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Causal Power Series and the Nonlinear Standard H^∞ Problem

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Abstract

In this note, using a power series approach [10, 11, 14] we describe a design procedure applicable to analytic nonlinear plants. Our technique is a generalization of the linear H^∞ theory. We can now use this theory to solve the full standard problem in robust control theory in the nonlinear framework.

1 Introduction

In this note, we extend our work on finding a suitable, implementable nonlinear extension of the linear H^∞ design methodology to the full standard problem. In what follows, we will just consider discrete-time systems, even though the techniques described below carry over to the continuous-time setting as well.

Our approach is valid for systems described by analytic input/output operators. As in [12, 13, 11, 10], our technique involves the expression of each n-linear term of a suitable Taylor expansion of the given operator as an equivalent linear operator acting on a certain associated tensor space which allowed us to iteratively apply the classical commutant lifting theorem in designing a compensator. (Our class of operators includes Volterra series.)

More precisely, in our approach we are reduced to applying the classical (linear) commutant lifting theorem to an H^2-space defined on some D^n (where D denotes the unit disc). Now when one applies the classical result to D^n ($n \geq 2$), even though time-invariance is preserved (that is, commutation with the appropriate shift), causality may be lost. Indeed, for systems described by analytic functions on the disc D (these correspond to stable, discrete-time, 1-D systems), time-invariance (that is, commutation with the unilateral shift) implies causality. For analytic functions on the $n-$disc ($n > 1$), this is not necessarily the case. For dynamical system control design and for any physical application, this is of course major drawback for such an approach.

Hence for a dilation result in $H^2(D^n)$ we need to include the causality constraint explicitly in the set-up of the dilation problem. It is precisely this problem which motivated the mathematical operator-theoretic work of [14] and [10] which incorporated Arveson theory [1] into the dilation, commutant lifting framework. In our paper [11], we show how these ideas lead to an explicit solution of a nonlinear extension of the weighted sensitivity minimization problem in linear H^∞ control.

While, the general method explicated in this note is based on a causal extension of the commutant lifting theorem, for the purposes of the operators and spaces which appear in control we will give a direct simple method for finding the optimal causal compensators. In fact, we will show that *the computation of an optimal causal nonlinear compensator may be reduced to a classical linear dilation problem whose solution is given by the Commutant Lifting Theorem.*

2 Preliminaries on the Causal Commutant Lifting Theorem

In order to make the presentation as self-contained as possible, we recall in this section the classical Classical Commutant Lifting Theorem [19] as well as the causal version [12], [10].

We let S and U denote isometries on the complex separable Hilbert spaces $\mathcal{G}$ and $\mathcal{K}$, respectively. Let $\mathcal{H} \subset \mathcal{K}$ denote a U^* invariant (closed) subspace. Let $P : \mathcal{K} \to \mathcal{H}$ denote orthogonal projection, and set

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$T = PU|\mathcal{H}$. For the convenience of the reader, we state the classical Commutant Lifting Theorem [19]:

Theorem 1 *Let $A : \mathcal{G} \to \mathcal{H}$ be a bounded linear operator such that $AS = TA$. Then there exists a bounded linear operator $B : \mathcal{G} \to \mathcal{K}$ such that*

$$BS = UB,\ A = PB,\ \|A\| = \|B\|.$$

Such an operator B is called an *intertwining dilation* (or *lifting*) of A. We now define *causality* in this framework. Roughly, causality means that for a given input/output system the past output is independent of the future inputs. This may be given precise mathematical formulation in terms of a family of projections which we shall now do. See also [1] and [9].

Let P_j, $j \geq 1$ be a sequence of orthogonal projections on $\mathcal{G}$ satisfying the following conditions:

$$P_1 \leq P_2 \leq \ldots \quad (1)$$
$$P_j \leq I - S^j S^{*j} \quad j = 1, 2, \ldots \quad (2)$$
$$P_{j+1} S(I - P_j) = 0 \quad j = 1, 2, \ldots. \quad (3)$$

We call such a family of projections a *causal structure* on $\mathcal{G}$.

Next $U : \mathcal{K} \to \mathcal{K}$ will denote a minimal isometric dilation of T. Without loss of generality (see [19]), we can take

$$\mathcal{K} = \mathcal{H} \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \mathcal{D}_T \oplus \ldots$$

and

$$U(h \oplus e_1 \oplus e_2 \oplus \ldots) = Th \oplus D_T h \oplus e_1 \oplus e_2 \ldots.$$

(Note that $\mathcal{D}_T := \overline{(D_T \mathcal{H})}$, where $D_T := (I - T^*T)^{1/2}$. D_T is the *defect operator* of T, and $\mathcal{D}_T$ the *defect space*.) We will identify $\mathcal{H}$ with the subspace $\mathcal{H} \oplus \{0\} \oplus \{0\} \oplus \ldots$ of $\mathcal{K}$.

Let $B : \mathcal{G} \to \mathcal{K}$ intertwine S with U, that is

$$UB = BS. \quad (4)$$

Note that this implies $U^j B = BS^j$ ($j \geq 1$), hence $U^j U^{*j} BS^j S^{*j} = BS^j S^{*j}$, and so

$$(I - U^j U^{*j})B = (I - U^j U^{*j})B(I - S^j S^{*j}) \quad j = 1, 2, \ldots. \quad (5)$$

We now make the following key definition:

Definition 1. An operator B satisfying (4) is called $(P_1, P_2, \ldots)$-*causal* (and if the sequence $\{P_j\}_{j=1}^{\infty}$ is fixed, *causal*) if

$$(I - U^j U^{*j})B = (I - U^j U^{*j})BP_j \quad j \geq 1 \quad (6)$$

or equivalently,

$$(I - P_j)B^* = (I - P_j)B^* U^j U^{*j} \quad j \geq 1. \quad (7)$$

Note by (5) that B is always $(I - SS^*, I - S^2 S^{*2}, \ldots)$-causal. In what follows the sequence $P_1, P_2, \ldots$ will be fixed and causality will always be defined relative to this causal structure.

We now set

$$\|A\|_c := \inf\{\|B\| : B \text{ is a causal intertwining dilation of } A\}. \quad (8)$$

By using the weak operator topology, we can easily prove that the infimum in (8) is actually a minimum.

Finally, let $\hat{S}$ denote the minimal unitary extension of S on $\hat{\mathcal{G}} \supset \mathcal{G}$, and

$$\mathcal{G}_c := \overline{\left(\cup_{j=0}^{\infty} \hat{S}^{*j} \mathcal{G}_j\right)} \subset \hat{\mathcal{G}}, \quad S_c := \hat{S}|\mathcal{G}_c. \quad (9)$$

We now recall the following result from [10]:

Theorem 2 (Causal Commutant Lifting Theorem) *1. If T is invertible, define*

$$A_c \hat{g} := T^{-j} A g_j, \quad (10)$$

for $\hat{g} = \hat{S}^{*j} g_j$, where $g_j \in \mathcal{G}_j$ ($j = 0, 1, \ldots$). Then the causal commutant lifting problem is solvable if and only if A_c is bounded. In this case

$$A_c S_c = T A_c, \quad \|A\|_c = \|A_c\|.$$

2. In general (i.e., we do not assume T is invertible), the causal commutant lifting problem is solvable if and only if there exists a linear, bounded operator

$$A' : \mathcal{G}_c \to \mathcal{H},\ A' S_c = T A',\ A'|\mathcal{G} = A. \quad (11)$$

If such an operator A' exists, then

$$\|A\|_c = \min\{\|A'\| : A' \text{ as in (11)}\}. \quad (12)$$

As we will see in the next section, the standard problem in robust control theory has a special structure which allows a direction construction of an operator A' as above. For completeness, we will give a direct proof based on [10] for the reduction of the "causal" standard problem to the "classical" standard problem.

3 Control Version of Causal Commutant Lifting Theorem

For the standard problem in robust control theory (see our discussion in Section 5 below), we may extract the following mathematical set-up. We are given complex separable Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$ equipped with the unilateral shifts $S_{\mathcal{E}_1}, S_{\mathcal{E}_2}, S_{\mathcal{F}_1}, S_{\mathcal{F}_2}$, respectively. Let $\Theta_1 : \mathcal{E}_1 \to \mathcal{F}_1$ be a co-isometry intertwining $S_{\mathcal{E}_1}$ with $S_{\mathcal{F}_1}$ (i.e., $\Theta_1 S_{\mathcal{E}_1} = S_{\mathcal{F}_1}\Theta_1$), and let $\Theta_2 : \mathcal{F}_2 \to \mathcal{E}_2$ be an isometry intertwining $S_{\mathcal{E}_2}$ with $S_{\mathcal{F}_2}$. We let $U_{\mathcal{E}_1}$, be the minimal unitary dilation of $S_{\mathcal{E}_1}$ on $\mathcal{K}_{\mathcal{E}_1}$, and similarly for $U_{\mathcal{E}_2}$ on $\mathcal{K}_{\mathcal{E}_2}$, $U_{\mathcal{F}_1}$ on $\mathcal{K}_{\mathcal{F}_1}$, and $U_{\mathcal{F}_2}$ on $\mathcal{K}_{\mathcal{F}_2}$.

Now let

$$P_{\mathcal{E}_2}^{(n)} := (I - S_{\mathcal{E}_2}^n S_{\mathcal{E}_2}^{*n}), \quad P_{\mathcal{F}_2}^{(n)} := (I - S_{\mathcal{F}_2}^n S_{\mathcal{F}_2}^{*n}), \quad n \geq 0.$$

We let the sequence $P_{\mathcal{E}_2}^{(n)}$ define the causal structure on $\mathcal{E}_2$, and similarly the causal structure of $\mathcal{F}_2$ is defined by the sequence $P_{\mathcal{F}_2}^{(n)}$. Moreover, the causal structure on $\mathcal{E}_1$ is defined by a general sequence of operators $P_1^{(n)}$, $n \geq 0$ satisfying the causal structure conditions given by (1, 2, 3), and similarly the causal structure on $\mathcal{F}_1$ is defined by a sequence of operators $P_2^{(n)}$, $n \geq 0$ satisfying these conditions as well. We assume that the two input/putput operators Θ_1, Θ_2, are causal with respect to the above structures. In Section 4 below, we will give precise definitions of the sequences $P_1^{(n)}$, $P_2^{(n)}$ for the relevant spaces appearing in the nonlinear H^∞ control problem.

We let $W : \mathcal{E}_1 \to \mathcal{E}_2$ denote a causal operator intertwining $S_{\mathcal{E}_1}$ with $S_{\mathcal{E}_2}$. Thus causality for W means that

$$P_{\mathcal{E}_2}^{(n)} W P_1^{(n)} = P_{\mathcal{E}_2}^{(n)} W, \quad \forall n \geq 0.$$

Finally, $Q : \mathcal{F}_1 \to \mathcal{F}_2$ will denote a causal operator intertwining $S_{\mathcal{F}_1}$ with $S_{\mathcal{F}_2}$.

Define now

$$\mathcal{E}_1^{(n)} := (I - P_1^{(n)})\mathcal{E}_1, \quad \forall n \geq 0.$$

Since by the causality of W we have

$$(I - S_{\mathcal{E}_2}^n S_{\mathcal{E}_2}^{*n}) W | \mathcal{E}_1^{(n)} = 0,$$

it follows that

$$W_n := S_{\mathcal{E}_2}^{*n} W | \mathcal{E}_1^{(n)},$$

is also unquely define by

$$S_{\mathcal{E}_2}^n W_n := W|\mathcal{E}_1^{(n)}, \quad W_n : \mathcal{E}_1^{(n)} \to \mathcal{E}_2.$$

Clearly,

$$\begin{aligned} W_n S_{\mathcal{E}_1} | \mathcal{E}_1^{(n)} &= S_{\mathcal{E}_2} W_n, \\ W_n &= W_{n+1} S_{\mathcal{E}_1} | S_{\mathcal{E}_1}^{(n)}. \end{aligned}$$

Now let

$$\mathcal{E}_1^{(c)} = \overline{\mathcal{E}_1^{(co)}}$$

where

$$\mathcal{E}_1^{(co)} := \bigcup_{j=0}^{\infty} U_{\mathcal{E}_1}^{*j} \mathcal{E}_1^{(j)} \subset \mathcal{K}_{\mathcal{E}_1}, \quad S_{\mathcal{E}_1}^{(c)} := U_{\mathcal{E}_1} | \mathcal{E}_1^{(c)}.$$

Finally, we define $W_c : \mathcal{E}_1^{(co)} \to \mathcal{E}_2$, by

$$W_c g := W_n g_n,$$

for $g = U_{\mathcal{E}_1}^{*n} g_n$, $g_n \in \mathcal{E}_1^{(n)}$, $n \geq 0$.

Note that we can make a similar construction on the spaces $\mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$. In particular, for a causal $Q : \mathcal{F}_1 \to \mathcal{F}_2$, such that $QS_{\mathcal{F}_1} = S_{\mathcal{F}_2}Q$, we can define $Q_c : \mathcal{F}_1^{(co)} \to \mathcal{F}_2$, where

$$\mathcal{E}_2^{(co)} := \bigcup_{j=0}^{\infty} U_{\mathcal{E}_2}^{*j} \mathcal{E}_2^{(j)}.$$

Next, it is easy to see both W_c and Q_c extend by continuity to the closure $\mathcal{E}_1^{(c)}$, respectively $\mathcal{F}_1^{(c)} = \overline{\mathcal{F}_1^{(co)}}$. Clearly, we also have

$$\|W_c\| = \|W\|, \quad W_c|\mathcal{E}_1 = W, \quad W_c S_{\mathcal{E}_1}^{(c)} = S_{\mathcal{E}_2} W_c,$$

and

$$\|W - \Theta_2 Q \Theta_1\| = \|(W - \Theta_2 Q \Theta_1)_c\|. \tag{13}$$

Now set

$$\mu(W, \Theta_1, \Theta_2) := \inf\{\|W - \Theta_2 Q \Theta_1\| : QS_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q\}.$$

This corresponds to the *classical standard control problem.* We also set

$$\mu_c(W, \Theta_1, \Theta_2) := \inf\{\|W - \Theta_2 Q \Theta_1\| : Q \text{ causal}, \ QS_{\mathcal{F}_1} = S_{\mathcal{F}_2} Q\}.$$

This is the *causal standard control problem.*

Let $\hat{\Theta}_1 : \mathcal{K}_{\mathcal{E}_1} \to \mathcal{K}_{\mathcal{F}_1}$ denote the extension of the co-isometry $\Theta_1 : \mathcal{E}_1 \to \mathcal{F}_1$, that is uniquely defined by

$$\hat{\Theta}_1 U_{\mathcal{E}_1}^{*n} e_1 = U_{\mathcal{F}_1}^{*n} \Theta_1 e_1, \quad \forall e_1 \in \mathcal{E}_1.$$

Note that $\hat{\Theta}_1$ is also isometric and $\hat{\Theta}_1 U_{\mathcal{E}_1} = U_{\mathcal{F}_1} \hat{\Theta}_1$. We can now state the following result:

Theorem 3 *Notation as above.*

1. $\mu_c(W, \Theta_1, \Theta_2) = \mu(W_c, \hat{\Theta}_1 | \mathcal{E}_1^{(c)}, \Theta_2)$.

2. *Q_{opt} is a causal optimal solution, i.e.,*

$$\mu_c(W, \Theta_1, \Theta_2) = \|W - \Theta_1 Q_{opt} \Theta_2\|$$

if and only if $Q_{opt,c}$ is such that

$$\mu(W_c, \hat{\Theta}_1 | \mathcal{E}_1^{(c)}, \Theta_2) = \|W_c - \Theta_2 Q_{opt,c} \hat{\Theta}_1 | \mathcal{E}_1^{(c)}\|.$$

Finally, let us recall how the classical standard problem can be solved using the Commutant Lifting Theorem. Set

$$\begin{aligned}\mathcal{H}_1 &:= \mathcal{E}_1^{(c)} \ominus (\hat{\Theta}_1|\mathcal{E}_1^{(c)})^*\mathcal{E}_1^{(c)},\\ \mathcal{H}_2 &:= \mathcal{E}_2 \ominus \Theta_2\mathcal{F}_2.\end{aligned}$$

Let $P : \mathcal{E}_2 \to \mathcal{H}_2$ denote orthogonal projection. Then we define the operator

$$\Lambda = \Lambda(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2) : \mathcal{H}_1 \to \mathcal{H}_2, \tag{14}$$

by

$$\Lambda h := PW_c h, \ \ h \in \mathcal{H}_1. \tag{15}$$

Then using the Commutant Lifting Theorem, one may show (see [19]) that

$$\|\Lambda\| = \mu(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2).$$

Thus from the above theorem, we have the following result:

Corollary 1 *Notation as above. Then*

$$\mu_c(W, \Theta_1, \Theta_2) = \|\Lambda(W_c, \hat{\Theta}_1|\mathcal{E}_1^{(c)}, \Theta_2)\|.$$

4 Causality in $H^2(D^n)$

In this section, we specialize the discussion on causality of the previous sections to the Hardy spaces which appear in the nonlinear control problem that we wish to study.

We first define the class of nonlinear input/output operators in which we will be interested. In order to do this, we will first need to discuss a few standard results about analytic mappings on Hilbert spaces. See [2], [3], [12], [13], and the references therein for complete details.

Let $H^2(D^n, \mathbf{C}^k)$ denote the standard Hardy space of $\mathbf{C}^k$–valued analytic functions on the n–disc D^n (D denotes the unit disc) with square integrable boundary values. We denote the shift on $H^2(D^n, \mathbf{C}^k)$ by $S_{(n)}$. Note that $S_{(n)}$ is defined by multiplication by the function $(z_1 \cdots z_n)$. (By abuse of notation, we will denote the shift on $H^2(D^n, \mathbf{C}^k)$ by $S_{(n)}$ for any k.)

We now consider an analytic map ϕ with $\mathcal{G} = H^2(D, \mathbf{C}^k) =: H_k$, and $\mathcal{H} = H^2(D^n, \mathbf{C}^m) =: H_m$. Note that

$$H_k \otimes \cdots \otimes H_k = (H_k)^{\otimes n} \cong H^2(D^n, \mathbf{C}^K) \ \text{ with } K = k^n \tag{16}$$

where we map $1 \otimes \cdots \otimes z \otimes \cdots \otimes 1$ (z in the i-th place) to z_i, $i = 1, \cdots, n$.

We will identify ϕ_n as a bounded linear map from $H^2(D^n, \mathbf{C}^K) \to H^2(D, \mathbf{C}^m)$ via the canonical isomorphism (16). Then we say that ϕ is *time-invariant* if

$$\phi_n S_{(n)} = S_{(1)}\phi_n, \quad \forall n \geq 1. \tag{17}$$

(We will also say each ϕ_n is *time-invariant.*) Equivalently, this means that $S_{(1)}\phi = \phi \circ S_{(1)}$ on some open ball about the origin in which ϕ is defined.

We say that $\phi_n : H^2(D^n, \mathbf{C}^K) \to H_m$ is *causal* if for $F(z_1, \ldots, z_n) \in H^2(D^n, \mathbf{C}^K)$,

$$F(z_1, \ldots, z_n) = \sum_{i_1, \ldots, i_n \geq 0} F_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n},$$

$$\phi_n(F)(z) := \sum_{p \geq 0} f_p z^p,$$

each f_p only depends on

$$\{F_{i_1, \ldots, i_n} : 0 \leq i_1, \ldots, i_n \leq p\}.$$

This means that for

$$F(z_1, \ldots, z_n) = \sum_{\max\{i_1, \ldots, i_n\} \geq p} F_{i_1, \ldots, i_n} z_1^{i_1} \cdots z_n^{i_n},$$

we have that

$$(I - S_{(1)}^p S_{(1)}^{*p})\phi_n(F(z_1, \ldots, z_n)) = 0. \tag{18}$$

It is easy to show that if Q and R are causal, then so is $Q \otimes R$.

5 Standard Nonlinear Problem

We will now describe the physical control problem in which we are interested. First, we will need to consider the precise kind of input/output operator we will be considering. As above, H_k denotes the standard Hardy space of $\mathbf{C}^k$-valued functions on the unit disc. We now make the following definition.

Then we say an analytic input/output operator $\phi : H_k \to H_m$ is *admissible* if it is causal, time-invariant, majorizable, and $\phi(0) = 0$. We denote

$$C_l := \{\text{space of admissible operators}\}.$$

Since the theory we are considering is local, the notion of admissibility is sufficient for all of the applications we have in mind. Again by abuse of notation, C_l will denote the set of admissible operators for any k and m.

We now begin to formulate our control problem. Referring to the standard feedback configuration, G represents the generalized plant which we assume is modelled by an admissible operator, and K the compensator. Let $\mathcal{F}(G, K)$ denote the input/output operator from w to z. Then we want to minimize the "size" of $\mathcal{F}(G,K)$ over all inputs of bounded energy (of fixed given bound) in the sense which will be given below. One can show that K stabilizes the closed loop if and only if

$$C = Q \circ (I - P \circ Q)^{-1}, \tag{19}$$

for some admissible operator Q. We will call such a Q, a *compensating parameter*. Then via this parametrization, we have

$$\mathcal{F}(G,K) = W - P \circ Q \circ R,$$

for admissible operators W, P, R which depend only on the generalized plant G. We now will say in what sense we wish to minimize the size of the operator $W - P \circ Q \circ R$ taken over all Q. We follow here our convention that for given $\phi \in \mathcal{C}_l$, ϕ_n will denote the bounded linear map on the space $(H_k^2)^{\otimes n} \cong H^2(D^n, \mathbf{C}^K)$ (with $K = k^n$) associated to the n-linear part of ϕ which we also denote by ϕ_n (and which we always assume without loss of generality is symmetric in its arguments). The context will always make the meaning of ϕ_n clear. We can now state the following definitions:

Definitions 2.

(i) For $W, P, R \in \mathcal{C}_l$, set

$$\mathcal{T}_{(W,P,R)}(Q)(\rho) := \mathcal{T}(Q)(\rho) := \sum_{n=1} \rho^n \|(W - P \circ Q \circ R)_n\|$$

for all $\rho > 0$ such that the sum converges.
(ii) We write $\mathcal{T}(Q) \preceq \mathcal{T}(\tilde{Q})$, if there exists a $\rho_o > 0$ such that $\mathcal{T}(Q)(\rho) \leq \mathcal{T}(\tilde{Q})(\rho)$ for all $\rho \in [0, \rho_o]$. If $\mathcal{T}(Q) \preceq \mathcal{T}(\tilde{Q})$ and $\mathcal{T}(\tilde{Q}) \preceq \mathcal{T}(Q)$, we write $\mathcal{T}(Q) \cong \mathcal{T}(\tilde{Q})$. This means that $\mathcal{T}(Q)(\rho) = \mathcal{T}(\tilde{Q})(\rho)$ for all $\rho > 0$ sufficiently small, i.e., $\mathcal{T}(Q)$ and $\mathcal{T}(\tilde{Q})$ are equal as germs of functions.
(iii) If $\mathcal{T}(Q) \preceq \mathcal{T}(\tilde{Q})$, but $\mathcal{T}(\tilde{Q}) \not\preceq \mathcal{T}(Q)$, we will say that Q *ameliorates* $\tilde{Q}$. Note that this means $\mathcal{T}(Q)(\rho) < \mathcal{T}(\tilde{Q})(\rho)$ for all $\rho > 0$ sufficiently small.

Now with Definitions 2, we can define a notion of "optimality:"

Definitions 3.

(i) $Q_o \in \mathcal{C}_l$ is called *optimal* if $\mathcal{T}(Q_o) \preceq \mathcal{T}(Q)$ for all $Q \in \mathcal{C}_l$.

(ii) We say $Q \in \mathcal{C}_l$ is *optimal with respect to its n-th term* Q_n, if for every n-linear $\hat{Q}_n \in \mathcal{C}_l$, we have

$$\mathcal{T}(Q_1 + \ldots + Q_n + Q_{n+1} \ldots) \preceq \mathcal{T}(Q_1 + \ldots + \hat{Q}_n + Q_{n+1} + \ldots).$$

If $Q \in \mathcal{C}_l$ is optimal with respect to all of its terms, then we say that it is *partially optimal*.

6 Iterated Causal Lifting Procedure for the Standard Problem

In this section, we discuss a construction from which we will derive partially optimal compensators relative to the closed loop operator $\mathcal{T}$ given in Definitions 3 above. As before, we are given the admissible operators W, P, R. We always suppose that P_1 (the linear part of P) is an isometry, and that R_1 is a co-isometry.

Using the notation of Section 4, we take $\mathcal{E}_1 := H^2(D^n, \mathbf{C}^{k_1})$, $\mathcal{E}_2 := H^2(D, \mathbf{C}^{k_2})$, $\mathcal{F}_1 := H^2(D^n, \mathbf{C}^{k_3})$, and $\mathcal{F}_2 := H^2(D, \mathbf{C}^{k_4})$.

We begin by noting the following key relationship:

$$(P \circ Q \circ R)_n = \sum_{1 \leq k \leq n} \sum_{i_1 + \cdots + i_k = n} P_k(Q_{i_1}(R^{\otimes i_1}), \ldots, Q_{i_k}(R^{\otimes i_k}))$$

Note that we may in term write that

$$Q_j(R^{\otimes j}) = \sum_{k_1, \ldots, k_j} Q_j(R_{k_1} \otimes \ldots \otimes R_{k_j}).$$

Thus we see that

$$(W - P \circ Q \circ R)_n = \hat{W}_n - P_1 Q_n(R_1^{\otimes n}),$$

where

$$\hat{W}_n = W_n + A(Q_1, \ldots, Q_{n-1}),$$

and $A(Q_1, \ldots, Q_{n-1})$ is an explicitly computable function of $Q_1, \ldots, Q_{n-1}$.

We are now ready to formulate the *iterative causal commutant lifting procedure*. ¿From the classical Commutant Lifting Theorem, we may choose Q_1 causal such that

$$\|W_1 - P_1 Q_1 R_1\| = \|\Lambda(W_1, R_1, P_1)\|.$$

Now given this Q_1, using Theorem 4, we choose a causal Q_2 such that

$$\|W_2 - P_1 Q_1 R_2 - P_2(Q_1 R_1 \otimes Q_1 R_1) - P_1 Q_2 (R_1 \otimes R_1)\| =$$

$$= \|\Lambda((W_2 - P_1 Q_1 R_2 - P_2(Q_1 R_1 \otimes Q_1 R_1)_c, \widehat{R_1 \otimes R_1} | \mathcal{E}_1^{(c)}, P_1)\|.$$

Inductively, given causal $Q_1, \ldots, Q_{n-1}$, we may choose Q_n causal such that

$$\|\hat{W}_n - P_1 Q_n(R_1^{\otimes n})\| = \|\Lambda((\hat{W}_n)_c, \widehat{R_1^{\otimes n}} | \mathcal{E}_1^{(c)}, P_1)\|. \tag{20}$$

Note that in each step of the procedure, the new "weight" $\hat{W}_n$ is determined by $W_n, P_1, R_1^{\otimes n}$, and the optimal causal parameters chosen. Thus, the iterative commutant lifting procedure is determined by the operator $\Lambda((\hat{W}_n)_c, \widehat{R_1^{\otimes n}} | \mathcal{E}_1^{(c)}, P_1)$, and so may be reduced to a classical dilation problem.

The following facts can be proven just as in [12] and [13] to which we refer the reader for the proofs. (See in particular [13], pages 849-853 .) First the causal iterative commutant lifting procedure converges. Next given any $Q \in \mathcal{C}_l$, we can apply the causal iterative commutant lifting procedure to $W - P \circ Q \circ R$. Now set

$$\mathcal{T}_C(q)(\rho) := \sum_{n=1} \rho^n \|\Lambda((\hat{W}_n)_c, \widehat{R_1^{\otimes n}}|\mathcal{E}_1^{(c)}, P_1)\|.$$

Then we have,

Proposition 1 *Given $Q \in \mathcal{C}_l$, there exists $\tilde{Q} \in \mathcal{C}_l$, such that $\mathcal{T}(\tilde{Q}) \equiv \mathcal{T}_C(Q)$. Moreover $\tilde{Q}$ may be derived from the causal iterated commutant lifting procedure.*

Moreover, as in [13] we have the following results:

Proposition 2 *Q is partially optimal if and only if $\mathcal{T}(Q) \cong \mathcal{T}_C(Q)$.*

Theorem 4 *For given P, R and W as above, any $Q \in \mathcal{C}_l$ is either partially optimal or can be ameliorated by a partially optimal compensating parameter.*

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