Weighted Sensitivity Minimization for Delay Systems

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Abstract

In this note we discuss the $H^\infty$-sensitivity minimization problem for linear time-invariant delay systems. While the unweighted case reduces to simple Nevanlinna-Pick interpolation, the weighted case turns out to be much more complicated and demands certain functional-analytic techniques for its solution.

Notation and Terminology

$D =$ open unit disc
$\tilde{D} =$ closed unit disc
$\partial D =$ unit circle
$H =$ open right half plane
$\mathcal{H} =$ closed right half plane
$H^p(X) =$ the standard Hardy $p$-space ($1 \leq p \leq \infty$) on $X$ where $X = D$ or $\mathcal{H}$. (See Duren [1] or Rudin [14] for details.) We will also use some elementary facts about $L^p$-spaces. Again see [1] or [14] for details. Finally if $u \in H^\infty(X)$ is an inner function, then $H^2(X) \oplus \mathbb{H}^2(X)$ will denote the orthogonal complement of $uH^2(X)$ in $H^2(X)$.

1. Introduction

Since the paper of Zames [21], there has been a large literature on weighted $H^\infty$-sensitivity minimization in control (see Francis [5] for an extensive list of references). Other than some recent work on the sensitivity minimization problem for time-varying systems [2,3] most of the research has been concentrated on the linear time-invariant (LTI) finite dimensional case. In the papers [6,9] explicit algorithms are derived for computing the optimal sensitivity and controllers in this case.

In this note, we will consider applying $H^\infty$-minimization techniques to systems with delays. Because of the infinite dimensional nature of the problem, in order to solve the weighted case, we will need infinite dimensional techniques. We will therefore use some of the results from Sarason [16] and Sz. Nagy-Foias [17,18]. In particular a special case of the commutant lifting theorem. For strictly proper weights and a plant consisting of a delay, we will derive explicit formulae for the minimal $H^\infty$-sensitivity, and the optimal controller.

In contrast to the above infinite-dimensional nature of weighted $H^\infty$-minimization of delay systems, we will also show that unweighted sensitivity minimization amounts to a simple interpolation problem for a large class of plants.

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2. Weighted Sensitivity Minimization

In this section we will show that the weighted-sensitivity $H^\infty$-minimization problem for even the simplest delay systems is non-trivial and demands infinite dimensional techniques. In contrast we will see in Section 3 that the unweighted analogue of this problem is much simpler and reduces to a well-known interpolation problem for a broad class of distributed systems.

We begin by recalling the general weighted sensitivity $H^\infty$-minimization problem for SISO, LTI plants (see e.g., [21,22,7] for details and motivation). We are given a SISO, LTI plant $P_0(s)$, and a stable "weighting" function $W(s)$. Let $C(s)$ be an internally stabilizing LTI controller for $P_0(s)$ in the feedback system of Figure 1. Then following Zames [21], we define the weighted sensitivity

\[ S_W(s) = W(s)(1 - P_0(s)C(s))^{-1}. \]

The problem we are interested in then, is in determining the existence of and computing

\[ \inf \{|S_W(s)| \infty : C \text{ stabilizing} \} \]

where $\| \cdot \|$ denotes the $H^\infty$-norm in the right half plane. In the finite dimensional case, sufficient conditions are given for this infimum to exist and an optimal controller (in general non-proper) is computed in [6,7].

In this section, we will be interested in taking $P_0(s) = e^{-hs}$ for $s > 0$. The general technique which we give below in the proof of (2.1) goes through immediately for any $W(s)$, a stable strictly proper real rational weighting function with stable inverse. In point of fact, the proof of (2.1) gives an explicit procedure for solving the Nehari problem of computing the distance of an $\ell^\infty$-function of the form $e^{j\omega}Q(s)$ (regarded as defined on the $j\omega$-axis) to $H^\infty$, where $Q(s)$ is a stable strictly proper real rational function. However in order to clearly illustrate our ideas and techniques we will take $W(s)$ to be the above linear weight. In Example (2.5) at the end of this section, we will write down a differential equation derived from the proof of (2.1) which allows one to solve the weighted sensitivity $H^\infty$-minimization problem for a quadratic weight.

We will now compute (in (2.1)) the infimum of (2.1), and then in (2.3) the corresponding optimal compensator. We thus begin with:

Theorem (2.1)

For $W(s) = 1/(as + 1)$, we have

\[ \inf \left\{ C \text{ stabilizing} \right\} \frac{1}{\| \frac{1}{W(1 - e^{-hs}C^{-1})} \|_\infty} \leq \frac{1}{\| \frac{1}{(as + 1) q^2(s) \}^{-\infty}}} \]
Proof

Following Zames 21, we are reduced to the following Neumann-type problem: Compute (H denotes the right half plane)

(4) \[
\inf_{s \in H^\infty(\mathbb{D})} \frac{1}{1-s} - e^{-\mu \frac{1}{1-z}}.
\]

From general theory (see e.g. Garnett \& page 137), the infimum exists, and there exists a unique optimal \( \tilde{q}, \tilde{q} \in H^\infty(H) \) which attains it. (In (2.2) we will compute \( \hat{h}_0 \).)

In order to compute (4), we transfer the problem to the unit disc \( \mathbb{D} \). Accordingly, set

\[
s := \frac{1 - z}{1 - \bar{z}}.
\]

Then

\[
W \left( \frac{1 - z}{1 - \bar{z}} \right) = \frac{1 - z}{(s - 1) z + (a - 1) \bar{z}} := \tilde{W}(z).
\]

Set moreover

\[
m(z) := e^{k \frac{z^2}{1 - z^2}}.
\]

Finally, let \( T \) denote the compression (i.e. the projection) of the unilateral shift operator on \( H^2 \) (defined by multiplication by \( z \) onto \( H^2 \subset mH^2 \), where \( H^2 := H^2(\mathbb{D}) \)). Then by Theorem 1 of 16 (a special case of the commutant lifting theorem 17, 18)

(5) \[
\inf_{s \in H^\infty(\mathbb{D})} \tilde{W}(z) - m(z)q(z)_{|\infty} = \Gamma
\]

where

\[
\Gamma := \left( 1 - T \right) \left( (s - 1) - (a - 1) \right)^{-1}.
\]

(Note that symbolically \( \tilde{W}(T) = \Gamma \).)

In order to compute \( \Gamma \), we will use some nice results from Sarason 15. First, we note that

\[
H^2 \subset mH^2 \cong L^2[0,1]
\]

(i.e. the two spaces are isometrically isomorphic). Moreover via this isometry \( 15 \),

\[
\frac{1 - T}{2} \cong (1 - V)^{-1}
\]

where

\[
V : L^2[0,1] \to L^2[0,1]
\]

is the Volterra operator

\[
V f(z) := \int_0^z f(t) dt
\]

and \( \cong \) denotes "is unitarily equivalent to".

Simple computations then show

\[
T = (1 - V)(1 - V)^{-1}
\]

and

\[
\Gamma \cong V(V - a)^{-1}
\]

Now clearly, \( V^{-1} = \hat{D} \) where \( \hat{D} \) is the derivative operator, \( \hat{D} f := f' \), with domain consisting of

\[
\{ f \in L^2[0,1], f' \in L^2[0,1], f(0) = 0 \}.
\]

Hence \( \Gamma \cong (a\hat{D} - 1)^{-1} \).

Now

\[
\Gamma = \hat{G} = \lambda \mu
\]

where \( \mu \) is the maximal eigenvalue of \( \hat{G} \). But then, \( \lambda_0 := \lambda \mu \) is the minimal positive eigenvalue of \( (\hat{G}^* \hat{G})^{-1} \). (Note that the operator \( \hat{G}^* \hat{G} \) has dense range which will be explicitly given below. The inverse of \( \hat{G} \) can then be identified with a differential closed operator with dense domain. For details about unbounded operators with dense domain see e.g. 14.) In other words, we are reduced to computing the minimal positive eigenvalue of the operator

\[
(\hat{G}^* \hat{G})^{-1} = (a\hat{D} - 1)(a\hat{D} - 1)^{-1}
\]

where (using integration by parts; see 11) \( \hat{D}^* = -\hat{D} \) with domain

\[
\{ f \in L^2[0,1], f' \in L^2[0,1], f(0) = 0 \}.
\]

Consequently, we may make the identification

\[
(\hat{G}^* \hat{G})^{-1} = -a^2 \hat{D}^2 - 1
\]

on the space of functions

\[
\{ f \in L^2[0,1], f' \in L^2[0,1], f(0) = 0 \}.
\]

We have therefore shown that in order to compute the minimal positive eigenvalue of \( (\hat{G}^* \hat{G})^{-1} \), we are required to solve the eigenvalue problem

\[
(-a^2 \hat{D}^2 - 1)f = \lambda f
\]

with boundary conditions

\[
\text{f(0) = 0, -af' + f(0) = 0.}
\]

Note that since \( \Gamma \) (and hence \( \hat{G} \)) is a contraction \( \lambda \sim 1 \). Since \( \lambda = 1 \) is clearly not an eigenvalue, we have \( \lambda > 1 \).

Therefore from the ordinary differential equation

\[
a^2 \hat{D}^2 f - (\lambda - 1)f = 0 \quad [\lambda > 1]
\]

we get

\[
f(t) = A \cos \left( \frac{\sqrt{\lambda - 1}}{a} t \right) - B \sin \left( \frac{\sqrt{\lambda - 1}}{a} t \right) t.
\]

From the boundary conditions, we see

\[
f(0) = B \sin \left( \frac{\sqrt{\lambda - 1}}{a} \right) = 0
\]

and

\[
f(0) = 0.
\]

Therefore

\[
f(0) = A \cos \left( \frac{\sqrt{\lambda - 1}}{a} \right) = 0
\]
Then from (e), we immediately derive the equation

\[ (\sqrt{\lambda - 1} \cos \left( \frac{\sqrt{\lambda - 1}}{\alpha} \right) + \sin \left( \frac{\sqrt{\lambda - 1}}{\alpha} \right) ) \theta = 0. \]

Set

\[ \theta := \frac{\sqrt{\lambda - 1}}{\alpha}. \]

Then from (6), we immediately derive the equation

\[ \tan y - ay - y = 0 \]

and from (7) the minimal strictly positive root \( y_2 \) of (8) (i.e. the unique root of (8) between \( T \) and \( T^2 ) \) corresponds to the required minimal eigenvalue \( \lambda_{\text{min}} \). Working our way back through the definitions, then shows \( \Gamma \) has the required valued. \( \square \)

Remarks (2.2)

Using the notation of (2.1), let \( \lambda = \lambda_{\text{min}} \) be the minimal positive eigenvalue of \( \left( \tilde{\Gamma}^* - \tilde{\Gamma} \right)^{-1} \) as computed above, and \( f_\lambda \) the corresponding eigenfunction.

\[ f_\lambda (t) = \sqrt{\lambda - 1} \cos \left( \frac{\sqrt{\lambda - 1}}{\alpha} \right) t - \sin \left( \frac{\sqrt{\lambda - 1}}{\alpha} \right) t. \]

It is well-known (see e.g. Sarason [16] or Power [13]) that the operator

\[ J : L^2(\theta, h) \rightarrow H^2 \subseteq mH^2 \]

\[ (H^2 = H^2(D), \quad m(z) = e^{i(\frac{2\pi}{2} - 1)}) \]

defined by

\[ Jf(z) = \frac{\sqrt{2}}{1 - z} \int_0^1 e^{i(\frac{x-1}{2} - 1)} f(t) dt \]

is an isometry (the isometry implicitly used in the above proof of (2.1)).

Set \( u := Jf_\lambda \). Finally, let \( q_{opt}(z) \) denote the function in \( H^\infty(D) \) which attains the infimum defined in (5). Using Zames [21] one can then easily compute the optimal compensator which attains the infimum given in the statement of (2.1) from \( \widetilde{q}_{opt}(z) := q_{opt}(z - 1)/(z - 1) \). Thus we are reduced to finding a formula for \( q_{opt}(z) \). This we do in the following:

**Theorem (2.3)**

Using the notation of (2.1) and (2.2), \( q_{opt}(z) \). \( f(z) \) \((z)\) for \( u(z) \) defined above, and where

\[ f(z) = \text{analytic part of } \tilde{\omega} \tilde{\mu} u \text{ in } H^2\{ - H^2(D) \}
\]

\[ = \text{image of } \tilde{\omega} \tilde{\mu} u \text{ in } H^2 \text{ under the orthogonal projection from } L^2 \text{ to } H^2. \]

**Proof**

In what follows by (standard) abuse of notation we let \( z \) denote the complex variable in the unit disc as well as \( \theta \). The context will make the meaning clear. Notice that for \( z = e^{i\theta}, \) \( z = 1, \) and \( \tilde{\omega} \tilde{\mu} = 1. \) Moreover we denote the (induced) Hilbert space norms on both \( H^2 \) and \( L^2,0, \) by \( \| \) \( \| \).

Then using the notation of (2.2) we have

\[ (\tilde{\Gamma}^* \tilde{\Gamma}) f_\lambda = \frac{1}{\lambda} f_\lambda. \]

Consequently,

\[ \| f_\lambda \|_{L^2}^2 = \| Jf_\lambda \|_{L^2}^2 = \| f_\lambda \|_{L^2}^2 = \frac{1}{\lambda} \| f_\lambda \|_{L^2}^2 = \| \tilde{\Gamma}^* f_\lambda \|_{L^2}^2 = \| \tilde{\Gamma} f_\lambda \|_{L^2}^2 = \| f_\lambda \|_{L^2}^2 \]

(In the terminology of [16], \( u \) is a maximal vector.)

Now by the commutant lifting theorem [16, 17], there exists \( \omega(z) \in H^\infty(D) \) such that

\[ \omega(T) = \Gamma \]

\[ \| \omega(z) \|_{\infty} = \| \Gamma \| \]

Moreover, since \( u \| u \|_2 \) is a maximal vector, \( \Gamma u = \omega u \) (see Sarason [16] page 188).

But from (9), we have that

\[ \omega(z) = \tilde{\omega} \tilde{\mu} \]

We must determine \( \omega(z) \). Clearly by (10) and (11), \( \omega(z) = -q_{opt}(z). \)

In order to do this, we first note that \( \tilde{\omega} \tilde{\mu} \) and \( \omega \tilde{\mu} \) are orthogonal to \( H^2 \) (in \( L^2 \)). Now we can write

\[ \tilde{\omega} \tilde{\mu} = \ell(z) - r(z) \]

(note \( z = z^{-1} \) on the unit circle) where

\[ \ell(z) = \text{the orthogonal projection of } \tilde{\omega} \tilde{\mu} \text{ onto } H^2(D) \]

\[ = \text{the analytic part of } \tilde{\omega} \tilde{\mu} \]

\[ = \omega(z) \]

\[ r(z) = \text{the anti-analytic part of } \tilde{\omega} \tilde{\mu}. \]

Multiplying (11) by \( \tilde{\mu} u \) and using (12) we see that

\[ \tilde{\mu} \omega = \ell(z) - r(z) = v(z) u(z). \]

Since \( \tilde{\mu} \omega \in H^2, \) \( u \omega = 0. \) Consequently

\[ v(z) = -\ell(z) u(z), \]

\[ \omega(z) = \tilde{\omega}(z) = \omega(z) u(z) \tilde{\mu} m(z). \]

Thus \( q_{opt}(z) = \ell(z) u(z) \) as required.
Remark (2.4)

For those who prefer working with the right half plane $H$, Laplace transforms, and differential equations, instead of the unit disc $D$ and elementary complex analysis, the above construction of $\hat{a}_{n}(s)$ amounts to a rigorous version of the following. First in the standard way we can regard $L^{2}(0,\infty)$ as the closed subspace of functions in $L^{2}(0,\infty)$ which are identically 0 outside the interval $0,\infty$. Hence with this identification, we can take the Laplace transform of elements of $L^{2}(0,\infty)$.

Now consider $g(t) := \hat{f}_{s}(t)$ (notation as in (2.1) and (2.2)). Then by definition $g(t) \in L^{2}(0,\infty)$, is the unique solution of the differential equation (with boundary condition)

$$\frac{d^{2}g}{dt^{2}} - g(t) = f_{s}(t), \quad g(0) = 0$$

The solution can then be computed to be

$$g(t) = \frac{a}{\sqrt{\lambda - 1}} e^{-\sqrt{\lambda - 1}t} - \frac{a}{\sqrt{\lambda - 1}} \cos \left( \frac{\sqrt{\lambda - 1}}{a} t \right).$$

Let $F_{s}(s), G(s)$ denote the Laplace transforms of the functions $f_{s}(t), g(t) \in L^{2}(0,\infty)$, respectively. Then it is easy to conclude as in (2.3) (using the notation of (2.1) and (2.2)) that

$$G(s), F_{s}(s) = W(s) - e^{-\sqrt{\lambda - 1}q_{s}(s)}.$$  

Example (2.5)

Let us compute $q_{s}(z)$ when $W(s) = 1 \{s - 1\}$. In this case we have $\hat{W}(z) = (1 + z)/2$. Following the notation of (2.3) we have that $z(\lambda) = (1 - z) - t(z)\omega(s)$, and $\hat{m} = \hat{m}/2 - \hat{m}z^2 - \hat{m}z$. Identifying $z^{1}$ and $z$ on the unit circle, and noting that $\hat{m}z^{2} = \hat{m}z^{2}$, we must have $\hat{m} = \hat{m} - \hat{m}z^{2} - \hat{m}z$. Thus it is easy to see that $z(\lambda)(z) = (1 - z)/2a$. Therefore in this case, $q_{s}(z) = (1 - h_{-1}) 2a$.

For $a = h = 1$, one can compute from (2.1) that the optimal sensitivity has norm approximately $4.42$.

Example (2.6)

We will now apply the procedure of (2.1) to show how to compute the infimum of (2) where $W(s) = 1 \{a^{2}z^{2} - 1\}$. Indeed following the argument of (2.1) one is reduced to the following elementary eigenvalue problem:

$$(a^{2}z^{2} - 2a^{2}z + 1)f = \lambda f$$

with boundary conditions:

(i) $f(1) = f'(1) = 0$

(ii) $a^{2}f''(0) - f(0) = 0$

(iii) $a^{2}f''(0) - f'(0) = 0$

Computing the minimal positive eigenvalue $\lambda_{\min}$ one gets that the required infimum is $1 \sqrt{\lambda_{\min}}$. (The interested reader can work out an explicit formula as done in (2.1)!)

3. Unweighted Sensitivity Minimization

In this section we would like to briefly consider the problem of unweighted sensitivity minimization for certain kinds of delay systems. The close connection between unweighted sensitivity minimization and robust system design in the finite dimensional case has already been considered in [11, 12]. In particular, in [11] it has been shown that for finite dimensional systems the problem of unweighted sensitivity minimization is equivalent to certain kinds of robust stabilization problems which in turn are equivalent to an interpolation problem of the Nevanlinna-Pick type.

We wish to show here that basically this phenomenon holds for certain kinds of distributed systems, i.e. even though these systems may be infinite dimensional, the unweighted sensitivity minimization problem amounts to a known finite interpolation problem. As we have seen, this is in striking contrast to the weighted case in which even for the simplest distributed systems, the solution of the weighted problem becomes highly non-trivial.

Since the results of this section will be given mainly to contrast with the much harder weighted problem, they will not be given in full generality. Indeed the same kinds of ideas used in [11, 10] for the solution of certain kinds of robust stabilization problems for broad classes of distributed systems go through immediately in the context of unweighted sensitivity minimization, and hence the arguments used below apply much more generally.

However for our modest heuristic purposes we will consider here a SISO, LTI plant $P_{0}(s)$ of the following form:

$$P_{0}(s) = e^{-hs}P_{0}(s)$$

where $h > 0$, and $P_{0}(s)$ is a strictly proper real rational function.

Given $C(s)$ an internally stabilizing proper LTI compensator for $P_{0}(s)$ in the feedback system of Figure 1, define the sensitivity

$$S(s) = (1 + P_{0}(s)C(s))^{-1}. $$

Then the problem we are interested in is computing

$$\inf \{ S(s) : C \text{ stabilizing} \}$$

where $\sup_{H}$ denotes the $H_{\infty}$-norm in the right half plane.

We now show why this is a simple interpolation problem (see also [22], [11, 19]). In order to do this, following 11 let us define $a_{\max}$. Suppose we are given $a_{1}, a_{2}, \ldots, a_{\ell}(\ell = \text{open unit disc})$, $a_{1}, a_{2}, \ldots, a_{\ell+1} \in \partial D(\ell = \text{the unit circle})$, and points $b_{1}, \ldots, C, a_{\ell+1} \in D$. (For simplicity we assume that all the $a_{i}$ are distinct. The general case of interpolation with multiplicities is similar [11]!)

Given $a \in R, a > 0$, we are interested in determining when there exists an analytic $f_{\omega} : D \rightarrow D$ such that $f_{\omega}(a_{i}) = a_{i}, i = 1, \ldots, \ell$. It is easy to compute the maximal $a_{\max}$ such that $f_{\omega}$ exists if and only if $a < a_{\max}$.

Briefly, define matrices

$$A = \frac{1}{1 - a_{i}a_{j}}, \quad B = \frac{b_{i}b_{j}}{1 - a_{i}a_{j}}.$$

Let $\omega_{\max}$ be the maximal eigenvalue of $1^{-1}B$. Then

$$a_{\max} = \min \left\{ \frac{1}{\omega_{\max}}, \frac{1}{b_{i+1}}, \ldots, \frac{1}{b_{\ell}} \right\}.$$  

Denote the zeros of $P_{0}(s)$ in $H(= H \setminus \{ \infty \})$ where $H$ denotes the closed right half plane by $z_{1}, \ldots, z_{m}$ and the poles in $H$ by $p_{1}, \ldots, p_{n}$. (Note since $P_{0}(s)$ is strictly proper one of the zeros will be $\infty$.) Let $\omega : H \rightarrow D$ be a fixed conformal equivalence.
exists an internally stabilizing proper compensator on since interested in computing the infimum over all activities is identical to that derived in

Let $\alpha_{\max}$ be computed with respect to this interpolation data.

**Theorem (3.1)**

$$\inf \{ \| S(s) \|_\infty : C \text{ stabilizing} \} \Rightarrow \frac{1}{\alpha_{\max}}$$

**Proof**

Basically we follow the same arguments and ideas in 11, 10, 19. Indeed suppose $r > 0$ is such that

$$(13) \quad S(s) : \hat{H} \rightarrow D ; \quad | z | < r.$$  

Note that since $P_0(s)$ is strictly proper $r > 1$. Clearly we are interested in computing the infimum over all $r$ such that there exists an internally stabilizing proper compensator $C(s)$ with $(13)$ holding. But $(13)$ holds if and only if

$$e^{-rz} P_0(s) C(s) : \hat{H} \rightarrow G$$

where $G := \{ z \in C : | z | > 1 \} > \frac{1}{2}$. Define now the conformal equivalence $\psi : G \rightarrow D$ by

$$\psi(z) := \frac{r(z - 1) - 1}{r},$$

Notice that $\psi(0) = 0,$ and $\psi(\infty) = \frac{1}{r}.$

Set $u(s) = \hat{P}_0(s) C(s).$ Then $\psi(e^{-rz} u(s)) = e^{-rz} q(s)$ and since $e^{-rz}$ is inner, we have that $q(s)$ is analytic and $q(s) : \hat{H} \rightarrow D.$ Moreover the interpolation conditions of internal stability ([20], [11]) translate into the following interpolation conditions on $q(s)$:

$$q(z_i) = 0 \quad i = 1, \ldots, m$$

$$q(p_j) = e^{h p_j} \quad j = 1, \ldots, n$$

Let now $\alpha_{\max}$ be as above. Then trivially from the definition of $\alpha_{\max}$,

$$\frac{1}{r} < \alpha_{\max}$$

from which we get the theorem. \( \Box \)

**Remark (3.2)**

The formula derived in (2.1) for the minimal unweighted sensitivity is identical to that derived in [11] in the finite dimensional case.

**Conclusions**

In this note, we have tried to illustrate some of the difficulties involved in the application of weighted sensitivity $H^\infty$ minimization techniques to even the simplest distributed systems. While the unweighted case is rather trivial and reduces to simple Nevanlinna-Pick interpolation the weighted problem seems to genuinely reflect the distributed nature of the system. However, we believe that the techniques from 15, 16, 18 which we used here, are certainly applicable to more complicated distributed plants, and in this sense this paper may be regarded as a first step in employing these methods to study the problem of weighted sensitivity $H^\infty$-minimization for infinite dimensional systems. However as our results indicate, there is still much research yet to be done in this direction.

**REFERENCES**

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![Block diagram](image)

**Fig. 1**