Hyperbolic "Smoothing" of Shapes

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Abstract

We have been developing a theory of generic 2-D shape based on a reaction-diffusion model from mathematical physics. The description of a shape is derived from the singularities of a curve evolution process driven by the reaction (hyperbolic) term. The diffusion (parabolic) term is related to smoothing and shape simplification. However, the unification of the two is problematic, because the slightest amount of diffusion dominates and prevents the formation of generic first-order shocks. The technical issue is whether it is possible to smooth a shape, in any sense, without destroying the shocks. We now report a constructive solution to this problem, by embedding the smoothing term in a global metric against which a purely hyperbolic evolution is performed from the initial curve. This is a new flow for shape, that extends the advantages of the original one. Specific metrics are developed, which lead to a natural hierarchy of shape features, analogous to the simplification one might perceive when viewing an object from increasing distances. We illustrate our new flow with a variety of examples.

1 Introduction

Look. Up in the sky; it's a bird; it's a plane; no, it's superman!

Although superman is a fictitious cartoon character in North America, the above quotation reveals two fundamental aspects of visual shape analysis. First, recognition proceeds from general to specific, and second, descriptions are more fully articulated as additional structure appears. The articulated appearance in this example depends on distance, with smaller details apparent only up close, while the generic descriptions are fundamental to cognitive psychology. Rosch, for example, has shown that entry level categorization precedes the recognition of specifics [13]. The temptation, however, is to use additional detail to refine the generic categories into specific instances (e.g., to recognize a *robin* as a specific *bird*), not to refine one generic category (*bird*) into another generic one (*plane*) before both are shown to be incorrect. The realization of these two competing aspects of the problem within a single theory is the focus of this paper.

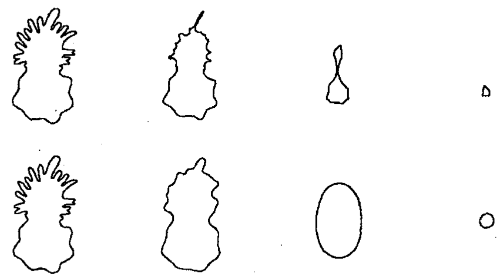

Figure 1: The pure reaction flow (TOP) and the pure diffusion flow (BOTTOM) for a pear shape adapted from Richards [12]. The reaction flow generates shocks whose loci are equivalent to Blums's skeleton; the diffusion flow continuously smoothes the shape, taking it to a circle in the limit.

There have been several attempts to develop generic shape descriptions within computer vision. One possibility is boundary based [3], and dictates that objects should be decomposed into parts at positions of maximal negative curvature. Boundary smoothing can then be used to blur noise and un-necessary detail away, resulting in scale spaces for shape [8, 9, 14]. A second approach is to use region based descriptions derived from Blum's medial axis transform, which can then be smoothed or pruned using various methods [10]. The challenge, for both approaches, is to simplify the description so as to support generic shape recognition.

In an attempt to do this, a curve evolution based reaction-diffusion space was proposed for shape analysis [5, 6, 7], see Section 2. This theory has the attractive property that natural components of shape are given

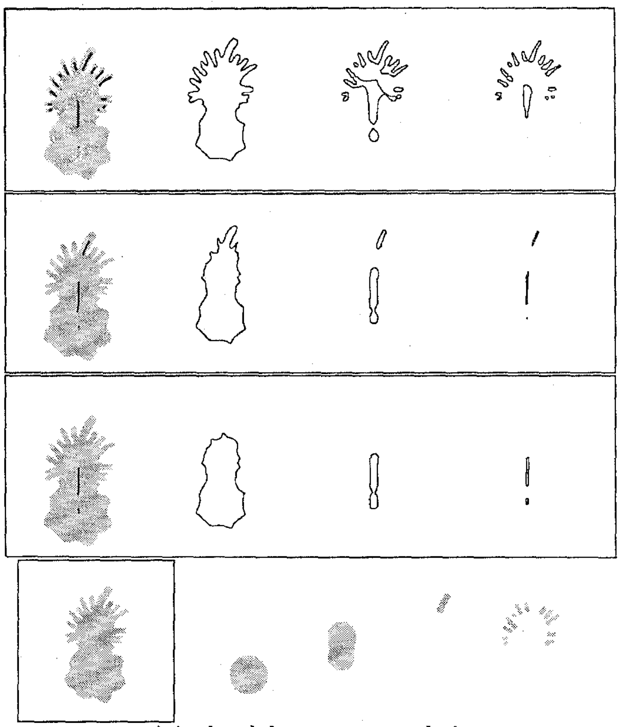

Figure 2: Simplification of the pear shape using our γ flow. TOP: The γ flow with no smoothing provides all the dominant structures. SECOND ROW: With some smoothing the "hairs" are removed, but the other structures remain. THIRD ROW: With further smoothing the stem is also removed, while the two lobes remain. BOTTOM: The reconstructed hierarchy.

by the singularities or shocks of the curve evolution equation. Further, it has recently been shown that it can be used to support recognition [17]. However, an unfortunate consequence of integrating a hyperbolic reaction term (required for generating shocks), with a parabolic diffusion term (required for simplification), see Figure 1, is that the slightest amount of diffusion will prevent the formation of generic first-order shocks which are crucial for shape description. The question of unification, then, is how to achieve a type of "smoothing" for hyperbolic equations. Whereas this may appear paradoxical, we have discovered a method to do this, which is the technical contribution of this paper.

The trick is to embed the "smoothing" information in a global metric, and then to perform the hyperbolic evolution from the initial curve against this metric. This has an interpretation as an area-based weighted gradient flow [16]. Specific metrics are developed according to the following intuition. Consider a shape to be built from a number of different clumps of material, with each initial clump describing a significant component. For example, in Figure 2 (top), the "hairy pair" is built from 15 distinct components, one for the base lobe, one for the upper lobe, one for the stem, and a dozen for the "hairs". The new evolution reveals precisely these components, together with any additional deformations implicit in the initial shape. To simplify this, however, we might want to eliminate the "hair", as in Figure 2 (second row), or the hair and the stem, as in Figure 2 (third row). The reconstructed hierarchy is depicted in Figure 2 (bottom). Notice how each simplification resembles a view of the pear from increasing distances. The results could be combined to define a type of scale-space for shapes.

2 Background

The foundation of our approach is the mathematical theory of curves flowing in the plane with speed a function of curvature. Using a model from mathematical physics (see [11] and the references therein), Kimia, Tannenbaum and Zucker introduced a reaction-diffusion space for visual shape analysis [5, 6, 7]. More precisely with κ the curvature, $\mathcal{N}$ the inward unit normal, and $\mathcal{C}$ the curve coordinates, families of plane curves flowing according the equation

$$\frac{\partial \mathcal{C}}{\partial t} = (\alpha + \beta\kappa)\mathcal{N} \tag{1}$$

were considered, with $\alpha, \beta \in \mathbf{R}$, $\beta \geq 0$. The key idea is to play off the hyperbolic reaction term α with the parabolic diffusion term β: the former leads to the formation of shocks from which a representation of shape can be derived (see Figure 3 for a summary and [7] for details), and the latter smoothes the front, which is important for distinguishing more significant shape features from less significant ones.

2.1 Hyperbolic Smoothing

Third order shocks indicate a symmetric axis for an extended region, as arises in images of objects with parallel sides. Thus they are also important for describing shapes, since they signify a natural type of part such as a tail or a leg, but therein lies the difficulty in working with them. Only first-order and second-order shocks are generic [7], in the sense that such singularities cannot be removed by a small perturbation; see Arnold [1]. True third-order shocks will almost *never* occur for natural or sampled images. Perfectly parallel sides would correspond to a 1-shock moving with infinite velocity. Instead, for real images minor variations will lead to a generic series of closely spaced 2-shocks with extremely rapid (but finite) 1-shocks between them. An example

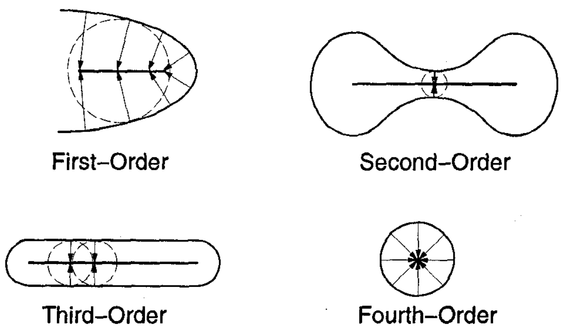

Figure 3: The four types of shocks. A 1-shock derives from a *protrusion*, and traces out a curve segment of 1-shocks. A 2-shock arises at a *neck*, and is immediately followed by two 1-shocks flowing away from it in opposite directions. 3-shocks correspond to an annihilation into a curve segment due to a *bend*, and a 4-shock an annihilation into a point or a *seed*. The loci of these shocks gives Blum's medial axis.

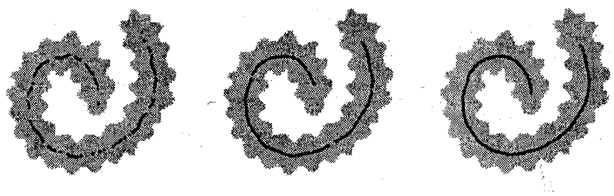

Figure 4: LEFT TO RIGHT: The high-order shocks are "regularized" by diffusion of the global metric, using the geometric heat equation, see Section 3. LEFT: An attempt to localize 3-shocks; the problem is that 2-shocks will appear between them and indicate distinct "parts". RIGHT: For sufficient diffusion in our γ flow, the worm appears to be a single part.

of this is shown in Figure 4; from a distance it resembles a worm but, up close, minor variations in the skin due to bending will induce multiple parts (left). As da Vinci put it: "the flesh which clothes the joints of the bones and other adjacent parts increases and decreases in thickness according to bending or stretching...". The task that we face is regularizing these natural irregularities in extended tails, or bends.

A first approach to regularizing 3-shocks might be to increase the amount of diffusion in the reaction-diffusion model. The difficulty with applying this directly to shape analysis is that the slightest bit of diffusion tends to dominate, preventing the formation of generic first-order shocks which are key to shape representation. In this paper we will modify the evolution equation so that the pde initially mimics parabolic behavior (allowing for degrees of smoothing) but in the limit becomes a pure constant motion flow, leading to the morphological skeleton. Specifically, we shall consider families of curves evolving under

$$\frac{\partial \mathcal{C}}{\partial t} = \gamma \mathcal{N}, \tag{2}$$

where $\gamma : \mathbf{R}^2 \to \mathbf{R}$ is a globally defined positive function on the plane. Note that in contrast to the reaction-diffusion equation 1, equation 2 is purely *hyperbolic*. A key development in this paper will be the choice of γ to effect an appropriate smoothing; hence the title of our paper.

Returning to the "bumpy" worm shape in Figure 4, but viewing it at a distance, the description of a smooth, coiled worm is more satisfying. This is precisely the type of simplification that we would like our notion of "smoothing" to effect; note that this couples the general problem of shape "smoothing" with the "regularization" of the high-order shocks. A γ function that does this is designed in Section 3. We first provide the necessary theoretical motivation for equation 2 (henceforth referred to as the γ flow), by interpreting it as a gradient flow.

2.2 Weighted Area Gradient Flows

We follow our treatment in [16], where we considered area (or volume) in a conformal metric and then derived the associated gradient flow. Let $\mathcal{C} = \mathcal{C}(p,t)$ be a smooth family of closed curves where t parametrizes the family and p the given curve. Modify the standard Euclidean metric $ds^2 = dx^2 + dy^2$ of the underlying space over which the evolution takes place to a conformal one $ds_\phi^2 = \phi^2(dx^2 + dy^2)$. The "$\phi$-area" of the curve is then defined as

$$A_\phi(t) = -\frac{1}{2}\int_0^{L(t)} \phi \langle \mathcal{C}, \mathcal{N}\rangle \, ds,$$

where $\phi : \mathbf{R}^2 \to \mathbf{R}$ is a positive differentiable function defined on the image plane. Differentiating the functional with respect to t, it can be shown that the fastest way to shrink the ϕ-area of the curve is via the equation

$$\mathcal{C}_t = \left\{ \phi + \frac{1}{2}\langle \mathcal{C}, \nabla\phi\rangle \right\} \mathcal{N}. \tag{3}$$

Note that since ϕ is a globally defined function on the plane, (3) defines a hyperbolic equation. As is standard in the subject, our implementation of equation (3) uses level sets [11]. An easy exercise shows that in level set form (3) may be written as

$$\Psi_t = \frac{1}{2}\mathrm{div}\left(\begin{pmatrix} x \\ y \end{pmatrix} \phi \right) \|\nabla\Psi\|, \tag{4}$$

and that equation (2) may be written as

$$\Psi_t = \gamma \, \|\nabla \Psi\|, \tag{5}$$

for the level set function Ψ.

Solving for ϕ from γ We now make the connection between equation (4) and equation (5), providing the necessary theoretical motivation for the γ flow. Equating the terms on the right hand side, we see that

$$x\phi_x + y\phi_y = 2(\gamma - \phi) \tag{6}$$

which is an Euler equation with forcing term γ, that can easily be solved (see [4], pp. 13-15.) The characteristic curves may be easily computed to be

$$\begin{aligned} x(t) &= se^t, \\ y(t) &= e^t, \\ z(t) &= e^{-2t}h(s) + 2e^{-2t}\int_0^t e^{2\tau}\gamma(x(\tau), y(\tau))d\tau, \\ h(x) &= \phi(x, 1), \end{aligned}$$

and so

$$\begin{aligned} z &= y^{-2}h(x/y) + 2y^{-2}\int_0^t e^{2\tau}\gamma(x(\tau), y(\tau))d\tau \\ &= \phi(x(t), y(t)). \end{aligned}$$

Thus for a given γ, we may recover ϕ, and conversely.

3 The γ Flow

3.1 General Properties

We now derive a number of properties of families of plane curves flowing according to equation (2). We will assume that $\gamma : \mathbf{R}^2 \to \mathbf{R}$ is a *globally* defined positive function on the plane $\mathbf{R}^2$. This is in contrast to the standard choices in which γ is a function defined only on the curve (typically taken to be some function of the curvature). As above for T the tangent, N the inward normal, and κ the curvature, one can compute that (see [2, 6])

$$\begin{aligned} N_t &= \gamma_s T, \\ T_t &= \gamma_s N, \\ \kappa_t &= \gamma_{ss} + \gamma\kappa^2, \\ \theta_t &= \gamma_s, \end{aligned}$$

where s is arc-length and θ is the angle parameter, i.e., $d\theta = \kappa ds$. Now if $\gamma_s = 0$ (that is, we are on a level curve $\gamma(s) = \mu$), notice that the normal and tangent don't change, i.e., they remain parallel. This indicates that a curve segment of the evolving curve that coincides with a level curve of γ remains as such, as long as the curve remains non-singular. (This is very similar to the classical skeleton flow.)

The evolution equation for κ is a Ricatti equation, which is well known in control theory. There is an essential relationship between the curvature, the minimal time till shock formation and the speed function γ, which can be made explicit. Let us assume that $\gamma_{ss} = 0$. For the γ functions we will be choosing (based on the signed distance function), the qualitative behavior when $\gamma_{ss} \neq 0$ will be similar. Separating variables, and integrating the evolution equation from 0 to t we get

$$\int_0^t \frac{1}{\kappa^2} d\kappa = \int_0^t \gamma dt.$$

Thus, we can solve for $\kappa(s, t)$:

$$\kappa(s, t) = \frac{\kappa(s, 0)}{1 - \kappa(s, 0)\Gamma},$$

where

$$\Gamma = \int_0^t \gamma dt.$$

Therefore the time of first shock formation, curvature along the initial curve, and the speed γ are related by

$$\int_0^t \gamma dt = \frac{1}{\kappa(s, 0)}.$$

What this formula roughly says is that the larger the speed γ, and the larger the maximal curvature of the initial curve, the shorter the time of shock formation. The well-known formula for $\gamma = 1$, $t = \frac{1}{\kappa(s,0)}$ for the time till first shock for the usual morphological (skeleton) flow, is a special case. As another case, when γ varies linearly (as in the case of a Euclidean distance function), e.g., $\gamma(s, t) = -\mu_1 t + \mu_0$, $\gamma(s, t)dt$ can be integrated to get the minimum time till shock formation:

$$t = \mu_0/\mu_1 - \sqrt{\mu_0^2 - 2\mu_1/\kappa(s, 0)}/\mu_1$$

Thus a shock forms provided $\mu_0^2 > 2\mu_1/\kappa(s, 0)$.

Remark.
The formulae for the evolution of curve length and area are also interesting:

$$\begin{aligned} L_t &= -\int_0^L \gamma\kappa ds = -\int_0^{2\pi} \gamma d\theta, \\ A_t &= -\int \gamma ds. \end{aligned}$$

In particular, we would like to see what happens to the isoperimetric inequality as a function of γ. Since we are interested in shock formation, we would like the isoperimetric inequality to get worse (that is, increase as a function of t) and least after some given time in the evolution. Recall that for closed simple curves we always have that

$$L^2/A \geq 2 \geq 4\pi, \tag{7}$$

with equality for the circle. Now for a family of curves as above, the isoperimetric inequality getting worse means that

$$\frac{d}{dt}\left(\frac{L^2}{A}\right) \geq 0. \tag{8}$$

But this is equivalent to

$$2L_t A \geq A_t L,$$

or from the above formula for L_t and A_t,

$$2\int_0^L \gamma\kappa ds A \leq \int_0^L \gamma ds L. \tag{9}$$

Notice that when $\gamma = 1$ (the usual skeleton), we have that (9), becomes

$$2\int_0^L \kappa ds A \leq \int_0^L ds L,$$

which is precisely the classical isoperimetric inequality

$$4\pi A \leq L^2$$

given above. Note that this follows since

$$\int_0^L \kappa ds = 2\pi.$$

We will be interested in studying (9) for other speed functions γ.

3.2 Construction of γ

We now turn to the design of γ functions that effect the type of smoothing we are after.

Requirement 1 *γ must be a globally defined function on the plane.*

This requirement follows from the interpretation of equation (2) as a weighted area gradient flow in Section 2.2.

Requirement 2 *γ must be a positive function on the plane.*

The second requirement keeps the curve moving in a fixed direction. Furthermore, we would like the γ flow to converge to globally salient features of the shape, which are typically captured by the central high order (2-,3-,4-) shocks (see Figure 3). Thus,

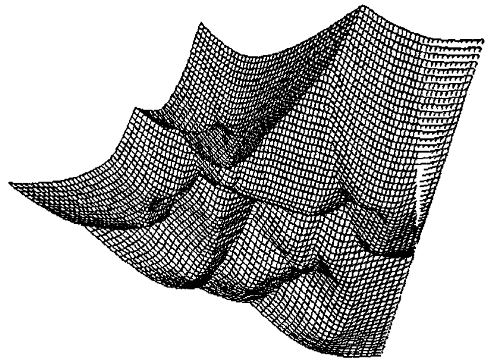

Figure 5: A surface plot of γ for the horse shape in Figure 7. The function is constructed as an outward distance map from the high order shocks.

Observation 1 *A sufficient condition for the γ flow to converge to the loci of high-order shocks is to let γ be identically zero at them.*

Let Ψ be the embedding surface of which the evolving curve C is its zero level set.

Observation 2 *The high-order shocks coincide with the zeros of $||\nabla\Psi||$.*

These zeros can be accurately detected using the algorithm in [15]. Thus,

Proposition 1 *For each $(x, y) \in \mathbf{R}^2$, let $\gamma(x, y)$ be the Euclidean distance to the nearest high-order shock.*

Formally, let S be the set of all high-order shocks and $d(.,.)$ the Euclidean distance function. Then, for all $(x, y) \in \mathbf{R}^2$

$$\gamma(x, y) = \min_{s \in S} d((x, y), (x(s), y(s))).$$

Therefore, γ is an outward distance map from the high-order shocks, see Figure 5. Note that the above construction satisfies the requirements that γ be positive and globally defined. Intuitively, it captures the notion that significance should be measured with respect to relative size (distance to the nearest high-order shock). To illustrate, a pimple on a nose is typically viewed as more salient than the same pimple on the cheek.

3.3 Smoothing γ

The above construction of γ is entirely determined by the initial shape. Thus, if the shape is made of many lumps, as in Figure 4, the γ flow will converge to these structures. In order to effect the form of smoothing we are after, our strategy will be to regularize γ by

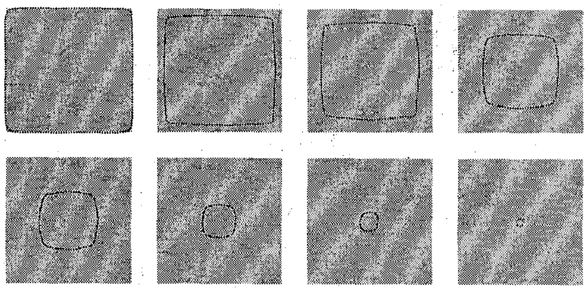

Figure 6: The front evolving under equation (2), with γ a distance function from the central 4-shock.

computing it from the high-order shocks of an increasingly smoothed distance map (to the original shape). A very natural form of smoothing is the geometric heat equation, since this flow simplifies the shock structure of the level curves. In particular, generic first-order shocks are associated with points of extremal curvature. It has been shown that under equation (1) with $\alpha = 0$, the number of curvature extrema and inflection points are strictly decreasing for all non-circular curves, and non-increasing for a perfect circle [6].

4 Examples

We provide several examples of shapes evolving under equation (2), with γ constructed as a distance function to the closest high-order shock, as described in Section 3. For all simulations the high-order shock locations are obtained using the the algorithm developed in [15], and the curve is evolved using level set techniques [11].

Figure 6 illustrates the γ flow on a square shape having a single 4-shock in its center. Observe that whereas shocks are initially formed (corners), they are eventually smoothed away. The limit shape appears to be a circle around the central 4-shock. Figure 7 (top) depicts a da Vinci horse, scanned from a book of his sketches, along with the high-order shocks of the silhouette. The associated γ flow is depicted in Figure 7 (middle and bottom rows). Notice how intuitive the captured components are, i.e., they correspond to the head, neck, torso, limbs, and tail. Finally, we consider a da Vinci study of the hind quarters of a horse, Figure 8 and zoom on the region of the right leg. Observe that with increased smoothing the γ flow effectively regularizes the undulating structure due to "the flesh which clothes the joints of the bones and other adjacent parts..."

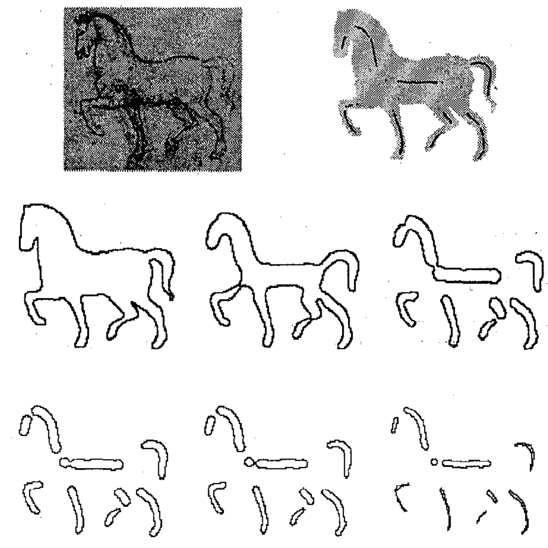

Figure 7: A da Vinci horse, scanned from a book of his sketches (top left) and the high-order shocks of the silhouette (top right). The front evolving under equation (2), with γ a distance function from the high-order shocks, is shown in the middle and bottom rows.

5 Conclusions

The problem of generic shape description through the singularities of a curve evolution process has been plagued by the mis-match between a reaction term, which causes the shocks that provide descriptors, and the diffusion term, which simplifies the shape via smoothing. In brief, diffusion destroys the first-order shock structure. We have discovered a solution to this problem, in which the "smoothing" information is embedded in a global metric, and a hyperbolic (reactive) flow is then effected against this metric. The result is a new flow that is amenable to smoothing but which still develops shocks. It is shown that increasing smoothing results in smoother flows, thereby leading to a hierarchical representation of structure. As a side effect, non-generic third-order shocks are regularized, facilitating the recognition of elongated "tails" and "bends" as distinct parts.

This technique is fundamentally different from traditional approaches (where some form of boundary melting occurs) in that, roughly speaking, the significance of a structure is determined by its size with respect to the structure to which it is attached. The global information is taken from the distance map, and is used to define the metric against which the hyperbolic evolution takes place. The smoothing is implemented on

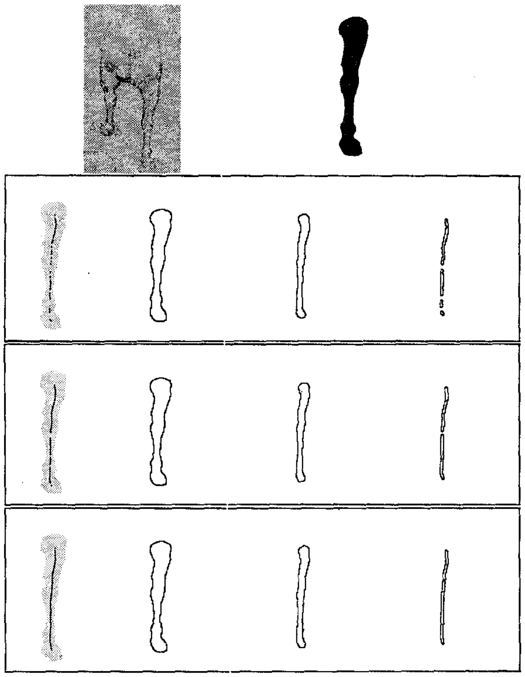

Figure 8: TOP ROW: A da Vinci study of the hind quarters of a horse. We zoom in on the region of the right leg. SECOND ROW: The high-order shocks of the original are overlayed (left), with the γ flow shown from left to right, yielding 4 components. THIRD ROW: With some amount of smoothing of the embedding surface, the γ flow now yields 2 components. BOTTOM ROW: With increased smoothing, the γ flow effectively regularizes the leg into a single structure.

the distance map and not on the evolution. The result is a purely hyperbolic process that "smooths" shapes.

Although the techniques developed in this paper are entirely two-dimensional, another advantage of this class of hyperbolic evolutions is that they are directly extensible to three dimensions. This will be extremely important for applications, especially those in biomedicine.

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References

[1] V. Arnold. *Mathematical Methods of Classical Mechanics.* Springer-Verlag, New York, 1989.

[2] M. Grayson. The heat equation shrinks embedded plane curves to round points. *J. of Differential Geometry*, 26:285–314, 1987.

[3] D. D. Hoffman and W. A. Richards. Parts of recognition. *Cognition*, 18:65–96, 1985.

[4] F. John. *Partial Differential Equations.* Springer-Verlag, New York, 1982.

[5] B. B. Kimia, A. Tannenbaum, and S. W. Zucker. Toward a computational theory of shape: An overview. *Lecture Notes in Computer Science*, 427:402–407, 1990.

[6] B. B. Kimia, A. Tannenbaum, and S. W. Zucker. On the evolution of curves via a function of curvature I: The classical case. *J. of Math. Analysis and Applications*, 163:438–458, 1992.

[7] B. B. Kimia, A. Tannenbaum, and S. W. Zucker. Shape, shocks, and deformations I: The components of two-dimensional shape and the reaction-diffusion space. *IJCV*, 15:189–224, 1995.

[8] J. J. Koenderink and A. J. van Doorn. Dynamic shape. *Biological Cybernetics*, 53:383–396, 1986.

[9] F. Mokhtarian and A. Mackworth. A theory of multiscale, curvature-based shape representation for planar curves. *PAMI*, 14(8):789–805, August 1992.

[10] R. L. Ogniewicz. *Discrete Voronoi Skeletons.* Hartung-Gorre, 1993.

[11] S. J. Osher and J. A. Sethian. Fronts propagation with curvature dependent speed: Algorithms based on hamilton-jacobi formulations. *Journal of Computational Physics*, 79:12–49, 1988.

[12] W. Richards, B. Dawson, and D. Whittington. Encoding contour shape by curvature extrema. *JOSA-A*, 3(9):1483–1489, 1986.

[13] E. Rosch, C. B. Mervis, W. D. Gray, D. M. Johnson, and P. Boyes-Braem. Basic objects in natural categories. *Cognitive Psychology*, 8:382–439, 1976.

[14] G. Sapiro and A. Tannenbaum. On affine plane curve evolution. *Journal of Functional Analysis*, 119:79–120, 1994.

[15] K. Siddiqi and B. B. Kimia. A shock grammar for recognition. In *CVPR*, pages 507–513, 1996.

[16] K. Siddiqi, Y. B. Lauzière, A. Tannenbaum, and S. W. Zucker. Area and length-minimizing flows for shape segmentation. *IEEE Transactions on Image Processing*, to appear, 1997.

[17] K. Siddiqi, A. Shokoufandeh, S. Dickinson, and S. W. Zucker. Shock graphs and shape matching. *International Conference on Computer Vision*, 1998.