H∞ Optimal Controllers for a Distributed Model of an Unstable Aircraft

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Abstract
We discuss the numerical computation of H∞ optimal controllers for an unstable aircraft model with a time delay. We show that the optimal H∞ controller for this infinite dimensional model can be computed from a finite determinantal formula. The properties of the optimal controller are examined in this paper. For a particular choice of weights, we illustrate with an example that the optimal controller is stable and continuous. We examine approximations of this controller, and illustrate suboptimality of the controllers obtained from these approximations.

1 Introduction

In this paper, we design an H∞ optimal compensator for the control of an unstable aircraft model with a delay. In our design, we will work directly with the distributed model and not a finite dimensional approximation. Our technique is based on so-called skew Toeplitz theory which allows one to design H∞ controllers for (possibly) infinite dimensional systems [3].

In Section 2 of this paper, we will review the aircraft dynamics and show that a transfer function with one unstable pole and a time delay forms an abstract model of the aircraft, for the purpose of controlling the longitudinal dynamics in the short period.

We design a non-trivial H∞ controller for this infinite dimensional model. This is a mixed 2-block S (sensitivity) and CS (controller times sensitivity) minimization problem which is also related to the robustness optimization in the gap metric. For a specific choice of the weights, modelling the uncertainties, we obtain the optimal H∞ controller, which is infinite dimensional.

We discuss the issue of numerical computation of this controller from a finite determinantal formula. This part of the paper is based on our work [1].

In this paper we also discuss approximations of the optimal controller. For the model we consider here it is possible (see [2]) to obtain finite dimensional suboptimal controllers by appropriately approximating the optimal controller. This fact is illustrated via an example.

2 Abstract Model of an Unstable Aircraft

An aircraft rigid body has 6 degrees-of-freedom and thus 12 states. The forces and moments are independent of the direction the aircraft is traveling and the 2 horizontal position coordinates. The dependence of atmospheric properties on altitude can be neglected. This leaves 8 states which can be further decomposed into 2 sets of 4 states because of the aircraft's plane of symmetry. The motions in the plane (two translations and one rotation) do not cause forces and moments in the out-of-plane motions (one translation and two rotations). The in-plane forces and moments caused by out-of-plane motions can be neglected. Thus the 2 sets of 4 states each are decoupled.

The in-plane motions are of interest for this paper. The four states model three degrees-of-freedom and several choices are used in practice. The specific state variables to be used in this paper are the magnitude of the velocity vector denoted by V, the flight path angle denoted by γ, the angle-of-attack denoted by α, and the pitch angular rate denoted by q.

The in-plane motions also known as the longitudinal motions can be further decomposed by a time scale or frequency separation. The slowest motions or lowest frequency motions involve the exchange of gravitational potential energy and kinetic energy. This is an oscillatory motion which is lightly damped and has a frequency which is within a factor of three of g/V where g is the gravitational acceleration. This frequency is usually a decade below the frequencies of interest in this paper. The states associated with this so-called phugoid motion are V and γ and will be assumed to be constant in the following.

The high frequency longitudinal motion is known as the short period. In this special case only one force and one moment will be of interest. The force is in the direction normal to the velocity and positive down. The moment is the pitching moment which is positive for nose up. The two variables which describe the motion of the aircraft are α and q. The equations of motion are given by

\[ mV_\gamma = mV(q - \dot{\alpha}) = L(\alpha, \delta, q) - mg \cos \gamma \]

\[ I_{\alpha}q = M_{\alpha}(\alpha, \delta, q) \]

where m is the mass of the aircraft, \( I_{\alpha} \) is the pitch axis moment of inertia, \( L \) is the aerodynamic lift force, \( M_{\alpha} \) is the aerodynamic pitching moment, and the propulsive force normal to the velocity vector and propulsion pitching moment have been neglected. The mass and moment of inertia are assumed to be constant. The aerodynamic terms depend on angle-of-attack, the control surface deflection denoted by δ, and the pitch rate.

For control development, these equations are linearized about the straight and level equilibrium condition where \( \alpha = q = 0 \) and \( \gamma = 0 \). The equilibrium condition is given by \( (\alpha, \delta) = (\alpha_0, \delta_0) \) such that \( L(\alpha_0, \delta_0, 0) = mg \) and \( M_{\alpha}(\alpha_0, \delta_0, 0) = 0 \).
After linearization the equations can be written
\[ \dot{\delta}_1 = Z_1 \alpha_1 + (1 + Z_1)q_1 + Z_p \delta_1, \]
\[ \dot{q}_1 = M_p \alpha_1 + M_p \dot{q}_1 + M_p \delta_1, \]
where the subscript 1 denotes a perturbation from the equilibrium condition. That is \( \alpha = \alpha_0 + \alpha_1, q = q_1, \) and \( \delta = \delta_1 + \delta_1. \)

The terms in the linearized equations arising from the partial derivatives of the aerodynamic functions are known as stability derivatives and are given by
\[ Z_1 = -\frac{1}{mV} \frac{\partial L}{\partial \alpha}(\alpha_0, \delta_0, 0), \]
\[ M_1 = \frac{1}{l_{eq}} \frac{\partial M_{arc}}{\partial \alpha}(\alpha_0, \delta_0, 0), \]
and similarly for \( Z_2, Z_3, M_3, M_4. \)

Typically the terms \( Z_3 \) and \( Z_4 \) are negligible and will be taken to be zero in the following. \( M_4 \) is an important term because it determines stability. This stability derivative can be positive or negative depending on the location of the aircraft center of mass. \( M_4 \) increases as the center of mass moves aft. The stability derivatives \( Z_1 \) and \( M_2 \) are always negative and provide damping. The characteristic equation is given by
\[ s^2 - (Z_0 + M_0)s - M_0 + Z_0 M_4 = 0, \]
thus if \( M_0 > Z_0 M_4 > 0, \) the characteristic equation can be factored as
\[ (s - a)(s + p) = 0 \]
where \( a > 0 \) is the positive root and \( -p < 0 \) is the negative root. Typically the aircraft is controlled by measuring pitch rate with a gyroscope and rate of change of flight path rate with an accelerometer. A common approach is to blend these two variables together to form a variable for feedback called
\[ y = q + K \dot{q} \]
and \( K > 0 \) is a constant and the subscript 1 (for linearization) was suppressed. In this case the transfer function from the control surface to the feedback variable is
\[ y(s) = M(s - Z_0(1 + K)) \]
\[ e(s) \]
Note that this transfer function has one negative zero because \( Z_0 < 0 \) and \( K > 0. \)

The next approximation is to assume that \( K \) is such that \( Z_0(1 + K) = -p. \) Then
\[ y(s) = M_2 \]
\[ e(s) = (s - a) \]

In the rigid aircraft which was just discussed, we will also consider the effects of the elastic correction, actuator, sensor, and computer. More precisely, the elastic correction is a high order transfer function with zeros and poles all larger than a frequency given by \( \omega_{elastic}. \) These poles and zeros will be located near the \( j \omega \) axis of the \( s \)-plane. It is reasonable to assume the poles have negative real parts but the zeros may have positive real parts depending on sensor and actuator locations relative to mode shapes for the elastic modes. The elastic effects can be compensated to some extent with control law elements such as notch filters. However, some effects such as those due to right-half plane zeros cannot be compensated without introducing effective time delay. Thus there is an effective delay in the feedback loop because of airframe elasticity.

An actuator is usually modeled for the purpose of control law design with a transfer function. The transfer function has a low pass characteristic with one to four poles typically. The poles all have magnitudes of at least a factor of three above the short period unstable pole. The actuator can be compensated by the control law to some extent but there will always remain some effective time delay in the feedback loop.

A sensor is usually modeled for the purpose of control law design with a transfer function. The transfer function has a low pass characteristic with one or two poles typically. The poles all have magnitudes of at least a factor of ten above the short period unstable pole. Typically the control law does not compensate for the sensor dynamics. Thus the sensor contributes effective time delay to the feedback loop.

Today a digital computer is typically used to implement the control law with a zero-order-hold and sampling function. The zero-order-hold function is well approximated by a pure time delay of one half the sample time of the digital implementation. There is also a pure time delay associated with the time it takes to perform the computations for the next actuator command update. Another source of delay associated with digital control is the prefiltering needed for anti-aliasing. It is not possible for the control law to compensate for the delays associated with the digital implementation. Thus the computer contributes time delay to the feedback loop.

The abstract model is now assembled from the pieces discussed above. The operation required is to sum the effective time delays associated with the individual components. The abstract model is given by
\[ P(s) = \frac{e^{-sh}}{\sigma a - 1} \]
where \( h \) is the total time delay in the feedback loop, and \( a = 1/\sigma, \) is the unstable pole; we have normalized \( M_0/a, \) so that \( |P(0)| = 1. \) The total time delay is given by
\[ h = h_{elastic} + h_{actuator} + h_{sensor} + h_{prefilter} + 1/2T_s + T_{cd} \]
where \( h_{elastic} \) is the effective delay associated with the airframe elasticity, \( h_{actuator} \) is the effective time delay associated with the actuator, \( h_{sensor} \) is the effective time delay associated with the sensor, \( h_{prefilter} \) is the effective time delay associated with the prefilter, \( T_s \) is the sample time, and \( T_{cd} \) is the computational delay.

The effective delays for elasticity, actuator, sensor, and prefilter are obtained by a limiting operation for low frequencies. Let \( G_i(s) \) for \( i = \) elastic, actuator, sensor, prefilter, denote an element of the feedback loop and let \( K_i(s) \) for \( i = \) elastic, actuator, sensor, prefilter, denote compensation if used. Note that \( K_i(s) = 1 \) for \( i = \) sensor, prefilter. Then the effective time delay is obtained from
\[ h = \lim_{\omega \to 0} \frac{1}{\omega} \arg(G_i(j\omega)K_i(j\omega)). \]
As an example, consider a first order low pass \( G_i(s)K_i(s) = \frac{1}{1+s}, \) then the effective delay is given by
\[ h_{effective} = \lim_{\omega \to 0} \frac{1}{\omega} \arg\left(\frac{1}{b} - \frac{1}{1 + j\omega}\right) = \lim_{\omega \to 0} \arctan\frac{\omega}{b} = \frac{1}{b} \]
where \( b > 0 \) is the break frequency of the first order low pass. The final abstract model can be mathematically parameterized as a one parameter family because it is only the product of the unstable pole and total time delay \( ah = h/\sigma \) that determines the difficulty of the control problem. The control problem difficulty increases with \( h/\sigma. \) The X-29 aircraft at its most unstable flight condition has a product of unstable pole and effective time delay \( h/\sigma \) as large as 0.37, the other conditions being as much as a factor of 6 smaller.
3 $H^\infty$ Control of the Unstable Aircraft Model

In the rest of this paper we will discuss the controller design for the abstract model (1) of an unstable aircraft. In particular we consider the following $2$-block $H^\infty$ control problem, which is related to the problems of stabilization, robustness to unmodelled dynamics and sensitivity reduction:

$$
\mu := \inf_{C \text{ stabilising}} \left\| W_1(1 + PC)^{-1} \right\|_{\infty},
$$

where $W_1$ and $W_2$ are weights describing desired sensitivity reduction and unmodelled dynamics respectively. Our purpose is to find the optimal performance level $\mu$ and the corresponding optimal controller $C_{opt}$, given data $P, W_1$ and $W_2$. We consider the following first order weights

$$W_1(s) = \frac{\alpha}{\rho} \left(\frac{1 + s\rho}{1 + \tau s}\right)^2 + \rho^2,$$

the optimal controller and the optimal performance are computed from (1) using the formulae given in [3]. The final results are summarized below, see the above references for complete details.

The optimal performance $\mu$ is given by

$$\mu = \frac{\alpha}{\rho \sqrt{1 + \rho^2}} \left(\frac{1 - \lambda}{1 + \lambda \sigma}\right)^2 + \rho^2,$$

where the quantity $\mu'$ is to be defined below.

We use the conformal map $z = \frac{\omega_f}{\omega_0}$ between the unit disk $D$ and the right half plane $RH P$, and define the inner function

$$m_\omega(z) = \frac{a_1 - z}{a_1 + z},$$

Let $a_2 = \frac{1 + \rho^2}{1 + \rho^2}$, and

$$t = \left(1 + \rho^2\right)^2 \frac{1}{\sigma} \frac{\rho}{\sqrt{1 + \rho^2}} \left(1 + \rho^2\right)^2 \Phi.$$
The Nyquist plot for this controller is also provided, see Figure 2, where the dotted line corresponds to the values of $P(j\omega)C_{opt}(j\omega)$ for negative values of $\omega$ and the solid line corresponds to positive values of $\omega$; the arrows show the increasing direction of $\omega$. Note that the critical point $-1$ is encircled once in the counterclockwise direction. Since by design $C_{opt}$ stabilizes the nominal plant, and $P$ has one unstable pole we deduce from this Nyquist plot that the optimal controller is stable.

It is important to remark that approximations of the $H^\infty$ optimal controller yield suboptimal controllers for the problem considered here, see [2]. To illustrate this point we look at the performance of the first and second order controllers obtained from $C_{opt}$:

$$C_1(s) = 2.141 \left( \frac{s + 0.2242}{s + 0.1} \right) \quad \text{and} \quad C_2(s) = C_1(s) \left( \frac{s/25 + 1}{s/32 + 1} \right).$$

Figure 3 shows the approximation errors $|C_{opt}(\omega) - C_i(\omega)|$, $i=1,2$. We deduce from Figure 4 that $C_1$ and $C_2$ stabilize the closed loop system with the infinite dimensional plant $P$. Moreover Figure 5 shows the performances $\|\Psi_1\|_{\infty} = 0.602$ and $\|\Psi_2\|_{\infty} = 0.5909$, where

$$\Psi_i(\omega) = \sigma_{max} \left( W_i(\omega)(1 + P(\omega)C_i(\omega))^{-1} \right).$$

So, we conclude that for $C_1$ deviation from the optimum performance $\mu = 0.5839$ is 3.09 percent, and this deviation is 1.19 percent for $C_2$.

Finite dimensional controllers can also be obtained by approximating the infinite dimensional plant directly. These controllers can then be checked if they do indeed stabilize the original system, and in such a case the corresponding performance can be compared to the optimum. Of course, even in such a method, it is very important to a priori compute the optimal, in order to have the standard with which to compare the putative suboptimal compensator.

Using state space methods via the new, powerful $\mu$-tools package of MATLAB, our dear friend and colleague Professor Gary Balas (President and CEO of MUSYN Inc.) performed the following amazing computations for us. A first order approximation of the unstable delay system gives the second order controller

$$C_{d2}(s) = \frac{2.34(s + 0.27)}{(0.00028 + 1)(s + 0.1)}.$$

A second order approximation of the unstable delay system (obtained via a first order Padé approximation of the delay) yields the third order controller

$$C_{d3}(s) = \frac{2.761(s + 20)(s + 0.23)}{(0.000588 + 1)(s + 26)(s + 0.1)}.$$

(Note that $W_1$ is first order with a pole at 0.1 and $W_2$ is approximately constant. Therefore the controller order is one greater than that of the plant.)

One can check that in fact each of the above controllers stabilize the original delay system (see Figure 6). One may also compute that the suboptimal performances of the second and third order controllers are 0.6627 and 0.5936, respectively. (See Figure 7.) Recall that the optimal performance is 0.5839. Finally, note the similarity between the controllers $C_1$ and $C_{d2}$, and between $C_2$ and $C_{d3}.$

5 Concluding Remarks

In this note, we have applied an $H^\infty$ control design to a two parameter abstract model of an unstable aircraft. The $H^\infty$ control problem considered here is a mixture of the problems of sensitivity reduction and robustness to additive unstructured perturbations in the plant model.

The skew Toeplitz method which we utilized gave the optimal compensator whose performance we were able to explicitly study as a function of the various system parameters. The advantage of using such a method is that it allows one to work directly with the delay, and moreover gives explicit numerical results in a straightforward rigorous way without the need for any approximations. For the specific example studied in this note, we have obtained equally good performance results either by approximating the optimal infinite dimensional controller, or by approximating the plant. (We do not claim this for a general fact!)

References


\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{a) Magnitude and b) Phase of $C_{opt}(j\omega)$}
\end{figure}
Figure 2: Nyquist plot for $C_{np}(j\omega)P(j\omega)$

Figure 6: Nyquist plot for $C_{np}(j\omega)P(j\omega)$, solid line $i = 2$, dashed line $i = 3$.

Figure 3: $|C_{np}(j\omega)|$, solid line $i = 1$, dashed line $i = 2$.

Figure 7: $|\Phi_{np}(j\omega)|$, solid line $i = 2$, dashed line $i = 3$.

Figure 4: Nyquist plot for $C_{np}(j\omega)P(j\omega)$, solid line $i = 1$, dashed line $i = 2$.

Figure 5: $|\Phi_{np}(j\omega)|$, solid line $i = 1$, dashed line $i = 2$. 