,T].\end{aligned} \tag{4.3.33}$$

Substituting (4.3.31)–(4.3.33) into (4.3.30), one has

$$\frac{\nu}{2}\frac{d}{dt}\|\nabla U_t\|^2 + \frac{1}{2}\|U_{tt}\|^2 \leq C + C\|\nabla U\|_{L^\infty}^2 + C\|\nabla U_t\|^2. \tag{4.3.34}$$

We note that all the terms on the RHS of (4.3.34) are integrable in time due to Lemmas 4.3.3 and 4.3.5. Therefore, we integrate (4.3.34) in time over $[0,T]$ to obtain the estimates in (4.3.29). This completes the proof of Lemma 4.3.7.

We are now ready to complete the regularity stated in Theorem 4.3.1.

Lemma 4.3.8. *Under the assumptions of Theorem 4.1.1, it holds that*

$$\|(\theta, U)\|_{C([0,T];H^3(\Omega))}^2 \leq C, \quad \textit{and} \quad \|U\|_{L^2([0,T];H^4(\Omega))}^2 \leq C. \tag{4.3.35}$$

Proof. Based on (4.3.22), (4.3.28) and (4.3.29), we see from Lemma 4.2.2 that,

$$\|U(\cdot,t)\|_{H^3}^2 \leq C(\|\theta\|_{H^1}^2 + \|U\cdot\nabla U\|_{H^1}^2 + \|U_t\|_{H^1}^2) \leq C, \quad \forall\ t \in [0,T], \tag{4.3.36}$$

which implies by Sobolev inequality that

$$\|U(\cdot,t)\|_{W^{2,p}}^2 \leq C\|U(\cdot,t)\|_{H^3}^2 \leq C, \quad \forall\ t\in[0,T], \quad \forall\ 1 \leq p < \infty, \tag{4.3.37}$$

and thus

$$\|\nabla U(\cdot,t)\|_{L^\infty} \leq C, \quad \forall\ t \in [0,T]. \tag{4.3.38}$$

Furthermore, for $t \in [0,T]$, it is easy to see that

$$\begin{aligned} &\|U_t\cdot\nabla U\|^2 \leq \|U_t\|^2\|\nabla U\|_{L^\infty}^2 \leq C, \\ &\|U\cdot\nabla U_t\|^2 \leq \|U\|_{L^\infty}^2\|\nabla U_t\|^2 \leq C, \\ &\|\theta_t\|^2 = \|U\cdot\nabla\theta\|^2 \leq \|U\|_{L^\infty}^2\|\nabla\theta\|^2 \leq C. \end{aligned} \tag{4.3.39}$$

From (4.3.13) and Lemma 4.2.2, we know

$$\int_0^T \|U_t\|_{H^2}^2 d\tau \leq C\int_0^T \left(\|U_{tt}\|^2 + \|U_t\cdot\nabla U\|^2 + \|U\cdot\nabla U_t\|^2 + \|\theta_t\|^2\right) d\tau, \tag{4.3.40}$$

which, together with (4.3.29) and (4.3.39), gives

$$\int_0^T \|U_t(\cdot,\tau)\|_{H^2}^2 d\tau \leq C. \tag{4.3.41}$$

In addition, Sobolev inequality and (4.3.36) yield

$$\begin{aligned}\|U\cdot\nabla U\|_{H^2}^2 &\le C\big(\|U\|_{L^\infty}^2\|U\|_{H^3}^2+\|\nabla U\|_{L^\infty}^2\|U\|_{H^2}^2\big)\\ &\le C\|U\|_{H^2}^2\|U\|_{H^3}^2\le C,\ \forall\, t\in[0,T].\end{aligned}\tag{4.3.42}$$

Now, it is clear that one needs higher order estimate on θ to complete the proof of this lemma. For this purpose, taking ∂_{xx} of $(4.1.1)_2$, we get

$$\theta_{xxt}+u_{xx}\theta_x+2u_x\theta_{xx}+v_{xx}\theta_y+2v_x\theta_{xy}+U\cdot\nabla\theta_{xx}=0.\tag{4.3.43}$$

For any $p\ge 2$, multiplying (4.3.43) by $|\theta_{xx}|^{p-2}\theta_{xx}$, integrating over Ω, and using Hölder's inequality, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega|\theta_{xx}|^p d\mathbf{x} &= -\int_\Omega\big(u_{xx}\theta_x+v_{xx}\theta_y+2u_x\theta_{xx}+2v_x\theta_{xy}\big)|\theta_{xx}|^{p-2}\theta_{xx}d\mathbf{x}\\ &\le\|\nabla\theta\|_{L^\infty}\|\nabla^2U\|_{L^p}\|\nabla^2\theta\|_{L^p}^{p-1}+2\|\nabla U\|_{L^\infty}\|\nabla^2\theta\|_{L^p}^p\\ &\le C\big(\|\nabla^2\theta\|_{L^p}^{p-1}+\|\nabla^2\theta\|_{L^p}^p\big),\end{aligned}\tag{4.3.44}$$

where we have used (4.3.25), (4.3.37) and (4.3.38). Similarly, one can show

$$\frac{1}{p}\frac{d}{dt}\int_\Omega|\theta_{xy}|^p dx\le C\big(\|\nabla^2\theta\|_{L^p}^{p-1}+\|\nabla^2\theta\|_{L^p}^p\big),\tag{4.3.45}$$

$$\frac{1}{p}\frac{d}{dt}\int_\Omega|\theta_{yy}|^p dx\le C\big(\|\nabla^2\theta\|_{L^p}^{p-1}+\|\nabla^2\theta\|_{L^p}^p\big).\tag{4.3.46}$$

Summing (4.3.44)–(4.3.46) together, we obtain

$$\frac{1}{p}\frac{d}{dt}\Big(\|\nabla^2\theta\|_{L^p}^p\Big)\le C\big(\|\nabla^2\theta\|_{L^p}^{p-1}+\|\nabla^2\theta\|_{L^p}^p\big).\tag{4.3.47}$$

It follows that

$$\frac{d}{dt}\Big(\|\nabla^2\theta\|_{L^p}\Big)\le C\big(1+\|\nabla^2\theta\|_{L^p}\big),\tag{4.3.48}$$

Applying Gronwall's inequality to (4.3.48), one has

$$\|\nabla^2\theta(\cdot,t)\|_{L^p}\le C,\quad\forall\ 2\le p<\infty,\ \forall\, t\in[0,T].\tag{4.3.49}$$

In a quite similar manner as in the derivation of (4.3.49), further estimates show that

$$
\begin{aligned}
\frac{d}{dt}\|\nabla^3\theta\|^2 &\le C\Big(\|\nabla U\|_{L^\infty}\|\nabla^3\theta\|^2 + \|\nabla\theta\|_{L^\infty}\|\nabla^3 U\|\|\nabla^3\theta\| + \|\nabla^2 U\|_{L^4}\|\nabla^2\theta\|_{L^4}\|\nabla^3\theta\|\Big)\\
&\le C(\|\nabla^3\theta\|^2 + \|\nabla^3\theta\|)\\
&\le C(\|\nabla^3\theta\|^2 + 1),
\end{aligned}
\tag{4.3.50}
$$

which implies

$$\|\theta(\cdot,t)\|_{H^3}^2 \le C,\ \forall\, t\in[0,T]. \tag{4.3.51}$$

Now, by Lemma 4.2.2, combining (4.3.41), (4.3.42) and (4.3.51), one has

$$\int_0^T \|U(\cdot,\tau)\|_{H^4}^2\, d\tau \le C\int_0^T \left(\|U_t\|_{H^2}^2 + \|U\cdot\nabla U\|_{H^2}^2 + \|\theta\|_{H^2}^2\right)\, d\tau \le C,$$

which completes the proof of Lemma 4.3.8.

For the proof of Theorem 4.3.1, it remains to prove the uniform bound of the kinetic energy (4.3.1).

Lemma 4.3.9. *Under the assumptions of Theorem 4.1.1, there is a uniform constant $\bar{C}$ independent of t, such that*

$$\|U(\cdot,t)\|^2 \le \max\left\{\|U(\cdot,0)\|^2\ ,\ \frac{\bar{C}^2}{\nu^2}\|\theta(\cdot,0)\|^2\right\},\quad \forall\, t\ge 0. \tag{4.3.52}$$

Proof. From the proof of Lemma 4.3.1, we observe that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|U\|^2 + \nu\|\nabla U\|^2 &= \int_\Omega \theta\mathbf{e}_2\cdot U\ d\mathbf{x}\\
&\le \frac{1}{2\delta\nu}\|\theta\|^2 + \delta\frac{\nu}{2}\|U\|^2,
\end{aligned}
\tag{4.3.53}
$$

for any positive δ. Poincaré's inequality says that there is a constant $\bar{C}=\bar{C}(\Omega)$ such that

$$\|U\| \le \bar{C}\|\nabla U\|.$$

Choosing $\delta = \frac{1}{\bar{C}}$, we know from (4.3.53) that

$$\frac{d}{dt}\|U\|^2 + \frac{\nu}{\bar{C}}\|U\|^2 \le \frac{\bar{C}}{\nu}\|\theta\|^2. \tag{4.3.54}$$

Solving the above differential inequality we get

$$\exp\Big\{\frac{\nu}{\bar{C}}t\Big\}\|U(\cdot,t)\|^2-\|U(\cdot,0)\|^2\le\frac{\bar{C}^2}{\nu^2}\|\theta_0\|^2\big(\exp\Big\{\frac{\nu}{\bar{C}}t\Big\}-1\big),\tag{4.3.55}$$

which implies

$$\|U(\cdot,t)\|^2\le\exp\Big\{-\frac{\nu}{\bar{C}}t\Big\}\Big(\|U(\cdot,0)\|^2-\frac{\bar{C}^2}{\nu^2}\|\theta_0\|^2\Big)+\frac{\bar{C}^2}{\nu^2}\|\theta_0\|^2,\quad\forall\, t>0.\tag{4.3.56}$$

Therefore, (4.3.52) follows immediately from (4.3.56). This completes the proof of Lemma 4.3.9.

Lemmas 4.3.8–4.3.9 conclude Theorem 4.3.1. With the global regularity established in Lemmas 4.3.1–4.3.8, we are able to prove the uniqueness of the solution.

Theorem 4.3.2. *Under the assumptions of Theorem 1.1, the solution of (4.1.1)–(4.1.2) is unique.*

Proof. Suppose there are two solutions (θ_1,U_1,P_1) and (θ_2,U_2,P_2) to (4.1.1)–(4.1.2). Setting $\tilde{\theta}=\theta_1-\theta_2$, $\tilde{U}=U_1-U_2$, and $\tilde{P}=P_1-P_2$, then $(\tilde{\theta},\tilde{U},\tilde{P})$ satisfy

$$\begin{cases}\tilde{U}_t+U_1\cdot\nabla\tilde{U}+\tilde{U}\cdot\nabla U_2+\nabla\tilde{P}=\nu\Delta\tilde{U}+\tilde{\theta}\mathbf{e}_2\\ \tilde{\theta}_t+U_1\cdot\nabla\tilde{\theta}+\tilde{U}\cdot\nabla\theta_2=0,\\ \nabla\cdot\tilde{U}=0\\ \tilde{U}|_{\partial\Omega}=0,\\ \tilde{U}(\mathbf{x},0)=0,\tilde{\theta}(\mathbf{x},0)=0,\ \mathbf{x}\in\Omega.\end{cases}\tag{4.3.57}$$

Since $\nabla\cdot U_1=0$ and $U_1|_{\partial\Omega}=0$, taking the L^2 inner products of $(4.3.57)_1$ with $\tilde{U}$ and $(4.3.57)_2$ with $\tilde{\theta}$, one has

$$\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\theta}\|^2+\|\tilde{U}\|^2\Big)+\nu\|\nabla\tilde{U}\|^2=-\int_\Omega\tilde{\theta}(\tilde{U}\cdot\nabla\theta_2)d\mathbf{x}-\int_\Omega\tilde{U}\cdot(\tilde{U}\cdot\nabla U_2)d\mathbf{x}+\int_\Omega\tilde{\theta}\tilde{v}d\mathbf{x},$$

where $\tilde{v}$ is the second component of $\tilde{U}$. Using the estimates for θ_2 and U_2, standard calculations give that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\theta}\|^2+\|\tilde{U}\|^2\Big)+\nu\|\nabla\tilde{U}\|^2\\&\le\|\nabla\theta_2\|_{L^\infty}(\|\tilde{\theta}\|^2+\|\tilde{U}\|^2)+\|\nabla U_2\|_{L^\infty}\|\tilde{U}\|^2+(\|\tilde{\theta}\|^2+\|\tilde{U}\|^2)\\&\le C(\|\tilde{\theta}\|^2+\|\tilde{U}\|^2),\end{aligned}$$

which implies that

$$e^{-2Ct}(\|\theta\|^2+\|U\|^2)\le\|\theta(0)\|^2+\|U(0)\|^2=0,$$

for any $t\ge 0$. So the solution of (4.1.1)–(4.1.2) is unique. This completes the proof of Theorem 4.3.2.

This theorem and Theorem 4.3.1 implies our main result, Theorem 4.1.1.

4.4 Inviscid Heat-Conductive Boussinesq Equations

To prove theorem 4.1.2, we first reformulate the IBVP (4.1.1) and (4.1.3). Let $\bar{P}=P-\bar{\theta}y$ and $\Theta=\theta-\bar{\theta}$, then we get from the original system that

$$\begin{cases}U_t+U\cdot\nabla U+\nabla\bar{P}=\Theta\mathbf{e}_2,\\\Theta_t+U\cdot\nabla\Theta=\kappa\Delta\Theta,\\\nabla\cdot U=0.\end{cases}\tag{4.4.1}$$

The initial and boundary conditions become

$$\begin{cases}(U,\Theta)(\mathbf{x},0)=(U_0,\Theta_0)(\mathbf{x}),\\U\cdot\mathbf{n}|_{\partial\Omega}=0,\quad\Theta|_{\partial\Omega}=0,\end{cases}\tag{4.4.2}$$

where $\Theta_0=\theta_0-\bar{\theta}$. It is clear that, for smooth solutions, (4.4.1)–(4.4.2) are equivalent to (4.1.1) and (4.1.3). By definition, the same is true for weak solutions. Hence, for the rest part of this subsection, we shall work on (4.4.1)–(4.4.2). The following theorem gives the key estimates.

Theorem 4.4.1. *Under the assumptions of Theorem 4.1.2, the solution obtained in Lemma 4.2.8 satisfies* $U\in C([0,T];H^3(\Omega)),\Theta\in C([0,T];H^3(\Omega))\cap L^2([0,T];H^4(\Omega))$

for any $T > 0$. Moreover, there exist constants $\eta > 0, \bar{C} > 0, C(p) > 0, \tilde{C} > 0$ independent of t such that for any fixed $p \in [2, \infty)$,

$$\begin{aligned} &\|\Theta(\cdot,t)\|_{H^3} \le \bar{C}\exp\{-\eta t\}, \quad \forall\ t \ge 0, \\ &\|U(\cdot,t)\|_{W^{1,p}} \le C(p), \quad \text{and} \quad \|\omega(\cdot,t)\|_{L^\infty} \le \tilde{C}, \quad \forall\ t \ge 0. \end{aligned} \tag{4.4.3}$$

Notation 4.4.1. *Unless specified, throughout this section, C and C_i will denote generic constants which are independent of θ, U and T. In addition, the values of the constants are different from those in previous chapters.*

The proof of Theorem 4.4.1 is divided into several steps of energy estimates which are stated as a sequence of lemmas. As mentioned in the Introduction, the decay rate will be achieved through careful coupling of weighted energy estimates. First, we give the decay estimate of $\|\Theta\|$. Indeed, we have

Lemma 4.4.1. *Under the assumptions of Theorem 4.1.2,, there exist constants $\alpha_0 > 0, \beta_0 > 0$ independent of t such that*

$$\begin{aligned} \|\Theta(\cdot,t)\|^2 \le \|\Theta_0\|^2\exp\{-2\beta_0 t\}, \quad \text{and} \\ \int_0^t \exp\{\beta_0\tau\}\|\nabla\Theta(\cdot,\tau)\|^2 d\tau \le \alpha_0\|\Theta_0\|^2, \quad \forall\ t \ge 0. \end{aligned} \tag{4.4.4}$$

Proof. First of all, by taking L^2 inner product of $(4.4.1)_2$ with Θ we get

$$\frac{1}{2}\frac{d}{dt}\|\Theta\|^2 + \kappa\|\nabla\Theta\|^2 = 0. \tag{4.4.5}$$

Since $\Theta|_{\partial\Omega} = 0$, Poincaré's inequality implies that

$$\|\Theta\|^2 \le C_0\|\nabla\Theta\|^2, \tag{4.4.6}$$

for some constant C_0 depending only on Ω. Combining (4.4.5) and (4.4.6) we get

$$\frac{d}{dt}\|\Theta\|^2 + \frac{2\kappa}{C_0}\|\Theta\|^2 \le 0,$$

which yields immediately that

$$\|\Theta(\cdot,t)\|^2 \le \|\Theta_0\|^2\exp\{-2\beta_0 t\}, \tag{4.4.7}$$

where $\beta_0 = \kappa/C_0$.

Multiplying (4.4.5) by $\exp\{\beta_0 t\}$ we have

$$\frac{d}{dt}\big(\exp\{\beta_0 t\}\|\Theta\|^2\big) + 2\kappa\exp\{\beta_0 t\}\|\nabla\Theta\|^2 = \beta_0\exp\{\beta_0 t\}\|\Theta\|^2. \tag{4.4.8}$$

Combining (4.4.7) and (4.4.8) we have

$$\frac{d}{dt}\big(\exp\{\beta_0 t\}\|\Theta\|^2\big) + 2\kappa\exp\{\beta_0 t\}\|\nabla\Theta\|^2 \leq \beta_0\exp\{-\beta_0 t\}\|\Theta_0\|^2. \tag{4.4.9}$$

For any $t \geq 0$, integrating (4.4.9) in time over $[0, t]$ we obtain

$$\begin{aligned}&\exp\{\beta_0 t\}\|\Theta(\cdot,t)\|^2 - \|\Theta_0\|^2 + 2\kappa\int_0^t \exp\{\beta_0\tau\}\|\nabla\Theta(\cdot,\tau)\|^2 d\tau\\ &\leq \big(1-\exp\{-\beta_0 t\}\big)\|\Theta_0\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\int_0^t \exp\{\beta_0\tau\}\|\nabla\Theta(\cdot,\tau)\|^2 d\tau &\leq \frac{1}{2\kappa}\big(2-\exp\{-\beta_0 t\}\big)\|\Theta_0\|^2\\ &\leq \alpha_0\|\Theta_0\|^2, \quad \forall\, t \geq 0,\end{aligned} \tag{4.4.10}$$

where $\alpha_0 = 1/\kappa$. This completes the proof of Lemma 4.4.1.

To improve the decay estimate of Θ to higher order norms, we proceed to find the uniform estimate of $\|U\|_{H^1}$. With the help of Lemma 4.4.1, we have

Lemma 4.4.2. *Under the assumptions of Theorem 4.1.2, there exists a constant $d_1 > 0$ independent of t such that*

$$\|U(\cdot,t)\|_{H^1}^2 \leq d_1, \quad \forall\, t \geq 0. \tag{4.4.11}$$

Proof. By taking L^2 inner product of $(4.4.1)_1$ with U we get

$$\frac{d}{dt}\|U\|^2 = 2\int_\Omega \Theta\mathbf{e}_2\cdot U d\mathbf{x}.$$

Cauchy-Schwartz inequality then implies that

$$\begin{aligned}\frac{d}{dt}\|U\|^2 &\leq \exp\{-\beta_0 t\}\|U\|^2 + \exp\{\beta_0 t\}\|\Theta\|^2\\ &\leq \exp\{-\beta_0 t\}\|U\|^2 + \exp\{-\beta_0 t\}\|\Theta_0\|^2,\end{aligned} \tag{4.4.12}$$

where we have used Lemma 4.4.1. Applying Gronwall's inequality to (4.4.12) we find

$$\begin{aligned} \|U(\cdot,t)\|^2 &\le \exp\left\{\int_0^t \exp\{-\beta_0\tau\}d\tau\right\}\Big(\|U_0\|^2 + \int_0^t \exp\{-\beta_0\tau\}\|\Theta_0\|^2 d\tau\Big) \\ &= \exp\left\{\frac{1}{\beta_0}\big(1-\exp\{-\beta_0 t\}\big)\right\}\Big(\|U_0\|^2 + \frac{\|\Theta_0\|^2}{\beta_0}\big(1-\exp\{-\beta_0 t\}\Big) \\ &\le \exp\{1/\beta_0\}\Big(\|U_0\|^2 + \frac{\|\Theta_0\|^2}{\beta_0}\Big), \quad \forall\, t\ge 0. \end{aligned} \tag{4.4.13}$$

To get the estimation of ∇U, we take the curl of $(4.4.1)_1$ to obtain

$$\omega_t + U\cdot\nabla\omega = \Theta_x, \tag{4.4.14}$$

where $\omega = v_x - u_y$ is the 2D vorticity. Taking the L^2 inner product of (4.4.14) with ω and using Cauchy-Schwartz inequality we get

$$\begin{aligned} \frac{d}{dt}\|\omega\|^2 &\le 2\|\omega\|\|\nabla\Theta\| \\ &\le \exp\{-\beta_0 t\}\|\omega\|^2 + \exp\{\beta_0 t\}\|\nabla\Theta\|^2. \end{aligned} \tag{4.4.15}$$

Applying Gronwall's inequality to (4.4.15) and using the second part of (4.4.4) we obtain

$$\begin{aligned} \|\omega(\cdot,t)\|^2 &\le \exp\left\{\int_0^t \exp\{-\beta_0\tau\}d\tau\right\}\Big(\|\omega_0\|^2 + \int_0^t \exp\{\beta_0\tau\}\|\nabla\Theta(\cdot,\tau)\|^2 d\tau\Big) \\ &\le \exp\left\{\frac{1}{\beta_0}\big(1-\exp\{-\beta_0 t\}\big)\right\}\Big(\|\omega_0\|^2 + \alpha_0\|\Theta_0\|^2\Big) \\ &\le \exp\{1/\beta_0\}\Big(\|\omega_0\|^2 + \alpha_0\|\Theta_0\|^2\Big), \quad \forall\, t\ge 0, \end{aligned}$$

which, together with (4.4.13) and Lemma 4.2.4 with $s=1, p=2$, implies that

$$\begin{aligned} \|U(\cdot,t)\|_{H^1}^2 &\le C(\|U(\cdot,t)\|^2 + \|\omega(\cdot,t)\|^2) \\ &\le C\exp\{1/\beta_0\}\Big(\|U_0\|^2 + \|\omega_0\|^2 + (\alpha_0 + 1/\beta_0)\|\Theta_0\|^2\Big) \equiv d_1, \quad \forall\, t\ge 0. \end{aligned}$$

This completes the proof of Lemma 4.4.2.

Now we prove the key lemma of this section, which gives the exponential decay of the H^1 norm of Θ and is a consequence of Lemma 4.4.2. The important role played by the uniform bound of $\|U\|_{H^1}^2$ will be revealed in the proof.

Lemma 4.4.3. *Under the assumptions of Theorem 4.1.2, there exist constants $\alpha_1 > 0, \beta_1 > 0$ independent of t such that*

$$\|\Theta(\cdot,t)\|_{H^1}^2 \le \alpha_1 \|\Theta_0\|_{H^1}^2 \exp\{-\beta_1 t\}, \quad \forall\ t \ge 0.$$

Proof. Taking L^2 inner product of $(4.4.1)_2$ with Θ_t we find

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta\|^2 + \|\Theta_t\|^2 = -\int_\Omega (U\cdot\nabla\Theta)\Theta_t d\mathbf{x}. \tag{4.4.16}$$

We estimate the RHS of (4.4.16) as follows: First, using Cauchy-Schwartz inequality we get

$$-\int_\Omega (U\cdot\nabla\Theta)\Theta_t d\mathbf{x} \le \|U\cdot\nabla\Theta\|^2 + \frac{1}{4}\|\Theta_t\|^2.$$

Using Lemmas 4.2.1 and 4.4.2 we have

$$\begin{aligned}\|U\cdot\nabla\Theta\|^2 &\le \|U\|_{L^4}^2\|\nabla\Theta\|_{L^4}^2\\ &\le C_1\|U\|_{H^1}^2\|\nabla\Theta\|_{L^4}^2\\ &\le C_1 d_1\|\nabla\Theta\|_{L^4}^2.\end{aligned}$$

Letting $C_2 = C_1 d_1$ we update (4.4.16) as

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta\|^2 + \frac{3}{4}\|\Theta_t\|^2 \le C_2\|\nabla\Theta\|_{L^4}^2. \tag{4.4.17}$$

For the RHS of (4.4.17), applying Ladyzhenskaya's inequality to $\nabla\Theta$ we get

$$\begin{aligned}\|\nabla\Theta\|_{L^4}^2 &\le C_3\big(\|\nabla\Theta\|\|D^2\Theta\| + \|\nabla\Theta\|^2\big)\\ &\le C(\delta)\|\nabla\Theta\|^2 + \delta\|D^2\Theta\|^2,\end{aligned} \tag{4.4.18}$$

where δ is a number to be determined. Now, using $(4.4.1)_2$ and Lemma 4.2.3 with $m = 0$ and $p = 2$ we have

$$\|\Theta\|_{H^2}^2 \le C_4\big(\|\Theta_t\|^2 + \|U\cdot\nabla\Theta\|^2\big). \tag{4.4.19}$$

For the second term on the RHS of (4.4.19), we use (4.4.18) to get

$$\|U\cdot\nabla\Theta\|^2 \le C_5\big(\|\nabla\Theta\|\|D^2\Theta\| + \|\nabla\Theta\|^2\big),$$

where $C_5 = C_1 d_1 C_3$. Then, using Cauchy-Schwartz inequality we update (4.4.19) as

$$\begin{aligned}
\|\Theta\|_{H^2}^2 &\leq C_4\Big(\|\Theta_t\|^2 + C_5\big(\|\nabla\Theta\|\|D^2\Theta\| + \|\nabla\Theta\|^2\big)\Big) \\
&\leq C_6\big(\|\Theta_t\|^2 + \|\nabla\Theta\|^2\big) + \frac{1}{2}\|\Theta\|_{H^2}^2.
\end{aligned}$$

So we have

$$\|\Theta\|_{H^2}^2 \leq C_7\big(\|\Theta_t\|^2 + \|\nabla\Theta\|^2\big), \tag{4.4.20}$$

where $C_7 = 2C_6$. By choosing $\delta = 1/(4C_2C_7)$ in (4.4.18), and by coupling the resulting inequality with (4.4.20) we obtain

$$\|\nabla\Theta\|_{L^4}^2 \leq C_8\|\nabla\Theta\|^2 + \frac{1}{4C_2}\|\Theta_t\|^2. \tag{4.4.21}$$

Combining (4.4.17) with (4.4.21) we get

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2 \leq C_9\|\nabla\Theta\|^2. \tag{4.4.22}$$

To explore the diffusive mechanism in the temperature equation, we multiply (4.4.5) by $2C_9/\kappa$ and add the resulting equation to (4.4.22) to get

$$\frac{d}{dt}\Big(\frac{C_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big) + C_9\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2 \leq 0. \tag{4.4.23}$$

Let $\beta_1 = \big(\frac{C_0}{\kappa} + \frac{\kappa}{2C_9}\big)^{-1}$. Then, with the help of Poincaré's inequality we have

$$\beta_1\Big(\frac{C_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big) \leq C_9\|\nabla\Theta\|^2. \tag{4.4.24}$$

Combining (4.4.23) with (4.4.24) we obtain

$$\frac{d}{dt}\Big(\frac{C_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big) + \beta_1\Big(\frac{C_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big) + \frac{1}{2}\|\Theta_t\|^2 \leq 0, \tag{4.4.25}$$

which implies that (where we dropped a positive term from the LHS)

$$\Big(\frac{C_9}{\kappa}\|\Theta(\cdot,t)\|^2 + \frac{\kappa}{2}\|\nabla\Theta(\cdot,t)\|^2\Big) \leq \Big(\frac{C_9}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\Big)\exp\{-\beta_1 t\}. \tag{4.4.26}$$

Therefore,

$$\|\Theta(\cdot,t)\|_{H^1}^2 \leq \alpha_1\|\Theta_0\|_{H^1}^2 \exp\{-\beta_1 t\}, \quad \forall\, t \geq 0, \tag{4.4.27}$$

where

$$\alpha_1 = \max\left\{\frac{C_9}{\kappa}, \frac{\kappa}{2}\right\}\Big/\min\left\{\frac{C_9}{\kappa}, \frac{\kappa}{2}\right\}.$$

This completes the proof of Lemma 4.4.3.

Remark 4.4.1. *The energy estimates coupling used in the proof of Lemma 4.4.3 will be repeated twice in Lemmas 4.4.5–4.4.6 to establish the exponential decay of* $\|\Theta\|_{H^2}$ *and* $\|\Theta\|_{H^3}$.

Before we proceed to improve the decay of Θ, we establish higher order uniform estimate of U. For this purpose, we have

Lemma 4.4.4. *Under the assumptions of Theorem 4.1.2, for any fixed* $p \in [2, \infty)$, *there exists a constant* $d_2 = d_2(p) > 0$ *independent of* t *such that*

$$\|U(\cdot, t)\|_{W^{1,p}} \le d_2, \quad \forall\ t \ge 0. \tag{4.4.28}$$

Proof. First, we establish an estimate similar to (4.4.10) for $\|\Theta\|_{H^2}$. Multiplying (4.4.23) by $e^{\beta_1 t/2}$ we get

$$\begin{aligned} &\frac{d}{dt}\left[e^{\beta_1 t/2}\Big(\frac{C_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big)\right] + e^{\beta_1 t/2}\Big(C_9\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2\Big) \\ &\le \frac{\beta_1}{2}e^{\beta_1 t/2}\Big(\frac{C_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big). \end{aligned} \tag{4.4.29}$$

Applying (4.4.26) to (4.4.29) we obtain

$$\begin{aligned} &\frac{d}{dt}\left[e^{\beta_1 t/2}\Big(\frac{C_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big)\right] + e^{\beta_1 t/2}\Big(C_9\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2\Big) \\ &\le \frac{\beta_1}{2}e^{-\beta_1 t/2}\Big(\frac{C_9}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\Big). \end{aligned} \tag{4.4.30}$$

Integrating (4.4.30) in time over $[0, t]$ we get

$$\begin{aligned} &e^{\beta_1 t/2}\Big(\frac{C_9}{\kappa}\|\Theta(\cdot, t)\|^2 + \frac{\kappa}{2}\|\nabla\Theta(\cdot, t)\|^2\Big) + \int_0^t e^{\beta_1\tau/2}\Big(C_9\|\nabla\Theta(\cdot, \tau)\|^2 + \frac{1}{2}\|\Theta_t(\cdot, \tau)\|^2\Big)d\tau \\ &\le 2\Big(\frac{C_9}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\Big), \end{aligned} \tag{4.4.31}$$

which yields

$$\int_0^t e^{\beta_1\tau/2}\Big(C_9\|\nabla\Theta(\cdot,\tau)\|^2+\frac{1}{2}\|\Theta_t(\cdot,\tau)\|^2\Big)d\tau \le 2\Big(\frac{C_9}{\kappa}\|\Theta_0\|^2+\frac{\kappa}{2}\|\nabla\Theta_0\|^2\Big). \quad (4.4.32)$$

Letting $C_{10}=2\big(\min\{C_9,1/2\}\big)^{-1}$ we get from (4.4.32) that

$$\int_0^t e^{\beta_1\tau/2}\Big(\|\nabla\Theta(\cdot,\tau)\|^2+\|\Theta_t(\cdot,\tau)\|^2\Big)d\tau \le C_{10}\Big(\frac{C_9}{\kappa}\|\Theta_0\|^2+\frac{\kappa}{2}\|\nabla\Theta_0\|^2\Big). \quad (4.4.33)$$

Using (4.4.20) and (4.4.33) we obtain the following estimate on $\|\Theta\|_{H^2}$:

$$\int_0^t e^{\beta_1\tau/2}\|\Theta(\cdot,\tau)\|_{H^2}^2 d\tau \le C_{11}, \quad \forall\, t\ge 0, \quad (4.4.34)$$

where $C_{11}=C_7C_{10}(\frac{C_9}{\kappa}\|\Theta_0\|^2+\frac{\kappa}{2}\|\nabla\Theta_0\|^2)$.

Now, for any fixed $p\in[2,\infty)$, multiplying the vorticity equation (4.4.14) by $|\omega|^{p-2}\omega$ and then integrating the resulting equation over Ω, we find, after integration by parts, that

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p=-\int_\Omega \Theta_x|\omega|^{p-2}\omega d\mathbf{x}, \quad \forall\, p\in[2,\infty). \quad (4.4.35)$$

Using Hölder's inequality, we estimate the RHS of (4.4.35) as

$$-\int_\Omega \Theta_x|\omega|^{p-2}\omega d\mathbf{x}\le \|\nabla\Theta\|_{L^p}\|\omega\|_{L^p}^{p-1}. \quad (4.4.36)$$

Combining (4.4.35) with (4.4.36) and using Sobolev embedding we get

$$\frac{d}{dt}\|\omega\|_{L^p}\le\|\nabla\Theta\|_{L^p}\le C_{12}(p)\|\Theta\|_{H^2},$$

which implies, after integrating in time and using Hölder's inequality and (4.4.34), that

$$\begin{aligned}\|\omega(\cdot,t)\|_{L^p} &\le \|\omega(\cdot,0)\|_{L^p}+C_{12}\int_0^t\|\Theta(\cdot,\tau)\|_{H^2}d\tau\\ &\le \|\omega(\cdot,0)\|_{L^p}+C_{12}\Big(\int_0^t e^{\beta_1\tau/2}\|\Theta(\cdot,\tau)\|_{H^2}^2d\tau\Big)^{\frac{1}{2}}\Big(\int_0^t e^{-\beta_1\tau/2}d\tau\Big)^{\frac{1}{2}}\\ &\le \|\omega(\cdot,0)\|_{L^p}+C_{12}\sqrt{C_{11}}\sqrt{2/\beta_2}\equiv C_{13}, \quad \forall\, t\ge 0.\end{aligned} \quad (4.4.37)$$

Therefore, Lemmas 4.2.1, 4.2.4, 4.4.2 and (4.4.37) imply that for any fixed $p \in [2, \infty)$,

$$\begin{aligned}
\|U(\cdot,t)\|_{W^{1,p}} &\le C\big(\|\omega(\cdot,t)\|_{L^p} + \|U(\cdot,t)\|_{L^p}\big) \\
&\le C\big(\|\omega(\cdot,t)\|_{L^p} + C(p)\|U(\cdot,t)\|_{H^1}\big) \\
&\le C\big(C_{13} + C(p)\sqrt{d_1}\big) \equiv d_2.
\end{aligned} \tag{4.4.38}$$

This completes the proof of Lemma 4.4.4.

With the help of Lemma 4.4.4 we are now ready to show the exponential decay of $\|\Theta\|_{H^2}$. Indeed, we have

Lemma 4.4.5. *Under the assumptions of Theorem 4.1.2, there exist constants* $\alpha_2 > 0, \beta_2 > 0$ *independent of* t *such that*

$$\|\Theta(\cdot,t)\|_{H^2}^2 \le \alpha_2 \exp\{-\beta_2 t\}, \quad \forall\ t \ge 0. \tag{4.4.39}$$

Proof. First, by taking L^2 inner product of $(4.4.1)_1$ with U_t we get

$$\|U_t\|^2 = -\int_\Omega U_t \cdot (U \cdot \nabla U) d\mathbf{x} + \int_\Omega \Theta v_t d\mathbf{x}, \tag{4.4.40}$$

from which we deduce, using Lemmas 4.4.2, 4.4.4 and (4.4.7), that

$$\begin{aligned}
\|U_t\|^2 &\le \frac{1}{4}\|U_t\|^2 + \|U \cdot \nabla U\|^2 + \|\Theta\|^2 + \frac{1}{4}\|U_t\|^2 \\
&\le \frac{1}{2}\|U_t\|^2 + \|U\|_{L^\infty}^2 \|\nabla U\|^2 + \|\Theta\|^2 \\
&\le \frac{1}{2}\|U_t\|^2 + C\|U\|_{W^{1,4}}^2 \|\nabla U\|^2 + \|\Theta\|^2 \\
&\le \frac{1}{2}\|U_t\|^2 + Cd_2 d_1 + \|\Theta_0\|^2.
\end{aligned} \tag{4.4.41}$$

Hence,

$$\|U_t\|^2 \le 2(Cd_2 d_1 + \|\Theta_0\|^2) \equiv C_{14}. \tag{4.4.42}$$

Taking temporal derivative of $(4.4.1)_2$ we have

$$\Theta_{tt} + U_t \cdot \nabla\Theta + U \cdot \nabla\Theta_t = \kappa\Delta\Theta_t. \tag{4.4.43}$$

Taking the L^2 inner product of (4.4.43) with Θ_t, we obtain, after integration by parts, that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta_t\|^2 + \kappa\|\nabla\Theta_t\|^2 &= -\int_\Omega (U_t\cdot\nabla\Theta)\Theta_t d\mathbf{x} \\ &= \int_\Omega \Theta(U_t\cdot\nabla\Theta_t)d\mathbf{x}.\end{aligned} \tag{4.4.44}$$

Using Cauchy-Schwartz inequality, (4.4.42) and Sobolev embedding we estimate the RHS of (4.4.44) as:

$$\begin{aligned}\int_\Omega \Theta(U_t\cdot\nabla\Theta_t)d\mathbf{x} &\le \frac{1}{2\kappa}\|\Theta U_t\|^2 + \frac{\kappa}{2}\|\nabla\Theta_t\|^2 \\ &\le \frac{1}{2\kappa}\|\Theta\|_{L^\infty}^2\|U_t\|^2 + \frac{\kappa}{2}\|\nabla\Theta_t\|^2 \\ &\le \frac{C_{14}}{2\kappa}\|\Theta\|_{L^\infty}^2 + \frac{\kappa}{2}\|\nabla\Theta_t\|^2 \\ &\le C_{15}\|\Theta\|_{H^2}^2 + \frac{\kappa}{2}\|\nabla\Theta_t\|^2.\end{aligned}$$

So we update (4.4.44) as

$$\frac{d}{dt}\|\Theta_t\|^2 + \kappa\|\nabla\Theta_t\|^2 \le C_{16}\|\Theta\|_{H^2}^2. \tag{4.4.45}$$

Coupling (4.4.45) with (4.4.20) we get

$$\frac{d}{dt}\|\Theta_t\|^2 + \kappa\|\nabla\Theta_t\|^2 \le C_{17}\big(\|\nabla\Theta\|^2 + \|\Theta_t\|^2\big). \tag{4.4.46}$$

Now, letting $C_{18} = \min\{C_9, 1/2\}$ we get from (4.4.23) that

$$\frac{d}{dt}\Big(\frac{C_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\Big) + C_{18}\big(\|\nabla\Theta\|^2 + \|\Theta_t\|^2\big) \le 0.$$

Multiplying the above inequality by $2C_{17}/C_{18}$, then adding the resulting inequality to (4.4.46) we obtain

$$\frac{d}{dt}\big(E(t)\big) + C_{17}\big(\|\nabla\Theta\|^2 + \|\Theta_t\|^2\big) + \kappa\|\nabla\Theta_t\|^2 \le 0, \tag{4.4.47}$$

where

$$E(t) = \frac{2C_{17}}{C_{18}}\left(\frac{C_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\right) + \|\Theta_t\|^2.$$

With the help of Poincaré's inequality we can easily see that there exists a constant $\beta_2 > 0$ independent of t such that

$$\beta_2 E(t) \leq C_{17}\big(\|\nabla\Theta\|^2 + \|\Theta_t\|^2\big).$$

Hence, we update (4.4.47) as

$$\frac{d}{dt}\big(E(t)\big) + \beta_2 E(t) + \kappa\|\nabla\Theta_t\|^2 \leq 0,$$

which implies that

$$E(t) \leq E(0)\exp\{-\beta_2 t\}. \tag{4.4.48}$$

In view of (4.4.20) we see that there exists a constant C_{19} independent of t such that

$$\|\Theta(\cdot,t)\|_{H^2}^2 \leq C_{19}E(t), \quad \forall\, t \geq 0,$$

which, together with (4.4.48), yields (4.4.39). This completes the proof of Lemma 4.4.5.

The next lemma is concerned with the decay of $\|\nabla\Theta_t\|^2$, based on which we can prove the decay of $\|\Theta\|_{H^3}^2$. For this purpose, we have

Lemma 4.4.6. *Under the assumptions of Theorem 4.1.2, there exist constants* $\alpha_3 > 0, \beta_3 > 0$ *independent of* t *such that*

$$\|\Theta(\cdot,t)\|_{H^1}^2 + \|\Theta_t(\cdot,t)\|_{H^1}^2 \leq \alpha_3 \exp\{-\beta_3 t\}, \quad \forall\, t \geq 0. \tag{4.4.49}$$

Proof. Taking L^2 inner product of (4.4.43) with Θ_{tt} we obtain

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta_t\|^2 + \|\Theta_{tt}\|^2 = -\int_\Omega \Theta_{tt}\big(U_t\cdot\nabla\Theta\big)d\mathbf{x} - \int_\Omega \Theta_{tt}\big(U\cdot\nabla\Theta_t\big)d\mathbf{x}. \tag{4.4.50}$$

Using (4.4.42), Lemmas 4.2.1 and 4.2.3, we estimate the first term on the RHS of (4.4.50) as:

$$\begin{aligned}
-\int_\Omega \Theta_{tt}(U_t\cdot\nabla\Theta)d\mathbf{x} &\leq \frac{1}{4}\|\Theta_{tt}\|^2 + \|U_t\|^2\|\nabla\Theta\|_{L^\infty}^2 \\
&\leq \frac{1}{4}\|\Theta_{tt}\|^2 + C\|\Theta\|_{W^{2,3}}^2 \\
&\leq \frac{1}{4}\|\Theta_{tt}\|^2 + C\big(\|\Theta_t\|_{L^3}^2 + \|U\cdot\nabla\Theta\|_{L^3}^2\big) \\
&\leq \frac{1}{4}\|\Theta_{tt}\|^2 + C\big(\|\Theta_t\|_{H^1}^2 + \|U\cdot\nabla\Theta\|_{L^3}^2\big).
\end{aligned} \tag{4.4.51}$$

From Lemma 4.4.4, Sobolev embedding and (4.4.20) we know

$$\begin{aligned}\|U\cdot\nabla\Theta\|_{L^3}^2 &\le \|U\|_{L^\infty}^2\|\nabla\Theta\|_{L^3}^2\\ &\le C\|\Theta\|_{H^2}^2\\ &\le C\big(\|\Theta_t\|^2+\|\nabla\Theta\|^2\big).\end{aligned}$$

Therefore, (4.4.51) becomes

$$-\int_\Omega \Theta_{tt}(U_t\cdot\nabla\Theta)d\mathbf{x} \le \frac{1}{4}\|\Theta_{tt}\|^2 + C\big(\|\Theta_t\|_{H^1}^2+\|\nabla\Theta\|^2\big). \tag{4.4.52}$$

For the second term on the RHS of (4.4.50), we have

$$\begin{aligned}-\int_\Omega \Theta_{tt}(U\cdot\nabla\Theta_t)d\mathbf{x} &\le \frac{1}{4}\|\Theta_{tt}\|^2 + \|U\|_{L^\infty}^2\|\nabla\Theta_t\|^2\\ &\le \frac{1}{4}\|\Theta_{tt}\|^2 + C\|\nabla\Theta_t\|^2.\end{aligned} \tag{4.4.53}$$

Combining (4.4.50) with (4.4.52)–(4.4.53) we get

$$\kappa\frac{d}{dt}\|\nabla\Theta_t\|^2 + \|\Theta_{tt}\|^2 \le \hat{C}\Big(\|\nabla\Theta\|^2+\|\Theta_t\|^2+\|\nabla\Theta_t\|^2\Big),$$

for some constant $\hat{C}>0$ independent of t. By applying the same idea used in the proof of Lemma 4.4.5, we absorb the RHS of the above inequality into the LHS of (4.4.47). Then it is straightforward to show that there exists a constant $\beta_3>0$ independent of t such that

$$\frac{d}{dt}\big(F(t)\big) + \beta_3 F(t) + \|\Theta_{tt}\|^2 \le 0, \tag{4.4.54}$$

where the quantity $F(t)$ is equivalent to $\|\Theta(\cdot,t)\|_{H^1}^2+\|\Theta_t(\cdot,t)\|_{H^1}^2$. By dropping $\|\Theta_{tt}\|^2$ we get

$$\frac{d}{dt}\big(F(t)\big) + \beta_3 F(t) \le 0,$$

which yields (4.4.49). This completes the proof of Lemma 4.4.6.

With the helps of Lemmas 4.4.2–4.4.6, we are now ready to prove the exponential decay of $\|\Theta\|_{H^3}^2$.

Lemma 4.4.7. *Under the assumptions of Theorem 4.1.2, there exist constants* $\alpha_4 > 0, \beta_4 > 0$ *independent of* t *such that*

$$\|\Theta(\cdot,t)\|_{H^3}^2 \le \alpha_4 \exp\{-\beta_4 t\}, \quad \forall\ t \ge 0. \tag{4.4.55}$$

Proof. First, since $\Theta|_{\partial\Omega} = 0$, using Lemma 4.2.3 with $m = 1$ and $p = 2$ we have

$$\|\Theta\|_{H^3}^2 \le C\big(\|\Theta_t\|_{H^1}^2 + \|U \cdot \nabla\Theta\|_{H^1}^2\big). \tag{4.4.56}$$

By virtue of Lemma 4.4.6, it suffices to estimate $\|U \cdot \nabla\Theta\|_{H^1}^2$ in order to prove the lemma. For this purpose, we observe, by Lemma 4.4.4 and Lemma 4.2.1 (ii), that

$$\begin{aligned}\|(U \cdot \nabla\Theta)(\cdot,t)\|_{H^1}^2 &\le \|U\|_{L^\infty}^2 \|\Theta\|_{H^2}^2 + \|\nabla U\|^2 \|\nabla\Theta\|_{L^\infty}^2 \\ &\le C\big(\|\Theta\|_{H^2}^2 + \|\Theta\|_{W^{2,3}}^2\big).\end{aligned} \tag{4.4.57}$$

From the derivations in (4.4.51) we have

$$\|\Theta\|_{W^{2,3}}^2 \le C\big(\|\Theta_t\|_{H^1}^2 + \|\nabla\Theta\|^2\big). \tag{4.4.58}$$

Substituting (4.4.58) into (4.4.57) we have

$$\|(U \cdot \nabla\Theta)(\cdot,t)\|_{H^1}^2 \le C\big(\|\Theta_t\|_{H^1}^2 + \|\Theta\|_{H^2}^2\big), \tag{4.4.59}$$

Plugging (4.4.59) into (4.4.56) we have

$$\|\Theta\|_{H^3}^2 \le C\big(\|\Theta\|_{H^2}^2 + \|\Theta_t\|_{H^1}^2\big), \tag{4.4.60}$$

which, together with Lemmas 4.4.5 and 4.4.6, implies (4.4.55). This completes the proof of Lemma 4.4.7.

As a consequence of Lemma 4.4.7, we show the uniform estimate of $\|\omega\|_{L^\infty}$.

Lemma 4.4.8. *Under the assumptions of Theorem 4.1.2, there exists a constant* $\tilde{C} > 0$ *independent of* t *such that*

$$\|\omega(\cdot,t)\|_{L^\infty} \le \tilde{C}, \quad \forall\ t \ge 0.$$

Proof. We note from (4.4.35) that for any $p \geq 2$, it holds that

$$\frac{d}{dt}\|\omega\|_{L^p} \leq \|\nabla\Theta\|_{L^p} \leq \|\nabla\Theta\|_{L^\infty}|\Omega|^{1/p} \leq \|\nabla\Theta\|_{L^\infty}\max\{1,|\Omega|\}. \tag{4.4.61}$$

By Sobolev embedding and Lemma 4.4.7 we have

$$\|\nabla\Theta\|_{L^\infty} \leq C\|\Theta\|_{H^3} \leq C\exp\{-\beta_4 t\}. \tag{4.4.62}$$

Plugging (4.4.62) into (4.4.61) we have

$$\frac{d}{dt}\|\omega\|_{L^p} \leq C\exp\{-\beta_4 t\}. \tag{4.4.63}$$

Upon integrating (4.4.63) in time we have

$$\|\omega(\cdot,t)\|_{L^p} \leq \|\omega(\cdot,0)\|_{L^p} + C/\beta_4 \leq \|\omega(\cdot,0)\|_{L^\infty}\max\{1,|\Omega|\} + C/\beta_4. \tag{4.4.64}$$

We note that the RHS of (4.4.64) is independent of t and $p \geq 2$. Therefore, letting $p \to \infty$ in (4.4.64) we complete the proof of Lemma 4.4.8.

Now we turn to the regularity of the velocity field. With the help of Lemma 4.4.7 we have

Lemma 4.4.9. *Under the assumptions of Theorem 4.1.2, for any $T > 0$, there exists a constant $M = M(T) > 0$ such that*

$$\|U\|^2_{C([0,T];H^3(\Omega))} \leq M. \tag{4.4.65}$$

Proof. We note, due to (4.4.55) and Sobolev embedding, that

$$\|\Theta\|^2_{C([0,T];C^{1+\gamma}(\bar{\Omega}))} \leq C\exp\{-\beta_4 t\},$$

for some $\gamma \in (0,1)$. Therefore, $(4.4.1)_1$ and Lemma 4.2.6 with $G = \Theta\mathbf{e}_2$ imply that for any fixed $T > 0$,

$$\|U\|^2_{C([0,T];C^1(\bar{\Omega}))} \leq C(T) < \infty. \tag{4.4.66}$$

To estimate $\|U\|_{H^3}$, we consider the vorticity equation (4.4.14). For any mixed spatial derivative D^α with $0 \le |\alpha| \le 2$, taking the L^2 inner product of D^α(4.4.14) with $D^\alpha\omega$ we get

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha\omega\|^2 = -\int_\Omega D^\alpha(U\cdot\nabla\omega)D^\alpha\omega\ d\mathbf{x} - \int_\Omega D^\alpha\Theta_x D^\alpha\omega\ d\mathbf{x}. \tag{4.4.67}$$

Since $\nabla\cdot U = 0$ and $U\cdot\mathbf{n}|_{\partial\Omega} = 0$, we rewrite the first term on the RHS of (4.4.67) as

$$\begin{aligned} -\int_\Omega D^\alpha(U\cdot\nabla\omega)D^\alpha\omega\ d\mathbf{x} &= -\int_\Omega D^\alpha\nabla\cdot(U\omega)D^\alpha\omega\ d\mathbf{x} \\ &= -\int_\Omega \big(D^\alpha\nabla\cdot(U\omega) - U\cdot\nabla D^\alpha\omega\big)D^\alpha\omega\ d\mathbf{x}. \end{aligned} \tag{4.4.68}$$

Combining (4.4.67) and (4.4.68), and using Cauchy-Schwartz inequality we get

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha\omega\|^2 \le \frac{1}{2}\|(D^\alpha\nabla\cdot(U\omega) - U\cdot\nabla D^\alpha\omega)\|^2 + \frac{1}{2}\|D^\alpha\Theta_x\|^2 + \|D^\alpha\omega\|^2. \tag{4.4.69}$$

Now, it is easy to see that

$$\|(D^\alpha\nabla\cdot(U\omega) - U\cdot\nabla D^\alpha\omega)\|^2 \le \|\nabla U\|_{L^\infty}^2\|\omega\|^2, \quad \text{for } |\alpha| = 0, \tag{4.4.70}$$

and

$$\begin{aligned} \|(D^\alpha\nabla\cdot(U\omega) - U\cdot\nabla D^\alpha\omega)\|^2 &= \|D^\alpha(\nabla\cdot U)\omega + (\nabla\cdot U)D^\alpha\omega + D^\alpha U\cdot\nabla\omega\|^2 \\ &\le \|U\|_{H^2}^2\|\omega\|_{L^\infty}^2 + 2\|\nabla U\|_{L^\infty}^2\|\omega\|_{H^1}^2, \quad \text{for } |\alpha| = 1. \end{aligned} \tag{4.4.71}$$

For $|\alpha| = 2$, with the help of Lemma 4.2.5 with $f = U, g = \omega$ and $|\beta| = 3$ we obtain

$$\|(D^\alpha\nabla\cdot(U\omega) - U\cdot\nabla D^\alpha\omega)\|^2 \le C(\|\nabla U\|_{L^\infty}^2\|\omega\|_{H^2}^2 + \|U\|_{H^3}^2\|\omega\|_{L^\infty}^2). \tag{4.4.72}$$

Combining (4.4.70)–(4.4.72) we see that for any multiindex α with $0 \le |\alpha| \le 2$ it holds that

$$\|(D^\alpha\nabla\cdot(U\omega) - U\cdot\nabla D^\alpha\omega)\|^2 \le C(\|\nabla U\|_{L^\infty}^2\|\omega\|_{H^2}^2 + \|U\|_{H^3}^2\|\omega\|_{L^\infty}^2). \tag{4.4.73}$$

Plugging (4.4.73) into (4.4.69) and using Lemma 4.2.4 with $s = 3, p = 2$ we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|D^\alpha\omega\|^2 &\le C\big(\|\nabla U\|_{L^\infty}^2\|\omega\|_{H^2}^2 + \|U\|_{H^3}^2\|\omega\|_{L^\infty}^2\big) + \|D^\alpha\omega\|^2 + \frac{1}{2}\|D^\alpha\Theta_x\|^2 \\ &\le C\|\nabla U\|_{L^\infty}^2(\|\omega\|_{H^2}^2 + \|U\|^2) + \|D^\alpha\omega\|^2 + \frac{1}{2}\|D^\alpha\Theta_x\|^2. \end{aligned} \tag{4.4.74}$$

Summing (4.4.74) over all α with $0 \le |\alpha| \le 2$ and using (4.4.55) and (4.4.66) we obtain

$$\begin{aligned} \frac{d}{dt}\|\omega\|_{H^2}^2 &\le C\|\nabla U\|_{L^\infty}^2(\|\omega\|_{H^2}^2 + \|U\|^2) + 2\|\omega\|_{H^2}^2 + \|\Theta\|_{H^3}^2 \\ &= \big(C\|\nabla U\|_{L^\infty} + 2\big)\|\omega\|_{H^2}^2 + C\|\nabla U\|_{L^\infty}^2\|U\|^2 + \|\Theta\|_{H^3}^2 \\ &\le C(T)\|\omega\|_{H^2}^2 + C(T). \end{aligned} \tag{4.4.75}$$

Then Gronwall's inequality implies that

$$\begin{aligned} \|\omega(\cdot,t)\|_{H^2}^2 &\le e^{CT}\Big(\|\omega(\cdot,0)\|_{H^2}^2 + \int_0^T C d\tau\Big) \\ &\le C(T), \quad \forall\, 0 \le t \le T, \end{aligned} \tag{4.4.76}$$

which, together with Lemmas 4.2.4 and 4.4.2, implies (4.4.65). This completes the proof of Lemma 4.4.9.

To complete the regularity stated in Theorem 4.4.1, it remains to estimate $\|\Theta\|_{H^4(\Omega)}^2$. Using the results obtained in previous lemmas, we can easily prove the following

Lemma 4.4.10. *Under the assumptions of Theorem 4.1.2, for any $T > 0$, there exists a constant $N = N(T) > 0$ such that*

$$\|\Theta\|_{L^2([0,T];H^4(\Omega))}^2 \le N. \tag{4.4.77}$$

Proof. First, we rewrite the equation (4.4.43) in terms of Θ_t as:

$$\kappa\Delta(\Theta_t) = \Theta_{tt} + U_t \cdot \nabla\Theta + U \cdot \nabla\Theta_t.$$

Since $\Theta_t|_{\partial\Omega} = 0$, applying Lemma 4.2.3 to the above equation we get

$$\|\Theta_t\|_{H^2}^2 \le C(\|\Theta_{tt}\|^2 + \|U_t \cdot \nabla\Theta\|^2 + \|U \cdot \nabla\Theta_t\|^2). \tag{4.4.78}$$

Using previous results we estimate the RHS of (4.4.78) as follows:

$$\begin{aligned} &C(\|\Theta_{tt}\|^2 + \|U_t \cdot \nabla\Theta\|^2 + \|U \cdot \nabla\Theta_t\|^2) \\ &\le C(\|\Theta_{tt}\|^2 + \|U_t\|^2\|\nabla\Theta\|_{L^\infty}^2 + \|U\|_{L^\infty}^2\|\nabla\Theta_t\|^2) \\ &\le C(\|\Theta_{tt}\|^2 + \|U_t\|^2\|\Theta\|_{H^3}^2 + \|U\|_{H^2}^2\|\nabla\Theta_t\|^2) \\ &\le C(\|\Theta_{tt}\|^2 + 1). \end{aligned} \tag{4.4.79}$$

So that, (4.4.78) is updated as

$$\|\Theta_t\|_{H^2}^2 \le C(\|\Theta_{tt}\|^2 + 1). \tag{4.4.80}$$

Now, for any $T > 0$, we integrate (4.4.54) in time over $[0, T]$ to get

$$F(T) + \beta_3 \int_0^T F(t)dt + \int_0^T \|\Theta_{tt}\|^2 dt \le F(0), \tag{4.4.81}$$

which, together with (4.4.76), implies that

$$\begin{aligned} \|\Theta_t\|_{L^2([0,T];H^2(\Omega))}^2 &\le C(\|\Theta_{tt}\|_{L^2([0,T];L^2(\Omega))}^2 + T) \\ &\le C(T). \end{aligned} \tag{4.4.82}$$

For the H^4 norm of Θ, Lemma 4.2.3 with $m = 2, p = 2$ and previous estimates indicate that

$$\begin{aligned} \|\Theta\|_{H^4}^2 &\le C\big(\|\Theta_t\|_{H^2}^2 + \|U \cdot \nabla\Theta\|_{H^2}^2\big) \\ &\le C\big(\|\Theta_t\|_{H^2}^2 + \|U\|_{L^\infty}^2\|\Theta\|_{H^3}^2 + \|\nabla U\|_{L^\infty}^2\|\Theta\|_{H^2}^2 + \|U\|_{H^2}^2\|\nabla\Theta\|_{L^\infty}^2\big) \\ &\le C\big(\|\Theta_t\|_{H^2}^2 + C(T)\big). \end{aligned} \tag{4.4.83}$$

Therefore, (4.4.77) follows from (4.4.82)–(4.4.83). This completes the proof of Lemma 4.4.10.

Lemmas 4.4.4, 4.4.7–4.4.10 conclude Theorem 4.4.1. Now we prove the uniqueness of the solution.

Theorem 4.4.2. *Under the assumptions of Theorem 4.1.2, the solution of (4.4.1)–(4.4.2) is unique.*

Proof. For any fixed $T > 0$, suppose there are two solutions $(\Theta_1, U_1, \bar{P}_1), (\Theta_2, U_2, \bar{P}_2)$ to (4.4.1)–(4.4.2). Setting $\tilde{\Theta} = \Theta_1 - \Theta_2, \tilde{U} = U_1 - U_2, \tilde{P} = \bar{P}_1 - \bar{P}_2$, then $(\tilde{\Theta}, \tilde{U}, \tilde{P})$

satisfy

$$\begin{cases} \tilde{U}_t + U_1 \cdot \nabla \tilde{U} + \tilde{U} \cdot \nabla U_2 + \nabla \tilde{P} = \tilde{\Theta}(0,1)^T, \\ \tilde{\Theta}_t + U_1 \cdot \nabla \tilde{\Theta} + \tilde{U} \cdot \nabla \Theta_2 = \kappa \Delta \tilde{\Theta}, \\ \nabla \cdot \tilde{U} = 0, \\ \tilde{U}(\mathbf{x}, 0) = 0, \quad \tilde{\Theta}(\mathbf{x}, 0) = 0, \ \mathbf{x} \in \Omega, \\ \tilde{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \tilde{\Theta}|_{\partial\Omega} = 0. \end{cases} \tag{4.4.84}$$

Taking the L^2 inner products of $(4.4.84)_1$ with $\tilde{U}$ and $(4.4.84)_2$ with $\tilde{\Theta}$ respectively we get

$$\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2\Big) + \kappa\|\nabla\tilde{\Theta}\|^2 = -\int_\Omega \tilde{\Theta}(\tilde{U}\cdot\nabla\Theta_2)d\mathbf{x} - \int_\Omega \tilde{U}\cdot(\tilde{U}\cdot\nabla U_2)d\mathbf{x} + \int_\Omega \tilde{\Theta}\tilde{v}d\mathbf{x}. \tag{4.4.85}$$

Using the estimates for Θ_2 and U_2, it follows from (4.4.85) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2\Big) + \kappa\|\nabla\tilde{\Theta}\|^2 \\ &\le \|\nabla\Theta_2\|_{L^\infty}(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2) + \|\nabla U_2\|_{L^\infty}\|\tilde{U}\|^2 + \frac{1}{2}(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2) \\ &\le C(T)(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2), \quad \forall\, t \in [0,T], \end{aligned} \tag{4.4.86}$$

which implies that

$$\big(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2\big) \le e^{-2C(T)t}\big(\|\tilde{\Theta}(0)\|^2 + \|\tilde{U}(0)\|^2\big) = 0,$$

for any $t \in [0,T]$. We conclude the theorem by noticing that $T > 0$ is arbitrary.

This theorem and Theorem 4.4.1 imply our main result, Theorem 4.1.2.

Remark 4.4.2. *The ideas applied in the proof of Theorem 4.1.2 can be adopted to study the initial-boundary value problem for (4.1.1) with $\nu = 0, \kappa > 0$ and the Neumann boundary condition on θ (i.e., $\frac{\partial\theta}{\partial\mathbf{n}}\big|_{\partial\Omega} = 0$). In this case, due to the conservation of total mass, the asymptotic state of θ is $\hat{\theta} = \frac{1}{|\Omega|}\int_\Omega \theta_0 d\mathbf{x}$. Similar results as in Theorem 4.1.2 hold for this case. We omit the details here.*

Remark 4.4.3. *It is interesting to study the 2D Boussinesq equations over bounded domains with non-smooth boundary, e.g., any polygonal domain. In that case, we* *have to introduce a weak solution. Similar to Navier-Stokes equations, one could use several formulations, e.g., velocity and pressure formulation, vorticity and stream function formulation or stream function formulation. In particular, the regularity of the solutions is an interesting problem when the domain is a polygon. We leave the study in the future.*

CHAPTER V

MATHEMATICAL STUDY OF MULTI-PHASE/MIXING FLOWS

5.1 Introduction

In this chapter we generalize the study of the 2D Boussinesq equations in the direction of multi-phase/mixing flows.

We consider the following initial-boundary value problem for a model of a two-phase flow under shear and the influence of gravitational force:

$$\begin{cases} \phi_t + U \cdot \nabla\phi = \Delta\mu, \\ \mu = -\alpha\Delta\phi + F'(\phi), \\ U_t + U \cdot \nabla U + \nabla P = \nu\Delta U + \mu\nabla\phi + \theta\overrightarrow{\mathbf{e_2}}, \\ \theta_t + U \cdot \nabla\theta = 0, \\ \nabla \cdot U = 0; \\ (\phi, \mu, U, \theta)(\mathbf{x}, 0) = (\phi_0, \mu_0, U_0, \theta_0)(\mathbf{x}), \\ \nabla\phi \cdot \mathbf{n}|_{\partial\Omega} = \nabla\mu \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad U|_{\partial\Omega} = 0, \end{cases} \tag{5.1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Here, ϕ is the order parameter and μ is a chemical potential derived from a coarse-grained study of the free energy of the fluid (c.f. [37]), U denotes the velocity and θ is the temperature. The constant $\nu > 0$ models viscosity.

We also study the IBVP for a simplified model of a two-component mixture, with a diffusive mass exchange among the medium particles of various density accounted for:

$$\begin{cases} \rho(U_t + U\cdot\nabla U) + \nabla P = \lambda\big(\nabla\rho\cdot\nabla U + U\cdot\nabla(\nabla\rho)\big) + \mu\Delta U + \vec{f}\rho, \\ \rho_t + U\cdot\nabla\rho = \lambda\Delta\rho, \\ \nabla\cdot U = 0; \\ (U,\rho)(\mathbf{x},0) = (U_0,\rho_0)(\mathbf{x}), \quad m \le \rho_0(\mathbf{x}) \le M; \\ U|_{\partial\Omega} = 0, \quad \nabla\rho\cdot\mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{5.1.2}$$

where ρ, U denote the density and velocity respectively, $\mu > 0$ is the coefficient of viscosity and $\lambda > 0$ models diffusion, and $m, M > 0$ are constants.

In this chapter, for the modeling equations of multi-phase flows, we generalize the study of [10] by considering additionally the effect of gravitational force in the motion of fluid. In the direction of mixing flows, we build up the regularity of the weak solution obtained in [5]. For both cases, we study global existence of smooth solutions to the initial-boundary value problems. For the global existence of smooth solutions, we require the following compatibility conditions:

For multi-phase flow model (5.1.1):

$$\begin{cases} \nabla\cdot U_0 = 0,\ \nabla\phi_0\cdot\mathbf{n}|_{\partial\Omega} = \nabla\mu_0\cdot\mathbf{n}|_{\partial\Omega} = U_0|_{\partial\Omega} = 0, \\ \nu\Delta U_0 + \mu_0\nabla\phi_0 + \theta_0\mathbf{e}_2 - \nabla P_0 = 0, \quad \mathbf{x}\in\partial\Omega,\ t = 0, \\ \mu_0 = -\alpha\Delta\phi_0 + F'(\phi_0), \end{cases} \tag{5.1.3}$$

where $P_0(\mathbf{x}) = P(\mathbf{x}, 0)$ is the solution to the Neumann boundary problem

$$\begin{cases} \Delta P_0 = \nabla\cdot[\theta_0\mathbf{e}_2 + \mu_0\nabla\phi_0 - U_0\cdot\nabla U_0], \quad \mathbf{x}\in\Omega, \\ \nabla P_0\cdot\mathbf{n}|_{\partial\Omega} = [\nu\Delta U_0 + \theta_0\mathbf{e}_2]\cdot\mathbf{n}|_{\partial\Omega}. \end{cases} \tag{5.1.4}$$

For mixing flow model (5.1.2):

$$\begin{cases} \nabla\cdot U_0 = 0,\ U_0|_{\partial\Omega} = 0, \quad \nabla\rho_0\cdot\mathbf{n}|_{\partial\Omega} = 0, \\ \lambda\nabla\rho_0\cdot\nabla U_0 + \mu\Delta U_0 + \vec{f}_0\rho_0 - \nabla P_0 = 0, \quad \mathbf{x}\in\partial\Omega, \end{cases} \tag{5.1.5}$$

where $P_0(\mathbf{x})$ is the solution to the Neumann boundary problem

$$\begin{cases} \nabla\cdot\left(\dfrac{\nabla P_0}{\rho_0}\right) = \nabla\cdot\left(\dfrac{\lambda}{\rho_0}\big(\nabla\rho_0\cdot\nabla U_0 + U_0\cdot\nabla(\nabla\rho_0)\big) - U_0\cdot\nabla U_0 + \dfrac{\mu}{\rho_0}\Delta U_0 + \vec{f}_0\right), \\ \nabla P_0\cdot\mathbf{n}|_{\partial\Omega} = [\lambda\nabla\rho_0\cdot\nabla U_0 + \mu\Delta U_0 + \vec{f}_0\rho_0]\cdot\mathbf{n}|_{\partial\Omega}. \end{cases} \tag{5.1.6}$$

Notation 5.1.1. *Unless specified, throughout this chapter, C and C_i will denote generic constants which are independent of the unknown function. In addition, the values of the constants are different from those in previous chapters.*

The following theorems are the main results of this chapter.

Theorem 5.1.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary and suppose that $F(\cdot)$ satisfies the following conditions:*

- (H_1) *F is of C^5 class and $F\geq 0$.;*
- (H_2) *There exist constants $C_1, C_2 > 0$ such that $|F^n(\phi)|\leq C_1|\phi|^{p-n}+C_2$,*
  $n=1,\cdots,5,\quad \forall\ 5\leq p<\infty$ and $\phi\in\mathbb{R}$;
- (H_3) *There exists a constant $F_1>0$ such that $F''\geq -F_1$.*

If $(\phi_0(\mathbf{x})\in H^4(\Omega)$, $\mu_0(\mathbf{x})\in H^2(\Omega)$, $(\theta_0(\mathbf{x}), U_0(\mathbf{x}))\in H^3(\Omega)$ satisfy the compatibility conditions (5.1.3)–(5.1.4), then there exists a unique solution (ϕ,μ,θ,U) of (5.1.1) globally in time such that $\phi\in C([0,T];H^4(\Omega))\cap L^2([0,T];H^6(\Omega))$, $\mu\in C([0,T];H^2(\Omega))\cap L^2([0,T];H^4(\Omega))$, $\theta\in C([0,T];H^3(\Omega))$ and $U\in C([0,T];H^3(\Omega))\cap L^2([0,T];H^4(\Omega))$ for any $T>0$.

Theorem 5.1.2. *Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with smooth boundary and suppose that the constant $\mu_1 = 2\mu-\lambda(M-m)>0$. If $(\rho_0(\mathbf{x}), U_0(\mathbf{x}))\in H^3(\Omega)$ satisfies the compatibility conditions (5.1.5)–(5.1.6) and $\vec{f}\in C([0,T];H^1(\Omega))\cap L^2([0,T];H^2(\Omega))$, $\vec{f}_t\in L^2([0,T];L^2(\Omega))$, then there exists a unique solution (ρ, U) of (5.1.2) globally in time such that $(\rho,U)(\mathbf{x},t)\in C([0,T];H^3(\Omega))\cap L^2([0,T];H^4(\Omega))$ for any $T>0$.*

Remark 5.1.1. *The assumptions* $(H_1) - (H_3)$ *in Theorem 5.1.1 are satisfied for a number of applications such as* $F(x) = (1-x^2)^2$*; see [10] and references therein. The condition* $2\mu - \lambda(M - m) > 0$ *in the statement of Theorem 5.1.2 suggests that, to ensure the global existence of smooth solution to (5.1.2), the rate of mass exchange between the two components can not exceed the threshold which is determined by the viscosity and the lower-upper bounds of the density. From the proof of Theorem 5.1.2 we will see that the number* $\frac{M-m}{2}$ *is optimal. It is not clear whether the condition can be removed till the date this thesis is written. The investigation is still underway.*

The proofs of the above theorems are in the spirit of the proof of Theorem 4.1.1. Still, there will be intensive applications of Sobolev and Ladyzhenskaya type inequalities. The standard results on Stokes equations still play an important role in the analysis. However, life is not that easy. For (5.1.1), due to the coupling of Cahn-Hilliard equation and Boussinesq equations, the nonlinear term $\mu\nabla\phi$ brings us a big challenge in the analysis. The regularity of U is much more difficult to build up than the one in the Boussinesq equations. More detailed applications of Sobolev type inequalities will be involved in the proof. We also observe that there are great differences between the Boussinesq equations and system (5.1.2). An obvious one is the appearance of the density in (5.1.2) which makes the complexity of analysis significantly increase. The reason is that, as the density is coupled with the velocity, when dealing with higher order estimates, more nonlinear terms will appear after taking derivatives in the velocity equations. Plus the second order derivative $U \cdot \nabla(\nabla)\rho$ standing in the velocity equations, the regularity of U is an substantial barrier to pass.

Since the global existence of weak solutions can be proved in similar fashion as in Chapter 4, we will focus our attention on the energy estimates which are essential for the global existence of smooth solutions. These will be done in the next section.

5.2 Multi-Phase Flow

In this section, we will prove Theorem 5.1.1. We first give the following lemma which can be proved using the arguments in [10] and Chapter 4.

Lemma 5.2.1. *Under the assumptions in Theorem 5.1.1, there exists a global weak solution* (U, θ) *of such that, for any* $T > 0$, $\phi \in C\big([0, T]; H^1(\Omega)\big) \cap L^2\big([0, T]; H^3(\Omega)\big)$, $\mu \in L^2([0, T]; H^1(\Omega))$, $U \in C([0, T); L^2(\Omega)) \cap L^2([0, T); H_0^1(\Omega))$ *and* $\theta \in C([0, T); L^p(\Omega))$ *for* $\forall\ 1 \leq p \leq \infty$.

Now we establish the regularity and uniqueness of the solution obtained in Lemma 5.2.1. The following theorem gives the key estimates.

Theorem 5.2.1. *Under the assumptions of Theorem 5.1.1, the solution obtained in Lemma 5.2.1 satisfies the following estimates:*

$$\begin{aligned} &\|\phi\|^2_{C([0,T);H^4(\Omega))} + \|\phi\|^2_{L^2([0,T];H^6(\Omega))} + \|\mu\|^2_{C([0,T);H^2(\Omega))} + \|\mu\|^2_{L^2([0,T];H^4(\Omega))} \leq C; \\ &\|U\|_{C([0,T);H^3(\Omega))} + \|U\|_{L^2([0,T);H^4(\Omega))} + \|\theta\|_{C([0,T);H^3(\Omega))} \leq C, \end{aligned} \tag{5.2.1}$$

for any $T > 0$.

First, due to the conservation of total mass, we have

Lemma 5.2.2. *Under the assumptions of Theorem 5.1.1, it holds that*

$$\|\theta\|_{L^p} = \|\theta_0\|_{L^p}, \quad 1 \leq p \leq \infty. \tag{5.2.2}$$

Next, we give some basic estimate of the solution.

Lemma 5.2.3. *Under the assumptions of Theorem 5.1.1, it holds that*

$$\begin{aligned} &\phi \in C\big([0, T]; H^1(\Omega)\big) \cap L^2\big([0, T]; H^3(\Omega)\big) \\ &U \in C\big([0, T]; L^2(\Omega)\big) \cap L^2\big([0, T]; H^1(\Omega)\big). \end{aligned} \tag{5.2.3}$$

Proof. Taking L^2 inner product of $(5.1.1)_3$ with U we have

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \nu\|\nabla U\|^2 = \int_\Omega \mu(\nabla\phi\cdot U)d\mathbf{x} + \int_\Omega \theta v d\mathbf{x}. \tag{5.2.4}$$

Taking L^2 inner product of $(5.1.1)_1$ with ϕ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\phi\|^2 &= \int_\Omega \phi\Delta\mu\mathbf{x}\\ &= -\int_\Omega \nabla\mu\cdot\nabla\phi d\mathbf{x}\\ &= -\alpha\|\Delta\phi\|^2 - \int_\Omega F''(\phi)|\nabla\phi|^2 d\mathbf{x},\end{aligned}$$

which yields

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2 + \alpha\|\Delta\phi\|^2 = -\int_\Omega F''(\phi)|\nabla\phi|^2 d\mathbf{x}. \tag{5.2.5}$$

Taking L^2 inner product of $(5.1.1)_1$ with μ we have

$$\frac{d}{dt}\left(\frac{\alpha}{2}\|\nabla\phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}\right) + \|\nabla\mu\|^2 = -\int_\Omega \mu(\nabla\phi\cdot U)d\mathbf{x}. \tag{5.2.6}$$

Adding (5.2.4) and (5.2.6) we get

$$\frac{d}{dt}\left(\frac{1}{2}\|U\|^2 + \frac{\alpha}{2}\|\nabla\phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}\right) + \nu\|\nabla U\|^2 + \|\nabla\mu\|^2 = \int_\Omega \theta v d\mathbf{x}. \tag{5.2.7}$$

Applying Cauchy-Schwartz inequality to the RHS of (5.2.7) and using (5.2.2) we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|U\|^2 + \frac{\alpha}{2}\|\nabla\phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}\right) + \nu\|\nabla U\|^2 + \|\nabla\mu\|^2 \le C\|\theta_0\|^2 + \frac{\nu}{2}\|\nabla U\|^2,$$

where we have used Poincaré's inequality to U. The preceding estimate implies that

$$\frac{d}{dt}\left(\frac{1}{2}\|U\|^2 + \frac{\alpha}{2}\|\nabla\phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}\right) + \frac{\nu}{2}\|\nabla U\|^2 + \|\nabla\mu\|^2 \le C. \tag{5.2.8}$$

Adding (5.2.5) and (5.2.8), using (H_2) we see that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\phi\|^2 + \frac{1}{2}\|U\|^2 + \frac{\alpha}{2}\|\nabla\phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}\right) + \alpha\|\Delta\phi\|^2 + \frac{\nu}{2}\|\nabla U\|^2 + \|\nabla\mu\|^2\\ &\le C - \int_\Omega F''(\phi)|\nabla\phi|^2 d\mathbf{x}\\ &\le C + F_1\|\nabla\phi\|^2.\end{aligned} \tag{5.2.9}$$

Letting

$$E_1(t) \equiv \frac{1}{2}\|\phi\|^2 + \frac{1}{2}\|U\|^2 + \frac{\alpha}{2}\|\nabla\phi\|^2 + \int_\Omega F(\phi)d\mathbf{x}, \tag{5.2.10}$$

since $F \geq 0$, we know from (5.2.9) that

$$\frac{d}{dt}E_1(t) + \alpha\|\Delta\phi\|^2 + \frac{\nu}{2}\|\nabla U\|^2 + \|\nabla\mu\|^2 \leq C + \frac{2F_1}{\alpha}E_1(t), \tag{5.2.11}$$

which implies, after applying Gronwall's inequality, that

$$E_1(t) \leq C, \ \forall\, 0 \leq t \leq T \quad \text{and} \quad \int_0^T \Big(\alpha\|\Delta\phi\|^2 + \frac{\nu}{2}\|\nabla U\|^2 + \|\nabla\mu\|^2\Big)d\tau \leq C. \tag{5.2.12}$$

Applying Lemma 4.2.4 to $F = \nabla\phi$ and using $(5.1.1)_2$ and (H_2) we see that

$$\begin{aligned}
\|\nabla\phi\|_{H^2}^2 &\leq C\big(\|\Delta\phi\|_{H^1}^2 + \|\nabla\phi\|^2\big) \\
&\leq C\big(\|\Delta\phi\|^2 + \|\nabla\phi\|^2 + \|\nabla\mu\|^2 + \|F''(\phi)\nabla\phi\|^2\big) \\
&\leq C\big(\|\Delta\phi\|^2 + \|\nabla\phi\|^2 + \|\nabla\mu\|^2 + \|\phi\|_{L^{4(p-2)}}^{2(p-2)}\|\nabla\phi\|_{L^4}^2 + \|\nabla\phi\|^2\big).
\end{aligned}$$

Since

$$\begin{aligned}
\|\phi\|_{L^{4(p-2)}}^{2(p-2)}\|\nabla\phi\|_{L^4}^2 &\leq C\|\phi\|_{H^1}^{2(p-2)}\|\nabla\phi\|_{H^1}^2 \\
&\leq C\|\phi\|_{H^1}^{2(p-2)}\big(\|\Delta\phi\|^2 + \|\nabla\phi\|^2\big),
\end{aligned}$$

we know that

$$\|\nabla\phi\|_{H^2}^2 \leq C\Big(\|\Delta\phi\|^2 + \|\nabla\phi\|^2 + \|\nabla\mu\|^2 + \|\phi\|_{H^1}^{2(p-2)}\big(\|\Delta\phi\|^2 + \|\nabla\phi\|^2\big)\Big),$$

which implies that

$$\|\phi\|_{H^3}^2 \leq C\Big(\|\Delta\phi\|^2 + \|\nabla\phi\|^2 + \|\phi\|^2 + \|\nabla\mu\|^2 + \|\phi\|_{H^1}^{2(p-2)}\big(\|\Delta\phi\|^2 + \|\nabla\phi\|^2\big)\Big).$$

Therefore, (5.2.10) and (5.2.12) yield (5.2.3). This completes the proof of Lemma 5.2.3.

The next lemma is the corner stone of this section which gives the estimate of $\|\phi\|_{C([0,T];H^2(\Omega))}^2$.

Lemma 5.2.4. *Under the assumptions of Theorem 5.1.1, it holds that*

$$\|\phi\|_{C([0,T];H^2(\Omega))}^2 \leq C. \tag{5.2.13}$$

Proof. Taking L^2 inner product of $(5.1.1)_1$ with ϕ_t we have

$$\|\phi_t\|^2 + \int_\Omega \phi_t(U\cdot\nabla\phi)d\mathbf{x} = \int_\Omega \phi_t\Delta\mu d\mathbf{x}. \tag{5.2.14}$$

Using the boundary condition we calculate the RHS of (5.2.14) as follows:

$$\begin{aligned}\int_\Omega \phi_t\Delta\mu d\mathbf{x} &= \int_\Omega \mu\Delta\phi_t d\mathbf{x}\\ &= -\frac{d}{dt}\left(\frac{\alpha}{2}\|\Delta\phi\|^2\right) + \int_\Omega F'(\phi)\Delta\phi_t d\mathbf{x}\\ &= -\frac{d}{dt}\left(\frac{\alpha}{2}\|\Delta\phi\|^2 + \frac{1}{2}\int_\Omega F''(\phi)|\nabla\phi|^2 d\mathbf{x}\right) + \frac{1}{2}\int_\Omega F'''(\phi)\phi_t|\nabla\phi|^2 d\mathbf{x}\end{aligned} \tag{5.2.15}$$

Plugging (5.2.15) into (5.2.14) we get

$$\frac{d}{dt}\left(\frac{\alpha}{2}\|\Delta\phi\|^2+\frac{1}{2}\int_\Omega F''(\phi)|\nabla\phi|^2 d\mathbf{x}\right)+\|\phi_t\|^2 = \frac{1}{2}\int_\Omega F'''(\phi)\phi_t|\nabla\phi|^2 d\mathbf{x}-\int_\Omega \phi_t(U\cdot\nabla\phi)d\mathbf{x}. \tag{5.2.16}$$

Using Cauchy-Schwartz inequality, Lemma 4.2.1, (H_2) and (5.2.3) we estimate the first term on the RHS of (5.2.16) as follows:

$$\begin{aligned}\frac{1}{2}\int_\Omega F'''(\phi)\phi_t|\nabla\phi|^2 d\mathbf{x} &\le \frac{1}{4}\|\phi_t\|^2 + \frac{1}{4}\int_\Omega |F'''(\phi)|^2|\nabla\phi|^4 d\mathbf{x}\\ &\le \frac{1}{4}\|\phi_t\|^2 + C\int_\Omega \big(|\phi|^{2(p-3)} + C\big)|\nabla\phi|^4 d\mathbf{x}\\ &\le \frac{1}{4}\|\phi_t\|^2 + C\|\nabla\phi\|_{L^4}^4 + C\|\phi\|_{L^{4(p-3)}}^{2(p-3)}\|\nabla\phi\|_{L^8}^4\\ &\le \frac{1}{4}\|\phi_t\|^2 + C\|\nabla\phi\|_{L^4}^4 + C\|\phi\|_{H^1}^{2(p-3)}\big(\|D^2\phi\|_{L^4}^2\|\nabla\phi\|^2 + \|\nabla\phi\|^4\big)\\ &\le \frac{1}{4}\|\phi_t\|^2 + C\|\phi\|_{H^3}^2 + C\end{aligned} \tag{5.2.17}$$

The second term on the RHS of (5.2.16) is estimated as

$$\begin{aligned}-\int_\Omega \phi_t(U\cdot\nabla\phi)d\mathbf{x} &\le \frac{1}{4}\|\phi_t\|^2 + \|U\cdot\nabla\phi\|^2\\ &\le \frac{1}{4}\|\phi_t\|^2 + C\|U\|\|\nabla U\|\big(\|\nabla\phi\|\|D^2\phi\| + \|\nabla\phi\|^2\big)\\ &\le \frac{1}{4}\|\phi_t\|^2 + C\|\nabla U\|^2 + C\|\phi\|_{H^2}^2 + C.\end{aligned} \tag{5.2.18}$$

Combining (5.2.16)–(5.2.18) we see that

$$\frac{d}{dt}\left(\frac{\alpha}{2}\|\Delta\phi\|^2 + \frac{1}{2}\int_\Omega F''(\phi)|\nabla\phi|^2 d\mathbf{x}\right) + \frac{1}{2}\|\phi_t\|^2 \le C_1(t), \tag{5.2.19}$$

where $C_1(t) = C\|\phi\|_{H^3}^2 + C\|\nabla U\|^2 + C$ satisfying $\int_0^T C_1(\tau)d\tau \le C(T)$ due to (5.2.3). Multiplying (5.2.9) by $\frac{2F_1}{\alpha}$ then adding the result to (5.2.19) we have

$$\frac{d}{dt}E_2(t) + \frac{2F_1}{\alpha}\left(\alpha\|\Delta\phi\|^2 + \frac{\nu}{2}\|\nabla U\|^2 + \|\nabla\mu\|^2\right) + \frac{1}{2}\|\phi_t\|^2 \le C_2(t), \tag{5.2.20}$$

where

$$E_2(t) \equiv \frac{F_1}{\alpha}\|\phi\|^2 + \frac{F_1}{\alpha}\|U\|^2 + \frac{2F_1}{\alpha}\int_\Omega F(\phi)d\mathbf{x} + \frac{\alpha}{2}\|\Delta\phi\|^2 + F_1\|\nabla\phi\|^2 + \frac{1}{2}\int_\Omega F''(\phi)|\nabla\phi|^2 d\mathbf{x},$$

and

$$C_2(t) = \frac{2CF_1}{\alpha} + \frac{2F_1^2}{\alpha}\|\nabla\phi\|^2 + C_1(t), \quad \text{satisfying} \quad \int_0^T C_2(\tau)d\tau \le C.$$

It is clear, since $F'' \ge -F_1$, that

$$E_2(t) \ge \frac{F_1}{\alpha}\|\phi\|^2 + \frac{F_1}{\alpha}\|U\|^2 + \frac{2F_1}{\alpha}\int_\Omega F(\phi)d\mathbf{x} + \frac{\alpha}{2}\|\Delta\phi\|^2 + \frac{F_1}{2}\|\nabla\phi\|^2. \tag{5.2.21}$$

Integrating (5.2.20) over $[0,T]$ we get

$$E_2(t) \le C, \quad \forall\ 0 \le t \le T \quad \text{and} \quad \int_0^T \|\phi_t\|^2 d\tau \le C, \tag{5.2.22}$$

which implies, in view of (5.2.21), that

$$\phi \in C\big([0,T]; H^2(\Omega)\big). \tag{5.2.23}$$

This completes the proof of Lemma 5.2.4.

With the help of Lemma 5.2.4 we are now ready to improve the regularity of U and build up higher order regularity of ϕ.

Lemma 5.2.5. *Under the assumptions of Theorem 5.1.1, it holds that*

$$\begin{aligned} &\|U\|_{C([0,T];H^1(\Omega))}^2 + \|U\|_{L^2([0,T];H^2(\Omega))}^2 \le C; \\ &\|\mu\|_{L^2([0,T];H^2(\Omega))}^2 + \|\phi\|_{L^2([0,T];H^4(\Omega))}^2 \le C. \end{aligned} \tag{5.2.24}$$

Proof. Taking L^2 inner product of $(5.1.1)_3$ with U_t we have

$$\begin{aligned} \frac{\nu}{2}\frac{d}{dt}\|\nabla U\|^2 + \|U_t\|^2 &= -\int_\Omega U_t \cdot (U \cdot \nabla U)d\mathbf{x} + \int_\Omega \theta v_t d\mathbf{x} + \int_\Omega \mu(\nabla\phi \cdot U_t)d\mathbf{x} \\ &\le \frac{1}{4}\|U_t\|^2 + 3\|\theta_0\|^2 + 3\|U \cdot \nabla U\|^2 + 3\|\mu\nabla\phi\|^2, \end{aligned} \tag{5.2.25}$$

where we have used Cauchy-Schwartz inequality. Similar arguments as in Chapter 4 yields

$$\|U\cdot\nabla U\|^2 \le C\big(\|\nabla U\|^2\|\nabla U\|^2 + C\big) + \frac{1}{12}\|U_t\|^2. \tag{5.2.26}$$

Similarly, by Lemma 5.2.4 we estimate the last term on the RHS of (5.2.25) as

$$\|\mu\nabla\phi\|^2 \le C + C\|\nabla\mu\|^2. \tag{5.2.27}$$

So we update (5.2.25) as

$$\nu\frac{d}{dt}\|\nabla U\|^2 + \|U_t\|^2 \le C\|\nabla U\|^2\|\nabla U\|^2 + C\|\nabla\mu\|^2 + C. \tag{5.2.28}$$

Applying Gronwall's inequality to (5.2.28) and using (5.2.3) we see that

$$U \in C\big([0,T]; H^1(\Omega)\big) \quad \text{and} \quad U_t \in L^2\big([0,T]; L^2(\Omega)\big), \tag{5.2.29}$$

which together with Lemma 4.2.2 and (5.2.27) implies that

$$U \in L^2\big([0,T]; H^2(\Omega)\big). \tag{5.2.30}$$

It is easy to see from $(5.1.1))_1$ and Lemma 5.2.4 that

$$\begin{aligned}\|\nabla\mu\|_{H^1} &\le C\big(\|\Delta\mu\| + \|\nabla\mu\|\big)\\ &\le C\big(\|\phi_t\| + \|U\cdot\nabla\phi\| + \|\nabla\mu\|\big)\\ &\le C\big(\|\phi_t\| + 1 + \|\nabla\mu\|\big).\end{aligned} \tag{5.2.31}$$

Then we see from (5.2.12) and (5.2.22) that

$$\mu \in L^2\big([0,T]; H^2(\Omega)\big). \tag{5.2.32}$$

Using Lemma 4.2.4, (H_2) and (5.2.23) we see that

$$\begin{aligned}
\|\phi\|_{H^4}^2 &\le C\big(\|\Delta\phi\|_{H^2}^2 + \|\phi\|_{H^3}^2\big)\\
&\le C\big(\|\mu\|_{H^2}^2 + \|\phi\|_{H^3}^2 + \|F'(\phi)\|_{H^2}^2\big)\\
&\le C\big(\|\mu\|_{H^2}^2 + \|\phi\|_{H^3}^2 + \|\phi\|_{L^{2(p-1)}}^{2(p-1)} + C + \|\phi\|_{L^{4(p-2)}}^{2(p-2)}\|\nabla\phi\|_{L^4}^2 + \|\nabla\phi\|^2 +\\
&\qquad \|\phi\|_{L^{4(p-2)}}^{2(p-2)}\|D^2\phi\|_{L^4}^2 + \|D^2\phi\|^2 + \|\phi\|_{L^{2(p-3)}}^{(p-3)}\|\nabla\phi\|_{L^8}^4 + \|\nabla\phi\|_{L^4}^4\big)\\
&\le C\big(\|\mu\|_{H^2}^2 + \|\phi\|_{H^3}^2 + \|\phi\|_{H^2}^4 + C\big)\\
&\le C\big(\|\mu\|_{H^2}^2 + \|\phi\|_{H^3}^2 + C\big),
\end{aligned} \tag{5.2.33}$$

which together with (5.2.3) and (5.2.32) implies that

$$\phi \in L^2\big([0,T];H^4(\Omega)\big).$$

This completes the proof of Lemma 5.2.5.

Now we improve the regularity of μ and ϕ.

Lemma 5.2.6. *Under the assumptions of Theorem 5.1.1, it holds that*

$$\|\mu\|_{C([0,T];H^2(\Omega))}^2 + \|\phi\|_{C([0,T];H^4(\Omega))}^2 \le C. \tag{5.2.34}$$

Proof. Taking L^2 inner product of $(5.1.1)_1$ with μ_t we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 + \alpha\|\nabla\phi_t\|^2 &= -\int_\Omega F''(\phi)\phi_t^2 d\mathbf{x} - \int_\Omega \mu_t(U\cdot\nabla\phi)d\mathbf{x}\\
&\le F_1\|\phi_t\|^2 + C\|U\cdot\nabla\phi\|^2 + \varepsilon\|\mu_t\|^2\\
&\le F_1\|\phi_t\|^2 + C + \varepsilon\alpha^2\|\Delta\phi_t\|^2 + \varepsilon\|F''(\phi)\phi_t\|^2\\
&\le C\|\phi_t\|^2 + C + \varepsilon\alpha^2\|\Delta\phi_t\|^2,
\end{aligned} \tag{5.2.35}$$

where we have used (H_3) and ε is a number to be determined. Differentiating $(5.1.1)_1$ with respect to t, multiplying the resulting equation by ϕ_t then integrating over Ω

yields

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\phi_t\|^2 + \alpha\|\Delta\phi_t\|^2 &= \int_\Omega F'(\phi)_t \Delta\phi_t d\mathbf{x} - \int_\Omega \phi_t (U_t \cdot \nabla\phi) d\mathbf{x} \\
&= \int_\Omega F'(\phi)_t \Delta\phi_t d\mathbf{x} + \int_\Omega \phi (U_t \cdot \nabla\phi_t) d\mathbf{x} \\
&\le \frac{\alpha}{4}\|\Delta\phi_t\|^2 + C\|F''(\phi)\phi_t\|^2 + C(\varepsilon_1)\|\phi U_t\|^2 + \varepsilon_1\|\nabla\phi_t\|^2 \\
&\le \frac{\alpha}{4}\|\Delta\phi_t\|^2 + C\|\phi_t\|^2 + C(\varepsilon_1)\|U_t\|^2 + \varepsilon_1\|\nabla\phi_t\|^2,
\end{aligned}
\tag{5.2.36}
$$

where ε_1 is a number to be determined. Choosing $\varepsilon = \frac{1}{4\alpha}$ and $\varepsilon_1 = \frac{\alpha}{2}$, then combining (5.2.35) and (5.2.36) we get

$$
\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\mu\|^2 + \|\phi_t\|^2\Big) + \frac{\alpha}{2}\Big(\|\nabla\phi_t\|^2 + \|\Delta\phi_t\|^2\Big) \le C\|\phi_t\|^2 + C\|U_t\|^2 + C. \tag{5.2.37}
$$

Since $\phi_t, U_t \in L^2\big([0,T]; L^2(\Omega)\big)$, integrating (5.2.37) over $[0,T]$ yields

$$
\nabla\mu, \phi_t \in C\big([0,T]; L^2(\Omega)\big) \quad \text{and} \quad \nabla\phi_t, \Delta\phi_t \in L^2\big([0,T]; L^2(\Omega)\big), \tag{5.2.38}
$$

which together with (5.2.31) and (5.2.33) implies that

$$
\mu \in C\big([0,T]; H^2(\Omega)\big) \;\text{ and }\; \phi \in C\big([0,T]; H^4(\Omega)\big).
$$

This completes the proof of Lemma 5.2.6.

As a consequence of Lemmas 5.2.4 and 5.2.6 and (5.2.38), we can show that

Lemma 5.2.7. *Under the assumptions of Theorem, it holds that*

$$
\|\mu\|^2_{L^2([0,T];H^4(\Omega))} + \|\phi\|^2_{L^2([0,T];H^6(\Omega))} \le C. \tag{5.2.39}
$$

The proof of Lemma 5.2.7 is straightforward, we omit the details. The next lemma is essential for improving the regularity of θ.

Lemma 5.2.8. *Under the assumptions of Theorem 5.1.1, it holds that*

$$
\|U\|^2_{C([0,T];H^2(\Omega))} + \|U\|^2_{L^2([0,T];W^{2,p}(\Omega))} \le C, \quad \forall\ 1 \le p < \infty. \tag{5.2.40}
$$

Proof. Differentiating $(5.1.1)_3$ with respect to t, then taking L^2 inner product of the resulting equation with U_t we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U_t\|^2 + \nu\|\nabla U_t\|^2 \\
=&\int_\Omega \Big[-(U_t\cdot\nabla U)\cdot U_t + \theta_t v_t + \mu_t(\nabla\phi\cdot U_t)d\mathbf{x} + \mu(\nabla\phi_t\cdot U_t)\Big]d\mathbf{x} \\
=&\int_\Omega \Big[-(U_t\cdot\nabla U)\cdot U_t + \theta U\cdot\nabla v_t + \mu_t(\nabla\phi\cdot U_t) + \mu(\nabla\phi_t\cdot U_t)\Big]d\mathbf{x} \\
\equiv& I_1+I_2+I_3+I_4
\end{aligned} \tag{5.2.41}$$

Using previous results we estimate I_i in (5.2.41) as follows:

$$\begin{aligned}
&I_1 \le \|\nabla U\|\|U_t\|_{L^4}^2 \le C\|U_t\|^2 + \frac{\nu}{4}\|\nabla U_t\|;\\
&I_2 \le \|\theta U\|^2 + \frac{\nu}{4}\|\nabla U_t\|^2 \le C + \frac{\nu}{4}\|\nabla U_t\|^2;\\
&I_3 \le \|\mu_t\nabla\phi\|^2 + \|U_t\|^2 \le C\big(\|\Delta\phi_t\|^2 + C\|\phi_t\|^2\big) + \|U_t\|^2;\\
&I_4 \le \|\mu\nabla\phi_t\|^2 + \|U_t\|^2 \le C\|\nabla\phi_t\|^2 + \|U_t\|^2.
\end{aligned} \tag{5.2.42}$$

Combining (5.2.41)–(5.2.42) we obtain

$$\frac{1}{2}\frac{d}{dt}\|U_t\|^2 + \frac{\nu}{2}\|\nabla U_t\|^2 \le C\big(\|U_t\|^2 + \|\phi_t\|^2 + \|\nabla\phi_t\|^2 + \|\Delta\phi_t\|^2\big) + C. \tag{5.2.43}$$

Integrating (5.2.43) over $[0,T]$ using (5.2.29) and (5.2.38) we see that

$$U_t \in C\big([0,T];L^2(\Omega)\big)\cap L^2\big([0,T];H^1(\Omega)\big), \tag{5.2.44}$$

which together with (5.2.26) implies that

$$U \in C\big([0,T];H^2(\Omega)\big). \tag{5.2.45}$$

As consequences of previous estimates we have the following:

$$\begin{gathered}
U_t \in L^2\big([0,T];L^p(\Omega)\big),\ \ (U\cdot\nabla U)\in C\big([0,T];L^p(\Omega)\big),\ \ \theta\in C\big([0,T];L^p(\Omega)\big);\\
\mu\nabla\phi \in C\big([0,T];L^p(\Omega)\big),\quad \forall\ 1<p\le\infty,
\end{gathered}$$

which imply that

$$U \in C\big([0,T];H^2(\Omega)\big)\cap L^2\big([0,T];W^{2,p}(\Omega)\big),\quad \forall\ 1\le p<\infty. \tag{5.2.46}$$

Recalling Lemma 4.2.1 we see from (5.2.26) that

$$U \in C\big([0,T]; C^0(\bar{\Omega})\big) \cap L^2\big([0,T]; C^1(\bar{\Omega})\big). \tag{5.2.47}$$

This completes the proof of Lemma 5.2.8.

With the help of Lemma 5.2.8, similar to Lemma 4.3.6, we have

Lemma 5.2.9. *Under the assumptions of Theorem 5.1.1, it holds that*

$$\|\nabla\theta\|_{L^\infty} \le C. \tag{5.2.48}$$

Using similar arguments as in the proof of Lemmas 4.3.7–4.3.8 and with the help of Lemmas 5.2.6–5.2.8 we can easily obtain the desired estimate of U:

Lemma 5.2.10. *Under the assumptions of Theorem 5.1.1, it holds that*

$$\begin{aligned} &\|U\|^2_{C([0,T];H^3(\Omega))} + \|U_t\|^2_{L^2([0,T];H^2(\Omega))} \le C;\\ &\|\theta\|^2_{C([0,T];H^3(\Omega))} + \|U\|^2_{L^2([0,T];H^4(\Omega))} \le C. \end{aligned} \tag{5.2.49}$$

This lemma and Lemmas 5.2.6–5.2.7 conclude the regularity stated in Theorem 5.1.1. Now we prove the uniqueness of the solution.

Theorem 5.2.2. *Under the assumptions of Theorem 5.1.1, the solution is unique.*

Proof. Suppose one has two solutions $(\phi_1, \theta_1, U_1, P_1), (\phi_2, \theta_2, U_2, P_2)$, setting $\tilde{\phi} = \phi_1 - \phi_2, \tilde{\theta} = \theta_1 - \theta_2, \tilde{U} = U_1 - U_2, \tilde{P} = P_1 - P_2$, then $(\tilde{\phi}, \tilde{\theta}, \tilde{U}, \tilde{P})$ satisfy

$$\begin{cases} \tilde{\phi}_t + U_1 \cdot \nabla\tilde{\phi} + \tilde{U} \cdot \nabla\phi_2 = \Delta\tilde{\mu}\\ \tilde{\mu} = \mu_1 - \mu_2 = -\alpha\Delta\tilde{\phi} + F'(\phi_1) - F'(\phi_2)\\ \tilde{U}_t + U_1 \cdot \nabla\tilde{U} + \tilde{U} \cdot \nabla U_2 + \nabla\tilde{P} = \nu\Delta\tilde{U} + \mu_1\nabla\tilde{\phi} - \tilde{\mu}\nabla\phi_2 + \tilde{\theta}(0,1)^T\\ \tilde{\theta}_t + U_1 \cdot \nabla\tilde{\theta} + \tilde{U} \cdot \nabla\theta_2 = 0\\ \nabla \cdot \tilde{U} = 0;\\ \nabla\Phi \cdot \mathbf{n}|_{\partial\Omega} = \nabla\mu \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \tilde{U}|_{\partial\Omega} = 0,\\ \tilde{\phi}(\mathbf{x}, 0) = 0, \tilde{U}(\mathbf{x}, 0) = 0, \tilde{\theta}(\mathbf{x}, 0) = 0, \ \mathbf{x} \in \Omega. \end{cases} \tag{5.2.50}$$

Using the incompressibility and boundary conditions for the solutions, after taking L^2 inner products and integrating by parts, we find

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\phi}\|^2 + \alpha\|\Delta\tilde{\phi}\|^2 \leq \|\nabla\phi_2\|_{L^\infty}\big(\|\tilde{U}\|^2 + \|\tilde{\phi}\|^2\big) + \frac{\alpha}{4}\|\Delta\tilde{\phi}\|^2 + \|F''(\bar{\phi})\|^2_{L^\infty}\|\tilde{\phi}\|^2,$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\theta}\|^2 + \|\tilde{U}\|^2\Big) + \nu\|\nabla\tilde{U}\|^2\\ &\leq \frac{\alpha}{4}\|\Delta\tilde{\phi}\|^2 + C\|F''(\bar{\phi})\|^2_{L^\infty}\|\tilde{\phi}\|^2 + C\|\nabla\phi_2\|^2_{L^\infty}\|\tilde{U}\|^2 + \frac{1}{2}(\|\tilde{\theta}\|^2 + \|\tilde{U}\|^2),\end{aligned}$$

where $\|F''(\bar{\phi})\|^2_{L^\infty} \leq \|F''(\phi_1)\|^2_{L^\infty} + \|F''(\phi_2)\|^2_{L^\infty}$. Using estimates of $(\phi_i, \theta_i, U_i), i = 1, 2$ we get

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\phi}\|^2 + \alpha\|\Delta\tilde{\phi}\|^2 \leq C\big(\|\tilde{U}\|^2 + \|\tilde{\phi}\|^2\big) + \frac{\alpha}{4}\|\Delta\tilde{\phi}\|^2, \tag{5.2.51}$$

and

$$\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\theta}\|^2 + \|\tilde{U}\|^2\Big) + \nu\|\nabla\tilde{U}\|^2 \leq C\big(\|\tilde{\theta}\|^2 + \|\tilde{U}\|^2\big) + \|\nabla\mu_1\|_{L^\infty}\big(\|\tilde{\phi}\|^2 + \|\tilde{U}\|^2\big) + \frac{\alpha}{4}\|\Delta\tilde{\phi}\|^2. \tag{5.2.52}$$

It is clear, by (5.2.39), that $\mu \in L^2\big([0,T]; H^3(\Omega)\big)$, which implies that

$$\nabla\mu \in L^2\big([0,T]; L^\infty(\Omega)\big). \tag{5.2.53}$$

Adding (5.2.51) and (5.2.52) we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\phi}\|^2 + \|\tilde{\theta}\|^2 + \|\tilde{U}\|^2\Big) + \nu\|\nabla\tilde{U}\|^2 + \frac{\alpha}{2}\|\Delta\tilde{\phi}\|^2 \leq C(t)\big(\|\tilde{\phi}\|^2 + \|\tilde{\theta}\|^2 + \|\tilde{U}\|^2\big), \tag{5.2.54}$$

where $C(t)$ satisfies $\int_0^T C(\tau)d\tau \leq C(T)$ for any $0 \leq T < \infty$. In particular, we have

$$\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\phi}\|^2 + \|\tilde{\theta}\|^2 + \|\tilde{U}\|^2\Big) \leq C(t)\big(\|\tilde{\phi}\|^2 + \|\tilde{\theta}\|^2 + \|\tilde{U}\|^2\big),$$

which implies that

$$\big(\|\tilde{\phi}(t)\|^2 + \|\tilde{\theta}(t)\|^2 + \|\tilde{U}(t)\|^2\big) \leq e^{2\int_0^T C(\tau)d\tau}\big(\|\tilde{\phi}(0)\|^2 + \|\tilde{\theta}(0)\|^2 + \|\tilde{U}(0)\|^2\big) = 0,$$

for any $0 \leq T < \infty$ and $0 \leq t \leq T$. So, the solution is unique. This completes the proof of Theorem 5.2.2.

5.3 Mixing Flow

In this section we will improve the regularity of the global weak solution established in [5]. First, we recall the system of equations:

$$\begin{cases} \rho(U_t + U\cdot\nabla U) + \nabla P = \lambda\big(\nabla\rho\cdot\nabla U + U\cdot\nabla(\nabla\rho)\big) + \mu\Delta U + \vec{f}\rho, \\ \rho_t + U\cdot\nabla\rho = \lambda\Delta\rho, \\ \nabla\cdot U = 0, \end{cases} \tag{5.3.1}$$

and the initial and boundary conditions:

$$\begin{cases} (U,\rho)(\mathbf{x},0) = (U_0,\rho_0)(\mathbf{x}), \quad m \le \rho_0(\mathbf{x}) \le M; \\ U|_{\partial\Omega} = 0, \quad \nabla\rho\cdot\mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{5.3.2}$$

where $m, M > 0$ are constants.

The first lemma gives the lower-upper bounds of the density.

Lemma 5.3.1. *Under the assumptions of Theorem 5.1.2, it holds that $m \le \rho(\mathbf{x},t) \le M$, for all $\mathbf{x}\in\Omega$ and $t \ge 0$.*

Proof. For any $p \ge 1$, Taking L^2 inner product of $(5.3.1)_2$ with $|\rho|^{p-2}\rho$ we have

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|\rho\|_{L^p}^p &= \lambda\int_\Omega \Delta\rho|\rho|^{p-2}\rho d\mathbf{x} \\ &= -\lambda(p-1)\int_\Omega |\rho|^{p-2}|\nabla\rho|^2 d\mathbf{x} \le 0, \end{aligned}$$

which implies that

$$\|\rho\|_{L^p}^p \le \|\rho_0\|_{L^p}^p, \quad \forall p \ge 1. \tag{5.3.3}$$

Letting $p\to\infty$ in (5.3.3) we have

$$\|\rho\|_{L^\infty} \le \|\rho_0\|_{L^\infty} \le M. \tag{5.3.4}$$

To find the lower bound of $\|\rho\|_{L^\infty}$, we consider the following initial-boundary value problem

$$\begin{cases} R_t + U\cdot\nabla R = \lambda\Delta R - \dfrac{2\lambda}{R}|\nabla R|^2, \\ \nabla\cdot U = 0; \\ R(\mathbf{x},0) = R_0(\mathbf{x} = \dfrac{1}{\rho_0}(\mathbf{x}) \le \dfrac{1}{m}, \\ \nabla R\cdot\mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{5.3.5}$$

where $R = \frac{1}{\rho}$. For any $p \ge 1$, Taking L^2 inner product of $(5.3.5)_1$ with $|R|^{p-2}R$ we have Ω we get

$$\frac{1}{p}\frac{d}{dt}\|R\|_{L^p}^p = -\lambda(p+1)\int_\Omega |R|^{p-2}|\nabla R|^2 d\mathbf{x} \le 0,$$

which implies that

$$\|R\|_{L^p}^p \le \|R_0\|_{L^p}^p, \quad \forall p \ge 1. \tag{5.3.6}$$

Letting $p\to\infty$ in (5.3.6) we have

$$\|R\|_{L^\infty} \le \|R_0\|_{L^\infty} \le \frac{1}{m}. \tag{5.3.7}$$

Therefore,

$$\|\rho\|_{L^\infty} \ge m, \tag{5.3.8}$$

this together with (5.3.4) concludes the proof of Lemma 5.3.1.

With the lower-upper bounds established in Lemma 5.3.1 we now deal with some lower order estimate of U.

Lemma 5.3.2. *Under the assumptions of Theorem 5.1.2, it holds that*

$$\|U\|_{C([0,T];L^2(\Omega))}^2 + \|U\|_{L^2([0,T];H_0^1(\Omega))}^2 \le C.$$

Proof. Taking L^2 inner product of $(1)_1$ with U we get

$$\begin{aligned}&\int_\Omega \rho\left(\frac{|U|^2}{2}\right)_t d\mathbf{x} + \int_\Omega \rho U\cdot\nabla\left(\frac{|U|^2}{2}\right)d\mathbf{x} + \mu\int_\Omega |\nabla U|^2 d\mathbf{x} \\ =&\lambda\int_\Omega \nabla\rho\cdot\nabla\left(\frac{|U|^2}{2}\right)d\mathbf{x} + \lambda\int_\Omega \big(U\cdot\nabla(\nabla\rho)\big)\cdot U d\mathbf{x} + \int_\Omega \rho\vec{f}\cdot U d\mathbf{x}.\end{aligned}$$

After integrating by parts we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|U|^2 d\mathbf{x} - \frac{1}{2}\int_\Omega \rho_t|U|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega \rho U\cdot\nabla|U|^2 d\mathbf{x} + \frac{\lambda}{2}\int_\Omega \Delta\rho|U|^2 d\mathbf{x}\\ =&\lambda\int_\Omega \big(U\cdot\nabla(\nabla\rho)\big)\cdot U d\mathbf{x} + \int_\Omega \rho\vec{f}\cdot U d\mathbf{x} - \mu\int_\Omega |\nabla U|^2 d\mathbf{x}.\end{aligned}$$

Using $(5.3.1)_2$ and $(5.3.1)_3$ we get from the above equation that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|U|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega \nabla(\rho U)|U|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega \rho U\cdot\nabla|U|^2 d\mathbf{x} + \mu\int_\Omega |\nabla U|^2 d\mathbf{x}\\ =&\lambda\int_\Omega \big(U\cdot\nabla(\nabla\rho)\big)\cdot U d\mathbf{x} + \int_\Omega \rho\vec{f}\cdot U d\mathbf{x},\end{aligned}$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|U|^2 d\mathbf{x} + \mu\int_\Omega |\nabla U|^2 d\mathbf{x} = \lambda\int_\Omega \big[U\cdot\nabla(\nabla\rho)\big]\cdot U d\mathbf{x} + \int_\Omega \rho\vec{f}\cdot U d\mathbf{x}. \quad (5.3.9)$$

For the first term on the RHS of (5.3.9), using $(5.3.1)_3$, after simple calculations we have

$$\big[U\cdot\nabla(\nabla\rho)\big]\cdot U = \nabla\cdot\big[U(U\cdot\nabla\rho) - (\rho U\cdot\nabla U)\big] + \rho(u_x^2 + 2u_y v_x + v_y^2). \quad (5.3.10)$$

Integrating (5.3.10) over Ω using the boundary condition we get

$$\int_\Omega \big[U\cdot\nabla(\nabla\rho)\big]\cdot U d\mathbf{x} = \int_\Omega \rho(u_x^2 + 2u_y v_x + v_y^2) d\mathbf{x}.$$

Using this equality we update (5.3.9) as

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U\|^2 + \mu\|\nabla U\|^2 = \lambda\int_\Omega \rho(u_x^2 + 2u_y v_x + v_y^2) d\mathbf{x} + \int_\Omega \rho\vec{f}\cdot U d\mathbf{x}. \quad (5.3.11)$$

Since

$$u_x^2 + 2u_y v_x + v_y^2 = \nabla\cdot(U\cdot\nabla U) - U\cdot\nabla(\nabla\cdot U),$$

we have

$$\int_\Omega (u_x^2 + 2u_y v_x + v_y^2) d\mathbf{x} = 0.$$

Then we have from (5.3.11) that

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U\|^2 + \mu\|\nabla U\|^2 = \lambda\int_\Omega \Big(\rho - \frac{M+m}{2}\Big)(u_x^2 + 2u_y v_x + v_y^2) d\mathbf{x} + \int_\Omega \rho\vec{f}\cdot U d\mathbf{x}. \quad (5.3.12)$$

Using Lemma 5.3.1 we estimate the RHS of (5.3.12) as follows:

$$\begin{aligned}&\left|\lambda\int_\Omega\Big(\rho-\frac{M+m}{2}\Big)(u_x^2+2u_yv_x+v_y^2)d\mathbf{x}+\int_\Omega\rho\vec{f}\cdot Ud\mathbf{x}\right|\\ \leq&\lambda\frac{M-m}{2}\|\nabla U\|^2+\frac{1}{2}\|\sqrt{\rho}U\|^2+\frac{M}{2}\|\vec{f}\|^2.\end{aligned}$$

So we update (5.3.12) as

$$\frac{d}{dt}\|\sqrt{\rho}U\|^2+\big[2\mu-\lambda(M-m)\big]\|\nabla U\|^2\leq\|\sqrt{\rho}U\|^2+M\|\vec{f}\|^2. \tag{5.3.13}$$

Applying Gronwall's inequality to (5.3.13) and using conditions on $\vec{f}$ in Theorem 5.1.2 we conclude that

$$\|\sqrt{\rho}U\|^2\leq C,\quad\text{and}\quad\|\nabla U\|^2\leq C. \tag{5.3.14}$$

Since $\|\rho\|_{L^\infty}\geq m$, we conclude the proof of the lemma immediately by (5.3.14).

As a consequence of Lemma 5.3.2 we have

Lemma 5.3.3. *Under the assumptions of Theorem 5.1.2, it holds that*

$$\|\rho\|^2_{C([0,T];H^1(\Omega))}+\|\rho\|^2_{L^2([0,T];H^2(\Omega))}\leq C.$$

Proof. Taking L^2 inner product of $(5.3.1)_2$ with $\Delta\rho$ we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho\|^2+\lambda\|\Delta\rho\|^2=\int_\Omega(U\cdot\nabla\rho)\Delta\rho d\mathbf{x}. \tag{5.3.15}$$

Using Cauchy-Schwartz inequality and Lemmas 5.3.1–5.3.2 we estimate the RHS of (5.3.15) as follows:

$$\begin{aligned}\int_\Omega(U\cdot\nabla\rho)\Delta\rho d\mathbf{x}&\leq C\|U\cdot\nabla\rho\|^2+\frac{\lambda}{4}\|\Delta\rho\|^2\\&\leq C\|U\|^2_{L^4}\|\nabla\rho\|^2_{L^4}+\frac{\lambda}{4}\|\Delta\rho\|^2\\&\leq C\|\nabla U\|\|\nabla\rho\|\|D^2\rho\|+C\|\nabla U\|\|\nabla\rho\|^2+\frac{\lambda}{4}\|\Delta\rho\|^2.\end{aligned} \tag{5.3.16}$$

From Lemma 4.2.4 we know that

$$\|D^2\rho\|\leq C\big(\|\Delta\rho\|+\|\nabla\rho\|\big).$$

So we update the first term on the RHS of (5.3.16) as

$$
\begin{aligned}
C\|\nabla U\|\|\nabla\rho\|\|D^2\rho\| &\le C\|\nabla U\|\|\nabla\rho\|\big(\|\Delta\rho\|+\|\nabla\rho\|\big)\\
&\le C\|\nabla U\|^2\|\nabla\rho\|^2+\frac{\lambda}{4}\|\Delta\rho\|^2+C\|\nabla U\|\|\nabla\rho\|^2 \qquad (5.3.17)\\
&\le C\big(1+\|\nabla U\|^2\big)\|\nabla\rho\|^2+\frac{\lambda}{4}\|\Delta\rho\|^2,
\end{aligned}
$$

where we have used Cauchy-Schwartz inequality. Combining (5.3.15)–(5.3.17) we have

$$
\frac{d}{dt}\|\nabla\rho\|^2+\lambda\|\Delta\rho\|^2\le C\big(1+\|\nabla U\|^2\big)\|\nabla\rho\|^2.
$$

Applying Gronwall's inequality and using Lemma 5.3.2 we have

$$
\|\nabla\rho\|^2\le C,\quad \text{and}\quad \int_0^T\|\Delta\rho\|^2d\tau\le C,
$$

which concludes the proof of the lemma 5.3.3.

The following estimate of $\|\rho_t\|^2_{L^2([0,T];L^2(\Omega))}$ is a direct consequence of Lemma 5.3.3.

Lemma 5.3.4. *Under the assumptions of Theorem 5.1.2, it holds that*

$$
\|\rho_t\|^2_{L^2([0,T];L^2(\Omega))}\le C.
$$

Proof. We observe that similar derivations used in (5.3.16)–(5.3.17) imply that

$$
\|U\cdot\nabla\rho\|^2\le C\big(1+\|\nabla U\|^2\big)\|\nabla\rho\|^2+C\|\Delta\rho\|^2. \qquad (5.3.18)
$$

Using Lemmas 5.3.2–5.3.3, $(5.3.1)_2$ and (5.3.18) we have

$$
\begin{aligned}
\|\rho_t\|^2_{L^2([0,T];L^2(\Omega))} &\le \lambda^2\|\Delta\rho\|^2_{L^2([0,T];L^2(\Omega))}+\|U\cdot\nabla\rho\|^2_{L^2([0,T];L^2(\Omega))}\\
&\le C\|\Delta\rho\|^2_{L^2([0,T];L^2(\Omega))}+C\int_0^T\big(1+\|\nabla U\|^2\big)d\tau\\
&\le C.
\end{aligned}
$$

This completes the proof of Lemma 5.3.4.

The next lemma is crucial for this section and is essential for building up the regularity of U.

Lemma 5.3.5. *Under the assumptions of Theorem 5.1.2, it holds that*

$$\|U\|_{H^2}^2 \le \tilde{C}\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla U\|^4 + \|D^2\rho\|^2\|\nabla U\|^2 + \|\nabla \rho_t\|^2 + \|D^2\rho\|^2 + \|\vec{f}\|^2 + 1\big).$$

Proof. We rewrite the velocity equation $(5.3.1)_1$ as the nonhomogeneous Stokes equations:

$$-\mu\Delta U + \nabla P = \vec{F},$$

where

$$\begin{aligned}\vec{F} &= -\rho U_t - \rho U\cdot\nabla U + \lambda\nabla\rho\cdot\nabla U + \lambda U\cdot\nabla(\nabla\rho) + \vec{f}\rho \\ &\equiv \sum_{i=1}^{5} F_i.\end{aligned}$$

From Lemma 4.2.2 we know that

$$\|U\|_{H^2}^2 \le C\|\vec{F}\|^2 \le C\sum_{i=1}^{5}\|F_i\|^2. \tag{5.3.19}$$

Now we estimate F_i as follows:

Using Lemma 5.3.1, it is easy to see that

$$\|F_1\|^2 = \|\rho U_t\|^2 \le M\|\sqrt{\rho}U_t\|^2. \tag{5.3.20}$$

Similarly, we have

$$\begin{aligned}\|F_2\|^2 &= \|\rho U\cdot\nabla U\|^2 \\ &\le C\|U\|\|\nabla U\|\big(\|\nabla U\|\|D^2U\| + \|\nabla U\|^2\big) \\ &\le \frac{1}{6}\|U\|_{H^2}^2 + C\big(\|\nabla U\|^4 + 1\big),\end{aligned} \tag{5.3.21}$$

where we have used Cauchy-Schwartz inequality.

Since $\rho \in C([0,T];H^1(\Omega))$, using the same idea we have

$$\begin{aligned}\|F_3\|^2 &= \lambda^2\|\nabla\rho\cdot\nabla U\|^2 \\ &\le C\big(\|\nabla\rho\|\|D^2\rho\| + \|\nabla\rho\|^2\big)\big(\|\nabla U\|\|D^2U\| + \|\nabla U\|^2\big) \\ &= C\big(\|D^2\rho\| + 1\big)\|\nabla U\|\|D^2U\| + C\big(\|D^2\rho\| + 1\big)\|\nabla U\|^2 \\ &\le \frac{1}{6}\|U\|_{H^2}^2 + C\big(\|D^2\rho\|^2 + 1\big)\|\nabla U\|^2.\end{aligned} \tag{5.3.22}$$

For the estimate of F_4, we have

$$\begin{aligned}\|F_4\|^2 &= \lambda^2\|U\cdot\nabla(\nabla\rho)\|^2\\ &\le C\|\nabla U\|\big(\|D^2\rho\|\|D^3\rho\| + \|D^2\rho\|^2\big).\end{aligned} \tag{5.3.23}$$

For the estimate of $\|D^3\rho\|$, from $(5.3.1)_2$ we have

$$\begin{aligned}\|D^3\rho\| &\le C\big(\|\Delta\rho\|_{H^1} + \|\nabla\rho\|\big)\\ &= C\big(\|\nabla\rho_t\| + \|\nabla(U\cdot\nabla\rho)\| + \|\Delta\rho\| + \|\nabla\rho\|\big)\\ &= C\big(\|\nabla\rho_t\| + \|\nabla U\cdot(\nabla\rho)^T\| + \|U\cdot\nabla(\nabla\rho)\| + \|\Delta\rho\| + 1\big).\end{aligned} \tag{5.3.24}$$

Plugging (5.3.24) into (5.3.23) we have

$$\begin{aligned}\lambda^2\|U\cdot\nabla(\nabla\rho)\|^2 \le{}& C\|\nabla U\|\|D^2\rho\|\Big[\|\nabla\rho_t\| + \|\nabla U\cdot(\nabla\rho)^T\| + \|\Delta\rho\| + 1\Big] +\\ & C\|\nabla U\|\|D^2\rho\|^2 + C\|\nabla U\|\|D^2\rho\|\|U\cdot\nabla(\nabla\rho)\|.\end{aligned} \tag{5.3.25}$$

Since

$$C\|\nabla U\|\|D^2\rho\|\|U\cdot\nabla(\nabla\rho)\| \le C\|\nabla U\|^2\|D^2\rho\|^2 + \frac{\lambda^2}{2}\|U\cdot\nabla(\nabla\rho)\|^2,$$

we have

$$\begin{aligned}\lambda^2\|U\cdot\nabla(\nabla\rho)\|^2 \le{}& C\|\nabla U\|\|D^2\rho\|\Big[\|\nabla\rho_t\| + \|\nabla U\cdot(\nabla\rho)^T\| + \|\Delta\rho\| + 1\Big] +\\ & C(\|\nabla U\| + \|\nabla U\|^2)\|D^2\rho\|^2\\ \le{}& C(\|\nabla U\|^2 + 1)\|D^2\rho\|^2 + C(\|\nabla\rho_t\|^2 + \|\nabla U\|^2 + \|\nabla U\cdot(\nabla\rho)^T\|^2),\end{aligned} \tag{5.3.26}$$

where we have used Cauchy-Schwartz inequality. Similar to (5.3.22) we have

$$\|\nabla U\cdot(\nabla\rho)^T\|^2 \le C\big(\|D^2\rho\| + 1\big)\|\nabla U\|\|D^2U\| + C\big(\|D^2\rho\| + 1\big)\|\nabla U\|^2,$$

which together with (5.3.26) yields

$$\|F_4\|^2 \le C\|\nabla\rho_t\|^2 + C(\|\nabla U\|^2 + 1)\|D^2\rho\|^2 + C(1 + \|\nabla U\|^4) + \frac{1}{6}\|D^2U\|^2, \tag{5.3.27}$$

where we have used Lemma 5.3.3 and Cauchy-Schwartz inequality.

Finally, using Lemma 5.3.1 we easily see that

$$\|F_5\|^2 = \|\vec{f}\rho\|^2 \leq M^2\|\vec{f}\|^2. \tag{5.3.28}$$

Collecting the above estimates of $F_i(i = 1, ..., 5)$ we have

$$\begin{aligned}\|U\|_{H^2}^2 \leq & C\|\nabla\rho_t\|^2 + C(\|\nabla U\|^2 + 1)\|D^2\rho\|^2 + C(1 + \|\nabla U\|^4) + \\ & \frac{1}{2}\|D^2U\|^2 + M\|\sqrt{\rho}U_t\|^2 + M^2\|\vec{f}\|^2,\end{aligned}$$

which implies that

$$\|U\|_{H^2}^2 \leq \tilde{C}\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla U\|^4 + \|D^2\rho\|^2\|\nabla U\|^2 + \|\nabla\rho_t\|^2 + \|D^2\rho\|^2 + \|\vec{f}\|^2 + 1\big).$$

This completes the proof of Lemma 5.3.5.

With the help of Lemma 5.3.5 we are able to improve the regularity of U and ρ_t.

Lemma 5.3.6. *Under the assumptions of Theorem 5.1.2, it holds that*

$$\begin{aligned}&\|U\|_{C([0,T];H^1(\Omega))}^2 + \|U_t\|_{L^2([0,T];L^2(\Omega))}^2 \leq C; \\ &\|\rho_t\|_{C([0,T];L^2(\Omega))}^2 + \|\rho_t\|_{L^2([0,T];H^1(\Omega))}^2 \leq C.\end{aligned}$$

Proof. Taking L^2 inner product of $(5.3.1)_1$ with U_t we have

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2 + \int_\Omega \rho|U_t|^2 d\mathbf{x} = & -\int_\Omega \rho(U\cdot\nabla U)U_t d\mathbf{x} + \\ & \lambda\int_\Omega \big[\nabla\rho\cdot\nabla U + U\cdot\nabla(\nabla\rho)\big]U_t d\mathbf{x} + \int_\Omega \rho\vec{f}\cdot U_t d\mathbf{x}.\end{aligned} \tag{5.3.29}$$

We estimate the RHS of (5.3.29) as follows:

Since $\|\rho\|_{L^\infty} \geq m$, by Cauchy-Schwartz inequality we have

$$\begin{aligned}& -\int_\Omega \rho(U\cdot\nabla U)U_t d\mathbf{x} + \lambda\int_\Omega \big[\nabla\rho\cdot\nabla U + U\cdot\nabla(\nabla\rho)\big]U_t d\mathbf{x} + \int_\Omega \rho\vec{f}\cdot U_t d\mathbf{x} \\ \leq & \frac{1}{8}\|\sqrt{\rho}U_t\|^2 + C\Big\|\big[\rho U\cdot\nabla U + \lambda\nabla\rho\cdot\nabla U + \lambda U\cdot\nabla(\nabla\rho) + \vec{f}\rho\big]\Big\|^2.\end{aligned} \tag{5.3.30}$$

From the proof of Lemma 5.3.5 we know that the RHS of (5.3.30) is bounded by $\sum_{i=2}^5 \|F_i\|^2$. Therefore, similar arguments as in (5.3.21), (5.3.22) and (5.3.27) imply that

$$\Big\| \big[\rho U\cdot\nabla U+\lambda\nabla\rho\cdot\nabla U+\lambda U\cdot\nabla(\nabla\rho)+\vec{f}\rho\big]\Big\|^2$$
$$\le\hat{C}(\eta,\xi)\big(\|\nabla U\|^4+\|D^2\rho\|^2\|\nabla U\|^2+\|D^2\rho\|^2+\|\vec{f}\|^2+1\big)+\eta\|\nabla\rho_t\|^2+\xi\|U\|_{H^2}^2,$$

where $\eta,\xi>0$ are constants to be determined. So we update (5.3.29) as

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2+\int_\Omega\rho|U_t|^2d\mathbf{x}\le&\hat{C}(\eta,\xi)\big(\|\nabla U\|^4+\|D^2\rho\|^2\|\nabla U\|^2+\|D^2\rho\|^2\\&+\|\vec{f}\|^2+1\big)+\frac{1}{8}\|\sqrt{\rho}U_t\|^2+\eta\|\nabla\rho_t\|^2+\xi\|U\|_{H^2}^2,\end{aligned}$$

Choosing $\eta=\lambda/8$ and $\xi=\min\{1/(8\tilde{C}),\lambda/(8\tilde{C})\}$, from Lemma 5.3.5 we have

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\|\nabla U\|^2+\int_\Omega\rho|U_t|^2d\mathbf{x}\le&\hat{C}\big(\|\nabla U\|^4+\|D^2\rho\|^2\|\nabla U\|^2+\|D^2\rho\|^2\\&+\|\vec{f}\|^2+1\big)+\frac{1}{4}\|\sqrt{\rho}U_t\|^2+\frac{\lambda}{4}\|\nabla\rho_t\|^2.\end{aligned}\tag{5.3.31}$$

Next, we take the temporal derivative of $(5.3.1)_2$ to get

$$\rho_{tt}+U_t\cdot\nabla\rho+U\cdot\nabla\rho_t=\lambda\Delta\rho_t.\tag{5.3.32}$$

Taking the L^2 inner product of (5.3.32) with ρ_t we have

$$\frac{1}{2}\frac{d}{dt}\|\rho_t\|^2+\lambda\|\nabla\rho_t\|^2=-\int_\Omega(U_t\cdot\nabla\rho)\rho_td\mathbf{x}.\tag{5.3.33}$$

Using Lemma 5.3.1 and Cauchy-Schwartz inequality we have

$$\begin{aligned}\left|-\int_\Omega(U_t\cdot\nabla\rho)\rho_td\mathbf{x}\right|&\le\frac{1}{4}\|\sqrt{\rho}U_t\|^2+C\|(\nabla\rho)\rho_t\|^2\\&\le\frac{1}{4}\|\sqrt{\rho}U_t\|^2+C\|\nabla\rho\|_{L^4}^2\|\rho_t\|_{L^4}^2.\end{aligned}\tag{5.3.34}$$

From (5.3.22) we know that

$$\begin{aligned}C\|\nabla\rho\|_{L^4}^2\|\rho_t\|_{L^4}^2&\le C(\|D^2\rho\|+1)(\|\rho_t\|\|\nabla\rho_t\|+\|\rho_t\|^2)\\&\le\frac{\lambda}{4}\|\nabla\rho_t\|^2+C(\|D^2\rho\|^2+1)\|\rho_t\|^2,\end{aligned}\tag{5.3.35}$$

where we have used Cauchy-Schwartz inequality. Combining (5.3.33)–(5.3.35) we have

$$\frac{1}{2}\frac{d}{dt}\|\rho_t\|^2+\frac{3}{4}\lambda\|\nabla\rho_t\|^2\le\frac{1}{4}\|\sqrt{\rho}U_t\|^2+C(\|D^2\rho\|^2+1)\|\rho_t\|^2.\tag{5.3.36}$$

Coupling (5.3.31) to (5.3.36) we have

$$
\begin{aligned}
&\frac{d}{dt}(\mu\|\nabla U\|^2 + \|\rho_t\|^2) + \|\sqrt{\rho}U_t\|^2 + \lambda\|\nabla\rho_t\|^2 \\
&\le \hat{C}\big(\|\nabla U\|^4 + \|D^2\rho\|^2\|\nabla U\|^2 + \|D^2\rho\|^2 + \|\vec{f}\|^2 + 1\big) + C(\|D^2\rho\|^2 + 1)\|\rho_t\|^2 \\
&\le C(\|D^2\rho\|^2 + 1)(\mu\|\nabla U\|^2 + \|\rho_t\|^2) + C\|\nabla U\|^4 + C(\|D^2\rho\|^2 + \|\vec{f}\|^2 + 1) \\
&\le C(\|D^2\rho\|^2 + \|\nabla U\|^2 + 1)(\mu\|\nabla U\|^2 + \|\rho_t\|^2) + C(\|D^2\rho\|^2 + \|\vec{f}\|^2 + 1) \\
&\equiv A(t)(\mu\|\nabla U\|^2 + \|\rho_t\|^2) + B(t).
\end{aligned}
\tag{5.3.37}
$$

According to Lemmas 5.3.2–5.3.3 we know that $A(t), B(t)$ are time integrable. Therefore, applying Gronwall's inequality to (5.3.37) we conclude that

$$
\mu\|\nabla U\|^2 + \|\rho_t\|^2 \le C, \ \forall\ t \in [0,T], \quad \text{and} \quad \int_0^T (\|\sqrt{\rho}U_t\|^2 + \lambda\|\nabla\rho_t\|^2)d\tau \le C,
$$

which together with Lemma 5.3.1 implies the lemma.

Using previous lemmas we improve the regularity of the solution to higher orders.

Lemma 5.3.7. *Under the assumptions of Theorem 5.1.2, it holds that*

$$
\begin{aligned}
&\|\rho\|^2_{C([0,T];H^2(\Omega))} + \|\rho\|^2_{L^2([0,T];H^3(\Omega))} \le C; \\
&\|U\|^2_{L^2([0,T];H^2(\Omega))} \le C.
\end{aligned}
$$

Proof. Using $(5.3.1)_2$ we have

$$
\begin{aligned}
\|\nabla\rho\|^2_{H^1} &\le C(\|\Delta\rho\|^2 + \|\nabla\rho\|^2) \\
&= C(\|\rho_t\|^2 + \|U\cdot\nabla\rho\|^2 + \|\nabla\rho\|^2) \\
&\le C(\|\rho_t\|^2 + \|U\|^2_{L^4}\|\nabla\rho\|^2_{L^4} + \|\nabla\rho\|^2) \\
&\le C\big(\|\rho_t\|^2 + \|U\|^2_{H^1}(\|\nabla\rho\|\|D^2\rho\| + \|\nabla\rho\|^2) + \|\nabla\rho\|^2\big).
\end{aligned}
$$

From Lemmas 5.3.3 and 5.3.6 we know that each term, except $\|D^2\rho\|$, is bounded by some constant. Therefore, we have

$$
\|\nabla\rho\|^2_{H^1} \le C(\|D^2\rho\| + 1),
$$

which implies that, by Cauchy-Schwartz inequality

$$\|\nabla\rho\|_{H^1}^2 \leq C. \tag{5.3.38}$$

As a consequence of (5.3.38) and Lemmas 5.3.5–5.3.6 we have

$$\begin{aligned}
\|U\|_{H^2}^2 &\leq \tilde{C}\big(\|\sqrt{\rho}U_t\|^2 + \|\nabla U\|^4 + \|D^2\rho\|^2\|\nabla U\|^2 + \|\nabla\rho_t\|^2 + \|D^2\rho\|^2 + \|\vec{f}\|^2 + 1\big) \\
&\leq C(\|U_t\|^2 + \|\nabla\rho_t\|^2 + \|\vec{f}\|^2 + 1),
\end{aligned}$$

which yields

$$\|U\|_{L^2([0,T];H^2(\Omega))}^2 \leq C(\|(U_t, \nabla\rho_t, \vec{f})\|_{L^2([0,T];L^2(\Omega))}^2 + 1) \leq C.$$

Similarly, we have

$$\begin{aligned}
\|\nabla\rho\|_{H^2}^2 &\leq C(\|\Delta\rho\|_{H^1}^2 + \|\nabla\rho\|^2) \\
&\leq C(\|\nabla\rho_t\|^2 + \|\nabla(U\cdot\nabla\rho)\|^2 + \|\nabla\rho\|^2) \\
&\leq C(\|\nabla\rho_t\|^2 + \|U\|_{H^2}^2 + \|\nabla\rho\|_{H^2} + 1) \\
&\leq C(\|\nabla\rho_t\|^2 + \|U\|_{H^2}^2 + 1) + \frac{1}{2}\|\nabla\rho\|_{H^2}^2,
\end{aligned}$$

which means

$$\|\nabla\rho\|_{H^2}^2 \leq C(\|\nabla\rho_t\|^2 + \|U\|_{H^2}^2 + 1). \tag{5.3.39}$$

Thus, (5.3.39) implies that

$$\|\nabla\rho\|_{C([0,T];H^2(\Omega))}^2 \leq C(\|\nabla\rho_t\|_{C([0,T];L^2(\Omega))}^2 + \|U\|_{C([0,T];H^2(\Omega))}^2 + 1) \leq C.$$

This completes the proof of Lemma 5.3.7.

Next we will work on temporal derivatives of the solution and improve the regularity of the solution to $C([0,T];H^s(\Omega))$.

Lemma 5.3.8. *Under the assumptions of Theorem 5.1.2, it holds that*

$$\begin{aligned}
&\|\rho\|_{C([0,T];H^3(\Omega))}^2 \leq C; \\
&\|U\|_{C([0,T];H^2(\Omega))}^2 \leq C.
\end{aligned}$$

Proof. Taking the temporal derivative of $(5.3.1)_1$ we have

$$\begin{aligned}&\rho_t(U_t + U\cdot\nabla U) + \rho(U_{tt} + U_t\cdot\nabla U + U\cdot\nabla U_t) + \nabla P_t\\ =&\mu\Delta U_t + \lambda\big(\nabla\rho_t\cdot\nabla U + \nabla\rho\cdot\nabla U_t + U_t\cdot\nabla(\nabla\rho) + U\cdot\nabla(\nabla\rho_t)\big) + \vec{f}\rho_t + \vec{f}_t\rho.\end{aligned} \tag{5.3.40}$$

Taking L^2 inner product of (5.3.40) with U_t we have, after integration by parts, that

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2 + \mu\|\nabla U_t\|^2 + \frac{1}{2}\int_\Omega(\rho_t - U\cdot\nabla\rho)|U_t|^2 d\mathbf{x} = \sum_{i=1}^{7} R_i + \lambda\int_\Omega(\nabla\rho\cdot\nabla U_t)\cdot U_t d\mathbf{x},$$

where

$$\begin{aligned}&R_1 = -\int_\Omega(\rho_t U\cdot\nabla U)\cdot U_t d\mathbf{x}, \quad R_2 = -\int_\Omega(\rho U_t\cdot\nabla U)\cdot U_t d\mathbf{x};\\ &R_3 = \lambda\int_\Omega(\nabla\rho_t\cdot\nabla U)\cdot U_t d\mathbf{x}, \quad R_4 = \lambda\int_\Omega(U_t\cdot\nabla(\nabla\rho))\cdot U_t d\mathbf{x},\\ &R_5 = -\lambda\int_\Omega\nabla\rho_t\cdot(U\cdot\nabla U_t)d\mathbf{x};\\ &R_6 = \lambda\int_\Omega\rho_t\vec{f}\cdot U_t d\mathbf{x}, \quad R_7 = \lambda\int_\Omega\rho\vec{f}_t\cdot U_t d\mathbf{x}.\end{aligned}$$

Using the boundary condition and $(5.3.1)_3$ we have

$$\lambda\int_\Omega(\nabla\rho\cdot\nabla U_t)\cdot U_t d\mathbf{x} = -\frac{\lambda}{2}\int_\Omega\Delta\rho|U_t|^2 d\mathbf{x}.$$

Moreover, since $\rho_t = \lambda\Delta\rho - U\cdot\nabla\rho$, we have

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2 + \mu\|\nabla U_t\|^2 = \sum_{i=1}^{9} R_i, \tag{5.3.41}$$

where

$$R_8 = \int_\Omega(U\cdot\nabla\rho)|U_t|^2 d\mathbf{x}, \quad R_9 = -\lambda\int_\Omega\Delta\rho|U_t|^2 d\mathbf{x}.$$

By Lemma and previous results we have:

$$\begin{aligned}R_1 &\le \|\rho_t\|_{L^4}\|U\|_{L^4}\|\nabla U\|_{L^4}\|U_t\|_{L^4}\\ &\le C\big(\|\nabla\rho_t\|^{1/2} + 1\big)\big(\|D^2U\|^{1/2} + 1\big)\big(\|\nabla U_t\| + \|U_t\|\big)\\ &\le C(\|\nabla\rho_t\|^2 + \|U\|_{H^2}^2 + \|\sqrt{\rho}U_t\|^2 + 1) + \varepsilon\|\nabla U_t\|^2,\end{aligned} \tag{5.3.42}$$

where $\varepsilon > 0$ is a constant to be determined. Similarly, we have

$$\begin{aligned}
R_2 &\leq C\|\sqrt{\rho}U_t\|^2 + \varepsilon\|\nabla U_t\|^2, \\
R_3 &\leq C(\|U\|_{H^2}^2 + 1)\|\sqrt{\rho}U_t\|^2 + C\|\nabla\rho_t\|^2 + \varepsilon\|\nabla U_t\|^2, \\
R_4 &\leq C\|\rho\|_{H^2}^2\|\sqrt{\rho}U_t\|^2 + \varepsilon\|\nabla U_t\|^2, \\
R_5 &\leq C\|U\|_{H^2}^2\|\nabla\rho_t\|^2 + \varepsilon\|\nabla U_t\|^2, \\
R_6 &\leq C\|\sqrt{\rho}U_t\|^2(\|\nabla\rho_t\|^2 + 1) + \|\vec{f}\|^2 + \varepsilon\|\nabla U_t\|^2, \\
R_7 &\leq C(\|\sqrt{\rho}U_t\|^2 + \|\vec{f}_t\|^2) + \varepsilon\|\nabla U_t\|^2, \\
R_8 &\leq C\|\sqrt{\rho}U_t\|^2 + \varepsilon\|\nabla U_t\|^2, \\
R_9 &\leq C(\|\rho\|_{H^2}^2 + 1)\|\sqrt{\rho}U_t\|^2 + \varepsilon\|\nabla U_t\|^2.
\end{aligned} \tag{5.3.43}$$

Collecting the results in (5.3.41)–(5.3.43) we have

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}U_t\|^2 + \mu\|\nabla U_t\|^2 \leq 9\varepsilon\|\nabla U_t\|^2 + D(t)(\|\sqrt{\rho}U_t\|^2 + \|\nabla\rho_t\|^2) + E(t), \tag{5.3.44}$$

where

$$\begin{aligned}
D(t) &= C\big(\|U\|_{H^2}^2 + \|\rho\|_{H^2}^2 + \|U_t\|^2 + 1\big), \\
E(t) &= C\big(\|U\|_{H^2}^2 + \|\vec{f}\|^2 + \|\vec{f}_t\|^2 + 1\big).
\end{aligned}$$

According to Lemmas 5.3.6–5.3.7 we know $D(t), E(t)$ are time integrable.

Next, taking L^2 inner product of (5.3.32) with $\Delta\rho_t$ we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla\rho_t\|^2 + \lambda\|\Delta\rho_t\|^2 &= \int_\Omega (U_t\cdot\nabla\rho + U\cdot\nabla\rho_t)\Delta\rho_t d\mathbf{x} \\
&\leq \frac{\lambda}{2}\|\Delta\rho_t\|^2 + C(\|U_t\cdot\nabla\rho\|^2 + \|U\cdot\nabla\rho_t\|^2) \\
&\leq \frac{\lambda}{2}\|\Delta\rho_t\|^2 + \varepsilon\|\nabla U_t\|^2 + C(\|U\|_{H^2}^2\|\nabla\rho_t\|^2 + \|U_t\|^2).
\end{aligned} \tag{5.3.45}$$

Combining (5.3.44) and (5.3.45) we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\sqrt{\rho}U_t\|^2 + \|\nabla\rho_t\|^2) + \mu\|\nabla U_t\|^2 + \frac{\lambda}{2}\|\Delta\rho_t\|^2 \\
\leq\ & 10\varepsilon\|\nabla U_t\|^2 + D(t)(\|\sqrt{\rho}U_t\|^2 + \|\nabla\rho_t\|^2) + E(t).
\end{aligned} \tag{5.3.46}$$

Choosing $\varepsilon = \mu/20$ we update (5.3.46) as

$$\frac{d}{dt}(\|\sqrt{\rho}U_t\|^2 + \|\nabla\rho_t\|^2) + \mu\|\nabla U_t\|^2 + \lambda\|\Delta\rho_t\|^2 \le D(t)(\|\sqrt{\rho}U_t\|^2 + \|\nabla\rho_t\|^2) + E(t). \tag{5.3.47}$$

Applying Gronwall's inequality to (5.3.47) we get that

$$\begin{aligned} &U_t \in C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H^1(\Omega)), \\ &\rho_t \in C([0,T]; H^1(\Omega)) \cap L^2([0,T]; H^2(\Omega)). \end{aligned} \tag{5.3.48}$$

Then it is easy to see from Lemmas 5.3.5–5.3.7 and (5.3.48) that

$$\|U\|_{H^2}^2 \le C,$$

which together with (5.3.39) yields

$$\|\rho\|_{H^3}^2 \le C.$$

This completes the proof of Lemma 5.3.8.

The next lemma gives the estimate of $\|U_t\|^2_{C([0,T];H^1(\Omega))}$ and $\|U_{tt}\|^2_{L^2([0,T];L^2(\Omega))}$ based on which we can get the desired estimate indicated in Theorem 5.1.2.

Lemma 5.3.9. *Under the assumptions of Theorem 5.1.2, it holds that*

$$\|U_t\|^2_{C([0,T];H^1(\Omega))} + \|U_{tt}\|^2_{L^2([0,T];L^2(\Omega))} \le C.$$

Proof. Taking L^2 inner product of (5.3.40) with U_{tt} we have

$$\begin{aligned} &\frac{\mu}{2}\frac{d}{dt}\|\nabla U_t\|^2 + \|\sqrt{\rho}U_{tt}\|^2 \\ &= \int_\Omega \big[-\rho_t(U_t + U\cdot\nabla U) - \rho(U_t\cdot\nabla U + U\cdot\nabla U_t) \\ &+\lambda\big(\nabla\rho_t\cdot\nabla U + \nabla\rho\cdot\nabla U_t + U_t\cdot\nabla(\nabla\rho) + U\cdot\nabla(\nabla\rho_t)\big) + \vec{f}\rho_t + \vec{f}_t\rho\big]\cdot U_{tt}d\mathbf{x}. \end{aligned} \tag{5.3.49}$$

Cauchy-Schwartz inequality then implies that

$$\mu\frac{d}{dt}\|\nabla U_t\|^2 + \|\sqrt{\rho}U_{tt}\|^2 \le C\sum_{j=1}^{10} I_j.$$

Using previous estimates, we have

$$I_1 = \|\rho_t U_t\|^2 \leq C\|\rho_t\|_{H^1}^2 \|U_t\|_{H^1}^2 \leq C\|\nabla U_t\|^2;$$

$$I_2 = \|\rho_t U \cdot \nabla U\|^2 \leq C\|U\|_{L^\infty}^2 \|\rho_t\|_{H^1}^2 \|U\|_{H^2}^2 \leq C\|U\|_{H^2}^2 \|\rho_t\|_{H^1}^2 \|U\|_{H^2}^2 \leq C;$$

$$I_3 = \|\rho U_t \cdot \nabla U\|^2 \leq C\|U_t\|_{H^1}^2 \|U\|_{H^2}^2 \leq C\|\nabla U_t\|^2;$$

$$I_4 = \|\rho U \cdot \nabla U_t\|^2 \leq C\|\rho\|_{L^\infty}^2 \|U\|_{L^\infty}^2 \|\nabla U_t\|^2 \leq C\|\nabla U_t\|^2;$$

$$I_5 = \lambda^2 \|\nabla \rho_t \cdot \nabla U\|^2 \leq C\|\rho_t\|_{H^2}^2 \|U\|_{H^2}^2 \leq C\|\rho_t\|_{H^2}^2;$$

$$I_6 = \lambda^2 \|\nabla \rho \cdot \nabla U_t\|^2 \leq C\|\nabla \rho\|_{L^\infty}^2 \|\nabla U_t\|^2 \leq C\|\rho\|_{H^3}^2 \|\nabla U_t\|^2 \leq C\|\nabla U_t\|^2;$$

$$I_7 = \lambda^2 \|U_t \cdot \nabla(\nabla \rho)\|^2 \leq C\|U_t\|_{H^1}^2 \|\rho\|_{H^3}^2 \leq C\|\nabla U_t\|^2;$$

$$I_8 = \lambda^2 \|U \cdot \nabla(\nabla \rho_t)\|^2 \leq C\|U\|_{L^\infty}^2 \|\rho_t\|_{H^2}^2 \leq C\|\rho_t\|_{H^2}^2;$$

$$I_9 = \|\vec{f}\rho_t\|^2 \leq C\|\vec{f}\|_{L^4}^2 \|\rho_t\|_{H^1}^2 \leq C\|\vec{f}\|_{L^4}^2;$$

$$I_{10} = \|\vec{f}_t \rho\|^2 \leq \|\vec{f}_t\|^2 \|\rho\|_{L^\infty}^2 \leq C\|\vec{f}_t\|^2.$$

Collecting the above results we get

$$\mu \frac{d}{dt}\|\nabla U_t\|^2 + \|\sqrt{\rho} U_{tt}\|^2 \leq C\|\nabla U_t\|^2 + C(\|\rho_t\|_{H^2}^2 + \|\vec{f}\|_{L^4}^2 + \|\vec{f}_t\|^2). \tag{5.3.50}$$

Integrating (5.3.50) over time from 0 to T and using (5.3.48) we have

$$\|\nabla U_t\|^2 \leq C, \quad \text{and} \quad \int_0^T \|U_{tt}\|^2 d\mathbf{x} \leq C.$$

This completes the proof of Lemma 5.3.9.

With the help of Lemma 5.3.9 we can easily prove the desired regularity. The proof is straightforward and we omit the details.

Lemma 5.3.10. *Under the assumptions of Theorem, it holds that*

$$U \in C([0,T]; H^3(\Omega)) \cap L^2([0,T]; H^4(\Omega));$$

$$U_t \in L^2([0,T]; H^2(\Omega)); \rho \in L^2([0,T]; H^4(\Omega)).$$

Finally, we show the uniqueness of the solution.

Theorem 5.3.1. *Under the assumptions of Theorem 5.1.2, the solution is unique.*

Proof. Suppose there are two solutions (ρ_1, U_1, P_1) and (ρ_2, U_2, P_2). Let $\tilde{\rho} = \rho_1 - \rho_2, \tilde{U} = U_1 - U_2, \tilde{P} = P_1 - P_2$. then the difference functions satisfy the following initial-boundary value problem:

$$\begin{cases} \tilde{\rho}(U_{1t} + U_1 \cdot \nabla U_1) + \rho_2(\tilde{U}_t + U_1 \cdot \nabla \tilde{U} + \tilde{U} \cdot \nabla U_2) + \nabla \tilde{P} = \\ \mu \Delta \tilde{U} + \lambda\big[\nabla \tilde{\rho} \cdot \nabla U_1 + \nabla \rho_2 \cdot \nabla \tilde{U} + U_1 \cdot \nabla(\nabla \tilde{\rho}) + \tilde{U} \cdot \nabla(\nabla \rho_2)\big] + \vec{f}\tilde{\rho}, \\ \tilde{\rho}_t + U_1 \cdot \nabla \tilde{\rho} + \tilde{U} \cdot \nabla \rho_2 = \lambda \Delta \tilde{\rho}, \\ \nabla \cdot \tilde{U} = 0; \\ (\tilde{\rho}, \tilde{U})(\mathbf{x}, 0) = \mathbf{0}; \\ \tilde{U}|_{\partial\Omega} = 0, \quad \nabla \tilde{\rho} \cdot \mathbf{n}|_{\partial\Omega} = 0. \end{cases} \tag{5.3.51}$$

Taking L^2 inner product of $(5.3.51)_2$ with $\tilde{\rho}$ we have

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\rho}\|^2 + \lambda\|\nabla\tilde{\rho}\|^2 = -\int_\Omega (\tilde{U} \cdot \nabla \rho_2)\tilde{\rho} d\mathbf{x}.$$

Since $\rho_2 \in C([0,T]; H^3(\Omega))$ for any $T \geq 0$, from Sobolev embedding we know $\nabla \rho_2 \in C([0,T]; L^\infty(\Omega))$. Therefore,

$$\begin{aligned} \frac{d}{dt}\|\tilde{\rho}\|^2 + 2\lambda\|\nabla\tilde{\rho}\|^2 &\leq 2\|\tilde{U}\|\|\nabla\rho_2\|_{L^\infty}\|\tilde{\rho}\| \\ &\leq C(\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2). \end{aligned} \tag{5.3.52}$$

Now, taking L^2 inner product of $(5.3.51)_1$ with $\tilde{U}$ and using Cauchy-Schwartz inequality we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\sqrt{\rho_2}\tilde{U}\|^2 + \mu\|\nabla\tilde{U}\|^2 \leq &(\|U_{1t}\|_{L^\infty} + \|U_1 \cdot \nabla U_1\|_{L^\infty} + \lambda\|\nabla U_1\|_{L^\infty} + \|\vec{f}\|_{L^\infty})\|\tilde{\rho}\|\|\tilde{U}\| \\ &+(\|\rho_2 U_1\|_{L^\infty} + \lambda\|\nabla\rho_2\|_{L^\infty})\|\nabla\tilde{U}\|\|\tilde{U}\| \\ &+(\|\rho_{2t}\|_{L^\infty} + \|\rho_2\nabla U_2\|_{L^\infty} + \lambda\|D^2\rho_2\|_{L^\infty})\|\tilde{U}\|^2 \\ &+\lambda\|U_1\|_{L^\infty}\|\nabla\tilde{\rho}\|\|\nabla\tilde{U}\|. \end{aligned}$$

In view of (5.3.48) and Lemma 5.3.10, by Cauchy-Schwartz inequality we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\sqrt{\rho_2}\tilde{U}\|^2 + \mu\|\nabla\tilde{U}\|^2 &\leq G(t)(\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2) + \frac{\mu}{4}\|\nabla\tilde{U}\|^2 + \lambda\bar{C}\|\nabla\tilde{\rho}\|\|\nabla\tilde{U}\| \\ &\leq G(t)(\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2) + \frac{\mu}{2}\|\nabla\tilde{U}\|^2 + \frac{\lambda^2\bar{C}^2}{\mu}\|\nabla\tilde{\rho}\|^2, \end{aligned} \tag{5.3.53}$$

where $G(t) \geq 0$ satisfies $\displaystyle\int_0^T G(\tau)d\tau \leq C(T) < \infty$ for any $T \geq 0$. Multiplying (5.3.52) by $\frac{\lambda \bar{C}^2}{\mu}$ and coupling the resulting inequality to (5.3.53) we have

$$\frac{d}{dt}\left(\frac{\lambda \bar{C}^2}{\mu}\|\tilde{\rho}\|^2 + \frac{1}{2}\|\sqrt{\rho_2}\tilde{U}\|^2\right) + \frac{\lambda^2 \bar{C}^2}{\mu}\|\nabla\tilde{\rho}\|^2 + \frac{\mu}{2}\|\nabla\tilde{U}\|^2 \leq (G(t)+C)(\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2),$$

which gives

$$\frac{d}{dt}\left(\frac{\lambda \bar{C}^2}{\mu}\|\tilde{\rho}\|^2 + \frac{1}{2}\|\sqrt{\rho_2}\tilde{U}\|^2\right) \leq (G(t)+C)(\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2).$$

Since $\rho_2 \geq m$, it is straightforward to show that

$$\|\tilde{\rho}\|^2 + \|\tilde{U}\|^2 \leq \alpha\left(\frac{\lambda \bar{C}^2}{\mu}\|\tilde{\rho}\|^2 + \frac{1}{2}\|\sqrt{\rho_2}\tilde{U}\|^2\right),$$

where $\alpha = (\min\{\frac{\lambda \bar{C}^2}{\mu}, \frac{m}{2}\})^{-1}$. Therefore we have

$$\frac{d}{dt}\left(\frac{\lambda \bar{C}^2}{\mu}\|\tilde{\rho}\|^2 + \frac{1}{2}\|\sqrt{\rho_2}\tilde{U}\|^2\right) \leq \alpha(G(t)+C)\left(\frac{\lambda \bar{C}^2}{\mu}\|\tilde{\rho}\|^2 + \frac{1}{2}\|\sqrt{\rho_2}\tilde{U}\|^2\right).$$

Gronwall's inequality then yields

$$\frac{\lambda \bar{C}^2}{\mu}\|\tilde{\rho}(t)\|^2 + \frac{1}{2}\|\sqrt{\rho_2}\tilde{U}(t)\|^2 \leq 0, \quad \forall\ t \geq 0.$$

where we take into account of the temporal integrability of $G(t)$ and the zero initial condition. Thus, the solution is unique.

CHAPTER VI

CONCLUSION

The results obtained in this thesis indicate that when the systems of nonlinear partial differential equations under consideration are set on bounded domains, the dissipative mechanisms usually produce global solution to the initial-boundary value problems, and the boundary effects force some of the solutions decay exponentially to equilibrium states which are normally constant states.

By a closer look at the results obtained in Chapters 2–3 we observe that the damping effect usually presents weak dissipation which can not prevent the development of singularity in the system for large data. But, it does prevent singularity for small smooth data.

The results in Chapters 4–5 suggest that, at least for 2D problems, viscosity and heat diffusion are strong enough to compensate the effects of large data, nonlinear convection, coupling and/or gravitational force in order to prevent the development of singularity.

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