ANALYSIS AND DESIGN OF ANISOTROPIC DIFFUSION FOR IMAGE PROCESSING

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ABSTRACT

Anisotropic diffusion is posed as a process of minimizing an energy function. Its global convergence behavior is determined by the shape of the energy surface, and its local behavior is described by an orthogonal decomposition with the decomposition coefficients being the eigenvalues of the local energy function. A sufficient condition for its convergence to a global minimum is given and is identified to be the same as the condition previously proposed for the well-posedness of 1-D diffusions. Some behavior conjectures are made for anisotropic diffusions not satisfying the sufficient condition. Finally, some well-behaved anisotropic diffusions are proposed and simulation results are shown.

1. INTRODUCTION

In an anisotropic diffusion, an image f defined in a domain Ω, is allowed to evolve over time via the following partial differential equation [6]

$$\frac{\partial f}{\partial t} = div(c(\|\nabla f\|)\nabla f). \tag{1}$$

The diffusion coefficient $c(\|\nabla f\|) \in [0,1]$ is required to be a decreasing function of the magnitude of local gradient such that (1) diffuses more in regions of small gradients and less around edges where the gradients are large. Let

$$\phi(s) = sc(s) \tag{2}$$

be the flux function widely used in 1-D diffusion equations. It has been shown that the desirable c(s) should be such that the flux function has the following thresholding property [6]

$$\phi'(s)\begin{cases} > 0, & s < T \\ < 0, & otherwise \end{cases}, \tag{3}$$

where T is the threshold. The motivation is that (1) will diffuse forward (smoothing) in smooth regions and backward (sharpening) around edges. Two such $c(s)$'s have been proposed [6]:

$$c(s) = e^{-(s/K)^2} \tag{4}$$

and

$$c(s) = \frac{1}{1+(s/K)^2}, \tag{5}$$

where K is a constant which controls the threshold T.

However, these anisotropic diffusions have been shown to be ill-posed processes in the sense that images close to each other are likely to diverge during the diffusion process. For example, the presence of noise, especially when the gradient generated by noise is comparable to that by image features, can drive the diffusion process to undesirable results [6, 9]. Even without noise, "stair-casing" effects can arise around smooth edges [9]. In practical implementation on computer, the diffusion process may diverge depending on difference schemes and grid sizes [5].

A sufficient and necessary condition for 1-D diffusions to be well-posed has been given [2, 1] :

$$\phi'(s) \geq 0, \tag{6}$$

which indicates that 1-D diffusions with their diffusion coefficients satisfying the thresholding property (3), such as (4) and (5), are ill-posed. Some arguments about the well-posedness of 2-D diffusions are made based on this condition, but no proof has previously appeared in the literature.

The ill-posedness of anisotropic diffusion is alleviated via introducing a smoothing operation to the variable of diffusion coefficient $c(s)$. One such example is [1]

$$\frac{\partial f}{\partial t} = div(c(\|\nabla G(s) * f\|)\nabla f). \tag{7}$$

where $G(s) * f$ denotes a convolution of the image at time t with a Gaussian kernel of scale s, which is to be given a priori. A properly selected s is critical to

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the success of the proposed anisotropic diffusion in the sense that the diffusion process would not be stable for too small an s, while image features are smeared for too large an s. One possible solution is to use a large s initially to suppress noise and then to reduce s so that image features are not further smeared [9]. Nevertheless, optimum selection of such an s is still an open problem. The convolution load involved in $G(s) * f$ is a serious problem because it is required at each time instant, whether it is implemented directly or by a separate isotropic diffusion. We also note that this peculiar scheme of isotropic diffusion within anisotropic diffusion is obviously against the spirit of anisotropic diffusion.

An alternative is to use curve evolution which is based on geometric heat flow of the level sets of the image. Diffusion schemes proposed include curvature motion [4], reaction-diffusion [3], and affine invariant scale-space [7].

This paper will pose anisotropic diffusion as a process of minimizing an energy function. The behaviors of anisotropic diffusion can then be best understood and described by the shape of the surface of the energy function, and well-behaved anisotropic diffusion can then be derived based on this analysis.

2. ANALYSIS

In order to understand the behaviors of the anisotropic diffusion (1), let us minimize the following energy function

$$E(f) = \int_{\Omega} F(||\nabla f||) d\Omega = \int_{\Omega} F(\sqrt{f_x^2 + f_y^2}) d\Omega. \quad (8)$$

The $F(||\nabla f||)$ is required to be a strictly increasing function of the magnitude of gradient such that

$$F'(||\nabla f||) > 0. \quad (9)$$

Consequently, the energy function (8) is a measure of smoothness and its minimization is equivalent to smoothing.

The the minima of (8) are at some of its stationary points given by the Euler-Lagrange equation [8]

$$div(\frac{F'(||\nabla f||)}{||\nabla f||} \nabla f) = 0. \quad (10)$$

Similar to gradient descent, (10) may be solved by the following parabolic equation

$$\frac{\partial f}{\partial t} = div(\frac{F'(||\nabla f||)}{||\nabla f||} \nabla f) \quad (11)$$

when $t \rightarrow \infty$. Obviously, equation (11) is the same as the anisotropic diffusion equation (1) if the diffusion coefficient is set to

$$c(||\nabla f||) = \frac{F'(||\nabla f||)}{||\nabla f||}. \quad (12)$$

As required of a diffusion coefficient, it is positive due to (9). It is also obvious that the flux function used for 1-D diffusion is

$$\phi(s) = sc(s) = F'(s). \quad (13)$$

For an interpretation of the anisotropic diffusion, let us first note that the eigenvalues of the Hessian matrix of $F(||\nabla f||)$, the integrand of (8), may be obtained as

$$\lambda_1 = \frac{F'(||\nabla f||)}{||\nabla f||}, \quad \lambda_2 = F''(||\nabla f||) \quad . \quad (14)$$

Note that $\lambda_1 = c$ by (12). We can then expand equation (11) into

$$\frac{\partial f}{\partial t} = \lambda_1 D_o + \lambda_2 D_g, \quad (15)$$

where

$$D_o = \frac{f_x^2 f_{yy} - 2 f_x f_y f_{xy} + f_y^2 f_{xx}}{f_x^2 + f_y^2}, \quad (16)$$

and

$$D_g = \frac{f_x^2 f_{xx} + 2 f_x f_y f_{xy} + f_y^2 f_{yy}}{f_x^2 + f_y^2}, \quad (17)$$

are the second order directional derivatives of f in directions orthogonal and parallel to the local gradient, respectively. Since the two second-order directional derivatives are in orthogonal directions, equation (15) represents an orthogonal decomposition of anisotropic diffusion, weighted by the eigenvalues.

Since the diffusion coefficient is nonnegative due to (9), the first term of (15) represents a degenerate forward diffusion in the direction orthogonal to the gradient, which tends to preserve edges since an edge is also orthogonal to the gradient. In addition, this directional smoothing can be encouraged within smooth regions and discouraged near edges if the diffusion coefficient is set to be a decreasing function of the magnitude of the gradient.

The second term of (15) represents a degenerate diffusion in the direction of the local gradient. It is forward (smoothing) if $\lambda_2 > 0$, and backward (sharpening) otherwise. In order to sharpen edges while smoothing small variations in intensity, it is desirable to have an $F(||\nabla f||)$ such that

$$\lambda_2 = F''(||\nabla f||) \begin{cases} > 0, & ||\nabla f|| < T \\ < 0, & otherwise \end{cases}, \quad (18)$$

where T is again a threshold. Since

$$F''(||\nabla f||) = \phi'(||\nabla f||), \tag{19}$$

equation (18) is actually the same as (3) except that it is obtained in a 2-D context.

With the above results in mind, let us now address the problem of ill-posedness of the anisotropic diffusion. The evolution of (11) can be interpreted as a descent process on the surface of the energy function (8), so its behavior is dependent on the shape of this energy surface and the initial conditions where it starts. If the energy surface has a single global minimum, the diffusion process will converge to that minimum starting from any image. The anisotropic diffusion is then well-posed. However, if the energy surface is rough and has many local minima, the diffusion process, starting from images close to each other, is likely to be caught in different local minima. The anisotropic diffusion may then be interpreted as ill-posed because images close to each other may diverge during diffusion process.

The energy surface of (8) is determined by its integrand when the image domain Ω is given. If the integrand is convex, then (8) is convex [8]. Consequently, the anisotropic diffusion (11),starting from any image, converges to the global minimum. This global minimum might correspond to many images continuously connected to each other and having the same degree of smoothness as measured by (8). The convexity of the integrand of (8) is guaranteed if

$$\lambda_1 \geq 0 \;\; and \;\; \lambda_2 \geq 0 \quad for \;\; all \;\; ||\nabla f|| \;\; . \tag{20}$$

Since $\lambda_1 \geq 0$ is always true for all $||\nabla f||$ due to (9), then the energy function (8) is convex if

$$\lambda_2 = F''(||\nabla f||) \geq 0 \;\; for \;\; all \;\; ||\nabla f||. \tag{21}$$

Consequently, anisotropic diffusion (11), starting from any initial image, always converges to the global minimum. Note that (21) is exactly the same as (6) due to (13), but it is only a sufficient condition for 2-D diffusions.

Other cases of λ_2 would lead to complicated energy surfaces. A still rather simple case is

$$\lambda_2 = F''(||\nabla f||) \leq 0 \;\; for \;\; all \;\; ||\nabla f||. \tag{22}$$

Then the integrand of (8) has a unique saddle point. It seems likely that the integration of it over the domain Ω also has a unique saddle point. Consequently, anisotropic diffusion has no local minima to converge to, instead, it converges to a boundary point on either side of the saddle depending on its initial position on the saddle. Images that are close to each other will stay close during the diffusion process as long as they are not distributed across the saddle ridge. The anisotropic diffusion may again be regarded as well-posed. Simulations on computers tend to confirm this.

When eigenvalue λ_2 satisfies the thresholding property (18), as the flux function (2) has been proposed to satisfy by Perona and Malik [6], the integrand of (8) is likely to have many local minima so that the energy function (8) would be more complicated with many local minima. Consequently, images close to each other are likely to diverge during the diffusion process and fall in different local minima. The anisotropic diffusion is thus ill-posed. Surprisingly, the idea of thresholding must be abandoned if anisotropic diffusion is to be well-posed.

3. DESIGN

The first type of anisotropic diffusion schemes are those which satisfy the condition (21). Their energy surfaces are convex, and their convergence is guaranteed. However, since $\lambda_2 > 0$ corresponds to a forward diffusion (smoothing) in the gradient direction, if we do not set $\lambda_2 = 0$, it will induce smoothing across image edges. Setting $\lambda_2 = 0$, we have

$$F(||\nabla f||) = ||\nabla f||. \tag{23}$$

This corresponds to the anisotropic diffusion that represents a good trade-off between well-posedness and performance, that is, it always converges to the same global minimum from any initial image and it smoothes images only in directions along image edges.

The second type of anisotropic diffusions include those satisfying (22). An important advantage of this type is that it has a backward diffusion in the gradient direction, hence tends to give sharper image edges. But they only have saddle-point type of convergence behavior. One example from this type may be obtained by extending (23) to

$$F(||\nabla f||) = ||\nabla f||^{1/n}, \quad n > 1. \tag{24}$$

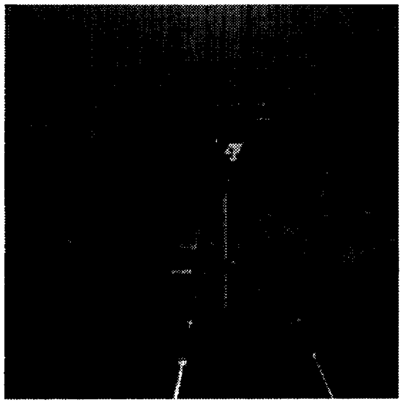
Other schemes in this type may be developed.

4. NUMERICAL SIMULATIONS

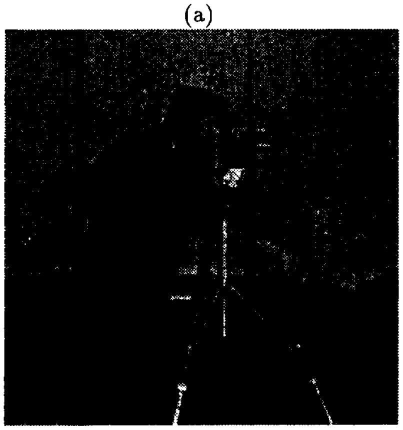
The original image in Figure 1(a) is degraded to generate Figure 1(b) by adding white Gaussian noise at SNR=10 dB, where SNR is defined by

$$SNR = \frac{Variance \;\; of \;\; image}{Variance \;\; of \;\; noise}. \tag{25}$$

The degraded image in Figure 1(b) is then used as initial conditions for anisotropic diffusion.



(a)



(b)

Figure 1: (a) Original cameraman image and (b) image degraded by white Gaussian noise at a SNR = 10 dB.

Figure 2(a), (b), and (c) are the scale-space images obtained using diffusion coefficient given by (4) with $K = 100$. It is obvious that noise is not removed. Figure 2(d), (e), and (f) shows the scale-space images obtained using (24) with $n = \frac{3}{2}$ (the corresponding diffusion coefficient $c(s) = \frac{2}{3}s^{-4/3}$). Noise is removed while edges are preserved.

5. CONCLUSION

Anisotropic diffusion is posed as a process of minimizing an energy function. Its global convergence behavior is determined by the shape of the energy surface, and its local behavior is described by an orthogonal decomposition with the decomposition coefficients being the eigenvalues of the local energy function. Consequently, the behavior analysis of anisotropic diffusion may be based on the eigenvalues of the Hessian matrix of the local energy function. A sufficient condition for its convergence to a global minimum is given and is identified to be the same as the condition previously proposed for the well-posedness of 1-D diffusions. The best diffusion scheme satisfying this sufficient condition is obtained. Other type of diffusion schemes which satisfy saddle-point type of well-posedness conditions are also proposed and are shown to be stable by simulation, but a strict analysis of well-posedness is not available. This constitutes one of our future tasks.

6. REFERENCES

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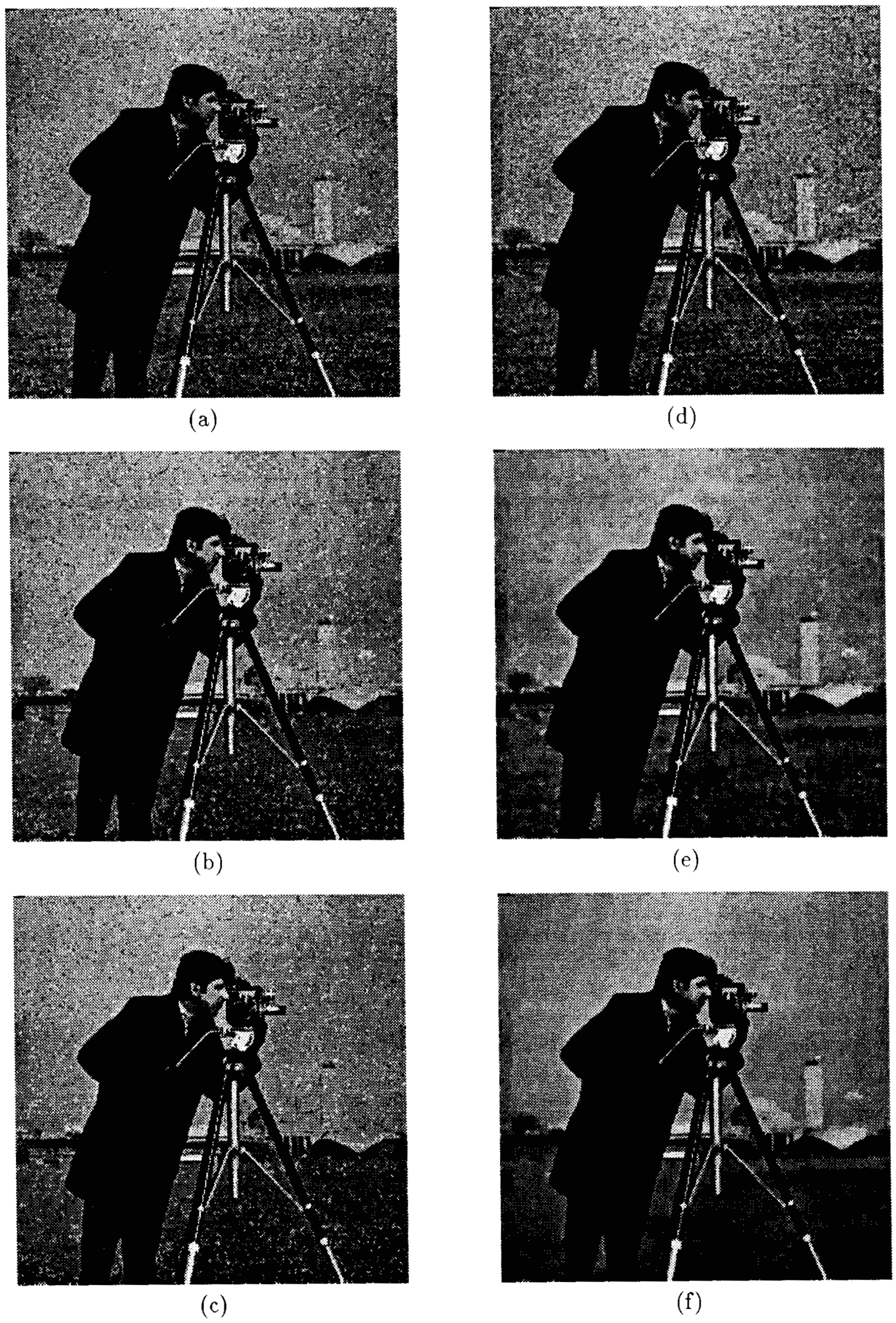

Figure 2: Scale spaces of Anisotropic Diffusion. Diffusion coefficient $c(s) = \exp^{-(s/k)^2}$ for images in the left column,(a) t=5, (b) t=45, and (c) t=150. Diffusion coefficient $c(s) = \frac{2}{3}s^{-4/3}$ for images in the right column, (d) t=10, (e) t=30, (f) t=50.