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Skew Toeplitz Theory and Pseudorational Transfer Functions

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Abstract

A state space version of the skew Toeplitz theory for (distributed) H^∞ optimization theory is presented. The approach combines this theory with the realization theory for pseudorational transfer functions.

1 Time-Domain Interpretation of Sarason Space

In [15], we established a strong link between the skew Toeplitz H^∞ theory of [2, 4, 5, 6, 9, 12] and its realization in the time domain. The fundamental motivation is that while the former is primarily a frequency domain theory it is nontrivial, but certainly desirable, to establish its time-domain counterpart for distributed parameter systems. Roughly speaking, the relationship is as follows: Consider the problem of weighted sensitivity H^∞-optimization:

$$\rho = \inf_{\phi \in H^\infty} \|W(s) - B(s)\phi(s)\|_\infty, \tag{1}$$

where $W \in H^\infty$ and $B(s)$ is an inner function. Let $X^B := (BH^2)^\perp = H^2 \ominus BH^2$. We know from Sarason's theorem [10] that ρ is precisely the norm of the compression of $W(s)$ to X^B. See [12] and the references therein.

The crucial step is to give a suitable (time-domain) representation for X^B. In the z domain, Ahern and Clerk [1] gave a representation for $H(B) := H^2 \ominus BH^2$, and the associated compressed shift $S(B) := \Pi S|H(B)$ where S denotes the unilateral right shift in H^2 and $\Pi : H^2 \to H(B)$ orthogonal projection. While it is possible to obtain results in the z domain and then transform them into the s domain, we here present a different route. We give a direct interpretation in the s domain using realization theory developed in [13, 14]. The key observation is that if

- $B(s)$ does not have any singular part, and
- its zeros satisfy certain growth conditions,

then it arises from the Laplace transform of a distribution with compact support. In other words, it falls into the category of pseudorational transfer functions, and we have a concrete realization procedure based on such a distribution [13]. In [15], using the eigenfunction completeness, we have shown that the above problem is reducible to the limiting case of Nevanlinna-Pick interpolation. A drawback is that it involves infinitely many interpolation conditions while in the skew Toeplitz theory, it can be reduced to essentially finitely many conditions. In this note we show that for delay-differential systems a more compact treatment is possible.

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2 Retarded Systems

Consider the following delay-differential system of retarded type:

$$\dot{x} = \alpha x(t) + \beta x(t-1) + bu(t) \tag{2}$$

which in turn is described by the functional differential equation

$$\frac{d}{dt}\begin{bmatrix} x \\ z(\theta) \end{bmatrix} = \begin{bmatrix} \alpha & \beta\delta_0 \\ 0 & \frac{d}{d\theta} \end{bmatrix}\begin{bmatrix} x \\ z(\theta) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u$$
$$= \begin{bmatrix} \alpha x + \beta z(0) \\ \frac{dz}{d\theta} \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u =: F\begin{bmatrix} x \\ z \end{bmatrix} + Gu$$

where $[x^T, z^T]^T \in \mathbb{R}^n \times (L^2[0,1])^n$ and the domain of F is given by

$$D(F) = \{[x^T, z^T]^T; \frac{dz}{d\theta} \in L^2[0,1], z(1) = x\}. \tag{3}$$

We note that i) this system is pseudorational in the sense of [13], and ii) if the approximate reachability condition ([14]) holds, then the state space $X = \mathbb{R}^n \times (L^2[0,1])^n$ is realized as the kernel space of the convolution operator with a certain distribution. In other words,

$$X \cong X^q = \{x(t) \in L^2_{loc}[0,\infty); \mathrm{supp}(q * x) \subset (-\infty, 0]\} \tag{4}$$

for some distribution q with compact support ([13]). This in turn implies that X is isomorphic to the space $X^B = H^2 \ominus BH^2$ for some inner $B(s)$ [15], when the system is stable, i.e., the zeros of $\det[sI - \alpha - e^{-s}\beta]$ lie in the closed left half complex plane. The precise form of B does not concern us here, so we omit it. It is the product of a delay transfer function and the Blaschke product consisting of the zeros of $\det[sI - \alpha - e^{-s}\beta]$. [1] In short, for a suitable B, $X^B = H^2 \ominus BH^2 \cong \mathbb{R}^n \times (L^2[0,1])^n$. To make use of Sarason's theorem, we need the compression of $W(s)$ to X^B. This compression of $W(s)$ is given by

$$W(-F^*)x = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} W(-\lambda)(\lambda I - F^*)^{-1} x d\lambda. \tag{5}$$

where F^* is the dual operator of F. The minus sign here accounts for the fact that F^* is the infinitesimal generator of the *right* shift operators, so that it corresponds to $-s$ rather than s.

Thus by computing F^* and invoking expression (5), we are led to a natural singular value equation to compute the optimal value ρ. Unfortunately the isomorphism $T : X^B \xrightarrow{\sim} \mathbb{R}^n \times (L^2[0,1])^n$ is not necessary an isometry with respect to the standard inner product of the latter. Therefore, we induce on X the following inner product:

$$\langle \begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \rangle_B := \langle T^{-1}\begin{bmatrix} x \\ z \end{bmatrix}, T^{-1}\begin{bmatrix} y \\ w \end{bmatrix} \rangle_{L^2[0,\infty)} \tag{6}$$

[1] The Laplace transform $\hat{q}(s)$ of the above distribution in (4) share the same zeros with $\det[sI - \alpha - e^{-s}\beta]$, and they are the mirror images of the zeros of $B(s)$ agaist the imaginary axis.

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With respect to this duality, we have the following:

Lemma 2.1 *The adjoint operator F^* with respect to the inner product (6) is*

$$F^* \begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} c(w(1)-y)+\alpha w(1) \\ -\frac{dw}{d\theta} \end{bmatrix} \qquad (7)$$

where $c = \|A\|_2^{-2}$, and $A(t)$ is the impulse response function of this system, and domain of F^* is

$$D(F^*) = \{w \in W_2^1[0,1]; w(0) = 0\} \qquad (8)$$

We omit the proof.

3 Sensitivity Computation

For simplicity, we take $W(s) = 1/(1+as)$, $a > 0$. When partial fraction expansion is available, the treatment is entirely similar. In this case the compression $W(-F^*)$ is precisely the resolvent operator $(I - aF^*)^{-1}$. Note that this resolvent always exists by the stability and $a > 0$ assumptions.

Then the norm $\rho = \|W(-F^*)\|$ is given by the maximal solution to the singular value equation

$$\left(\rho^2 I - W(-F)W(-F^*)\right) v = 0$$

which in turn leads to

$$\rho^2 (I - aF^*)(I - aF)\, v = v.$$

Straightforward calculation, along with (3), (8), yields

$$\begin{aligned} x &= \rho^2((1+ac)\mu - a(c+\alpha)(x - az'(1))) \\ \mu &= x - a(\alpha x + \beta z(0)) \\ z(\theta) &= \rho^2(z(\theta) - a^2 z''(\theta)) \end{aligned}$$

where $v = [x^T, z^T]^T$ and

$$\begin{aligned} z(1) &= x \\ z(0) - az'(0) &= 0. \end{aligned}$$

Now $\rho \le \|W\|_\infty = 1$, and one can check 1 is not a singular value here. So we may assume $\rho < 1$. Solving the differential equation and arranging the terms, we get the following:

Theorem 3.1 *Let*

$$\rho = \inf_{\phi \in H^\infty} \|W(s) - B(s)\phi(s)\|_\infty,$$

where $B(s)$ be the inner function associated to the delay-differential equation (2) as in Section 2, and $W(s)$ is $1/(1+as)$ Then ρ is the maximal solution such that the finite-dimensional eigenvalue problem

$$(P - EQ) \begin{bmatrix} x \\ z(0) \end{bmatrix} = 0 \qquad (9)$$

admits a nontrivial solution $[x^T, z^T(0)]^T$ where

$$\begin{aligned} P &= \begin{bmatrix} 1 & 0 \\ (1/\rho^2 + a\alpha(2+ac) - 1)/a^2(c+\alpha) & \beta(1+ac)/a(c+\alpha) \end{bmatrix} \\ E &= \begin{bmatrix} \cos\gamma & \sin\gamma/\gamma \\ -\gamma\sin\gamma & \cos\gamma \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 0 & 1/a \end{bmatrix} \end{aligned}$$

and $\gamma = \sqrt{1-\rho^2}/\rho a$.

As noted above, for a more general weight $W(s)$ one can invoke a partial fraction expansion, and get accordingly modified matrices P and Q. Hence the essential *finite-dimensionality of this problem, in spite of the underlying infinite-dimensionality of the plant, is unchanged.*

It is interesting to see the special case of pure delay. In this case, conditions on x become void, and we get

$$\begin{aligned} z''(\theta) - \gamma^2 z &= 0 \\ z(1) &= 0 \\ z(0) - az'(0) &= 0 \end{aligned}$$

which yields

$$a\gamma\cos\gamma + \sin\gamma = 0.$$

so that the optimal value is obtained from the unique root of $\tan\gamma + a\gamma = 0$ in $(\pi/2, \pi]$. This is precisely the solution obtained in [4, 12]. It is of interest to investigate the relationship with the other related works [3, 7, 16], etc., as well as to obtain formulae for general weights $W(s)$ as in [6, 8, 11] as indicated above.

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