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TRIGGERING AXIAL INSTABILITIES IN SOLID ROCKETS: NUMERICAL PREDICTIONS

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TRIGGERING AXIAL INSTABILITIES IN SOLID ROCKETS; NUMERICAL PREDICTIONS

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Abstract

A theoretical model describing the nonlinear behavior of axial-mode intermediate frequency combustion instability in a solid rocket is developed. Numerical solutions have been obtained to the time-dependent conservation equations describing the one-dimensional two-phase flow in a solid propellant combustion chamber and choked nozzle system. The solution includes the unsteady flow in the nozzle, eliminating the need for an artificial boundary condition. Permissible flow field solutions include shock wave initial disturbances as well as shock waves resulting from the interaction of continuous disturbances with the combustion process. The equations governing the time-dependent pressure-coupled combustion process are solved simultaneously with the unsteady flow field at a series of burning stations along the propellant grain. Predictions of the engine response to various amplitude disturbances are discussed. Under certain conditions, the propellant burning rate exhibits sharp, finite-amplitude spikes which lead to large amplitude axial instabilities in the engine flow field.

Introduction

Nonlinear axial-mode combustion instability remains a serious problem in the development of solid propellant rocket motors. Although the use of metal loaded propellants which produce solid particles in the flow has reduced the occurrence of high frequency instabilities, it has not eliminated the axial-mode intermediate frequency (100-1000 Hz) problem. In certain instances, solid rocket engines have shown an increased susceptibility to intermediate frequency instabilities as the metal content of the propellant is increased. If such an instability reaches a large amplitude limit cycle, it may lead to an increase in mean chamber pressure and burning rate, excessive heat transfer rates, and a severe vibration level. The occurrence of any one of these may result in malfunction or destruction of the engine.

Longitudinal instabilities normally arise spontaneously, presumably triggered by the presence of small-amplitude unorganized disturbances in the combustion chamber, or by a momentary flow blockage due to a solid fragment exhausting through the nozzle. Attempts to isolate controlling parameters and establish stability trends by firing full-scale engines become prohibitively expensive. Outside of a few reported cases of instability in full-scale engines, most experimental data has been gathered from laboratory tests of small-scale motors. Extensive experimental programs have been conducted to determine the effect of chamber diameter, length, throat-to-port area ratio, initial propellant temperature and propellant composition on the instability in the motor. Much useful information has been obtained but the experimental data has resisted all efforts to find simple correlation laws applicable to all engines.

It was observed in References 7 and 8 that once a motor is operating in an unstable manner, its behavior is independent of the process which initiated the instability. Thus, except for a short transient, artificial triggering produced the same engine response found in the spontaneous cases. In the above experiments, the ultimate test for stability was the response of the engine flow field to a sharp "pulse" created by an explosive charge placed at the head end of the combustion chamber. Several charges were exploded at predetermined times during the motor firing while the pressure time-history at the head and aft ends was recorded. With this technique the results were a measure of the "nonlinear stability" of the motor; i.e., does a large amplitude disturbance decay and the flow field return to the steady state, or does it lead to an increasing mean chamber pressure and burning rate, and instability in the chamber? Using schlieren techniques with a windowed rocket motor, Brownlee and Kimbell verified that the traveling compression wave axial instabilities can exist at shock strength (M \approx 1.2). This fact along with the observed behavior of the motors subjected to pulsing indicate that the controlling mechanisms are definitely nonlinear.

The majority of theoretical investigations to date on solid rocket combustion instabilities have been limited to considerations of the behavior of small-amplitude disturbances. As a result, these analyses cannot predict the conditions under which a finite-amplitude disturbance can trigger instability in a linearly-stable engine, nor can they predict the characteristics of the final instabilities in linearly-stable and unstable engines. The solutions to these problems will require the development of appropriate nonlinear theories. Various aspects of the nonlinear problem have been treated in References 2, 11, and 16 through 20.

A theoretical description of the nonlinear behavior of axial instabilities will require a flow field model with certain minimum capabilities. The need for a solution of the full unsteady conservation equations for a two-phase flow is obvious. Traveling discontinuities are acceptable solutions to these nonlinear equations, and hence the method of solution must allow for shock waves, if they appear in the flow field. The boundary conditions imposed upon the flow field model are very important since the growth or decay of disturbances in the chamber is often decided by a slight imbalance.

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between the overall "gain" and "loss" processes of the system. In this respect, the choked throat of the nozzle provides the physical downstream boundary condition to disturbances in the engine. The theoretical model must account for this effect without imposing restrictions on the magnitude of the mean flow Mach number or the amplitude of the disturbance. Equally important is the solution to the solid propellant combustion problem which has boundary conditions dependent on the chamber flow properties. This solution must account for the unsteadiness of the burning process without making assumptions about the type of disturbance in the chamber.

It is the objective of this paper to develop a theoretical model for solid propellant rocket engines with a tubular grain which can predict the time-dependent behavior of the combustion chamber and nozzle flow field after the introduction of an arbitrary disturbance. The solution will include the unsteady flow in the choked nozzle, eliminating the need for an artificial nozzle boundary condition. The combustion of the solid propellant will be described by a time-dependent solution obtained simultaneously with the chamber flow field. No restrictions will be placed on the amplitude or type of disturbance in the engine. Shock waves resulting from "combustion driving" or an initial disturbance simulating the experimental pulsing technique, will be acceptable solutions to the developed theoretical model.

Analysis

The development of the mathematical model is discussed in two parts in this section; the determination of the chamber and nozzle flow field will be considered first, followed by a discussion of the unsteady propellant combustion process.

Chamber and Nozzle Flow Field

Solid propellant rocket motors exhibiting axial mode instability are usually characterized by a ratio of chamber length to radius much greater than unity. Hence, radial property variations are sufficiently small that a one-dimensional formulation provides an adequate description of the flow field. The system under consideration is a converging-diverging nozzle attached to a combustor containing a tubular solid propellant (Figure 1). The propellant is assumed to be the metal-loaded type (typically, aluminum) which leaves solid particles (e.g., \( \text{Al}_2\text{O}_3 \)) suspended in the gas flow when combustion is complete. Thus, the flow in the chamber and nozzle is two phase. The gas medium itself is assumed to be thermally perfect, inviscid, and non-heat-conducting. The solid particles entrained in the gas are assumed to be spherical, inert and monodisperse, and the drag force between gas and particles is determined by a coefficient proportional to the Reynolds number. Increases in chamber volume as a result of propellant surface regeneration are neglected. These volume changes are unimportant in the present instability problem but could easily be incorporated in the analysis to study ignition transient phenomena, etc.

It is recognized that the assumptions regarding the solid particle behavior result in neglecting some important details of the metal-loaded solid propellant combustion process. As Price\(^{(21)}\) has emphasized, solid particles are not produced directly in the flame zone. They are formed by complicated condensed phase reactions between the gas and agglomerates of molten metal which are forming on the propellant surface; these reactions may take place well outside of the flame zone while droplets of molten metal are entrained in the gas flow. However, the complexity of these processes and the inability to properly model them mathematically forced their omission from the present flow field model. The present analysis also assumes that the heat transfer coefficient between gas and solid particles is vanishingly small, i.e., the particles remain at the temperature achieved in the flame zone. This assumption could easily be removed with the addition of a particle energy equation, as was done in Reference \(20\) which analyzes a similar solid rocket stability problem. However, the influence on motor stability will be secondary unless the model allows the particles to burn while entrained in the gas flow, as suggested by experimental observations.

With the above assumptions, the conservation laws are applied to the stream tube control volume shown in Figure 1. The mass flow rates of gas and particles entering the control volume at the edge of the flame zone are determined by the propellant combustion model derived in the next section. It is assumed that these mass flows enter the perimeter boundary with negligible velocity, and are then accelerated to their local respective axial velocities within the control volume. In a one-dimensional formulation, the integral of the mass flow rate over the perimeter boundary becomes a mass source within the control volume, and the acceleration of this mass flow to the local axial velocity results in a momentum loss as well as an energy dissipation. The integral approach, however, obscures the exact details of this process.

The equations obtained by applying the conservation laws to the general control volume could be solved directly in their finite-element form. Finite-element methods\(^{(22)}\) using MacCormack's\(^{(23)}\) integration scheme have been used to compute steady flow fields which include shock waves. Although the shock wave is "smears" over several elements, no special computational procedures are required to...
treat the discontinuity. An alternative approach (24-26) is to integrate the conservation equations in their partial differential form with a finite-difference technique. Then the information contained in the associated characteristic equations can be used to track discrete discontinuities precisely. In the present combustion instability problem, traveling shock waves are anticipated as possible solutions; if present their instantaneous location is important in obtaining the proper combustion response. Since finite-element methods are unproven in unsteady flows with waves traveling in both directions, the discrete discontinuity method of Moretti (24) is adopted in this analysis.

Shrinking the control volume shown in Figure 1 to zero thickness results in the following hyperbolic system of partial differential equations for the flow field under consideration:

Continuity (gas phase)

\[ \frac{\partial p}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = R_1 = \frac{2\dot{h}}{R} - \rho \frac{u \, d\ln A}{dx} \]  

Continuity (solid particles)

\[ \frac{\partial p}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = R_2 = \frac{2\dot{h}}{R} - \rho \frac{u \, d\ln A}{dx} \]  

Momentum (gas phase)

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} + \frac{1}{\gamma} \frac{\partial p}{\partial x} = R_3 = \left\{ K_p (u - u_p) + \frac{2\dot{h} \, u_p}{R} \right\} / \rho \]  

Momentum (solid particles)

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = R_2 = K(u - u_p) + \frac{2\dot{h}}{R} \frac{u_p}{\rho} \]  

Energy (gas phase)

\[ \frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = R_4 = \frac{y(y-1)}{\rho} \left\{ \dot{h} \left( \frac{c - h}{y - 1} \right) + \frac{1}{2} \left( \dot{a}_g \, u_g^2 + \dot{a}_p \, u_p^2 \right) + K_p (u - u_p)^2 \right\} \]  

where \( h = p/\rho \)

\( S = \ln p - y \ln \rho \)

These equations are employed in a computational coordinate system which contains two fixed boundaries (the head end of the combustor, and the nozzle supersonic exit plane) and any number of internal moving boundaries as shown in Figure 2. A shock wave discontinuity is treated as a moving boundary, and the flow field in a region between any two boundaries is assumed continuous. In the general case, the two boundaries of a flow field region may move independently of each other. To avoid having to compute the movement of a boundary over a system of fixed grid points and to simplify the calculation procedure, the independent variables are "normalized" for each region. Let the instantaneous location of the right hand boundary be given by \( c(t) \), and the left hand boundary by \( b(t) \). Then for a typical region of continuous flow, define

\[ X = \frac{f(x) - b(t)}{c(t) - b(t)} \]  

\[ T = t \]  

where the stretched coordinate, \( \xi \), allows equally spaced grid points to represent an uneven distribution in the physical coordinate \( x \); e.g., closely spaced points in the throat region. Using Equation (6), Equations (1) through (5) transform to,

\[ \frac{\partial p}{\partial T} + U \frac{\partial p}{\partial X} + \eta \frac{\partial u}{\partial X} = R_1 \]  

\[ \frac{\partial u}{\partial T} + U \frac{\partial u}{\partial X} + \eta \frac{\partial p}{\partial X} = R_2 \]  

\[ \frac{\partial u}{\partial T} + U \frac{\partial u}{\partial X} + \frac{\partial p}{\partial X} = R_3 \]  

\[ \frac{\partial u}{\partial T} + U \frac{\partial u}{\partial X} = R_4 \]  

\[ \frac{\partial u}{\partial T} + U \frac{\partial u}{\partial X} = R_5 \]

where

\[ U = \eta u + \beta (c - b) \]  

\[ \eta = \frac{\partial \xi / \partial x}{c - b} \]  

\[ \beta = (X - 1) \frac{\partial b}{\partial t} - X \frac{\partial c}{\partial t} \]
B. Equation (18) then determines \( p \) at point B. In practice, an iteration process is required to locate the point C by using the average speed of sound along the left-running characteristic BC.

The second term on the right hand side follows directly from Equation (7), and the third term can be computed from an additional time differentiation of Equation (7). Evaluating all spatial derivatives with centered differencing, the right hand side of Equation (13) can be expressed entirely in terms of known quantities at time \( T \). Given the increment \( \Delta t \), Equation (13) and analogous equations for the remaining four dependent variables are used to calculate the flow field variables at the next time step. Forward marching stability is assured by the C.F.L. (27) condition which requires that the domain of dependence of each point in the numerical flow field must include its physical domain of dependence. The present analysis uses this stability criterion in the form (in the \( t-x \) plane)

\[
\Delta t = 0.7 \left( \frac{\Delta x}{|u| + a} \right)_{\text{minimum over the}} \text{flow field} \tag{14}
\]

The known boundary conditions for the present problem are \( u = u_p = 0 \) at the head end of the combustion chamber and a specified flame temperature of the combustion products which enter the control volume from the flame zone. However, the numerical computation requires values of all the dependent variables on all boundaries at each time step. These values must be determined (25), in a manner consistent with the internal flow field computation. The method-of-characteristics (MOC) can accomplish this without the arbitrariness of backward differencing (see the recent work of Abbott (28) for added proof). The MOC procedure can be illustrated with the following example of a flow field containing one right-running shock wave (see Figure 3).

Equations (7) through (11) can be rewritten (29) in the following characteristic form:

\[
\left( \frac{a}{w_p} \right) d\rho = \left\{ \pm R_3 + a \left( \frac{R_1}{\rho} + \frac{R_2}{\rho} \right) \right\} dT \tag{15}
\]

along \( dx \) \( = \pm a \) (physical plane)

\[
dS = R_4 dT \tag{16}
\]

along \( dx \) \( = u \) (physical plane)

\[
\left\{ \begin{align*}
\frac{du_p}{dT} &= R_5 dT \\
\frac{dp}{dT} &= (R_2 - \eta_p) \frac{\partial u_p}{\partial x} dT
\end{align*} \right. \tag{17}
\]

along \( dx \) \( = u_p \) (physical plane)

An explanation of the unorthodox form of Equation (15) can be found in Reference 29.

Boundary values at the head end of the combustion chamber \( (X = 0 \) of region 1) are obtained at the next time step with the procedure illustrated in Figure 3b. Because of the degenerate nature of the streamline \( AB \) (i.e., \( u = u_p = 0 \)), Equation (18) is uncoupled from Equations (15) and (16). Then the solution of Equation (15) written along the left-running characteristic which reaches the new wall point \( B \) from an interior point \( C \), and Equation (16) written along the gas path streamline \( AB \) determines the values of \( p \) and \( S \) (and hence \( p \) and \( a \)) at point

Figure 3. Example Problem With Right-Running Shock Wave.
values, based only on information ahead of the shock at time \( t = 0 \), are assigned to the boundary \( X = 0 \) in region 2. The solid particle flow undergoes a relaxation process in the region behind the shock wave but is unaffected by the discontinuity. Hence, \( u_0 \) and \( p_0 \) at \( A' \) are equal to their upstream values at \( B' \). Four variables remain to be determined: \( p_0, u_0 \) downstream of the shock wave \( (A') \) and the new shock wave velocity. Three equations are provided by the Riemann-Hugoniot "jump conditions" across the discontinuity. The final equation follows from the compatibility relation (Equation 15) written along the right-running characteristic which reaches the shock from the high pressure side. In the present analysis, the simultaneous solution to these four equations was obtained with a minimization scheme due to Davidson[30]. Thus, in tracking shock waves, no continuous differential equation is integrated across the discontinuity.

The final boundary in the example of Figure 3 is the supersonic exit plane of the nozzle; that is, \( X = 1 \) in region 2. In a single gas phase flow, a common procedure to obtain boundary values is to linearly extrapolate all variables from their values at the two adjacent upstream grid points, assuming they are in supersonic flow. Errors introduced by this extrapolation never influence the upstream calculations since the flow velocity exceeds the local speed of propagation of a disturbance. This method yields acceptable results in the present analysis when the particle flow is deleted. With the particle flow included, however, the small errors in \( u_0 \) and \( p_0 \) at the boundary do propagate upstream and eventually destroy the solution. Attempts to explain this on the basis of an effective speed of propagation have not been successful. The problem has been avoided, however, with a full method-of-characteristics solution at the exit plane for all five flow field variables.

It is anticipated that, under certain conditions, continuous disturbances in the engine could steepen and eventually form shock waves. Hence, use of the above computational method is contingent upon recognizing the formation of an embedded shock wave in a region of continuous flow. Any attempt to compute an infinite pressure gradient to a "large" number computed from finite-difference values is erroneous at best. Hence, a more sensitive method is required to predict the onset of the discontinuity. Figure 4 illustrates a typical situation in the numerical flow field when a compression wave is steepening. Following Hornik's work[52], a third-order polynomial fitted to the values at four discrete grid points is assumed to represent the pressure distribution in this interval. If this polynomial predicts an inflection point (i.e., an infinite gradient), its location is taken as the origin of the discontinuity. Thus, given the polynomial

\[
x = x(p) = ap^3 + bp^2 + cp + d
\]

where \( a, b, c, \) and \( d \) are constants, the point at which \( dx/dp = 0 \) (i.e., \( dp/dx \to \infty \)) is specified by

\[
dp\frac{dx}{dp} = 0 = 3ap^2 + 2bp + c
\]

Real roots of Equation (20) are possible only for non-negative values of the discriminant,

\[
D = b^2 - 3ac
\]

In the present problem, \( D \) is negative for any combination of four grid point values except when compression waves steepen. Thus, at each time step, a computer search routine checks for a non-negative value of \( D \) in all continuous regions; if one is found, an interior boundary is inserted at this location and the computation proceeds tracking a Mach wave as a discontinuity. Further details of the computational method can be found in Reference 29.

**Figure 4. Pressure Waveform During Steepening of a Compression Wave in the Numerical Flow Field.**

**Solid Propellant Combustion Model**

The combustion of metal-loaded solid propellants is an extremely complex process which may be sensitive to both pressure and velocity fluctuations in the chamber flow field. Since there is no available theoretical model which properly accounts for the velocity coupling phenomena, this analysis will concentrate on the nonlinear time-dependent behavior of a typical pressure-coupled model. The tubular solid propellant in Figure 1 is assumed to be represented by a series of fixed "burning stations" along the length of the grain (Figure 5). The burning rate at each station is determined from the solution of a one-dimensional combustion model, whose boundary conditions are dependent only on the chamber flow properties directly "below," i.e., at the same axial location. Thus, no prior knowledge of the type of disturbance in the chamber is required.
The physical one-dimensional model used at each burning station is shown in Figure 5. With the standard assumptions of a homogeneous solid material, simple pyrolysis from solid to gas, no condensed phase reactions, a quasi-steady gas-phase flame with a one-step reaction (oxidizer + fuel \rightarrow products), the analysis separates into three regions \( \text{(31)} \).

The energy equations for the resultant thermal theory can be written as,

**Solid Phase** \( (-\infty \leq y^* \leq 0) \)

\[
\rho_s c_s \frac{\partial T^*}{\partial t^*} + \rho_s r_s \frac{\partial T^*}{\partial y^*} - k_s \frac{\partial^2 T^*}{\partial y^*^2} = 0
\]

where \( r^* = \frac{E^*}{R_T} \frac{T^*}{C_T} \). \( (22) \)

**Interface** \( (y^* = 0) \)

\[
k_s \frac{\partial T^*}{\partial y^*} = -\dot{m}_s \frac{\partial T^*}{\partial y^*} + k_s \frac{\partial T^*}{\partial y^*} \]

where \( \dot{m}_s \) is positive for an endothermic heat release in the surface reaction \( (23) \)

**Flame Zone** \( (0 \leq y^* \leq +\infty) \)

\[
\frac{\partial T^*}{\partial t^*} + \frac{\partial T^*}{\partial y^*} - \frac{\partial^2 T^*}{\partial y^*^2} = -Q_s \omega^* \]

where \( \omega^* \) is the gas reaction rate \( \sim \frac{E^*}{R_T} \frac{T^*}{C_T} \). \( (24) \)

This system reduces \( (29) \) to an initial value - boundary value problem for the temperature distribution in the unburned solid; i.e.,

\[
\frac{\partial T}{\partial t} + r \frac{\partial T}{\partial y} - \frac{\partial^2 T}{\partial y^2} = 0 \quad (-\infty \leq y \leq 0) \]

**IC:** \( T(y,0) = T_0(y) \) \( (a) \)

**BC:**

\[
\frac{\partial T}{\partial y}(0,t) = -Z_s(\omega_s^*)r + Z_f(\omega_f^*)r
\]

reaction reaction at in surface flame \( (c) \)

The numerical solution to this system is obtained with a method of invariant imbedding outlined by Meyer \( (32) \). The instantaneous value of the surface temperature predicted by this solution determines the surface regression rate \( r \),

\[
r = \left( \frac{E^*/R_T}{C_T} \right) \frac{\dot{m}_s}{\dot{m}_f} \]

which specifies the mass flow rate from the surface according to

\[
\dot{m}_f = \dot{m}_s \frac{r}{\rho_s} \frac{T^*}{C_T} \frac{E^*}{R_T}
\]

Since this theory assumes a homogeneous solid phase, it is unable to distinguish between mass flow rates of gas and particles. The present analysis assumes that the weight percent loading of solids in the unburned propellant can be used directly to determine \( \dot{m}_s \) and \( \dot{m}_f \) which enter the control volume perimeter.

The temperature gradient boundary condition, Equation \( (25c) \), depends on the heat released in the surface reaction as well as the heat released in the flame reaction. If the net heat released in the surface reaction is exothermic \( (33) \), the first term in Equation \( (25c) \) becomes positive and the potential then exists for very large burning rates under certain dynamic conditions \( (34) \). This phenomenon is discussed in the next section.

**Results**

The capabilities of the chamber and nozzle flow field analysis are demonstrated in the following example (Case 1, Appendix A) where particle flow is neglected. In place of the time-dependent combustion analysis, the propellant burning rate \( r \) at each station is assumed to follow the familiar steady state law,

\[
r = a_p n \]

where \( n = 0.3 \) for this example. Starting with the assumption that the propellant was burning in equilibrium at ambient pressure, an "ignition transient" was computed until steady state was achieved (see Figure 6). This calculation does not model all the processes involved in solid rocket ignition, but it demonstrates that the time integration in the computer program yields a stable asymptotic limit. The flow field at \( t = \infty \) was altered with a shock wave disturbance (40% pressure increase) near the head end of the combustion chamber and the computation was restarted. Figure 7 illustrates the evolution of the spatial pressure distribution in the engine for increasing time. The distribution
Two-Parameter Response Function

A = 7.54
B = 0.59
n = 0.79

The symbols, representing the value of the response function computed from the time-dependent mass flow rate, are compared to the real part of the well-known two-parameter response function from linear theory. A rocket engine using this propellant system was designed so that its fundamental mode operates at the non-dimensional frequency \( \bar{\omega} = 10 \). To investigate the sensitivity of the time-dependent combustion process under actual motor conditions, the computed steady state flow field was altered with two

Figure 7. Pressure Distributions in Rocket Engine as the Result of Altering the Steady State Flow Field (Case 1) with a Shock Wave Disturbance (40% pressure increase).

Figure 8. Pressure Time-History at Head End of Combustion Chamber as the Result of Altering the Steady State Flow Field (Case 1) with a Shock Wave Disturbance (40% pressure increase).

Figure 9. Combustion Response as a Function of Frequency for the Propellant System in Case 2 (\( q^*_p = -122 \text{ cal/gm} \)).

Similar calculations have been made for engine flow fields containing solid particles (typically, 10\( \mu \)m diameter, 10% weight loading). The results are not shown here for the sake of brevity, but they can be summarized with two observations. First, the decay rate of a disturbance at the fundamental longitudinal frequency of the engine is virtually unaffected by the presence of the particles. And second, disturbances at the higher harmonics will be substantially damped if the average particle diameter is "tuned" to these frequencies. These results would be anticipated for inert, spherical particles which contribute drag and dissipate energy. However, this model of the particle flow is incapable of explaining the increased susceptibility of metal-loaded propellants to axial instabilities. For these reasons, the examples to follow have ignored the presence of solid particles in the flow field.

The nonlinear time-dependent combustion model derived earlier was used to predict values of the response function, \( (m^2/\rho)/(p/\rho) \), for a propellant system representative of AP propellants (Case 2, Appendix A).
different amplitude continuous disturbances as shown in Figure 10. The engine response to the 20% amplitude disturbance (see Figure 11) is displayed as the pressure time-history at the head end of the chamber along with the instantaneous propellant surface mass flow rate at the same location. As might be anticipated, the mass flow response is too far out-of-phase (lag) with the pressure disturbance to overcome the convective losses through the nozzle. The effect of initial disturbance amplitude is evident in Figure 12 which shows that a 40% amplitude disturbance grows to more than twice its initial magnitude in less than four cycles. However, the propellant mass flow rate now includes sharp, finite amplitude "spikes" which are responsible for similar sharp rises in pressure. This type of burning rate behavior was not anticipated as the numerical analysis was developed; hence, a spike leads to an unavoidable truncation error associated with the time integration scheme in continuous flow field regions. It is believed that the numerical computation predicts the correct trend for engine response in these cases, but the accuracy of numerical values is undetermined. No attempt was made to compute a limit cycle oscillation under these conditions.

The nonlinear behavior of the combustion model alone was investigated for several propellant systems and various frequencies. The response leading to the spike formation in the following example is based on the same propellant system as in Case 2 above, but with the exothermic heat release in the surface reaction adjusted to 120 cal (Case 3, Appendix A). The frequency of the disturbance is chosen to be \( \Omega = 5.5 \) corresponding to the maximum response (see Figure 13) when the mass flow rate is in-phase with a small amplitude pressure oscillation at this frequency. Figure 14a shows the sinusoidal mass flow rate due to a
sineoidal pressure oscillation in the flame with an amplitude of 1% of the assumed chamber pressure. Increasing this amplitude to 5% and repeating the calculation results in the nonlinear wave form shown in Figure 14b. A 7.5% amplitude oscillation results in intermittent spikes in the mass flow rate response as indicated in Figure 14c. Finally, a 10% amplitude sine wave in pressure results in the repeating pattern of spikes shown in Figure 14d. In the intermittent pattern (Figure 14c), the spikes lag the pressure maximums, but in the repeating pattern (Figure 14d) the spikes are exactly in-phase with the maximums. The latter two figures imply the existence of a threshold amplitude criterion, which must be exceeded before a spike will occur on every cycle of the pressure oscillation. The spike can be seen in greater detail in Figure 15 where the mass flow rate shown in Figure 14d between t = 3.20 and 3.30 is replotted. The temperature distribution in the unburned propellant is illustrated in Figure 16 for three instants of time during the occurrence of the spike (t = 3.0, 3.25, 3.30 as indicated on Figure 15). At the peak of the spike, the temperature profile sustains a high surface temperature and severe surface temperature gradient, but these effects influence only the region near the propellant surface.

![Figure 15. Expanded View of "Spike" in Propellant Mass Flow Rate Occurring Between t = 3.20 and t = 3.30 in Figure 14d.](image)

The final example provides a description of how this propellant system would behave in a rocket engine (Case 3, Appendix A) whose fundamental longitudinal frequency corresponds to Ω = 5.5 on the response curve of Figure 13. A continuous disturbance with a maximum amplitude of 10% of the chamber pressure (similar to Figure 10) was added to the steady state flow field and the computation was restarted. The pressure time-history and propellant mass flow rate at the head end of the chamber are shown in Figure 17. The trend toward a large amplitude instability is quite evident.

**Conclusions**

A mathematical model of a solid propellant rocket engine with a tubular propellant grain has been developed to predict the time-dependent response of the engine flow field to arbitrary
1) The unsteady flow in the choked nozzle is important in determining how a disturbance is reflected in the throat region. Specifically, the Mach number in the geometric throat cannot be assumed to remain at unity.

2) The ability of typical diameter, inert, mono-disperse solid particles to damp low frequency longitudinal disturbances is minimal.

3) For all cases examined, the steady state propellant burning rate law \( r = ap^n \) \( (n < 1) \), is incapable of sustaining longitudinal instabilities.

4) A pressure-coupled solid propellant combustion model based on commonly employed assumptions predicts large amplitude burning rate spikes as the result of small amplitude pressure oscillations, under certain conditions. The quasi-steady flame zone assumption must be reexamined in light of these results.

5) With this combustion model, the numerical solution for engine response predicts the possibility that small amplitude disturbances may grow into large amplitude axial instabilities.

Appendix A

Engine and Propellant Parameters Used in Computations

**Case 1**

- Chamber radius \( = 1 \text{ ft} \)
- Engine length \( = 10 \text{ ft} \)
- Grain length \( = 9 \text{ ft} \)
- Nozzle area ratio \( = 4.0 \)
- Chamber temperature \( = 4500^\circ \text{R} \)
- Chamber pressure \( \sim 1950 \text{ psia} \)
- Propellant mass flow rate \( = .1 \text{ lb/hr} \)
- Isentropic index \( = 1.20 \)

**Case 2**

- Chamber radius \( = 1 \text{ ft} \)
- Engine length \( = 10 \text{ ft} \)
- Grain length \( = 8 \text{ ft} \)
- Nozzle area ratio \( = 11.7 \)
- Chamber temperature \( = 4500^\circ \text{R} \)
- Chamber pressure \( \sim 400 \text{ psia} \)
- Mean molecular weight \( = 22 \text{ lb mole}^{-1} \)
- Isentropic index \( = 1.20 \)

\[
\begin{align*}
Q_S &= -122 \text{ cal gm}^{-1} \\
\sigma_s &= 0.275 \text{ cal gm}^{-2} \text{K} \\
k_S &= 1.2 \times 10^{-3} \text{ cal cm}^{-1} \text{ sec}^{-1} \text{ K} \\
c_{p} &= 1.95 \text{ gm cm}^{-3} \\
E_S &= 20 \text{ kcal/mole} \\
Q_T &= 523 \text{ cal gm}^{-1} \\
\sigma_p &= 0.30 \text{ cal gm}^{-2} \text{K} \\
k_T &= 2.0 \times 10^{-4} \text{ cal cm}^{-1} \text{ sec}^{-1} \text{ K} \\
E_T &= 30 \text{ kcal/mole}
\end{align*}
\]
Case 3 (same as Case 2, except for the following)

Grain length = 8.5 ft
Nozzle area ratio = 12.1
Chamber pressure ~ 600 psia

\[ Q_s^* = -120 \text{ cal/gm} \]
\[ Q_f^* = 525 \text{ cal/gm} \]
\[ r^* = 0.685 \text{ cm/sec} @ p^* = 600 \text{ psia}, \bar{t}_{\text{sur}} = 880^\circ \text{K} \]

Nomenclature

- \( A \): cross sectional area
- \( A_{\text{ch}}^* \): reference area
- \( a \): \( (a^* / a^*, \sqrt{p/p}) \text{ speed of sound} \)
- \( B_r \): \( (b_r^* / b_r^*) \text{, frequency factor in Arrhenius surface reaction} \)
- \( b(t) \): \( (\xi - \text{space}) \text{ location of left hand boundary of flow field region} \)
- \( c(t) \): \( (\xi - \text{space}) \text{ location of right hand boundary of flow field region} \)
- \( c_p^* \): specific heat of gaseous combustion products
- \( c_s^* \): specific heat of solid propellant
- \( D \): discriminant defined in Equation (21)
- \( E_s^* \): activation energy of Arrhenius surface reaction
- \( E_f^* \): activation energy of Arrhenius flame reaction
- \( h \): \( (h^*/c_{\text{ch}}^* T_{\text{ch}}^*) \text{ enthalpy of gas} \)
- \( h_{\text{c}} \): enthalpy of combustion products at flame temperature
- \( K \): \( \left( \frac{Q^*}{2 p_s^* / s^*} \right) \text{ constant which follows from} \)
  \( \text{Stokes Flow Drag Law} \)
- \( k_{\text{g}} \): thermal conductivity of combustion products
- \( k_s^* \): thermal conductivity of solid propellant
- \( L^* \): axial reference length
- \( \dot{\theta} \): \( (\dot{\theta}^* / \dot{\theta}^* r_m^*) \text{, propellant surface mass flow rate} \)
- \( \dot{m}_g \): \( (\dot{m}^* / \dot{m}^* r_m^*) \text{, mass flow rate of gas} \)
- \( \dot{m}_f \): \( (\dot{m}_f^* / \dot{m}_f^* r_f^*) \text{, mass flow rate of particles} \)
- \( n \): reaction index
\( p_p \) (\( \frac{\rho_p}{\rho_s} \)), density of solid particles

\( \rho_s^* \) solid propellant density

\( a^* \) average particle diameter

\( \Omega (\omega/\Omega^*) \) frequency

\( \omega (\omega^* L^*/a^* \), reaction rate in gas phase flame

Superscripts

* dimensional quantity

\( \dot{\cdot} \) time-varying fluctuation

Subscripts

g gas phase

p solid particles

r reference state

s solid propellant material

References:


A theoretical model describing the nonlinear behavior of axial-mode intermediate frequency combustion instability in a solid rocket is developed. Numerical solutions have been obtained to the time-dependent conservation equations describing the one-dimensional two-phase flow in a solid propellant combustion chamber and choked nozzle system. The solution includes the unsteady flow in the nozzle, eliminating the need for an artificial boundary condition. Permissible flow field solutions include shock wave initial disturbances as well as shock waves resulting from the interaction of continuous disturbances with the combustion process. The equations governing the time-dependent pressure-coupled combustion process are solved simultaneously with the unsteady chamber flow at a series of burning stations along the propellant grain. Predictions of the engine response to various amplitude disturbances are discussed. Under certain conditions, the propellant burning rate exhibits sharp, finite-amplitude spikes which lead to large amplitude axial instabilities in the engine flow field.