

Pseudorational Functions and H^∞ Theory

Yutaka Yamamoto *

Division of Applied Systems Science
Faculty of Engineering
Kyoto University
Kyoto 606-01
JAPAN

Allen Tannenbaum †

Department of Electrical Engineering
University of Minnesota
Minneapolis, Minnesota 55455
UNITED STATES

Abstract

The H^∞ optimization problem for a class of distributed parameter systems is studied. This class is called *pseudorational*, and is particularly in close relationship with Sarason's interpolation theorem. A general state space representation for Sarason's theorem is obtained. It is shown that for the case of the plant represented by a Blaschke product in this class, the optimal sensitivity computation is reduced to the limiting case of the Nevanlinna-Pick solutions.

Notation and Conventions

As usual, $L[\psi](s)$ or $\hat{\psi}(s)$ denotes the Laplace transform of ψ . The Laplace transforms will always be considered to be two-sided, i.e., taken over all real line $(-\infty, \infty)$. If X is a space of functions on the real line, then the space consisting of the Laplace transforms of elements in X , provided they exist, is denoted by \hat{X} . Let $L^2[0, \infty)$ and $L^2(-\infty, 0]$ be the standard Lebesgue square integrable spaces. Then their Laplace transformed spaces are $H^2 = \hat{L}^2[0, \infty)$, $H^2_- = \hat{L}^2(-\infty, 0]$. By the Paley-Wiener theorem, $L^2(-j\infty, j\infty) = H^2 \oplus H^2_-$. This induces natural projections $\pi_+ : L^2(-j\infty, j\infty) \rightarrow H^2$ and $\pi_- : L^2(-j\infty, j\infty) \rightarrow H^2_-$. They in turn yield $\pi_+ : L^2(-\infty, \infty) \rightarrow L^2[0, \infty)$ and $\pi_- : L^2(-\infty, \infty) \rightarrow L^2(-\infty, 0]$ via Fourier transform. We will make no notational distinction between them; the context will always tell which. We will usually write π for π_+ . \mathbb{C}_σ denotes the subset $\{s; \operatorname{Re} s > \sigma\}$ of the complex numbers. As usual, H^∞ stands for the space $H^\infty(\mathbb{C}_0)$. On the other hand, H^∞_- is the space of bounded analytic functions on the open left complex plane. While multiplication by elements in H^∞ leaves H^2 invariant, that by H^∞_- leaves H^2_- invariant. For a function $g(s)$ of a

complex variable, $\tilde{g}(s) := \overline{g(-\bar{s})}$.

1 Sarason's Theorem

Sarason's Theorem, and its generalized version, the Sz.-Nagy-Foias commutant lifting theorem, associate the H^∞ optimal value with the norm of a certain operator in a quotient space of the H^2 space. It does not, however, directly lead to a computational procedure of such a norm. This is where a connection with time-domain realization becomes more efficient.

The objective of the present paper is to establish such a link for a certain class of systems. Although certainly not most general, this class of systems, called *pseudorational* has the following desirable properties:

1. It embraces all delay-differential systems.
2. The spectrum of the system agrees with the poles of the transfer function.

1.1 Pseudorational Transfer Functions

We now present some basic notions for pseudorational transfer functions. For the simplicity of exposition, we confine ourselves to the single-input/single-output systems. The general treatment can be found in [15, 16] and references therein. Consider the following convolution relationship between input and output functions u and y :

$$q * y = p * u. \quad (1)$$

We are interested in the behavior where such a relation is determined based on bounded-time data. One way of doing this is to require that q and p have compact support.

Definition 1.1 An impulse response $g(t)$ is said to be *pseudorational* if there exist two distributions q and p with compact support such that

1. q is invertible over the convolution algebra \mathcal{D}'_+ of distributions with support bounded on the left,
2. $\operatorname{ord} q^{-1} = -\operatorname{ord} q$, and
3. $g = q^{-1} * p$.

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The transfer function $G(s) = \hat{g}(s)$ (later seen to be meromorphic) is also said to be pseudorational if the conditions above are satisfied.

Without loss of generality we may take q and p to have support contained in $(-\infty, 0]$. Then by the well-known theorem of Paley-Wiener, the two-sided Laplace transform $\hat{q}(s)$ of such a q (and also p) is an entire function with growth estimate of type

$$\begin{aligned} |\varphi(s)| &\leq C(1+|s|)^m e^{a \operatorname{Re} s}, \operatorname{Re} s \geq 0 \\ &\leq C(1+|s|)^m, \operatorname{Re} s \leq 0. \end{aligned} \quad (2)$$

for some constants $C, a > 0$ and nonnegative integer m . Thus the transfer function $G(s)$ is the ratio of two entire functions of exponential type.

There are at least two good reasons to study pseudorational transfer functions. One is that a fairly large class of systems can be studied in this category. Delay systems are naturally in this class. For example, the transfer function $G(s) = 1/(se^s - 1)$ yields the convolution equation $(\delta'_{-1} - \delta) * y = u$ and this input/output relation is easily seen to be pseudorational (δ_a is the Delta function placed at a).

The second reason is that a fairly complete realization theory, analogous to the finite-dimensional theory, is available. This leads to a state space realization of the H^∞ norm problem.

Let $G = q^{-1} * p$ be pseudorational. As above, we assume $\operatorname{supp} q \subset (-\infty, 0]$. Define a subspace $X^q \subset L^2_{loc}[0, \infty)$ by

$$X^q := \{x(t) \in L^2_{loc}[0, \infty); \operatorname{supp}(q * x) \subset (-\infty, 0]\}. \quad (3)$$

This amounts to requiring that the distribution $q * x$ must vanish on $(0, \infty)$. Let $\{\sigma_t\}$ be the left-shift semigroup in $L^2_{loc}[0, \infty)$, i.e.,

$$(\sigma_t \gamma)(\tau) := \gamma(\tau + t).$$

It is easily checked that X^q is a left shift invariant closed subspace of $L^2_{loc}[0, \infty)$. A standard observable realization can be constructed with X^q as a state space and σ_t , restricted to X^q , being a state generation semigroup. We denote its infinitesimal generator by T^* . (Its adjoint T is the generator of the right shift, and will be discussed later.)

Not only does this space give a realization, it will also play a central role in the computation of the H^∞ norm. The following properties are known [15, 16]:

Theorem 1.2 Let $G(s)$, q and X^q be as above.

1. X^q is a Hilbert space.
2. The spectrum of the infinitesimal generator T^* is given by

$$\sigma(T^*) = \{\lambda; \hat{q}(\lambda) = 0\} \quad (4)$$

The corresponding eigenvector is $\exp(\lambda t)$.

3. The semigroup σ_t in X^q is exponentially stable if and only if there exists $c < 0$ such that $\sigma(T^*) \subset \{s; \operatorname{Re} s < c\}$.
4. X^q is a (closed) subspace of $L^2[0, \infty)$ if and only if 3 holds. In this case, $\hat{X}^q \subset H^2$ is a closed subspace.

1.2 Sarason's Theorem in X^q

We give a time-domain version of Sarason's theorem in terms of the space X^q defined in the previous section. To this end, we need a few preliminaries from complex analysis.

Let $g = q^{-1} * p$ be pseudorational. Then by (2) $\hat{q}(s)$ is an entire function of exponential type. It then follows from the classical Lindelöf theorem that it admits the following infinite-product representation [17].

$$\hat{q}(s) = C e^{as} \prod_{n=1}^{\infty} \left(1 + \frac{s}{\lambda_n}\right) \exp\left(-\frac{s}{\lambda_n}\right). \quad (5)$$

Here $-\bar{\lambda}_1, -\bar{\lambda}_2, \dots$ are the zeros of $\hat{q}(s)$ (the reason for complex conjugation will become clear shortly). Furthermore, if all these zeros have negative real parts (i.e., $\operatorname{Re} \lambda_n > 0$), we have ([17])

$$\sum_{n=1}^{\infty} \frac{|\operatorname{Re} \lambda_n|}{|\lambda_n|^2} < \infty. \quad (6)$$

Now let L be the largest real number such that $\delta_{-L} * q$ has support contained in $(-\infty, 0]$. We have the following result:

Theorem 1.3 Let q and L be as above.

$$X^q \cong L^2[0, L] \oplus X_1$$

where X_1 is the closure of the space spanned by the eigenfunctions $\{e^{-\bar{\lambda}_1 t}, e^{-\bar{\lambda}_2 t}, \dots, e^{-\bar{\lambda}_n t}, \dots\}$.

We now consider the inner function associated with the above q . Let $B(s)$ be the inner function

$$B(s) = e^{-Ls} \prod_{n=1}^{\infty} \frac{s - \lambda_n}{s + \bar{\lambda}_n}, \quad (7)$$

where $\operatorname{Re} \lambda_n > 0$ for all n . Then by (6), the above Blaschke product is easily seen to converge.

The space BH^2 is a closed right-shift invariant subspace of H^2 . Sarason's theorem relates the H^∞ minimum distance problem to the computation of an operator norm in the orthogonal complement of BH^2 . We have the following result:

Theorem 1.4 Let $B(s)$, q , L be as above. Then

$$(BH^2)^\perp = \hat{X}^q \cong L^2[0, L] \oplus X_1$$

where $X_1 = \overline{\text{span}\{e^{-\lambda_1 t}, e^{-\lambda_2 t}, \dots\}}$. Furthermore,

$$\pi^B : H^2 \rightarrow \hat{X}^g : x(s) \mapsto B(s)\pi_- B^- x(s) \quad (8)$$

is the orthogonal projection along BH^2 .

Proof The subspace $L^2[0, L]$ corresponds to the pure-delay part, and we can assume $L = 0$ without loss of generality. Since $L[e^{-\lambda_n t}] = 1/(s + \bar{\lambda}_n)$, it is readily seen that $g \in H^2$ is orthogonal to all $1/(s + \bar{\lambda}_n)$ if $g(\lambda_n) = 0$ for all n , i.e., $g \in BH^2$. Conversely, any such g is clearly orthogonal to \hat{X}^g . Since $L = 0$, \hat{X}^g is densely spanned by eigenfunctions $1/(s + \bar{\lambda}_n)$, $n = 1, 2, \dots$, and hence the first equality follows. The second equality is Theorem 1.3. Denote this space \hat{X}^g by X^B . From what we have just seen, X^B is precisely the set

$$X^B = \{x \in H^2; B^- x \in H_-^2\}. \quad (9)$$

Now write $B^- x = x_1 + x_2$, $x_1 \in H_-^2$, $x_2 \in H^2$. It follows that $\pi^B x = Bx_1 = BB^- x + Bx_2 = x + Bx_2 \in H^2$. By (9), this belongs to X^B . This also yields $x - \pi^B x = -Bx_2 \in BH^2$, so that π^B is the orthogonal projection onto X^B along BH^2 . \square

Remark 1.5 The projection (8) was first introduced by Fuhrmann mainly for discrete-time systems (see [10] for details).

Let σ_t^r be the right shift operator in $L^2[0, \infty)$:

$$\sigma_t^r \phi(\tau) := \begin{cases} \phi(\tau - t), & \tau \geq t \\ 0 & \tau < t. \end{cases}$$

Its counterpart in H^2 is simply the multiplication operator by e^{-ts} . We define the compressed shift $T_t : X^B \rightarrow X^B$ by

$$T_t x := \pi^B e^{-ts} x$$

It is easily seen that $\{T_t\}$ constitutes the dual semigroup of the left shift σ_t , and we denote its infinitesimal generator by T .

Let us now give Sarason's theorem, in the form specialized to X^B . Let f be any function in H^∞ , and $M_f : H^2 \rightarrow H^2$ be the multiplication operator induced by f . Define

$$f(T) := \pi^B M_f|_{X^B}.$$

It is easy to see that $f(T)$ commutes with T_t for every $t \geq 0$. Sarason's theorem gives the converse in the following sense.

Theorem 1.6 (Sarason [13]) *Let $A : X^B \rightarrow X^B$ be any bounded linear operator such that $AT_t = T_t A$ for all $t \geq 0$. Then there exists a function $f \in H^\infty$ such that $A = f(T)$ and $\|A\| = \|f\|_\infty$.*

In particular, as in [14], if

$$\mu = \inf\{\|W - B\psi\|_\infty; \psi \in H^\infty\},$$

we have $\mu = \|W(T)\|$, so that the optimal sensitivity problem for stable pseudorational plants as above is reduced to the computation of the norm of the Sarason operator.

2 Optimal Sensitivity Computation

We give a solution to the model matching problem

$$\mu = \inf\{\|W - B\psi\|_\infty; \psi \in H^\infty\},$$

where $B(s)$ is as given in (7), and $W(s)$ is a strictly proper stable rational function. For simplicity, we will assume that all zeros λ_n are simple.

We make use of Sarason's theorem. To this end, we must give a representation for $W(T)$ in the state space X^B :

Proposition 2.1 *Let $A : X^B \rightarrow X^B$ be the operator that interpolates T , i.e., $A = W(T)$ for some H^∞ function W . Then it is represented by the generalized Dunford integral*

$$Ax = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} W(-\lambda)(\lambda I - T)^{-1} x d\lambda. \quad (10)$$

Sketch of Proof. Recall $X^B \cong L^2[0, L] \oplus X_1$, the latter being eigenfunction complete by Theorem 1.4. We can thus consider these two spaces separately.

First take any $x \in L^2[0, L]$. Since T_t is the right shift operator, its infinitesimal generator T is the differential operator $-d/dt$ with domain

$$D(T) := \{x \in L^2[0, L]; \dot{x} \in L^2[0, L], x(0) = 0\}.$$

Hence in the s -domain, this T corresponds to the multiplication by $-s$. Let W be a strictly proper stable rational function. For brevity, assume $W(s) = 1/(1 + as)$. It readily follows from the above that $W(T) = (I - aT)^{-1}$; note the negative sign due to the correspondence $s \leftrightarrow -T$. It can be shown that this resolvent always exists. Now consider the contour integral

$$\frac{1}{2\pi j} \oint \frac{1}{1 - a\lambda} (\lambda I - T)^{-1} x d\lambda$$

where the contour travels from $-j\infty$ to $j\infty$, and goes through a large semicircle in the positive half plane. It can be shown that $(\lambda I - T)^{-1}$ is bounded and analytic here. Then the only singularity is at $\lambda = -1/a$. Hence by the residue theorem, the integral (10) is equal to $(I - aT)^{-1}$.

Now take an $x \in X_1$. It is easier to work with the time domain. Recall that X_1 is densely spanned by the eigenfunctions $g_n = \exp(-\bar{\lambda}_n t)$ corresponding to T^* . Hence in this case it is easier to work with the dual operator A^* which should take the form

$$A^* x = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \overline{W(-\bar{\lambda})} (\lambda I - T^*)^{-1} x d\lambda \quad (11)$$

Take $z = g_n$. It is known ([14]) that $W(T)^*g_n$ should satisfy $W(T)^*g_n = \overline{W(\lambda_n)}g_n$. It is also known ([15]) that

$$(\lambda I - T^*)^{-1}g_n = \frac{e^{\lambda t}}{\lambda + \bar{\lambda}_n} - (e^{\lambda t} * e^{-\bar{\lambda}_n t}).$$

Again by the residue theorem, the integral corresponding to the first term yields $\overline{W(\lambda_n)}e^{-\bar{\lambda}_n t}$ while the second is analytic so that it yields zero, as expected. \square

Let us now show how we can compute the optimal sensitivity. For simplicity, we take $W(s) = 1/(1 + as)$. By Sarason's theorem 1.6, this amounts to computing the operator norm of $A = W(T)$. This leads to the computation of the maximal singular value of $W(T)$ (in this case, $W(T)$ is even compact [15]). The singular value equation

$$(\gamma^2 I - A^*A)\bar{v} = 0 \quad (12)$$

leads to

$$\gamma^2(I - aT)(I - aT^*)\bar{v} - \bar{v} = 0. \quad (13)$$

It is known that the solution to this problem breaks into the two singular value equations

$$\begin{aligned} \gamma^2(I - aT)(I - aT^*)z - z &= 0 \\ \gamma^2(I - aT^*)(I - aT)v - v &= 0. \end{aligned}$$

plus a coupling condition ([8]), where $\bar{v}^T = [z \ v]^T$, $z \in L^2[0, L]$, $v \in X_1$. The solution to the first problem is well elaborated in [7, 14]. So let us concentrate on the second equation, i.e., we assume $L = 0$ from here on.

Let X_N denote the finite-dimensional subspace of X^B spanned by $\{g_n; n = 1, 2, \dots, N\}$. Suppose v be a solution to (12). Truncate v into X_N . As $N \rightarrow \infty$, the truncated v_N approaches the real solution v . Hence, the optimal sensitivity γ is given by γ_N where γ_N is the maximal singular value for the truncated v_N . Therefore, we need only solve this problem in X_N and take the limit. Let us solve (13) in X_N .

Let us first specify the action of F^* on g_n . Put $F^*e^{-\bar{\lambda}_j t} = \sum_{k=1}^N a_k^j e^{-\bar{\lambda}_k t}$. Then $\langle e^{-\bar{\lambda}_i t}, F^*e^{-\bar{\lambda}_j t} \rangle = \sum_{k=1}^N \bar{a}_k^j / (\bar{\lambda}_i + \lambda_k)$. Combining this with

$$\langle F^*e^{-\bar{\lambda}_i t}, e^{-\bar{\lambda}_j t} \rangle = \langle -\bar{\lambda}_i e^{-\bar{\lambda}_i t}, e^{-\bar{\lambda}_j t} \rangle = \frac{-\bar{\lambda}_i}{\bar{\lambda}_i + \lambda_j}$$

we get $\Lambda A = \text{diag}[-\lambda_1, \dots, -\lambda_N]\Lambda$ where $A = (a_k^j)$, $\Lambda = (1/(\lambda_i + \bar{\lambda}_j))$. Substituting these into (13) yields

$$\mu^2 \Lambda^{-1} \text{diag}[b_1, \dots, b_N] \Lambda \text{diag}[\bar{b}_1, \dots, \bar{b}_N] v - v = 0$$

where $\mu = \gamma^{-1}$, and $b_i = W(\lambda_i) = 1 + a\lambda_i$. This readily implies that v should be a solution of the eigenvalue problem

$$(\Lambda - \mu^2 \Gamma)v = 0, \quad \Gamma = \frac{b_i \bar{b}_j}{\lambda_i + \bar{\lambda}_j}.$$

This is precisely the Nevanlinna-Pick solution [4]. We need only take

$$\gamma_{opt} = \sup_N \Lambda_N^{-1/2} \Gamma_N \Lambda_N^{-1/2}. \quad (14)$$

Note that the solution obtained in (14) makes use of the representation (10) and reduces the computation of γ to the approximation by the finite-dimensional Nevanlinna-Pick solution via Sarason's theorem. This connection is well known for finite-dimensional systems (e.g., [4]). The passage to the limit here is guaranteed by the eigenfunction completeness for X^B . A possible drawback is that it involves infinitely many interpolation conditions. However, this infinite-dimensionality is superfluous when $W(s)$ is rational. Via an interesting duality, by the skew Toeplitz theory ([2, 9]), one can reduce the present problem to a finite-dimensional problem. (See also [3, 5, 6, 12, 18] and the references therein.) The precise connection with this and the present continuous-time state space formalism is to be reported in a subsequent work.

3 Time-Domain Structure of $H(m)$

All of our Hardy spaces in this section will be defined on the unit disc D in the usual way. We let $m \in H^\infty$ denote an inner function. In this setting, in order to solve the H^∞ problem using Sarason theory [13], the key objects we must understand are the space $H(m) := H^2 \ominus mH^2$, and the associated compressed shift $S(m) := \Pi S|_{H(m)}$ where S denotes the unilateral right shift in H^2 and $\Pi : H^2 \rightarrow H(m)$ orthogonal projection. See [14] and the references therein.

In this section, following [1] we will describe time-domain interpretations of $H(m)$ and the compressed shift operator $S(m)$. In principle, this will allow us to perform infinite dimensional H^∞ presently carried out in the frequency domain using skew Toeplitz theory [2, 9], in the time-domain via algebro-differential equations, and give a more system-theoretic (based on realization theory) insight into the various manipulations of this functional calculus.

First of all as is well-known [11], the inner function m admits the decomposition $m = m_1 m_2 m_3$ where:

$$m_1(z) = \prod_{k=1}^{\infty} \left(-\frac{\bar{z}_k}{|z_k|} \right) \frac{z - z_k}{1 - \bar{z}_k z}, \quad \text{Blaschke product,}$$

$$m_2(z) = \exp \left\{ - \int_0^{2\pi} \frac{e^{jt} + z}{e^{-jt} - z} d\sigma(t) \right\}, \text{ singular function,}$$

and

$$m_3(z) = \exp \left\{ - \sum_{k=1}^{\infty} r_k \frac{e^{jtk} + z}{e^{jtk} - z} \right\}, \text{ delay function.}$$

Here z_k denotes a Blaschke sequence with

$$\sum (1 - |z_k|) < \infty, \quad \frac{\bar{z}_k}{|z_k|} := 1, \text{ for } z_k = 0.$$

Moreover, σ denotes a finite, positive, continuous, singular measure, and $r_k \geq 0$, $\sum r_k < \infty$.

Next set $\mathcal{H}_i := H^2 \ominus m_i H^2$, $i = 1, 2, 3$. Then it is easy to see that $H(m) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Therefore, to get our time-domain representation of $H(m)$ it is enough to consider each \mathcal{H}_i separately. Then Ahern-Clark [1] prove the following:

Theorem 3.1 Set $\mathcal{L}_1 := L^2(d\sigma_{m_1})$, where σ_{m_1} is the measure on the positive integers that assigns a mass $1 - |a_k|$ to the integer k . Moreover, let $\mathcal{L}_2 = L^2(d\sigma)$, and let \mathcal{L}_3 be the sum of the L^2 -spaces of Lebesgue measure on the real intervals of length r_k . There exists an explicit unitary operator $V : \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \rightarrow H(m)$ such that $V|_{\mathcal{L}_i} : \mathcal{L}_i \rightarrow \mathcal{H}_i$, $i = 1, 2, 3$.

One may also give an explicit time-domain description of $S(m)$, that is one may write down the action of the operator $V^* S(m) V$ on L^2 . One may also show that this operator is the sum of a unitary operator and a finite rank operator [9]. This fact is the key, in making the skew Toeplitz theory work either in the time or frequency domains.

Once we have computed the time-domain version of $S(m)$, we can reduce the computation of the norm of the Sarason operator $W(S(m))$ to an eigenvalue algebro-differential equation problem on the realization space $H(m)$, and thus solve the H^∞ problem. In fact, this was exactly the approach taken in [7] for a simple delay system. The full details of this will be carried out in the full version of this paper.

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