Title: STUDY OF THE EFFECTS OF ROTATING FRAME TURBULENCE (RFT) ON HELICOPTER...

PROJECT ADMINISTRATION DATA

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Security class (U,C,S,TS) : ONR resident rep. is ACO (Y/N): N
Defense priority rating  N/A supplemental sheet
Equipment title vests with: Sponsor GIT
PURCHASE OF EQUIPMENT OR FACILITIES IS NOT AN ALLOWABLE COST.

Administrative comments -
AN INTERCHANGE FOR JOINT RESEARCH UNDER THE NASA-AMES UNIVERSITY CONSORTIUM - INCLUDES A SUBCONTRACT TO FLORIDA ATLANTIC UNIVERSITY.
NOTICE OF PROJECT CLOSEOUT

Closeout Notice Date 11/26/90

Project No. E-16-606 Center No. R6421-0A0
Project Director SCHRAGE D P School/Lab AERO ENGR
Sponsor NASA/AMES RESEARCH CTR, CA
Contract/Grant No. NCA2-266 Contract Entity GTRC
Prime Contract No. 

Title STUDY OF THE EFFECTS OF ROTATING FRAME TURBULENCE (RFT) ON HELICOPTER...

Effective Completion Date 900331 (Performance) 900331 (Reports)

Closeout Actions Required: Date Y/N Submitted

Final Invoice or Copy of Final Invoice Y 900723
Final Report of Inventions and/or Subcontracts Y 
Government Property Inventory & Related Certificate Y 900630
Classified Material Certificate N 
Release and Assignment N 
Other N

Comments

Subproject Under Main Project No. 

Continues Project No. 

Distribution Required:

- Project Director Y
- Administrative Network Representative Y
- GTRI Accounting/Grants and Contracts Y
- Procurement/Supply Services Y
- Research Property Management Y
- Research Security Services Y
- Reports Coordinator (OCA) Y
- GTRC Y
- Project File Y
- Other ____________________________ 

NOTE: Final Patent Questionnaire sent to PDPI.
GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION

NOTICE OF PROJECT CLOSEOUT (SUBPROJECTS)

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Closeout Notice Date 11/26/90

Project No. E-16-606

Center No. R6421-0A0

Project Director SCHRAGE D P

School/Lab AERO ENGR

Principal NASA/AMES RESEARCH CTR, CA

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Project # E-25-M48

Contract # NCA2-266

# R6421-0A1

PD TONGUE B H

Unit 02.010.126 T

MOD# ADMIN MECH ENGR *

Main proj # E-16-606

OCA CO IRL

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/AMES RESEARCH CTR, CA

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Start 880104 End 880930 Funded 9,989.00 Contract 9,989.00

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END

* indicates the project is a subproject.
I indicates the project is active and being updated.
A indicates the project is currently active.
T indicates the project has been terminated.
R indicates a terminated project that is being modified.
A STUDY OF THE EFFECTS OF ROTATING FRAME TURBULENCE (RFT) ON HELICOPTER FLIGHT MECHANICS

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ABSTRACT

The turbulence actually experienced by a helicopter blade-element significantly differs from the space-fixed free atmospheric turbulence. The turbulence in the rotor disk requires a rotationally sampled description in a rotating frame of reference. It is referred to as the rotating frame turbulence or RFT, which exhibits a striking phenomenon. The RFT spectral density versus frequency shows high peak values at 1P, 2P, 3P, etc., frequencies. The energy increase at these peaks is balanced by an energy decrease primarily at the lower-than-1P frequency range. Particularly for low altitude flight regimes of pure helicopters, such as the nap-of-the-earth maneuvers, the conventional space-fixed description of turbulence is not a good approximation, since the turbulence scale length can have values comparable to the rotor radius. Accordingly the flight mechanics characteristics with RFT description are compared with those based on the conventional space-fixed turbulence description. The results demonstrate that the RFT qualitatively and quantitatively affects the prediction of helicopter flight mechanics characteristics in turbulence. Such comparisons should play an important role in the new development of handling qualities specifications for helicopters.

INTRODUCTION

The modern helicopter is no longer a vehicle used for simple missions where only its hovering capability is required. The helicopters are ever-increasingly faced with complex missions which push the aircraft to its design limits. One emerging requirement is stabilized flight through moderate, or even severe atmospheric turbulence to accomplish high workload mission tasks.

Rotorcraft, both civilian and military, now compete on a commercial basis with many other forms of transportation and have often shown greater reliability and productivity in mission performance. For example, as stated in [1], nearly all of the lighthouses and lightships around the British coastline are now relieved by helicopters. This is because they have demonstrated higher mission reliability, that is, completing the specified mission on time, than the former relief boats, which were often up to seven days late. Helicopters are also based on off-shore oil platforms in the North Sea to provide daily routine inter-rig support and to provide rescue services. These helicopters commonly operate to and from landing pads with restricted access in severely turbulent atmospheric winds as high as 55 knots. Military rotorcraft of the U.S. Navy and Marine Corps operating from ships frequently encounter harsh turbulent winds. These rotorcraft are required to perform missions such as hovering over and landing on moving ships, in moderate or severe turbulence, at speeds up to 50 knots while stormy seas induce ship motion up to 15 degrees yielding oscillations on the landing deck in excess of 20 feet [2]. Given these flight conditions which both military and civilian rotorcraft encounter routinely, it is essential to accurately predict rotorcraft performance in a turbulent atmosphere. This capability would allow both passive and active methods of control to be considered early in the preliminary design process. Thus, rotorcraft designers will be able to address the influence of atmospheric turbulence adequately.

We now address the lack of an adequate low-altitude turbulence model to assess helicopter flight mechanics characteristics, loads and vibrations. In fact this lack, particularly for the nap-of-the-earth maneuvers, has been identified as a critical gap in the recent NASA/Army study to develop new han-
TURBULENCE MODELS

For helicopter applications, the vertical turbulence velocity \( g(t) \) is the most dominant component. Therefore, in the present treatment, the fore-to-aft and side-to-side turbulence velocity components in the rotor plane are neglected.

In the stochastic treatment, a stationary vertical turbulence velocity \( g(t) \) is described by (auto) spectral density function \( S_p(\omega) \) or autocorrelation function \( R_p(\tau) \). There are many forms of atmospheric turbulence models quoted in the literature. The two widely used models are the von Karman and the Dryden models. The following equations describe the power spectral density function for the vertical turbulence velocity component according to von Karman and Dryden:

**von Karman:**

\[
S_p(\omega) = \sigma_g^2 \frac{L}{2\pi} \frac{1 + \frac{6}{5}(1.34 L \omega)^2}{[1 + (1.34 L \omega)^2]^4}
\]  

**Dryden:**

\[
S_p(\omega) = \sigma_g^2 \frac{L}{2\pi} \frac{1 + 3(\omega L)^2}{[1 + (\omega L)^2]^2}
\]

A noteworthy feature of wind turbine studies is the concomitant corroboration of predictions by test data turbulence excitations and turbulence induced vibrations and loads. This wind turbine experience shows that atmospheric turbulence can contribute decisively to the life-time load spectrum and that the RFT effects cannot be neglected. Concerning low speed conventional or pure helicopters, the past studies of turbulence effects on flight mechanics have all neglected the RFT effects. It has been assumed that the entire disk experiences a spatially uniform turbulence velocity field identical to that felt at the rotor hub center. Outside the earth's boundary layer where the turbulence length scale is 600 feet or more as compared to a rotor diameter which is 80 feet or less, this assumption seemed to be reasonable. With this assumption, random loads and vibration were found to be relatively insignificant as compared to the deterministic dynamic loads and vibrations from steady and maneuvering flights. Within the earth's boundary layer and depending on the ground texture, the turbulence length scale has values that are comparable to the rotor diameter so that the assumption of space fixed turbulence is not a good approximation. Therefore, the treatment of turbulence effects for low-altitude flight regimes such as the nap-of-the-earth maneuvers require inclusion of RFT effects. Accordingly this study addresses the RFT effects on low frequency helicopter response with particular emphasis on handling qualities. Such a study should serve as a valuable reference point for the future development of handling qualities specifications for helicopters in turbulence.
In equations 1 and 2, $\omega$ is the spacewise circular frequency given by $\omega = 2 \pi k$, where $k$ is the wavenumber per unit length, and $L$ is the scale length of the fore-to-aft or longitudinal turbulence component. This turbulence scale length $L$ is given by

$$L = \frac{2}{\sigma^2} \int_0^\infty R_q(x) \, dx$$  \hspace{1cm} (3)

It is mentioned in passing that the scale length of the vertical turbulence velocity is equal to $L/2$.

The von Karman power spectrum (equation 1) is generally preferred but the analysis simplifies considerably when the Ornstein-Unlenbeck model is used without appreciable sacrifice in accuracy of results. In this work, the Ornstein-Unlenbeck model is used according to which

$$S_q(\omega) = \frac{\sigma^2}{2\pi} \frac{2L}{\left[4 + (L\omega)^2\right]}$$  \hspace{1cm} (4)

In the development of the RFT model, we define the following rotorcraft parameters.

$$\mu = \frac{V \cos(\alpha)}{\Omega R}$$  \hspace{1cm} (5)

$$\lambda = \frac{V \sin(\alpha) + v}{\Omega R} = \mu \tan(\alpha) + \lambda_i$$  \hspace{1cm} (6)

$$\lambda_i = \frac{C_T}{2 \sqrt{\mu^2 + \lambda^2}}$$  \hspace{1cm} (7)

In equations 5, 6, and 7 $\mu$ is the advance ratio, $\lambda$ is the total inflow, and $\lambda_i$ is the induced inflow (see Figure 1). For hover we have, $\alpha \approx 0.0$, $V = 0.0$, $\lambda = \lambda_i$, and $T \approx W$. Thus, we can simply compute the thrust coefficient, downwash velocity, and the induced inflow using the following equations.

$$C_T = \frac{W}{\rho A V^2}$$  \hspace{1cm} (8)

$$v = \Omega R \sqrt{\frac{C_T}{2}}$$  \hspace{1cm} (9)

$$\lambda_i = \frac{C_T}{2 \lambda}$$  \hspace{1cm} (10)

Then, as done in reference [6], the out of plane velocity through the rotor, $w$, is approximated as

$$w = K \lambda_i \Omega R$$  \hspace{1cm} (11)

In hover, we set $K = 1.0$. Thus, in the present exploratory study, we neglect the role of mean turbulence velocity and axial flight velocity. Then, in non-dimensional form, $w$ is given by $\tilde{w}$.

$$\tilde{w} = \lambda_i$$  \hspace{1cm} (12)

Using Taylor’s hypothesis, i.e. the frozen field concept, in conjunction with the Ornstein-Unlenbeck power spectrum, we can write the autocorrelation function as a function of the spatial separation between two time intervals. Consider the rotor disk shown in Figure 1. At the 0.7R blade station, the blade section encounters the following velocity components:

$$\frac{dx}{dt} = V + 0.7 \Omega R \sin \Omega t$$  \hspace{1cm} (13)

$$\frac{dy}{dt} = 0.7 \Omega R \cos \Omega t$$  \hspace{1cm} (14)

$$\frac{dz}{dt} = w$$  \hspace{1cm} (15)
Integration of equations 13, 14, and 15 from $t_1$ to $t_2$ results in the following equations.

\[
x(t_2) - x(t_1) = V (t_2 - t_1) - 0.7R (\cos \Omega t_2 - \cos \Omega t_1) \\
y(t_2) - y(t_1) = 0.7R (\sin \Omega t_2 - \sin \Omega t_1) \\
z(t_2) - z(t_1) = w (t_2 - t_1)
\]  

(16) \hspace{1cm} (17) \hspace{1cm} (18)

In equation 18, $w$ represents the total mean airflow perpendicular to the rotor disk in the $z$-direction due to axial flight velocity, mean vertical gust velocity and downwash velocity. From the frozen field concept, the turbulence autocorrelation function is simply a function of the spatial separation for the lapsed time $(t_2 - t_1)$:

\[
R_w(t_1, t_2) = \sigma_w^2 g(r)
\]  

(19)

Using the Ornstein-Unlenbeck power spectrum the vertical turbulence autocorrelation function at the $0.7R$ blade station is now given by [3]-[5]

\[
R_w(t_1, t_2) = \sigma_w^2 e^{-r/(L/2)}
\]  

(20)

In equation 20, $r$ is the spatial separation of the $0.7R$ blade station during the lapsed time $(t_2 - t_1)$ and is given by

\[
r = \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2 + (z(t_2) - z(t_1))^2}
\]  

(21)

Substituting equations 16, 17, and 18 into equation 21 we get

\[
r = \sqrt{[V(t_2 - t_1) - 0.7R(\cos \Omega t_2 - \cos \Omega t_1)]^2 + [0.7R(\sin \Omega t_2 - \sin \Omega t_1)]^2 + [w(t_2 - t_1)]^2}
\]  

(22)

In terms of non-dimensional time, $\tau = \Omega t$, non-dimensional flight velocity, $\mu = V \cos(\alpha)/\Omega R$, and non-dimensional total mean airflow, $\bar{\omega} = \omega/\Omega R$, we can define the following constants.

\[
a = \frac{V \cos(\alpha)}{\Omega R (L/2) R} = \frac{2\mu}{L/R} = 2 \frac{R}{L} \mu
\]  

(23)

\[
b = 2 \frac{R}{L} \bar{\omega}
\]  

(24)

\[
c = 1.4 \frac{R}{L}
\]  

(25)

Also, for simplicity of notation, the following definitions are introduced.

\[
\tau = \bar{t}_2 - \bar{t}_1
\]  

(26)

\[
t = \frac{\bar{t}_2 + \bar{t}_1}{2}
\]  

(27)

Plugging into the expression for $r$ given by equation 22, and dividing by $L/2$, results in the following expression after trigonometric simplification.

\[
\frac{r}{L/2} = \sqrt{(a^2 + b^2) \tau^2 + 4ac \tau \sin t \sin \frac{\tau}{2} + 4c^2 \sin^2 \frac{\tau}{2}}
\]  

(28)

Substituting equation 28 into equation 20 yields the final expression for the nonstationary vertical turbulence autocorrelation function accounting for the rotating frame effects.

\[
R_w(t, \tau) = \sigma_w^2 \exp[-\sqrt{(a^2 + b^2) \tau^2 + 4ac \tau \sin t \sin \frac{\tau}{2} + 4c^2 \sin^2 \frac{\tau}{2}}]
\]  

(29)

In a space fixed turbulence formulation, the stationary model for the vertical turbulence velocities simplifies to

\[
R_w(t, \tau) = \sigma_w^2 \exp[-\sqrt{(a^2 + b^2) \tau^2}]
\]  

(30)

In hover $\mu = 0$, hence $a = 0$, and the stationary rotating frame vertical turbulence model is given by

\[
R_w(t, \tau) = \sigma_w^2 \exp[-\sqrt{b^2 \tau^2 + 4c^2 \sin^2 \frac{\tau}{2}}]
\]  

(31)

The corresponding space fixed vertical turbulence model in hover is simply

\[
R_w(t, \tau) = \sigma_w^2 \exp[-\sqrt{b^2 \tau^2}]
\]  

(32)
TURBULENCE FILTER IMPLEMENTATION

The main goal of the present research is to investigate in hover the effects of RFT on flight mechanics characteristics and compare them with those based on space fixed formulation. For the hover case, the autocorrelation function of the RFT, \( R_w(r) \), is stationary [6, 4, 5, 3] and hence, the simplicity of this analysis facilitates an improved appreciation of the RFT vis-a-vis conventional space fixed turbulence.

In order to effectively assess different aircraft turbulence models, a combination of pilot, gust and helicopter model needs to be chosen such that all three models are of the same texture or level of detail. The helicopter model used for this investigation is the UH-60A Black Hawk helicopter. A generic blade element analysis flight simulation program [9] is used for response simulation.

The gust model used in this investigation is the continuous stochastic turbulence model approach as previously described. Basically, we begin with a power spectrum for the vertical atmospheric turbulence using the Ornstein-Unlenbeck model given by equation 4. From the vertical turbulence power spectral density function, we seek to derive a turbulence filter system driven by white noise.

Consider, the space fixed turbulence autocorrelation function given by equation 32. The power spectral density function or Fourier transform is easily computed analytically. The space fixed power spectrum is given by

\[
S_p(\omega) = \frac{2b}{\pi(\omega^2 + b^2)}
\]  

(33)

The turbulence filter is also easily deduced by decomposing \( S_p(\omega) \) into a complex function multiplied by its conjugate. The turbulence filter is given by

\[
F(i\omega) = \frac{1}{b + i\omega}
\]  

(34)

The final filter for the space fixed turbulence is obtained by normalizing the power spectral density function. The normalized filter, in transfer function form, is given by

\[
F(s) = \frac{A}{s + b}
\]  

(35)

where \( A \) is the normalization constant.

For turbulence modeling the level of power must be parametric. The level of power is given by \( \sigma^2 \) where \( \sigma \) is the standard deviation representing the intensity of the turbulence. In order to compare different turbulence models, the total power spectrum must be normalized [10]. Following reference [10], the normalization requirement is:

\[
\frac{T}{\pi} \int_0^\infty |F(i\omega)|^2 \, d\omega = 1
\]  

(36)

where \( T \) is the sampling time. The normalization requirement for the first order filter becomes

\[
\frac{A^2 T}{2b} = 1
\]  

(37)

Thus, the constant \( A \) is given by

\[
A = \sqrt{\frac{2b}{T}}
\]  

(38)

Consider the RFT autocorrelation function given by equation 31. The power spectral density function of equation 31 is difficult to compute analytically. Thus, a numerical technique is employed to compute the power spectral density function using fast Fourier transforms. The nature of the power spectral density function has been investigated in several papers [7, 6], and all find that the power spectral density function for the turbulence process contains spikes occurring at integer multiples of the rotor rotational speed. For handling qualities work, \( 1P \) modes are most important. Hence, when approximating a power spectral density function using the RFT approach, it is most important to accurately model the first two peaks in the power spectral density function (a low frequency approximation). To capture the first two peaks in the the power spectral density function a third order turbulence filter is employed where the natural frequency of the turbulence filter is chosen to be the frequency of the second peak in the power spectral density function and the damping of the filter system is given by

\[
\zeta = \frac{1}{2Q}
\]  

(39)
In equation 39, \( Q \) is the amplitude of the first spike in the power spectral density function. The filter gain is chosen to match the dc-gain characteristics of the numerically computed power spectrum.

**RESULTS**

This section primarily compares the helicopter response results obtained using the RFT model and the space-fixed turbulence model.

Figure 2 shows the spectral density of vertical turbulence as experienced by a 0.7R blade station according to RFT and space-fixed formulations for \( L/R = 4 \), that is when the turbulence scale length \( L \) is four times the rotor radius. It is significant that the turbulence spectral density has sharp peaks at \( 1P, 2P, 3P, \) etc. It is equally significant that the conventional space-fixed distribution fails to capture these peaks. The consequence of the presence or absence of these peaks on an isolated rigid blade flapping response is shown in Figure 3. The blade is flexibly hinged at the hub center and a Lock number of 8 is used. Since the rigid blade can respond only to \( 1P \) variation in the excitation, the strong peak at \( 1P \) in Figure 3 for the case of RFT is noteworthy. Equally noteworthy is the fact that the occurrence of such response peaks cannot be captured by the space-fixed turbulence model.

The effect of turbulence on the Black Hawk helicopter response is considered next using a generic blade element analysis flight simulation program \([9]\). The helicopter is initially trimmed for zero wind hover. Then with the controls held fixed and with the flight control system turned off, the helicopter response due to turbulence excitation is obtained.

For the Black Hawk helicopter, the rotor diameter is 53.66 ft and the thrust coefficient \((C_T)\) in hover is 0.00532. The turbulence scale length to rotor radius \((L/R)\) is set to 1. For these values, the power spectral density function of the RFT is obtained by taking the fast Fourier transform of the autocorrelation function of equation 31 and the same is shown in Figure 4. As expected, the RFT spectral density function has sharp peaks at \( 1P, 2P, 3P, \) etc. Following the procedure described in the previous section, a third order filter is designed to approximate the first two peaks in the power spectral density function of the RFT.

With turbulence intensity set to 10 ft/sec, the sample functions of the RFT and the space fixed cases are obtained and the same are shown in Figure 5. It is interesting to note that, in general, the sample function for the RFT exhibits much larger peak-to-peak amplitude as compared to that of the space fixed case. The helicopter response to turbulence is obtained by assuming that the turbulence is represented by the sample functions of Figure 5. The effect of turbulence on the flap response of a reference blade is shown in Figure 6, wherein the change in flap response from trim is plotted versus time. From Figure 6, it is clear that the blade flap response for the RFT case is significantly different from that of the space fixed case.

The body accelerations and normal velocity in the body axes frame of reference are shown in Figures 7 through 10. It is to be noted that the body axis system used in this study has its x-axis to the front, y-axis to the right, and the z-axis down. The effect of turbulence on the body normal acceleration response is shown in Figure 7. The body normal acceleration response is strikingly different for the RFT case as compared to that of the space fixed case. The peak-to-peak amplitude of the normal acceleration is much larger than that of the space fixed case. The difference in response between the two cases, i.e., RFT and space fixed cases, highlights the necessity of treating turbulence in a rotating frame of reference for helicopter applications. The body longitudinal and lateral accelerations are shown in Figures 8 and 9, respectively. Though the general level of magnitude of longitudinal and lateral accelerations are small compared to the normal acceleration response, it is clear from Figures 8 and 9 that these accelerations are quite different for the RFT case as compared to the corresponding accelerations for the space fixed turbulence. The body normal velocity response is shown in Figure 10 from which it is clear that the normal velocity response is significantly different for the RFT case as compared to that of the space fixed case.

In order to assess the effect of turbulence scale length on the helicopter flight mechanics, the Black Hawk helicopter response simulation is repeated for a turbulence scale length \((L/R)\) of 10. The change in flap response from trim of the reference blade due to vertical turbulence with \( L/R = 10 \) is shown in Figure 11 and the body normal acceleration response is shown in Figure 12. Though the turbulence
intensity is the same, the general magnitude of response for the L/R = 10 case is reduced as compared to that of the L/R = 1 case. Also, from comparison of Figures 13 and 14 of the L/R = 10 case with the corresponding figures of L/R = 1 case (Figures 6 and 7), it is clear that the difference in response between the RFT and the space fixed case is reduced as L/R is increased. As noted in the introduction, with increasing L/R, the difference in responses between the RFT and the space fixed cases decreases.

CONCLUSIONS

It has been known for a long time that near-the-ground hovering in turbulence causes a loss of performance requiring great pilot skill and it causes additional loads and vibrations. A treatment of this phenomenon using the concept of RFT demonstrates the following:

1. blade response as well as the body response to turbulence excitations is strongly affected by RFT, and
2. the conventional space-fixed description fails to capture those effects.

Thus, the present treatment of turbulence in the rotating frame provides a means of describing turbulence effects on low frequency blade response and helicopter handling qualities both qualitatively and quantitatively.

REFERENCES


ACKNOWLEDGEMENTS

We wish to acknowledge the help received from Mark Costello and Jamshed Riaz during the course of this study. This work is performed under the NASA-Ames University Consortium Grant No. NCA2-266.
Figure 1. Rotor Disk Velocity Diagram.

Figure 2. Comparison of RFT and Space Fixed Turbulence Models
Figure 3. Effect of Turbulence on Isolated Blade Flap Response.

Figure 4. Turbulence Models for the Black Hawk Helicopter in Hover for L/R = 1.
Figure 5. Vertical Turbulence Sample Functions for L/R = 1.

Figure 6. Effect of Turbulence on the Black Hawk Helicopter Rotor Blade Flap Response for L/R = 1.
Figure 7. Effect of Turbulence on the Black Hawk Helicopter Normal Acceleration Response for L/R = 1.

Figure 8. Effect of Turbulence on the Black Hawk Helicopter Longitudinal Acceleration Response for L/R = 1.
Figure 9. Effect of Turbulence on the Black Hawk Helicopter Lateral Acceleration Response for L/R = 1.

Figure 10. Effect of Turbulence on the Black Hawk Helicopter Body Vertical Velocity Response for L/R = 1.
Figure 11. Effect of Turbulence on the Black Hawk Helicopter Rotor Blade Flap Response for L/R = 10.

Figure 12. Effect of Turbulence on the Black Hawk Helicopter Normal Acceleration Response for L/R = 10.
STUDY OF THE EFFECTS OF ROTATING FRAME TURBULENCE (RFT) ON HELICOPTER FLIGHT MECHANICS, LOADS, AND VIBRATION

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October 1, 1987 - September 30, 1989

Funds for the support of this study have been allocated by the U. S. Army Aeroflightdynamics Directorate through NASA Ames Research Center, Moffett Field, California, under Interchange No. NCA2-159.
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ABSTRACT

This report pertains to noneulerian description and rotational sampling of space-fixed turbulence to portray the actual turbulence experienced by the helicopter blades, popularly known as rotating frame turbulence. It is shown that in the stationary case of axial flight the spectrum of the rotating frame turbulence has peaks centered at integer multiples of rotational speed. In forward flight, however, the peaks are scattered in a more complicated fashion. The spectrum in the forward flight is shown to be cyclostationary. The rotating frame turbulence effects on the statistics of the blade response rms values and average threshold crossing rates are presented as well. The helicopter simulation model used in this study includes rotor flapping dynamics and a first harmonic representation of inflow. Based on the simulation results obtained using the generic blade element analysis program, it is shown that the rotating frame turbulence effects are appreciable for low-advance ratio and low-altitude flight conditions. Also, a methodology for obtaining a comprehensive model of the rotating frame turbulence is presented.

INTRODUCTION

The atmospheric velocity vector varies with time, space, local weather condition, and ground texture, an obviously complicated phenomenon to describe mathematically. There are currently two basic approaches for modeling atmospheric turbulence, namely, discrete gust analysis and stochastic turbulence analysis [1].

The discrete gust model is a deterministic input which is injected into the relative wind vector. Various types of time inputs such as step functions, ramp functions, and one-minus-cosine functions are typically used to disturb the aircraft, hopefully in a manner similar to atmospheric turbulence. Designers have found this method of atmospheric turbulence analysis to be useful during preliminary design for the calculation of maximum flight loads. In a more detailed analysis, sinusoid discrete gusts are used to investigate resonance modes of an aircraft which may be excited in flight through atmospheric turbulence by varying the frequency content of the input gust. For gust analysis particular to rotorcraft, there are generally two types of discrete gusts which
are commonly used, namely, gradually penetrating gusts and instantaneously engulfing gusts. The instantly engulfing discrete gust changes the relative wind vector over the entire aircraft instantly, so that at any time the aircraft only feels one gust velocity. The gradually penetrating gust distributes a spatially dependent gust over the aircraft by engulfing the aircraft gradually as the gust convects past the aircraft at the speed of the relative velocity between the mean wind and the aircraft's center of gravity. Researchers using discrete gust analysis for helicopters have noticed that it is important to use gradually penetrating discrete gusts to obtain more realistic rotor and rigid body response to atmospheric turbulence [1,2]. One of the most notable features of atmospheric turbulence is its "randomness." A disadvantage of the discrete gust approach for flying qualities analysis is that it is deterministic and does not possess this random feature. Also, the discrete gust method does not provide a feasible means of computing aircraft response statistics needed for calculating probability of exceedence statistics.

The continuous stochastic atmospheric turbulence model constructs a random signal which over a period of time contains the amplitude and frequency content characteristic of atmospheric conditions. Several stochastic atmospheric turbulence models are available which represent atmospheric turbulence in terms of average wind speeds and turbulence levels. Although quantifying atmospheric wind conditions is primarily a meteorological activity, care must be taken to correctly use the available models for particular altitude ranges. While there are no accurate atmospheric wind models for some flight conditions that rotorcraft encounters, especially atmospheric turbulence in the proximity of man made and natural obstacles, a large percentage of flight conditions encountered in typical helicopter usage can be accurately modeled. For continuous stochastic atmospheric turbulence analysis particular to rotorcraft, two basic types of models are currently employed, denoted body frame sampling and rotating frame sampling models.

The atmospheric wind velocity vector is generally modeled as a superposition of a deterministic low frequency component, called the mean wind, and a stochastic high frequency component, denoted as atmospheric turbulence. As shown in Figure 1, at low altitude this assumption seems to be justified for investigating the impact of the atmospheric wind velocity on phenomena having a
characteristic frequency higher than the mean wind frequency. The atmospheric
turbulence component of the atmospheric wind velocity vector can be adequately
described by a turbulence intensity, \( \sigma \), a turbulence length scale, \( L \), and a
distance metric, \( r_{dm} \), in many cases applicable to rotorcraft. The turbulence
intensity and the length scale depend on meteorological conditions and the
distance metric is the distance between the aerodynamic surfaces of interest at
two points in time. Most rotorcraft atmospheric turbulence models directly use
the statistics which were derived for fixed wing aircraft flying straight and
level at a constant speed. These models will be denoted as body fixed sampling
models. However, for a typical rotary wing vehicle the majority of forces and
moments are created by rotating aircraft elements. Atmospheric turbulence
directly affects the forces and moments generated by rotating and non-rotating
aircraft elements by altering the relative wind seen by the lifting surface.
Hence, it is important to consider the effects of a rotating aircraft element
when computing the helicopter rotor blade distance metric for use in describing
atmospheric turbulence on the rotating aircraft elements. Figure 2 shows the
path taken by a point on the rotor hub and the path taken by a point on the tip
of the rotor blade at a moderate advance ratio. As can be seen in Figure 2, the
position vector at two points on a rotor blade at two different times can be
dramatically different from the corresponding distance vector taken at the rotor
hub. This is the basic difference between the current rotorcraft body frame
sampling models which use the distance metric computed from a fixed point on the
rigid body, usually the aircraft center of gravity, and the new rotorcraft
atmospheric turbulence models called rotating frame sampling models which compute
the distance metric from a point fixed on a blade element of the rotor.

To illustrate the underlying physics of rotorcraft atmospheric turbulence
dynamics and to explain the information contained in the abstract quantity, \( r_{dm} \),
the distance metric, we shall consider a simplified gust scenario. The
stochastic description of atmospheric turbulence consists of an infinite sum of
sinusoids spanning all frequencies, each wave with a different amplitude, hence
the results obtained for one wave in the summation will apply to the entire
turbulence field. With this in mind, consider a helicopter with a rotor radius,
\( R \), flying straight and level at a constant speed, \( V_{a/c} \), through a temporally
constant and spatially varying vertical sinusoidal gust with a period, \( L \),
convecting towards the aircraft at the mean wind speed, \( V_{mw} \), as shown in Figure

3
3. The atmospheric turbulence velocity experienced by the rotor hub as a function of time is given by the equation (1)

\[ w_{\text{turb}} = \sigma \sin \left( \frac{2\pi(V_a/c + V_{mw})t}{L} \right) \]  

where \( \sigma \) represents the turbulence intensity.

The atmospheric turbulence velocity experienced by the aircraft center of gravity as a function of time is the same as that observed at the rotor hub except for a shift in phase, as given in equation (2)

\[ w_{\text{turb}} = \sigma \sin \left( \frac{2\pi(V_a/c + V_{mw})t}{L} + \phi \right) \]  

If the rotor rotational speed is \( \Omega \), then the turbulence velocity experienced by the rotor blade tip is given by equation (3), assuming that at \( t = 0 \) the blade position and the hub position in the flight path direction are coincident, and ignoring flap, lag, and structural dynamics.

\[ w_{\text{turb}} = \sigma \sin \left( 2\pi \frac{(V_a/c + V_{mw})}{L} t + 2\pi \frac{R}{L} \sin (\Omega t) \right) \]  

Equations (1) and (2) are examples of body frame sampling models while equation (3) represents a rotating frame sampling model. The distance metric for the body frame sampling and rotating frame sampling models are given by equations (4) and (5), respectively.

\[ r_{\text{dm body fixed}} = \frac{2\pi(V_a/c + V_{mw})}{L} (t_2 - t_1) \]  

\[ r_{\text{dm rotor blade fixed}} = \frac{2\pi(V_a/c + V_{mw})}{L} (t_2 - t_1) + R(\sin \Omega t_2 - \sin \Omega t_1) \]

From equations (4) and (5) we see that the distance metric represents the motion of the aerodynamic point of interest with respect to the turbulence field. The turbulence length scale, \( L \), represents the characteristic period of gust velocities and the turbulence intensity, \( \sigma \), represents a characteristic amplitude.
of the gust field. The gust velocities experienced by the aircraft using the body frame sampling and rotating frame sampling models are plotted in Figure 4 for a forward speed, $V = 40.0$ feet/second, a rotor radius, $R = 28.0$ feet, a rotor rotational speed, $\Omega = 25.0$ rad/second, a gust length scale, $L = 50.0$ feet, and a mean wind speed, $V_{mw} = 10.0$ feet/second. As can be seen in Figure 4, the gust velocity felt by the rotor blade using the rotating frame sampling model has much higher frequency content than the rotor blade gust velocity generated using the body frame sampling model. The above example, while oversimplifying the gust problem, illustrates two important points regarding body frame sampling and rotating frame sampling atmospheric turbulence models. First, time histories of gusts encountered by a rotor blade when flying through identical gusts are very different. While the body frame sampling is used extensively throughout the rotorcraft industry due to its simple structure, the rotating frame sampling model is theoretically more accurate for defining gust velocities felt by a blade element of a rotor. Also, in general, the rotating frame sampling model will yield rotor blade gust velocities which have more energy at higher frequency than the corresponding gust velocities using a body frame sampling model. Figure 5 shows the power spectral density function of atmospheric turbulence velocity perpendicular to the rotor disk of a horizontal axis wind turbine at the blade tip obtained from three different theoretical models of the atmosphere using both body and rotating frame sampling models and test data. Note that the test data closely follows the rotating frame sampling model and the power spectral density function contains peaks at integer multiplies of the rotor rotational speed. This figure illustrates the inadequacy of the body frame sampling models for predicting the frequency response of atmospheric turbulence experienced by a rotating blade.

Atmospheric turbulence time histories are usually created by exciting a differential equation with white noise. These stochastic differential equations, commonly called shaping filters, are determined such that the statistics of the output match the theoretically derived atmospheric turbulence statistics. It can be shown that the shaping filter describing atmospheric turbulence felt by the aircraft center of gravity while flying straight and level represents a stationary process and can be exactly modeled using a first order filter. It will be shown that the shaping filter for a rotor blade using a rotating frame sampling model is periodic with a period equal to the rotor rotational speed with
the statistics of this process being cyclostationary.

SINGLE POINT AUTO-CORRELATION MODEL AND SIMULATION

Turbulence Model: A single point auto-correlation model has been developed to assess the effects of rotating frame turbulence and dynamic stall on gust response of helicopter blades. In this model only the auto-correlation of the vertical component of gust at 0.7R radial station is considered. The details of the model and its various manifestations are explained in the Appendix. The salient conclusions derived from this model are:

a. Rotating frame turbulence is a (weakly) stationary process in hover. In forward flight this process is (wide sense) cyclostationary or periodically nonstationary (not mean square periodic).

b. The instantaneous spectrum of rotating frame turbulence simultaneously predicts both the periodically varying nonstationarity with respect to time as well as the occurrence of split peaks centered at P/2, P, 3P, 2P, etc. and transfer of energy with respect to frequencies.

c. The response statistics comprising spectral density, rms values, and expected threshold crossing rates are significantly affected by rotating frame turbulence.

d. Dynamic stall affects the response due to both rotating frame and space-fixed excitations and hence aggravates gust sensitivity.

A closed-form solution of rotating frame spectrum for a space-fixed turbulence model is developed in the report which provides a qualitative and parametric investigation of the characteristics with respect to both frequency and time.

Simulation: The turbulence model described in the report was used in the generic blade element analysis program to investigate the effects of vertical turbulence. The basic problem in this regard is creating turbulence samples which contain statistics in accordance with the model. Whereas the hovering case presents no difficulty due to stationary nature of correlation between the sample values, the forward flight case involves non-stationarity and, therefore, needs a special technique to generate sample vertical turbulence.

Since the auto-correlation in forward flight is cyclo-stationary, the
problem somewhat simplifies to designing a filter structure suitable for one cycle. This structure would then repeat itself after each cycle and remain in step with cyclo-stationarity. Within the cycle, the total cycle time was divided into 12 equal intervals. In each interval it was assumed that the frequency spectrum was stationary and the same was approximated by the instantaneous spectrum at the beginning of the interval. A linear phase Finite Impulse Response (FIR) filter was designed to correspond to the frequency spectrum, and 16 sample values were generated using this filter within the stationary interval. Twelve FIR filters were required to cater for one cycle. In actual implementation, one FIR filter was created whose coefficients changed 12 times within the cycle, and these coefficients repeated themselves in every subsequent cycle in the same order. This technique, therefore, ensured that both the non-stationary and cyclo-stationary effects were captured. The FIR filter was driven by white noise input. The same filter structure was used for the hovering case. In this case, however, the filter coefficients remain constant due to the stationarity of the process.

The FIR filter was designed using the frequency-sampling technique as follows:

a. For a given value of \( t \), and a range of values of \( \tau \), a row vector of auto-correlation values was calculated. The Black Hawk parameters were used to calculate these values.

b. The above vector was fast-Fourier transformed to get the instantaneous spectral density.

c. The instantaneous frequency response was then obtained by taking the square root of the spectral density, and inverse fast Fourier transform was applied to it to get the impulse response.

d. The impulse response was windowed using hamming window to the desired filter length.

e. The resulting impulse sequence was used as the FIR filter coefficients pertaining to the particular interval.

f. The above procedure was repeated for different values of \( t \) and the row vector for different values of \( \tau \) to obtain a matrix of FIR filter coefficients. Each row of the matrix corresponded to a particular interval within the cycle.
Having designed the FIR coefficient matrix, a sample of white noise was generated. Convolution of white noise with the first row of filter coefficients was carried out to create sixteen sample values. At this point, the filter coefficients were changed by switching to the second row and continuing with the convolution. Similar procedure was adopted for the remaining rows. After using the last row, the first row was picked up again and the cycle was started anew. The process was terminated after a suitable sample length for vertical turbulence was obtained. This sample was then used as input to the generic blade element analysis flight simulation program to study the effects of vertical turbulence on the Black Hawk helicopter.

The helicopter model in the flight simulation program is a total force, non-linear large angle representation in six rigid body degrees of freedom. In addition, rotor rigid blade flapping, lagging, and hub rotational degrees of freedom are represented. A blade element approach is used to model each main rotor blade. Downwash is approximated to have a first harmonic distribution as a function of skew angle. The underlying framework of the program is modular with each major force and moment producing element(s) of the aircraft treated as a separate entity.

The simulation was carried out using the value of $L/R = 1$, where $L$ is the turbulence length scale, and $R$ the blade radius. The intensity of turbulence was assumed to be 10 ft/sec. First, the rotating frame turbulence velocities were generated for hover (Fig. 6) and 30 knots (Fig. 7) using identical parameters. The power spectral density curve of the velocities for the hover case and an instantaneous spectrum curve for 30 knots case are given in Fig. 8 and 9, respectively. Figures 8 and 9 demonstrate the validity of the FIR filter approach in creating sample functions for the vertical turbulence. The essential statistical characteristics of the RFT models are present in the sample functions, which is also evident from Figures 8 and 9. These velocities were, then, introduced at the hub to excite the helicopter system without any corrective input from the pilot and with the stability augmentation system off. All blades were, therefore, subject to the same turbulent velocity at a particular instant of time—the effects of rotational sampling being included in the turbulence sample. After the aircraft reached its trim position, it was suddenly engulfed in the turbulence corresponding to the flight regime. The time series of the response of the helicopter were recorded in each case and the power spectral density computed.
The flapping angle and the vertical acceleration response of the helicopter in the hover case are shown in Fig. 10 and 11, and the respective power density curves in Fig. 12 and 13. From the power spectral density curve of the rotating frame turbulence at hover (Fig. 8), it may be noted that a significant amount of energy shift to 1 per rev, 2 per rev, and 3 per rev modes is present in the sample function. A decay in energy content with increasing frequency can also be noted. Since the hover case is basically a stationary case, this power spectral density curve is representative of the frequency content in the sample at all times. If now the power density curves of the response of the helicopter during hover are studied, it will be noted that this frequency content re-emerges in the output. Both the vertical acceleration and flapping angle curves show a deterioration from the steady-state condition, with fluctuations centered at 1 per rev, 2 per rev, etc. This will have significant effect on the handling and ride qualities of the helicopter and is an area which needs to be addressed, especially in the case of landing and take-off from landing pads with severely restricted access in severely turbulent atmospheric winds.

Turning attention to the case of the rotating frame turbulence in the forward flight case, it needs to be emphasized that it is a non-stationary process. The instantaneous spectrum curve for a particular time is shown in Fig. 9. Here one can notice the phenomenon of split peaks around 1 per rev as described in the Appendix. The frequency spectrum in this case varies with time and hence the helicopter is subjected to quite a complex frequency content over the period of excitation. The time responses in terms of flapping angle and vertical acceleration in this case are shown in Fig. 14 and 15, with respective power spectral densities in Fig. 16 and 17. The response of the helicopter and the respective power spectral densities in this case depict average frequency content of the output. Again it is noticed that on the average the flapping and the vertical acceleration response is concentrated around 1 per rev, 2 per rev, etc., but the intensity is diminished compared to the hover case. This is consistent with the theory which predicts a diminishing effect of rotating frame turbulence as the advance ratio increases.

**IMPROVED ROTATING FRAME TURBULENCE MODEL**

**Basic Turbulence Theory:** To set the stage for the new model, consider the following definition for turbulent flow given by Hinze in reference [3]:

---

9
Turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned.

According to this definition, the flow has to satisfy the condition of irregularity which highlights an important feature of turbulence. Because of the irregular nature of any turbulent flow, it is impossible to describe it by deterministic functional relationships. Fortunately, turbulence velocities are generally random in the sense that it is possible to describe it by the laws of probability, hence a statistical approach is often adopted to describe atmospheric turbulence velocities.

To illustrate the manner in which the mean and turbulent velocities are coupled we shall discuss the general governing equations for flow velocities and pressure of an incompressible turbulent flow. For incompressible flow, the fluid dynamics and thermodynamics are uncoupled, and since we are interested in velocity information we will be primarily concerned with the equations which govern the fluid velocity dynamics, namely, conservation of mass and momentum. The continuity equation for incompressible flow written in differential form is given by equation (6).

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]  

(6)

The conservation of momentum equations for incompressible flow are given by equations (7), (8), and (9), neglecting gravity.

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} u^2 + \frac{\partial v}{\partial y} uv + \frac{\partial w}{\partial z} uw = -\frac{1}{Q} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)
\]  

(7)

\[
\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} uv + \frac{\partial}{\partial y} v^2 + \frac{\partial}{\partial z} vw = -\frac{1}{Q} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)
\]  

(8)

\[
\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} uw + \frac{\partial}{\partial y} vw + \frac{\partial}{\partial z} w^2 = -\frac{1}{Q} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\]  

(9)

Typically, turbulent flows are modeled as a superposition of a deterministic
mean velocity component and a random turbulent velocity component.

\[ u = u_{\text{mean}} + u_{\text{turbulent}} = \bar{u} + u \]  

\[ v = v_{\text{mean}} + v_{\text{turbulent}} = \bar{v} + v \]  

\[ w = w_{\text{mean}} + w_{\text{turbulent}} = \bar{w} + w \]  

To obtain the equations of motion for turbulent flow, equations (10), (11), and (12) are substituted into the governing equations. Then, the equations are time averaged to obtain the governing equations for the average flow velocities. The resulting equations of motion for the average flow velocities are given by equations (13) through (16).

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]  

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (R_{uu} + u^2) + \frac{\partial}{\partial y} (R_{uv} + \bar{uv}) + \frac{\partial}{\partial z} (R_{uw} + \bar{uw}) = -\frac{1}{Q} \frac{\partial}{\partial x} \left( \frac{a^2 u}{\partial x^2} + \frac{a^2 u}{\partial y^2} + \frac{a^2 u}{\partial z^2} \right) \]  

\[ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} (R_{vu} + \bar{vu}) + \frac{\partial}{\partial y} (R_{vv} + \bar{vv}) + \frac{\partial}{\partial z} (R_{vw} + \bar{vw}) = -\frac{1}{Q} \frac{\partial}{\partial y} \left( \frac{a^2 v}{\partial x^2} + \frac{a^2 v}{\partial y^2} + \frac{a^2 v}{\partial z^2} \right) \]  

\[ \frac{\partial w}{\partial t} + \frac{\partial}{\partial x} (R_{wu} + \bar{wu}) + \frac{\partial}{\partial y} (R_{wv} + \bar{wv}) + \frac{\partial}{\partial z} (R_{ww} + \bar{ww}) = -\frac{1}{Q} \frac{\partial}{\partial z} \left( \frac{a^2 w}{\partial x^2} + \frac{a^2 w}{\partial y^2} + \frac{a^2 w}{\partial z^2} \right) \]  

Examining these equations, we see that the continuity equation for the average flow velocities is identical to the laminar flow case. However, the momentum equations for the average flow velocities are slightly different form
the laminar flow equations. Each component of the momentum equation, after time averaging, contains mean values plus three turbulence component products or correlation functions. The correlation matrix for turbulent flow velocities is formally defined by equation (17), where $E[\cdot]$ denotes the expectation operator.

$$R(t_2,t_1) = 
\begin{bmatrix}
E[u(t_2)u(t_1)] & E[u(t_2)v(t_1)] & E[u(t_2)w(t_1)] \\
E[v(t_2)u(t_1)] & E[v(t_2)v(t_1)] & E[v(t_2)w(t_1)] \\
E[w(t_2)u(t_1)] & E[w(t_2)v(t_1)] & E[w(t_2)w(t_1)]
\end{bmatrix}$$

(17)

The correlation terms arise from the convective acceleration of the fluid although they are usually termed turbulent stresses throughout the literature. The point of the above derivation was to illustrate the coupling which exists between the mean flow velocities or mean wind and the turbulence statistics in the most fundamental fluid flow equations.

Isotropic turbulence is the simplest type of turbulence because no preference in space occurs and a minimum number of quantities and relations are required to describe its stochastic structure and behavior. However, it is also a hypothetical type of turbulence, because no actual flow shows true isotropy, though conditions may be such that isotropy is more or less closely approached. Both theoretical and experimental evidence show that the fine structure of most actual non-isotropic flows is nearly isotropic.

At high altitude, atmospheric turbulence is for the most part isotropic [4]. At low altitude there is evidence that atmospheric turbulence is non-isotropic [5], however, in the following development we shall assume that the structure of the atmospheric turbulence correlation matrix is isotropic at all altitudes.

For flying qualities analysis we are interested in modeling the atmospheric mean wind and atmospheric turbulence. The information contained in the turbulence correlation matrix is sufficient to form a stochastic state space model for the atmospheric turbulence velocities, therefore it is important to ascertain their general form. To this end, note that isotropic turbulence
assumes the turbulence correlation matrix to be independent of translations, rotations, and reflections of the coordinate system chosen. Consider a simple situation where we are interested in the correlation matrix of turbulence velocities along the x axis as shown in Figure 18. Using the definition of isotropy we can restrict the form of the correlation matrix noting the following relations.

\[ E[u_A v_B] = - E[u_A v_B] = 0 \]  (18)

\[ E[u_A w_B] = - E[u_A w_B] = 0 \]  (19)

\[ E[v_A w_B] = - E[v_A w_B] = 0 \]  (20)

\[ E[v_A v_B] = - E[w_A w_B] \]  (21)

Hence, for \( r_A - r_B \) in the x direction, the correlation matrix must take on the following form to satisfy the isotropy constraints.

\[
R(t_2, t_1) = \begin{bmatrix}
E[u(t_2)u(t_1)] & 0 & 0 \\
0 & E[v(t_2)v(t_1)] & 0 \\
0 & 0 & E[v(t_2)v(t_1)]
\end{bmatrix}
\]  (22)

The correlation component \( E[u_A v_B] \) is termed the longitudinal correlation for isotropic turbulence and will be denoted as \( R_a \) while the correlation components \( E[v_A v_B] \) and \( E[w_A w_B] \) are termed the lateral correlation component for isotropic turbulence and will be denoted by \( R_x \). The longitudinal correlation function is the correlation function between quantities parallel to the two points of interest and the lateral correlation is the correlation between quantities perpendicular to the vector drawn between the two points of interest. Both correlations are schematically drawn in Figure 19. If we desire the correlation matrix in a reference frame which is not aligned with the vector drawn from the two points of interest, a vector transformation can be used to obtain the turbulence statistics in the desired reference frame from the turbulence statistics parallel and orthogonal to the distance metric. Without going through the trigonometry, the correlation matrix in the desired reference
frame is given by equation (23) [3].

\[
R(t_2,t_1) = \frac{R_a - R_x}{r_{dm}}^2 \begin{bmatrix} x^2 & xy & xz \\
                        xy & y^2 & yz \\
xz & yz & z^2 \end{bmatrix} + R_\xi \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}
\]  

(23)

The term \( r_{dm} \) is the distance metric with components \( x, y, \) and \( z \) expressed in the desired reference frame. This matrix can be further reduced by the application of the continuity equation which yields a relationship between the longitudinal and lateral isotropic correlation functions, given as equation (24).

\[
R_\xi = R_a + \frac{|r_{dm}|}{2} \frac{\partial R_a}{\partial r_{dm}}
\]  

(24)

The above spatial correlation matrix given by equation (23) is valid for any turbulent flow field which satisfies the condition of isotropy or any turbulent flow which is being approximated as isotropic. We shall particularize this correlation matrix to the rotorcraft atmospheric turbulence problem in the subsequent development by specializing the longitudinal and lateral correlation functions, and the distance metric to a rotor blade in an arbitrary flight condition.

The Stochastic Structure of Atmospheric Turbulence: It is obvious that if we wish to study flight in turbulent air theoretically, we must know enough about atmospheric turbulence to construct a reasonable mathematical model of it. In the above section, the three dimensional correlation matrix is a function of two fundamental correlation functions, a turbulence length scale, a turbulence intensity, and the distance metric. The distance metric is determined from the geometry of the atmospheric turbulence field and the aircraft flight condition. The length scale and intensity depend on atmospheric conditions. It has been found that the atmospheric structure in the boundary layer of the earth is different from higher altitude atmospheric turbulence and is described separately in the next two sections.
Low Altitude Structure of Atmospheric Turbulence: Figure 1 obtained from reference [6] is a plot of the power spectrum of the response of an anemometer fixed, with respect to the earth. It is seen that measured winds contain high frequency energy, peaking about 70 cycles/hour, and low frequency energy, peaking about 0.01 cycles/hour and that there is a wide frequency region near 1 cycle/hour where there is little wind energy. This figure justifies dividing the atmospheric wind velocity into a deterministic slowly varying, mean wind velocity component and a stochastic rapidly varying turbulence wind velocity. For time periods as long as several minutes, the mean wind is taken to be a constant and determined by the atmospheric conditions which the aircraft is operating within. At low altitude, the mean atmospheric wind is similar to boundary layer flow over a flat surface. Because of the earth's boundary layer, the mean wind increases with increasing height. Reference [7] provides several models which are used for the atmospheric mean wind velocity, the atmospheric turbulence variance, and the atmospheric turbulence length scale valid under an altitude of 300 feet. These models can be quite different and empirical formulas should be used when possible. However, in the absence of empirical formulas for a particular area, the following models from reference [7] should be representative and useful for deducing basic trends.

In engineering practice, it is common to use a power law formulation for the mean wind velocity profile [7]. The atmospheric mean wind velocity direction at low altitude is assumed parallel to the earth's surface and is given by equation (25).

\[ V_{mw} = V_r \left( \frac{z}{z_r} \right)^\alpha \]  \hspace{1cm} (25)

In this expression for the mean wind velocity, \( V_r \) is the mean wind at reference height, \( z_r \) is the reference height, and \( \alpha \) is the power law index which is usually between 0.05 and 0.3. The power law index is usually higher at night (0.2-0.3) due to stable atmospheric conditions, and lower during the day (0.05-0.2) in unstable air.

In some circumstances, rather than compute the mean wind for a given atmospheric condition, we are interested in the probability of encountering a particular mean wind velocity at a certain altitude. This probability can be
calculated using the Wind Duration Curve which generally follows a Weibull probability distribution given in equation (26).

\[
Pr(V_{mw} > V) = e^{-\left(V_{mw}/V_o\right)^k}
\]  

(26)

A representative model for the atmospheric turbulence intensity at low altitude, under 300 feet from the surface, is given by equation (27).

\[
\sigma = 0.3V_{mw} \ln(z/z_0)
\]

(27)

In equation (27), \(z\) is the altitude and \(z_0\) is the terrain roughness length which is detailed in Table 1.

A representative model for the turbulence length scale is given in equation (28).

\[
L = \frac{2000z}{2500+z}
\]

(28)

High Altitude Structure of Atmospheric Turbulence: Atmospheric turbulence at high altitudes has been reviewed in [8] where the turbulence length scale and intensity have been examined. The general finding is that high altitude atmospheric turbulence is isotropic. A representative model for the length scale and intensity is given by equations (29) and (30).

\[
L = 1000 - 5000 \text{ feet}
\]

(29)

\[
\sigma = \begin{cases} 
4 \text{ ft/s (light turbulence, clean air)} \\
8 \text{ ft/s (moderate turbulence, cumulous cloud)} \\
16 \text{ ft/s (severe turbulence, thunderstorm)}
\end{cases}
\]

(30)

The mean wind at altitude is taken to be the value given using the low altitude model evaluated at 300 feet.

Rotorcraft Atmospheric Turbulence Correlation Matrix: It should be noted that in the following development the inertial, body, and blade reference frames are defined in the conventional manner. The atmospheric reference frame is a frame
which is attached to the atmospheric mean wind and is oriented in such a manner that the atmospheric x-axis is parallel to the mean wind. For mean wind parallel to the surface of the earth, the atmosphere z-axis points down into the earth's surface and the atmosphere y-axis forms a right handed coordinate system.

Before proceeding about how the statistics of atmospheric turbulence are formed for a rotor blade, a note on a generally accepted assumption when modeling the atmosphere for aircraft applications is in order. Atmospheric turbulence is a very complicated process and varies in both time and space. However, it turns out that turbulence velocities at any point in the atmosphere frame vary slowly in time. As one changes spatial coordinates in the atmosphere frame, turbulence velocities change quickly as compared to temporal changes. These observations led Taylor [9] to simplify the atmospheric turbulence problem by assuming the turbulence field in the atmosphere reference frame to be constant in time. This simplification is known as "Taylor's Hypothesis" or "The Frozen Field Concept," so named because a patch of turbulence moving with the mean wind speed which is frozen in time models atmospheric turbulence. Clearly, this assumption cannot be made when the aircraft's lifting surfaces remain fixed in the atmospheric frame because in this case turbulence gradients arise only from temporal changes. At first glance, it seems that rotorcraft would fall into this situation often, for example in hover. The major lifting surfaces for a rotorcraft are the rotor blades which are always moving in the atmospheric frame further justifying the applicability of the frozen field concept for a rotorcraft in an arbitrary flight condition. Also, as we saw in the previous section, with turbulence comes mean wind, so that in hover with respect to an earth fixed cue the aircraft continuously changes position in the atmosphere frame.

The manner in which the generally applicable isotropic turbulence statistics and the specific atmospheric turbulence statistics are particularized to the rotorcraft atmospheric turbulence problem is through the proper computation of the distance metric, $r_{dm}$, and specialization of the fundamental isotropic correlation function to atmospheric turbulence.

There are many forms of the fundamental correlation functions. The longitudinal correlation function for the von Karman model is given by equations (31) through (33),
\[ R_a = b \left( \frac{|r_{dm}|}{aL} \right)^{\frac{1}{3}} K_{\frac{1}{3}} \left( \frac{|r_{dm}|}{aL} \right) \] (31)

where,

\[ a = \frac{\Gamma\left(\frac{1}{3}\right)}{\sqrt{\pi} \, \Gamma\left(\frac{5}{6}\right)} - 1.339 \] (32)

and,

\[ b = \frac{2^{2/3}}{\Gamma\left(\frac{1}{3}\right)} = 0.5295 \] (33)

and \( L \) is the integral length scale of turbulence, \( \sigma^2 \) is the variance of atmospheric turbulence, \( \Gamma \) is the gamma function, and \( K_i \) is a modified Bessel function of the second kind of fractal order \( i \). An efficient method for calculating the above correlation function is given in reference [10]. The Rosenbrock model [11] assumes, without supporting evidence, the longitudinal correlation to be given by equation (34).

\[ R_a = \sigma^2 e^{-|r_{dm}|^2/L^2} \] (34)

The Dryden model, based on an approximation to the von Karman model, is given by equation (35).

\[ R_a = \sigma^2 e^{-|r_{dm}|/L} \] (35)

Theoretically, the von Karman model should provide the most accurate results at the expense of higher computation time. If it is important to quickly compute the fundamental correlations, the Dryden model may be employed.

The distance metric is defined as the relative distance between the two points of interest on the lifting surfaces at two points in time with respect to the atmosphere frame. More quantitatively, it is defined as the distance from the atmosphere frame to point 2 and time \( t_2 \) minus the distance from the
atmosphere frame to point 1 at time $t_1$, as given in equation (36). This is illustrated in Figure 8.

$$r_{dm} = \phi_{in/blade}(t)(r_{in/blade}(t_2) - r_{in/at}(t_2) - r_{in/blade}(t_1) + r_{in/at}(t_1)) \quad (36)$$

In the above equation, $\phi_{in/blade}(t)$ is the transformation matrix from the inertial frame to the blade frame at time $t$ and is later defined by $T_6^T$, $r_{in/blade}(t)$ is the vector from the inertial frame to the blade frame at time $t$ expressed in inertial coordinates, and $r_{in/at}(t)$ is the vector from the inertial frame to the atmosphere frame at time $t$ also expressed in inertial coordinates. The distance metric is written in the blade frame (Fig. 20) since later we shall seek to obtain a model for atmospheric turbulence velocities along the rotor blade in the blade system. For an articulated rotor, the general form of the vector, $r_{in/blade}$, is given by equation (37).

$$r_{in/blade} = r_{cg} + T_1e_1 + T_2e_2 + T_3e_3 + T_4e_4 + T_5e_5 + T_6e_6 \quad (37)$$

where the $T_i$ matrices are given by the following matrix multiplications,

$$T_i = \prod_{k=1}^{t} \Phi_k \quad (38)$$

and the $\Phi_i$ transformation matrices are given by equations (39) through (44).

$$\Phi_1 = \begin{bmatrix}
\cos\theta\cos\psi & -\cos\phi\sin\theta + \sin\phi\sin\theta\cos\psi & \sin\phi\sin\theta + \cos\phi\sin\theta\cos\psi \\
\cos\phi\sin\theta & \cos\phi\cos\theta + \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\theta + \cos\phi\sin\theta\sin\psi \\
-\sin\theta & -\sin\phi\cos\theta & \cos\phi\cos\theta
\end{bmatrix} \quad (39)$$

$$\Phi_2 = \begin{bmatrix}
\cos \Gamma_x & 0 & \sin \Gamma_y \\
\sin \Gamma_x & \sin \Gamma_y & \cos \Gamma_x & -\sin \Gamma_x \cos \Gamma_y \\
-\cos \Gamma_x & \sin \Gamma_y & \sin \Gamma_x & \cos \Gamma_x \cos \Gamma_y
\end{bmatrix} \quad (40)$$
In the above equations, $\phi_1$ represents the conventional Euler angle transformation from the inertial frame to the body frame, $\phi_2$ represents shaft tilt transformations, $\phi_3$ represents the transformation from the nonrotating shaft axis to the rotating shaft axis, $\phi_4$ through $\phi_6$ represent the rotor blade flap, lag, and pitch transformations. The $e_i$ vectors in equation (37) represent distances from the hub to a blade section in the appropriate frame. The vector,
The distance metric given by equation (36) models an articulated rotor with a flap-lag-pitch hinge sequence. An efficient method for computing the above strings of orthogonal transformation matrices was found by Miller and White [12] using exponential basis functions. Different hinge sequences can be easily accommodated using this methodology. If we assume the rigid body aircraft to be flying at constant linear and angular velocity and the Fourier coefficients of the rotor degrees of freedom to be constant, then the rotorcraft atmospheric turbulence distance metric is given by equation (47).

\[
\begin{align*}
    r_{\text{in/at}} &= S_1 \begin{bmatrix}
        V_{\text{mw}} & t \\
        0 & \\
        0 & 
    \end{bmatrix} \\
    S_1 &= \begin{bmatrix}
        \cos \psi_a \cos \theta_a & -\sin \psi_a & \cos \psi_a \sin \theta_a \\
        \sin \psi_a \cos \theta_a & \cos \psi_a & \sin \psi_a \sin \theta_a \\
        -\sin \theta_a & 0 & \cos \theta_a
    \end{bmatrix}
\end{align*}
\]
In order to compute equation (47), the rotation angles for the corresponding transformation matrices must be calculated. Since constant body rates have been assumed, the rigid body Euler angles can be written using equations (48), (49), and (50).

\[ \psi(t) = \psi_0 + \dot{\psi}_0 t \]  
\[ \theta(t) = \theta_0 + \dot{\theta}_0 t \]  
\[ \phi(t) = \phi_0 + \dot{\phi}_0 t \]  

Generally, we are interested in sustained maneuvers with given body rates, hence \( p_b, q_b, \) and \( r_b \) are specified. Equation (51) relates the Euler angular rates with the body angular rates.

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

(51)

Inserting equation (51) into equations (48) through (50) yields transcendental equations for the Euler angles.

\[ \psi(t) = \psi_0 + (q_b \sin \phi \sec \theta + r_b \cos \phi \sec \theta) t \]  
(52)
\[ \theta(t) = \theta_0 + (q_b \cos \phi - r_b \sin \phi) t \]  
(53)
\[ \phi(t) = \phi_0 + (p_b + q_b \sin \phi \tan \theta + r_b \cos \phi \tan \theta) t \]  
(54)

Upon inspection we see that only the bank angle, \( \phi(t) \), needs to be solved by iteration since the equations for heading angle, \( \psi(t) \), and pitch angle \( \theta(t) \), are not dependent upon themselves. Equation (54) can be solved by Newton's iteration method which is given by equation (55)
\[ \phi_{i+1}(t) = \phi_i(t) - \frac{f(t)}{f(t)} \]  

where, the quantities \( f(t) \) and \( f(t) \) are given by equations (56) and (57).

\[ f(t) = \phi - \phi_0 - (p_b + \sin \phi \tan \theta_b + \cos \phi \tan \theta_r) t \]  

\[ f(t) = 1 - (q_b t \cos \phi - r_b t \sin \phi) \tan \theta + (q_b t \sin \phi + r_b t \cos \phi)^2 \sec^2 \theta \]  

The above expression for the distance metric along with its components can be inserted into the correlation matrix given in equation (23) to form the rotorcraft atmospheric turbulence correlation matrix \( R(r_1, r_2, t_2, t_1) \).

We have previously mentioned that the correlation matrix for a rotor blade represents a cyclostationary process. A process is cyclostationary if and only if the mean function of the random process is periodic with a period \( T \),

\[ \mu_x(t + T) = E[x(t + T)] = E[x(t)] = \mu_x(t) \]  

and if the correlation matrix of the process is jointly periodic in \( t_2 \) and \( t_1 \),

\[ R_{xx}(t_2 + T, t_1 + T) = E[x(t_2 + T)x(t_1 + T)^*] = R_{xx}(t_2, t_1) \]  

Notice that it is possible for cyclostationary processes, satisfying equations (58) and (59), to have constant mean and variance yet have correlations which display periodic fluctuations.

In the simplified atmospheric turbulence example presented earlier, the rotating frame sampling model was realized by a time scale transformation,

\[ w_{turb}(t) = w_{turb}( \tilde{t}(t) ) \]  

which takes the general form given by equation (34).

\[ w_{turb}(t) = w_{turb}( at + p(t) ) \]
In equation (61), \( a \) is a constant and \( p(t) \) is a periodic function. If the input process to equation (61) is stationary with a constant mean function, then the resulting random process also has a constant mean function and variance.

\[
\begin{align*}
  m_{\text{w turb}}(t + T) &= E[w_{\text{turb}}(a(t + T) + p(t + T))] \\
  &= m_{\text{w input}}, \text{ a constant} \\
  &= m_{\text{w turb}}(t) \tag{62}
\end{align*}
\]

\[
R_{ww}(t_2 + T, t_1 + T) = E[w(a(t_2 + T) + p(t_2 + T)) w(a(t_1 + T) + p(t_1 + T))]
\]

\[
= E[w(a(t_2 + T) + p(t_2)) w(a(t_1 + T) + p(t_1))]
\]

\[
= R_{gg}(a(t_2 + T) + p(t_2) - a(t_1 + T) - p(t_1))
\]

\[
= R_{ww}(t_2, t_1) \tag{63}
\]

The variance is then given by equation (64).

\[
\sigma^2 = R_{ww}(t, t) - m_{\text{w turb}}^2 = R_{gg} - m_g^2, \text{ a constant} \tag{64}
\]

A realization of any random waveform which is transmitted or received from a body where the spatial frame of reference varies with respect to that of an observer of the random waveform will undergo a time scale transformation which is frequently referred to as a Doppler shift or Doppler effect. If the spatial variation is an oscillation in the sense of equation (61), and the original process is stationary, then the resultant Doppler shifted process is cyclostationary. The atmospheric turbulence correlation matrix has been derived from a Doppler shifted process in the form of equation (61) where the inner function represents the distance metric and the outer function represents the atmospheric turbulence field. Therefore, the mean and variance of the
atmospheric turbulence correlation matrix are constant regardless of the severity of the periodic fluctuations in the distance metric.

The correlation matrix has been derived by considering two arbitrary points in time, \( t_2 \) and \( t_1 \). To obtain differential equations excited by white noise which describe this statistical process a change of variables on the time coordinates must be made and is given by equation (65) and (66).

\[
\begin{align*}
\tau &= \frac{t_2 + t_1}{2} \\
\tau &= t_2 - t_1
\end{align*}
\]

The resulting correlation matrix is a function of \( \tau \) which represents the midpoint of the two arbitrary times and \( t \) which represents the time difference. In contrast to the previous correlation matrix with arguments \( t_2 \) and \( t_1 \), the correlation with coordinates \( t \) and \( \tau \) is a decaying function in \( \tau \) and periodic in \( t \). The atmospheric turbulence dynamic equations are determined by \( R \) as a function of \( \tau \), with the periodicity of the system being determined by the behavior of \( R \) with \( t \).

**CONCLUSIONS**

A model derived from first principles describing atmospheric turbulence velocities as experienced in the rotating frame of reference has been presented. The physics of the rotorcraft atmospheric turbulence problem are highlighted through an oversimplified problem which illustrates the basic differences between conventional atmospheric turbulence models and the more sophisticated rotating frame turbulence models. The time histories of gusts encountered by a rotor blade element and the rotor hub are very different when flying through identical gusts. The rotating frame sampling model will yield rotor blade gust velocities which have more energy at higher frequency, concentrated at integer multiplies of the rotor rotational speed, than the corresponding gust velocities using a body frame sampling model. To derive the rotating frame turbulence model, basic turbulence theory is reviewed and the known statistics on the atmosphere, suitable for engineering calculations, are given. The rotorcraft atmospheric turbulence correlation is derived for an arbitrary constant rate flight condition and is generalized to most rotorcraft configurations. The statistics in the rotating frame are found to be cyclostationary due to the periodic Doppler shift.
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ABSTRACT

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Title: Effects of Rotating Frame Turbulence and Dynamic Stall on Gust Response of Helicopter Blades
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Degree: Master of Science in Engineering
Year: 1990

Rotating frame turbulence, or RFT, refers to the actual turbulence experienced by the helicopter blades and requires noneulerian description and rotational sampling of measurements. In the stationary case of axial flight, its spectra has peaks centered at integer multiples of rotational speed $P$, as in wind turbines. In forward flight, its instantaneous or frequency-time spectra has split peaks centered at $P/2$, $P$, $3P/2$, $2P$ etc. Though nonstationary, it is wide sense cyclostationary in that its autocorrelation function $R(t_1, t_2) = R(t_1 + 2m\pi, t_2 + 2n\pi)$ for integers $m = n$ only. The major RFT characteristics --- spectral peaks, the consequent transfer of energy essentially from the low-frequency region ($\leq 1P$) to the high-frequency region ($>1P$) and cyclostationarity --- cannot be predicted by conventional space-fixed description. However, these characteristics are simultaneously predicted by the instantaneous spectra, and for their qualitative and parametric investigation, a closed-form solution of an instantaneous spectrum is presented for a space-fixed turbulence model. The RFT effects on the blade response statistics of rms values and average threshold crossing rates are presented as well. The blade model includes flap bending degrees of freedom and dynamic stall effects. The blade response statistics demonstrate that RFT effects are appreciable for low-advance ratio and low-altitude flight conditions.
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Chapter 1

Introduction

In the presence of turbulence, the flight dynamics requirements of helicopters are becoming increasingly demanding. Typical examples include nap-of-the-earth (NOE) maneuvers, lighthouse and offshore oil-platform missions and ship landing [1-3]. Turbulence also affects the flight dynamics of other air vehicles that combine the hovering efficiency of helicopters with the cruising efficiency of airplanes [4,5]. The tilt rotor “chugging” problem, which is basically a fore-to-aft low-frequency acceleration of the rigid body mode coupling with the rotor torque mode, is a case in point. It occurs at relatively low speeds of about 150 knots and aggravates during descent [5]. Although turbulence may not be the dominant issue in helicopter structural dynamics, the loads on some rotor components are sensitive to gust-induced loads, e.g. pitch-link loads when the rotor is stalled [6]. Such cases require that measurements and predictions based on smooth air are isolated from those based on random sources of turbulence. Recently, particularly at low altitudes and low speeds there has been an improved appreciation of turbulence effects concerning
both flight dynamics (low-frequency response and stability, pilot's work load and flight control systems [1, 7-9]), and structural dynamics (fatigue, ride quality and vibrations [4, 10-12]).

An earlier study [12], with reference to high-speed compound helicopters (nondimensional speed \( \mu > 0.8 \)), treats RFT covariance functions, demonstrates the non-stationarity and concludes that RFT effects on blade response are negligible. It must be emphasized that Ref. 12 neglects the effects of flow velocity in the axial direction on the turbulence correlation distance and that the RFT, though nonstationary in forward flight, is stationary in axial flight, as in wind turbines. Moreover, the time-domain treatment of Ref. 12 cannot predict the frequency-dependent RFT properties of occurrence of peaks and the consequent transfer of energy from the low-frequency region to the high-frequency region (redistribution of spectra to higher frequencies). For such high-speed conditions (\( \mu > 0.8 \)), the correlation distance is virtually determined by the dominant flight speed; contributions from the rotational and axial-flow velocities are negligible, and RFT ceases to be an issue. In contrast, investigations of wind turbines have shown dominant RFT effects on turbulence characteristics and turbine response [13,14]. Recently, Ref. 15 presents a generalized RFT model (autocorrelation function) that is valid in axial as well as in forward flight. Such a model qualitatively explains the contrasting findings of negligible RFT effects on turbulence modeling for high-speed flights and dominant RFT effects for wind turbines and low-speed helicopters (\( \mu \leq 0.3 \)).

For conventional helicopters (\( \mu < 0.4 \)), the RFT effects can be strikingly different from those of space-fixed turbulence, and these differences increase with decreasing
speed $\mu$ and turbulence scale length $L/R$ [15]. However, in Ref. 15, the treatment of RFT characteristics of occurrence of peaks and transfer of energy with respect to frequencies is restricted to stationary hovering conditions. The spectral density description of Ref. 15 is the conventional description done in wind turbine studies and is not applicable to the nonstationary forward-flight conditions, nor is it feasible to predict frequency-dependent RFT effects from the covariance-function description therein. Although in earlier studies [12, 15, 16] the nonstationarity of RFT is emphasized, the special properties of this nonstationarity have not been demonstrated. That demonstration and a method of predicting simultaneously the RFT characteristics with respect to both time and frequency form a major part of this investigation.

Ref. 3, a follow-up to Ref. 15, demonstrates RFT effects on low-frequency flight dynamics response in hover, including corroboration by the GENHEL flight dynamics simulation program. As shown in Ref. 3, RFT effects increase with decreasing values of turbulence scale length $L/R$. It is emphasized that $L$ decreases with decreasing altitude, and that near obstacles, $L$ can have values comparable to the rotor radius $R$ [1].

In forward flight, a stochastic description of RFT properties is demanding since it is necessary to predict both the nonstationarity and the properties of this nonstationarity with respect to time as well as the occurrence of peaks and transfer of energy with respect to frequencies. Accordingly, the instantaneous spectra of RFT are formulated, including a closed-form solution for qualitative and parametric studies. In this context, a few comments are in order concerning the instantaneous
spectra vis-a-vis covariance functions and double-frequency spectra as a means of
describing RFT. This will also provide a better appreciation of the novelty of formu-
lating RFT by the instantaneous spectra. The double-frequency spectrum, though
a mathematically equivalent description of the covariance, has a diminished physi-
cal meaning with its real co-spectrum and imaginary quad-spectrum; and it cannot
predict the special properties of RFT nonstationarity with respect to time. Sim-
ilarly, the frequency-dependent properties of peaks and energy transfer cannot be
predicted by the covariance-function description used in earlier studies [4, 11, 12,
15]. On the other hand, both the time- and frequency-dependent properties of RFT
can be simultaneously predicted by the instantaneous spectrum, which is physi-
cally realizable since it represents the distribution of energy in the frequency-time
domain. Moreover, the RFT autocorrelation functions, as presented in Ref. 15
for example, show that the RFT characteristics are strikingly different from the
space-fixed turbulence characteristics for low speed $\mu$ and turbulence scale length
$L/R$ values. Therefore, it is important to investigate how these differences affect
response statistics; hence RFT effects on the blade response statistics of rms values
and average threshold crossing rates are investigated as well.
Chapter 2

Modeling Turbulence

Turbulence is typically modeled on the assumption that it is uniform over the rotor disk, with turbulence at the hub taken as representative. Conventional eulerian or space-fixed description of turbulence is assumed satisfactory [4, 12]. Further, the rotor disk turbulence is represented essentially by the free atmospheric turbulence, with negligible contributions from self-induced turbulence, such as that due to random shedding of trailing vortices [17]. Therefore, as is done in the study of airplanes, the Taylor’s hypothesis or frozen-turbulence approximation is invoked, and the turbulence excitation at a blade station is represented by the eulerian-description models, the von Karman model being the baseline model (Taylor-von Karman theory). To put this investigation in perspective, we first review the characteristics of RFT vis-a-vis space-fixed turbulence, including experimental validation.

Experimental and analytical investigations of RFT effects on wind turbines are fairly extensive [13, 14]. Fig. 2.1 from Ref. 14 includes the predicted spectral density of the longitudinal or fore-to-aft turbulence velocity at a blade station according
Fig. 21: Spectrum of longitudinal wind speed fluctuations, as seen from a rotating blade station (RFT), compared with the space-fixed spectrum.
to the von Karman model. Some minor quantitative differences between the actual Bessel function model used in Ref. 14 and the von Karman Bessel function model are bypassed. The fore-to-aft component, perpendicular to the turbine plane, is the most dominant component, as is the axial or vertical component for rotorcraft. To capture the RFT effects, the data-base was generated from the noneulerian sampling of measurements along a circular array of effectively rotating anemometers (rotational sampling), for details, see Refs. 13 and 14. The monotonically decaying behaviour is exhibited by the conventional eulerian description, which fails to capture the two basic features of data: transfer of energy from the essentially low-frequency region (<1P; P: rotational speed) to the higher-than-1P region and the occurrence of peaks at 1P, 2P etc. By comparison, the RFT description dramatically improves the correlation. The correlation also shows the overall viability of the Taylor-von Karman theory. Fig. 2.2 from Ref. 18 also shows that the von Karman model agrees well with the other two models due to Frost and Kaimal, which are representative of a wide range of turbulence models used, respectively, in flight vehicles and meteorology, wind turbines and structural engineering.

A comprehensive RFT analysis that explains the physics of turbulence in the rotor disk with respect to parameters such as advance ratio $\mu$ and turbulence scale length $L/R$ is presented in this chapter. In the specific context of turbulence modeling for rotorcraft applications at low altitudes and low speeds ($\mu < 0.3$), assuming the validity of the Taylor-von Karman theory is reasonable. There is no test data to refute this assumption, which in fact, is used in virtually every study on rotorcraft response to turbulence [4, 11, 15]. This means, as noted earlier, that in addition to
Fig. 2.2: Turbulence PSD models by Frost, Kaimal and von Karman
the dominance of free atmospheric turbulence, there is also the presence of a mean convective turbulence velocity that justifies the frozen-turbulence approximation. In the present development, Taylor-von Karman theory is assumed valid even for the hovering case, in which the frozen-turbulence approximation is far less plausible. This is done mainly for two reasons. First, the spectral density results in hover, both of stationary RFT and vehicle response, can be viewed as a continuation of instantaneous spectral density results in forward flight in the frequency-time plane for discrete time values. Second, the results as presented here under the stationary conditions in hover are qualitatively similar to those in wind turbines. Thus, the RFT and vehicle-response spectral density results in hover provide a natural transition to and better appreciation of nonstationary RFT and vehicle-response results in forward flight. The results also bring out the similarity with the experimental and analytical wind turbine studies.

2.1 Modeling Approach

The vertical turbulence velocity perpendicular to the rotor disk is the dominant component compared to the components in the flight (longitudinal) and lateral directions, as is the longitudinal component for HAWT (Horizontal Axis Wind Turbines). For simplicity, we consider only the autocorrelation of this vertical component at an isolated blade station, say 0.7R from the hub center. The schematic is shown in Fig. 2.3(a), where \(0.7\Omega R \sin \Omega t\) and \(0.7\Omega R \cos \Omega t\) are the periodic rotational velocity components in the X or flight direction and in the Y or lateral
Fig. 2-3: Rotor in forward flight (momentum theory)
direction, respectively. The Z or shaft direction follows the right-handed and rotor-fixed XYZ system. With respect to the space-fixed vertical axis, the rotor shaft is tilted forward in the flight direction with a positive rotor disk angle of attack $\alpha_s$. Fig. 2.3(b) shows the flight velocity component $V \sin \alpha_s$, and induced velocity $v$ due to thrust, in the Z direction. Therefore, for a given flight speed $V$ and induced velocity $v$, we introduce the advance ratio $\mu$ and the total induced velocity $\lambda$ as follows:

$$\mu = \frac{V \cos \alpha_s}{\Omega R} \quad (2.1)$$

$$\lambda = \frac{V \sin \alpha_s + v}{\Omega R} = \mu \tan \alpha_s + \lambda_i \quad (2.2)$$

From the actuator-disk momentum theory, we can relate the dimensionless induced velocity $\lambda_i$ due to thrust $C_T$ and the total induced velocity $\lambda$ as follows:

$$\lambda_i = \frac{C_T}{2\sqrt{\mu^2 + \lambda^2}} \quad (2.3)$$

In the time interval $t_2 - t_1$, the spatial separation $r$ of the 0.7 R blade station can be written in terms of the following velocity components (see Fig. 2.3):

$$\frac{dX}{dt} = V + 0.7\Omega R \sin \Omega t \quad (2.4)$$

$$\frac{dY}{dt} = 0.7\Omega R \cos \Omega t \quad (2.5)$$

$$\frac{dZ}{dt} = U \quad (2.6)$$

In Eq. 2.6, $U$ represents the total mean air-flow velocity perpendicular to the rotor disk in the Z direction due to axial flight velocity and downwash. In the present analysis, we take $U$ as equal to $k\lambda_i\Omega R$ ($k$: a scaling constant) in axial
flight and $k\lambda \Omega R$ in forward flight for computational purposes (also see comments on the applicability of Taylor's frozen-wave hypothesis at the beginning of this chapter). Following the earlier studies of modeling turbulence for wind turbines and helicopters, most of which are based on the Taylor-von Karman hypothesis, we also assume that the autocorrelation function $R_w(t_1, t_2)$ of the vertical turbulence $w(t)$ is a function of $r$ only:

$$R_w(t_1, t_2) = \sigma_w^2 g(r) \quad (2.7)$$

where $\sigma_w$ is the rms value or intensity of vertical turbulence and

$$r = \left[ \left( X(t_2) - X(t_1) \right)^2 + \left( Y(t_2) - Y(t_1) \right)^2 + \left( Z(t_2) - Z(t_1) \right)^2 \right]^{\frac{1}{2}} \quad (2.8)$$

Thus, $r$ can be calculated using Eqs. 2.4 to 2.6.

We present three models of autocorrelation function in the next section. They will be referred to as Exponential, Dryden [19] and Rosenbrock [20] models. Before proceeding to find the autocorrelation functions of these three models, we need to determine the ratio $r/L$. Nondimensionalizing with time unit $1/\Omega$ and tip speed $\Omega R$, we obtain

$$\tilde{t}_1 = \Omega t_1, \quad \tilde{t}_2 = \Omega t_2, \quad \tilde{w} = w/\Omega R, \quad \tilde{U} = U/\Omega R \quad (2.9)$$

Setting

$$\frac{V \cos \alpha}{\Omega R (L/2) R} = \frac{\mu}{(L/2) R} = \frac{2\mu}{(L/R)} = a \quad (2.10)$$

we obtain

$$\frac{r}{L} (\tilde{t}_1, \tilde{t}_2) = \frac{1}{2} \left[ \left\{ a(\tilde{t}_2 - \tilde{t}_1) - 1.4 (R/L) \cos \tilde{t}_2 + 1.4 (R/L) \cos \tilde{t}_1 \right\}^2 \right. \]$$
We observe that
\[
\frac{r}{L}(\tilde{t}_1, \tilde{t}_2) = \frac{r}{L}(\tilde{t}_1 + 2m\pi, \tilde{t}_2 + 2n\pi) \text{ for integer } m = n \text{ only} \quad (2.12)
\]
Eq. 2.12 will be used to study the periodic structure of the autocorrelation function in the next section.

For simplicity of notation, we introduce
\[
\tilde{t}_2 - \tilde{t}_1 = \tau, \quad \frac{(\tilde{t}_1 + \tilde{t}_2)}{2} = t, \quad c = 1.4\frac{R}{L} \text{ and } b = (2\frac{R}{L})\bar{U} \quad (2.13)
\]
Then the ratio \( r/L \) simplifies to
\[
\frac{r}{L}(t, \tau) = \frac{1}{2}[r^2(a^2 + b^2) + 4c\sin r/2(c\sin r/2 + a\sin t)]^{\frac{1}{2}} \quad (2.14)
\]
In axial flight (e.g. hovering), we have \( \mu = 0 \) and consequently \( a = 0 \). Hence, with RFT effects,
\[
\frac{r}{L} = \frac{1}{2}[r^2b^2 + 4c^2\sin^2 r/2]^{\frac{1}{2}} \quad (2.15)
\]
Neglecting RFT effects,
\[
\frac{r}{L} = \frac{1}{2}[r^2b^2]^{\frac{1}{2}} \quad (2.16)
\]
The ratio \( r/L \) given by Eq. 2.14 can be used in any space-fixed turbulence model for which Eq. 2.7 holds (Taylor's hypothesis), as illustrated in the next section.
2.2 Turbulence Models

For the vertical turbulence velocity \( w(t) \), the Exponential, Dryden and Rosenbrock models are, respectively, given by Eqs. 2.17 to 2.19:

\[
R_w(t, \tau) = \sigma_w^2 e^{-|\tau|/(L/2)} \]  
(2.17)

\[
R_w(t, \tau) = \sigma_w^2 \left(1 - \frac{\tau}{2L}\right) e^{-\tau/L} \]  
(2.18)

\[
R_w(t, \tau) = \sigma_w^2 e^{-4\tau^2/L^2} \]  
(2.19)

where \( L \) is the scale length of the longitudinal turbulence. We note that \( L/2 \) is the turbulence scale length of the vertical component \( w(t) \) [19].

For example, the autocorrelation function of the nonstationary RFT, for the Exponential model, can be written from Eq. 2.14 as

\[
R_w(t, \tau) = \sigma_w^2 \exp \left\{-[\tau^2(a^2 + b^2) + 4c\sin \tau/2(c\sin \tau/2 + a\tau \sin t)]^{\frac{1}{2}} \right\} \]  
(2.20)

which is nonstationary. The autocorrelation function of the stationary RFT for the Exponential model can be written from Eq. 2.15 as

\[
R_w(\tau) = \sigma_w^2 \exp \left\{-[\tau^2b^2 + 4c^2\sin^2 \tau/2]^{\frac{1}{2}} \right\} \]  
(2.21)

Thus \( R_w(\tau) \) is stationary. The RFT contribution is reflected by the \( \sin^2 \tau/2 \) term in Eq. 2.21. Neglecting RFT effects, Eq. 2.21 simplifies to

\[
R_w(\tau) = \sigma_w^2 e^{-b|\tau|} \]  
(2.22)

and the corresponding spectral density function is

\[
S_w(f) = \sigma_w^2 \frac{b}{\pi} \frac{1}{b^2 + (2\pi f)^2} \]  
(2.23)
As a consequence of Eqs. 2.7 and 2.12, we have

\[ R_w(t_1 + 2m\pi, t_2 + 2n\pi) = R_w(t_1, t_2) \quad \text{only for} \quad m = n \quad (2.24) \]

Thus, the process is wide sense cyclostationary on the diagonal of the \((t_1, t_2)\) plane [21]. However, it must be emphasized that the process is not mean square periodic since the above equation does not hold for \(m \neq n\). From Eq. 2.20, the process is also not weakly stationary, since \(R_w(t_1, t_2)\) is not a function of \(t_2 - t_1\) only.

### 2.3 Results and Discussion

We present the characteristics of RFT over a comprehensive range of flight regimes comprising axial flight (\(\mu = 0\)), transitional flight (\(\mu \leq 0.1\)), and steady cruising when the advance ratio \(\mu\) is about 0.3 for pure rotorcraft and \(\mu \geq 0.6\) for high-speed compound rotorcraft. The emphasis here is to compare the characteristics of RFT with those of space-fixed turbulence and to study their possible impact on blade response. To do that, two parameters have been chosen — \(L/R\) and \(\mu\). The assumed values of other parameters are as follows: shaft tilt \(\alpha_s = 8^\circ\) and thrust coefficient \(C_T = 0.006\). Turbulence intensity \(\sigma_w\) and mean turbulence scaling parameter \(k\) are taken as unity.

We begin with the simpler hover (\(\mu = 0\)) cases since the RFT is a stationary process (see Eqs. 2.21 to 2.23). The spectral density results are presented as \(f^*S(f)\) versus \(f\) for a better presentation of low-frequency results. It is noteworthy that the area under the curve \(\int f S(f)d(\ln f)\) gives the variance, as in the conventional \(S(f)\)-
versus-\(f\) representation. The spectral density functions are computed numerically using a Fast Fourier Transform routine. Figs. 2.4 to 2.6 are based on the exponential model, Eq. 2.17, and they show the spectral density functions along with their autocorrelation functions. In these figures, the solid lines refer to RFT and dotted lines refer to space-fixed turbulence. We consider cases with \(L/R = 1\) and 4 in Figs. 2.4 and 2.5, respectively, representing typical low-altitude hovering. The upper-left figures are the autocorrelation functions, whose Fourier transforms are the spectral density functions, presented as the main figure. The upper-right figure gives the spectral density function in the conventional \(S(f)\)-versus-\(f\) representation. The important feature in Figs. 2.4 and 2.5 is that the RFT shows high peak values at frequencies of \(1P, 2P\) as compared to the monotonically decaying feature of the space-fixed turbulence. Since the total energy of the turbulence spectrum remains the same in the rotating system, the energy decrease at lower-than-\(1P\) frequencies is balanced by the energy increase at the peaks. Thus, it can be seen that there is considerable transfer of energy from the low-frequency (\(\leq 1P\)) region to the high-frequency (\(> 1P\)) region. The RFT has less energy when compared to space-fixed turbulence below \(1P\) frequencies. Fig. 2.6 represents results for \(L/R=12.0\), say, under typical steady cruising conditions. It shows that the transfer of energy and the dominance of peaks at higher frequencies are more marked with decreasing values of \(L/R\). As \(L/R\) increases, the difference between RFT and space-fixed turbulence decreases. This diminishing effect of RFT is evident from autocorrelation functions represented as upper-left inset figures in Figs. 2.4 to 2.6. The same characteristics can be observed with the Dryden and Rosenbrock models (Figs. 2.7 and 2.8).
Fig. 2.4: RFT effects on space-fixed exponential model for the vertical turbulence at 0.7R blade station
Fig. 2.5: RFT effects on space-fixed exponential model for the vertical turbulence at 0.7R blade station
Fig. 2.6: RFT effects on space-fixed exponential model for the vertical turbulence at 0.7R blade station
Fig. 2.7: RFT effects on space-fixed Dryden model for the vertical turbulence at 0.7R blade station
Fig. 2.8: RFT effects on space-fixed Rosenbrock model for the vertical turbulence at 0.7R blade station
In forward flight, the RFT becomes nonstationary though the space-fixed turbulence is stationary. This can be seen, for example, by comparing Eq. 2.20 with Eq. 2.21. As before, we first consider the Exponential model to explain the characteristics of RFT in forward flight. Fig. 2.9 represents a very low-speed ($\mu=0.05$) transition flight. The upper inset Fig. 2.9(a) shows the rapidly decaying nature of space-fixed turbulence. Fig. 2.9 also shows the increasing effect of RFT as $L/R$ decreases. Fig. 2.10 gives the autocorrelation function in forward flight for $\mu=0.2$ and 0.8, both for $L/R=4$. While the first case, $\mu=0.2$, refers to a typical cruising condition of a low-speed conventional helicopter, the second case refers to a high-speed compound helicopter. It can be seen from Figs. 2.10(a) and (b) that RFT effects rapidly decrease as $\mu$ increases. In the present analysis, the rotational speed $\Omega$ is held constant, as is generally the case for helicopters in steady cruising. With increasing $\mu$, the flight speed increases and the contribution of rotational velocity $\Omega R$ becomes less dominant. This explains why RFT effects were found to be negligible for high-speed compound helicopters [16]. Figs. 2.9 and 2.10 show that RFT is significant for low-advance-ratio ($\mu \leq 0.3$) and low-altitude conditions ($L/R \approx 4$). The characteristics observed earlier for the Exponential model are typified in Figs. 2.11 and 2.12, which are based on the Dryden and Rosenbrock models, respectively.
Fig. 2.9: RFT in transition flight for exponential model
Fig. 2.10: RFT in forward flight for exponential model

\[ \mu=0.2, \ L/R=4 \]

\[ \mu=0.8, \ L/R=4 \]
Fig. 2.11: RFT in forward flight for Dryden model
Fig. 2.12: RFT in forward flight for Rosenbrock model
Chapter 3

Instantaneous Spectrum

3.1 Spectral Analysis

The instantaneous or frequency-time spectral density function $S_w(f,t)$ provides a means of describing both the energy transfer with respect to frequency $f$ and the periodically varying nonstationarity with respect to time $t$. To this end, the auto-correlation function $R_w(t,\tau)$ can be expressed as the inverse Fourier transform of $S_w(f,t)$ [22, 23]:

$$R_w(t,\tau) = E \left[ w(t - \frac{\tau}{2})w(t + \frac{\tau}{2}) \right] = \int_{-\infty}^{\infty} S_w(f,t)e^{i2\pi f\tau} df$$

(3.1)

where $E$ represents the averaging or expectation operation. For $\tau=0$,

$$\sigma_w^2(t) = E \left[ w^2(t) \right] = R_w(t,0) = \int_{-\infty}^{\infty} S_w(f,t)dt \geq 0$$

(3.2)

Thus, $S_w(f,t)$ is real valued, a noteworthy feature. Since the variance $\sigma_w^2(t)$ can be identified as a measure of turbulence energy, the instantaneous spectrum $S_w(f,t)$ in Eq. 3.2 shows the distribution of energy in the (f-t) plane. Further, $S_w(f,t)$ is
physically realizable when compared to the conventional double-frequency spectrum $S_w(f_1, f_2)$, which, having both a real part (co-spectrum) and imaginary part (quad spectrum), has diminished physical meaning. From Eq. 3.1,

$$S_w(f, t) = \int_{-\infty}^{\infty} R_w(t, \tau) e^{-i2\pi f \tau} d\tau$$

(3.3)

Since $R_w(t, \tau)$ is an even function of $\tau$, Eq. 3.3 simplifies to

$$S_w(f, t) = 2 \int_{0}^{\infty} R_w(t, \tau) \cos 2\pi f \tau d\tau$$

(3.4)

With $t$ replaced by $(t_1 + t_2)/2$, and as a consequence of Eq. 2.24, we have

$$S_w(f, t) = S_w(f, t + 2m\pi)$$

Hence,\n
$$S_w(f, t) = S_w(f, t + 2m\pi)$$

(3.5)

Eq. 3.5 shows that $S_w(f, t)$ is periodic with respect to time.

### 3.2 Closed-Form Solutions

In Section 2.2 we introduced three space-fixed turbulence models whose autocorrelation functions are given by Eqs. 2.17, 2.18 and 2.19, respectively. The autocorrelation functions of the corresponding RFT models are obtained when $r/L$ in these equations is substituted from Eq. 2.14. The Rosenbrock model permits a closed-form solution of its instantaneous spectral density $S_w(f, t)$ with RFT effects and qualitatively agrees with the von Karman model [13]. For continuity of presentation, we begin with Eq. 2.19, which defines the autocorrelation function of the Rosenbrock model:
Substituting for \( r/L \) from Eq. 2.14, we get

\[
R_w(t, \tau) = \exp \left\{ -(a^2 + b^2) \tau^2 + 4c^2 \sin^2 \frac{T}{2} + 4ac \sin t \tau \sin \frac{T}{2} \right\} \tag{3.6}
\]

Similarly, for the stationary case, as defined in Eq. 2.15, we have

\[
R_w(\tau) = \exp \left\{ - \left[ b^2 \tau^2 + 4c^2 \sin^2 \frac{T}{2} \right] \right\} \tag{3.7}
\]

### 3.2.1 Stationary Case

The spectral density function for the stationary case can be derived as follows:

\[
S(\omega) = \frac{2}{\pi} \int_0^\infty \exp \left\{ - \left[ b^2 \tau^2 + 4c^2 \sin^2 \frac{T}{2} \right] \right\} \cos \omega \tau d\tau
\]

\[
= \frac{2}{\pi} \int_0^\infty \exp \left\{ -b^2 \tau^2 \right\} \exp \left\{ -4c^2 \sin^2 \frac{T}{2} \right\} \cos \omega \tau d\tau
\]

\[
= \frac{2}{\pi} \int_0^\infty e^{-b^2 \tau^2} \cos \omega \tau \left\{ 1 - (4c^2) \sin^2 \frac{T}{2} + \frac{1}{2!} (4c^2)^2 \sin^4 \frac{T}{2} - \cdots \right\} d\tau
\]

\[
= \frac{2}{\pi} \int_0^\infty e^{-b^2 \tau^2} \cos \omega \tau \sum_{r=0}^\infty \frac{(-1)^r}{r!} (4c^2)^r \sin^{2r} \frac{T}{2} d\tau
\]

\[
= \frac{2}{\pi} \sum_{r=0}^\infty \frac{(-1)^r}{r!} (4c^2)^r \int_0^\infty e^{-b^2 \tau^2} \cos \omega \tau \sin^{2r} \frac{T}{2} d\tau \tag{3.8}
\]

We have

\[
\cos \omega \tau \sin^{2r} \frac{T}{2} d\tau = \left( \frac{e^{i\omega \tau} + e^{-i\omega \tau}}{2} \right) \left( \frac{e^{i\tau/2} - e^{-i\tau/2}}{2i} \right)^{2r}
\]

\[
= \left( \frac{e^{i\omega \tau} + e^{-i\omega \tau}}{2} \right) \sum_{s=0}^{2r} \frac{(-1)^s(2r)! \left( e^{i\tau/2} \right)^{2r-s} \left( e^{-i\tau/2} \right)^s}{(2i)^{2r}}
\]

\[
= \sum_{s=0}^{2r} \frac{(-1)^{r+s}(2r)!}{2^{2r+1}(2r-s)!s!} \left\{ \exp \left[ i(r - s + \omega)\tau \right] + \exp \left[ i(r - s - \omega)\tau \right] \right\}
\]
With $p = r - s$, we have

$$
\cos \omega \tau \sin^{2r} \frac{\tau}{2} \, d\tau = \sum_{p=-r}^{r} \frac{(-1)^p(2r)!}{2^{2r+1}(r + p)!(r - p)!} \times \\
\times \{ \exp [i(p + \omega)\tau] + \exp [i(p - \omega)\tau] \}
$$

$$
= \sum_{p=-r}^{r} \frac{(-1)^p(2r)!}{2^{2r}(r + p)!(r - p)!} \cos(p + \omega)\tau
$$

$$
= \sum_{p=0}^{r} \frac{(-1)^p(2r)!T(p)}{2^{2r}(r + p)!(r - p)!} \{ \cos(\omega + p)\tau + \cos(\omega - p)\tau \}
$$

where

$$
T(p) = \begin{cases} 
\frac{1}{2}, & p = 0 \\
1, & p = 1, 2, 3, \ldots 
\end{cases}
$$

Substituting the above expression in $S(\omega)$, we get

$$
S(\omega) = \frac{2}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left( \frac{4\alpha^2}{r} \right)^r \sum_{p=0}^{r} \frac{(-1)^p(2r)!T(p)}{2^{2r}(r + p)!(r - p)!} \times \\
\times \int_{0}^{\infty} e^{-\beta^2 \tau^2} \{ \cos(\omega + p)\tau + \cos(\omega - p)\tau \} \, d\tau
$$

$$
= \frac{2}{\pi} \sum_{r=0}^{\infty} \sum_{p=0}^{r} \frac{(-1)^{r+p}(2r)!}{r!2^{2r}(r + p)!(r - p)!} \left( \frac{4\alpha^2}{r} \right)^r T(p) \times \\
\times \int_{0}^{\infty} e^{-\beta^2 \tau^2} \{ \cos(\omega + p)\tau + \cos(\omega - p)\tau \} \, d\tau 
$$

(3.9)

From the identity

$$
\int_{0}^{\infty} e^{-\alpha^2 \tau^2} \cos \beta \tau d\tau = \frac{\sqrt{\pi}}{2\alpha} e^{-\beta^2/4\alpha^2}
$$

Eq. 3.9 simplifies to

$$
S(\omega) = \frac{2}{\pi} \sum_{r=0}^{\infty} \sum_{p=0}^{r} \frac{(-1)^{r+p}(2r)!}{r!2^{2r}(r + p)!(r - p)!} \left( \frac{4\alpha^2}{r} \right)^r T(p) \sqrt{\pi} \frac{\sqrt{\pi}}{2\alpha} \times
$$
\[
\times \left\{ \exp \left[ -\frac{(\omega + p)^2}{4b^2} \right] + \exp \left[ -\frac{(\omega - p)^2}{4b^2} \right] \right\}
\]
\[
= \frac{1}{b\sqrt{\pi}} \sum_{p=0}^{\infty} \sum_{r=p}^{\infty} \frac{(-1)^r p!(2r)!}{r!2^r(r + p)!(r - p)!} \times
\times \left\{ \exp \left[ -\frac{(\omega + p)^2}{4b^2} \right] + \exp \left[ -\frac{(\omega - p)^2}{4b^2} \right] \right\}
\]

With \( r - p = q \), the above expression reduces to
\[
S(\omega) = \frac{1}{b\sqrt{\pi}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^q (2p + 2q)! (4c^2)^r T(p)}{2^{2r + 2q}(p + q)!(2p + q)!q!} \times
\times \left\{ \exp \left[ -\frac{(\omega + p)^2}{4b^2} \right] + \exp \left[ -\frac{(\omega - p)^2}{4b^2} \right] \right\}
\]

This treatment, essentially that given in Ref. 20 sets the stage for the generalized treatment of the nonstationary instantaneous spectrum, as presented in the next subsection.

### 3.2.2 Nonstationary Case

The RFT autocorrelation function according to Eq. 2.19 (Rosenbrock model) is given by Eq. 3.6. For simplicity of notation, let
\[ a^2 + b^2 = A^2, \quad 4c^2 = B^2, \quad \text{and} \quad 4ac \sin t = C^2 \]

Therefore,
\[
R_{\omega}(t, \tau) = \exp \left\{ - \left[ A^2 \tau^2 + B^2 \sin^2 \frac{\tau}{2} + C^2 \tau \sin \frac{\tau}{2} \right] \right\}
\]

The corresponding instantaneous spectral density function from Eq. 3.4 is
\[
S(\omega, t) = \frac{2}{\pi} \int_0^\infty \exp \left\{ - \left[ A^2 \tau^2 + B^2 \sin^2 \frac{\tau}{2} + C^2 \tau \sin \frac{\tau}{2} \right] \right\} \cos \omega \tau \, d\tau
\]
\[
= \frac{2}{\pi} \int_0^\infty \exp \left\{-A^2 \tau^2 \right\} \exp \left\{- \left[ B^2 \sin^2 \frac{\tau}{2} + C^2 \tau \sin \frac{\tau}{2} \right] \right\} \cos \omega \tau \, d\tau \quad (3.10)
\]

In Eq. 3.10, \( \exp \left\{- \left[ B^2 \sin^2 \frac{\tau}{2} + C^2 \tau \sin \frac{\tau}{2} \right] \right\} \) can be expressed as

\[
= 1 - \left( B^2 \sin^2 \frac{\tau}{2} + C^2 \tau \sin \frac{\tau}{2} \right) + \frac{1}{2!} \left( B^2 \sin^2 \frac{\tau}{2} + C^2 \tau \sin \frac{\tau}{2} \right)^2 - \cdots
\]

\[
= \sum_{r=0}^\infty \frac{(-1)^r}{r!} \left( B^2 \sin^2 \frac{\tau}{2} + C^2 \tau \sin \frac{\tau}{2} \right)^r
\]

\[
= \sum_{r=0}^\infty \frac{(-1)^r}{r!} \sum_{s=0}^r \frac{r!}{(r-s)!s!} \left( C^2 \tau \sin \frac{\tau}{2} \right)^{r-s} \left( B^2 \sin^2 \frac{\tau}{2} \right)^s
\]

Substituting this expression in Eq. 3.10, we get

\[
S(\omega, t) = \frac{2}{\pi} \sum_{r=0}^\infty \sum_{s=0}^r \frac{(-1)^r}{(r-s)!s!} B^{2s} C^{2r-2s} \int_0^\infty \tau^{r-s} e^{-A^2 \tau^2} \cos \omega \tau \sin^{r+s} \frac{\tau}{2} \, d\tau
\]

\[
= \frac{2}{\pi} \sum_{s=0}^\infty \sum_{q=0}^\infty \frac{(-1)^{s+q}}{q!s!} B^{2s} C^{2q} \int_0^\infty \tau^q e^{-A^2 \tau^2} \cos \omega \tau \sin^{s+q} \frac{\tau}{2} \, d\tau
\]

With \( q = r - s \), we get

\[
S(\omega, t) = \frac{2}{\pi} \sum_{s=0}^\infty \sum_{q=0}^\infty \frac{(-1)^{s+q}}{q!s!} B^{2s} C^{2q} \int_0^\infty \tau^q e^{-A^2 \tau^2} \cos \omega \tau \sin^{s+q} \frac{\tau}{2} \, d\tau \quad (3.11)
\]

Let \( q + 2s = 2u \); then,

\[
\cos \omega \tau \sin^{s+q} \frac{\tau}{2} = \cos \omega \tau \sin^{2u} \frac{\tau}{2}
\]

\[
= \left( \frac{e^{i\omega \tau} + e^{-i\omega \tau}}{2} \right) \left( \frac{e^{i\tau/2} - e^{-i\tau/2}}{2i} \right)^{2u}
\]

\[
= \left( \frac{e^{i\omega \tau} + e^{-i\omega \tau}}{2} \right)^{2u} \sum_{g=0}^{2u} \frac{(-1)^g (2u)!}{(2u-g)!g!} \left( \frac{e^{i\tau/2}}{2i} \right)^{2u-g} \left( \frac{e^{-i\tau/2}}{2i} \right)^g
\]

Therefore,
\[
\cos\omega\tau \sin^{q+2s} \frac{\tau}{2} = \sum_{g=0}^{2u} \frac{(-1)^{g+u} (2u)!}{2^{2u+1} (2u-g)! g!} \times \\
\times \left\{ \exp \left[ i(u-g+\omega)\tau \right] + \exp \left[ i(u-g-\omega)\tau \right] \right\}
\]

Substituting \( p = u - g \), we have

\[
\cos\omega\tau \sin^{2u} \frac{\tau}{2} = \sum_{p=-u}^{u} \frac{(-1)^p (2u)!}{2^{2u} (u+p)! (u-p)!} \times \\
\times \left\{ \frac{\exp \left[ i(p+\omega)\tau \right] + \exp \left[ i(p-\omega)\tau \right]}{2} \right\}
\]

Let

\[
T(q_1) = 1, \quad T(q_2) = 0, \quad \text{for } q = 0, 2, 4, 6, \ldots \text{ even}
\]

\[
T(q_1) = 0, \quad T(q_2) = 1, \quad \text{for } q = 1, 3, 5, 7, \ldots \text{ odd}
\]

Therefore,

\[
\cos\omega\tau \sin^{2u} \frac{\tau}{2} = \sum_{p=-u}^{u} \frac{(2u)!}{2^{2u} (u+p)! (u-p)!} \times \\
\times \left\{ (-1)^p T(q_1) \cos(\omega + p)\tau + (-1)^\frac{2q+1}{2} T(q_2) \sin(\omega + p)\tau \right\}
\]

\[
= \sum_{p=-(\frac{s}{2})}^{p=(\frac{s}{2}+s)} \frac{(q + 2s)!}{2^{q+2s} (\frac{s}{2} + s + p)! (\frac{s}{2} + s - p)!} \times \\
\times \left\{ (-1)^p T(q_1) \cos(\omega + p)\tau + (-1)^\frac{2q+1}{2} T(q_2) \sin(\omega + p)\tau \right\}
\]

Substituting the expression for \( \cos\omega\tau \sin^{q+2s} \frac{\tau}{2} \), Eq. 3.11 takes the form:

\[
S(\omega, t) = \frac{2}{\pi} \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \sum_{p=-(\frac{s}{2})}^{p=(\frac{s}{2}+s)} \frac{(-1)^{q+s} (q + 2s)! B^2s C^{2q}}{2^{q+2s} q! s! (\frac{s}{2} + s + p)! (\frac{s}{2} + s - p)!} \times
\]
The integral in Eq. 3.12 can be represented in closed-form in terms of degenerative hypergeometric functions, which in the notation of Ref. 24 are as follows:

$$iF_1(v; \kappa; Z) = \Phi(v, \kappa; Z)$$  \hspace{1cm} (3.13)

where the \( \Phi \) function is given as

$$\Phi(v, \kappa; Z) = 1 + \frac{v}{\kappa} Z + \frac{v(v + 1)}{\kappa(\kappa + 1)} \frac{Z}{2} + \frac{v(v + 1)(v + 2)}{\kappa(\kappa + 1)(\kappa + 2)} \frac{Z^3}{3!} + \cdots$$  \hspace{1cm} (3.14)

which satisfies the following identities:

$$\int_0^{\infty} x^{\eta-1} e^{-\beta x^2} \cos m x d x = \frac{\Gamma \left( \frac{\eta}{2} \right)}{2 \beta^{n/2}} \left( \frac{\eta}{2} \right)^{1/2} \left( \frac{m^2}{4\beta} \right)$$  \hspace{1cm} (3.15)

and

$$\int_0^{\infty} x^{\eta-1} e^{-\beta x^2} \sin m x d x = \frac{m e^{-m^2/4\beta}}{2 \beta^{n/2}} \left( \frac{\eta + 1}{2} \right)^{1/2} \left( \frac{m^2}{4\beta} \right)$$  \hspace{1cm} (3.16)

Upon substituting Eqs. 3.13 to 3.16, Eq. 3.12 takes the final form:

$$S(\omega, t) = \frac{2}{\pi} \sum_{s=0}^{\infty} \sum_{q=0}^{\infty} \sum_{p=-(\frac{3}{4}+s)}^{p=\left(\frac{3}{4}+s\right)} \frac{(-1)^{q+s} (q + 2s)! B^{2s} C^{2q}}{2^{q+2s} q! s! (\frac{3}{2} + s + p)! (\frac{3}{2} + s - p)!} \times \left[ (-1)^p T(q_1) \frac{\Gamma \left( \frac{q+1}{2} \right)}{2 A^{q+1}} \Phi \left( \frac{q+1}{2}, \frac{1}{2}, -\frac{(\omega + p)^2}{4 A^2} \right) \right. \left. \right.$$  

$$\left. + (-1)^{2p+1} T(q_2) \frac{(\omega + p) e^{-\frac{(\omega + p)^2}{4 A^2}}}{2 A^{q+2}} \Gamma \left( \frac{q+1}{2} \right) \Phi \left( \frac{1-q}{2}, \frac{3}{2}, -\frac{(\omega + p)^2}{4 A^2} \right) \right]$$  \hspace{1cm} (3.17)
Eq. 3.17 gives the complete solution for the instantaneous spectrum of the nonstationary autocorrelation function. For brevity, the series solution of the $\Phi$ function, given by Eq. 3.14, is not shown in Eq. 3.17. To further simplify Eq. 3.17, we set

$$S(\omega, t) = G_1 [G_2 \Phi_1 + G_3 \Phi_2]$$  \hspace{1cm} (3.18)

where

$$\Phi_1 = T(q_1) \Phi \left( \frac{q + 1}{2}, \frac{1}{2}; \frac{-(\omega + p)^2}{4A^2} \right)$$

$$\Phi_2 = T(q_2) \Phi \left( \frac{1 - q}{2}, \frac{3}{2}; \frac{(\omega + p)^2}{4A^2} \right)$$

To facilitate subsequent discussion of results, we express $S(\omega, t)$ as

$$S(\omega, t) = M_1 + M_2$$  \hspace{1cm} (3.19)

where

$$M_1 = G_1 G_2 \Phi_1$$

$$M_2 = G_1 G_3 \Phi_2$$

To obtain the stationary solution, as in Section 3.2.1, we need to substitute $a=0$ and hence $C=0$ (see the notation at the beginning of this subsection) in Eq. 3.17. In other words, we need to eliminate the q-summation in Eq. 3.17, because of the $C^{2q}$ term. Thus, by eliminating the q-summation, we can arrive at the stationary solution as in the preceding subsection. Note that the elimination of q-summation leads to $T(q_2)=0$ and hence $\Phi_2=0$ in Eq. 3.18. Hence, for the stationary case, Eq. 3.18 reduces to
\[ S(w) = G_1 G_2 \Phi_1 \]  

(3.20)

Eqs. 3.18, 3.19 and 3.20 will be used to show the effect of degenerative hypergeometric functions, as in the next section.

### 3.3 Results and Discussion

We begin with RFT instantaneous spectra for the Rosenbrock model since for that model we have a closed-form expression. Fig. 3.1 represents the RFT instantaneous spectra for \( \mu = 0.05 \) and 0.2 both at \( L/R = 4 \). The peaks occur at \( P/2, P, 3P/2, 2P \) etc. and their height decreases with increasing frequency. For clarity, only the first two dominant split peaks centered at \( P/2 \) and \( P \) are shown in Fig. 3.1. We need to determine why the peaks are centered at \( P/2, P, 3P/2, 2P \) etc. why there is a split. First we consider the simplified expression given by Eq. 3.19. This equation is represented in Fig. 3.2, in which the main figure gives the complete solution of \( S(f,t) \) or \( (M_1 + M_2) \). The upper left figure gives \( M_1 \) and the upper right figure gives \( M_2 \). It can be seen that the peaks at \( P, 2P \) etc. arise due to \( M_1 \) (upper left figure) and the peaks at \( P/2, 3P/2 \) etc. arise due to \( M_2 \) (upper right figure). The plots of the degenerative hypergeometric functions \( \Phi_1 \) and \( \Phi_2 \), from Eq. 3.18, also show why the peaks are centered at \( P/2, P, 3P/2, 2P \) etc., as shown in Fig. 3.3. The split peaks can be explained using the phenomenon of beating sinusoids [23]. It is worth noting that Eq. 3.17 contains these beating sinusoids in the form of more complicated degenerative hypergeometric functions (compare with sine and cosine functions in Eq. 3.12). Inspite of the complexity of the present example, an insight
Fig. 3.1: Instantaneous spectra for Rosenbrock model

(a) $\mu=0.05, \frac{L}{R}=4$

(b) $\mu=0.2, \frac{L}{R}=4$
Fig. 3.2: Representation of $S(f,t)$ from Eq. 3.19 at a discrete time
Fig. 3.3: Degenerative Hypergeometric functions $\Phi_1$ and $\Phi_2$
into the structure of peak splitting can be obtained by considering the waveform:

\[ x(t) = D \cos 2\pi f_1 t + D \cos 2\pi f_2 t \]  \hspace{1cm} (3.21)

Using the trigonometric relation, the above equation can be represented as

\[ x(t) = 2D \cos 2\pi \left( \frac{f_2 - f_1}{2} \right) t \cos 2\pi \left( \frac{f_1 + f_2}{2} \right) t \]  \hspace{1cm} (3.22)

In Eq. 3.22, the amplitude modulation frequency is \( \frac{1}{2} (f_2 - f_1) \) and the carrier frequency is \( \frac{1}{2} (f_1 + f_2) \). The split peaks may be caused by the difference in frequencies \( (f_2 - f_1) \) \cite{23, 25}.

We now present the property of the RFT instantaneous spectra with respect to time. Fig. 3.4 shows the sections of \( S(f, t) \) versus \( t \) for two discrete frequencies and the periodicity with respect to time (see Eq. 3.5), thus exhibiting the time-wise periodicity of the cyclostationary RFT. The effect of advance ratio \( \mu \) on \( S(f, t) \) at a discrete time for the Rosenbrock model can be seen from Figs. 3.5 and 3.6. For the zero advance ratio case (\( \mu = 0 \)), Fig. 3.5(a) shows that the spectrum has peaks at \( P, 2P \) etc. This is expected since for \( \mu = 0 \) we have the stationary case, in which the hypergeometric function \( \Phi_1 \) (see Eq. 3.20) has regular peaks (without splitting). Fig. 3.7 is a reflection of this phenomenon. In Fig. 3.5(b), as \( \mu \) is increased to 0.01, the split peaks begin to appear at \( P/2, P, 3P/2, 2P \) etc. This split becomes increasingly wider as \( \mu \) increases and eventually disappears for high advance ratio of \( \mu = 0.3 \) (see Fig. 3.6(c)).

The accuracy of the closed-form solution of the RFT instantaneous spectrum \( S(f, t) \) for the Rosenbrock model is shown in Figs. 3.8 and 3.9. Fig. 3.8 compares the closed-form solution vis-a-vis numerical solution for \( \mu = 0.05 \) and \( L/R = 4 \) at discrete
Fig. 3.4: Sections of RFT Instantaneous Spectra at discrete frequencies for Rosenbrock model
Fig. 3.5: Section of Instantaneous Spectrum at various advance ratios $\mu$ for Rosenbrock model
Fig. 3.6: Section of Instantaneous Spectrum at various advance ratios $\mu$
for Rosenbrock model
Fig. 3.7: Degenerative Hypergeometric function $\Phi_1$ in hover.
Fig. 3.8: Numerical and Closed-form solutions of the Instantaneous Spectrum for Rosenbrock model
Fig. 3.9: Numerical and Closed-form solutions of the Instantaneous Spectrum for Rosenbrock model
times. As can be seen, the closed-form solution agrees with the numerical solution
to several significant figures. It is mentioned that in calculating the summations
in Eq. 3.17, only the first five terms are enough to obtain the exact solution. The
agreement of the closed-form solution with the numerical solution is also presented
for $\mu=0.1$, $L/R=4$ and for $\mu=0.1$, $L/R=12$ at a discrete time, as in Fig. 3.9. Thus,
Figs. 3.8 and 3.9 show that the closed-form solution agrees with the numerical
solution over a range of $\mu$ and $L/R$ values.

Finally, the characteristics --- peaks and splitting of peaks --- observed for the
Rosenbrock model, as in Fig. 3.1, are qualitatively seen in Figs. 3.10 and 3.11 as
well, which are based on the Exponential and Dryden models, respectively.
Fig. 3.10: Instantaneous spectra for exponential model
Fig. 3.11: Instantaneous spectra for Dryden model
Chapter 4

Response Analysis

We begin this chapter with a brief account of the theory of stationary and nonstationary responses of linear or perturbed linear systems for completion. This also facilitates discussion of numerical results and avoids ambiguities in terminology. For the case of axial flight (e.g. hovering) the system has constant coefficients [26] and the gust excitation without and with RFT effects is stationary (see Eq. 2.21). For, this stationary case, the power spectral density technique is applied. In forward flight, the gust excitation without RFT effects is stationary (see Eq. 2.20 with c=0), and it is nonstationary with RFT effects. However, the response is nonstationary with or without RFT effects, since the system has periodic coefficients. For this case of nonstationary response, the state transition matrix approach is used. Finally, threshold crossing statistics, which are useful in fatigue and ride quality studies, are outlined.
4.1 Stationary Excitation

If $x$ is the vector representing the generalized coordinates of the blade motion, the equations can be represented in matrix form as

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{y\}$$  \hspace{1cm} (4.1)

where $[m]$, $[c]$ and $[k]$ are the mass, damping and stiffness matrices, respectively. Taking the Fourier transform of both sides, the response vector in the frequency domain can be expressed as

$$\{X(f)\} = [H(f)] \{Y(f)\}$$  \hspace{1cm} (4.2)

where $H(f)$ is the frequency-response matrix. Taking the complex conjugate and transpose of Eq. 4.2, we get

$$[X^*(f)] = [Y^*(f)] [H^*(f)]^T$$  \hspace{1cm} (4.3)

Multiplying Eqs. 4.2 and 4.3, and taking the expectation on both sides, we get

$$[E\{X(f)\} [X^*(f)]]] = [H(f)] \ [E\{Y(f)\} [Y^*(f)]]] \ [H^*(f)]^T$$  \hspace{1cm} (4.4)

The frequency-response matrix for the system defined by Eq. 4.2 is

$$[H(f)] = \left[-(2\pi f)^2[m] + i(2\pi f)[c] + [k]\right]^{-1} ; \omega = 2\pi f$$  \hspace{1cm} (4.5)

The complex natural frequencies are obtained as the roots of the characteristic equation

$$|\lambda^2[m] + \lambda[c] + [k]| = 0$$  \hspace{1cm} (4.6)
Under this stationary case, the input power spectral density and the frequency response matrix completely describe the output power spectral density. For example, the input spectral density matrix \( S^y(f) \) with elements \( S^y_{jk}(f) \) is

\[
[S^y_{jk}(f)] = [E\{Y_j(f)\}Y_k^*(f)]
\]

(4.7)

where \( Y_j(f) \) is the sample Fourier transform of the random input for the \( j \)th generalized coordinate. From Eq. 4.4, the output power spectral density matrix \( S^x(f) \) with elements \( S^x_{jk}(f) \) is given by:

\[
[S^x(f)] = [H(f)] [S^y(f)] [H^*(f)]^T
\]

(4.8)

In principle, given the input power spectral density matrix \( S^y(f) \) and the frequency-response matrix \( H(f) \), the output power spectral density matrix \( S^x(f) \) can be computed by Eq. 4.8.

### 4.2 Nonstationary Excitation

We consider a linear or perturbed linear system with state vector \( X \) and nonstationary excitation or input vector \( \lambda \) whose expected value is zero, \( E\{\lambda\} = 0 \). For this general case, the equation of motion can be represented as

\[
\dot{X} = [A(t)]X + [B(t)]\lambda
\]

(4.9)

As stated earlier, the inputs are idealized at 0.7R blade station as a point excitation, and the one-point input autocorrelation function \( R_\lambda(t_1, t_2) \) is given by Eq. 2.20. If \( \Phi(t, \theta) \) is the state transition matrix of Eq. 4.9, the state vector \( X(t) \) can be expressed as
\[ X(t) = \int_0^t \Phi(t, \theta) B(\theta) \lambda(\theta) d\theta \quad (4.10) \]

where \( \Phi(t, \theta) \) is the solution of the matrix differential equation

\[ \frac{d}{dt} \Phi(t, \theta) = A(t) \Phi(t, \theta) \quad (4.11) \]

with

\[ \Phi(\theta, \theta) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad (4.12) \]

For the above method, we need \( \Phi(t, \theta) \) for variable \( \theta \) whereas the solution of Eq. 4.11 gives the state transition matrix for fixed \( \theta \) and variable \( t \). It is often convenient to evaluate \( \Phi(t, \theta) \) from the adjoint system equation

\[ \frac{d}{dt} \Phi(t, \theta) = -A^T(\theta) \Phi(t, \theta) \quad (4.13) \]

with initial conditions

\[ \Phi(t, t) = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad (4.14) \]

and with the solution \( \Phi(t, \theta) \) for fixed \( t \) and variable \( \theta \). The state variance matrix or \( E[X(t)X(t)^T] \) from Eq. 4.10 takes the form:

\[ \sigma_X^2(t) = \int_0^t \int_0^t \Phi(t, \theta_1) B(\theta_1) E[\lambda(\theta_1)\lambda^T(\theta_2)] B^T(\theta_2) \Phi^T(t, \theta_2) d\theta_1 d\theta_2 \quad (4.15) \]

where \( E[\lambda(\theta_1)\lambda^T(\theta_2)] \) is the input correlation function \( R_{\lambda}(\theta_1, \theta_2) \).

### 4.3 Threshold Crossing Statistics

In addition to the output spectral density and mean square response, a probabilistic description of the response includes average rates of threshold crossings. This is
primarily used in fatigue and ride-quality studies [27]. Most often, it is the threshold crossings for a given magnitude with positive slope, also called up-crossings, that are of interest. For any stochastic process, the expected rate of crossings of magnitude \( \xi \) can be given as follows:

\[
E[N_+(\xi, t)] = \int_0^\infty \dot{\beta} P_{\beta \dot{\beta}}(\xi, \dot{\beta}, t) \, d\dot{\beta}
\]  

(4.16)

where the joint probability of deflection \( \beta \) can be expressed in closed-form:

\[
P_{\beta \dot{\beta}}(\beta, \dot{\beta}, t) = \frac{1}{2 \pi \sigma_\beta \sigma_{\dot{\beta}} \sqrt{1 - \rho_{\beta \dot{\beta}}^2}} \exp \left[ -\frac{\beta^2 \sigma_{\dot{\beta}}^2 - 2 \sigma_\beta \sigma_{\dot{\beta}} \rho_{\beta \dot{\beta}} \beta \dot{\beta} + \sigma_{\dot{\beta}}^2 \dot{\beta}^2}{2 \sigma_\beta^2 \sigma_{\dot{\beta}}^2 (1 - \rho_{\beta \dot{\beta}}^2)} \right]
\]  

(4.17)

where the correlation coefficient \( \rho_{\beta \dot{\beta}} \) is defined by:

\[
\rho_{\beta \dot{\beta}} = \frac{R_{\beta \dot{\beta}}}{\sigma_\beta \sigma_{\dot{\beta}}}
\]  

(4.18)

Substituting Eq. 4.17 in Eq. 4.16 and performing the integration for \( \beta = \xi \) gives the expected rate of up-crossings of threshold level \( \xi \):

\[
E[N_+(\xi, t)] = \frac{\sqrt{1 - \rho_{\beta \dot{\beta}}^2} \sigma_{\dot{\beta}}}{2 \pi \sigma_\beta} \times
\]

\[
\times \left\{ \exp \left[ -\frac{h^2}{2 (1 - \rho_{\beta \dot{\beta}}^2)} \right] + \sqrt{\pi} \nu \exp \left[ -\frac{h^2}{2} \right] [1 + erf(v)] \right\}
\]  

(4.19)

where

\[
h = \frac{\xi}{\sigma_\beta} \quad \text{and} \quad \nu = \frac{h \rho_{\beta \dot{\beta}}}{\sqrt{2 (1 - \rho_{\beta \dot{\beta}}^2)}}
\]

and the error function is given by

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\theta^2} d\theta
\]

For zero-level up-crossing, Eq. 4.19 simplifies to
$E[N_+(\xi, t)] = \frac{1}{2\pi} \sqrt{1 - \rho^2_{\beta\beta}} \frac{\sigma_{\beta}}{\sigma_{\beta}}$ \hspace{1cm} (4.20)

In subsequent chapters, the preceding three sections have been applied to rigid blade models with and without dynamic stall and also to elastic blade with higher modes.
Chapter 5

Rigid Blade Response

5.1 Linear Analysis

The rigid flapping equation of motion can be represented as follows [12]:

\[ \ddot{\beta} + \frac{\gamma}{2} \left[ \frac{1}{4} + \frac{\mu}{3} \sin \psi \right] \dot{\beta} + \left[ P_\beta^2 + \frac{\gamma}{2} \left( \frac{\mu}{3} \cos \psi + \frac{\mu^2}{2} \sin \psi \cos \psi \right) \right] \beta = \]

\[ \frac{\gamma}{2} \left[ \frac{1}{4} + \frac{2\mu}{3} \sin \psi + \frac{\mu^2}{2} \sin^2 \psi \right] \theta + \frac{\gamma}{2} \left[ \frac{1}{3} + \frac{\mu}{2} \sin \psi \right] \lambda \]  

or in a compact form:

\[ \ddot{\beta} + C(\psi) \dot{\beta} + K(\psi) \beta = m_\theta \theta + m_\lambda \lambda \]  

(5.1)

where

\[ \theta = \theta_0 + \theta_c \cos \psi + \theta_s \sin \psi \]

Here \( \theta_0 \) is the collective pitch, \( \theta_c \) and \( \theta_s \) are the cyclic or sine-and-cosine pitch components, \( \psi \) is azimuth angle, \( \mu \) is advance ratio, \( \gamma \) is Lock number, \( P_\beta \) is the
rotating flap frequency and $\lambda$ is inflow. Eq. 5.2 is of the form given in Eq 4.1. In hover ($\mu=0$), we have

$$\ddot{\beta} + \frac{\gamma}{8} \dot{\beta} + \rho_\beta^2 \beta = \frac{\gamma}{6} \lambda$$

(5.3)

for which the frequency-response function is

$$H(f) = 1/\left[-(2\pi f)^2 + i2\pi f \left(\frac{\gamma}{8} \right) + \rho_\beta^2\right]$$

Applying Eq. 4.8, the response spectral density function of the rigid flapping blade is

$$S_\beta(f) = S_\lambda(f)/\left\{ (2\pi f)^4 + \left[ \frac{\gamma^2}{64} - 2\rho_\beta^2 \right] (2\pi f)^2 + \rho_\beta^4 \right\}$$

(5.4)

The input spectral density function is the Fourier transform of the autocorrelation function of the Exponential model (see Eq. 2.20).

Eq. 5.2 can be represented in state variable form as

$$\dot{X} = \begin{bmatrix} \beta \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K(\psi) & -C(\psi) \end{bmatrix} \begin{bmatrix} \beta \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ m_\theta \theta & m_\lambda \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$$

(5.5)

This is of the form shown in Eq. 4.9. Applying Eq. 4.15, we get the rms value of flapping response. For example, if the $\Phi$ matrix elements are $\Phi_{11}, \Phi_{12}, \Phi_{21}, \Phi_{22}$ and $B$ matrix elements are $B_{11}, B_{12}, B_{21}, B_{22}$, the mean square value of $\beta$ is

$$\sigma_{\beta \dot{\beta}}^2(t) = \int_0^1 \int_0^1 \Phi_{12}(\theta_1) (m_\theta_1 m_\theta_2 + m_\lambda_1 R_\lambda(\theta_1, \theta_2) m_\lambda_2) \Phi_{12}(\theta_2) d\theta_1 d\theta_2$$

Once $\sigma_{\beta \dot{\beta}}^2, \sigma_{\dot{\beta} \dot{\beta}}^2$ and $R_{\beta \dot{\beta}}$ are known, Eq. 4.19 can be used to find the expected rate of up-crossings of threshold level $\xi$. We mention in passing that the cross-covariance $R_{\beta \dot{\beta}}$ is equal to $\rho_{\beta \dot{\beta}} \sigma_\beta \sigma_\dot{\beta}$, where $\rho_{\beta \dot{\beta}}$ is the instantaneous correlation coefficient between $\beta$ and $\dot{\beta}$. 
5.2 Dynamic Stall Analysis

Conventional rotor dynamic analysis does not include the effects of blade stall at high angles of attack and is based on a linearized aerodynamic model that assumes a constant lift-curve slope throughout the operating range of angle of attack. Thus, such analyses cannot accurately represent blade response at high lift or high speed. The unified ONERA dynamic stall lift model, which is a modified version of the ONERA model due to Peters [28], is used in this analysis. The stall model is semi-empirically derived using measured wind-tunnel data for an oscillating airfoil in conjunction with a parameter identification scheme. The result is a set of differential equations that relate the normal lift coefficient of an airfoil to angle of attack and its time derivatives.

The flapping motion and the lift (coefficient) dynamic stall equations with respect to azimuth angle $\psi$ and the i-th element are given by the following three equations [28]:

\[ \ddot{\beta} + P_\beta^2 \beta = \frac{\gamma}{2a} \sum_{i=1}^{N} (X_i + \mu \sin \psi)^2 \Delta_i X_i (C_{Z_{1i}} + C_{Z_{2i}}) \quad (5.6) \]

\[ C_{Z_{1i}} = -\frac{X}{K_i} C_{Z_{1i}} + \frac{X}{K_i} \alpha \dot{\theta}_i + \vartheta \dot{\theta}_i \quad (5.7) \]

\[ C_{Z_{2i}} = -\frac{2\alpha \bar{\gamma}}{K_i} C_{Z_{2i}} - \frac{\bar{\gamma}^2 (1 + \alpha^2)}{K_i^2} C_{Z_{2i}} - \frac{\bar{\gamma}^2 (1 + \alpha^2)}{K_i^2} \left\{ \Delta C_Z + \bar{c} K_i \frac{\partial \Delta C_Z}{\partial \theta} \dot{\theta}_i \right\} \quad (5.8) \]

where $C_{Z_1}$ and $C_{Z_2}$ represent the lift (coefficient) due to linear or unstalled lift and nonlinear or stalled lift, respectively, $\theta$ is the total aerodynamic angle of attack and $\Delta C_Z$ is the difference between the extended linear lift curve ($C_{Z_1}$) and the actual static lift curve $C_{Z_*}$. The parameters $\chi$, $s$, $\vartheta$, $\alpha$, $\bar{\gamma}$, and $\bar{c}$ are functions of blade angle.
of attack and were determined from wind-tunnel tests. For example, the specific parameters for the ONERA OA212 airfoil used in this study are

\[ \chi = 0.20 \]

\[ \vartheta = \frac{\partial C_{Z_i}}{\partial \theta} - \frac{4\pi}{180} [1 + 1.43 \Delta C_z] \]

\[ \gamma = 0.10 + 0.023 (|\theta| - 13^\circ) u(|\theta| - 13^\circ) \]

\[ \alpha = \frac{0.105}{\gamma} \]

\[ \bar{\theta} = 2.0 - 5.1 \tan^{-1} \{1.21 (|\theta| - 13^\circ)\} u(|\theta| - 13^\circ) \]

where \( \chi \) is a time delay parameter, \( \vartheta \) is a parameter that relates the lift to the pitch rate of the airfoil, \( \gamma \) is natural frequency, \( \alpha \) is damping factor, and \( \bar{\theta} \) is phase-shift parameter. The term \( u(|\theta| - 13^\circ) \) is a unit step function that is zero for angle of attack less than 13° and 1 for angles of attack greater than or equal to 13°.

The static lift curve of the OA212 airfoil is presented in Fig. 5.1. The lift coefficient is linear between 0° and 13°. For larger angles, the magnitude shows a deviation from linearity. \( C_{Z_i} \) is the actual static stall lift coefficient. \( C_{Z_l} \) is the linear lift (static) coefficient, which can be expressed as

\[ C_{Z_l} = a \sin \theta \cos \theta \]

where \( a \) is the lift curve slope, which for the OA212 airfoil is equal to 7.1. The static stall lift coefficient is approximated by the following functions for angles of attack between 0° and 360°:

\[ C_{Z_s} = C_{Z_l} \quad 0^\circ \leq \theta < 13^\circ \]
Fig. 5.1: Static and Linear Lift Coefficients vs. Angle of Attack
\[ C_{Z_t} = 1.55 \quad 13^\circ \leq \theta < 77^\circ \]
\[ C_{Z_t} = C_{Z_i} \quad 77^\circ \leq \theta < 103^\circ \]
\[ C_{Z_t} = -1.55 \quad 103^\circ \leq \theta < 167^\circ \]
\[ C_{Z_t} = C_{Z_i} \quad 167^\circ \leq \theta < 193^\circ \]
\[ C_{Z_t} = 1.55 \quad 193^\circ \leq \theta < 257^\circ \]
\[ C_{Z_t} = C_{Z_i} \quad 257^\circ \leq \theta < 283^\circ \]
\[ C_{Z_t} = -1.55 \quad 283^\circ \leq \theta < 347^\circ \]
\[ C_{Z_t} = C_{Z_i} \quad 347^\circ \leq \theta < 360^\circ \]

Therefore, \( \Delta C_Z \) can be obtained as a difference between the extended linear static lift \( C_{Z_t} \) and the actual static stall lift \( C_{Z_s} \) (see Fig. 5.1). For the flapping blade, the total angle of attack of the blade is

\[ \theta_t = \theta_0 + \theta_c \sin \psi + \theta_e \cos \psi - \frac{X_i \dot{\beta} + \mu \beta \cos \psi + \lambda}{X_i + \mu \sin \psi} \quad (5.9) \]

where \( \theta_0 \) is the collective pitch, \( \theta_c \) and \( \theta_e \) are the cyclic pitch and \( \lambda \) is the inflow.

Since the dynamic stall equations are nonlinear in nature, we need to linearize them about a periodic equilibrium position. The state variable \( \beta \) and the stall parameters to be perturbed are as follows:

\[ \beta = \overline{\beta} + \delta(\beta) \]
\[ C_{Z_{t_i}} = \overline{C_{Z_{t_i}}} + \delta(C_{Z_{t_i}}) \]
\[ C_{Z_{i}} = \overline{C_{Z_{i}}} + \delta(C_{Z_{i}}) \]
\[ \theta_{i} = \overline{\theta_{i}} + \delta(\theta_{i}) \]
\[ \dot{\theta}_{i} = \overline{\dot{\theta}_{i}} + \delta(\dot{\theta}_{i}) \]
\[ \Delta C_{Z_{i}} = \overline{\Delta C_{Z_{i}}} + \delta(\Delta C_{Z_{i}}) \]
\[ \frac{\partial \Delta C_{Z_{i}}}{\partial \theta} = \frac{\partial \overline{\Delta C_{Z_{i}}}}{\partial \theta} + \delta \left( \frac{\partial \Delta C_{Z_{i}}}{\partial \theta} \right) \]
\[ \lambda = \overline{\lambda} + \delta(\lambda) \]
\[ \vartheta = \overline{\vartheta} + \delta(\vartheta) \]
\[ \overline{\gamma} = \overline{\gamma} + \delta(\overline{\gamma}) \]
\[ \overline{\varepsilon} = \overline{\varepsilon} + \delta(\overline{\varepsilon}) \]

where the barred quantities are equilibrium values and \( \delta \) quantities are small perturbations. The equilibrium inflow \( \overline{\lambda} \) is computed by solving the following two equations simultaneously:

\[ \overline{\lambda} = \mu \tan \alpha + \frac{C_{T}}{2\sqrt{\lambda^2 + \mu^2}} \]  \hspace{1cm} (5.10)

and

\[ C_{T} = \frac{a \sigma}{2} \left( \frac{\theta_{0}}{3} - \frac{\overline{\lambda}}{2} \right) \]  \hspace{1cm} (5.11)

Substituting the perturbed quantities in Eqs. 5.6 to 5.8, we obtain a set of perturbation equations for the blade with stall dynamics:
\[
\delta (\dot{\beta}) = -P_\beta^2 \delta (\beta) + \frac{\gamma}{2a} \sum_{i=1}^{N} (X_i + \mu \sin \psi)^2 \Delta_i X_i [\delta (C_{z_i}) + \delta (C_{\theta_i})] \tag{5.12}
\]

\[
\delta (\dot{C}_{z_i}) = -\frac{X_i}{K_i} \delta (C_{z_i}) + \frac{X_i}{K_i} a \delta (\theta_i) + \overline{\delta} \dot{\delta} (\theta_i) + \overline{\delta} \theta_i \delta (\vartheta) \tag{5.13}
\]

\[
\delta (\dot{C}_{\theta_i}) = -\frac{2\alpha \overline{\delta}}{K_i} \delta (\dot{C}_{z_i}) - \frac{(\frac{\overline{\delta^2}}{K_i^2} + \alpha^2 \gamma^2)}{K_i^2} \delta (C_{z_i}) - \frac{2\overline{\delta}}{K_i^2} \dot{C}_{z_i} \delta (\gamma)
\]

\[
\frac{(\frac{\overline{\delta^2}}{K_i^2} + \alpha^2 \gamma^2)}{K_i^2} \left\{ \frac{\delta \Delta C_{z_i}}{\partial \theta} \delta (\theta_i) + \overline{\delta K_i} \frac{\delta \Delta C_{z_i}}{\partial \theta} \delta (\theta_i) \right\}
\]

\[
+ \overline{\delta K_i} \theta_i \frac{\partial^2 \Delta C_{z_i}}{\partial \theta^2} \delta (\theta_i) + K_i \theta_i \frac{\partial \Delta C_{z_i}}{\partial \theta} \delta (\theta_i)
\]

\[
- \frac{2\overline{\delta}}{K_i^2} \left\{ \Delta C_{z_i} + \overline{\delta K_i} \theta_i \frac{\partial \Delta C_{z_i}}{\partial \theta} \right\} \delta (\gamma) \tag{5.14}
\]

Expressions for the perturbed dynamic stall quantities are given by

\[
\delta (\gamma) = 0.023 u (|\theta_i| - 13^\circ) \delta (\theta_i)
\]

\[
\delta (\varepsilon) = \frac{-5.1(1.21)}{1 + (1.21)^2 (|\theta_i| - 13^\circ)^3} u (|\theta_i| - 13^\circ) \delta (\theta_i)
\]

\[
\delta (\vartheta) = \frac{-4\pi}{180} (1.43) \frac{\partial \Delta C_{z_i}}{\partial \theta_i} \delta (\theta_i)
\]

\[
\delta (\theta_i) = -\left\{ \frac{X_i \delta (\dot{\beta}) + \mu \delta (\beta) \cos \psi + \delta (\lambda)}{X_i + \mu \sin \psi} \right\}
\]

The equilibrium quantities are

\[
\chi = 0.20
\]

\[
\overline{\delta} = \frac{\partial C_{z_i}}{\partial \theta_i} - \frac{4\pi}{180} [1 + 1.43 \Delta C_{z} (\overline{\theta_i})]
\]

\[
\alpha \overline{\delta} = 0.105
\]
\[
\overline{\eta} = 0.10 + 0.023 (|\overline{\theta}| - 13^\circ) u (|\overline{\theta}| - 13^\circ)
\]
\[
\overline{\epsilon} = 2.0 - 5.1 \tan^{-1} \left\{ 1.21 (|\overline{\theta}| - 13^\circ) \right\} u (|\overline{\theta}| - 13^\circ)
\]
\[
\overline{\theta}_i = \theta_0 + \theta_\ast \sin \psi + \theta_c \cos \psi - \left\{ \frac{X_i \overline{\beta} + \mu \overline{\beta} \cos \psi + X}{X_i + \mu \sin \psi} \right\}
\]

The analysis begins with the computation of steady state \(\overline{\beta}\) (\(\overline{\beta}\) replaced by \(\beta\) in Eq. 5.6) and \(\overline{\lambda}\) (Eqs. 5.10 and 5.11). Then, the perturbed Eqs. 5.12 to 5.14 are used in conjunction with Sections 4.2 and 4.3 to get the rms value of flapping response and the threshold crossing statistics. Note that the perturbed inflow, \(\delta (\lambda)\), is treated as the random input whose mean value is assumed to be zero. The flapping equation along with the two dynamic stall equations constitute the system of equations for rigid-blade analysis. If \(N\) is the number of elements, the lift equations will have \(3N\) state variables and the flapping equation has two state variables. Therefore, the total number of state variables is \(3N+2\).

### 5.3 Results and Discussion

The response statistics are generated for the following baseline parameters:

- \(0 \leq \mu \leq 0.3\), \(1 \leq L/R \leq 4\), thrust coefficient \(C_T=0.006\), rotating flap frequency \(1.1 \leq P_\beta \leq 1.4\), Lock number \(\gamma=8\), solidity ratio \(\sigma=0.1\), shaft tilt \(\alpha_s = 8^\circ\), collective pitch \(\theta_0 = 0^\circ\) (unless specified in the figure), cyclic pitch \(\theta_c, \theta_\ast = 0^\circ\), blade semi-chord=0.05, blade root cut-out=20% and the number of blade elements \(N=3\).
5.3.1 Linear Aerodynamics

We begin with the output or the (rigid flapping) response spectral density function in hover ($\mu=0$), as shown in Fig. 5.2. The response spectral density is computed from Eq. 5.4. Fig. 5.2 shows both $f S(f)$-versus-$f$ and conventional $S(f)$-versus-$f$ representations. It is reiterated that the $f S(f)$-versus-$f$ representation is to focus on the impact of energy transfer due to RFT effects from the low- to high-frequency regions. Fig. 5.2 is better appreciated in conjunction with Fig. 2.3, which shows the input spectral density. From Fig. 5.2, the response spectral density with RFT excitation is less than that with the space-fixed turbulence excitation in the low-frequency region. This is due to the energy transfer to the high-frequency region in the RFT description. Thus, compared with the response due to RFT excitation, the response due to space-fixed turbulence is more severe in the low-frequency region. This is expected because, in the space-fixed description, almost all of the turbulence energy is concentrated in the low-frequency region (see upper-right figure in Fig. 2.3) and, consequently, the response spectral density rapidly approaches zero with increasing frequency. Since the rigid-blade flapping frequency $P_\beta$ is close to the dominant $1P$ peak in the input RFT spectral description, we observe a strong peak at $1P$ in response in Fig. 5.2.

The response statistics of rms values and expected rate of threshold crossings are computed using Eq. 5.5. Only the time interval during the third rotor revolution ($4\pi \leq t \leq 6\pi$) is selected when the system reaches steady state. Fig. 5.3 shows that the rms values with RFT effects are lower than the rms values without RFT effects. This is because of the energy transfer to the high-frequency region. Hence,
Fig. 5.2: Rigid Flapping Response Spectral Density in Hover ($\mu=0.0$, $L/R=4$, $P_\beta=1.1$)
Fig. 5.3: Rigid Flapping Response RMS Values
($\mu=0.1, \ L/R=1, \ P_\beta=1.1$)
RFT effects diminish the rms response for this case with $L/R=1$ and $P_{\beta}=1.1$. As the turbulence scale length $L/R$ increases, the RFT effects decrease (see Figs. 2.4 to 2.6). This is reflected for $L/R=4$ in Fig. 5.4, which shows that the difference between the response rms values due to RFT and space-fixed turbulence excitations decreases (compare with Fig. 5.3 for $L/R=1$). When compared with Fig. 5.3, Fig. 5.5 shows that as the advance ratio $\mu$ increases, the rms value increases, a consequence of increasing gust sensitivity with increasing speed with or without RFT effects. We now come back to the earlier observed $L/R$ effect for $\mu=0.1$ --- decreasing RFT effects on response with increasing $L/R$. This observation is further substantiated by comparing Fig. 5.5 with Fig. 5.6, both for $\mu=0.3$, a typical cruising speed condition for conventional helicopters. The effect of rotating flap frequency $P_{\beta}$ on the rms flap response $\sigma_{\beta}$ is presented in Figs. 5.7, 5.8 and 5.9 for $P_{\beta}=1.1$, 1.3 and 1.4, respectively, and for each case $\mu=0.2$ and $L/R=4$. We observe that both the global maximum of $\sigma_{\beta}$ over one rotor revolution and its amplitude decrease with increasing $P_{\beta}$. This is expected in the space-fixed case since with increasing $P_{\beta}$ value, the energy in the excitation frequency decreases rapidly with increasing frequency (see Fig. 2.5). In the RFT case, this effect is not as clear since there is transfer of energy toward the high-frequency range. Nevertheless, we observe that with increasing $P_{\beta}$ values, the system frequency moves farther from the dominant 1P peak and the flapping blade becomes less sensitive to the dominant 1P peak of the excitation. It is emphasized that for unrealistically high values of $P_{\beta}$, the rms flapping response due to RFT is higher than that due to space-fixed turbulence.

Finally, Figs. 5.10, 5.11 and 5.12 show the RFT effects on the expectation
Fig. 5.4: Rigid Flapping Response RMS Values
\( (\mu=0.1, \ L/R=4, \ P_\beta=1.1) \)
Fig. 5.5: Rigid Flapping Response RMS Values
($\mu=0.3$, $L/R=1$, $P_\beta=1.1$)
Fig. 5.6: Rigid Flapping Response RMS Values
\((\mu=0.3, \ L/R=4, \ P_\beta=1.1)\)
Fig. 5.7: Rigid Flapping Response RMS Values
($\mu=0.2$, $L/R=4$, $P_\beta=1.1$)
Fig. 5.8: Rigid Flapping Response RMS Values
($\mu=0.2$, $L/R=4$, $P_\beta=1.3$)
Fig. 5.9: Rigid Flapping Response RMS Values

\( \sigma_\beta(t) \)

(\( \mu=0.2, \ L/R=4, \ P_\beta=1.4 \))
Fig. 5.10: Rigid Flapping Response Threshold Crossing Rates
($\mu=0.2$, $L/R=4$, $P_\beta=1.1$, $\xi=0$)
Fig. 5.11: Rigid Flapping Response Threshold Crossing Rates
($\mu=0.2$, $L/R=4$, $P_\beta=1.1$, $\xi=2$)
Fig. 5.12: Rigid Flapping Response Threshold Crossing Rates
($\mu=0.2$, $L/R=4$, $P_\beta=1.1$, $\xi=4$)
of up-crossings of the threshold level $\xi$. For $\xi=0$, Fig. 5.10 shows that the total number of zero-level up-crossings over one rotor revolution \(\left( \int_0^{2\pi} E[N_{+\beta}(0,t)] \right)\) substantially increases due to RFT effects. This is due to the decreased RFT effects on $\rho_{\beta\dot{\beta}}$ and $\sigma_{\beta}$, and increased RFT effects on $\sigma_{\dot{\beta}}$, as seen from Figs. 5.7, 5.13 and 5.14 (see Eq. 4.20). Moreover, from Figs. 5.11 and 5.12, it can be seen that as $\xi$ increases, the expected rate decreases rapidly, as expected.

5.3.2 Dynamic Stall

The effect of dynamic stall on response is presented in Figs. 5.15 to 5.21 for turbulence scale length $L/R=4$ and $P_{\beta}=1.1$. We begin with Fig. 5.15, which gives the flapping response rms values for $\mu=0.1$ and $\theta_0 = 0^\circ$. The solid line refers to results with dynamic stall, the dashed line to those without dynamic stall or the linear case. As seen from Fig. 5.15, dynamic stall substantially affects the response due to both RFT and space-fixed turbulence inputs. Moreover, Fig. 5.16, when compared with Fig. 5.15, shows that as $\mu$ increases, the effect of dynamic stall increases. For a given advance ratio, say $\mu=0.3$, the dynamic stall effect increases as the collective pitch $\theta_0$ increases, as seen from Figs. 5.17 and 5.18. The results in Figs. 5.15 to 5.18 are also expected since, with increasing advance ratio and collective pitch, the rotor blade experiences increasing stalled angle of attack. The effect of dynamic stall on the expected rate of threshold crossings is given in Figs. 5.19, 5.20 and 5.21. They show a substantial increase due to dynamic stall effects (compare these three figures with Figs. 5.10, 5.11 and 5.12, respectively).
Fig. 5.13: RMS Values of flap response $\dot{\beta}$
($\mu=0.2$, L/R=4, $P_\beta=1.1$)
Fig. 5.14: Correlation Coefficient between $\beta$ and $\dot{\beta}$
($\mu=0.2$, \(L/R=4\), \(P_\beta=1.1\))
Fig. 5.15: Rigid Flapping Response RMS Values with Dynamic Stall ($\mu=0.1$, $L/R=4$, $P_\beta=1.1$, $\theta_o=0^\circ$)
Fig. 5.16: Rigid Flapping Response RMS Values with Dynamic Stall ($\mu=0.2$, $L/R=4$, $P_\beta=1.1$, $\theta_o=0^\circ$)
Fig. 5.17: Rigid Flapping Response RMS Values with Dynamic Stall ($\mu=0.3$, $L/R=4$, $P_\beta=1.1$, $\theta_o=5^\circ$)
Fig. 5.18: Rigid Flapping Response RMS Values with Dynamic Stall ($\mu=0.3$, $L/R=4$, $P_\beta=1.1$, $\theta_o=10^\circ$)
Fig. 5.19: Rigid Flapping Response Threshold Crossing Rates with Dynamic Stall ($\mu=0.2$, $L/R=4$, $P_\beta=1.1$, $\theta_o=0^0$, $\xi=0$)
Fig. 5.20: Rigid Flapping Response Threshold Crossing Rates with Dynamic Stall ($\mu=0.2$, $L/R=4$, $P_\beta=1.1$, $\theta_0=0^\circ$, $\xi=2$)
Fig. 5.21: Rigid Flapping Response Threshold Crossing Rates with Dynamic Stall ($\mu=0.2$, $L/R=4$, $P_\beta=1.1$, $\theta_0=0^0$, $\xi=4$)
Chapter 6

Flap Bending Analysis

6.1 Mathematical Model

In the earlier chapter, the helicopter rotor blade was considered as a rigid blade with a root spring restraint. Only rigid body mode due to flap degree of freedom was considered. In this chapter, an elastic blade model with several flap bending degrees of freedom is presented. Following Ref. 29, we represent the flapping motion $\ddot{w}$ of the blade as follows:

$$\ddot{w} + \gamma \dot{w} (x + \mu \sin \psi) - \left( \tau \dot{w}^+ \right)^+ + \frac{\gamma}{2} \ddot{w}^+ (x + \mu \sin \psi) \mu \cos \psi + \Lambda_2 \ddot{w}^{+++}$$

$$= \frac{\gamma}{2} (x + \mu \sin \psi)^2 \theta - \frac{\gamma}{2} (x + \mu \sin \psi) \lambda$$

(6.1)

where the nondimensional tension $\tau$ is

$$\tau = \frac{1 - \dddot{x}^2}{2}$$
The quantities \( \dot{\phi}, (\phi)^+ \) are \( \partial/\partial \psi \) and \( \partial/\partial \bar{z} \), respectively. \( \theta \) is the pitch angle, \( \lambda \) is inflow and \( \Lambda_2 \) is the elastic blade parameter, which for typical helicopters is 0.002.

The flap equation presented in Eq. 6.1 is a set of linear partial differential equations with variable coefficients. The Galerkin procedure is used to convert the set of linear partial differential equations to a set of ordinary differential equations. The method consists of assuming a solution in the form of a series composed of a linear combination of admissible functions \( W_j \) multiplied by the time dependent functions (generalized coordinates) \( \beta_j \). That is,

\[
\bar{w} = \sum_{j=1}^{3} W_j(\bar{z})\beta_j(\psi)
\]  

For a blade with no hinge offset and no offsets regarding centers of mass and areas, the boundary conditions are [29]

\[
\bar{w}(0) = 0 \tag{6.3}
\]

\[
\bar{K}_\beta \bar{w}^+(0) - \Lambda_2 \bar{w}^{++}(0) = 0 \tag{6.4}
\]

\[
\bar{w}^{++}(1) = \bar{w}^{+++}(1) = 0 \tag{6.5}
\]

Before proceeding to apply Galerkin's scheme, the virtual work at the root can be obtained from the boundary condition as follows:

\[
\bar{K}_\beta W_j^+(0) = \Lambda_2 W_j^{++}(0)
\]

or

\[
\bar{K}_\beta W_i^+(0) W_j^+(0) = \Lambda_2 W_i^+(0) W_j^{++}(0) \tag{6.6}
\]
Multiplying Eq. 6.1 by $W_i(\ddot{x})$, integrating between 0 and 1 and finally substituting Eq. 6.6, we get

$$\sum_{j=1}^{3} \left[ F_{ij} \ddot{\beta}_j + \frac{\gamma}{6} (E_{ij} + F_{ij}\mu \sin \psi) \dot{\beta}_j \right. $$

$$+ \left\{ D_{ij} + A_2 Q_{ij} + K_\beta S_{ij} + \frac{\gamma}{6} (O_{ij} + P_{ij}\mu \sin \psi) \mu \cos \psi \right\} \hat{\beta}_j \right]$$

$$= \frac{\gamma}{6} \left[ \left( C_i + 2B_i \mu \sin \psi + A_i \mu^2 \sin^2 \psi \right) \theta - (B_i + A_i \mu \sin \psi) \lambda \right] \quad (6.7)$$

After selecting the three comparison functions, it is possible to compute the coefficients of Eq. 6.7. This results in three linear ordinary differential equations to be solved for the generalized coordinates in the time domain.

### 6.2 Comparison functions

The comparison functions can be trigonometric, power series or polynomials. Among these choices, a set of modified Duncan polynomials was successfully applied in Ref. 29, according to which the polynomials are

$$W_1 = x \quad (6.8)$$

$$W_2 = \frac{1}{16} \left( 15x^7 - 63x^5 + 105x^3 - 41x \right) \quad (6.9)$$

$$W_3 = 10.81453x^9 - 41.19704x^7 + 56.23063x^5 - 28.83052x^3 + 3.982398x \quad (6.10)$$

These polynomials are orthogonal and symmetric in the interval 0 to 1. They satisfy all the geometric boundary conditions and the natural boundary conditions at the tip of the blade. However, only the geometric boundary conditions are satisfied at
the root of the blade. That is why the virtual work, given by Eq. 6.6, was added to the final Eq. 6.7. The rotating frequencies of the three modes can be computed using Southwell's formula [30]:

$$\omega_i^2 = \frac{\omega_{nr}^2}{\Omega^2} + \alpha_i$$ \hspace{1cm} (6.11)

where $\omega_i$ is the rotating frequency of the modes and $\omega_{nr}$ is the non-rotating frequency. The terms $\frac{\omega_{nr}^2}{\Omega^2}$ and $\alpha_i$ can be computed from the following two equations:

$$\frac{\omega_{nr}^2}{\Omega^2} = \frac{\lambda_2 \int_0^1 \left( \frac{d^2W_i}{dx^2} \right)^2 dx}{\int_0^1 W_i^2 dx}$$ \hspace{1cm} (6.12)

$$\alpha_i = \frac{\int_0^1 \int_0^1 \left( \frac{dW_i}{dx} \right)^2 dx dx}{\int_0^1 W_i^2 dx}$$ \hspace{1cm} (6.13)

Using Eqs. 6.11, 6.12 and 6.13, the computed rotating frequencies of the first, second and third modes are 1.0, 2.61 and 4.69, respectively.

When the polynomials given by Eqs. 6.8 to 6.10 are substituted in the integral coefficients of Eq. 6.7, the resulting equation can be solved for the three flap bending modes. In the present analysis, the fundamental flap frequency $P_\beta$ is calculated from the constant stiffness coefficient of Eq. 6.7 as follows:

$$P_\beta^2 = \left( D_{11} + \lambda_2 Q_{11} + \bar{K}_\beta S_{11} \right) / F_{11}$$

We have $D_{11}=1/3$, $Q_{11}=0$, $S_{11}=1$ and $F_{11}=1/3$, hence

$$\bar{K}_\beta = \frac{P_\beta^2 - 1}{3}$$

Note that the rigid blade model treated in Chapter 5 can be recovered by using the first polynomial $W_1$, Eq. 6.8. The Galerkin integrals are given in Appendix
II. Eq. 6.7 is used in conjunction with the theory presented in Chapter 4 (e.g. Eqs. 4.8, 4.15 and 4.19) to obtain the output spectral density, rms flapping response and threshold crossing rates.

6.3 Results and Discussion

The response statistics are generated for the following baseline parameters:
$0 \leq \mu \leq 0.3$, $1 \leq L/R \leq 4$, thrust coefficient $C_T=0.006$, rotating flap frequency $1.1 \leq P_\beta \leq 1.4$, Lock number $\gamma=8$, solidity ratio $\sigma=0.1$, shaft tilt $\alpha_s = 8^\circ$, collective pitch $\theta_0 = 0^\circ$ (unless specified in the figure) and cyclic pitch $\theta_c, \theta_s = 0^\circ$.

The emphasis is on the low-frequency first flap bending mode, treated earlier on the basis of rigid-blade modeling. However, the response statistics of the second and third modes are also included for completeness. Moreover, turbulence model with or without RFT effects is a "point-approximation" to include vertical turbulence excitation at the 0.7R blade station only. It does not include spatial modes or shape functions, and it appears that to fully capture the effect of high-frequency modes, further refinement of the turbulence model is required with inclusion of such shape functions.

We consider the flap bending response spectral densities in hover, Fig. 6.1, in which $f S(f)$-versus-$f$ representation is given for all three modes. Since the first mode is a rigid blade representation (see the chosen polynomial in Eq. 6.8), much of the discussion corresponding to the rigid flap response spectral density in Fig. 5.2 can be applied here as well. As before, the first mode RFT spectral density
Fig. 6.1: Flap Bending Response Spectral Densities in Hover ($\mu=0.0$, $L/R=4$, $P_\beta=1.1$)
has a sharp peak at the 1P region, as compared to the rapidly decaying nature of the space-fixed turbulence spectrum for the same mode. Though of much smaller magnitude when compared to the first mode, RFT does excite the second and third modes. The response peak at 1P for all three modes is due to the dominant RFT spectral density peak at 1P.

The flap bending response statistics of rms values and expected rate of threshold crossings are presented in Figs. 6.2 to 6.12. As in the rigid blade analysis (Chapter 5), only the time interval during the third rotor revolution ($4\pi \leq t \leq 6\pi$) is selected, when the system reaches steady state. First, let us consider Fig. 6.2, which gives the flap bending rms values of the three modes for both RFT and space-fixed turbulence excitations. The global maximum of the response rms value of the first mode is reduced by RFT. This decrease is due to the transfer of energy in RFT from the low-frequency region to the high-frequency region. The second- and third-mode responses are too small to pass a judgement on RFT effects, meriting further investigation. Fig. 6.2, along with Fig. 6.3, shows that for all three modes, as L/R increases, the difference between the flap bending rms values due to RFT and space-fixed turbulence excitations decreases, as was the case for the rigid-blade model (see Fig. 5.3 for L/R=1 and Fig. 5.4 for L/R=4). For $\mu=0.1$, the effect of fundamental flap frequency $P_\beta$ on the flap bending rms values for the three modes is presented in Figs. 6.3, 6.4 and 6.5 for $P_\beta=1.1$, 1.2 and 1.4, respectively. It is observed that as $P_\beta$ increases, the rms values of the first mode decrease whereas the rms values of the second and third modes increase. Figs. 6.6 and 6.7, along with Fig. 6.3, show that as the advance ratio $\mu$ increases, the rms values of all three modes increase.
Fig. 6.2: Flap Bending Response RMS Values
($\mu=0.1$, L/R=1, $P_\beta=1.1$)
Fig. 6.3: Flap Bending Response RMS Values
($\mu=0.1$, $L/R=4$, $P_\beta=1.1$)
Fig. 6.4: Flap Bending Response RMS Values
($\mu=0.1, \ L/R=4, \ P_\beta=1.2$)
Fig. 6.5: Flap Bending Response RMS Values
($\mu = 0.1$, $L/R = 4$, $P_\beta = 1.4$)
Fig. 6.6: Flap Bending Response RMS Values
($\mu=0.2, \ L/R=4, \ P_\beta=1.1$)
Fig. 6.7: Flap Bending Response RMS Values
(\(\mu=0.3, \ L/R=4, \ P_\beta=1.1\))
Figs. 6.7 and 6.8 for \( \mu = 0.3 \) once again show that as \( P_\beta \) increases, the rms values of the first mode decrease and the rms values of the higher modes increase. Finally, from Figs. 6.9 and 6.10, it can be observed that RFT effects are appreciable on the expected rate of zero-level up-crossings for all three modes. As was the case for the rigid blade model (Figs. 5.7, 5.13 and 5.14), this is due to the decreased RFT effects on \( \rho_{\theta\dot{\theta}} \) and \( \sigma_\beta \), and increased RFT effects on \( \sigma_\dot{\theta} \), as shown in Figs. 6.6, 6.11 and 6.12. Thus, the results in this chapter further corroborate the results based on the rigid-blade model presented in Chapter 5.
Fig. 6.8: Flap Bending Response RMS Values
($\mu=0.3, \ L/R=4, \ P_\beta=1.4$)
Fig. 6.9: Flap Bending Response Threshold Crossing Rates ($\mu=0.1$, $L/R=4$, $P_\beta=1.1$, $\xi=0$)
Fig. 6.10: Flap Bending Response Threshold Crossing Rates (μ=0.2, L/R=4, Pβ=1.1, ξ=0)
Fig. 6.11: RMS Values of Flap Bending Response $\dot{\beta}$
($\mu=0.2, \ L/R=4, \ P_\beta=1.1$)
Fig. 6.12: Correlation Coefficient between $\beta$ and $\dot{\beta}$
($\mu=0.2$, $L/R=4$, $P_\beta=1.1$)
Chapter 7

Conclusions

The preceding study concerns the stochastic structure of RFT and its effects on blade flapping response statistics for low-speed and low-altitude conditions. It is an extension of earlier studies restricted to high-speed and low-thrust conditions of compound helicopters, for which the RFT effects are negligible. The study leads to the following conclusions:

1. RFT, or rotating frame turbulence, is the actual turbulence experienced by a blade station and requires noneulerian or moving frame description. It is (weakly) stationary in hover. In forward flight, though it is nonstationary, it is (wide sense) cyclostationary or periodically nonstationary (not mean square periodic), i.e. \( R(t_1, t_2) = R(t_1 + 2m\pi, t_2 + 2n\pi) \) for integers \( m=n \) only.

2. The instantaneous or frequency-time spectrum \( S(f, t) \) simultaneously predicts both the periodically varying nonstationarity with respect to time as well as the occurrence of split peaks centered at \( P/2, P, 3P/2, 2P \) etc. and transfer
of energy with respect to frequencies. The developed closed-form solution of
RFT instantaneous spectrum for a space-fixed turbulence model provides a
qualitative and parametric investigation of the characteristics with respect to
both frequency and time.

3. The response statistics comprising spectral density, rms values and expected
threshold crossing rates are significantly affected by RFT.

4. Dynamic stall affects the response due to both RFT and space-fixed excita-
tions and hence aggravates gust sensitivity.
Appendix I
Nomenclature

\( \alpha \)  
linear static lift curve slope, /radian

\( \bar{c} \)  
phase shift parameter

\( C_T \)  
thrust coefficient

\( C(\psi) \)  
damping coefficient

\( C_{Z_1} \)  
lift coefficient in linear regime

\( C_{Z_2} \)  
lift coefficient in nonlinear regime

\( C_{Z_s} \)  
static lift coefficient

\( C_{Z_l} \)  
static lift coefficient in linear region

\( E[\_] \)  
mathematical expectation

\( f \)  
frequency \((f=\omega/2\pi)\), rad/sec

\( _1F_1 \)  
degenerative hypergeometric function

\( K(\psi) \)  
stiffness coefficient

\( K_i \)  
ratio of blade semichord to element radius

\( \tilde{K}_\beta \)  
blade flapping root spring, N/m

\( L \)  
turbulence scale length, m

\( m_\theta \)  
coefficient of pitch angle

\( m_\lambda \)  
coefficient of inflow

\( N \)  
number of blade elements
\( P \) rotational speed, /revolution
\( P_\beta \) nondimensional flap natural frequency, /revolution
\( q_i \) generalized coordinates
\( R \) rotor radius, m
\( R_{w(t_1,t_2)} \) autocorrelation function due to vertical turbulence \( w(t) \)
\( R_{\beta\dot\beta} \) cross covariance function
\( S_{w(f,t)} \) frequency-time spectrum of vertical turbulence \( w(t) \)
\( t \) time, sec
\( V \) flight speed, m/sec
\( \bar{\omega} \) nondimensional vertical deflection of the blade
\( W_j \) admissible functions
\( \bar{x} \) nondimensional distance along the blade
\( X_i \) nondimensional radius of the blade element
\( \alpha \) damping parameter
\( \alpha_s \) shaft tilt, degrees
\( \beta \) blade flapping angle, radians
\( \gamma \) Lock number
\( \tilde{\gamma} \) stall natural frequency
\( \Gamma \) gamma function
\( \delta() \) perturbed quantity
\( \Delta_i \) nondimensional blade element width
\( \vartheta \) parameter relating lift and airfoil pitch rate
\( \theta_0 \quad \text{collective pitch, degrees} \)
\( \theta_c, \theta_s \quad \text{cyclic pitch, degrees} \)
\( \lambda \quad \text{total inflow} \)
\( \lambda_i \quad \text{induced inflow} \)
\( \Lambda_2 \quad \text{elastic blade parameter} \)
\( \mu \quad \text{advance ratio} \ (V/\Omega R) \)
\( \xi \quad \text{threshold level} \)
\( \sigma \quad \text{solidity ratio} \)
\( \sigma_\lambda^2 (t) \quad \text{state variance matrix} \)
\( \sigma_w \quad \text{rms value of vertical turbulence} \)
\( \sigma_\beta \text{ or } \sigma_\beta \quad \text{rms value of } \beta \)
\( \sigma_{\dot{\beta}} \text{ or } \sigma_{\dot{\beta}} \quad \text{rms value of } \dot{\beta} \)
\( \tau \quad \text{nondimensional tension, time difference } t_2 - t_1 \)
\( \Phi \quad \text{degenerative hypergeometric function, state transition matrix} \)
\( \chi \quad \text{time delay parameter} \)
\( \psi \quad \text{azimuth angle, radians} \)
\( \Omega \quad \text{rotor angular speed, rad/sec} \)
\( \dot{()}, \ddot{()}, (...) \quad \text{steady state quantity} \)
\( \partial \quad \partial / \partial \psi \)
\( ()^+ \quad \partial / \partial \bar{x} \)
Appendix II
Galerkin Integrals

In Chapter 6, Galerkin's method is used to obtain an approximate solution to the flap equations. The coefficients of these equations have, in general, integral forms. The integrands consist of a product of comparison functions or their derivatives. The domain of integration is from 0 to 1 for hingeless or centrally hinged articulated rotor. The Galerkin integrals are as follows:

\[ F_{ij} = \int_0^1 W_i W_j d\bar{x} \]
\[ E_{ij} = \int_0^1 \bar{x} W_i W_j d\bar{x} \]
\[ P_{ij} = \int_0^1 W_i W_j^+ d\bar{x} \]
\[ O_{ij} = \int_0^1 \bar{x} W_i W_j^+ d\bar{x} \]
\[ D_{ij} = \frac{1}{2} \int_0^1 \left(1 - \bar{x}^2\right) W_i^+ W_j^+ d\bar{x} \]
\[ Q_{ij} = \int_0^1 W_i^{++} W_j^{++} d\bar{x} \]
\[ S_{ij} = W_i^+(0) W_j^+(0) \]
\[ A_i = \int_0^1 W_i d\bar{x} \]
\[ B_i = \int_0^1 \bar{x} W_i d\bar{x} \]
\[ C_i = \int_0^1 \bar{x}^2 W_i d\bar{x} \]
References


14. Anderson, M.B. and Fordham, E.J., “An Analysis of Results From an Atmospheric Experiment to Examine the Structure of the Turbulent Wind As Seen


